Pascal's Triangle, Pascal's Pyramid, and the Trinomial Triangle

Antonio Saucedo Jr.

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Pascal’s Triangle, Pascal’s Pyramid, and the Trinomial Triangle

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Antonio Saucedo Jr.

June 2019
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June 2019

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Many properties have been found hidden in Pascal’s triangle. In this paper, we will present several known properties in Pascal’s triangle as well as the properties that lift to different extensions of the triangle, namely Pascal’s pyramid and the trinomial triangle. We will tailor our interest towards Fermat numbers and the hockey stick property. We will also show the importance of the hockey stick properties by using them to prove a property in the trinomial triangle.
Acknowledgements

Foremost, I would like to thank Dr. Joseph Chavez for introducing me to the material discussed on this paper, proofreading the material, and for his patience and support throughout both my undergraduate and graduate career. When I first took a class with Dr. Chavez, he quickly became one of my favorite professors due to his interest in combinatorics. The techniques and skills I learned through his problem solving class not only helped me through this paper, but also throughout my classes required to graduate. I would also like to thank Dr. Rolland Trapp and Dr. Belisario Ventura for proofreading and commenting on the paper.
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Chapter 1

Introduction

The main goal of this paper is to see which patterns in Pascal’s triangle apply to extensions of Pascal’s triangle, such as Pascal’s pyramid and the trinomial triangle. We will give an emphasis on where Fermat numbers appear in Pascal’s triangle and its extensions as well as the existence of hockey sticks. Pascal’s triangle is an infinite array of numbers which is constructed by beginning with the number 1. To obtain a new row, we must sum the two adjacent values of the preceding row. Repeating this algorithm eight times, we can create the triangle in Figure 1.1 [Tuc12].

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
\end{array}
\]

Figure 1.1: Pascal’s Triangle

The entries in Pascal’s triangle represent the coefficients of the binomial expansion \((a+b)^n\), where \(a, b\) are any real numbers and \(n\) is any nonnegative integer. Expanding
the first few cases gives the following
\[
(a + b)^0 = 1 \\
(a + b)^1 = a + b \\
(a + b)^2 = a^2 + 2ab + b^2 \\
(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.
\]
When \(n = 0\), the coefficient of the binomial expansion is the initial row of Pascal’s triangle, which we’ll call the 0\(^{th}\) row. If \(n = 1, 2, 3, 4\), the coefficients of each binomial expansion yield the first, second, third, and fourth row, respectively. Furthermore, using similar notation introduced in [Tuc12], the entries in Pascal’s triangle can be expressed by
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n, k \in \mathbb{Z}_{\geq 0}. \tag{1.1}
\]
\((n\ k)\) is the entry in the \(n^{th}\) row and \(k^{th}\) column of Pascal’s triangle.

Pascal’s Identity (1.2) was also introduced in [Tuc12] and states that any entry in Pascal’s triangle is the sum of the two adjacent values in the preceding row:
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad n, k \in \mathbb{Z}_{\geq 0}. \tag{1.2}
\]
Pascal’s identity plays a major roll in proving many properties in Pascal’s triangle. It is a key factor to completing some induction proofs in the triangle. Also,
\[
\binom{r}{0} = \binom{r}{r} = 1, \quad r \in \mathbb{Z}_{\geq 0}. \tag{1.3}
\]
Using the notation in (1.1), Pascal’s triangle may be expressed in the following form.
Figure 1.2: Pascal’s Triangle in Combinatorial Notation

In Chapter 2, we will start by looking at elementary properties in Pascal’s Triangle and later introduce some which may not be well known to some readers. In Chapters 3 and 4, we will look at which patterns in Pascal’s triangle extend to Pascal’s pyramid and the trinomial triangle, respectively.
Chapter 2

Pascal’s Triangle

2.1 Elementary Properties

We will begin by mentioning several well known patterns hidden in Figure 1.1. Firstly, the triangle is symmetrical about the vertical axis. That is, the numbers on the left side of the triangle have matching numbers on the right side. Also, the first diagonal is just 1’s, the second diagonal is known as the natural numbers, the third diagonal is the triangular numbers, the fourth diagonal is the tetrahedral numbers, and so on.

The existence of powers of 2 appear in each row of Pascal’s triangle [Pol09]. That is, the sum of row \( n \) is \( 2^n \). Consider the sum of the fourth row

\[
1 + 4 + 6 + 4 + 1 = 16 = 2^4.
\]

Given any row \( n \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad n, k \geq 0. \tag{2.1}
\]

Proof. Equation (2.1) can be easily proven by mathematical induction, and Pascal’s identity (1.2) plays a major roll in completing the proof. The base case, when \( n = 0 \), is obvious since \( 2^0 = 1 \). Now assume the equation is true for \( n = l \), that is \( \sum_{k=0}^{l} = 2^l \). Our goal is to show the equation is true for \( n = l + 1 \). Consider

\[
\binom{l+1}{0} + \binom{l+1}{1} + \binom{l+1}{2} + \ldots + \binom{l+1}{l+1}.
\]

Using Pascal’s identity and the fact that \( \binom{l+1}{0} = \binom{l}{0} \) and \( \binom{l+1}{l+1} = \binom{l}{l} \), we can rewrite the
expression as
\[
\binom{l}{0} + \binom{l}{1} + \binom{l}{1} + \cdots + \binom{l}{l-1} + \binom{l}{l} + \binom{l}{l} = 2 \left[ \binom{l}{0} + \binom{l}{1} + \cdots + \binom{l}{l} \right].
\]

Notice the sum is what we assumed to be true. Therefore, the expression is
\[
2 \cdot 2^l = 2^{l+1}.
\]

This completes the induction proof and thus (2.1) holds.

Polya introduced the sum of the squares of each entry in row \( n \) is equal to the middle entry of row \( 2n \) in [Pol09]. That is,
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \quad n, k \geq 0.
\] (2.2)

Consider the sum of the squares of the entries in row 4
\[
\sum_{k=0}^{4} \binom{4}{k}^2 = 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70.
\]

Also,
\[
\binom{8}{4} = \frac{8!}{4! \cdot 4!} = 70.
\]

An illustration of this example is shown in Figure 2.1.

Figure 2.1: Example of (2.2) for \( n = 4 \)
We can show (2.2) is true by using the block walking technique, the idea of this proof is shown by Polya in [Pol09] For simplicity, we will outline the proof by considering the specific example of the sum of the squares of row 2. By (2.2) we know the sum is \( \binom{2}{2} = 6 \).

Using the block walking technique, suppose we start at the top of the triangle, meaning the 0\(^{th}\) row. Our goal is to end up at 6. Figure 2.2 (a) shows 3 ways of walking from the 0\(^{th}\) row to 6. Since Pascal’s triangle is symmetrical about the vertical axis, we know there exists 3 more paths in the opposite direction which is shown in Figure 2.2 (b).

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 4 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 4 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

(a) \hspace{2cm} (b)

Figure 2.2: Block Walking Example

Notice these paths combined create an outer path of a diamond with the middle row being row 2. That is, to land on the entry 6 we must cross each entry in row 2. Observe, there is only one way to get to the entry 1 in row 2, there are two ways to get to the entry 2 in row 2, and there is only one way to get to the last entry 1 in row 2. Since the diamond is symmetrical about the horizontal axis, there is only one way to get to the entry 6 from entry 1 of row 2, there are two ways to get to entry 6 from entry 2 of row 2, and there is only one way to get to entry 6 from the last entry 1 of row 2. In all, there are \( 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 6 \) total ways of walking from the 0\(^{th}\) row to entry 6.

This can be generalized as follows: starting with the 0\(^{th}\) row, there are \( \binom{n}{0} \) ways to get to entry \( \binom{n}{0} \) of row \( n \), \( \binom{n}{1} \) ways to get to entry \( \binom{n}{1} \) of row \( n \), ..., and \( \binom{n}{n} \) ways to get to entry \( \binom{n}{n} \) of row \( n \). Since the diamond shape is symmetrical about the \( n^{th} \) row, there are \( \binom{n}{0} \) ways to get to entry \( \binom{2n}{n} \) from the entry \( \binom{n}{0} \) of row \( n \), \( \binom{n}{1} \) ways to get to entry \( \binom{2n}{n} \) from entry \( \binom{n}{1} \) of row \( n \), ..., and \( \binom{n}{n} \) ways to get to entry \( \binom{2n}{n} \) from entry \( \binom{n}{n} \) of row \( n \). In all, there are \( \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 \) ways to get to entry \( \binom{2n}{n} \) from the 0\(^{th}\) row.

Also hidden in each row are the powers of 11. Consider each entry in row \( n \)
being in its corresponding place value, that is, ones, tens, hundreds, and so on. Since place values are just multiples of 10, we can easily calculate each row as follows:

row 0: \(1 \cdot 10^0 = 1 = 11^0\)

row 1: \(1 \cdot 10^1 + 1 \cdot 10^0 = 11 = 11^1\)

row 2: \(1 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 = 121 = 11^2\)

row 3: \(1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + 1 \cdot 10^0 = 1331 = 11^3\).

This property applies to each row in Pascal’s triangle. Koshy introduces a formula in [Kos11] which yields \(11^n\) for any row \(n\) by

\[
\sum_{k=0}^{n} \binom{n}{k} \cdot 10^{n-k} = 11^n, \quad n \geq 0. \tag{2.3}
\]

The square of any entry in diagonal 2 is equal to the sum of the two adjacent entries on the right [Kos11]. That is,

\[
\binom{n}{1}^2 = \binom{n}{2} + \binom{n+1}{2}, \quad n \geq 1. \tag{2.4}
\]

As an example, say \(n = 5\). From (2.4),

\[
\binom{5}{1}^2 = 5^2 = \binom{5}{2} + \binom{6}{2} = 25.
\]

Figure 2.3 shows which entries are being considered for \(n = 5\).
The Star of David centered at \( \binom{n}{k} \) is another fascinating pattern introduced in [HP87], which states

\[
\binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1} = \binom{n-1}{k-1} \binom{n}{k} \binom{n+1}{k}.
\] (2.5)

In Figure 2.4, we see the Star of David centered at \( \binom{5}{2} \). Using (2.5),

\[
\binom{4}{2} \binom{5}{1} \binom{6}{3} = \binom{4}{1} \binom{5}{3} \binom{6}{2}.
\]

Which simplifies to

\[
6 \cdot 5 \cdot 20 = 4 \cdot 10 \cdot 15.
\]

Notice both products equal 600. We can show this is always true by starting with the left-hand side of (2.5) and rearranging the denominators of the products as follows:

\[
\binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1} = \frac{(n-1)!}{k!(n-1-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{(n+1)!}{k!(n-k+1)!} = \binom{n-1}{k-1} \binom{n}{k} \binom{n+1}{k}.
\]

---

**Boscarol’s identity**, as stated in [Kos11], is given by

\[
\sum_{k=0}^{i} \frac{\binom{n+k}{k}}{2^{n+k}} + \sum_{l=0}^{n} \frac{\binom{n+i-l}{i}}{2^{n+i-l}} = 2.
\] (2.6)
Beginning at any point of the form \( \binom{n}{0} \), (2.6) will hold. Consider the path in Figure 2.4, using Boscarol’s identity, we have:

\[
\frac{1}{2^3} + \frac{4}{2^4} + \frac{10}{2^5} + \frac{20}{2^6} + \frac{35}{2^7} + \frac{35}{2^7} + \frac{15}{2^5} + \frac{5}{2^4} + \frac{1}{2^4} = 2.
\]

![Figure 2.5: Boscarol’s Identity Example](image)

The last property we’ll discuss is the teardrop property, which states the sum of all the entries in a parallelogram is equal to one less than the entry below it [Pol09]. Notice the sum of the entries in the parallelogram in Figure 2.6 is equal to 5, which is one less than 6.

![Figure 2.6: Teardrop Example](image)
2.2 Fermat Numbers

Let us explore some patterns which may not be well known to some readers. We will begin by looking at the existence of Ferrmat numbers [Hew77], which are of the form

\[ F(m) = 2^{2^m} + 1, \quad m \in \mathbb{Z}_{\geq 0}. \]  

Clearly, the first few Fermat numbers are

\[ F(0) = 3 \]
\[ F(1) = 5 \]
\[ F(2) = 17 \]
\[ F(3) = 257. \]

We can find these numbers hidden in Pascal’s triangle, to do so we must consider each entry in Pascal’s triangle modulo 2.

\[ \begin{array}{cccccccc}
\text{row } 0 & & & & & & & 1 \\
\text{row } 1 & & & 1 & & & 1 & 1 \\
\text{row } 2 & & 1 & & 0 & & 1 & \\
\text{row } 3 & & & 1 & & 1 & & 1 & 1 \\
\text{row } 4 & & & & 1 & & 0 & & 0 & 1 \\
\text{row } 5 & & & & & 1 & & 1 & & 0 & 0 & 1 & 1 \\
\text{row } 6 & & & & & & 1 & & 0 & 1 & 0 & 1 & 0 & 1 \\
\text{row } 7 & & & & & & & 1 & & 1 & 1 & 1 & 1 & 1 & 1 \\
\text{row } 8 & & & & & & & & 1 & & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]

Figure 2.7: Pascal’s Triangle Modulo 2

As shown in [Hew77], if we consider each row a binary number, we are able to translate to decimal notation. For simplicity, let

\[ a(n) = \sum_{k=0}^{n} \binom{n}{k}^* \cdot 2^{n-k} \]  

be the sequence of numbers constructed when row \( n \) is translated from binary to decimal. In this equation, \( \binom{n}{k}^* \) is the entry in row \( n \) and column \( k \) of Pascal’s triangle modulo 2.
For example, using (2.8) to translate row 5 yields the following:

\[ a(5) = \sum_{k=0}^{5} \binom{5}{k} \cdot 2^{5-k} \]

\[ = \binom{5}{0} \cdot 2^5 + \binom{5}{1} \cdot 2^4 + \binom{5}{2} \cdot 2^3 + \cdots + \binom{5}{5} \cdot 2^0 \]

\[ = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \]

\[ = 51. \]

The figure below shows rows 0 – 8 of Figure 2.8 translated to decimal using equation (2.8).

<table>
<thead>
<tr>
<th>Row</th>
<th>Binary</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>101</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1111</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>10001</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>110011</td>
<td>51</td>
</tr>
<tr>
<td>6</td>
<td>1010101</td>
<td>85</td>
</tr>
<tr>
<td>7</td>
<td>11111111</td>
<td>255</td>
</tr>
<tr>
<td>8</td>
<td>100000001</td>
<td>257</td>
</tr>
</tbody>
</table>

Figure 2.8: Decimal Representation of Pascal’s Triangle Modulo 2

The decimal representation of the binary numbers in rows 1, 2, 4, and 8 are Fermat numbers. That is, if we translate row 2\(^m\) to decimal notation, the result is \(F(m)\). Let us look closely at the decimal representations from Figure 2.8, excluding row 0.
Each value is either a Fermat number or a product of Fermat numbers. As we can see, the gap between each Fermat number increases rapidly. Consider two consecutive Fermat numbers $F(m)$ and $F(m + 1)$, which appear in rows $2^m$ and $2^{m+1}$, respectively. The gap between these two rows contains each product of $F(m)$ with each value of every row before $F(m)$, excluding row 0, in consecutive order. For example, $F(3) = 257$ and $F(4) = 65,537$. Between the gap of these two numbers exists $3 \cdot F(3)$, $5 \cdot F(3)$, $15 \cdot F(3)$, $17 \cdot F(3)$, $51 \cdot F(3)$, $85 \cdot F(3)$, and $255 \cdot F(3)$, respectively. Figure 2.9 shows the binary and decimal representation of rows 8 – 16 in Pascal’s triangle modulo 2.

<table>
<thead>
<tr>
<th>Row</th>
<th>Binary</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>100000001</td>
<td>257 = $F(3)$</td>
</tr>
<tr>
<td>9</td>
<td>1100000011</td>
<td>771 = 3 · 257</td>
</tr>
<tr>
<td>10</td>
<td>10100000101</td>
<td>1285 = 5 · 257</td>
</tr>
<tr>
<td>11</td>
<td>111100001111</td>
<td>3855 = 15 · 257</td>
</tr>
<tr>
<td>12</td>
<td>1000100010001</td>
<td>4369 = 17 · 257</td>
</tr>
<tr>
<td>13</td>
<td>11001100110011</td>
<td>13107 = 51 · 257</td>
</tr>
<tr>
<td>14</td>
<td>101010101010101</td>
<td>21845 = 85 · 257</td>
</tr>
<tr>
<td>15</td>
<td>11111111111111111</td>
<td>65535 = 255 · 257</td>
</tr>
<tr>
<td>16</td>
<td>100000000000000001</td>
<td>65537 = $F(4)$</td>
</tr>
</tbody>
</table>

Figure 2.9: Decimal Representation of Rows 8-16 in Pascal’s Triangle Modulo 2

[Hew77] nicely tied the relationship between the rows in Pascal's triangle modulo 2 and the Fermat numbers. Since each row is a Fermat number or a product of Fermat
numbers, we can let
\[ b(n) = F(m^{j_0} \cdot F(m - 1)^{j_1} \cdots \cdot F(0)^{j_m}) \] (2.9)
be the number found in row \( n \), translated to decimal, where \( n = j_0j_1 \ldots j_m \) is in binary expansion. Using (2.9) we can find any decimal representation of row \( n \) in terms of Fermat numbers. For example, the number in row \( n = 5 \), that is \( n = 101 \) can be found by
\[
\begin{align*}
    b(101) &= F(2)^1 \cdot F(1)^0 \cdot F(0)^1 \\
          &= 17^1 \cdot 5^0 \cdot 3^1 \\
          &= 51.
\end{align*}
\]
Notice \( n = 101 \) implies \( j_0 = 1, j_1 = 0, \) and \( j_2 = 1 \) and therefore \( m = 2 \).

### 2.3 Hockey Stick

The hockey stick derived its name from the interesting path it creates in the triangle. There are several hockey sticks that are found in Pascal’s triangle. The following three theorems are shown in [CL18].

**Theorem 2.1. Hockey Stick Theorem**

\[
\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}, \quad n \geq 0.
\]

As an example, consider the case when \( n = 2 \) and \( k = 3 \). Then,
\[
\sum_{i=0}^{3} \binom{2+i}{i} = \binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3}.
\]
Simplifying yields the following:
\[
1 + 3 + 6 + 10 = 20 = \binom{6}{3}.
\]
The path for this example is shown in Figure 2.10.
Theorem 2.2. The Little Hockey Stick and Puck Theorem

\[ \binom{n}{0} + \binom{n + 2}{1} + \binom{n + 4}{2} = \binom{n + 5}{2} - \binom{n + 5}{0}, \quad n \geq 0. \]

If \( n = 2 \), then

\[ \binom{2}{0} + \binom{4}{1} + \binom{6}{2} = 20. \]

Also,

\[ \binom{7}{2} - \binom{7}{0} = 20. \]

The path used in this example is shown in Figure 2.11.
Theorem 2.3. The Big Hockey Stick and Puck Theorem

\[
\binom{n}{0} + \binom{n+2}{1} + \binom{n+4}{2} + \binom{n+6}{3} = \binom{n+7}{3} - \binom{n+6}{1}, \quad n \geq 0.
\]

If \( n = 1 \), we have

\[
\binom{1}{0} + \binom{3}{1} + \binom{7}{3} = 49.
\]

Also,

\[
\binom{8}{3} - \binom{7}{1} = 49.
\]

A path for the Big Hockey Stick and Puck Theorem, for \( n = 1 \), is shown in Figure 2.12.

![Figure 2.12: The Big Hockey Stick and Puck Example](image)

A general representation of Theorems 2.2 and 2.3 is shown in (2.10) [CL18]. For any entry of the form \( \binom{n}{0} \) in Pascal’s triangle, the following holds

\[
\sum_{i=0}^{k} \binom{n+2i}{i} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{n+2k-j+1}{k-2j}.
\] (2.10)

Suppose we begin the hockey stick at \( \binom{1}{0} \) and we limit \( k \) to 5. Using equation (2.10), we have

\[
\sum_{i=0}^{5} \binom{1+2i}{i} = \sum_{j=0}^{2} (-1)^j \binom{12-j}{5-2j}.
\]

Expanding the summation yields

\[
\binom{1}{0} + \binom{3}{1} + \binom{5}{2} + \binom{7}{3} + \binom{9}{4} + \binom{11}{5} = \binom{12}{5} - \binom{11}{3} + \binom{10}{1}.
\]
Simplifying,

\[1 + 3 + 10 + 35 + 126 + 462 = 792 - 165 + 10.\]

A path for this hockey stick and puck example is shown in Figure 2.11. A sketch for the proof of (2.10) can be found in [CL18].

Figure 2.13: Hokey Stick Example
Chapter 3

Pascal’s Pyramid

3.1 Introduction

Pascal’s pyramid is a three dimensional extension of Pascal’s triangle which is generated by starting with a 1 on the $0^{th}$ layer; a new layer is obtained by adding the three adjacent values in the previous layer. Repeating this algorithm four times will generate the following pyramid [HP12].

![Figure 3.1: Pascal’s Pyramid](image)

To help visualize, we will partition the tetrahedron into layers. Note the $0^{th}$ layer has a value of 1, which we will omit.
The entries in layer $n$ represent the coefficients of the expansion $(a + b + c)^n$, where $a$, $b$, $c$ are real numbers and $n$ is any non negative integer. Expanding the first few cases gives the following

$$(a + b + c)^0 = 1$$
$$(a + b + c)^1 = a + b + c$$
$$(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$
$$(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3acb^2 + 3ac^2 + 3bc^2 + c^3.$$

Notice the coefficients match the entries in each layer. They may be easily viewed below.

From [CO99], we can construct any layer in Pascal’s pyramid. To obtain layer $n$, we must first construct Pascal’s triangle up to row $n$. We then write row $n$ as a column, next to Pascal’s triangle and multiply each entry in the column to the corresponding row in Pascal’s triangle. For example, to obtain layer 5 we first write down rows 0 – 5 of Pascal’s triangle, additionally we will write row 5 on the far left.
Figure 3.4: Construction of Pascal’s Pyramid Layer 5

The product of the entries in the left column with the entries in the corresponding row of Pascal’s triangle yields the fifth layer of Pascal’s pyramid.

```
1  1  1  1  1  1  1
5  1  1  1  1  1  1
10 1  2  1  1  1  1  1
10 1  3  3  1  1  1  1  1
5  1  4  6  4  1  1  1  1
1  1  5  10 10 5  1  1
```

Figure 3.5: Fifth Layer of Pascal’s Pyramid

Similar to the binomial coefficients, we can represent the coefficients in Pascal’s pyramid by [Wal00]

\[
\binom{n}{i,j,k} = \frac{n!}{i!j!k!}, \quad i + j + k = n, \quad n, i, j, k \geq 0.
\]  \hfill (3.1)

In addition,

\[
\binom{n}{0,n,0} = \binom{n}{n,0,0} = \binom{n}{0,0,n} = 1, \quad n \geq 0.
\]  \hfill (3.2)

Notice \(\binom{n}{i,j,k}\) is the coefficient of \(a^i b^j c^k\) in the expansion \((a + b + c)^n\). Similar to Pascal’s identity, the following holds [HP12]

\[
\binom{n}{i,j,k} = \binom{n-1}{i-1,j,k} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}.
\]  \hfill (3.3)
3.2 Elementary Properties

We will now consider patterns in Pascal’s triangle that are also applicable in Pascal’s pyramid. Notice the sum of each entry in layer \( n \) is \( 3^n \). In Figure 3.2, the sum of the entries in layer 3 should be \( 3^3 \). Observe

\[
1 + 1 + 1 + 3 + 3 + 3 + 3 + 3 + 6 = 27 = 3^3,
\]
as desired.

*Powers of 111* are found hidden in each layer of the pyramid. Let \( n \) be a layer in Pascal’s pyramid. \( 111^n \) can be found by translating the sum of each vertical column to decimal. In Figure 3.6, we have \( 111^0 = 1, 111^1 = 111, \) and \( 111^2 = 12321 \), for layers 0, 1, and 2, respectively.

![Figure 3.6: Powers of 111](image)

Similarly, the sum of each diagonal in layer \( n \) is \( 12^n \). Examples of *powers of 12*, for \( n = 0, 1, 2 \) can be found on Figure 3.7.

![Figure 3.7: Powers of 12](image)

A 3-dimensional version of the Star of David was shown in [HP12]. Given the
center \(_{(i,j,k)}\), the following products are equal:

\[
\begin{aligned}
&\left(\begin{array}{c} n-1 \\ i, j-1, k \end{array}\right) \left(\begin{array}{c} n \\ i+1, j, k-1 \end{array}\right) \left(\begin{array}{c} n \\ i, j +1, k \end{array}\right) \left(\begin{array}{c} n+1 \\ i, j-1, k \end{array}\right) \\
=&\left(\begin{array}{c} n-1 \\ i-1, j, k \end{array}\right) \left(\begin{array}{c} n \\ i, j-1, k+1 \end{array}\right) \left(\begin{array}{c} n \\ i+1, j, k \end{array}\right) \left(\begin{array}{c} n+1 \\ i, j, k \end{array}\right) \\
=&\left(\begin{array}{c} n-1 \\ i, j, k-1 \end{array}\right) \left(\begin{array}{c} n \\ i-1, j, k+1 \end{array}\right) \left(\begin{array}{c} n \\ i+1, j, k \end{array}\right) \left(\begin{array}{c} n+1 \\ i, j, k+1 \end{array}\right).
\end{aligned}
\] (3.4)

In Figure 3.8, we see the layers in which the product takes into consideration, namely layers \(n-1\), \(n\), and \(n+1\), respectively.

\[
\begin{array}{ccc}
\left(\begin{array}{c} n-1 \\ i-1, j, k \end{array}\right) & \quad & \left(\begin{array}{c} n-1 \\ i, j-1, k \end{array}\right) \\
\left(\begin{array}{c} n \\ i, j, k-1 \end{array}\right) & \quad & \left(\begin{array}{c} n \\ i, j, k \end{array}\right) \\
\left(\begin{array}{c} n \\ i-1, j, k+1 \end{array}\right) & \quad & \left(\begin{array}{c} n \\ i, j, k+1 \end{array}\right) \\
\left(\begin{array}{c} n \\ i+1, j, k-1 \end{array}\right) & \quad & \left(\begin{array}{c} n \\ i+1, j, k \end{array}\right) \\
\left(\begin{array}{c} n+1 \\ i, j, k-1 \end{array}\right) & \quad & \left(\begin{array}{c} n+1 \\ i, j, k \end{array}\right)
\end{array}
\]

Figure 3.8: 3 Layers Used and Entries for Center \(_{(i,j,k)}\)

Using the center \(_{(1,1,1)}\), we have the following:

\[
\begin{aligned}
&\left(\begin{array}{ccc} 2 & 3 & 4 \\ 1,0,1 & 0,1,2 & 1,2,1 \end{array}\right) = \left(\begin{array}{ccc} 2 & 3 & 4 \\ 0,1,1 & 1,0,2 & 1,2,0 \end{array}\right) \\
=&\left(\begin{array}{ccc} 2 & 3 & 4 \\ 1,1,0 & 0,2,1 & 2,0,1 \end{array}\right) \left(\begin{array}{ccc} 2 & 3 & 4 \\ 1,1,0 & 0,2,1 & 2,0,1 \end{array}\right).
\end{aligned}
\]

By (3.1), the products simplify to 216.

### 3.3 Fermat Numbers

Analogous to Chapter 2, Fermat numbers also exist in Pascal’s pyramid. We will consider each entry in Pascal’s pyramid modulo 2.
Figure 3.9: Layers 1-4 Modulo 2

Examining each layer individually and translating each row from binary to decimal gives us the following tables.

<table>
<thead>
<tr>
<th>Layer 1</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>Binary</td>
<td>Decimal</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Layer 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>Binary</td>
<td>Decimal</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>00</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>101</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Layer 3</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>Binary</td>
<td>Decimal</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>101</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1111</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Layer 4</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>Binary</td>
<td>Decimal</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>00</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>000</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0000</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>10001</td>
<td>17</td>
</tr>
</tbody>
</table>

Figure 3.10: Layers 1-4 Translated to Decimal

The last row of each layer is either a Fermat number or a product of Fermat numbers.

Notice row $m$ of layer $m$ has the same decimal value as row $m$ in Pascal’s triangle modulo 2 translated to decimal. For example, row 3 of layer 3 has a decimal representation of 15. In Figure 2.7, we see the decimal representation of row 3 in Pascal’s triangle module 2 is also 15. Therefore, $F(m)$ will make its first appearance in layer $2^m$. 
3.4 Hockey Stick

A 3-dimensional representation of the hockey stick, as stated in [Hew77], can be found by

\[
\sum_{u=0}^{v} \sum_{i,j,i-n}^{2i+j=2n+u} (n+u) = \sum_{w=0}^{\lfloor v/2 \rfloor} \left( \sum_{2i+j=2n+v+2w+2}^{n+v+1} (-1)^w \right) (3.5)
\]

As an example, let us begin with layer \( n = 1 \) and limit \( v \) to 3. To help us visualize, we will first consider the left hand side of (3.5). Using these parameters, (3.5) reads

\[
\sum_{u=0}^{3} \sum_{i,j,i-1}^{2i+j=2n+1+u} \left( \frac{1+u}{n+1} \right).
\]

We can find our sum by letting \( u \) vary from 0 to 3 and finding the corresponding parameters for the inner sum. Table 3.1 shows the corresponding sum and parameters, given \( u \) is fixed.

<table>
<thead>
<tr>
<th>( u )</th>
<th>Summation Notation</th>
<th>Inner Sum Parameters</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sum_{i,j,i-1}^{2i+j=2} )</td>
<td>( i = 1, j = 0 )</td>
<td>( (1,0,0) )</td>
</tr>
<tr>
<td>1</td>
<td>( \sum_{i,j,i-1}^{2i+j=3} )</td>
<td>( i = 1, j = 1 )</td>
<td>( (2,1,0) )</td>
</tr>
<tr>
<td>2</td>
<td>( \sum_{i,j,i-1}^{2i+j=4} )</td>
<td>( i = 2, j = 0 )</td>
<td>( (3,0,1) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( i = 1, j = 2 )</td>
<td>( + (3,1,0) )</td>
</tr>
<tr>
<td>3</td>
<td>( \sum_{i,j,i-1}^{2i+j=5} )</td>
<td>( i = 2, j = 1 )</td>
<td>( (4,1,1) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( i = 1, j = 3 )</td>
<td>( + (4,1,3,0) )</td>
</tr>
</tbody>
</table>

Table 3.1: Sum of Left-hand Side of (3.4)

Thus, the left-hand side of (3.5) simplifies to

\[
\left( \frac{1}{1,0,0} \right) + \left( \frac{2}{1,1,0} \right) + \left( \frac{3}{2,0,1} \right) + \left( \frac{3}{1,2,0} \right) + \left( \frac{4}{2,1,1} \right) + \left( \frac{4}{1,3,0} \right)
\]

\[
= 1 + 2 + 3 + 3 + 4 + 12
\]

\[
= 25.
\]
Figure 3.11 shows the layers and values in which this left-hand side of (3.5) takes into consideration, in this example.

The right-hand side of (3.5), with the parameters stated above, is

\[\left\lfloor \frac{3}{2} \right\rfloor \sum_{w=0}^{1} \left( (-1)^w \sum_{2i+j=2+1+3+2w+2} \left( \begin{array}{c} 1 + 3 + 1 \\ i, j, i - 1 - 2w - 1 \end{array} \right) \right) \]

\[= \sum_{w=0}^{1} \left( (-1)^w \sum_{2i+j=7+2w} \left( \begin{array}{c} 5 \\ i, j, i - 2w - 2 \end{array} \right) \right). \]

We can find our sum by letting \( w \) vary from 0 to 1 and finding the corresponding parameters for the inner sum. Table 3.2 shows the corresponding sum and parameters, given \( w \) is fixed.

<table>
<thead>
<tr>
<th>( w )</th>
<th>Summation Notation</th>
<th>Inner Sum Parameters</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sum_{2i+j=7} \left( \begin{array}{c} 5 \ i,j,i-2 \end{array} \right) )</td>
<td>( i = 3, j = 1 ) ( \left( \begin{array}{c} 5 \ 3,1,1 \end{array} \right) )</td>
<td>((\frac{5}{3,1,1}))</td>
</tr>
<tr>
<td>1</td>
<td>( -\sum_{2i+j=9} \left( \begin{array}{c} 5 \ i,j,i-4 \end{array} \right) )</td>
<td>( i = 4, j = 1 ) ( \left( \begin{array}{c} 5 \ 4,1,0 \end{array} \right) )</td>
<td>( -(\frac{5}{4,1,0}) )</td>
</tr>
</tbody>
</table>

Table 3.2: Sum of Right-hand Side of 3.4

The right hand-side of (3.5) yields
\[
\begin{pmatrix}
5 \\
3, 1, 1
\end{pmatrix} + \begin{pmatrix}
5 \\
2, 3, 0
\end{pmatrix} - \begin{pmatrix}
5 \\
4, 1, 0
\end{pmatrix}
= 20 + 10 - 5
= 25
\]

Figure 3.12 shows the layer and values in which this right-hand side of (3.5) takes into consideration, in this example.

\[
\begin{array}{cccc}
1 \\
5 & 5 \\
10 & 20 & 10 \\
10 & 30 & 30 & 10 \\
5 & 20 & 30 & 20 & 5 \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
\]

Figure 3.12: Right-hand Side of 3.4

Notice

\[1 + 2 + 3 + 3 + 4 + 12 = 10 + 20 - 5,\]

as desired.
Chapter 4

Trinomial Triangle

4.1 Introduction

The trinomial triangle is another extension of Pascal’s triangle which is constructed by beginning with the constant term 1 in the 0th row. To obtain a new row, we sum the three adjacent values in the preceding row. Repeating this algorithm 4 times yields the following triangle. Notice, entries in each row are symmetrical about the vertical axis, known as the apex [Pol09].

\[
\begin{array}{cccccc}
  & & & & & \\
  & 1 & & & & \\
 1 & 1 & 1 & & & \\
  & 1 & 2 & 3 & 2 & 1 \\
 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
  & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
\end{array}
\]

Figure 4.1: Trinomial Triangle

The entries in row \(n\) represent the coefficients of the expansion \((1 + x + x^2)^n\). Expanding the first few cases yields:

\[
(1 + x + x^2)^0 = 1 \\
(1 + x + x^2)^1 = 1 + x + x^2 \\
(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4 \\
(1 + x + x^2)^3 = 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6. 
\]
Using similar notation as [Wei], the coefficients in this triangle may be expressed as

\[ \binom{n}{k} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2n - 2i}{n - k - i}, \quad n \geq 0, \quad -n \leq k \leq n. \quad (4.1) \]

In addition,

\[ \binom{n}{-k} = \binom{n}{k}. \quad (4.2) \]

Also,

\[ \binom{0}{0} = 1. \quad (4.3) \]

Calculating entries in the trinomial triangle is more tedious than calculating entries in Pascal’s triangle. Using (4.1) we can calculate a few entries of the trinomial triangle.

Notice

\[ \binom{2}{0} = \sum_{i=0}^{2} (-1)^i \binom{2}{i} \binom{4 - 2i}{2 - i}. \]

Expanding the sum yields

\[ \sum_{i=0}^{2} (-1)^i \binom{2}{i} \binom{4 - 2i}{2 - i} = \binom{2}{0} \binom{4}{2} - \binom{2}{1} \binom{2}{1} + \binom{2}{2} \binom{0}{0}. \]

Simplifying using (1.1), the coefficient is

\[ \binom{2}{0} = 1 \cdot 6 - 2 \cdot 2 + 1 \cdot 1 = 3. \]

Similarly, we can compute \( \binom{4}{1} \) as follows:

\[ \binom{4}{1} = \sum_{i=0}^{4} (-1)^i \binom{4}{i} \binom{8 - 2i}{3 - i}. \]

Expanding and simplifying

\[ \sum_{i=0}^{4} (-1)^i \binom{4}{i} \binom{8 - 2i}{3 - i} = \binom{4}{0} \binom{8}{3} - \binom{4}{1} \binom{6}{2} + \binom{4}{2} \binom{4}{1} - \binom{4}{3} \binom{2}{0} + \binom{4}{4} \binom{0}{-1} \]

\[ = 1 \cdot 56 - 4 \cdot 15 + 6 \cdot 4 - 4 \cdot 1 + 1 \cdot 0 = 16. \]

Notice \( \binom{n}{k} \) is the coefficient of \( x^{n+k} \) in the expansion of \( (1+x+x^2)^n \). Therefore, we can express the trinomial triangle in the following form.
4.2 Elementary Properties

As mentioned in Section 4.1, the values of the triangle are symmetrical about the apex. Notice each value in the apex is an odd number. Using (4.1), the middle entry of the \( n \)th row can be found by:

\[
\binom{n}{0}_2 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2n-2i}{n-i}.
\] (4.4)

Similar to Pascal’s identity, the trinomial coefficients satisfy the following [Wei]

\[
\binom{n}{k}_2 = \binom{n-1}{k-1}_2 + \binom{n-1}{k}_2 + \binom{n-1}{k+1}_2.
\] (4.5)

Meaning each entry is the sum of the three adjacent entries in the previous row.

\textit{Triangular numbers} are hidden on the third diagonal of the triangle.

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 2 & 3 & 2 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1
\end{array}
\]

Figure 4.3: Triangular Numbers in the Trinomial Triangle

\textit{Powers of 3} are found by taking the sum of the entries in a single row of the trinomial triangle. That is, the sum of the entries in row \( n \) is \( 3^n \). For example, the sum
of the entries in row 4:

\[ 1 + 4 + 10 + 16 + 19 + 16 + 10 + 4 + 1 = 81 = 3^4. \]

That is, for any row \( n \)

\[
\sum_{k=-n}^{n} \binom{n}{k} = 3^n, \quad n \geq 0.
\]  \tag{4.6}

A similar pattern is the sum of the squares of the entries in row \( n \) is equal to the middle entry of row \( 2n \) [Pol09]. That is,

\[
\sum_{k=-n}^{n} \left( \binom{n}{k} \right)^2 = \binom{2n}{0}, \quad n \geq 0.
\]  \tag{4.7}

Consider the sum of the squares of the entries in row 2

\[
\sum_{k=-2}^{2} \left( \binom{2}{k} \right)^2 = \binom{2}{-2}^2 + \binom{2}{-1}^2 + \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 19.
\]

Also,

\[
\binom{4}{0} = 19.
\]

An illustration of this path is shown in Figure 4.4.

\begin{figure}[h]
\centering
\begin{tabular}{ccccccc}
  & 1 &  &  &  &  &  \\
1 & 1 & 1 & 1 &  &  &  \\
1 & 2 & 3 & 2 & 1 &  &  \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & \textbf{19} & 16 & 10 & 4 & 1 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
\end{tabular}
\caption{Sum Of Squares for \( n = 2 \)}
\end{figure}

Similar to the powers of 11 introduced in Section 2.1, the powers of 111 are hidden in the trinomial triangle. Translating row \( n \) to decimal will result in 111\(^n\). That is,

\[
\sum_{k=-n}^{n} \binom{n}{k} \cdot 10^{n-k} = 111^n, \quad n \geq 0.
\]  \tag{4.8}
Translating row 3, (4.8) reads:

\[ \sum_{k=-3}^{3} \binom{3}{k} \cdot 10^{3-k}. \]

Expanding the sum yields

\[
\binom{3}{-3} \cdot 10^{3+3} + \binom{3}{-2} \cdot 10^{3+2} + \binom{3}{-1} \cdot 10^{3+1} + \binom{3}{0} \cdot 10^{3-0} \\
+ \binom{3}{1} \cdot 10^{3-1} + \binom{3}{2} \cdot 10^{3-2} + \binom{3}{3} \cdot 10^{3-3}.
\]

Simplifying we have

\[ 1 \cdot 10^6 + 3 \cdot 10^5 + 6 \cdot 10^4 + 7 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 1 \cdot 10^0 = 1367631 = 111^3, \]

as desired.

The sum of the entries in row \( n \) with alternating signs is equal to 1 [Pol09]. That is,

\[ \sum_{k=-n}^{n} (-1)^{n-k} \binom{n}{k} = 1, \quad n \geq 0. \]  \hspace{1cm} (4.9)

The sum of the entries in row 4, with alternating signs, yields

\[ \sum_{k=-4}^{4} (-1)^{4-k} \binom{4}{k}. \]

Expanding the sum

\[
(-1)^{4+4} \binom{4}{4} + (-1)^{4+3} \binom{4}{3} + (-1)^{4+2} \binom{4}{2} + (-1)^{4+1} \binom{4}{1} + (-1)^{4+0} \binom{4}{0} \\
+ (-1)^{4-1} \binom{4}{1} + (-1)^{4-2} \binom{4}{2} + (-1)^{4-3} \binom{4}{3} + (-1)^{4-4} \binom{4}{4},
\]

and simplifying

\[ 1 - 4 + 10 - 16 + 19 - 16 + 10 - 4 + 1 = 1. \]

The teardrop property also applies to the trinomial triangle. The parallelogram can be created with even or odd number of rows. If the parallelogram consists of even number of rows, the sum of the parallelogram is equal to one less than the entry below the parallelogram. The sum of the parallelogram in Figure 4.5 is 18, which is one less than 19.
If the parallelogram consists of odd number of rows, the sum of the parallelogram is equal to the entry below it. Notice the sum of the parallelogram in Figure 4.6 is 7, which is also the entry right below it.

4.3 Hockey Stick

Several hockey stick properties are shown in [CL18]. The Trinomial Diagonal Hockey Stick and Multipuck has two different cases. For the first case, consider beginning the diagonal hockey stick path at \( \binom{n}{k} \), then

\[
\sum_{i=0}^{k} \binom{n+i}{-n+i} = \sum_{j=0}^{2k} \left( (-1)^{j} \binom{n+k+1}{-n+k-1-j} \right), \quad n, k \geq 0. \tag{4.10}
\]
Beginning at \( \binom{1}{-1} \) and setting \( k = 3 \), then

\[
\sum_{i=0}^{3} \left( \frac{1+i}{-1+i} \right) = \binom{1}{-1}^2 + \binom{2}{0}^2 + \binom{3}{1}^2 + \binom{4}{2}^2.
\]

Also,

\[
\sum_{j=0}^{6} \left( (-1)^j \left( \frac{5}{1-j} \right) \right) = \binom{5}{1}^2 - \binom{5}{0}^2 + \binom{5}{-1}^2 - \binom{5}{-2}^2 + \binom{5}{-3}^2 - \binom{5}{-4}^2 + \binom{5}{-5}^2.
\]

Notice

\[
\binom{1}{-1}^2 + \binom{2}{0}^2 + \binom{3}{1}^2 + \binom{4}{2}^2 = 1 + 3 + 6 + 10 = 20
\]

and

\[
\binom{5}{1}^2 - \binom{5}{0}^2 + \binom{5}{-1}^2 - \binom{5}{-2}^2 + \binom{5}{-3}^2 - \binom{5}{-4}^2 + \binom{5}{-5}^2 = 45 - 51 + 45 - 30 + 15 - 5 + 1 = 20.
\]

An illustration of this path is shown in Figure 4.7.

![Figure 4.7: Diagonal Hockey Stick and Multipuck Case 1](image)

For the second case of the diagonal hockey stick and multipuck property, consider beginning at \( \binom{n}{-n+1} \), then

\[
\sum_{i=0}^{k} \left( \frac{n+i}{-n+1+i} \right) = \sum_{j=0}^{2k+1} \left( (-1)^j \left( \frac{n+k+1}{-n+k-j} \right) \right), \quad n, k \geq 0.
\]
Beginning at \( \binom{1}{0} \) and setting \( k = 3 \),

\[
\sum_{i=0}^{3} \binom{1+i}{i}^2 = \binom{1}{0}^2 + \binom{2}{1}^2 + \binom{3}{2}^2 + \binom{4}{3}^2.
\]

Furthermore,

\[
\sum_{j=0}^{7} (-1)^j \binom{5}{2-j}^2
= \binom{5}{2}^2 - \binom{5}{1}^2 + \binom{5}{0}^2 - \binom{5}{-1}^2 + \binom{5}{-2}^2 - \binom{5}{-3}^2 + \binom{5}{-4}^2 - \binom{5}{-5}^2.
\]

In both equations,

\[
\binom{1}{0}^2 + \binom{2}{1}^2 + \binom{3}{2}^2 + \binom{4}{3}^2 = 1 + 2 + 3 + 4 = 10
\]

and

\[
\binom{5}{2}^2 - \binom{5}{1}^2 + \binom{5}{0}^2 - \binom{5}{-1}^2 + \binom{5}{-2}^2 - \binom{5}{-3}^2 + \binom{5}{-4}^2 - \binom{5}{-5}^2
= 30 - 45 + 51 - 45 + 30 - 15 + 5 - 1 = 10.
\]

An illustration of this path can be found in Figure 4.8.

![Figure 4.8: Diagonal Hockey Stick and Multipuck Case 2](image-url)

A generalized hockey stick and multipuck formula, for the trinomial triangle, was introduced in [Hew77]

\[
\sum_{i=0}^{k} \binom{n+i}{n}^2 = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \left( -1 \right)^{j} \binom{n+k+1}{n+2j+1}^2 \quad n, k \geq 0.
\]
As an example, let \( n = 0 \) and \( k = 4 \). A path for this example is shown in Figure 4.9.

Using these parameters,

\[
\sum_{i=0}^{4} \binom{i}{0}_2 = \binom{0}{0}_2 + \binom{1}{0}_2 + \binom{2}{0}_2 + \binom{3}{0}_2 + \binom{4}{0}_2.
\]

Also,

\[
\sum_{j=0}^{2} \left((-1)^j \binom{5}{2j+1}_2\right) = \binom{5}{1}_2 - \binom{5}{3}_2 + \binom{5}{5}_2.
\]

Notice

\[
\binom{0}{0}_2 + \binom{1}{0}_2 + \binom{2}{0}_2 + \binom{3}{0}_2 + \binom{4}{0}_2 = 1 + 1 + 3 + 7 + 19 = 31
\]

and

\[
\binom{5}{1}_2 - \binom{5}{3}_2 + \binom{5}{5}_2 = 45 - 15 + 1 = 31,
\]

as desired.

![Trinomial Triangle](image)

**Figure 4.9: Hockey Stick Example in the Trinomial Triangle**

Not only are the hockey stick properties fascinating, but they may also be used to prove other properties in the triangle. Let us revisit the teardrop property introduced in Section 4.2. We will consider the parallelogram with odd number of rows, the sum of the parallelogram is stated in the following theorem.

**Theorem 4.1.** If \( k \) is the number of rows in the parallelogram and if \( k \) is odd, then

\[
\sum_{j=0}^{k-1} \sum_{i=0}^{k-1} \left(i + j\right)_2 \binom{i}{j}_2 + \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} \left(i + 1 + j\right)_2 \binom{i}{j}_2 = \binom{k}{0}_2.
\]

We will show that the teardrop property exists for an odd number of rows in the trinomial triangle. The idea of the proof is as follows. We will assume the sum of the
parallelogram is 7 when \( k = 3 \) rows, as shown in Figure 4.10. Then we will show it also holds for \( k = 5 \) rows by adding four hockey hockey sticks which we will call \( H_1, H_1, H_3, \) and \( H_4. \)

![Figure 4.10: Example of Teardrop](image)

Observe the \( H_1 + H_2 = 51 \) and \( H_3 + H_4 = 16. \) As shown in Figure 4.11 (a), by adding \( H_1 \) and \( H_2 \) we are required to borrow 16. The 16 that we are over counting by is replaced by the sum of \( H_3 \) and \( H_4, \) as shown in Figure 4.11 (b). Notice \( H_1 \) and \( H_3 \) overlap at 7, but one of these 7s is replaced by the parallelogram which we stated in our induction hypothesis. Therefore, by adding these four hockey sticks, we can create a parallelogram with \( k = 5 \) rows. The sum of this parallelogram is the sum of the hockey sticks which is 51, as shown in Figure.

![Figure 4.11: Adding Hockey Sticks to Parallelogram with 3 Rows](image)

The proof of the general case will flow similar to the outlined proof above.

**Proof.** We will prove Theorem 4.1 by using induction on the number of rows in the parallelogram, for odd number of rows. The base case is obvious and can be seen in Figure 4.12.
We will now assume the sum of the parallelogram is true for \( k \) rows, where \( k \) is odd. Therefore, the sum of the parallelogram is \( \binom{k}{0} \). Our goal is to show the sum is true for the next number of odd rows, namely a parallelogram with \( k + 2 \) rows.

Similar to Figure 4.11, we have to add four hockey sticks to create the next parallelogram in the sequence. We start by adding \( H_1 \) and \( H_2 \) which are

\[
H_1 : \sum_{i=0}^{\frac{k+1}{2}-1} \left( \frac{k+1}{2} + i \right) \left( \frac{k+1}{2} - 1 - i \right) = \sum_{j=0}^{2(\frac{k+1}{2}-1)+1} (-1)^j \left( \frac{k+1}{2} + (\frac{k+1}{2} - 1) + 1 \right)
\]

\[
H_2 : \sum_{i=0}^{\frac{k+1}{2}} \left( \frac{k+1}{2} + i \right) \left( -\frac{k+1}{2} + i \right) = \sum_{j=0}^{2(\frac{k+1}{2})} (-1)^j \left( \frac{k+1}{2} + (\frac{k+1}{2} - 1) + 1 \right)
\]

The sum of these two hockey sticks is \( \binom{\frac{k+1}{2}+1}{0} = \binom{\frac{k+2}{2}}{0} \), since all other entries add to 0. \( H_1 + H_2 \) provide most of the entries needed to create the next parallelogram, however we borrowed the term \( \binom{\frac{k+1}{2}}{1} \) which derived from \( H_1 \).

The last two hockey sticks are given by

\[
H_3 : \sum_{i=0}^{\frac{k+1}{2}-1} \left( \frac{k+1}{2} + i \right) \left( \frac{k+1}{2} - 1 - i \right) = \sum_{j=0}^{2(\frac{k+1}{2}-1)-1} (-1)^j \left( \frac{k+1}{2} + (\frac{k+1}{2} - 1) + 1 \right)
\]

\[
H_4 : \sum_{i=0}^{\frac{k+1}{2}-1} \left( \frac{k+1}{2} + i \right) \left( \frac{k+1}{2} - 1 - i \right) = \sum_{j=0}^{2(\frac{k+1}{2})-1} (-1)^j \left( \frac{k+1}{2} + (\frac{k+1}{2} - 1) + 1 \right)
\]

But the sum of these hockey sticks is \( \binom{\frac{k+1}{2}}{1} \), since all other entries add to 0. Not only does \( H_3 + H_4 \) sum to the value we borrowed in \( H_1 + H_2 \), but it also provides the last entries needed to complete the next parallelogram.
Observe, when $i = \frac{k+1}{2} - 1$, $H_1$ and $H_3$ share the same term, namely $\binom{k}{0}_2$. This extra term is used to complete the parallelogram, since $\binom{k}{0}_2$ is the sum of the parallelogram in our induction hypothesis. Thus, by adding adding hockey sticks $H_1$, $H_2$, $H_3$, and $H_4$ to our parallelogram from our induction hypothesis we can create a new parallelogram whose sum is $\binom{k+2}{0}_2$, as desired.

\□
Chapter 5

Conclusion

As seen in Chapters 2 through 4, many properties in Pascal’s triangle have analogous representation in Pascal’s pyramid and the trinomial triangle. The hockey stick property translates to both Pascal’s pyramid and the trinomial triangle. However, we were not able to find if Fermat numbers exist in the trinomial triangle. That need not mean that it does not, it may just be encrypted deep in the trinomial triangle that it will require more time to find. Also, different types of hockey sticks exist in the trinomial triangle, but we were only able to find one representation for Pascal’s pyramid. Is it possible for other types of hockey sticks to exist in the pyramid? Furthermore, in the trinomial triangle, we were able prove the sum of an odd number of rows of a parallelogram is equal to the entry below it using hockey stick properties. It may be possible to prove other properties of the triangles or pyramid using the hockey sticks introduced.

It may be convincing that each property found in Pascal’s triangle should be able to translate to any extension, it may just be a matter of searching. For future research, we will see if there are any properties that do not exist in Pascal’s triangle, but do in the extensions of it. Something to consider is if we are able to generalize every property to n-dimensional pyramids or triangles for multinomial expansions. Also, we may want to consider how these properties translate to other triangles such as the harmonic triangle and the tribonacci convolution triangle.
Bibliography


