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Symmetric Presentations and Double Coset Enumeration

Charles Seager

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

In this project, we demonstrate our discovery of original symmetric presentations and constructions of important groups, including nonabelian simple groups, and groups that have these as factor groups. The target nonabelian simple groups include alternating, linear, and sporadic groups. We give isomorphism types for each finite homomorphic image that has been found. We present original symmetric presentations of M_{12}, $M_{21} : (2 \times 2)$, $L_3(4) : 2^2$, $2 :^{\cdot} L_3(4) : 2$, $S(4,3)$, and S_7 as homomorphism images of the progenitors $2^{*20} : A_5$, $2^{*10} : PGL(2,9)$, $2^{*10} : Aut(A_6)$, $2^{*10} : A_6$, $2^{*10} : A_5$, and $2^{*24} : S_5$, respectively. We also construct M_{12}, $M_{21} : (2 \times 2)$, $L_3(4) : 2^2$, $L_3(4) : 2^2$, $2 :^{\cdot} L_3(4) : 2$, $S(4,3)$, and S_7 over A_5, $PGL(2,9)$, $Aut(A_6)$, A_6, A_5, and S_5, respectively, using our technique of double coset enumeration. All of the symmetric presentations given are original to the best of our knowledge.

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Chapter 1

Introduction

In the following chapters we will discuss how we went about obtaining homomorphic images, their isomorphism types, and their Cayley diagrams. We will begin our discussion of how a progenitor of infinite order is constructed from a control group N. We used the computer based program MAGMA to help expedite the construction of such progenitors to obtain homomorphic images of various interesting groups, and to research these groups in more detail. We perform double coset enumeration (DCE) on groups such as M_{21} over the control group PGL(2,9). We also proved the isomorphism types for these groups.

We begin by defining the progenitor. A progenitor is a semi-direct product of the form: $P \cong 2^{*n} : N = \{\pi\omega \mid \pi \in N\}$, where ω is a reduced word in the t_i, 2^{*n} denotes the free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, \cdots, n$. N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators. We refer to the subgroup N as the control subgroup and to the involutory generators of the free product as the symmetric generators. The unique progenitor is then factored by the relations that produce finite homomorphic images.

The progenitor $m^{*n} : N$, where m is the order of the $t_i's$, n is the number of $t_i's$, and N is the control group, has infinite order. Thus we need to write a permutation progenitor where we take N to be transitive on n letters, with the general form of a permutation progenitor given by:

$< x, y, t| < x, y > \cong N$, t^m, $(t, N^i)>$, where N^i is the stabiliser of i in N

Since t commutes with the stabiliser of i in N, (t, N^i), we can obtain the number of conjugates of t. We have $[G : C_g(a)]$ the number of conjugates of H in G. So to find the index of the centraliser of N and t also denoted as *Centraliser*$[N, t]$, we are going to compute $[G : C_g(a)]$. The index of the *Centraliser* $[N, t]$ is equal to the number of conjugates of t and also equal to the stabiliser of a single point in N. Using this, we will find the permutation progenitor of the following example, $2^{*10} : A_5$.

EXAMPLE

For this example, we will illustrate how to write a symmetric presentation for the infinite progenitor $2^{*24} : S_5$. Our control group $N = S_5$ is transitive on 24 letters and the generators for N are $x \sim$ (3,17,11,7,5)(4,18,12,8,6)(9,14,22,20,15)(10,13,21,19,16) and y $\sim$ (1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24). A presentation is given by

$$< x^5, y^2, (x^{-1} * y)^4, (x * y * x^{-2} * y * x)^2 >$$

Now we let t be a symmetric generator where $t \sim t_1$ and is of order 2. Since t $\sim t_1$, we must compute the stabiliser of the single point 1 in N, denoted N^1. So $N^1 =$ <(3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13)¿ $\cong \mathbb{Z}_5$. Notice that all powers of the generator fix 1. Then we write $(t_1, N^1) = (t_1, \mathbb{Z}_5)$ to denote that t_1 commutes with our point stabiliser. Thus, our permutation progenitor $2^{*24} : S_5$ is given as follows:

$$< x^5, y^2, (x^{-1} * y)^4, (x * y * x^{-2} * y * x)^2, t^2, (t, x) >$$

From here we ask MAGMA for orbits, find first order relations, and run the program a background job. Periodically checking the composition factors, we are looking for faithful groups, that is, groups having $|ker f| = 1$. When performing DCE on a specific group, we use MAGMA. We create a Cayley Diagram for the group, which summarizes the DCE information.

We then find relations, by observing patterns and by trial and error. Once relations have been found and simplified, the Lemmas will be proved. Then we can perform manual DCE, followed by proving isomorphism types.

1.1 Definitions

Definition 1. *A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1, \pi_2\omega_2, \cdots}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relations $\pi_1\omega_1, \pi_1\omega_1, \cdots$ have been factored out.

Definition 2. *A* **group** *G $(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*
- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$. [Rot95]*

Definition 3. *Let G be a set. A (binary)* **operation** *on G is a function that assigns each ordered pair of elements of G an element on G. [Rot95]*

Definition 4. *For group G, a* **subgroup** *S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$. [Rot95]*

Definition 5. *Let H be a subgroup of group G. H is a* **proper** *subgroup of G if $H \neq G$. We denote this as $H < G$. [Rot95]*

Definition 6. *A* **symmetric group***, S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions. [Rot95]*

Definition 7. *If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$. [Rot95]*

Definition 8. *A* **semigroup** *$(G, *)$ is a nonempty set G equipped with an associative operation. [Rot95]*

Definition 9. *If $x \in X$ and $\phi \in S_X$, then ϕ* **fixes** *x if $\phi(x) = x$ and ϕ* **moves** *x if $\phi(x) \neq x$. [Rot95]*

Definition 10. *For permutations $\alpha, \beta \in S_X$, α and β are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(a) = a \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y. \text{ [Rot95]}$$

Definition 11. *A permutation which interchanges a pair of elements is a* **transposition**. *[Rot95]*

Definition 12. *In group G, if a,b $\in G$, a and b* **commute** *if $a * b = b * a$. [Rot95]*

Definition 13. *A group G is* **abelian** *if every pair of elements in G commutes with one another. [Rot95]*

Definition 14. *Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G. [Rot95]*

Definition 15. *Let G be a group and $T = t_1, t_2, ..., t_n$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the* **progenitor** *to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_n. [Cur07]*

Definition 16. *Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by $N_G(H) = \{a \in G | aHa^{-1} = H\}$. [Rot95]*

Definition 17. *Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}. \text{ [Rot95]}$$

Definition 18. *Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we say G is* **elementary abelian**. *[Rot95]*

Definition 19. *Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \to H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all a,b $\in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to H, $G \cong H$, if there is exists an isomorphism $f : G \to H$. [Rot95]*

Definition 20. *Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *. [Rot95]*

Definition 21. *Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X. [Rot95]*

Definition 22. *Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$. [Rot95]*

Definition 23. *The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$. [Rot95]*

Definition 24. *A* **general linear group***, $GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$. [Rot95]*

Definition 25. *A* **special linear group***, $SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$. [Rot95]*

Definition 26. *A* **projective special linear group***, $PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}. \text{ [Rot95]}$$

Definition 27. *A* **projective general linear group***, $PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}. \text{ [Rot95]}$$

Definition 28. (**Monomial Character**) *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H. [Led77]*

Definition 29. (**Character**) *Let $A(x) = (A_{ij}(x))$ be a matrix representation of G of degree m. We consider the character polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the **character** *of $A(x)$. [**?**]*

Definition 30. *The sum of squares of the degrees of the s=distinct irreducible characters of G is equal to $|G|$. The* **degree of a character** *χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character. [**?**]*

Definition 31. (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x)$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. Then lifting process preserves irreducibility. [**?**]*

Definition 32. (**Induced Character**) *Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then*

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

*is an induced character of G. [**?**]*

Definition 33. *Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let C_α, $\alpha = 1, 2, ..., m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, 3, ..., m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \alpha = 1, 2, 3, ..., m.[\textbf{?}]$$

Definition 34. *Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$. [Rot95]*

Definition 35. *Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \lhd G$ to denote K as being normal in G. [Rot95]*

Definition 36. *Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St. [Rot95]*

Definition 37. *Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G. [Rot95]*

Definition 38. *Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$. [Rot95]*

Definition 39. *Let N be a group. The* **point stabilizer** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s. [Rot95]}$$

Definition 40. *Let N be a group. The* **coset stabilizer** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s. [Rot95]}$$

Definition 41. *Let G be a group. The* **center** *of G, $Z(G)$, is the set of all elements in G that commute with all elements of G. [Rot95]*

Definition 42. *A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1, \pi_2\omega_2, \ldots}$$

*where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1\omega_1, \pi_2\omega_2, \ldots$ have been factored out. [**?**]*

Definition 43. *We define*

$$\mathcal{N}^i = C_{\mathcal{N}}(t_i); \mathcal{N}^{ij} = C_{\mathcal{N}}(\langle t_i, t_j \rangle) \text{ etc,}$$

single point and two point stabilizer in $\mathcal{N}$ respectively. The coset stabilizing subgroup, $\mathcal{N}^{(w)}$, of $\mathcal{N}$ is given by

$$\mathcal{N}^{(w)} = \pi \in \mathcal{N} : \mathcal{N}w\pi = \mathcal{N}w,$$

for w a word in the symmetric generators. Clearly $\mathcal{N}^{w} \leq \mathcal{N}^{(w)}$, and the number of cosets in the double coset $[w] = \mathcal{N}w\mathcal{N}$ is given by $|\mathcal{N}|/|\mathcal{N}^{(w)}|$, since $\mathcal{N}w\pi_1 \neq \mathcal{N}w\pi_2$
$\iff \mathcal{N}w\pi_1\pi_2^{-1} \neq \mathcal{N}w$
$\iff \pi_1\pi_2 \notin \mathcal{N}^{(w)}$
$\iff \mathcal{N}^{(w)}\pi_1\pi_2^{-1} \neq \mathcal{N}^{(w)}$
$\iff \mathcal{N}^{(w)}\pi_1 \neq \mathcal{N}^{(w)}\pi_2$.
DoubleCosetEnumerationArithmetic *In order to obtain the index of $\mathcal{N}$ in $\mathcal{G}$ we shall perform a manual double coset enumeration of $\mathcal{G}$ over $\mathcal{N}$; thus we must find all double cosets $[w]$ and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $\mathcal{N}^{(w)}$ on the symmetric generators. We need only identify, for each $[w]$, the double coset to which $\mathcal{N}wt_i$ belongs for one symmetric generator t_i from each orbit. [Cur07]*

Definition 44. First Isomorphism Theorem(F.I.T). *Let $\phi : G \to H$ is a homomorphism with $Ker\phi$. Then,*

$$\bullet Ker\phi \trianglelefteq G$$
$$\bullet G/Ker\phi \cong img\phi \text{ [Rot95]}$$

1.2 Theorems

Theorem 1. *The number of irreducible character of G is equal to the number of conjugacy classes of G. [Cur07]*

Theorem 2. *Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$. [Rot95]*

Theorem 3. *Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]*

Theorem 4. *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles. [Rot95]*

Theorem 5. *Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*
- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*
- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$. [Rot95]*

Theorem 6. *The intersection of any family of subgroups of a group G is again a subgroup of G. [Rot95]*

Theorem 7. *If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 8. *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. [Rot95]*

Theorem 9. *If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|. \text{ [Rot95]}$$

Theorem 10. *If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$. [Rot95]*

Theorem 11. *The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$. [Rot95]*

Theorem 12. *Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$. [Rot95]*

Theorem 13. *If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite. [Rot95]

Theorem 14. *If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$. [Rot95]*

Theorem 15. *Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n. [Rot95]*

Theorem 16. *If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker\rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H. [Rot95]*

Theorem 17. *If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set. [Rot95]*

Theorem 18. *Every two composition series of a group G are equivalent.*

We will refer to this Theorem as the **Jordan-Hölder Theorem**. *[Rot95]*

Theorem 19. *Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$. [Rot95]*

1.3 Lemmas

Lemma 20. *Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*
- *If $H \lhd G$, then the subsets Hx are blocks of X. [Rot95]*

Chapter 2

A Symmetric Presentation of the Mathieu Sporadic Group M_{12}

Consider $N = < xx, yy > \cong A_5$, where xx=(1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19) and yy=(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)

A symmetric presentation for the progenitor 2^{*20}: N is given by

$G < x, y, t >$:=Group$< x, y, t | x^5, y^2, (y * x^{-1})^3, t^2, (t, y * x) >$

where x $\sim$ xx, y $\sim$ yy, and t $\sim$ t_1. We factor this progenitor by the three relations

$(y * t)^4, (y * t^{(x*y*x^2)})^2, (y * t^{(x^3 y)})^5$ and prove that $G < x, y, t >$:=
Group$< x, y, t | x^5, y^2, (y * x^{-1})^3, t^2, (t, y * x), (y * t)^4, (y * t^{(x*y*x^2)})^2, (y * t^{(x^3*y)})^5 > \cong M_{12}$.

In order to do this, we need to perform manual double coset enumeration of G over N. However, the number of NwN - double cosets of N in G is 47. Thus, we select the maximal subgroup

$$M = < x, y, x^2 * t * x * y * t * x^2 * y * t * x^2 * t, y * x * t * x * y * t * x^2 * y * t * x^2 * t * x^{-2} >$$

and perform double coset enumeration of MwN - double cosets.

2.1 Relations, Expansions, and Their Conjugation

Our relations are $(y*t)^4 = e \implies$

$$y^4{*}t_1^{y^3}{*}t_1^{y^2}{*}t_1^{y}{*}t_1 = e \implies$$

$$e{*}t_1^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}*$$

$$t_1^e{*}t_1^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}t_1 = e \implies$$

$$t_{17}t_1t_{17}t_1 = e \ (1)$$

$$(y*t^{x*y*x^2})^2 = e \implies$$

$$(y*t_1^{(1,15,18,5,8)(2,16,17,6,7)(3,13,10,20,11)(4,14,9,19,12)})^2 = e \implies$$

$$(y*t_{15})^2 = e \implies$$

$$y^2*t_{15}^y*t_{15} = e \implies$$

$$e*t_{15}^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}*t_{15} = e \implies$$

$$t_{16}t_{15} = e \ (2)$$

$$(y*t^{(x^3*y)})^5 = e \implies$$

$$(y*t_1^{(1,12,4,15,6)(2,11,3,16,5)(7,10,17,19,14)(8,9,18,20,13)})^5 \ e \implies$$

$$(y*t_{12})^5 = e \implies$$

$$y^5*t_{12}^{y^4}*t_{12}^{y^3}*t_{12}^{y^2}*t_{12}^{y}*t_{12} = e \implies$$

$$y^5*t_{12}^e*t_{12}^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}*t_{12}^e*$$

$$t_{12}^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}*t_{12} = e \implies$$

$$y^5t_{12}t_{13}t_{12}t_{13}t_{12} = e \ (3)$$

$$(x^2*t*x*y*t*x^2*y*t*x^2*t) \in M \implies$$

$$x^2 * t_1 * x * y * t_1 * x^2 * y * t_1 * \underline{x^2} * t_1 \in M \implies$$

$$x^2 * t_1 * x * y * t_1 * \underline{x^2 * y * x^2} * t_{10} * t_1 \in M \implies$$

$$x^2 * t_1 * \underline{x * y * x^2 * y * x^2} * t_7 * t_{10} t_1 \in M \implies$$

$$\underline{x^2 * x * y * x^2 * y * x^2} * t_4 * t_7 * t_{10} t_1 \in M \implies$$

$$M t_4 t_7 = M t_1 t_{10} \ (4)$$

$$(y * x * t * x * y * t * x^2 * y * t * x^2 * t * x^{-2}) \in M \implies$$

$$y * x * t_1 * x * y * t_1 * x^2 * y * t_1 * x^2 * t_1 * \underline{x^{-2}} \in M \implies$$

$$y * x * t_1 * x * y * t_1 * x^2 * y * t_1 * \underline{x^2 * x^{-2}} * t_{13} \in M \implies$$

$$y * x * t_1 * x * y * t_1 * \underline{x^2 * y * e} * t_1 * t_{13} \in M \implies$$

$$y * x * t_1 * \underline{x * y * x^2 * y} * t_{19} * t_1 * t_{13} \in M \implies$$

$$\underline{y * x * x * y * x^2 * y} * t_{16} * t_{19} * t_1 * t_{13} \in M \implies$$

$$M t_{16} t_{19} = M t_{13} t_1 \ (5)$$

Additionally,

$$x * y * x^{-1} * t * x^2 * t * x * y * x^{-1} * t * x^2 * y * t * t^{(x*y)} * t^x *$$

$$t^{(x*y*x^{-1}*y*x*y*x*y)} * t^y * t^{(x*y*x^{-1})} * t = e \implies$$

$$x * y * x^{-1} * t_1 * x^2 * t_1 * x * y * x^{-1} * t_1 * x^2 * y * t_1 *$$

$$t_1^{(1,8,13)(2,7,14)(3,6,19)(4,5,20)(9,11,15)(10,12,16)} *$$

$$t_1^{(1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19)} *$$

$$t_1^{(1,20,16)(2,19,15)(3,17,10)(4,18,9)(5,12,7)(6,11,8)} *$$

$$t_1^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)} *$$

$$t_1^{(1,3)(2,4)(5,15)(6,16)(7,9)(8,10)(11,12)(13,17)(14,18)(19,20)} * t_1 = e \implies$$

$$x * y * x^{-1} * t_1 * x^2 * t_1 * x * y * x^{-1} * t_1 * \underline{x^2 * y} * t_1 * t_8 * t_6 * t_{20} * t_{17} * t_3 * t_1 = e \implies$$

$$x * y * x^{-1} * t_1 * x^2 * t_1 * \underline{x * y * x * y} * t_{19} * t_1 * t_8 * t_6 * t_{20} * t_{17} * t_3 * t_1 = e \implies$$

$$x * y * x^{-1} * t_1 * \underline{x^3 * y * x * y} * t_{13} * t_{19} * t_1 * t_8 * t_6 * t_{20} * t_{17} * t_3 * t_1 = e \implies$$

$$\underline{x * y * x^2 * y * x * y} * t_{16} * t_{13} * t_{19} * t_1 * t_8 * t_6 * t_{20} * t_{17} * t_3 * t_1 = e \implies$$

$$(1,10,11,5,19)(2,9,12,6,20)(3,16,8,17,13)(4,15,7,18,14)t_{16}t_{13}t_{19}t_1t_8t_6t_{20}t_{17}t_3t_1 = e \ (6)$$

$$y * x * t * x * y * t * x^2 * y * t * x^2 * t * x^{-2} * t^{(x^2)} * t^{(x*y*x)} *$$

$$t^{(x^2*y)} * t^{(x*y*x^-1)} * t^{(x*y*x^-1*y*x*y)} * t = e \implies$$

$$(1,16,14)(2,15,13)(3,18,6)(4,17,5)(7,10,11)(8,9,12)t_{16}t_{19}t_1t_{13}t_{10}t_{12}t_{19}t_3t_5t_1 = e \ (7)$$

$$x^{-1} * t * x^{-2} * t * y * x^{-2} * t * y * x^{-1} * t * x * y * t^{(x*y*x^{-1}*y*x*y*x^2)} * t^{(x*y*x^{-1}*y*x^2)} *$$

$$t^{(x^2)} * t^{(x*y*x^{-1}*y*x*y)} * t^{(x*y*x^{-1})} * t = e \implies$$

$$(1,11)(2,12)(3,6)(4,5)(7,8)(9,14)(10,13)(15,16)(17,20)(18,19)t_3t_{10}t_1t_8t_{14}t_{11}t_{10}t_5t_3t_1 = e \ (8)$$

$$t * x^{-2} * t * y * x^{-2} * t * y * x^{-1} * t * x^{-1} * y * x * t^{(x*y*x^{-1}*y*x)} * t^{(x*y*x^{-1})} *$$

$$t^{(x*y*x^2*y)} * t^{(x*y*x^{-1}*y*x*y*x)} * t^{(x*y*x^{-1}*y*x*y)} * t = e \implies$$

$$(1,20,14)(2,19,13)(3,5,8)(4,6,7)(9,17,11)(10,18,12)t_{20}t_{18}t_4t_6t_7t_3t_{16}t_9t_5t_1 = e \ (9)$$

$$t * x * t * x * y * t * x * t * x * t * y * t * x^{-2} * t * t^{(x*y*x^{-1})} * t^{(x*y*x^{-1}*y*x*y)} *$$

$$t^{(x*y*x^2)} * t^{(x*y*x^{-1})} * t^{(x^2*y)} * t^{(x*y*x^{-1}*y*x)} = e \implies$$

$$(1,18,8,15,5)(2,17,7,16,6)(3,10,11,13,20)(4,9,12,14,19)t_{18}t_7t_{11}t_{20}t_{10}t_{13}t_1t_3t_5t_{15}t_3t_{19}t_7 = e \ (10)$$

$$y * x^{-2} * t * x^2 * t * x * y * x^{-1} * t * x^{-2} * t * x^{-2} * t * t^{(x*y*x^{-1})} *$$

$$t^{(x*y*x^{-1}*y*x*y)} * t^y * t^{(x^2*y)} * t^{(x^3)} = e \implies$$

$$(1,17,8,19,12)(2,18,7,20,11)(3,5,14,15,10)(4,6,13,16,9)t_{12}t_8t_6t_{13}t_1t_3t_5t_{17}t_{19}t_{13} = e \ (11)$$

$$t*x*y*x^2*t*x*y*t*x*y*x^{-1}*t*y*t*t^{(x*y*x^{-1})}*t^{(x*y*x^{-1}*y*x^2)}*$$

$$t^{(x*y*x^{-1}*y*x*y*x*y)}*t^{(x*y*x^{-1}*y*x)}*t^{(x*y*x^{-1}*y*x*y*x)} = e \implies$$

$$(1,5)(2,6)(3,4)(7,20)(8,19)(9,17)(10,18)(11,15)(12,16)(13,14)t_5t_{19}t_4t_{17}t_1t_3t_{11}t_{20}t_7t_9 = e \ (12)$$

$$x*y*x^{-1}*t*x^2*t*x*y*x^{-1}*t*x^2*y*t \in M \implies$$

$$Mt_{16}t_{13} = Mt_1t_{19} \ (13)$$

$$y*x*t*x*y*t*x^2*y*t*x^2*t*x^{-2} \in M \implies$$

$$Mt_{16}t_{19} = Mt_{13}t_1 \ (14)$$

$$x^{-1}*t*x^{-2}*t*y*x^{-2}*t*y*x^{-1}*t*x*y \in M \implies$$

$$Mt_3t_{10} = Mt_8t_1 \ (15)$$

$$t*x^{-2}*t*y*x^{-2}*t*y*x^{-1}*t*x^{-1}*y*x \in M \implies$$

$$Mt_{20}t_{18} = Mt_6t_4 \ (16)$$

$$t*x*t*x*y*t*x*t*x*t*y*t*x^{-2} \in M \implies$$

$$Mt_{18}t_7t_{11} = Mt_{13}t_{10}t_{20} \ (17)$$

$$y*x^{-2}*t*x^2*t*x*y*x^{-1}*t*x^{-2}*t*x^{-2} \in M \implies$$

$$Mt_{12}t_8 = Mt_{13}t_6 \ (18)$$

$$t*x*y*x^2*t*x*y*t*x*y*x^{-1}*t*y \in M \implies$$

$$Mt_5t_{19} = Mt_{17}t_4 \ (19)$$

$$Mt_1t_3t_5t_7 = Mt_4t_{17}t_{16}t_{20} \ (20)$$

$$Mt_1t_5t_7t_{19} = Mt_6t_9t_{11}t_4 \ (21)$$

$$Mt_1t_5t_7t_{19} = Mt_{17}t_7t_5t_{10} \ (22)$$

$$Mt_1t_3t_5t_7t_1 = Mt_{12}t_{10}t_6 \ (23)$$

$$Mt_1t_3t_5t_{15} = Mt_7t_{19}t_3 \ (24)$$

$$Mt_1t_5t_7t_1 = Mt_{16}t_{12}t_6 \ (25)$$

$$Mt_1t_3t_{11} = Mt_9t_7t_{20} \ (26)$$

$$Mt_1t_3t_{11} = Mt_{17}t_{14}t_{15} \ (27)$$

$$Mt_1t_3t_5 = Mt_{13}t_{19}t_{17} \ (28)$$

$$Mt_1t_3t_{13} = Mt_{17}t_5t_{16} \ (29)$$

$$Mt_1t_3t_{13} = Mt_{20}t_4t_{15} \ (30)$$

$$Mt_1t_5t_3 = Mt_{10}t_{12}t_{19} \ (31)$$

$$Mt_1t_5t_7 = Mt_3t_{15}t_9 \ (32)$$

$$Mt_1t_3t_5t_3 = Mt_{13}t_2t_{18} \ (33)$$

$$Mt_1t_3t_{11}t_{13} = Mt_8t_{14}t_{15} \ (34)$$

$$Mt_1t_3t_{11}t_5 = Mt_1t_7t_{18}t_{11} \ (35)$$

$$Mt_1t_3t_{11}t_5 = Mt_7t_1t_{17}t_{19} \ (36)$$

$$Mt_1t_3t_5t_7 = Mt_4t_{17}t_{16}t_{20} \ (37)$$

$$Mt_1t_3t_{13}t_1 = Mt_{17}t_4t_7t_5 \ (38)$$

$$Mt_1t_3t_5 = Mt_1t_7t_{18}t_{11} \ (39)$$

$$Mt_1t_5t_9 = Mt_7t_3t_{16} \ (40)$$

are consequences of the additional relations.

Conjugating (2) by N gives us the following relations:

$$t_1t_2 = e \ (41)$$

$$t_3t_4 = e \ (42)$$

$$t_5t_6 = e \ (43)$$

$$t_7t_8 = e \ (44)$$

$$t_9t_{10} = e \ (45)$$

$$t_{11}t_{12} = e \ (46)$$

$$t_{13}t_{14} = e \ (47)$$

$$t_{17}t_{18} = e \ (48)$$

$$t_{19}t_{20} = e \ (49)$$

Conjugating (1) by N gives us the following relations:

$$t_1t_6t_1t_6 = e \ (50)$$

$$t_1t_{12}t_1t_{12} = e \ (51)$$

$$t_3t_{11}t_3t_{11} = e \ (52)$$

Conjugating (3) by N gives us the following relations:

$$(xyx^{-1}) \ t_2t_4t_2t_4t_2 = e \ (53)$$

$$(y) \ t_5t_7t_5t_7t_5 = e \ (54)$$

$$(yx^2yx) \ t_6t_{19}t_6t_{19}t_6 = e \ (55)$$

Conjugating (14) by N gives us the following relation:

$$Mt_2t_3 = Mt_7t_{10} \ (56)$$

Conjugating (15) by N gives us the following relations:

$$Mt_5t_{19} = Mt_{18}t_4 \ (57)$$

$$Mt_{17}t_3 = Mt_6t_{20} \ (58)$$

Conjugating (18) by N gives us the following relation:

$$Mt_{16}t_{13} = Mt_1t_{19} \ (59)$$

2.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas.

Lemma 1: $Mt_1 = Mt_2$

Proof: $M\ \underline{t_1} = M\ \underline{t_2} = Mt_2$ (by (41)).

Lemma 2: $Mt_1t_1 \in$ [*]

Proof: Since our “t's” are of order two, $t_1t_1 = e$.

Therefore $Mt_1t_1 = Me \in$ [*].

Lemma 3: $Mt_1t_2 \in$ [*]

Proof: $Mt_1\ \underline{t_2} = Mt_1\ \underline{t_1} = Mt_1t_1$ (by (41))

$Mt_1t_1 \in$ [*] (by Lemma 2).

Lemma 4: $Mt_1t_3 \in$ [1,3]

Proof: $(Mt_1t_3)^e = Mt_1t_3 \in$ [1,3]

since $e \in N$

Therefore $Mt_1t_3 \in$ [1,3].

Lemma 5: $Mt_1t_4 \in$ [1,3]

Proof: $Mt_1\ \underline{t_4} = Mt_1\ \underline{t_3} = Mt_1t_3$ (by (42))

$Mt_1t_3 \in$ [1,3] by Lemma 4.

Lemma 6: $Mt_1t_5 \in$ [1,5]

Proof: $(Mt_1t_5)^e = Mt_1t_5 \in$ [1,5]

since $e \in N$

Therefore $Mt_1t_5 \in$ [1,5].

Lemma 7: $Mt_1t_3 = Mt_7t_9$

Proof: $M\ \underline{t_1}\ t_3 = M\ \underline{t_2}\ t_3 = Mt_2t_3$ (by (41))

Now $Mt_7\ \underline{t_9} = Mt_7\ \underline{t_{10}} = Mt_7t_{10}$ (by (45))

$Mt_2t_3 = Mt_7t_{10}$ (by (42)).

Lemma 8: $Mt_1t_3t_1 \in$ [1,3]

Proof: $M\ \underline{t_1}\ t_3\ \underline{t_1} = M\ \underline{t_2}\ t_3\ t_2 = Mt_2t_3t_2$ (by (41))

$Mt_2\ \underline{t_3}\ t_2 = Mt_2\ \underline{t_4}\ t_2 = Mt_2t_4t_2$ (by (42))

$M\ \underline{(xyx^{-1})t_2t_4t_2} = M\ \underline{t_2t_4} = Mt_2t_4$ (by (53))

$M\ \underline{t_2}\ t_4 = M\ \underline{t_1}\ t_4 = Mt_1t_4$ (by (41))

$Mt_1\ \underline{t_4} = Mt_1\ \underline{t_3} = Mt_1t_3$ (by (42))

$Mt_1t_3 \in$ [1,3].

Lemma 9: $Mt_1t_3t_2 \in$ [1,3]

Proof: $Mt_1t_3\ \underline{t_2} = Mt_1t_3\ \underline{t_1} = Mt_1t_3t_1$ (by (41))

$Mt_1t_3t_1 \in$ [1,3] (by Lemma 8).

Lemma 10: $Mt_1t_3t_3 \in [1]$.

Proof: Since our "t's" are of order two, $Mt_1\ \underline{t_3t_3} = Mt_1\ \underline{e} = Mt_1 \in [1]$

Therefore $Mt_1t_3t_3 \in [1]$.

Lemma 11: $Mt_1t_3t_4 \in [1]$

Proof: $Mt_1t_3\ \underline{t_4} = Mt_1t_3\ \underline{t_3} = Mt_1t_3t_3$ (by (42))

$Mt_1t_3t_3 \in [1]$ (by Lemma 10).

Lemma 12: $Mt_1t_3t_5 \in [1,3,5]$

Proof: $(Mt_1t_3t_5)^e = Mt_1t_3t_5 \in [1,3,5]$

since $e \in N$

Therefore $Mt_1t_3t_5 \in [1,3,5]$.

Lemma 13: $Mt_1t_3t_6 \in [1,3,5]$

Proof: $Mt_1t_3\ \underline{t_6} = Mt_1t_3\ \underline{t_5} = Mt_1t_3t_5$ (by (43))

$Mt_1t_3t_5 \in [1,3,5]$.

Lemma 14: $Mt_1t_3t_{11} \in [1,3,11]$

Proof: $(Mt_1t_3t_{11})^e = Mt_1t_3t_{11} \in [1,3,11]$

since $e \in N$

Therefore $Mt_1t_3t_{11} \in [1,3,11]$.

Lemma 15: $Mt_1t_3t_{12} \in [1,3,11]$

Proof: $Mt_1t_3\ \underline{t_{12}} = Mt_1t_3\ \underline{t_{11}} = Mt_1t_3t_{11}$ (by (46))

$Mt_1t_3t_{11} \in [1,3,11]$.

Lemma 16: $Mt_1t_3t_{13} \in [1,3,13]$

Proof: $(Mt_1t_3t_{13})^e = Mt_1t_3t_{13} \in [1,3,13]$

since $e \in N$

Therefore $Mt_1t_3t_{13} \in [1,3,13]$.

Lemma 17: $Mt_1t_3t_{17} \in [1,3,5]$.

Proof: $M\ \underline{t_1t_3t_{17}} = M\ \underline{(yx^2yx)t_{16}t_{13}t_{19}t_1t_8t_6t_{20}} =$

$Mt_{16}t_{13}t_{19}t_1t_8t_6t_{20}$ (by (6))

$\underline{Mt_{16}t_{13}t_{19}}\ t_1t_8t_6t_{20} = \underline{Mt_1}\ t_1t_8t_6t_{20} = Mt_8t_6t_{20}$ (by (59))

$Mt_8t_6t_{20} \in [1,3,5]$ since

$(Mt_8t_6t_{20})^{(1,13,8)(2,14,7)(3,19,6)(4,20,5)(9,15,11)(10,16,12)} = Mt_1t_3t_5$

where $(1,13,8)(2,14,7)(3,19,6)(4,20,5)(9,15,11)(10,16,12) \in N$.

Lemma 18: (1) $Mt_1t_5 = Mt_5t_1$ and (2) $Mt_1t_5 = Mt_2t_6$

(1) Proof: $Mt_1\ \underline{t_5} = Mt_1\ \underline{t_6} = Mt_1t_6$ (by (43))

$M\ \underline{t_1t_6} = M\ \underline{t_6t_1} = Mt_6t_1$ (by (50))

$M\ \underline{t_6}\ t_1 = M\ \underline{t_5}\ t_1 = Mt_5t_1$ (by (43)).

(2) Proof: $M\ \underline{t_1}\ t_5 = M\ \underline{t_2}\ t_5 = Mt_2t_5$ (by (41))

$Mt_2\ \underline{t_5} = Mt_2\ \underline{t_6} = Mt_2t_6$ (by (43)).

Lemma 19: $Mt_1t_5t_1 \in [1]$

Proof: $Mt_1\ \underline{t_5}\ t_1 = Mt_1\ \underline{t_6}\ t_1 = Mt_1t_6t_1$ (by (43))

$M\ \underline{t_1t_6t_1} = M\ \underline{t_1} = Mt_1$ (by (50))

$Mt_1 \in [1]$.

Lemma 20: $Mt_1t_5t_3 \in$ [1,3,11].

Proof: $\underline{Mt_1t_5t_3} = \underline{Mt_{10}t_{12}t_{19}} = Mt_{10}t_{12}t_{19}$ (by (31))

$Mt_{10}t_{12}t_{19} \in$ [1,3,11] since

$(Mt_{10}t_{12}t_{19})^{(1,13,6,17,10)(2,14,5,18,9)(3,15,8,20,12)(4,16,7,19,11)} = Mt_1t_3t_{11}$

where (1,13,6,17,10)(2,14,5,18,9)(3,15,8,20,12)(4,16,7,19,11) $\in N$.

Lemma 21: $Mt_1t_5t_7 \in$ [1,5,7]

Proof: $(Mt_1t_5t_7)^e = Mt_1t_5t_7 \in$ [1,5,7]

since $e \in N$

Therefore $Mt_1t_5t_7 \in$ [1,5,7].

Lemma 22: $Mt_1t_5t_9 \in$ [1,3,11]

Proof: $Mt_1\ \underline{t_5}\ t_9 = Mt_1\ \underline{t_6}\ t_9 = Mt_1t_6t_9$ (by (43))

$Mt_1t_6\ \underline{t_9} = Mt_1t_6\ \underline{t_{10}} = Mt_1t_6t_{10}$ (by (45))

$Mt_1t_6t_{10} \in$ [1,3,11] since

$M(t_1t_6t_{10})^{(1,14,20)(2,13,19)(3,8,5)(4,7,6)(9,11,17)(10,12,18)} = Mt_1t_4t_{12}$

where (1,14,20)(2,13,19)(3,8,5)(4,7,6)(9,11,17)(10,12,18) $\in N$

$Mt_1\ \underline{t_4}\ t_{12} = Mt_1\ \underline{t_3}\ t_{12} = Mt_1t_3t_{12}$ (by (42))

$Mt_1t_3\ \underline{t_{12}} = Mt_1t_3\ \underline{t_{11}} = Mt_1t_3t_{11}$ (by (46)).

Lemma 23: $Mt_1t_5t_{10} \in$ [1,3,11]

Proof: $Mt_1t_5\ \underline{t_{10}} = Mt_1t_5\ \underline{t_9} = Mt_1t_5t_9$ (by (45))

$Mt_1t_5t_9 \in$ [1,3,11] (by Lemma 22).

Lemma 24: $Mt_1t_3t_5 = Mt_{14}t_{11}t_{10}$

Proof: $M\ \underline{t_1t_3t_5} = M\ \underline{(x^2yx^{-2})t_3t_{10}t_1t_8t_{14}t_{11}t_{10}} = Mt_3t_{10}t_1t_8t_{14}t_{11}t_{10}$ (by (8))

$\underline{Mt_3t_{10}t_1}\ t_8t_{14}t_{11}t_{10} = \underline{Mt_8}\ t_8t_{14}t_{11}t_{10} = Mt_{14}t_{11}t_{10}$ (by (15)).

Lemma 25: $Mt_1t_3t_5t_1 \in$ [1,3,5]

Proof: $Mt_1t_3\ \underline{t_5}\ t_1 = Mt_1t_3\ \underline{t_6}\ t_1 = Mt_1t_3t_6t_1$ (by (43))

$Mt_1t_3\ \underline{t_6t_1} = Mt_1t_3\ \underline{t_1t_6} = Mt_1t_3t_1t_6$ (by (50))

$Mt_1t_3t_1\ \underline{t_6} = Mt_1t_3t_1\ \underline{t_5} = Mt_1t_3t_1t_5$ (by (43))

$M\ \underline{t_1}\ t_3\ \underline{t_1}\ t_5 = M\ \underline{t_2}\ t_3\ \underline{t_2}\ t_5 = Mt_2t_3t_2t_5$ (by (41))

$Mt_2\ \underline{t_3}\ t_2t_5 = Mt_2\ \underline{t_4}\ t_2t_5 = Mt_2t_4t_2t_5$ (by (42))

$M\ \underline{(xyx^{-1})t_2t_4t_2}\ t_5 = M\ \underline{t_2t_4}\ t_5 = Mt_2t_4t_5$ (by (53))

$M\ \underline{t_2}\ t_4t_5 = M\ \underline{t_1}\ t_4t_5 = Mt_1t_4t_5$ (by (41))

$Mt_1\ \underline{t_4}\ t_5 = Mt_1\ \underline{t_3}\ t_5 = Mt_1t_3t_5$ (by (42))

$Mt_1t_3t_5 \in$ [1,3,5].

Lemma 26: $Mt_1t_3t_5t_2 \in$ [1,3,5]

Proof: $Mt_1t_3t_5\ \underline{t_2} = Mt_1t_3t_5\ \underline{t_1} = Mt_1t_3t_5t_1$ (by (41))

$Mt_1t_3t_5t_1 \in$ [1,3,5] (by Lemma 25).

Lemma 27: $Mt_1t_3t_5t_3 \in$ [1,3,11]

Proof: $\underline{Mt_1t_3t_5t_3} = \underline{Mt_{13}t_2t_{18}} = Mt_{13}t_2t_{18}$ (by (33))

$Mt_{13}t_2t_{18} \in$ [1,3,11] since

$(Mt_{13}t_2t_{18})^{(1,4,18,11,13)(2,3,17,12,14)(5,16,8,19,9)(6,15,7,20,10)} = Mt_1t_3t_{11}$

where (1,4,18,11,13)(2,3,17,12,14)(5,16,8,19,9)(6,15,7,20,10) $\in N$.

Lemma 28: $Mt_1t_3t_5t_4 \in$ [1,3,11]

Proof: $Mt_1t_3t_5\ \underline{t_4} = Mt_1t_3t_5\ \underline{t_3} = Mt_1t_3t_5t_3$ (by (42))

$Mt_1t_3t_5t_3 \in$ [1,3,11] (by Lemma 27).

Lemma 29: $Mt_1t_3t_5t_5 \in$ [1,3]

Proof: Since our "t's" are of order two, $t_5t_5 = e$.

Therefore $Mt_1t_3\ \underline{t_5t_5} = Mt_1t_3\ \underline{e} = Mt_1t_3 \in$ [1,3].

Lemma 30: $Mt_1t_3t_5t_6 \in$ [1,3]

Proof: $Mt_1t_3t_5\ \underline{t_6} = Mt_1t_3t_5\ \underline{t_5} = Mt_1t_3t_5t_5$ (by (43))

$Mt_1t_3\ \underline{t_5t_5} = Mt_1t_3\ \underline{e} = Mt_1t_3 \in$ [1,3].

Lemma 31: $Mt_1t_3t_5t_7 \in$ [1,3,5,7]

Proof: $(Mt_1t_3t_5t_7)^e = Mt_1t_3t_5t_7 \in$ [1,3,5,7]

since $e = N$

Therefore $Mt_1t_3t_5t_7 \in$ [1,3,5,7].

Lemma 32: $Mt_1t_3t_5t_{15} \in$ [1,5,7]

Proof: $\underline{Mt_1t_3t_5t_{15}} = \underline{Mt_7t_{19}t_3} = Mt_7t_{19}t_3$ (by (24))

$Mt_7t_{19}t_3 \in$ [1,5,7] since

$(Mt_7t_{19}t_3)^{(1,3,7)(2,4,8)(5,14,19)(6,13,20)(11,15,17)(12,16,18)} = Mt_1t_5t_7$

where (1,3,7)(2,4,8)(5,14,19)(6,13,20)(11,15,17)(12,16,18) $\in N$.

Lemma 33: $Mt_1t_3t_5t_{16} \in$ [1,5,7]

Proof: $Mt_1t_3t_5\ \underline{t_{16}} = Mt_1t_3t_5\ \underline{t_{15}} = Mt_1t_3t_5t_{15}$ (by (2))

$Mt_1t_3t_5t_{15} \in$ [1,5,7] (by Lemma 32).

Lemma 34: $Mt_1t_3t_5t_{17} \in$ [1,3]

Proof: $\underline{Mt_1t_3t_5t_{17}} = \underline{Mt_{13}t_{19}} = Mt_{13}t_{19}$ (by (28))

$Mt_{13}t_{19} \in$ [1,3] since

$(Mt_{13}t_{19})^{(1,8,13)(2,7,14)(3,6,19)(4,5,20)(9,11,15)(10,12,16)} = Mt_1t_3$

where (1,8,13)(2,7,14)(3,6,19)(4,5,20)(9,11,15)(10,12,16) $\in N$.

Lemma 35: $Mt_1t_3t_{11} = Mt_9t_7t_{20}$

Proof: $\underline{Mt_1t_3t_{11}} = \underline{Mt_9t_7t_{20}} = Mt_9t_7t_{20}$ (by (26)).

Lemma 36: $Mt_1t_3t_{11}t_1 \in$ [1,3,11]

Proof: $Mt_1t_3\ \underline{t_{11}}\ t_1 = Mt_1t_3\ \underline{t_{12}}\ t_1 = Mt_1t_3t_{12}t_1$ (by (46))

$Mt_1t_3\ \underline{t_{12}t_1} = Mt_1t_3\ \underline{t_1t_{12}} = Mt_1t_3t_1t_{12}$ (by (51))

$M\ \underline{t_1}\ t_3\ \underline{t_1}\ t_{12} = M\ \underline{t_2}\ t_3\ \underline{t_2}\ t_{12} = Mt_2t_3t_2t_{12}$ (by (41))

$Mt_1\ \underline{t_3}\ t_1t_{12} = Mt_2\ \underline{t_4}\ t_2t_{12} = Mt_2t_4t_2t_{12}$ (by (42))

$M\ \underline{(xyx^{-1})t_2t_4t_2}\ t_{12} = M\ \underline{t_2t_4}\ t_{12} = Mt_2t_4t_{12}$ (by (53))

$M\ \underline{t_2}\ t_4t_{12} = M\ \underline{t_1}\ t_4t_{12} = Mt_1t_4t_{12}$ (by (41))

$Mt_1\ \underline{t_4}\ t_{12} = Mt_1\ \underline{t_3}\ t_{12} = Mt_1t_3t_{12}$ (by (42))

$Mt_2t_4\ \underline{t_{12}} = Mt_1t_3\ \underline{t_{11}} = Mt_1t_3t_{11}$ (by (46))

$Mt_1t_3t_{11} \in$ [1,3,11].

Lemma 37: $Mt_1t_3t_{11}t_2 \in [1,3,11]$

Proof: $Mt_1t_3t_{11}\ \underline{t_2} = Mt_1t_3t_{11}\ \underline{t_1} = Mt_1t_3t_{11}t_1$ (by (24))

$Mt_1t_3t_{11}t_1 \in [1,3,11]$ (by Lemma 36).

Lemma 38: $Mt_1t_3t_{11}t_3 \in [1,5]$

Proof: $Mt_1t_3\ \underline{t_{11}t_3} = Mt_1t_3\ \underline{t_3t_{11}}$ (by (34)) $= Mt_1t_{11} \in [1,5]$

since $(Mt_1t_{11})^{(3,9,7)(4,10,8)(5,18,11)(6,17,12)(13,19,15)(14,20,16)} = Mt_1t_5$,

where $(3,9,7)(4,10,8)(5,18,11)(6,17,12)(13,19,15)(14,20,16) \in N$.

Lemma 39: $Mt_1t_3t_{11}t_4 \in [1,5]$

Proof: $Mt_1t_3t_{11}\ \underline{t_4} = Mt_1t_3t_{11}\ \underline{t_3} = Mt_1t_3t_{11}t_3$ (by (42))

$Mt_1t_3t_{11}t_3 \in [1,5]$ (by Lemma 38).

Lemma 40: $Mt_1t_3t_{11}t_5 \in [1,3,11,5]$

Proof: $(Mt_1t_3t_{11}t_5)^e = Mt_1t_3t_{11}t_5 \in [1,3,11,5]$

since $e \in N$.

Therefore $Mt_1t_3t_{11}t_5 \in [1,3,11,5]$.

Lemma 41: $Mt_1t_3t_{11}t_{11} \in [1,3]$

Proof: $Mt_1t_3\ \underline{t_{11}t_{11}} = Mt_1t_3\ \underline{e} = Mt_1t_3 \in [1,3]$.

Lemma 42: $Mt_1t_3t_{11}t_{12} \in [1,3]$

Proof: $Mt_1t_3t_{11}\ \underline{t_{12}} = Mt_1t_3t_{11}\ \underline{t_{11}} = Mt_1t_3$ (by (46))

$Mt_1t_3 \in [1,3]$.

Lemma 43: $Mt_1t_3t_{11}t_{13} \in [1,3,5]$

Proof: $\underline{Mt_1t_3t_{11}t_{13}} = \underline{Mt_8t_{14}t_{15}} = Mt_8t_{14}t_{15}$ (by (34))

$Mt_8t_{14}t_{15} \in [1,3,5]$ since

$(Mt_8t_{14}t_{15})^{(1,19,17,12,8)(2,20,18,11,7)(3,15,5,10,14)(4,16,6,9,13)} = Mt_1t_3t_5$

where $(1,19,17,12,8)(2,20,18,11,7)(3,15,5,10,14)(4,16,6,9,13) \in N$

Lemma 44: $Mt_1t_3t_{11}t_{14} \in [1,3,5]$

Proof: $Mt_1t_3t_{11}\ \underline{t_{14}} = Mt_1t_3t_{11}\ \underline{t_{13}} = Mt_1t_3t_{11}t_{13}$ (by (47))

$Mt_1t_3t_{11}t_{13} \in [1,3,5]$ (by Lemma 43).

Lemma 45: $Mt_1t_3t_{11}t_{15} \in [1,5]$

Proof: $\underline{Mt_1t_3t_{11}t_{15}} = \underline{Mt_{17}t_{14}} = Mt_{17}t_{14}$ (by (27))

$Mt_{17}t_{14} \in [1,5]$ since

$(Mt_{17}t_{14})^{(1,12,19,8,17)(2,11,20,7,18)(3,10,15,14,5)(4,9,16,13,6)} = Mt_1t_5$

where $(1,12,19,8,17)(2,11,20,7,18)(3,10,15,14,5)(4,9,16,13,6) \in N$.

Lemma 46: (1) $Mt_1t_3t_{13} = Mt_{17}t_5t_{16}$ and (2) $Mt_1t_3t_{13} = Mt_{20}t_4t_{15}$

(1) Proof: $\underline{Mt_1t_3t_{13}} = \underline{Mt_{17}t_5t_{16}} = Mt_{17}t_5t_{16}$ (by (29)).

(2) Proof: $\underline{Mt_1t_3t_{13}} = \underline{Mt_{20}t_4t_{15}} = Mt_{20}t_4t_{15}$ (by (30)).

Lemma 47: $Mt_1t_3t_{13}t_1 \in [1,3,5,7]$

$\underline{Mt_1t_3t_{13}t_1} = \underline{Mt_{17}t_4t_7t_5} = Mt_{17}t_4t_7t_5$ (by (38))

$Mt_{17}t_4t_7t_5 \in [1,3,5,7]$ since

$(Mt_{17}t_4t_7t_5)^{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)} = Mt_1t_3t_5t_7$

where $(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16) \in N$

$Mt_1t_3t_5t_7 \in [1,3,5,7]$.

Lemma 48: $Mt_1t_3t_{13}t_{13} \in [1,3]$

Proof: $Mt_1t_3\ \underline{t_{13}t_{13}} = Mt_1t_3\ \underline{e} = Mt_1t_3 \in [1,3]$.

Lemma 49: (1) $Mt_1t_5t_7 = Mt_3t_{15}t_9$ and (2) $Mt_1t_5t_7 = Mt_6t_2t_8$

(1) Proof: $\underline{Mt_1t_5t_7} = \underline{Mt_3t_{15}t_9} = Mt_3t_{15}t_9$ (by (32)).

(2) Proof: $Mt_1\ \underline{t_5}\ t_7 = Mt_1\ \underline{t_6}\ t_7 = Mt_1t_6t_7$ (by (43))

$M\ \underline{t_1t_6}\ t_7 = M\ \underline{t_6t_1}\ t_7 = Mt_6t_1t_7$ (by (50))

$Mt_6\ \underline{t_1}\ t_7 = Mt_6\ \underline{t_2}\ t_7 = Mt_6t_2t_7$ (by (41))

$Mt_6t_2\ \underline{t_7} = Mt_6t_2\ \underline{t_8} = Mt_6t_2t_8$ (by (44)).

Lemma 50: $Mt_1t_5t_7t_{19} \in [1,5,7,19]$

Proof: $(Mt_1t_5t_7t_{19})^e = Mt_1t_5t_7t_{19}$

since $e \in N$

Therefore $Mt_1t_5t_7t_{19} \in [1,5,7,19]$.

Lemma 51: $Mt_1t_5t_7t_1 \in [1,3,5]$

Proof: $\underline{Mt_1t_5t_7t_1} = \underline{Mt_{16}t_{12}t_6} = Mt_{16}t_{12}t_6$ (by (25))

$Mt_{16}t_{12}t_6 \in [1,3,5]$ since

$(Mt_{16}t_{12}t_6)^{(1,16)(2,15)(3,12)(4,11)(5,6)(7,18)(8,17)(9,10)(13,19)(14,20)} = Mt_1t_3t_5$

where $(1,16)(2,15)(3,12)(4,11)(5,6)(7,18)(8,17)(9,10)(13,19)(14,20) \in N$.

Lemma 52: $Mt_1t_5t_7t_2 \in [1,3,5]$

Proof: $Mt_1t_5t_7\ \underline{t_2} = Mt_1t_5t_7\ \underline{t_1} = Mt_1t_5t_7t_1$ (by (41))

$Mt_1t_5t_7t_1 \in$ [1,3,5] by (Lemma 51).

Lemma 53: $Mt_1t_5t_7t_7 \in$ [1,5]

Proof: Since our "t's" are of order two, $t_7t_7 = e$

$Mt_1t_5\ \underline{t_7t_7} = Mt_1t_5\ \underline{e} = Mt_1t_5$.

Lemma 54: (1) $Mt_1t_3t_5t_7 = Mt_{17}t_4t_7t_5$ and (2) $Mt_1t_3t_5t_7 = Mt_4t_{17}t_{16}t_{20}$

(1) Proof: $Mt_1t_3(y)\ \underline{(y)t_5t_7} = Mt_1t_3(y)\ \underline{t_5t_7t_5}$ (by (54))

$Mt_1t_3\ \underline{(y)}\ t_5t_7t_5 =$

$Mt_1t_3\ \underline{(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16)}$ *

$t_5t_7t_5 = Mt_{17}t_4t_5t_7t_5$

$Mt_{17}\ \underline{t_4}\ t_5t_7t_5 = Mt_{17}\ \underline{t_3}\ t_5t_7t_5 = Mt_{17}t_3t_5t_7t_5$ (by (42))

$\underline{Mt_{17}t_3}\ t_5t_7t_5 = \underline{Mt_6t_{20}}\ t_5t_7t_5 = Mt_6t_{20}t_5t_7t_5$ (by (58))

$Mt_6t_{20}\ \underline{t_5}\ t_7t_5 = Mt_6t_{20}\underline{t_6}\ t_7t_5 = Mt_6t_{20}t_6t_7t_5$ (by (43))

$Mt_6\ \underline{t_{20}}\ t_6t_7t_5 = Mt_6\ \underline{t_{19}}\ t_6t_7t_5 = Mt_6t_{19}t_6t_7t_5$ (by (49))

$M\ \underline{t_6t_{19}t_6}\ t_7t_5 = M\ \underline{(yx^2yx)t_6t_{19}}\ t_7t_5 = Mt_6t_{19}t_7t_5$ (by (55))

$M\ \underline{t_6}\ t_{19}t_7t_5 = M\ \underline{t_5}\ t_{19}t_7t_5 = Mt_5t_{19}t_7t_5$ (by (43))

$\underline{Mt_5t_{19}}\ t_7t_5 = \underline{Mt_{18}t_4}\ t_7t_5 = Mt_{18}t_4t_7t_5$ (by (57))

$M\ \underline{t_{18}}\ t_4t_7t_5 = M\ \underline{t_{17}}\ t_4t_7t_5 = Mt_{17}t_4t_7t_5$ (by (48)).

(2) Proof: $\underline{Mt_1t_3t_5t_7} = \underline{Mt_4t_{17}t_{16}t_{20}} = Mt_4t_{17}t_{16}t_{20}$ (by (37)).

Lemma 55: $Mt_1t_3t_5t_7t_1 \in$ [1,3,13]

$\underline{Mt_1t_3t_5t_7t_1} = \underline{Mt_{12}t_{10}t_6} = Mt_{12}t_{10}t_6$ (by (23))

$Mt_{12}t_{10}t_6 \in$ [1,3,13] since

$(Mt_{12}t_{10}t_6)^{(1,17,8,19,12)(2,18,7,20,11)(3,5,14,15,10)(4,6,13,16,9)} = Mt_1t_3t_{13}$

where (1,17,8,19,12)(2,18,7,20,11)(3,5,14,15,10)(4,6,13,16,9) $\in N$.

Lemma 56: $Mt_1t_3t_5t_7t_7 \in$ [1,3,5]

Proof: Since our "t's" are of order two, $t_7t_7 = e$

$Mt_1t_3t_5\ \underline{t_7t_7} = Mt_1t_3t_5\ \underline{e} = Mt_1t_3t_5 \in$ [1,3,5]

Therefore $Mt_1t_3t_5t_7t_7 \in$ [1,3,5].

Lemma 57: (1) $Mt_1t_3t_{11}t_5 = Mt_1t_7t_{18}t_{11}$ and (2) $Mt_1t_3t_{11}t_5 = Mt_7t_1t_{17}t_{19}$

(1) Proof: $\underline{Mt_1t_3t_{11}t_5} = \underline{Mt_1t_7t_{18}t_{11}} = Mt_1t_7t_{18}t_{11}$ (by (35)).

(2) Proof: $\underline{Mt_1t_3t_{11}t_5} = \underline{Mt_7t_1t_{17}t_{19}} = Mt_7t_1t_{17}t_{19}$ (by (36)).

Lemma 58: $Mt_1t_3t_{11}t_5t_1 \in$ [1,3,11,5]

$Mt_1t_3t_{11}\ \underline{t_5}\ t_1 = Mt_1t_3t_{11}\ \underline{t_6}\ t_1 = Mt_1t_3t_{11}t_6t_1$ (by (43))

$Mt_1t_3\ \underline{t_{11}}\ t_6t_1 = Mt_1t_3\ \underline{t_{12}}\ t_6t_1 = Mt_1t_3t_{12}t_6t_1$ (by (46))

$Mt_1t_3t_{12}\ \underline{t_6t_1} = Mt_1t_3t_{12}\ \underline{t_1t_6} = Mt_1t_3t_{12}t_1t_6$ (by (50))

$Mt_1t_3\ \underline{t_{12}t_1}\ t_6 = Mt_1t_3\ \underline{t_1t_{12}}\ t_6 = Mt_1t_3t_1t_{12}t_6$ (by (51))

$M\ \underline{t_1}\ t_3\ \underline{t_1}\ t_{12}t_6 = M\ \underline{t_2}\ t_3\ \underline{t_2}\ t_{12}t_6 = Mt_2t_3t_2t_{12}t_6$ (by (41))

$Mt_2\ \underline{t_3}\ t_2t_{12}t_6 = Mt_2\ \underline{t_4}\ t_2t_{12}t_6 = Mt_2t_4t_2t_{12}t_6$ (by (42))

$M\ \underline{t_2t_4t_2}\ t_{12}t_6 = M\ \underline{(xyx^{-1})t_2t_4}\ t_{12}t_6 = Mt_2t_4t_{12}t_6$ (by (53))

$M\ \underline{t_2}\ t_4t_{12}t_6 = M\ \underline{t_1}\ t_4t_{12}t_6 = Mt_1t_4t_{12}t_6$ (by (41))

$Mt_1\ \underline{t_4}\ t_{12}t_6 = Mt_1\ \underline{t_3}\ t_{12}t_6 = Mt_1t_3t_{12}t_6$ (by (42))

$Mt_1t_3\ \underline{t_{12}}\ t_6 = Mt_1t_3\ \underline{t_{11}}\ t_6 = Mt_1t_3t_{11}t_6$ (by (46))

$Mt_1t_3t_{11}\ \underline{t_6} = Mt_1t_3t_{11}\ \underline{t_5} = Mt_1t_3t_{11}t_5$ (by (43))

$Mt_1t_3t_{11}t_5 \in$ [1,3,11,5].

Lemma 59: $Mt_1t_3t_{11}t_5t_2 \in$ [1,3,11,5]

Proof: $Mt_1t_3t_{11}t_5\ \underline{t_2} = Mt_1t_3t_{11}t_5\ \underline{t_1} = Mt_1t_3t_{11}t_5t_1$ (by (41))

$Mt_1t_3t_{11}t_5t_1 \in$ [1,3,11,5] (by Lemma 58).

Lemma 60: $Mt_1t_3t_{11}t_5t_5 \in$ [1,3,11]

Proof: Since our "t's" are of order two, $t_5t_5 = e$

$Mt_1t_3t_{11}\ \underline{t_5t_5} = Mt_1t_3t_{11}\ \underline{e} = Mt_1t_3t_{11} \in$ [1,3,11].

Lemma 61: (1) $Mt_1t_5t_7t_{19} = Mt_6t_9t_{11}t_4$ and (2) $Mt_1t_5t_7t_{19} = Mt_{17}t_7t_5t_{10}$

(1) Proof: $\underline{Mt_1t_5t_7t_{19}} = \underline{Mt_6t_9t_{11}t_4} = Mt_6t_9t_{11}t_4$ (by (21)).

(2) Proof: $\underline{Mt_1t_5t_7t_{19}} = \underline{Mt_{17}t_7t_5t_{10}} = Mt_{17}t_7t_5t_{10}$ (by (22)).

Lemma 62: $Mt_1t_5t_7t_{19}t_{19} \in$ [1,5,7]

Proof: Since our "t's" are of order two, $t_{19}t_{19} = e$

$Mt_1t_5t_7t_{19}t_{19} = Mt_1t_5t_7(e) = Mt_1t_5t_7 \in$ [1,5,7]

Therefore $Mt_1t_5t_7t_{19}t_{19} \in$ [1,5,7].

2.3 Double Coset Enumeration of G over M and N

$MeN = \{M\}$. Since N is transitive on $\Omega = \{1, 2, \ldots, 20\}$ and $Mt_1 \in Mt_1N$, 20 "$t_i's$" extend the coset representative N to the double coset $Mt_1N =$ [1]. Now

$$N^{(1)} \geq <(3,7,9)(4,8,10)(5,11,18)(6,12,17)(13,15,19)(14,16,20)> \cong \mathbb{Z}_3.$$

Since $Mt_1 = Mt_2$ (Lemma 1),

$$Mt_1^{(1,2)(3,13)(4,14)(5,6)(7,19)(8,20)(9,15)(10,16)(11,17)(12,18)} = Mt_2 \text{ where}$$

$$(1,2)(3,13)(4,14)(5,6)(7,19)(8,20)(9,15)(10,16)(11,17)(12,18) \in N \implies$$

$$N^{(1)} \geq <(3,7,9)(4,8,10)(5,11,18)(6,12,17)(13,15,19)(14,16,20),$$

$$(1,2)(3,13)(4,14)(5,6)(7,19)(8,20)(9,15)(10,16)(11,17)(12,18)> \cong S_3.$$

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{60}{6} = 10$.

$N^{(1)}$ has orbits $\{1,2\}$, $\{3,7,13,9,19,15\}$, $\{4,8,14,10,20,16\}$, and $\{5,11,6,18,17,12\}$ on Ω. We note that $Mt_1t_1 \in$ [*], $Mt_1t_3 \in$ [1,3] (Lemma 4), $Mt_1t_4 \in$ [1,3] (Lemma 5) and $Mt_1t_5 \in$ [1,5] (Lemma 6).

We now consider the double coset [1,3].

$N^{(13)} \geq N^{13} \geq 1$.

However, since $Mt_1t_3 = Mt_7t_9$ (Lemma 7),

$$<(1,7)(2,8)(3,9)(4,10)(5,16)(6,15)(11,19)(12,20)(13,14)(17,18)> \in N^{(13)}, \implies$$

$$N^{(13)} \geq <(1,7)(2,8)(3,9)(4,10)(5,16)(6,15)(11,19)(12,20)(13,14)(17,18)> \cong \mathbb{Z}_2.$$

Thus, [1,3] contains $\frac{60}{2} = 30$ double cosets.

The orbits of $N^{(13)}$ on Ω are $\{1,7\}$, $\{2,8\}$, $\{3,9\}$, $\{4,10\}$, $\{5,16\}$, $\{6,15\}$, $\{11,19\}$, $\{12,20\}$, $\{13,14\}$, $\{17,18\}$. We note that $Mt_1t_3t_1 \in$ [1,3] (Lemma 8), $Mt_1t_3t_2 \in$ [1,3] (Lemma 9), $Mt_1t_3t_3 \in$ [1] (Lemma 10), $Mt_1t_3t_4 \in$ [1] (Lemma 11), $Mt_1t_3t_5 \in$ [1,3,5] (Lemma 12), $Mt_1t_3t_6 \in$ [1,3,5] (Lemma 13), $Mt_1t_3t_{11} \in$ [1,3,11] (Lemma 14), $Mt_1t_3t_{12} \in$ [1,3,11] (Lemma 15), $Mt_1t_3t_{13} \in$ [1,3,13] (Lemma 16) and $Mt_1t_3t_{17} \in$ [1,3,5] (Lemma 17).

Next, we consider the double coset [1,5].

Note $N^{1,5} = 1$.

Now $Mt_1t_5 = Mt_5t_1$ (Lemma 18), so

(1,5)(2,6)(3,4)(7,20)(8,19)(9,17)(10,18)(11,15)(12,16)(13,14) $\in N^{(1,5)}$ and

$Mt_1t_5 = Mt_2t_6$ (Lemma 18) so

(1,2)(3,13)(4,14)(5,6)(7,19)(8,20)(9,15)(10,16)(11,17)(12,18) $\in N^{(1,5)}$.

Thus, $N^{(1,5)} \geq < (1,5)(2,6)(3,4)(7,20)(8,19)(9,17)(10,18)(11,15)(12,16)(13,14),$

$(1,2)(3,13)(4,14)(5,6)(7,19)(8,20)(9,15)(10,16)(11,17)(12,18) > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

So [1,5] contains $\frac{60}{4} = 15$ right cosets.

The orbits of $N^{(1,5)}$ on Ω are $\{1,6,5,2\}$, $\{3,14,4,13\}$, $\{7,8,20,19\}$, $\{9,11,17,15\}$ and $\{10,12,18,16\}$. $Mt_1t_5t_1 \in [1]$ (Lemma 19), $Mt_1t_5t_3 \in$ [1,3,11] (Lemma 20), $Mt_1t_5t_7 \in$ [1,5,7] (Lemma 21), $Mt_1t_5t_9 \in$ [1,3,11] (Lemma 22) and $Mt_1t_5t_{10} \in$ [1,3,11] (Lemma 23).

We look at the double coset [1,3,5].

Note $N^{1,3,5} = 1$.

Now $Mt_1t_3t_5 = Mt_{14}t_{11}t_{10}$ (Lemma 24).

Therefore, (1,14)(2,13)(3,11)(4,12)(5,10)(6,9)(7,8)(15,19)(16,20)(17,18) $\in N^{(1,3,5)}$. So $N^{(1,3,5)} \geq < (1,14)(2,13)(3,11)(4,12)(5,10)(6,9)(7,8)(15,19)(16,20)(17,18) > \cong \mathbb{Z}_2$. Thus [1,3,5] contains $\frac{60}{2} = 30$ double cosets.

The orbits of $N^{(1,3,5)}$ on Ω are $\{1,14\}$, $\{2,13\}$, $\{3,11\}$, $\{4,12\}$, $\{5,10\}$, $\{6,9\}$, $\{7,8\}$, $\{15,19\}$, $\{16,20\}$ and $\{17,18\}$.

$Mt_1t_3t_5t_1 \in$ [1,3,5] (Lemma 25), $Mt_1t_3t_5t_2 \in$ [1,3,5] (Lemma 26), $Mt_1t_3t_5t_3 \in$ [1,3,11] (Lemma 27), $Mt_1t_3t_5t_4 \in$ [1,3,11] (Lemma 28), $Mt_1t_3t_5t_5 \in$ [1,3] (Lemma 29), $Mt_1t_{13}t_5t_6 \in$ [1,3] (Lemma 30), $Mt_1t_{13}t_5t_7 \in$ [1,3,5,7] (Lemma 31), $Mt_1t_3t_5t_{15} \in$ [1,5,7] (Lemma 32), $Mt_1t_3t_5t_{16} \in$ [1,5,7] (Lemma 33) and $Mt_1t_{13}t_5t_{17} \in$ [1,3] (Lemma 34).

The next double coset to be considered is [1,3,11].

Now $N^{(1,3,11)} = 1$.

But $Mt_1t_3t_{11} = Mt_9t_7t_{20}$ (Lemma 35).

Then (1,9)(2,10)(3,7)(4,8)(5,6)(11,20)(12,19)(13,18)(14,17)(15,16) $\in N^{(1,3,11)}$.

Thus [1,3,11] contains $\frac{60}{2} = 30$ double cosets.

The orbits of $N^{(1,3,11)}$ on Ω are $\{1, 9\}$, $\{2, 10\}$, $\{3, 7\}$, $\{4, 8\}$, $\{5, 6\}$, $\{11, 20\}$, $\{12, 19\}$, $\{13, 18\}$, $\{14, 17\}$ and $\{15, 16\}$.

We have $Mt_1t_3t_{11}t_1 \in$ [1,3,11] (Lemma 36), $Mt_1t_3t_{11}t_2 \in$ [1,3,11] (Lemma 37), $Mt_1t_3t_{11}t_3 \in$ [1,5] (Lemma 38), $Mt_1t_3t_{11}t_4 \in$ [1,5] (Lemma 39), $Mt_1t_{13}t_{11}t_5 \in$ [1,3,11,5] (Lemma 40), $Mt_1t_3t_{11}t_{11} \in$ [1,3] (Lemma 41), $Mt_1t_3t_{11}t_{12} \in$ [1,3] (Lemma 42), $Mt_1t_3t_{11}t_{13} \in$ [1,3,5] (Lemma 43), $Mt_1t_{13}t_{11}t_{14} \in$ [1,3,5] (Lemma 44) and $Mt_1t_{13}t_{11}t_{15} \in$ [1,5] (Lemma 45).

The next double coset we consider now is [1,3,13].

Now $N^{1,3,13} = 1$.

However, $Mt_1t_3t_{13} = Mt_{17}t_5t_{16}$ (Lemma 46) $\implies$

(1,17,8,19,12)(2,18,7,20,11)(3,5,14,15,10)(4,6,13,16,9) $\in N^{(1,3,13)}$

and $Mt_1t_3t_{13} = Mt_{20}t_4t_{15}$ (Lemma 46) $\implies$

(1,20)(2,19)(3,4)(5,9)(6,10)(7,17)(8,18)(11,12)(13,15)(14,16) $\in N^{(1,3,13)}$.

Thus, $N^{(1,3,13)} \geq < (1, 17, 8, 19, 12)(2, 18, 7, 20, 11)(3, 5, 14, 15, 10)(4, 6, 13, 16, 9),$

$(1, 20)(2, 19)(3, 4)(5, 9)(6, 10)(7, 17)(8, 18)(11, 12)(13, 15)(14, 16) > \cong D_{10}$.

The number of right cosets in [1,3,13] is $\frac{|N|}{|D_{10}|} = \frac{60}{10} = 6$.

The orbits of $N^{(1,3,11)}$ on Ω are $\{1, 12, 20, 19, 11, 7, 8, 2, 18, 17\}$ and

$\{3, 10, 4, 15, 6, 9, 14, 13, 16, 5\}$.

We have $Mt_1t_3t_{13}t_1 \in$ [1,3,5,7] (Lemma 47) and $Mt_1t_3t_{13}t_{13} \in$ [1,3] (Lemma 48).

The next double coset to be considered is [1,5,7].

Now $N^{(1,5,7)} = 1$.

But $Mt_1t_5t_7 = Mt_3t_{15}t_9$ (Lemma 49). Then

(1,3)(2,4)(5,15)(6,16)(7,9)(8,10)(11,12)(13,17)(14,18)(19,20) $\in N^{(1,5,7)}$. Also,

$Mt_1t_5t_7 = Mt_6t_2t_8$ (Lemma 49) $\implies$

(1,6)(2,5)(3,14)(4,13)(7,8)(9,11)(10,12)(15,17)(16,18)(19,20) $\in N^{(1,5,7)}$.

So, $N^{(1,5,7)} \geq < (1,3)(2,4)(5,15)(6,16)(7,9)(8,10)(11,12)(13,17)(14,18)(19,20),$

$(1,6)(2,5)(3,14)(4,13)(7,8)(9,11)(10,12)(15,17)(16,18)(19,20) > \cong S_3$.

Then, the number of right cosets in the double coset [1,5,7] are $\frac{|N|}{|S_3|} = \frac{60}{6} = 10$.

The orbits of $N^{(1,5,7)}$ on Ω are $\{19,20\}$, $\{1,3,6,14,16,18\}$, $\{2,4,5,13,15,17\}$ and $\{7,9,8,11,10,12\}$.

We have: $Mt_1t_5t_7t_{19} \in$ [1,5,7,19] (Lemma 50), $Mt_1t_5t_7t_1 \in$ [1,3,5] (Lemma 51), $Mt_1t_5t_7t_2 \in$ [1,3,5] (Lemma 52) and $Mt_1t_5t_7t_7 \in$ [1,5] (Lemma 53).

The next double coset we consider is [1,3,5,7].

Now $N^{1,3,5,7} = 1$.

$Mt_1t_3t_5t_7 = Mt_{17}t_4t_7t_5$ (Lemma 54). Then

(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16) $\in N^{(1,3,5,7)}$.

Also, $Mt_1t_3t_5t_7 = Mt_4t_{17}t_{16}t_{20}$ (Lemma 54) $\implies$

(1,4,18,11,13)(2,3,17,12,14)(5,16,8,19,9)(6,15,7,20,10) $\in N^{(1,3,5,7)}$.

Thus, $N^{(1,3,5,7)} \geq < (1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16),$

$(1,4,18,11,13)(2,3,17,12,14)(5,16,8,19,9)(6,15,7,20,10) > \cong D_{10}$.

Therefore, the number of right cosets in [1,3,5,7] is $\frac{|N|}{|D_{10}|} = \frac{60}{10} = 6$.

The orbits of $N^{(1,3,5,7)}$ on Ω are $\{1,17,4,12,3,18,13,14,2,11\}$ and

$\{5,7,16,20,15,8,9,10,6,19\}$.

We have: $Mt_1t_3t_5t_7t_1 \in$ [1,3,13] (Lemma 55)

and $Mt_1t_3t_5t_7t_7 \in$ [1,3,5] (Lemma 56).

The next double coset we consider is [1,3,11,5].

Now $N^{1,3,11,5} = 1$.

But $Mt_1t_3t_{11}t_5 = Mt_1t_7t_{18}t_{11}$ (Lemma 57). Then

(3,7,9)(4,8,10)(5,11,18)(6,12,17)(13,15,19)(14,16,20) $\in N^{(1,3,11,5)}$. Also,

$Mt_1t_3t_{11}t_5 = Mt_7t_1t_{17}t_{19}$ (Lemma 57) $\implies$

(1,7,3)(2,8,4)(5,19,14)(6,20,13)(11,17,15)(12,18,16) $\in N^{(1,3,11,5)}$.

Thus, $N^{(1,3,11,5)} \geq < (3,7,9)(4,8,10)(5,11,18)(6,12,17)(13,15,19)(14,16,20),$

$(1,7,3)(2,8,4)(5,19,14)(6,20,13)(11,17,15)(12,18,16) > \cong A_4$.

Therefore, the number of right cosets in [1,3,11,5] is $\frac{|N|}{|A_4|} = \frac{60}{12} = 5$.

The orbits of $N^{(1,3,11,5)}$ on Ω are $\{1,7,9,3\}$, $\{2,8,10,4\}$ and

$\{5,11,19,18,17,13,14,16,6,15,20,12\}$.

We have: $Mt_1t_3t_{11}t_5t_1 \in$ [1,3,11,5] (Lemma 58), $Mt_1t_3t_{11}t_5t_2 \in$ [1,3,11,5] (Lemma 59) and $Mt_1t_3t_{11}t_5t_5 \in$ [1,3,11] (Lemma 60).

Finally, we consider the double coset [1,5,7,19].

Now $N^{1,5,7,19} = 1$.

But $Mt_1t_5t_7t_{19} = Mt_6t_9t_{11}t_4$ (Lemma 61). Then

(1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19) $\in N^{(1,5,7,19)}$. Also,

$Mt_1t_5t_7t_{19} = Mt_{17}t_7t_5t_{10}$ (Lemma 61) $\implies$

(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16) $\in N^{(1,5,7,19)}$. Thus,

$N^{(1,5,7,19)} \geq < (1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19),$

$(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16) > \cong A_5$.

Therefore, the number of right cosets in [1,5,7,19] is $\frac{|N|}{|A_5|} = \frac{60}{60} = 1$.

$N^{(1,5,7,19)}$ on Ω has a single orbit

$\{1, 16, 11, 5, 9, 4, 15, 13, 10, 8, 3, 14, 7, 19, 18, 2, 20, 17, 6, 12\}$. We have:

$Mt_1t_5t_7t_{19}t_{19} \in$ [1,5,7] (Lemma 62), the set of right cosets of G over M is closed under right multiplication by $t_i's$. Therefore we must have found all of the MwN double cosets of G.

2.4 Proof of Isomorphism of G_1

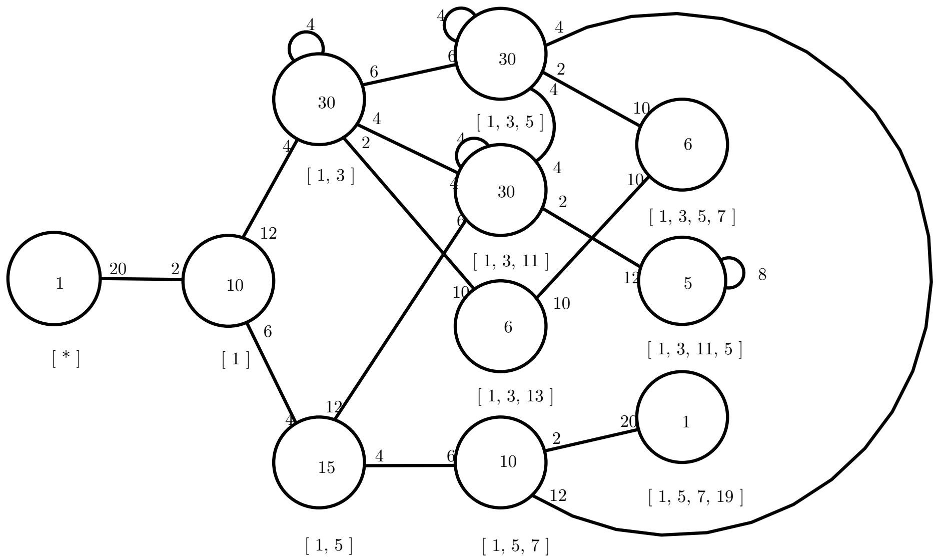
Our argument shows that $|G| \geq$ (1+10+30+15+30+30+6+10+6+5+1) $\times$ $|N|$ = 144 $\times$ 60 = 8640. Now we show that $|G| \leq 8640$.

Since G acts on the set of 112 cosets

$\{[*], [1], [1,3], [1,5], [1,3,5], [1,3,11], [1,3,13], [1,5,7], [1,3,5,7], [1,3,11,5], [1,5,7,19]\}$, the mapping $f : G \longrightarrow S_{112}$ is a homomorphism. Thus $G \,/\, ker f \cong f(G) = G_1$ $\implies |G| \,/\, |ker f| = |G_1| = 8640 \implies |G| = 8640 \times |ker f| \leq 8640$, since $|ker f| \geq 1$.

Now we have $|G| \geq 8640$ and $|G| \leq 8640$, therefore $|G| = 8640$, so $ker f = 1$ and $G \cong G_1$. We see that a composition series for G is
$G \trianglelefteq M_{12} \trianglelefteq 1 \implies G \cong M_{12}$.

Figure 2.1: Cayley Diagram for M_{12}



Chapter 3

A Symmetric Presentation of $M_{21} : (2 \times 2)$

Consider $N = < xx, yy, zz > \cong$ PGL(2,9) where $xx = (1,2,10)(3,4,5)(6,7,8)$, $yy = (1,7,3,4,2,5,6,8)$, and $zz = (1,2)(4,7)(5,8)(9,10)$. A symmetric presentation for the progenitor $2^{*10} : N$ is given by

$$G < x, y, z, t >:= \text{Group}< x, y, z, t | x^3, y^8, z^2, (y^{-1} * z)^2, (y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x),$$
$$(z * x^{-1})^3, (x * y^{-2} * x^{-1} * y * x^{-1} * y), t^2, (t, y^{-1} * x * z * y^2), (t, y^{-2} * x * y^{-1}) >$$

where $x \sim xx$, $y \sim yy$, $z \sim zz$, and $t \sim t_1$.

We prove that this progenitor factored by the relations $(z * t)^4 = (y^4 * z)$ and $(z * x * y * t)^6$ gives $M_{21} : (2 \times 2)$. as a true homomorphic image. Thus, we prove that

$$G < x, y, z, t >:= \text{Group}< x, y, z, t | x^3, y^8, z^2, (y^{-1} * z)^2, (y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x),$$
$$(z * x^{-1})^3, (x * y^{-2} * x^{-1} * y * x^{-1} * y), t^2, (t, y^{-1} * x * z * y^2), (t, y^{-2} * x * y^{-1}),$$
$$(z * x * y * t)^6,\ (y^4 * z) * (z * t)^4 > \cong M_{21} : (2 \times 2).$$

In order to do this, we need to perform manual double coset enumeration of G over N.

3.1 Relations, Expansion, and Their Conjugation

Our two relations are $(z*x*y*t)^6 = e$ (1)

$$(z*t)^4 = (y^4*z) \ (2)$$

$$(z*x*y*t)^6 = e \implies$$

$$(z*x*y)^6 * t_1^{(z*x*y)^5} * t_1^{(z*x*y)^4} * t_1^{(z*x*y)^3} * t_1^{(z*x*y)^2} * t_1^{(z*x*y)} * t_1 = e \implies$$

$$(z*x*y)^6 * t_1^{(1,3)(2,10)(4,6)(5,9)(7,8)} * t_1^{(1,6,8,9,2)(3,4,7,5,10)} * t_1^{(1,7,2,4,9,3,8,10,6,5)} *$$

$$t_1^{(1,9,6,2,8)(3,5,4,10,7)} * t_1^{(1,10,9,7,6,3,2,5,8,4)} * t_1 = e \implies$$

$$(zxy)^6 t_3 t_6 t_7 t_9 t_{10} t_1 = e \ (1)$$

$$(y^4*z)*(z*t)^4 = e \implies$$

$$y^4 * z * z^4 * t_1^{z^3} * t_1^{z^2} * t_1^{z} * t_1 = e \implies$$

$$y^4 * z * e * t_1^{(1,2)(4,7)(5,8)(9,10)} * t_1^{e} * t_1^{(1,2)(4,7)(5,8)(9,10)} * t_1 = e \implies$$

$$(y^4 z) t_2 t_1 t_2 t_1 = e \implies$$

$$(3,6)(4,5)(7,8)(9,10) t_1 t_2 t_1 t_2 = e \ (2)$$

Conjugating (1) by N gives us the following relation:

$$(1,6,8,9,2)(3,4,7,5,10) t_6 t_3 t_2 t_5 t_8 t_4 = e \ (3)$$

Conjugating (2) by N gives us the following relations:

$$(2,6)(4,10)(5,7)(8,9) t_1 t_3 t_1 t_3 = e \ (4)$$

$$(2,5)(3,10)(6,8)(7,9) t_1 t_4 t_1 t_4 = e \ (5)$$

$$(2,10)(3,8)(4,7)(5,6) t_1 t_9 t_1 t_9 = e \ (6)$$

$$(2,9)(3,4)(5,8)(6,7) t_1 t_{10} t_1 t_{10} = e \ (7)$$

$$(1,6)(4,8)(5,10)(7,9)t_2t_3t_2t_3 = e \ (8)$$

$$(1,5)(3,8)(6,9)(7,10)t_2t_4t_2t_4 = e \ (9)$$

$$(1,9)(3,5)(4,7)(6,8)t_2t_{10}t_2t_{10} = e \ (10)$$

$$(1,9)(2,4)(5,6)(7,10)t_3t_8t_3t_8 = e \ (11)$$

$$(1,2)(3,9)(6,10)(7,8)t_4t_5t_4t_5 = e \ (12)$$

$$(1,6)(2,3)(5,7)(9,10)t_4t_8t_4t_8 = e \ (13)$$

$$(1,3)(2,5)(4,10)(6,7)t_8t_9t_8t_9 = e \ (14)$$

$$(1,2)(3,6)(4,8)(5,7)t_9t_{10}t_9t_{10} = e \ (15)$$

3.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_2t_1$

Proof: $N\ \underline{(3,6)(4,5)(7,8)(9,10)t_1t_2} = N\ \underline{t_2t_1} = Nt_2t_1$ (by (2)).

Lemma 2: $Nt_1t_2t_3 = Nt_2t_1t_3$

Proof: $N\ \underline{t_1t_2}\ t_3 = N\ \underline{t_2t_1}\ t_3 = Nt_2t_1t_3$ (by Lemma (1)).

Lemma 3: $Nt_1t_2t_3 = Nt_7t_5t_4$

Proof: Nt_1(1,6)(4,8)(5,10)(7,9) $\underline{(1,6)(4,8)(5,10)(7,9)t_2t_3} =$

$Nt_1(1,6)(4,8)(5,10)(7,9)\ \underline{t_3t_2} = Nt_6t_3t_2$ (by (8))

$N\ \underline{(1,6,8,9,2)(3,4,7,5,10)t_6t_3t_2} = N\ \underline{t_4t_8t_5} = Nt_4t_8t_5$ (by (3))

$N\ \underline{(1,6)(2,3)(5,7)(9,10)t_4t_8}\ t_5 = N\ \underline{t_8t_4}\ t_5 = Nt_8t_4t_5$ (by (13))

$Nt_8(1,2)(3,9)(6,10)(7,8)\ \underline{(1,2)(3,9)(6,10)(7,8)t_4t_5} =$

$Nt_8(1,2)(3,9)(6,10)(7,8)\ \underline{t_5t_4} = Nt_7t_5t_4$ (by (12)).

Lemma 4: $Nt_1t_2t_3t_9 = Nt_7t_5t_4t_9$

Proof: $\underline{Nt_1t_2t_3}\ t_9 = \underline{Nt_7t_5t_4}\ t_9 = Nt_7t_5t_4t_9$ (by Lemma 3).

Lemma 5: $Nt_1t_2t_3t_9 = Nt_2t_9t_8t_1$

Proof: $Nt_1t_2t_3t_9 = Nt_1t_2t_3t_9\ \underline{t_1t_8t_9t_2t_2t_9t_8t_1} = Nt_1t_2t_3t_9t_1t_8t_9t_2t_2t_9t_8t_1$

$Nt_1t_2t_3(2,10)(3,8)(4,7)(5,6)\ \underline{(2,10)(3,8)(4,7)(5,6)t_9t_1}\ t_8t_9t_2t_2t_9t_8t_1 =$

$Nt_1t_2t_3(2,10)(3,8)(4,7)(5,6)\ \underline{t_1t_9}\ t_8t_9t_2t_2t_9t_8t_1 =$

$Nt_1t_{10}t_8\ \underline{t_1t_9}\ t_8t_9t_2t_2t_9t_8t_1 = Nt_1t_{10}t_8t_1t_9t_8t_9t_2t_2t_9t_8t_1$ (by (6))

$Nt_1t_{10}t_8t_1\ \underline{(1,3)(2,5)(4,10)(6,7)t_9t_8t_9}\ t_2t_2t_9t_8t_1 =$

$Nt_1t_{10}t_8t_1\ \underline{t_8}\ t_2t_2t_9t_8t_1 = Nt_3t_4t_8t_3t_8t_2t_2t_9t_8t_1$ (by (14))

$Nt_3t_4(1,9)(2,4)(5,6)(7,10)\ \underline{(1,9)(2,4)(5,6)(7,10)t_8t_3t_8}\ t_2t_2t_9t_8t_1 =$

$Nt_3t_4(1,9)(2,4)(5,6)(7,10)\ \underline{t_3}\ t_2t_2t_9t_8t_1 = Nt_3t_2t_3t_2t_2t_9t_8t_1$ (by (11))

$N\ \underline{(1,6)(4,8)(5,10)(7,9)t_3t_2t_3t_2}\ t_2t_9t_8t_1 =$

$N\ \underline{e}\ t_2t_9t_8t_1 = Nt_2t_9t_8t_1$ (by (8)).

Lemma 6: $Nt_1t_2t_3t_9t_{10} = Nt_2t_{10}t_4t_9t_1$

Proof: $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_3t_9t_{10}\ \underline{t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1} =$

$Nt_1t_2t_3t_9t_{10}t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1$

$Nt_1t_2t_3(1,2)(3,6)(4,8)(5,7)\ \underline{(1,2)(3,6)(4,8)(5,7)t_9t_{10}}\ t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_1t_2t_3(1,2)(3,6)(4,8)(5,7)\ \underline{t_{10}t_9}\ t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_2t_1t_6t_{10}t_9t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (15))

$Nt_2t_1t_6t_{10}$(2,10)(3,8)(4,7)(5,6) $\underline{(2,10)(3,8)(4,7)(5,6)t_9t_1t_9}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_2t_1t_6t_{10}$(2,10)(3,8)(4,7)(5,6) $\underline{t_1}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_{10}t_1t_5t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (6))

$Nt_{10}t_1t_5$(3,6)(4,5)(7,8)(9,10) $\underline{(3,6)(4,5)(7,8)(9,10)t_2t_1}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_{10}t_1t_5$(3,6)(4,5)(7,8)(9,10) $\underline{t_1t_2}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_9t_1t_4t_1t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (2))

Nt_9(2,5)(3,10)(6,8)(7,9) $\underline{(2,5)(3,10)(6,8)(7,9)t_1t_4t_1}\ t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

Nt_9(2,5)(3,10)(6,8)(7,9) $\underline{t_4}\ t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_7t_4t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (5))

Nt_7(1,5)(3,8)(6,9)(7,10) $\underline{(1,5)(3,8)(6,9)(7,10)t_4t_2t_4}\ t_{10}t_2t_2t_{10}t_4t_9t_1 =$

Nt_7(1,5)(3,8)(6,9)(7,10) $\underline{t_2}\ t_{10}t_2t_2t_{10}t_4t_9t_1 = Nt_{10}t_2t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (9))

$N\ \underline{(1,9)(3,5)(4,7)(6,8)t_{10}t_2t_{10}t_2}\ t_2t_{10}t_4t_9t_1 = N\ \underline{e}\ t_2t_{10}t_4t_9t_1 =$

$Nt_2t_{10}t_4t_9t_1$ (by (10)).

Lemma 7: $Nt_1t_2t_3t_9t_{10} = Nt_7t_5t_4t_9t_{10}$

Proof: $N\ \underline{t_1t_2t_3t_9}\ t_{10} = N\ \underline{t_7t_5t_4t_9}\ t_{10} = Nt_7t_5t_4t_9t_{10}$ (by Lemma 4).

Lemma 8: $Nt_1t_2t_3t_9t_{10} = Nt_2t_1t_3t_{10}t_9$

Proof: $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_3t_9t_{10}\ \underline{t_9t_{10}t_3t_1t_2t_2t_1t_3t_{10}t_9} =$

$Nt_1t_2t_3t_9t_{10}t_9t_{10}t_3t_1t_2t_2t_1t_3t_{10}t_9$

$Nt_1t_2t_3$(1,2)(3,6)(4,8)(5,7) $\underline{(1,2)(3,6)(4,8)(5,7)t_9t_{10}t_9t_{10}}\ t_3t_1t_2t_2t_1t_3t_{10}t_9 =$

$Nt_1t_2t_3$(1,2)(3,6)(4,8)(5,7) $\underline{e}\ t_3t_1t_2t_2t_1t_3t_{10}t_9 =$

$Nt_2t_1t_6t_3t_1t_2t_2t_1t_3t_{10}t_9$ (by (15))

$Nt_2t_1t_6$(2,6)(4,10)(5,7)(8,9) $\underline{(2,6)(4,10)(5,7)(8,9)t_3t_1}$ $t_2t_2t_1t_3t_{10}t_9$ =

$Nt_2t_1t_6$(2,6)(4,10)(5,7)(8,9) $\underline{t_1t_3}$ $t_2t_2t_1t_3t_{10}t_9$ =

$Nt_6t_1t_2t_1t_3t_2t_2t_1t_3t_{10}t_9$ (by (4))

Nt_6(3,6)(4,5)(7,8)(9,10) $\underline{(3,6)(4,5)(7,8)(9,10)t_1t_2t_1}$ $t_3t_2t_2t_1t_3t_{10}t_9$ =

Nt_6(3,6)(4,5)(7,8)(9,10) $\underline{t_2}$ $t_3t_2t_2t_1t_3t_{10}t_9 = Nt_3t_2t_3t_2t_2t_1t_3t_{10}t_9$ (by (2))

N $\underline{(1,6)(4,8)(5,10)(7,9)t_3t_2t_3t_2}$ $t_2t_1t_3t_{10}t_9$ =

N $\underline{e}$ $t_2t_1t_3t_{10}t_9 = Nt_2t_1t_3t_{10}t_9$ (by (8)).

3.3 Double Coset Enumeration of G over N

$NeN = \{N\}$. Since N is transitive on $\Omega = \{1, 2,\ldots, 10\}$ and $Nt_1 \in Nt_1N$, 10 $t_i's$ extend the coset representative N to the double coset $Nt_1N = [1]$.

Now $N^{(1)} \geq <(2,5,9,8)(3,7,4,6),(2,8,4,5,10,3,7,6)> \cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$.

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{720}{72} = 10$.

$N^{(1)}$ has orbits $\{1\}$ and $\{2,5,8,9,10,4,3,6,7\}$ on Ω. We note that $Nt_1t_1 \in [*]$, and $Nt_1t_2 \in$ [1,2].

We now consider the double coset [1,2].

We note that $N^{1,2} = 1$.

$N^{(1,2)} \geq N^{\{1,2\}} = <(3,7,10,5,6,8,9,4)> \cong \mathbb{Z}_8$. However, since $Nt_1t_2 = Nt_2t_1$ (Lemma 1), (1,2)(4,7)(5,8)(9,10) $\in N^{(1,2)}$ Thus,

$N^{(1,2)} \geq <(3,7,10,5,6,8,9,4),(1,2)(4,7)(5,8)(9,10)> \cong \mathbb{Z}_8{:}\mathbb{Z}_2$

Therefore, [1,2] contains $\frac{720}{16} = 45$ right cosets.

The orbits of $N^{(12)}$ on Ω are $\{1, 2\}$ and $\{3, 6, 7, 4, 9, 10, 5, 8\}$. We pick representatives 2 and 3 from the 1-orbit and 3-orbit, respectively, and determine the double cosets to which $Nt_1t_2t_2$ and $Nt_1t_2t_3$ belong. It is clear that $Nt_1t_2t_2 = Nt_1 \in [1]$.

Now consider the double coset [1,2,3].

We note that $N^{1,2,3} = 1$.

We have $Nt_1t_2t_3 = Nt_2t_1t_3$ (Lemma 2). So $< (1, 2)(4, 7)(5, 8)(9, 10) > \in N^{(1,2,3)}$. Also, $Nt_1t_2t_3 = Nt_7t_5t_4$ (Lemma 3). So (1,7,3,4,2,5,6,8) $\in N^{(1,2,3)}$. Thus, $N^{(1,2,3)} \geq\ < (1, 2)(4, 7)(5, 8)(9, 10), (1, 7, 3, 4, 2, 5, 6, 8) > \cong D_{16}$. Therefore [1,2,3] contains $\frac{|N|}{|N^{(123)}|} = \frac{|N|}{|D_{16}|} = \frac{720}{16} = 45$ right cosets.

$N^{(123)}$ has orbits $\{9, 10\}$ and $\{1, 7, 2, 3, 4, 5, 6, 8\}$ on Ω. We pick representatives 9 and 3 from the 9-orbit and 1-orbit, respectively, and determine the double cosets that contain the right cosets $Nt_1t_2t_3t_9$ and $Nt_1t_2t_3t_3$. Si$Nt_1t_2t_3t_3 = Nt_1t_2 \in$ [1,2], and $Nt_1t_2t_3t_9$ $\in$ [1,2,3,9].

Now we consider the double coset [1,2,3,9].

We note that $N^{1,2,3,9} = 1$.

From Lemma 4, $Nt_1t_2t_3t_9 = Nt_7t_5t_4t_9$. So (1,7,3,4,2,5,6,8) $\in N^{(1,2,3,9)}$. From Lemma 5, $Nt_1t_2t_3t_9 = Nt_2t_9t_8t_1$. So (1,2,9)(3,8,7)(4,6,5) $\in N^{(1,2,3,9)}$. Thus, $N^{(1,2,3,9)} \geq\ < (1, 7, 3, 4, 2, 5, 6, 8), (1, 2, 9)(3, 8, 7)(4, 6, 5) > \cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$. Thus, [1,2,3,9] contains $\frac{|N|}{|N^{(1,2,3,9)}|} = \frac{|N|}{|\mathbb{Z}_3^2:\mathbb{Z}_8|} = \frac{720}{72} = 10$ right cosets.

The orbits of $N^{(1,2,3)}$ on Ω are $\{10\}$ and $\{1, 7, 2, 3, 5, 9, 4, 8, 6\}$. We pick representatives 10 and 9 from the 10-orbit and 1-orbit, respectively, and determine the double cosets that contain the right cosets $Nt_1t_2t_3t_9t_{10}$ and $Nt_1t_2t_3t_9t_9$. Since $Nt_1t_2t_3t_9t_9 = Nt_1t_2t_3 \in$ [1,2,3] and $Nt_1t_2t_3t_9t_{10} \in$ [1,2,3,9,10], we need only consider the double coset [1,2,3,9,10].

We note that $N^{1,2,3,9,10} = 1$.

Now $Nt_1t_2t_3t_9t_{10} = Nt_2t_{10}t_4t_9t_1$ (Lemma 6) so (1,2,10)(3,4,5)(6,7,8) $\in N^{(1,2,3,9,10)}$, $Nt_1t_2t_3t_9t_{10} = Nt_7t_5t_4t_9t_{10}$ (Lemma 7) so (1,7,3,4,2,5,6,8) $\in N^{(1,2,3,9,10)}$, and $Nt_1t_2t_3t_9t_{10}$

$= Nt_2t_1t_3t_{10}t_9$ (Lemma 8) so (1,2)(4,7)(5,8)(9,10) $\in N^{(1,2,3,9,10)}$. Then $N^{(1,2,3,9,10)} \geq\ < (1,2,10)(3,4,5)(6,7,8), (1,7,3,4,2,5,6,8), (1,2)(4,7)(5,8)(9,10) > = N$. Therefore, the number of right cosets in [1,2,3,9,10] is $\frac{|N|}{|N|} = \frac{720}{720}$ =1.

$N^{(1,2,3,9,10)}$ on Ω has a single orbit $\{1,2,3,4,5,6,7,8,9,10\}$ We have: $Nt_1t_2t_3t_9t_{10}t_{10}$ $= Nt_1t_2t_3t_9 \in$ [1,2,3,9], the set of right cosets of G over N is closed under right multiplication by $t_i{}'s$. Therefore we must have found all of the NwN double cosets of G.

3.4 Proof of Isomorphism of G_1

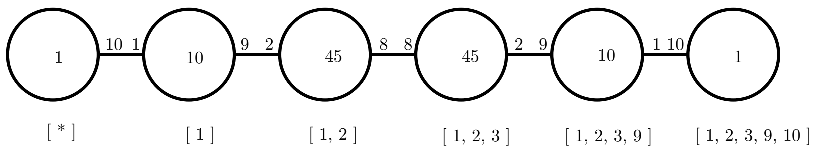
Our argument shows that $|G| \geq$ (1+10+45+45+10+1) $\times$ $|N|$ = 112 $\times$ 720 = 80,640. Now we show that $|G| \leq$ 80,640.

Since G acts on the set of 112 cosets $\{[*], [1], [1,2], [1,2,3], [1,2,3,9], [1,2,3,9,10]\}$, the mapping $f : G \longrightarrow S_{112}$ is a homomorphism. Thus G / $kerf \cong f(G) = G_1 \implies |G|$ / $|kerf| = |G_1|$ = 80,640 $\implies$ $|G|$ = 80,640 $\times$ $|kerf| \leq$ 80,640, since $|kerf| \geq 1$.

Now we have $|G| \geq$ 80,640 and $|G| \leq$ 80,640, therefore $|G|$ = 80,640, so $kerf$ = 1 and $G \cong G_1$. We see that a composition series for G is

$$G \trianglelefteq \frac{M_{21}:(2\times 2)}{PGL(2,9)} \trianglelefteq 1 \implies G \cong \frac{M_{21}:(2\times 2)}{PGL(2,9)}.$$

Figure 3.1: Cayley Diagram for M_{21} (2×2)



Chapter 4

Construction of $L_3(4):2^2$ over $PGL(2,9)$

Consider $N = < xx, yy, zz > \cong PGL(2,9)$ where xx = (1,2,10)(3,4,5)(6,7,8), $yy \sim$ (1,7,3,4,2,5,6,8), and zz = (1,2)(4,7)(5,8)(9,10). A symmetric presentation of the progenitor $2^{*10} : N$ is given by

$$G < x, y, z, t >:= \text{Group}< x, y, z, t | x^3, y^8, z^2, (y^{-1} * z)^2, (y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x),$$

$$(z * x^{-1})^3, (x * y^{-2} * x^{-1} * y * x^{-1} * y), t^2, (t, y^{-1} * x * z * y^2), (t, y^{-2} * x * y^{-1}) >$$

When we factor the above progenitor by the relation

$$(t * t^x)^2 = y^4 * z$$

we obtain $L_3(4) : 2^2$ as a homomorphic image.

```
> G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
>
> (z*x^-1)^3,y^-1*x^-1*y^-1*x^-1*y^2*x),
>
> (x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
>
```

```
> (t*t^x)^2=y^4*z>;
>
>  Index(G,sub<G|x,y,z>);
>
> 224
>
> f,G1,k:=CosetAction(G,sub<G|x,y,z>);
>
> CompositionFactors(G1);
>
>     G
>
>     |  Cyclic(2)
>
>     *
>
>     |  Cyclic(2)
>
>     *
>
>     |  A(2, 4)                    = L(3, 4)
>
>     *
>
>     |  Cyclic(2)
>
>     1
```

The center of the image is of order 2 and is generated by $(z * x * y * t)^6$

```
> NL:=NormalLattice(G1);
>
> NL;
>
>
> Normal subgroup lattice
>
> -----------------------
>
>
```

```
> [7]  Order 161280  Length 1  Maximal Subgroups: 4 5 6
>
> ---
>
> [6]  Order 80640   Length 1  Maximal Subgroups: 3
>
> [5]  Order 80640   Length 1  Maximal Subgroups: 3
>
> [4]  Order 80640   Length 1  Maximal Subgroups: 3
>
> ---
>
> [3]  Order 40320   Length 1  Maximal Subgroups: 2
>
> ---
>
> [2]  Order 2       Length 1  Maximal Subgroups: 1
>
> ---
>
> [1]  Order 1       Length 1  Maximal Subgroups:
>
>
> NL[2] eq sub<G1|f((z*x*y*t)^6)>;
>
> true
```

We factor by the center and obtain $L_3(4):2^2$. The relation $(z*x*y*t)^6$ implies the relation $(t*t^x)^2 = y^4*z$. Hence, $G \cong L_3(4):2^2$.

```
>  G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
>  (z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x), (x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*\
x*z*y^2),(t,y^-2*x*y^-1),(z*x*y*t)^6>;
> f,G1,k:=CosetAction(G,sub<G|x,y,z>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  A(2, 4)                    = L(3, 4)
    1
```

In order to do this, we need to perform manual double coset enumeration of G over N.

4.1 Relations, Expansions, and Their Conjugations

We will use $(z * x * y * t)^6$ as well as $(t * t^x)^2 = y^4 * z$ to construct $L_3(4) : 2^2$ over N.

Our relation is $(z * x * y * t)^6$ (1).

Additionally, $(t * t^x)^2 = y^4 * z$ (2) is a consequence of the above relation.

$$(z * x * y * t)^6 = e \implies$$

$$(z * x * y)^6 t_1^{(z*x*y)^5} t_1^{(z*x*y)^4} t_1^{(z*x*y)^3} t_1^{(z*x*y)^2} t_1^{(z*x*y)} t_1 = e \implies$$

$$(z * x * y)^6 t_1^{(1,3)(2,10)(4,6)(5,9)(7,8)} t_1^{(1,6,8,9,2)(3,4,7,5,10)} t_1^{(1,7,2,4,9,3,8,10,6,5)} *$$

$$t_1^{(1,9,6,2,8)(3,5,4,10,7)} t_1^{(1,10,9,7,6,3,2,5,8,4)} t_1 = e \implies$$

$$(xzxyx^{-1}y)\ t_3 t_6 t_7 t_9 t_{10} t_1 = e \implies$$

$$(1,2,9,8,6)(3,10,5,7,4)\ t_3 t_6 t_7 t_9 t_{10} t_1 = e \ (1)$$

$$(t * t^x)^2 = y^4 * z \implies$$

$$(y^4 * z)\ (t * t^x)^2 = e \implies$$

$$(y^4 * z)\ (t_1 t_1^{(1,2,10)(3,4,5)(6,7,8)})^2 = e \implies$$

$$(y^4 * z)\ (t_1 t_2)^2 = e \implies$$

$$(y^4 z)\ t_1 t_2 t_1 t_2 = e \implies$$

$$(3,6)(4,5)(7,8)(9,10)\ t_1 t_2 t_1 t_2 = e \ (2)$$

Conjugating (2) by N gives us the following relations:

$$(2,4)(3,7)(6,9)(8,10)\ t_1 t_5 t_1 t_5 = e \ (3)$$

$$(2,10)(3,8)(4,7)(5,6)\ t_1t_9t_1t_9 = e\ (4)$$

$$(1,6)(4,8)(5,10)(7,9)\ t_2t_3t_2t_3 = e\ (5)$$

$$(1,5)(3,8)(6,9)(7,10)\ t_2t_4t_2t_4 = e\ (6)$$

$$(1,10)(3,7)(4,6)(5,8)\ t_2t_9t_2t_9 = e\ (7)$$

$$(1,9)(3,5)(4,7)(6,8)\ t_2t_{10}t_2t_{10} = e\ (8)$$

$$(1,2)(4,7)(5,8)(9,10)\ t_3t_6t_3t_6 = e\ (9)$$

$$(1,9)(2,4)(5,6)(7,10)\ t_3t_8t_3t_8 = e\ (10)$$

$$(1,8)(2,7)(4,5)(6,10)\ t_3t_9t_3t_9 = e\ (11)$$

$$(1,3)(2,5)(4,10)(6,7)\ t_8t_9t_8t_9 = e\ (12)$$

$$(1,2)(3,6)(4,8)(5,7)\ t_9t_{10}t_9t_{10} = e\ (13)$$

Conjugating (1) by N gives us the following relation:

$$(1,7,8,4,10)(2,5,9,3,6)\ t_2t_1t_3t_4t_5t_7 = e\ (14)$$

4.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_2t_1$

Proof: $N\ \underline{(3,6)(4,5)(7,8)(9,10)t_1t_2} = N\ \underline{t_2t_1} = Nt_2t_1$ (by (2)).

Lemma 2: $Nt_1t_2t_3t_4 \in$ [1,2]

Proof: $\underline{Nt_1t_2}\ t_3t_4 = \underline{Nt_2t_1}\ t_3t_4 = Nt_2t_1t_3t_4$ (by Lemma 1)

$N\ \underline{(1,7,8,4,10)(2,5,9,3,6)t_2t_1t_3t_4} = N\ \underline{t_7t_5} = Nt_7t_5$ (by (3))

and $Nt_7t_5 \in$ [1,2] since $(Nt_7t_5)^{(1,3,6,9,7)(2,10,8,4,5)} = Nt_1t_2$, where (1,3,6,9,7)(2,10,8,4,5) $\in N$.

Lemma 3: $Nt_1t_2t_3 = Nt_3t_6t_2$

Proof: $Nt_1t_2t_3 = Nt_1t_2t_3\ \underline{t_2t_6t_3t_3t_6t_2} = Nt_1t_2t_3t_2t_6t_3t_3t_6t_2$

$Nt_1\ \underline{t_2t_3t_2}\ t_6t_3t_3t_6t_2 = Nt_1\ \underline{(1,6)(4,8)(5,10)(7,9)t_3}\ t_6t_3t_3t_6t_2 =$

$Nt_6t_3t_6t_3t_3t_6t_2$ (by (5))

$N\ \underline{(1,2)(4,7)(5,8)(9,10)t_6t_3t_6t_3}\ t_3t_6t_2 = N\ \underline{e}\ t_3t_6t_2 = Nt_3t_6t_2$ (by (9)).

Lemma 4: $Nt_1t_2t_3 = Nt_2t_1t_3$

Proof: $\underline{Nt_1t_2}\ t_3 = \underline{Nt_2t_1}\ t_3 = Nt_2t_1t_3$ (by Lemma 1).

Lemma 5: $Nt_1t_2t_3t_9 = Nt_3t_6t_2t_9$

Proof: $\underline{Nt_1t_2t_3}\ t_9 = \underline{Nt_3t_6t_2}\ t_9 = Nt_3t_6t_2t_9$ (by Lemma 3).

Lemma 6: $Nt_1t_2t_3t_9 = Nt_2t_9t_8t_1$

Proof: $Nt_1t_2t_3t_9 = Nt_1t_2t_3t_9\ \underline{t_1t_8t_9t_2t_2t_9t_8t_1}$

$Nt_1t_2t_3t_9t_1(1,3)(2,5)(4,10)(6,7)\ \underline{(1,3)(2,5)(4,10)(6,7)t_8t_9}\ t_2t_2t_9t_8t_1 =$

$Nt_1t_2t_3t_9t_1$(1,3)(2,5)(4,10)(6,7) $\underline{t_9t_8}\ t_2t_2t_9t_8t_1 =$

$Nt_3t_5t_1t_9t_3t_9t_8t_2t_2t_9t_8t_1$ (by (12))

$Nt_3t_5t_1(1,8)(2,7)(4,5)(6,10)\ \underline{(1,8)(2,7)(4,5)(6,10)t_9t_3t_9}\ t_8t_2t_2t_9t_8t_1 =$

$Nt_3t_4t_8\ \underline{t_3}\ t_8t_2t_2t_9t_8t_1 = Nt_3t_4t_8t_3t_8t_2t_2t_9t_8t_1$ (by (11))

$Nt_3t_4(1,9)(2,4)(5,6)(7,10)\ \underline{(1,9)(2,4)(5,6)(7,10)t_8t_3t_8}\ t_2t_2t_9t_8t_1 =$

$Nt_3t_2\ \underline{t_3}\ t_2t_2t_9t_8t_1 = Nt_3t_2t_3t_2t_2t_9t_8t_1$ (by (10))

N(1,6)(4,8)(5,10)(7,9) $\underline{(1,6)(4,8)(5,10)(7,9)t_3t_2t_3t_2}\ t_2t_9t_8t_1 =$

$N \underline{e}\ t_2t_9t_8t_1 = Nt_2t_9t_8t_1$ (by (5)).

Lemma 7: $Nt_1t_2t_3t_9t_{10} = Nt_2t_{10}t_4t_9t_1$

$Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_3t_9t_{10}\ \underline{t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1} =$

$Nt_1t_2t_3t_9t_{10}t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1$

$Nt_1t_2t_3t_9t_{10}(2,10)(3,8)(4,7)(5,6)\ \underline{(2,10)(3,8)(4,7)(5,6)t_1t_9}$ *

$t_4t_{10}t_2t_2t_{10}t_4t_9t_1 = t_1t_{10}t_8t_9t_2\ \underline{t_9t_1}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_1t_{10}t_8t_9t_2t_9t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (14))

$Nt_1t_{10}t_8(1,10)(3,7)(4,6)(5,8)\ \underline{(1,10)(3,7)(4,6)(5,8)t_9t_2t_9}$ *

$t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1 = Nt_{10}t_1t_5\ \underline{t_2}\ t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_{10}t_1t_5t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (7))

$Nt_{10}(2,4)(3,7)(6,9)(8,10)\ \underline{(2,4)(3,7)(6,9)(8,10)t_1t_5}\ t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_8\ \underline{t_5t_1}\ t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1 = Nt_8t_5t_1t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (4))

$Nt_8t_5(3,6)(4,5)(7,8)(9,10)\ \underline{(3,6)(4,5)(7,8)(9,10)t_1t_2t_1}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_7t_4\ \underline{t_2}\ t_4t_{10}t_2t_2t_{10}t_4t_9t_1 = Nt_7t_4t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (2))

$Nt_7(1,5)(3,8)(6,9)(7,10)\ \underline{(1,5)(3,8)(6,9)(7,10)t_4t_2t_4}\ t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Nt_{10}\ \underline{t_2}\ t_{10}t_2t_2t_{10}t_4t_9t_1 = Nt_{10}t_2t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (6))

$N\ \underline{(1,9)(3,5)(4,7)(6,8)t_{10}t_2t_{10}t_2}\ t_2t_{10}t_4t_9t_1 =$

$N\ \underline{e}\ t_2t_{10}t_4t_9t_1 = Nt_2t_{10}t_4t_9t_1$ (by (8)).

Lemma 8: $Nt_1t_2t_3t_9t_{10} = Nt_3t_6t_2t_9t_{10}$

Proof: $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_3t_9t_{10}\ \underline{t_{10}t_9t_2t_6t_3t_3t_6t_2t_9t_{10}} =$

$Nt_1t_2t_3\ \underline{t_9t_{10}t_{10}t_9}\ t_2t_6t_3t_3t_6t_2t_9t_{10} = Nt_1t_2t_3t_2t_6t_3t_3t_6t_2t_9t_{10}$

$Nt_1(1,6)(4,8)(5,10)(7,9)\ \underline{(1,6)(4,8)(5,10)(7,9)t_2t_3t_2}\ t_6t_3t_3t_6t_2t_9t_{10} =$

$Nt_6\ \underline{t_3}\ t_6t_3t_3t_6t_2t_9t_{10} = Nt_6t_3t_6t_3t_3t_6t_2t_9t_{10}$ (by (5))

N(1,2)(4,7)(5,8)(9,10) $\underline{(1,2)(4,7)(5,8)(9,10)t_3t_6t_3t_6}\ t_3t_6t_2t_9t_{10} =$

$N\ \underline{e}\ t_3t_6t_2t_9t_{10} = Nt_3t_6t_2t_9t_{10}$ (by (9)).

Lemma 9: $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_6t_{10}t_9$

Proof: $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_3t_9t_{10}\ \underline{t_9t_{10}t_6t_2t_1t_1t_2t_6t_{10}t_9} =$

$Nt_1t_2t_3t_9t_{10}t_9t_{10}t_6t_2t_1t_1t_2t_6t_{10}t_9$

$Nt_1t_2t_3(1,2)(3,6)(4,8)(5,7)\ \underline{(1,2)(3,6)(4,8)(5,7)t_9t_{10}t_9t_{10}}\ t_6t_2t_1t_1t_2t_6t_{10}t_9 =$

$Nt_2t_1t_6\ \underline{e}\ t_6t_2t_1t_1t_2t_6t_{10}t_9 = Nt_2t_1\ \underline{t_6t_6}\ t_2t_1t_1t_2t_6t_{10}t_9 =$

$Nt_2t_1t_2t_1t_1t_2t_6t_{10}t_9$ (by (13))

$N\ \underline{(3,6)(4,5)(7,8)(9,10)t_2t_1t_2t_1}\ t_1t_2t_6t_{10}t_9 = N\ \underline{e}\ t_1t_2t_6t_{10}t_9 =$

$Nt_1t_2t_6t_{10}t_9$ (by (2)).

4.3 Double Coset Enumeration of G over N

$NeN = \{N\}$. Since N is transitive on $\Omega = \{1, 2, \ldots, 10\}$ and $Nt_1 \in Nt_1N$, 10 $t_i's$ extend the coset representative N to the double coset $Nt_1N = [1]$.

Now $N^{(1)} \geq\ <(2,4,5)(3,9,8)(6,7,10), (2,3,6)(4,9,7)(5,8,10), (2,8,4,5,10,3,7,6)> \cong \mathbb{Z}_3^2 : \mathbb{Z}_8$.

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{720}{72} = 10$.

$N^{(1)}$ has orbits $\{1\}$ and $\{2, 10, 8, 5, 7, 6, 3, 4, 9\}$ on Ω. We note that $Nt_1t_1 \in$ [*], and $Nt_1t_2 \in$ [1,2].

We now consider the double coset [1,2].

$N^{(1,2)} \geq N^{1,2} = < (3, 8, 10, 4, 6, 7, 9, 5) > \cong \mathbb{Z}_8$. However, since $Nt_1t_2 = Nt_2t_1$ (Lemma 1), (1,2)(3,9)(6,10)(7,8) $\in N^{(1,2)}$ Thus,

$N^{(1,2)} \geq < (3, 8, 10, 4, 6, 7, 9, 5), (1, 2)(3, 9)(6, 10)(7, 8) > \cong \mathbb{Z}_8{:}\mathbb{Z}_2$

Therefore, [1,2] contains $\frac{720}{16} = 45$ right cosets.

The orbits of $N^{(12)}$ on Ω are $\{1, 2\}$ and $\{3, 8, 7, 4, 9, 10, 5, 6\}$. We pick representatives 2 and 3 from the 1-orbit and 3-orbit, respectively, and determine the double cosets to which $Nt_1t_2t_2$ and $Nt_1t_2t_3$ belong. It is clear that $Nt_1t_2t_2 = Nt_1 \in [1]$ and $Nt_1t_2t_3 \in$ [1,2,3].

Now consider the double coset [1,2,3].

We note that $N^{1,2,3} = 1$.

We have $Nt_1t_2t_3 = Nt_3t_6t_2$ (Lemma 3), then (1,3,2,6)(4,5,8,7) $\in N^{(1,2,3)}$. Also $Nt_1t_2t_3 = Nt_2t_1t_3$ (Lemma 4), then (1,2)(4,7)(5,8)(9,10) $\in N^{(1,2,3)}$. Thus, $N^{(1,2,3)} \geq$ <(1,3,2,6)(4,5,8,7),(1,2)(4,7)(5,8)(9,10)> $\cong D_{16}$. Therefore [1,2,3] contains $\frac{|N|}{|N^{(123)}|} = \frac{|N|}{|D_{16}|}$ $= \frac{720}{16} = 45$ right cosets.

$N^{(123)}$ has orbits $\{9, 10\}$, $\{1, 3, 2, 6\}$, and $\{4, 5, 7, 8\}$ on Ω. We pick representatives 9 from the 2-orbit, and 3 and 4 from the 4-orbits, and determine the double cosets that contain the right cosets $Nt_1t_2t_3t_9$, $Nt_1t_2t_3t_3$, and $Nt_1t_2t_3t_4$. We have $Nt_1t_2t_3t_3 = Nt_1t_2 \in$ [1,2], $Nt_1t_2t_3t_4 \in$ [1,2] (Lemma 2), and $Nt_1t_2t_3t_9 \in$ [1,2,3,9].

We consider the double coset [1,2,3,9].

We note that $N^{1,2,3,9} = 1$.

From Lemma 4, $Nt_1t_2t_3t_9 = Nt_7t_5t_4t_9$. So (1,7,3,4,2,5,6,8) $\in N^{(1,2,3,9)}$. From Lemma 6, $Nt_1t_2t_3t_9 = Nt_2t_9t_8t_1$. So (1,2,9)(3,8,7)(4,6,5) $\in N^{(1,2,3,9)}$. Thus, $N^{(1,2,3,9)}$ $\geq N^{1,2,3,9}$ = <(1,7,3,4,2,5,6,8),(1,2,9)(3,8,7)(4,6,5)> $\cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$. Thus, [1,2,3,9] contains $\frac{|N|}{|N^{(1,2,3,9)}|} = \frac{|N|}{|\mathbb{Z}_3^2{:}\mathbb{Z}_8|} = \frac{720}{72} = 10$ right cosets.

The orbits of $N^{(1,2,3,9)}$ on Ω are $\{10\}$ and $\{1,7,2,3,5,9,4,8,6\}$. We pick representatives 10 and 9 from the 10-orbit and 1-orbit, respectively, and determine the double cosets that contain the right cosets $Nt_1t_2t_3t_9t_{10}$ and $Nt_1t_2t_3t_9t_9$. Since $Nt_1t_2t_3t_9t_9 = Nt_1t_2t_3 \in$ [1,2,3] and $Nt_1t_2t_3t_9t_{10} \in$ [1,2,3,9,10], we need only consider the double coset [1,2,3,9,10].

We note that $N^{1,2,3,9,10} = 1$.

Now $Nt_1t_2t_3t_9t_{10} = Nt_2t_{10}t_4t_9t_1$ (Lemma 7) $\implies$ (1,2,10)(3,4,5)(6,7,8) $\in N^{(1,2,3,9,10)}$, $Nt_1t_2t_3t_9t_{10} = Nt_3t_6t_2t_9t_{10}$ (Lemma 8) $\implies$ (1,3,2,6)(4,5,8,7) $\in N^{(1,2,3,9,10)}$, and $Nt_1t_2t_3t_9t_{10} = Nt_1t_2t_6t_{10}t_9$ (Lemma 9) $\implies$ (3,6)(4,5)(7,8)(9,10) $\in N^{(1,2,3,9,10)}$. Then $N^{(1,2,3,9,10)} \geq$ <(1,2,10)(3,4,5)(6,7,8),(1,3,2,6)(4,5,8,7),(3,6)(4,5)(7,8)(9,10)> $\cong S_6$. Therefore, the number of right cosets in [1,2,3,9,10] is $\frac{|N|}{|N|} = \frac{720}{720}$ =1.

$N^{(1,2,3,9,10)}$ on Ω has a single orbit $\{1,2,7,10,5,8,3,4,9,6\}$ We have: $Nt_1t_2t_3t_9t_{10}t_{10} = Nt_1t_2t_3t_9 \in$ [1,2,3,9], the set of right cosets of G over N is closed under right multiplication by $t_i's$. Therefore we must have found all of the NwN double cosets of G.

4.4 Proof of Isomorphism of G_1

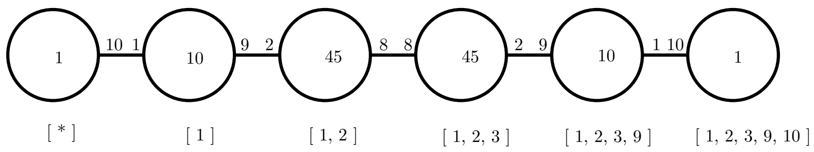
Our argument shows that $|G| \geq$ (1+10+45+45+10+1) $\times$ $|N|$ = 112 $\times$ 720 = 80,640. Now we show that $|G| \leq$ 80,640.

Since G acts on the set of 112 cosets $\{[*], [1], [1,2], [1,2,3], [1,2,3,9], [1,2,3,9,10]\}$, the mapping $f : G \longrightarrow S_{112}$ is a homomorphism. Thus G / $kerf \cong f(G) = G_1 \implies |G|$ / $|kerf|$ = $|G_1|$ = 80,640 $\implies$ $|G|$ = 80,640 $\times$ $|kerf| \leq$ 80,640, since $|kerf| \geq 1$.

Now we have $|G| \geq$ 80,640 and $|G| \leq$ 80,640, therefore $|G|$ = 80,640, so $kerf$ = 1 and $G \cong G_1$. We see that a composition series for G is

$G \trianglelefteq PSL(3,4) \trianglelefteq 1 \implies G \cong L_3(4) : 2^2$.

Figure 4.1: Cayley Diagram for $L_3(4) : 2^2$ over $PGL(2, 9)$



Chapter 5

Construction of $L_3(4) : 2^2$ over $Aut(A_6)$

Consider $N = < xx, yy, zz, ww > \cong Aut(A_6)$ where xx = (1,2,10)(3,4,5)(6,7,8), yy = (1,7,3,4,2,5,6,8), zz = (1,2)(4,7)(5,8)(9,10), and ww = (3,6)(4,7)(5,8). A symmetric presentation for the progenitor is given by

$$G < x, y, z, w, t > = Group < x, y, z, w, t | x^3, y^8, z^2, w^2, (y^{-1} * z)^2, (x^{-1} * w * x * w),$$

$$(z * w)^2, (w * y^3 * w * y^{-1}), (z * x^{-1})^3, y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x), t^2,$$

$$(t, x^{-1} * y^{-1} * x), (t, y * z * w * y^{-1}) >$$

where $x \sim xx$, $y \sim yy$, $z \sim zz$, $w \sim ww$, and $t \sim t_1$. We factor this progenitor by the two relations $(t * t^x)^2 = y^4 * z$ and $(x * y * z * w * t * y^{-1} * t * x * t * y^{-1} * t * z * t)$ and prove that $G < x, y, z, w, t >:= Group < x, y, z, w, t | x^3, y^8, z^2, w^2, (y^{-1} * z)^2, (x^{-1} * w * x * w)$,$(z * w)^2, (w * y^3 * w * y^{-1}), (z * x^{-1})^3, (y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x), t^2, (t, x^{-1} * y^{-1} * x), (t, y * z * w * y^{-1})$,$(t * t^x)^2 = y^4 * z$, $(x * y * z * w * t * y^{-1} * t * x * t * y^{-1} * t * z * t) >$ $\cong L_3(4) : 2^2$. In order to do this, we need to perform manual double coset enumeration of G over N.

5.1 Relations, Expansions, and Their Conjugations

Our relations are $(t * t^x)^2 = y^4 * z$ (1)

$$(x * y * z * w * t * y^{-1} * t * x * t * y^{-1} * t * z * t) = e \ (2)$$

$$(t * t^{x})^2 = (y^4 * z) \implies$$

$$(y^4 * z) * (t_1 * t_1^{x})^2 = e \implies$$

$$(y^4 * z) * (t_1 * t_1^{(1,2,10)(3,4,5)(6,7,8)})^2 = e \implies$$

$$(y^4 * z) * (t_1 * t_2)^2 = e \implies$$

$$(y^4 z)\ t_1 t_2 t_1 t_2 = e \implies$$

$$(3,6)(4,5)(7,8)(9,10) t_1 t_2 t_1 t_2 = e \ (1)$$

$$(x * y * z * w * t * y^{-1} * t * x * t * y^{-1} * t * z * t) = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * t_1 * x * t_1 * y^{-1} * t_1 * z * t_1 = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * t_1 * x * t_1 * y^{-1} * \underline{z * z^{-1}} * t_1 * z * t_1 = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * t_1 * x * t_1 * (y^{-1} * z) * t_1^{z} * t_1 = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * t_1 * (x * \underline{y^{-1} * z) * (y^{-1} * z)^{-1}} * t_1 *$$

$$y^{-1} * z * t_1^{(1,2)(4,7)(5,8)(9,10)} * t_1 = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * t_1 * (x * y^{-1} * z) * t_1^{(y^{-1} * z)} * t_2 * t_1) = e \implies$$

$$x * y * z * w * t_1 * (y^{-1} * x * y^{-1} * z) * (x * y^{-1} * z)^{-1} * t_1 *$$

$$x * y^{-1} * z * t_1^{(1,5)(2,7)(3,4)(6,8)(9,10)} * t_2 * t_1 = e \implies$$

$$x * y * z * w * t_1 * y^{-1} * x * y^{-1} * z * t_1^{(x * y^{-1} * z)} * t_5 * t_2 * t_1 = e \implies$$

$$x * y * z * w * y^{-1} * \underline{x * y^{-1} * z * (y^{-1} * x * y^{-1} * z)^{-1}} * t_1 *$$

$$y^{-1} * x * y^{-1} * z * t_1^{(1,7,6,2,9,10,5,4)} * t_5 * t_2 * t_1 = e \implies$$

$$x * y * z * w * y^{-1} * x * y^{-1} * z * t_1^{(y^{-1} * x * y^{-1} * z)} * t_7 * t_5 * t_2 * t_1 = e \implies$$

$$x * y * z * w * y^{-1} * x * y^{-1} * z * t_1^{(1,8,2)(3,6,4)(5,9,10)} * t_7 * t_5 * t_2 * t_1 = e \implies$$

$$(x^{-1}zxy^{-1}w)\ t_8t_7t_5t_2t_1 = e \implies$$

$$(1,9,5,3,8,2,10,7)(4,6)\ t_8t_7t_5t_2t_1 = e\ (2)$$

Conjugating (2) by N gives us the following relation.

$$(1,9,5,3,8,2,10,7)(4,6)\ t_2t_1t_3t_{10}t_9 = e \implies$$

$$(x^{-1}zxy^{-1}w)\ t_2t_1t_3t_{10}t_9 = e\ (3)$$

5.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_2t_1$

Proof: $N\ \underline{(3,6)(4,5)(7,8)(9,10)t_1t_2} = N\ \underline{t_2t_1} = Nt_2t_1$ (by (1)).

Lemma 2: $Nt_1t_2t_3 \in$ [1,2]

Proof: $\underline{Nt_1t_2}\ t_3 = \underline{Nt_2t_1}\ t_3 = Nt_2t_1t_3$ (by Lemma 1)

$N\ \underline{(1,9,5,3,8,2,10,7)(4,6)t_2t_1t_3} = N\ \underline{t_9t_{10}} = Nt_9t_{10}$ (by (3))

$(1,6,9)(2,3,10)(4,8,7) \in N \implies$

$(Nt_9t_{10})^{(1,6,9)(2,3,10)(4,8,7)} = Nt_1t_2 \in$ [1,2]

since $(1,6,9)(2,3,10)(4,8,7) \in N$.

5.3 Double Coset Enumeration of G over N

$NeN = \{N\}$. Since N is transitive on $\Omega = \{1, 2, \ldots, 10\}$ and $Nt_1 \in Nt_1N$, 10 $t_i's$ extend the coset representative N to the double coset $Nt_1N = [1]$.

$$N^{(1)} \geq N^1 = < (2,9,5,10,6,8,7,4), (2,9,10)(3,4,5,6,7,8) >$$

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{1440}{144} = 10$.

$N^{(1)}$ has orbits $\{1\}$ and $\{2,5,8,9,10,4,3,6,7\}$ on Ω. We note that $Nt_1t_1 \in$ [*], and $Nt_1t_2 \in$ [1,2].

We now consider the double coset [1,2].

Since $Nt_1t_2 = Nt_2t_1$ (Lemma 2),

$N^{(1,2)} \geq N^{1,2} = < (3,7,10,5,6,8,9,4), (3,5,6,4)(7,9,8,10) >$.

Thus [1,2] contains $\frac{1440}{32} = 45$ right cosets.

The orbits of $N^{(12)}$ on Ω are $\{1,2\}$ and $\{3,6,7,4,9,10,5,8\}$. We pick representatives 2 and 3 from the 1-orbit and 3-orbit, respectively, and determine the double cosets to which $Nt_1t_2t_2$ and $Nt_1t_2t_3$ belong. It is clear that $Nt_1t_2t_2 = Nt_1 \in$ [1] and $Nt_1t_2t_3 \in$ [1,2] (Lemma 2), the set of right cosets of G over N is closed under right multiplication by $t_i's$. Therefore we must have found all of the NwN double cosets of G.

5.4 Proof of Isomorphism of G_1

Our argument shows that $|G| \geq$ (1+10+45) $\times$ $|N|$ = 56 $\times$ 1440 = 80,640. Now we show that $|G| \leq$ 80,640.

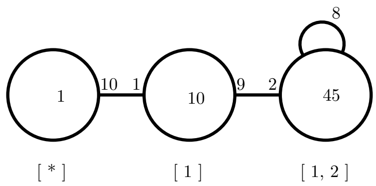
Since G acts on the set of 56 cosets $\{[*], [1], [1,2]\}$, the mapping

$f : G \longrightarrow S_{56}$ is a homomorphism. Thus G / $kerf \cong f(G) = G_1 \implies |G|$ / $|kerf| = |G_1|$ = 80,640 $\implies$ $|G|$ = 81,640 $\times$ $|kerf| \leq$ 81,640, since $|kerf| \geq 1$.

Now we have $|G| \geq 81{,}640$ and $|G| \leq 81{,}640$, therefore $|G| = 81{,}640$, so $kerf = 1$ and $G \cong G_1$. We see that a composition series for G is

$$G \trianglelefteq L(3,4) \trianglelefteq 1 \implies G \cong L(3,4) : \mathbb{Z}_2^2.$$

Figure 5.1: Cayley Diagram for $L_3(4) : 2^2$ over $Aut(A_6)$



Chapter 6

Construction of $2^{\cdot}L_3(4)$:2

We first note that $L_3(4) = M_{21}$, the one point stabilizer of the Mathieu sporadic simple group M_{21}. Consider $N = < xx, yy, zz > \cong A_6$, where $xx =$ (1,2,10)(3,4,5)(6,7,8), $yy =$ (1,3,2,6)(4,5,8,7), and $zz =$ (1,2)(4,7)(5,8)(9,10).

A symmetric presentation for the progenitor $2^{\star 10} : N$ is given by

$$G < x, y, z, t >:= Group < x, y, z, t | x^3, y^4, z^2, (y^{-1} * z)^2, y^{-2} * x^{-1} * y^2 * x^{-1},$$

$$(z * x^{-1})^3, x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^{-1} * y * x * y, (x^{-1} * y^{-1} * x * z)^3,$$

$$t^2, (t, y * x^{-1} * z * x * y), (t, z * x * y * x^{-1} * z), (t, y^x), (t, x * y^2) >,$$

where $x{\sim}xx$, $y{\sim}yy$, and $t{\sim}t_1$. We factor this progenitor by the 3 relations

$(y^2 * z) * (t * t^x)^2$, $(z * x * y * x * t^{(y*x)})^6$, and $((z * x * y)^2 * t^x)^8$ and prove that

$$G < x, y, z, t >:= Group < x, y, z, t | x^3, y^4, z^2, (y^{-1} * z)^2, y^{-2} * x^{-1} * y^2 * x^{-1},$$

$$(z * x^{-1})^3, x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^{-1} * y * x * y, (x^{-1} * y^{-1} * x * z)^3,$$

$$t^2, (t, y * x^{-1} * z * x * y), (t, z * x * y * x^{-1} * z), (t, y^x), (t, x * y^2),$$

$$(y^2 * z) * (t * t^x)^2, (z * x * y * x * t^{(y*x)})^6, ((z * x * y)^2 * t^x)^8 > \cong 2^{\cdot L_3(4)}{:}2.$$

In order to do this, we need to perform manual double coset enumeration of G over N. The number of NwN - double cosets of N in G is 12. A maximal subgroup of

G containing N is isomorphic to S_7. However, G does not have a faithful permutation representation on the right cosets of S_7 in G. Thus, we select the subgroup M:= sub< $G|x, y, z * t * y * x * t * x * t * y^{-1} * t * x * t$ > and perform double coset enumeration of MwN - double cosets.

6.1 Relations, Expansions, and Their Conjugation

Our relations are $(y^2 * z) * (t * t^x)^2 = e$ (1)

$$(z * x * y * x * t^{(y*x)})^6 = e \ (2)$$

$$((z * x * y)^2 * t^x)^8 = e \ (3)$$

$$(y^2 * z) * (t * t^x)^2 = e \implies$$

$$(y^2 * z) * (t_1 * t_1^{(1,2,10)(3,4,5)(6,7,8)})^2 = e \implies$$

$$y^2 * z * t_1 * t_2 * t_1 * t_2 = e \implies$$

$$(3,6)(4,5)(7,8)(9,10)t_1t_2t_1t_2 = e \ (1)$$

$$(z * x * y * x * t^{(y*x)})^6 = e \implies$$

$$(z * x * y * x * t_1^{(1,4,3,10)(2,7,5,6)})^6 = e \implies$$

$$(z * x * y * x * t_4)^6 = e \implies$$

$$((z * x * y * x)^6 t_4^{(z*x*y*x)^5} t_4^{(z*x*y*x)^4} t_4^{(z*x*y*x)^3} t_4^{(z*x*y*x)^2} t_4^{(z*x*y*x)} t_4 = e \implies$$

$$((z * x * y * x)^6 t_4^{(2,7,6,5)(4,8,10,9)} t_4^e t_4^{(2,5,6,7)(4,9,10,8)} t_4^{(2,6)(4,10)(5,7)(8,9)} t_4^{(2,7,6,5)(4,8,10,9)} t_4 = e \implies$$

$$(z * x * y * x)^6 t_8t_4t_9t_{10}t_8t_4 = e \implies$$

$$(2,6)(4,10)(5,7)(8,9)t_8t_4t_9t_{10}t_8t_4 = e \ (2)$$

$$((z * x * y)^2 * t^x)^8 = e \implies$$

$$((z * x * y)^2 * t_1^{(1,2,10)(3,4,5)(6,7,8)})^8 = e \implies$$

$$(z*x*y)^2*t_2)^8 = e \implies$$

$$(z*x*y)^6{*}t_2^{(z*x*y)^4}{*}t_2^{(z*x*y)^2}{*}t_2^e{*}t_2^{(z*x*y)^8}{*}t_2^{(z*x*y)^6}{*}t_2^{(z*x*y)^4}{*}t_2^{(z*x*y)^2}{*}t_2 = e \implies$$

$$(z*x*y)^6{*}t_2^{(1,5,3,9,10)(2,8,7,4,6)}{*}t_2^{(1,9,5,10,3)(2,4,8,6,7)}{*}t_2{*}t_2^{(1,3,10,5,9)(2,7,6,8,4)}*$$

$$t_2^{(1,10,9,3,5)(2,6,4,7,8)}{*}t_2^{(1,5,3,9,10)(2,8,7,4,6)}{*}t_2^{(1,9,5,10,3)(2,4,8,6,7)}{*}t_2 = e \implies$$

$$(z*x*y)^6{*}t_8{*}t_4{*}t_2{*}t_7{*}t_6{*}t_8{*}t_4{*}t_2 = e \implies$$

$$(z*x*y)^6{*}t_8{*}t_4{*}t_2{*}t_7{*}t_6{*}t_8{*}t_4{*}t_2 = e \implies$$

$$(1{,}10{,}9{,}3{,}5)(2{,}6{,}4{,}7{,}8)t_8t_4t_2t_7t_6t_8t_4t_2 = e \ (3)$$

Moreover, $(z*t*y*x*t*x*t*y^{-1}*t*x*t) \in M \implies$

$$z*t_1*y*x*t_1*x*t_1*y^{-1}*t_1{*}\ \underline{x}*t_1 \in M \implies$$

$$z*t_1*y*x*t_1*x*t_1{*}\ \underline{y^{-1}*x}*t_2*t_1 \in M \implies$$

$$z*t_1*y*x*t_1{*}\ \underline{x*y^{-1}*x}*t_7*t_2*t_1 \in M \implies$$

$$z*t_1{*}\ \underline{y*x*x*y^{-1}*x}*t_4*t_7*t_2*t_1 \in M \implies$$

$$z*y*x^2*y^{-1}*x*t_5*t_4*t_7*t_2*t_1 \in M \implies$$

$$Mt_5t_4t_7 = Mt_1t_2 \ (4)$$

Conjugating (4) by N gives us the following relation:

$$Mt_1t_2t_4 = Mt_8t_7 \ (5)$$

Conjugating (1) by N gives us the following relations:

$$(2,4)(3,7)(6,9)(8,10)t_1t_5t_1t_5 = e \ (6)$$

$$(2,10)(3,8)(4,7)(5,6)t_1t_9t_1t_9 = e \ (7)$$

$$(1,6)(4,8)(5,10)(7,9)t_2t_3t_2t_3 = e \ (8)$$

$$(1,5)(3,8)(6,9)(7,10)t_2t_4t_2t_4 = e \ (9)$$

$$(1,10)(3,7)(4,6)(5,8)t_2t_9t_2t_9 = e \ (10)$$

$$(1,9)(3,5)(4,7)(6,8)t_2t_{10}t_2t_{10} = e \ (11)$$

$$(1,10)(2,8)(5,9)(6,7)t_3t_4t_3t_4 = e \ (12)$$

$$(1,2)(4,7)(5,8)(9,10)t_3t_6t_3t_6 = e \ (13)$$

$$(1,9)(2,4)(5,6)(7,10)t_3t_8t_3t_8 = e \ (14)$$

$$(1,3)(2,5)(4,10)(6,7)t_8t_9t_8t_9 = e \ (15)$$

$$\text{(1,2)(3,6)(4,8)(5,7)}t_9t_{10}t_9t_{10} = e \ (16)$$

6.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas.

Lemma 1: $Mt_1t_2 = Mt_2t_1$

Proof: $M\ \underline{(3,6)(4,5)(7,8)(9,10)t_1t_2} = M\ \underline{t_2t_1} = Mt_2t_1$ (by (1)).

Lemma 2: $Mt_1t_2t_4 \in$ [1,2]

Proof: $\underline{Mt_1t_2t_4} = \underline{Mt_8t_7} = Mt_8t_7$ (by (5))

$Mt_8t_7 \in$ [1,2] since

$(Mt_8t_7)^{(1,4,8)(2,5,7)(3,10,9)} = Mt_1t_2$ where (1,4,8)(2,5,7)(3,10,9) $\in N$.

Lemma 3: $Mt_1t_2t_3 = Mt_3t_6t_2$

Proof: $Mt_1t_2t_3 = Mt_1t_2t_3\ \underline{t_2t_6t_3t_3t_6t_2} = Mt_1t_2t_3t_2t_6t_3t_3t_6t_2$

Mt_1t_2(1,6)(4,8)(5,10)(7,9) $\underline{(1,6)(4,8)(5,10)(7,9)t_2t_3t_2}\ t_6t_3t_3t_6t_2 =$

Mt_1t_2(1,6)(4,8)(5,10)(7,9) $\underline{t_3}\ t_6t_3t_3t_6t_2 =$

$Mt_6t_3t_6t_3t_3t_6t_2$ (by (8))

$M\ \underline{(1,2)(4,7)(5,8)(9,10)t_3t_6t_3t_6}\ t_3t_6t_2 = M\ \underline{e}\ t_3t_6t_2 =$

$Mt_3t_6t_2$ (by (13)).

Lemma 4: $Mt_1t_2t_3 = Mt_2t_1t_3$

Proof: $\underline{Mt_1t_2}\ t_3 = \underline{Mt_2t_1}\ t_3 = Mt_2t_1t_3$ (by Lemma 1).

Lemma 5: $Mt_1t_2t_3t_4 \in$ [1,2,3]

$\underline{Mt_1t_2}\ t_3t_4 = \underline{Mt_8t_7t_4}\ t_3t_4 =$

$Mt_8t_7t_4t_3t_4$ (by (5))

$Mt_8t_7(1,10)(2,8)(5,9)(6,7)\ \underline{(1,10)(2,8)(5,9)(6,7)t_4t_3t_4} =$

$Mt_8t_7(1,10)(2,8)(5,9)(6,7)\ \underline{t_3} = Mt_2t_6t_3$ (by (12))

$Mt_2t_6t_3 \in$ [1,2,3] since $(Mt_2t_6t_3)^{(1,6,2)(4,5,9)(7,10,8)} = Mt_1t_2t_3$,

where (1,6,2)(4,5,9)(7,10,8) $\in N$.

Lemma 6: $Mt_1t_2t_3t_9 = Mt_3t_6t_2t_9$

$\underline{Mt_1t_2t_3}\ t_9 = \underline{Mt_3t_6t_2}\ t_9 = Mt_3t_6t_2t_9$ (by Lemma 3).

Lemma 7: $Mt_1t_2t_3t_9 = Mt_2t_9t_8t_1$

Proof: $Mt_1t_2t_3t_9 = Mt_1t_2t_3t_9\ \underline{t_1t_8t_9t_2t_2t_9t_8t_1} = Mt_1t_2t_3t_9t_1t_8t_9t_2t_2t_9t_8t_1$

$Mt_1t_2t_3$(2,10)(3,8)(4,7)(5,6) $\underline{(2,10)(3,8)(4,7)(5,6)t_9t_1}\ t_8t_9t_2t_2t_9t_8t_1 =$

$Mt_1t_2t_3$(2,10)(3,8)(4,7)(5,6) $\underline{t_1t_9}\ t_8t_9t_2t_2t_9t_8t_1 =$

$Mt_1t_{10}t_8t_1t_9t_8t_9t_2t_2t_9t_8t_1$ (by (7))

$Mt_1t_{10}t_8t_1$(1,3)(2,5)(4,10)(6,7) $\underline{(1,3)(2,5)(4,10)(6,7)t_9t_8t_9}\ t_2t_2t_9t_8t_1 =$

$Mt_1t_{10}t_8t_1$(1,3)(2,5)(4,10)(6,7) $\underline{t_8}$ $t_2t_2t_9t_8t_1$ =

$Mt_3t_4t_8t_3t_8t_2t_2t_9t_8t_1$ (by (15))

Mt_3t_4(1,9)(2,4)(5,6)(7,10) $\underline{(1,9)(2,4)(5,6)(7,10)t_8t_3t_8}$ $t_2t_2t_9t_8t_1$ =

Mt_3t_4(1,9)(2,4)(5,6)(7,10) $\underline{t_3}$ $t_2t_2t_9t_8t_1$ =

$Mt_3t_2t_3t_2t_2t_9t_8t_1$ (by (14))

M $\underline{(1,6)(4,8)(5,10)(7,9)t_3t_2t_3t_2}$ $t_2t_9t_8t_1 = M$ $\underline{e}$ $t_2t_9t_8t_1$ =

$Mt_2t_9t_8t_1$ (by (8)).

Lemma 8: $Mt_1t_2t_3t_9t_{10} = Mt_2t_{10}t_4t_9t_1$

Proof: $Mt_1t_2t_3t_9t_{10} = Mt_1t_2t_3t_9t_{10}$ $\underline{t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1}$ =

$Mt_1t_2t_3t_9t_{10}t_1t_9t_4t_{10}t_2t_2t_{10}t_4t_9t_1$

$Mt_1t_2t_3t_9t_{10}(2,10)(3,8)(4,7)(5,6)$ $\underline{(2,10)(3,8)(4,7)(5,6)t_1t_9}$ *

$t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

$Mt_1t_2t_3t_9t_{10}(2,10)(3,8)(4,7)(5,6)$ $\underline{t_9t_1}$ $t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

$Mt_1t_{10}t_8t_9t_2t_9t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (7))

$Mt_1t_{10}t_8(1,10)(3,7)(4,6)(5,8)$ $\underline{(1,10)(3,7)(4,6)(5,8)t_9t_2t_9}$ *

$t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

$Mt_1t_{10}t_8(1,10)(3,7)(4,6)(5,8)$ $\underline{t_2}$ $t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

$Mt_{10}t_1t_5t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (10))

Mt_{10}(2,4)(3,7)(6,9)(8,10) $\underline{(2,4)(3,7)(6,9)(8,10)t_1t_5}$ $t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

Mt_{10}(2,4)(3,7)(6,9)(8,10) $\underline{t_5t_1}$ $t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ =

$Mt_8t_5t_1t_2t_1t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (6))

Mt_8t_5(3,6)(4,5)(7,8)(9,10) $\underline{(3,6)(4,5)(7,8)(9,10)t_1t_2t_1}$ $t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

Mt_8t_5(3,6)(4,5)(7,8)(9,10) $\underline{t_2}$ $t_4t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Mt_7t_4t_2t_4t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (1))

Mt_7(1,5)(3,8)(6,9)(7,10) $\underline{(1,5)(3,8)(6,9)(7,10)t_4t_2t_4}$ $t_{10}t_2t_2t_{10}t_4t_9t_1 =$

Mt_7(1,5)(3,8)(6,9)(7,10) $\underline{t_2}$ $t_{10}t_2t_2t_{10}t_4t_9t_1 =$

$Mt_{10}t_2t_{10}t_2t_2t_{10}t_4t_9t_1$ (by (9))

M $\underline{(1,9)(3,5)(4,7)(6,8)t_{10}t_2t_{10}t_2}$ $t_2t_{10}t_4t_9t_1 = M$ $\underline{e}$ $t_2t_{10}t_4t_9t_1 =$

$Mt_2t_{10}t_4t_9t_1$ (by (11)).

Lemma 9: $Mt_1t_2t_3t_9t_{10} = Mt_3t_6t_2t_9t_{10}$

Proof: $Mt_1t_2t_3t_9t_{10} = Mt_1t_2t_3t_9t_{10}$ $\underline{t_{10}t_9t_2t_6t_3t_3t_6t_2t_9t_{10}}$ $=$

$Mt_1t_2t_3$ $\underline{t_9t_{10}t_{10}t_9}$ $t_2t_6t_3t_3t_6t_2t_9t_{10} = Mt_1t_2t_3t_2t_6t_3t_3t_6t_2t_9t_{10}$

$Mt_1(1,6)(4,8)(5,10)(7,9)$ $\underline{(1,6)(4,8)(5,10)(7,9)t_2t_3t_2}$ $t_6t_3t_3t_6t_2t_9t_{10} =$

$Mt_1(1,6)(4,8)(5,10)(7,9)$ $\underline{t_3}$ $t_6t_3t_3t_6t_2t_9t_{10} =$

$Mt_6t_3t_6t_3t_3t_6t_2t_9t_{10}$ (by (8))

M $\underline{(1,2)(4,7)(5,8)(9,10)t_6t_3t_6t_3}$ $t_3t_6t_2t_9t_{10} =$

M $\underline{e}$ $t_3t_6t_2t_9t_{10} = Mt_3t_6t_2t_9t_{10}$ (by (13)).

Lemma 10: $Mt_1t_2t_3t_9t_{10} = Mt_1t_2t_6t_{10}t_9$

Proof: $Mt_1t_2t_3t_9t_{10} = Mt_1t_2t_3t_9t_{10}$ $\underline{t_9t_{10}t_6t_2t_1t_1t_2t_6t_{10}t_9}$ $=$

$Mt_1t_2t_3t_9t_{10}t_9t_{10}t_6t_2t_1t_1t_2t_6t_{10}t_9$

$Mt_1t_2t_3(1,2)(3,6)(4,8)(5,7)\ \underline{(1,2)(3,6)(4,8)(5,7)t_9t_{10}t_9t_{10}}\ t_6t_2t_1t_1t_2t_6t_{10}t_9 =$

$Mt_1t_2t_3(1,2)(3,6)(4,8)(5,7)\ \underline{e}\ t_6t_2t_1t_1t_2t_6t_{10}t_9 = Mt_2t_1\ \underline{t_6t_6}\ t_2t_1t_1t_2t_6t_{10}t_9 =$

$Mt_2t_1t_2t_1t_1t_2t_6t_{10}t_9$ (by (16))

$M\ \underline{(3,6)(4,5)(7,8)(9,10)t_2t_1t_2t_1}\ t_1t_2t_6t_{10}t_9 =$

$M\ \underline{e}\ t_1t_2t_6t_{10}t_9 = Mt_1t_2t_6t_{10}t_9$ (by (1)).

6.3 Double Coset Enumeration of G over M and N:

$MeN = M$. Since N is transitive on $\Omega = \{1, 2, \ldots, 10\}$ and $Mt_1 \in Mt_1N$, 10 $t_i's$ extend the coset representative N to the double coset $Mt_1N = [1]$.

Now $N^{(1)} \geq < (2,4,5)(3,9,8)(6,7,10), (2,7,6,5)(4,8,10,9), (2,7,10,4)(3,5,8,6) > \cong \mathbb{Z}_3^2{:}\mathbb{Z}_4$.

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{360}{36} = 10$.

$N^{(1)}$ has orbits $\{1\}$ and $\{2,4,10,7,6,5,8,9,3\}$ on Ω. We take the representative 1 from the 1-orbit and 2 from the 2-orbit and determine the double cosets that contain Mt_1t_1 and Mt_1t_2. Now $Mt_1t_1 = Mt_1^2 = Me = M \in$ [*] and $Mt_1t_2 \in$ [1,2].

We now consider the double coset [1,2].

We note that $N^{1,2} = 1$.

$N^{(1,2)} \geq N^{1,2} = < (3,10,6,9)(4,7,5,8) > \cong \mathbb{Z}_4$. Now $Mt_1t_2 = Mt_2t_1$ (Lemma 1). Then (1,2)(4,7)(5,8)(9,10) $\in N^{(12)}$.

Thus $N^{(12)} \geq < (3,10,6,9)(4,7,5,8), (1,2)(4,7)(5,8)(9,10) > \cong D_8$. Therefore, the number of right cosets in the double coset [1,2] is $\frac{|N|}{|D_8|} = \frac{360}{8} = 45$.

The orbits of $N^{(12)}$ on Ω are $\{1,2\}$, $\{3,10,9,6\}$ and $\{4,7,5,8\}$. We select the representatives 1, 3, and 4 from the 1-orbit, 3-orbit, and 4-orbit, respectively, and determine the double cosets that contain $Mt_1t_2t_2$, $Mt_1t_2t_3$, and $Mt_1t_2t_4$. Now $Mt_1t_2t_2 = Mt_1t_2^2 = Mt_1(e) = Mt_1 \in$ [1], $Mt_1t_2t_4 \in$ [1,2] (Lemma 2), and $Mt_1t_2t_3 \in$ [1,2,3].

We consider the double coset [1,2,3] next.

We see that $N^{1,2,3} = 1$.

But we note that $Nt_1t_2t_3 = Nt_3t_6t_2$ (Lemma 3). Thus (1,3,2,6)(4,5,8,7) $\in N^{(1,2,3)}$. Also $Mt_1t_2t_3 = Mt_2t_1t_3$. Then (1,2)(4,7)(5,8)(9,10) $\in N^{(1,2,3)}$. Therefore $N^{(1,2,3)}$ $\geq <(1,3,2,6)(4,5,8,7),(1,2)(4,7)(5,8)(9,10)> \cong D_8$. So the number of right cosets in the double coset [1,2,3] is $\frac{|N|}{|N^{(1,2,3)}|} = \frac{360}{8} = 45$.

The orbits of $N^{(1,2,3)}$ on Ω are $\{9,10\}$, $\{1,3,2,6\}$ and $\{4,5,7,8\}$. We choose the representatives 9, 3, and 4 from the 1-orbit, 3-orbit, and 4-orbit, respectively, and determine the double cosets that contain $Mt_1t_2t_2t_3t_9$, $Mt_1t_2t_3t_3$, and $Mt_1t_2t_3t_4$. $Mt_1t_2t_3t_9 \in$ [1,2,3,9], $Mt_1t_2t_3t_3 \in$ [1,2,3], and $Mt_1t_2t_3t_4 \in$ [1,2,3] (Lemma 5).

Thus we must consider the new double coset $Mt_1t_2t_2t_3t_9 =$ [1,2,3,9].

We note that $N^{1,2,3,9} = 1$.

But $Mt_1t_2t_3t_9 = Mt_3t_6t_2t_9$ (Lemma 6). Then (1,3,2,6)(4,5,8,7) $\in N^{(1,2,3,9)}$. We also have $Mt_1t_2t_3t_9 = Mt_2t_9t_8t_1$ (Lemma 7). So (1,2,9)(3,8,7)(4,6,5) $\in N^{(1,2,3,9)}$.

Thus $N^{(1,2,3,9)} \geq <(1,3,2,6)(4,5,8,7),(1,2,9)(3,8,7)(4,6,5)> \cong \mathbb{Z}_3^2{:}\mathbb{Z}_4$.

The number of right cosets in the double coset [1,2,3,9] is $\frac{|N|}{|N^{(1,2,3,9)}|} = \frac{360}{36} = 10$.

The orbits of $N^{(1,2,3,9)}$ on Ω are $\{10\}$ and $\{1,3,2,8,6,9,7,5,4\}$. We choose representatives 10 and 9 from the two orbits and determine the double cosets to which $Mt_1t_2t_3t_9t_{10}$ and $Mt_1t_2t_3t_9t_9$ belong. We have $Mt_1t_2t_3t_9t_9 = Mt_1t_2t_3t_9^2 = Mt_1t_2t_3(e) = Mt_1t_2t_3 \in$ [1,2,3]. $Mt_1t_2t_3t_9t_{10}$ belongs to the new double coset [1,2,3,9,10].

We now consider the double coset [1,2,3,9,10].

Now $N^{1,2,3,9,10} = 1$.

But $Mt_1t_2t_3t_9t_{10} = Mt_2t_{10}t_4t_9t_1$ (Lemma 8). This gives (1,2,10)(3,4,5)(6,7,8) $\in N^{(1,2,3,9,10)}$. We have $Mt_1t_2t_3t_9t_{10} = Mt_3t_6t_2t_9t_{10}$ (Lemma 9). Therefore (1,3,2,6)(4,5,8,7)

$\in N^{(1,2,3,9,10)}$. We also have $Mt_1t_2t_3t_9t_{10} = Mt_1t_2t_6t_{10}t_9$ (Lemma 10). Then (3,6)(4,5)(7,8)(9,10) $\in N^{(1,2,3,9,10)}$. Therefore

$$N^{(1,2,3,9,10)} \geq < (1,2,10)(3,4,5)(6,7,8), (1,3,2,6)(4,5,8,7), (3,6)(4,5)(7,8)(9,10) > \cong A_6.$$

$N^{(1,2,3,9,10)} \cong A_6$ is transitive on $\Omega = \{1, 2, \ldots, 10\}$. Since $Mt_1t_2t_3t_9t_{10}t_{10} = Mt_1t_2t_3t_9t_{10}^2 = Mt_1t_2t_3t_9(e) = Mt_1t_2t_3t_9 \in$ [1,2,3,9], the set of right cosets of G over M is closed under right multiplication by $t_i{}'s$. Therefore we must have found all of the MwN double cosets of G.

6.4 Proof of Isomorphism of G_1

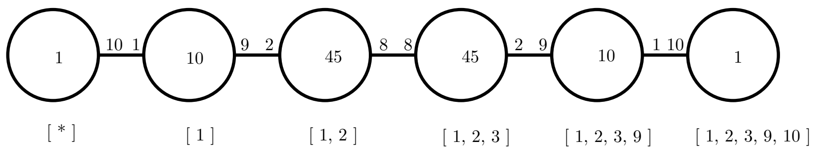
Our argument shows that $|G| \geq$ (1+10+45+45+10+1) $\times$ $|N|$ = 112 $\times$ 360 = 40,320. Now we show that $|G| \leq$ 40,320.

Since G acts on the set of 112 cosets $\{[*], [1], [1,2], [1,2,3], [1,2,3,9], [1,2,3,9,10]\}$, the mapping $f : G \longrightarrow S_{112}$ is a homomorphism. Thus $G \; / \; kerf \cong f(G) = G_1 \implies |G| \; / \; |kerf| = |G_1| =$ 40,320 $\implies |G| =$ 40,320 $\times$ $|kerf| \leq$ 40,320, since $|kerf| \geq 1$.

Now we have $|G| \geq$ 40,320 and $|G| \leq$ 40,320, therefore $|G| =$ 40,320, so $kerf = 1$ and $G \cong G_1$. We see that a composition series for G is

$$G \trianglelefteq L_3(4) \trianglelefteq 1 \implies G \cong L_3(4).$$

Figure 6.1: Cayley Diagram for $2^{\cdot}L_3(4){:}2$



Chapter 7

Construction of $S(4,3)$

Consider $N = < xx, yy > \cong A_5$, where

xx=(1,9)(3,4)(5,10)(6,7) and yy=(1,3,5,7,9)(2,4,6,8,10).

A symmetric presentation for the progenitor is $2^{*10} : N$ is given by

$$G < x, y, t >:=\text{Group}< x, y, t | x^2, y^5, (x * y^{-1})^3, t^2, (t, y^{-1} * x),$$

$$(t, y^2 * x * y^{-2} * x * y^2), (t, t^{(y*x*y^4)}), (t * t^{(y*x*y^{-1})})^2 >$$

where $x \sim xx$, $y \sim yy$, and $t \sim t_1$. We factor this progenitor by the three

relations $(y * x * y^{-1} * t)^4$, $(y * x * t^{y^2})^6$, $(y * t^y)^8$ and prove that

$$G < x, y, t >:=\text{Group}< x, y, t | x^2, y^5, (x * y^{-1})^3, t^2, (t, y^{-1} * x), (t, y^2 * x * y^{-2} * x * y^2),$$

$$(t, t^{(y*x*y^4)}), (t * t^{(y*x*y^{-1})})^2, (y * x * y^{-1} * t)^4, (y * x * t^{y^2})^6, (y * t^y)^8 > \cong S(4,3).$$

In order to do this, we need to perform manual double coset enumeration of G over N. However, the number of NwN - double cosets of N in G is 40. Thus, we select the maximal subgroup $M = < (y * x * t * y * x * t * y)^2, (y * t * x * y^{-1} * t * y^{-2} * t * x * y^{-1} * t * y^{-2} * t), (t * x * y^{-1} * t * x * t * y^{-1} * t * y * t * x * t * y * x * t) >$ and perform double coset enumeration of MwN - double cosets.

7.1 Relations Expansions and Their Conjugations

Our relations are $(y*x*y^{-1}*t)^4 = e$ (1)

$$(y*x*t^{y^2})^6 = e \ (2)$$

$$(y*t^y)^8 = e \ (3)$$

$$(y*x*t*y*x*t*y)^2 \in M \ (4)$$

$$(y*t*x*y^{-1}*t*y^{-2}*t*x*y^{-1}*t*y^{-2}*t) \in M \ (5)$$

$$(t*x*y^{-1}*t*x*t*y^{-1}*t*y*t*x*t*y*x*t) \in M \ (6)$$

$$(y*x*y^{-1}*t)^4 = e \implies$$

$$(y*x*y^{-1})^4 t_1^{(y*x*y^{-1})^3} t_1^{(y*x*y^{-1})^2} t_1^{(y*x*y^{-1})} t_1 = e \implies$$

$$e\ t_1^{(1,2)(3,8)(4,5)(7,9)}\ t_1^e\ t_1^{(1,2)(3,8)(4,5)(7,9)}\ t_1 = e \implies$$

$$t_2t_1t_2t_1 = e \ (1)$$

$$(y*x*t^{y^2})^6 = e \implies$$

$$(y*x*t^{(1,5,9,3,7)(2,6,10,4,8)})^6 = e \implies$$

$$(y*x*t_5)^6 = e \implies$$

$$(y*x)^6\ t_5^{(y*x)^5}\ t_5^{(y*x)^4}\ t_5^{(y*x)^3}\ t_5^{(y*x)^2}\ t_5^{(y*x)}\ t_5 = e \implies$$

$$e\ t_5^{(1,7,4)(2,10,3)(5,8,6)}\ t_5^{(1,4,7)(2,3,10)(5,6,8)}\ t_5^e\ t_5^{(1,7,4)(2,10,3)(5,8,6)}\ t_5^{(1,4,7)(2,3,10)(5,6,8)}\ t_5 = e \implies$$

$$t_8t_6t_5t_8t_6t_5 = e \ (2)$$

$$(y*t^y)^8 = e \implies$$

$$(y*t^{(1,3,5,7,9)(2,4,6,8,10)})^8 = e \implies$$

$$(y*t_3)^8 = e \implies$$

$$y^8\ t_3^{y^7}\ t_3^{y^6}\ t_3^{y^5}\ t_3^{y^4}\ t_3^{y^3}\ t_3^{y^2}\ t_3^{y}\ t_3 = e \implies$$

$$y^8 * t_3^{(1,5,9,3,7)(2,6,10,4,8)} * t_3^{(1,3,5,7,9)(2,4,6,8,10)} * t_3^{e} * t_3^{(1,9,7,5,3)(2,10,8,6,4)} *$$

$$t_3^{(1,7,3,9,5)(2,8,4,10,6)} * t_3^{(1,5,9,3,7)(2,6,10,4,8)} * t_3^{(1,3,5,7,9)(2,4,6,8,10)} * t_3 = e \implies$$

$$y^8 t_7 t_5 t_3 t_1 t_9 t_7 t_5 t_3 = e\ (3)$$

$$(y * x * t * y * x * t * y)^2 \in M \implies$$

$$(y * x * t_1 * y * x * t_1 * y)(y * x * t_1 * y * x * t_1 * \underline{y}) \in M \implies$$

$$t_1 * y * x * t_1 * y^2 * x * t_1 * y * x * \underline{y} * t_3 \in M \implies$$

$$t_1 * y * x * t_1 * y^2 * x * t_1 * \underline{y * x * y} * t_3 \in M \implies$$

$$t_1 * y * x * t_1 * y^2 * x * \underline{y * x * y} * t_6 * t_3 \in M \implies$$

$$t_1 * y * x * \underline{y^2 * x * y * x * y} * t_4 * t_6 * t_3 \in M \implies$$

$$\underline{y * x * y^2 * x * y * x * y} * t_7 * t_4 * t_6 * t_3 \in M \implies$$

$$M t_7 t_4 t_6 = M t_3\ (4)$$

$$(y * t * x * y^{-1} * t * y^{-2} * t * x * y^{-1} * t * y^{-2} * t) \in M \implies$$

$$y * t_1 * x * y^{-1} * t_1 * y^{-2} * t_1 * x * y^{-1} * t_1 * \underline{y^{-2}} * t_1 \in M \implies$$

$$y * t_1 * x * y^{-1} * t_1 * y^{-2} * t_1 * \underline{x * y^{-1} * y^{-2}} * t_7 * t_1 \in M \implies$$

$$y * t_1 * x * y^{-1} * t_1 * \underline{y^{-2} * x * y^2} * t_3 * t_7 * t_1 \in M \implies$$

$$y * t_1 * \underline{x * y^{-1} * y^{-2} * x * y^2} * t_{10} * t_3 * t_7 * t_1 \in M \implies$$

$$y * \underline{x * y^2 * x * y^2} * t_8 * t_{10} * t_3 * t_7 * t_1 \in M \implies$$

$$M t_8 t_{10} t_3 = M t_1 t_7\ (5)$$

$$(t * x * y^{-1} * t * x * t * y^{-1} * t * y * t * x * t * y * x * t) \in M \implies$$

$$t * x * y^{-1} * t * x * t * y^{-1} * t * y * t * x * t * y * x * t \in M \implies$$

$$t_1 * x * y^{-1} * t_1 * x * t_1 * y^{-1} * t_1 * y * t_1 * x * t_1 * \underline{y * x} * t_1 \in M \implies$$

$$t_1 * x * y^{-1} * t_1 * x * t_1 * y^{-1} * t_1 * y * t_1 * \underline{x * y * x} * t_4 * t_1 \in M \implies$$

$$t_1 * x * y^{-1} * t_1 * x * t_1 * y^{-1} * t_1 * \underline{y * x * y * x} * t_9 * t_4 * t_1 \in M \implies$$

$$t_1 * x * y^{-1} * t_1 * x * t_1 * \underline{y^{-1} * y * x * y * x} * t_7 * t_9 * t_4 * t_1 \in M \implies$$

$$t_1 * x * y^{-1} * t_1 * \underline{x * x * y * x} * t_9 * t_7 * t_9 * t_4 * t_1 \in M \implies$$

$$t_1 * \underline{x * y^{-1} * x^2 * y * x} * t_4 * t_9 * t_7 * t_9 * t_4 * t_1 \in M \implies$$

$$\underline{e} * t_1 * t_4 * t_9 * t_7 * t_9 * t_4 * t_1 \in M \implies$$

$$Mt_1t_4t_9t_7 = Mt_1t_4t_9 \ (6)$$

Conjugating (1) by N gives us the following relations:

$$t_1t_8t_1t_8 = e \ (7)$$

$$t_3t_6t_3t_6 = e \ (8)$$

$$t_4t_5t_4t_5 = e \ (9)$$

$$t_4t_6t_4t_6 = e \ (10)$$

$$t_5t_6t_5t_6 = e \ (11)$$

$$t_6t_8t_6t_8 = e \ (12)$$

$$t_8t_9t_8t_9 = e \ (13)$$

Conjugating (4) by N gives us the following relations:

$$Mt_1t_2 = Mt_4t_5 \ (14)$$

$$Mt_4t_6 = Mt_3t_7 \ (15)$$

$$Mt_6t_5 = Mt_9t_8 \text{ (16)}$$

$$Mt_{10}t_7 = Mt_8t_1 \text{ (17)}$$

Conjugating (5) by N gives us the following relations:

$$Mt_1t_2t_3 = Mt_5t_4 = \text{ (18)}$$

$$Mt_1t_8t_6 = Mt_{10}t_7 \text{ (19)}$$

Conjugating (6) by N gives us the following relation:

$$Mt_1t_2t_6t_{10} = Mt_1t_2t_6 \text{ (20)}$$

7.2 Lemmas

Lemma 1: $Mt_1t_6 \in [1]$

Proof: $Mt_1\ \underline{t_6} = Mt_1\ \underline{t_8t_6t_8} = Mt_1t_8t_6t_8$ (by (12))

$\underline{Mt_1t_8t_6}\ t_8 = \underline{Mt_{10}t_7}\ t_8 = Mt_{10}t_7t_8$ (by (19))

$\underline{Mt_{10}t_7}\ t_8 = \underline{Mt_8t_1}\ t_8 = Mt_8t_1t_8$ (by (17))

$M\ \underline{t_8t_1t_8} = M\ \underline{t_1} = Mt_1$ (by (7)) $\in [1]$.

Lemma 2: $Mt_1t_2t_3 \in [1,2]$

Proof: $\underline{Mt_1t_2t_3} = \underline{Mt_5t_4} = Mt_5t_4$ (by (18))

$Mt_5t_4 \in [1,2]$ since

$(Mt_5t_4)^{(1,5)(2,4)(6,10)(7,9)} = Mt_1t_2,$

where $(1,5)(2,4)(6,10)(7,9) \in N$.

Lemma 3: $Mt_1t_2t_6t_{10} \in [1,2,6]$

Proof: $\underline{Mt_1t_2t_6t_{10}} = \underline{Mt_1t_2t_6} = Mt_1t_2t_6$ (by (16)) $\in [1,2,6]$.

Lemma 4: $Mt_1t_2t_6t_3 \in$ [1,2,6]

Proof: $Mt_1t_2\ \underline{t_6t_3} = Mt_1t_2\ \underline{t_3t_6} = Mt_1t_2t_3t_6$ (by (8))

$\underline{Mt_1t_2t_3}\ t_6 = \underline{Mt_5t_4}\ t_6 = Mt_5t_4t_6$ (by (18))

$M\ \underline{t_5t_4}\ t_6 = M\ \underline{t_4t_5}\ t_6 = Mt_4t_5t_6$ (by (9))

$M\ \underline{t_4t_5}\ t_6 = M\ \underline{t_1t_2}\ t_6 = Mt_1t_2t_6$ (by (14)) $\in$ [1,2,6].

Lemma 5: $Mt_1t_2 = Mt_2t_1$

Proof: $M\ \underline{t_1t_2} = M\ \underline{t_2t_1} = Mt_2t_1$ (by (1)).

Lemma 6: $Mt_1t_2 = Mt_4t_5$

Proof: $\underline{Mt_1t_2} = \underline{Mt_4t_5} = Mt_4t_5$ (by (14)).

Lemma 7: $Mt_1t_2t_6 = Mt_2t_1t_6$

Proof: $M\ \underline{t_1t_2}\ t_6 = M\ \underline{t_2t_1}\ t_6 = Mt_2t_1t_6$ (by (1)).

Lemma 8: $Mt_1t_2t_6 = Mt_3t_7t_5$

Proof: $\underline{Mt_1t_2}\ t_6 = \underline{Mt_4t_5}\ t_6 = Mt_4t_5t_6$ (by (14))

$Mt_4\ \underline{t_5t_6} = Mt_4\ \underline{t_6t_5} = Mt_4t_6t_5$ (by (11))

$\underline{Mt_4t_6}\ t_5 = \underline{Mt_3t_7}\ t_5 = Mt_3t_7t_5$ (by (15)).

Lemma 9: $Mt_1t_2t_6 = Mt_8t_9t_4$

Proof: $\underline{Mt_1t_2}\ t_6 = \underline{Mt_4t_5}\ t_6 = Mt_4t_5t_6$ (by (14))

$M\ \underline{t_4t_5}\ t_6 = M\ \underline{t_5t_4}\ t_6 = Mt_5t_4t_6$ (by (9))

$Mt_5\ \underline{t_4t_6} = Mt_5\ \underline{t_6t_4} = Mt_5t_6t_4$ (by (10)

$M\ \underline{t_5t_6}\ t_4 = M\ \underline{t_6t_5}\ t_4 = Mt_6t_5t_4$ (by (11))

$\underline{Mt_6t_5}\ t_4 = \underline{Mt_9t_8}\ t_4 = Mt_9t_8t_4$ (by (16))

$M\ \underline{t_9t_8}\ t_4 = M\ \underline{t_8t_9}\ t_4 = Mt_8t_9t_4$ (by (13)).

Lemma 10: $Mt_1t_2t_7 \in$ [1,2]

$\underline{Mt_1t_2t_7} = \underline{Mt_2t_1} = Mt_2t_1$ (by (17))

$M\ \underline{t_2t_1} = M\ \underline{t_1t_2} = Mt_1t_2$ (by (1)) $\in$ [1,2].

7.3 Double Coset Enumeration of G over M and N

$MeN = \{M\}$. Since N is transitive on $\Omega = \{1, 2, \ldots, 10\}$ and $Nt_1 \in Nt_1N$, 10 $t_i's$ extend the coset representative N to the double coset $Mt_1N =$ [1].

$$N^{(1)} \geq <(2,10)(3,9)(4,8)(5,7),(2,5,4)(3,9,6)(7,10,8)> \cong \mathbb{Z}_2{:}\mathbb{Z}_3.$$

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{60}{6} = 10$.

$N^{(1)}$ has orbits $\{1\}$, $\{3,6,9\}$, and $\{2,8,5,7,10,4\}$ on Ω. We note that $Mt_1t_1 \in$ [*], $Mt_1t_6 \in$ [1] (Lemma 1), and $Mt_1t_2 \in$ [1,2].

We now consider the double coset [1,2].

$$N^{(1,2)} \geq N^{1,2} \geq 1.$$

However, since $Mt_1t_2 = Mt_2t_1$ (Lemma 5), (1,2)(3,8)(4,5)(7,9) $\in N^{(1,2)}$, and $Mt_1t_2 = Mt_4t_5$ (Lemma 6), (1,4)(2,5)(3,8)(6,10),(1,2)(3,8)(4,5)(7,9) $\in N^{(1,2)}$. Thus,

$$N^{(1,2)} \geq <(1,4)(2,5)(3,8)(6,10),(1,2)(3,8)(4,5)(7,9)> \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The number of right cosets in [1,2] is $\frac{|N|}{|N^{(1,2)}|} = \frac{60}{4} = 15$.

The orbits of $N^{(12)}$ on Ω are $\{3,8\}$, $\{6,10\}$, $\{7,9\}$, and $\{1,2,4,5\}$. We pick representatives 3, 6, and 7 from the 2-orbits, respectively, and 2 from the 4-orbit and determine the double cosets to which $Nt_1t_2t_3$, $Nt_1t_2t_6$, $Nt_1t_2t_7$ and $Nt_1t_2t_2$ belong.

$Nt_1t_2t_3 \in$ [1,2] (Lemma 2), $Nt_1t_2t_6 \in$ [1,2,6], and $Nt_1t_2t_7 \in$ [1,2] (Lemma 10). It is clear that $Nt_1t_2t_2 = Nt_1 \in$ [1].

Now consider the double coset [1,2,6].

We note that $N^{1,2,6} = 1$.

We have $Nt_1t_2t_6 = Nt_2t_1t_6$ (Lemma 7), then $< (1,2)(3,8)(4,5)(7,9) > \in N^{(1,2,6)}$, $Nt_1t_2t_6 = Nt_3t_7t_5$ (Lemma 8) so $< (1,3)(2,7)(5,6)(8,9) > \in N^{(1,2,6)}$ and $Nt_1t_2t_6$ $= Nt_8t_9t_4$ (Lemma 9), so (1,8,7)(2,9,3)(4,5,6) $\in N^{(1,2,6)}$. Thus

$N^{(1,2,6)} \geq < (1,2)(3,8)(4,5)(7,9), (1,8,7)(2,9,3)(4,5,6) >$.

The number of right cosets in [1,2,6] is $\frac{|N|}{|N^{(1,2,6)}|} = \frac{60}{6} = 10$.

$N^{(1,2,6)}$ has orbits $\{10\}$, $\{4,5,6\}$, and $\{1,2,3,8,7,9\}$ on Ω. We pick representatives 10 from the 1-orbit, 6 from the 3-orbit, and 1 from the 6-orbit. We note that $Nt_1t_2t_6t_6 \in$ [1,2], $Nt_1t_2t_6t_{10} \in$ [1,2,6] (Lemma 3), $Nt_1t_2t_6t_1 \in$ [1,2,6] (Lemma 4), the set of right cosets of G over M is closed under right multiplication by $t_i{}'s$. Therefore we must have found all of the MwN double cosets of G.

7.4 Proof of Isomorphism of G_1

Our argument shows that $|G| \geq$ (1+10+15+10) $\times$ $|N|$ = 36 $\times$ 60 = 2160. Now we show that $|G| \leq 2160$.

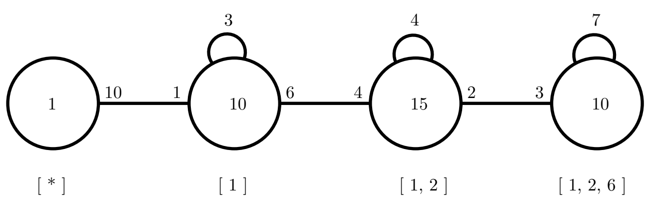
Since G acts on the set of 36 cosets $\{[*], [1], [1,2], [1,2,6]\}$, the mapping

$f : G \longrightarrow S_{36}$ is a homomorphism. Thus $G \ / \ kerf \cong f(G) = G_1 \implies |G| \ / \ |kerf| = |G_1| = 2160 \implies |G| = 2160 \times |kerf| \leq 2160$, since $|kerf| \geq 1$.

Now we have $|G| \geq 2160$ and $|G| \leq 2160$, therefore $|G| = 2160$, so $kerf = 1$ and $G \cong G_1$. We see that a composition series for G is

$G \trianglelefteq S(4,3) \trianglelefteq 1 \implies G \cong S(4,3)$.

Figure 7.1: Cayley Diagram for $S(4,3)$



Chapter 8

Construction of S_7 over S_5

Consider $N = < xx, yy > \cong S_5$, where

xx = (3,17,11,7,5)(4,18,12,8,6)(9,14,22,20,15)(10,13,21,19,16) and

yy = (1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24).

A symmetric presentation for the progenitor 2^{*24}: N is given by

$$G < x, y, t > := Group < x, y, t | x^5, y^2, (x^{-1} * y)^4, (x * y * x^{-2} * y * x)^2 >$$

where $x \sim xx$, $y \sim yy$, and $t \sim t_1$. We factor this progenitor by the three relations and prove that $G < x, y, t > := Group < x, y, t | x^5, y^2, (x^{-1} * y)^4, (x * y * x^{-2} * y * x)^2, t^2, (t, x), (y * x^2 * t)^7, (y * x * t)^5, (x * t^{(y*x)})^6 > \cong S_7$.

In order to do this, we need to perform manual double coset enumeration of G over N.

8.1 Relations, Expansions, and Their Conjugation

Our relations are $((y * x)^2 * t^{(y*x^4)})^2 = e$ (1)

$$y * x^{-1} * y * x * y * x^2 * t * t^{(y*x^3)} * t^{(y*x^4)} * t^y * t = e \ (2)$$

$$(x^{-1} * y * t * t^{(y*x)} * t^{(y*x^3)} * t^y * t = e \ (3)$$

$$x^2 * y * x^{-1} * y * x^2 * t^{(y*x^2)} * t^{(y*x^2*y*x^2)} * t^{(y*x^2)} * t * t^y * t = e \ (4)$$

$$(t * t^y)^3 = e \ (5)$$

$$(t * t^{(y*x^2*y*x^{-1}*y)} = e \ (6)$$

Additionally,

$$(x * t^{(y*x)})^6 = e \ (7)$$

$$(y * x * t)^5 = e \ (8)$$

$$(y * x^2 * t)^7 = e \ (9)$$

are consequences of the additional relations.

$$((y * x)^2 * t^{(y*x^4)})^2 = e \implies$$

$$((y * x)^2 * t_1^{(1,5,15,3)(2,6,16,4)(7,10,8,9)(11,21,23,13)(12,22,24,14)(17,20,18,19)})^2 = e \implies$$

$$((y * x)^2 * t_5)^2 = e \implies$$

$$(y * x)^2 * t_5 * (y * x)^2 * t_5 = e \implies$$

$$\underline{(y * x)^2 * (y * x)^2} * t_6 * t_5 = e \implies$$

$$\underline{e} * t_6 * t_5 = e \implies$$

$$t_5 t_6 = e \ (1)$$

$$(y * x^{-1} * y * x * y * x^2) * t * t^{(y*x^3)} * t^{(y*x^4)} * t^y * t = e \implies$$

$$(y * x^{-1} * y * x * y * x^2 * t_1 * t_1^{(1,7,16,6,19,3)(2,8,15,5,20,4)(9,11,13,17,22,24)(10,12,14,18,21,23)} *$$

$$t_1^{(1,5,15,3)(2,6,16,4)(7,10,8,9)(11,21,23,13)(12,22,24,14)(17,20,18,19)} *$$

$$t_1^{(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24)} * t_1 = e \implies$$

$$(yx^{-1}yxyx^2)t_1t_7t_5t_3t_1 = e \ (2)$$

$$x^{-1} * y * t * t^{(y*x)} * t^{(y*x^3)} * t^y * t = e \implies$$

$$x^{-1} * y * t_1 * t_1^{(1,17,9,3)(2,18,10,4)(5,14,6,13)(7,21,23,19)(8,22,24,20)(11,16,12,15)} *$$

$$t_1^{(1,7,16,6,19,3)(2,8,15,5,20,4)(9,11,13,17,22,24)(10,12,14,18,21,23)} *$$

$$t_1^{(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24)} * t_1 = e \implies$$

$$(x^{-1}y)t_1t_{17}t_7t_3t_1 = e \ (3)$$

$$x^2 * y * x^{-1} * y * x^2 * t^{(y*x^2)} * t^{(y*x^2*y*x^2)} * t^{(y*x^2)} * t * t^y * t = e \implies$$

$$x^2 * y * x^{-1} * y * x^2 * t_1^{(y*x^2)} * t_1^{(y*x^2*y*x^2)} * t_1^{(y*x^2)} * t_1 * t_1^y * t_1 = e \implies$$

$$x^2 * y * x^{-1} * y * x^2 * t_1^{(1,11,10,18,13,3)(2,12,9,17,14,4)(5,22,24,15,7,19)(6,21,23,16,8,20)} *$$

$$t_1^{(1,10,13)(2,9,14)(3,11,18)(4,12,17)(5,24,7)(6,23,8)(15,19,22)(16,20,21)} *$$

$$t_1^{(1,11,10,18,13,3)(2,12,9,17,14,4)(5,22,24,15,7,19)(6,21,23,16,8,20)} * t_1 *$$

$$t_1^{(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24)} * t_1 = e \implies$$

$$(x^{-1}yxyx^{-2})t_{11}t_{10}t_{11}t_1t_3t_1 = e \ (4)$$

$$(t * t^y)^3 = e \implies$$

$$(t_1 * t_1^y)^3 = e \implies$$

$$(t_1 * t_1^{(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24)})^3 = e \implies$$

$$(t_1 * t_3)^3 = e \implies$$

$$t_1t_3t_1t_3t_1t_3 = e \ (5)$$

$$(t * t^{(y*x^2*y*x^{-1}*y)} = e \implies$$

$$t_1t_{23} = e \ (6)$$

$$(x * t^{(y*x)})^6 = e \implies$$

$$(x * t_1^{(1,17,9,3)(2,18,10,4)(5,14,6,13)(7,21,23,19)(8,22,24,20)(11,16,12,15)})^6 = e \implies$$

$$(x * t_{17})^6 = e \implies$$

$$x^6 \ t_{17}^{x^5} \ t_{17}^{x^4} \ t_{17}^{x^3} \ t_{17}^{x^2} \ t_{17}^{x} \ t_{17} = e \implies$$

$$x^6 \ t_{17}^{e} \ t_{17}^{(3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13)} * t_{17}^{(3,7,17,5,11)(4,8,18,6,12)(9,20,14,15,22)(10,19,13,16,21)} *$$

$$t_{17}^{(3,11,5,17,7)(4,12,6,18,8)(9,22,15,14,20)(10,21,16,13,19)} * t_{17}^{(3,17,11,7,5)(4,18,12,8,6)(9,14,22,20,15)(10,13,21,19,16)} \ t_{17} = e \implies$$

$$(x) \ t_{17} \ t_3 \ t_5 \ t_7 \ t_{11} \ t_{17} = e \ (7)$$

$$(y * x * t)^5 = e \implies$$

$$(y * x)^5 * t_1^{(y*x)^4} * t_1^{(y*x)^3} * t_1^{(y*x)^2} * t_1^{(y*x)} * t_1 = e \implies$$

$$(y * x)^5 * t_1^e * t_1^{(1,3,9,17)(2,4,10,18)(5,13,6,14)(7,19,23,21)(8,20,24,22)(11,15,12,16)} *$$

$$t_1^{(1,9)(2,10)(3,17)(4,18)(5,6)(7,23)(8,24)(11,12)(13,14)(15,16)(19,21)(20,22)} *$$

$$t_1^{(1,17,9,3)(2,18,10,4)(5,14,6,13)(7,21,23,19)(8,22,24,20)(11,16,12,15)} * t_1 = e \implies$$

$$(yx) \ t_1 t_3 t_9 t_{17} t_1 = e \ (8)$$

$$(y * x^2 * t)^7 = e \implies$$

$$(y * x^2)^7 * t_1^{(y*x^2)^6} * t_1^{(y*x^2)^5} * t_1^{(y*x^2)^4} * t_1^{(y*x^2)^3} * t_1^{(y*x^2)^2} * t_1^{(y*x^2)} * t_1 = e \implies$$

$$(y * x^2)^7 * t_1^e * t_1^{(1,3,13,18,10,11)(2,4,14,17,9,12)(5,19,7,15,24,22)(6,20,8,16,23,21)} *$$

$$t_1^{(1,13,10)(2,14,9)(3,18,11)(4,17,12)(5,7,24)(6,8,23)(15,22,19)(16,21,20)} *$$

$$t_1^{(1,18)(2,17)(3,10)(4,9)(5,15)(6,16)(7,22)(8,21)(11,13)(12,14)(19,24)(20,23)} *$$

$$t_1^{(1,10,13)(2,9,14)(3,11,18)(4,12,17)(5,24,7)(6,23,8)(15,19,22)(16,20,21)} *$$

$$t_1^{(1,11,10,18,13,3)(2,12,9,17,14,4)(5,22,24,15,7,19)(6,21,23,16,8,20)} * t_1 = e \implies$$

$$(yx^2) \ t_1 t_3 t_{13} t_{18} t_{10} t_{11} t_1 = e \ (9)$$

Conjugating (1) by N gives us the following relations:

$$t_1t_2 = e \ (11)$$

$$t_3t_4 = e \ (12)$$

$$t_7t_8 = e \ (13)$$

$$t_7t_9 = e \ (14)$$

$$t_9t_{10} = e \ (15)$$

$$t_{21}t_{22} = e \ (16)$$

Conjugating (6) by N gives us the following relations:

$$t_3t_{21} = e \ (17)$$

$$t_{23}t_{24} = e \ (18)$$

8.2 Lemmas

Useful Lemmas: Based on the above relations, we prove the following lemmas:

Lemma 1: $Nt_1 = Nt_2$

Proof: $N\ \underline{t_1} = N\ \underline{t_2} = Nt_1 = Nt_2$ (by (11)).

Lemma 2: $Nt_1 = Nt_{23}$

Proof: $N\ \underline{t_1} = N\ \underline{t_{23}} = Nt_{23}$ (by (6)).

Lemma 3: $Nt_1 = Nt_{24}$

Proof: $N\ \underline{t_1} = N\ \underline{t_{23}} = Nt_{23}$ (by Lemma 2)

and $N\ \underline{t_{23}} = N\ \underline{t_{24}} = Nt_{24}$ (by (18)).

Lemma 4: $Nt_1t_3 \in [1,3]$

Proof: $e \in N \implies$

$(Nt_1t_3)^e = Nt_1t_3 \in$ [1,3].

Lemma 5: $Nt_1t_4 \in$ [1,3]

Proof: $Nt_1\ \underline{t_4} = Nt_1\ \underline{t_3}$ (by (12)) $= Nt_1t_3 \in$ [1,3] (by Lemma 4).

Lemma 6: $Nt_1t_9 \in$ [1,3]

Proof: $Nt_1\ \underline{t_9} = Nt_1\ \underline{t_7} = Nt_1t_7$ (by (14))

and $Nt_1t_7 \in$ [1,3] since

$(Nt_1t_7)^{(3,11,5,17,7)(4,12,6,18,8)(9,22,15,14,20)(10,21,16,13,19)} = Nt_1t_3,$

where (3,11,5,17,7)(4,12,6,18,8)(9,22,15,14,20)(10,21,16,13,19) $\in N$.

Lemma 7: $Nt_1t_{10} \in$ [1,3]

$Nt_1\ \underline{t_{10}} = Nt_1\ \underline{t_9} = Nt_1t_9$ (by (15)) $\in$ [1,3] (by Lemma 6).

Lemma 8: $Nt_1t_3 = Nt_{23}t_{22}$

$N\ \underline{t_1}\ t_3 = N\ \underline{t_{23}}\ t_3 = Nt_{23}t_3$ (by (6))

$Nt_{23}\ \underline{t_3} = Nt_{23}\ \underline{t_{21}} = Nt_{23}t_{21}$ (by (17))

$Nt_{23}\ \underline{t_{21}} = Nt_{23}\ \underline{t_{22}} = Nt_{23}t_{22}$ (by (16)).

Lemma 9: $Nt_1t_3t_1 \in$ [1,3,1]

Proof: $(Nt_1t_3t_1)^e = Nt_1t_3t_1 \in$ [1,3,1] since $e \in N$.

Lemma 10: $Nt_1t_3t_3 \in$ [1]

Proof: Since our "t's" are of order two, $t_3t_3 = e$.

Therefore $Nt_1\ \underline{t_3t_3} = Nt_1\ \underline{e} = Nt_1 \in$ [1].

Lemma 11: $Nt_1t_3t_5 \in$ [1,3]

Proof: $N\ \underline{t_1t_3t_5} = N\ \underline{(yx^{-1}yxyx^2)t_1t_7} = Nt_1t_7$ (by (2))

$(Nt_1t_7)^{(3,11,5,17,7)(4,12,6,18,8)(9,22,15,14,20)(10,21,16,13,19)} = Nt_1t_3 \in$ [1,3]

since (3,11,5,17,7)(4,12,6,18,8)(9,22,15,14,20)(10,21,16,13,19) $\in N$.

Lemma 12: $Nt_1t_3t_6 \in$ [1,3]

Proof: $Nt_1t_3\ \underline{t_6} = Nt_1t_3\ \underline{t_5} = Nt_1t_3t_5$ (by (1))

$Nt_1t_3t_5 \in$ [1,3] (by Lemma 11).

Lemma 13: $Nt_1t_3t_7 \in$ [1,3]

Proof: $N\ \underline{t_1t_3t_7} = N\ \underline{(x^{-1}y)t_1t_{17}} = Nt_1t_{17}$ (by (3))

$(Nt_1t_{17})^{(3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13)} = Nt_1t_3 \in$ [1,3]

since (3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13) $\in N$.

Lemma 14: $Nt_1t_3t_8 \in$ [1,3]

Proof: $Nt_1t_3\ \underline{t_8} = Nt_1t_3\ \underline{t_7} = Nt_1t_3t_7$ (by (13))

$Nt_1t_3t_7 \in$ [1,3] (by Lemma 13).

Lemma 15: $Nt_1t_3t_1 = Nt_3t_1t_3$

Proof: $N\ \underline{t_1t_3t_1} = N\ \underline{t_3t_1t_3} = Nt_3t_1t_3$ (by (5)).

Lemma 16: $Nt_1t_3t_1 = Nt_{11}t_{10}t_{11}$

Proof: $N\ \underline{t_1t_3t_1} = N\ \underline{(x^2yx^{-1}yx^2)t_{11}t_{10}t_{11}} = Nt_{11}t_{10}t_{11}$ (by (4)).

Lemma 17: $Nt_1t_3t_1t_1 \in$ [1,3]

Proof: Since our "t's" are of order two, $t_1t_1 = e$.

Therefore $Nt_1t_3\ \underline{t_1t_1} = Nt_1t_3\ \underline{e} = Nt_1t_3 \in$ [1,3].

8.3 Double Coset Enumeration of G over N

$NeN = \{N\}$. Since N is transitive on $\Omega = \{1, 2, \ldots, 24\}$ and $Nt_1 \in Nt_1N$, 24 $t_i's$ extend the coset representative N to the double coset $Nt_1N = [1]$.

$$N^{(1)} \geq < (3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13) > \cong \mathbb{Z}_5$$

Since $Nt_1 = Nt_2$ (Lemma 1),

$$N^{(1)} \geq < (3,5,7,11,17)(4,6,8,12,18)(9,15,20,22,14)(10,16,19,21,13),$$

$$(1,2)(3,8)(4,7)(5,6)(9,21)(10,22)(11,18)(12,17)(13,14)(15,19)(16,20)(23,24) >$$

The number of right cosets in [1] is $\frac{|N|}{|N^{(1)}|} = \frac{120}{20} = 6$.

$N^{(1)}$ has orbits $\{1,2\}$, $\{23,24\}$, $\{3,17,12,18,8,4,6,11,5,7\}$, and

$\{9,14,13,10,21,16,19,22,15,20\}$ on Ω.

We note that $Nt_1t_1 \in$ [*] and $Nt_1t_{23} \in$ [*] (Lemma 2), so four elements return to [*]. $Nt_1t_3 \in$ [1,3] (Lemma 4) and $Nt_1t_9 \in$ [1,3] (Lemma 6) so twenty elements go to [1,3].

Next, we consider the double coset [1,3].

Note $N^{1,3} = 1$.

Now $Nt_1t_3 = Nt_{23}t_{22}$ (Lemma 8).

So (1,23,2,24)(3,22,4,21)(5,15,18,10)(6,16,17,9)(7,14,12,13,11,20) $\in N^{(1,3)}$ and

(1,2)(3,4)(5,18)(6,17)(7,12)(8,11)(9,16)(10,15)(13,20)(14,19)(21,22)(23,24) $\in N^{(1,3)}$.

Thus, $N^{(1,3)} \geq$

$< (1,23,2,24)(3,22,4,21)(5,15,18,10)(6,16,17,9)(7,14,12,13,11,20),$

(1,2)(3,4)(5,18)(6,17)(7,12)(8,11)(9,16)(10,15)(13,20)(14,19)(21,22)(23,24) $>$

The number of right cosets in [1,3] is $\frac{|N|}{|N^{(1,3)}|} = \frac{120}{4} = 30$.

The orbits of $N^{(1,3)}$ on Ω are $\{1,23,2,24\}$, $\{3,22,4,21\}$, $\{5,15,18,10\}$, $\{6,16,17,9\}$, $\{7,14,12,19\}$, and $\{8,13,11,20\}$.

$Nt_1t_3t_1 \in$ [1,3,1] (Lemma 9), $Nt_1t_3t_3 \in$ [1] (Lemma 10), $Nt_1t_3t_5 \in$ [1,3] (Lemma 11), $Nt_1t_3t_6 \in$ [1,3] (Lemma 12), $Nt_1t3t_7 \in$ [1,3] (Lemma 13), and $Nt_1t_3t_8 \in$ [1,3] (Lemma 14).

We look at the double coset [1,3,1].

Note $N^{1,3,1} = 1$.

Now $Nt_1t_3t_1 = Nt_3t_1t_3$ (Lemma 15) so

(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24) $\in N^{(1,3,1)}$

and $Nt_1t_3t_1 = Nt_{11}t_{10}t_{11}$ (Lemma 16) $\implies$

(1,11,22,7)(2,12,21,8)(3,10,23,16)(4,9,24,15)(5,14,6,13)(17,19,18,20) $\in N^{(1,3,1)}$.

Thus $N^{(1,3,1)} \geq$

$< (1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)(21,23)(22,24),$
$(1,11,22,7)(2,12,21,8)(3,10,23,16)(4,9,24,15)(5,14,6,13)(17,19,18,20) >$.

The number of right cosets in [1,3,1] is $\frac{|N|}{|N^{(1,3,1)}|} = \frac{120}{24} = 5$.

$N^{(1,3,1)}$ on Ω has a single orbit

$\{1,3,19,6,11,16,10,7,24,18,21,13,22,17,12,23,14,9,15,20,8,5,2,4\}$.

We have $Nt_1t_3t_1t_1 \in$ [1,3] (Lemma 17), so twenty four elements return to [1,3], the set of right cosets of G over N is closed under right multiplication by $t_i's$. Therefore we must have found all of the NwN double cosets of G.

8.4 Proof of Isomorphism of G_1

Our argument shows that $|G| \geq (1{+}6{+}30{+}5) \times |N| = 42 \times 120 = 5040$. Now we show that $|G| \leq 5040$.

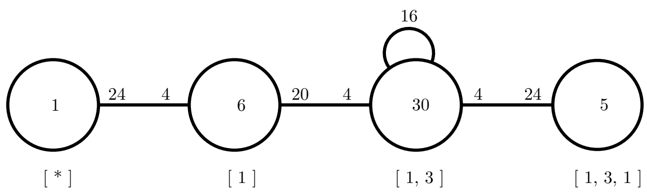
Since G acts on the set of 42 cosets $\{[*], [1], [1,3], [1,3,1]\}$, the mapping

$f : G \longrightarrow S_{42}$ is a homomorphism. Thus $G \;/\; kerf \cong f(G) = G_1 \implies |G| \;/\; |kerf| = |G_1| = 5040 \implies |G| = 5040 \times |kerf| \leq 5040$, since $|kerf| \geq 1$.

Now we have $|G| \geq 5040$ and $|G| \leq 5040$, therefore $|G| = 5040$, so $kerf = 1$ and $G \cong G_1$. We see that a composition series for G is

$$G \trianglelefteq S_7 \trianglelefteq 1 \implies G \cong S_7.$$

Figure 8.1: Cayley Diagram for S_7 over S_5



Chapter 9

Isomorphism Tables

9.1 Transitive Groups (10,3) $\cong D_{20}$

$$G < x, y, t >:= Group < x, y, t | x^2, (y^{-1} * x)^2, y^{10}, t^2, (t, x * y^3),$$

$$(t, t^{(y^5)}), (t, t^y), (t, t^{(y^2)}), (t, t^{(y^3)}), (t, t^{(y^4)}), (x * y * t^{(y^2)})^a,$$

$$(y^2 * t^y)^b, (y^-4 * t^y)^c, (y^{-2} * t^y)^d, (y * t^y)^e, (y^3 * t^y)^f >,$$

where $x \sim xx = (1,8)(2,7)(3,6)(4,5)(9,10)$ and
$y \sim yy = (1,2,3,4,5,6,7,8,9,10)$.

Table 9.1: Some Finite Images of $2^{*10} : D_{20}$

$\#G$	a	b	c	d	e	f	$Isomorphism\ Type$
5120	0	0	0	5	0	0	D_{20}
$10,240$	0	0	0	0	0	10	D_{20}
$20,480$	8	10	0	10	0	0	D_{20}

9.2 Transitive Groups (10,7) $\cong A_5$

$$G < x, y, t >:= Group < x, y, t | x^5, y^2, (x * y^{-1})^3, t^2,$$
$$(t, x^2 * y * x^2 * y * x^{-1}), (t, x * y * x^{-2} * y * x^2 * y),$$
$$(y * x * t^x)^a, (y * x * y^{-1} * t)^b, (y * x * y^{-1} * t^{(y^3)})^c,$$
$$(y * x * y^{-1} * t^{(y*x*y)})^d, (y * x * y^{-1} * t^{(y^2)})^e >;$$

where $x \sim xx =$ (1,3,5,7,9)(2,4,6,8,10) and

$y \sim yy =$ (1,9)(3,4)(5,10)(6,7).

Table 9.2: Some Finite Images of $2^{*10} : A_5$

660	5	0	0	0	5	$PSL_2(11)$
960	3	0	0	0	0	$\mathbb{Z}_2^4 : A_5$
6840	0	0	0	0	4	$PGL_2(19)$
7680	4	0	0	0	6	$\mathbb{Z}_2^6 : S_5$
$1,267,200$	0	0	0	0	5	$(\mathbb{Z}_4 : \mathbb{Z}_2^3) : (A_5 \times L_2(11))$

9.3 Transitive Groups (10,26) $\cong A_6$

$$G < x, y, z, t >:= Group < x, y, z, t || x^3, y^4, z^2, (y^{-1} * z)^2,$$
$$(y^{-2} * x^{-1} * y^2 * x^{-1}), (z * x^{-1})^3, (x^{-1} * y^{-1} * x * z)^3,$$
$$(x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^{-1} * y * x * y), t^2, (t, x * y^2),$$
$$(t, y * x^{-1} * z * x * y), (t, z * x * y * x^{-1} * z), (t, y^x), (z * t^{(y*x)})^a,$$
$$(z * x * y * t)^b, (z * x * t^x)^c, (z * t^x)^d, (z * t^y)^e, (z * x * t)^f >,$$

where $x \sim xx =$ (1,2,10)(3,4,5)(6,7,8),

$y \sim yy =$ (1,3,2,6)(4,5,8,7), and

$z \sim zz =$ (1,2)(4,7)(5,8)(9,10).

Table 9.3: Some Finite Images of $2^{*10} : A_6$

$\#G$	a	b	c	d	e	f	$Isomorphism\ Type$
$368,640$	0	0	0	4	0	0	$\mathbb{Z}_2 : A_6$
$14,171,760$	0	0	0	6	2	6	$\mathbb{Z}_3^9 :^\cdot (\mathbb{Z}_2 \times A_6)$
$188,743,680$	8	10	0	0	0	6	$\mathbb{Z}_2^{18} : PGL(2,9)$

9.4 Transitive Groups (10,30) $\cong A_6$

$$G < x, y, z, t >:= Group < x, y, z, t | x^3, y^8, z^2, (y^{-1} * z)^2, (z * x^{-1})^3,$$
$$(y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x), (x * y^{-2} * x^{-1} * y * x^{-1} * y), t^2,$$
$$(t, y^{-1} * x * z * y^2), (t, y^{-2} * x * y^{-1}), (x * z * x * y * x * z * t)^a,$$
$$(z * t^{(y^2)})^b, (z * x * y * t)^c, (z * x * t^x)^d, (z * x * t)^e, (z * t)^f >,$$

where $x \sim xx =$ (1,2,10)(3,4,5)(6,7,8),

$y \sim yy =$ (1,7,3,4,2,5,6,8), and

$z \sim zz =$ (1,2)(4,7)(5,8)(9,10).

Table 9.4: Some Finite Images of $2^{*10} : A_6$

$\#G$	a	b	c	d	e	f	$Isomorphism\ Type$
$80,640$	0	0	6	0	0	0	$L_3(4) : \mathbb{Z}_2^2$
$368,640$	0	0	10	0	0	4	$\mathbb{Z}_2^9 : PGL(2,9)$
$737,280$	0	0	0	0	0	4	$\mathbb{Z}_2^{10} : PGL(2,9)$
$28,343,520$	0	2	0	0	6	6	$PSL(3,4)$

9.5 Transitive Groups (10,35) $\cong Aut(A_6)$

$$G < x, y, z, w, t >:= Group < x, y, z, w, t | x^3, y^8, z^2, w^2,$$
$$(y^{-1} * z)^2, (x^{-1} * w * x * w), (w * y^3 * w * y^{-1}),$$
$$(z * x^{-1})^3, (y^{-1} * x^{-1} * y^{-1} * x^{-1} * y^2 * x), t^2,$$
$$(t, x^{-1} * y^{-1} * x), (t, y * z * w * y^{-1}), (z * w)^2,$$
$$(y * w * t)^a, (z * x * t)^b, (z * t^{(y^2)})^c, (z * t)^d >;$$

where $x \sim xx$ = (1,2,10)(3,4,5)(6,7,8),
$y \sim yy$ = (1,7,3,4,2,5,6,8),
$z \sim zz$ = (1,2)(4,7)(5,8)(9,10), and
$w \sim ww$ = (3,6)(4,7)(5,8).

Table 9.5: Some Finite Images of $2^{*10} : Aut(A_6)$

$\#G$	a	b	c	d	$Isomorphism\ Type$
$80,640$	7	0	0	0	$L_3(4) : \mathbb{Z}_2^2$
$1,474,560$	0	0	0	4	$\mathbb{Z}_2^{10} \cdot (PGL(2,9) : \mathbb{Z}_2)$
$2,949,120$	8	0	0	0	$\mathbb{Z}_4 \cdot (\mathbb{Z}_2^9 : Aut(A_6))$

9.6 Transitive Groups (15,5) $\cong A_5$

$$G < x, y, t >:= Group < x, y, t | x^5, y^3, (x * y * x)^2,$$
$$t^2, (t, x * y^{-1} * x^{-1} * y * x), (t, x^2 * y),$$
$$(x^{-1} * t^{(y*x*y^2)})^a, (x^{-1} * t^y)^b >,$$

where $x \sim xx = $ (1,9,10,3,14)(2,15,7,12,6)(4,5,11,13,8) and
$y \sim yy = $ (1,4,10)(2,5,8)(3,7,11)(6,9,15)(12,14,13).

Table 9.6: Some Finite Images of $2^{*15} : A_5$

$\#G$	a	b	$Isomorphism\ Type$
1920	3	5	$\mathbb{Z}_2^5 : A_5$
3840	4	3	$\mathbb{Z}_2^6 : A_5$
$245,760$	8	3	$\mathbb{Z}_2^{12} : A_5$

9.7 Transitive Groups (20,15) $\cong A_5$

$$G < x, y, t >:= Group < x, y, t | x^5, y^2, (y * x^{-1})^3,$$
$$t^2, (t, y * x), (y * t^{(x^2 * y)})^a, (y * t^x)^b,$$
$$(y * t^{(x^3 * y)})^c, (y * t^{(x * y * x^2)})^d, (y * t)^e >,$$

where $x \sim xx =$

(1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19) and

$y \sim yy =$ (1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16).

Table 9.7: Some Finite Images of $2^{*20} : A_5$

$\#G$	a	b	c	d	e	$Isomorphism\ Type$
1920	0	0	2	4	0	$\mathbb{Z}_2^5 : A_5$
7560	3	0	0	0	3	$\mathbb{Z}_3 \times A_7$
$61,440$	0	0	4	2	4	$\mathbb{Z}_2^9 : S_5$
$95,040$	0	0	5	2	4	M_{12}
$160,380$	0	0	5	0	3	$\mathbb{Z}_3^5 \because L_2(11)$
$3,870,720$	0	0	6	0	3	$(\mathbb{Z}_2^5 \times \mathbb{Z}_3) \because (\mathbb{Z}_2^4 : A_5)$

9.8 Transitive Groups (24,202) $\cong S_5$

$$G < x, y, t >:= Group < x, y, t | x^5, y^2, (x^{-1} * y)^4,$$
$$(x * y * x^{-2} * y * x)^2, t^2, (t, x), (y * x^2 * t)^a,$$
$$(y * x * t)^b, (x * t^{(y*x)})^c >,$$

where $x \sim xx =$

(3,17,11,7,5)(4,18,12,8,6)(9,14,22,20,15)(10,13,21,19,16) and

$y \sim yy =$

(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)

(12,20)(15,17)(16,18)(21,23)(22,24).

Table 9.8: Some Finite Images of $2^{*24} : S_5$

$\#G$	a	b	c	*Isomorphism Type*
5040	7	5	6	S_7
$30,000$	6	4	0	$\mathbb{Z}_5^3 : (S_5 \times \mathbb{Z}_2)$
$31,200$	4	6	0	$PGL(2,25) \times \mathbb{Z}_2$
$62,400$	4	0	4	$PGL(2,25) \times \mathbb{Z}_2^2$
$46,800,000$	0	5	4	$\mathbb{Z}_5 \times (S(4,5) : \mathbb{Z}_2)$

Appendix A

MAGMA Code for M_{12}

```
S:=Sym(20);
xx:=S!(1,6,10,13,17)(2,5,9,14,18)(3,8,12,15,20)(4,7,11,16,19);

yy:=S!(1,17)(2,18)(3,4)(5,7)(6,8)(9,20)(10,19)(11,14)(12,13)(15,16);

N:=sub<S|xx,yy>;

#N;

G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),

(y*t)^4,

(x*y*x^-1*t*x^2*t*x*y*x^-1*t*x^2*y*t*t^(x*y)*t^x*
t^(x*y*x^-1*y*x*y*x*y)*t^y*t^(x*y*x^-1)*t),

(y*x*t*x*y*t*x^2*y*t*x^2*t*x^-2*t^(x^2)*t^(x*y*x)*
t^(x^2*y)*t^(x*y*x^-1)*t^(x*y*x^-1*y*x*y)*t),

(x^-1*t*x^-2*t*y*x^-2*t*y*x^-1*t*x*y*t^(x*y*x^-1*y*x*y*x^2)*
t^(x*y*x^-1*y*x^2)*t^(x^2)*t^(x*y*x^-1*y*x*y)*t^(x*y*x^-1)*t),

(t*x^-2*t*y*x^-2*t*y*x^-1*t*x^-1*y*x*t^(x*y*x^-1*y*x)*t^(x*y*x^-1)*
t^(x*y*x^2*y)*t^(x*y*x^-1*y*x*y*x)*t^(x*y*x^-1*y*x*y)*t),

(t*x*t*x*y*t*x*t*x*t*y*t*x^-2*t*t^(x*y*x^-1)*t^(x*y*x^-1*y*x*y)*
t^(x*y*x^2)*t^(x*y*x^-1)*t^(x^2*y)*t^(x*y*x^-1*y*x)),
```

```
(y*x^-2*t*x^2*t*x*y*x^-1*t*x^-2*t*x^-2*t*t^(x*y*x^-1)*
t^(x*y*x^-1*y*x*y)*t^y*t^(x^2*y)*t^(x^3)),

(t*x*y*x^2*t*x*y*t*x*y*x^-1*t*y*t*t^(x*y*x^-1)*t^(x*y*x^-1*y*x^2)*
t^(x*y*x^-1*y*x*y*x*y)*t^(x*y*x^-1*y*x)*t^(x*y*x^-1*y*x*y*x)),

(y*t^(x*y*x^2))^2,

(y*t^(x^3*y))^5>;

NN<a,b>:=Group<a,b|a^5,b^2,(b*a^-1)^3>;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);

ArrayP:=[Id(N): i in [1..#N]];

for i in [2..#N] do

P:=[Id(N): l in [1..#Sch[i]]];

for j in [1..#Sch[i]] do

if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;

if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;

if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;

end for;

PP:=Id(N);

for k in [1..#P] do

PP:=PP*P[k]; end for;

ArrayP[i]:=PP;

end for;

N1:=Stabiliser(N,1);
```

```
N1 eq sub<N|yy*xx>;

f,G1,k:=CosetAction(G,sub<G|x,y,
x^2 * t * x * y * t * x^2 * y * t * x^2 * t,
x*y*x^-1*t*x^2*t*x*y*x^-1*t*x^2*y*t,
y*x*t*x*y*t*x^2*y*t*x^2*t*x^-2,
x^-1*t*x^-2*t*y*x^-2*t*y*x^-1*t*x*y,
t*x^-2*t*y*x^-2*t*y*x^-1*t*x^-1*y*x,
t*x*t*x*y*t*x*t*x*t*y*t*x^-2,
y*x^-2*t*x^2*t*x*y*x^-1*t*x^-2*t*x^-2,
t*x*y*x^2*t*x*y*t*x*y*x^-1*t*y>);

Index(G,sub<G|x,y>);

IN:=sub<G1|f(x),f(y)>;

M:=sub<G|x,y,(x^2 * t * x * y * t * x^2 * y * t * x^2 * t),
(x*y*x^-1*t*x^2*t*x*y*x^-1*t*x^2*y*t),
(y*x*t*x*y*t*x^2*y*t*x^2*t*x^-2),
(x^-1*t*x^-2*t*y*x^-2*t*y*x^-1*t*x*y),
(t*x^-2*t*y*x^-2*t*y*x^-1*t*x^-1*y*x),
(t*x*t*x*y*t*x*t*x*t*y*t*x^-2),
(y*x^-2*t*x^2*t*x*y*x^-1*t*x^-2*t*x^-2),
(t*x*y*x^2*t*x*y*t*x*y*x^-1*t*y)>;

IM:=sub<G1|f(x),f(y),f(x^2 * t * x * y * t * x^2 * y * t * x^2 * t),
f(x*y*x^-1*t*x^2*t*x*y*x^-1*t*x^2*y*t),
f(y*x*t*x*y*t*x^2*y*t*x^2*t*x^-2),
f(x^-1*t*x^-2*t*y*x^-2*t*y*x^-1*t*x*y),
f(t*x^-2*t*y*x^-2*t*y*x^-1*t*x^-1*y*x),
f(t*x*t*x*y*t*x*t*x*t*y*t*x^-2),
f(y*x^-2*t*x^2*t*x*y*x^-1*t*x^-2*t*x^-2),
f(t*x*y*x^2*t*x*y*t*x*y*x^-1*t*y)>;


#IM;

CompositionFactors(IM);

#DoubleCosets(G,sub<G|x,y,x^2 * t * x * y * t * x^2 * y * t *x^2 * t>,
sub<G|x,y>);

DoubleCosets(G,sub<G|x,y,x^2 * t * x * y * t * x^2 * y * t * x^2 * t>,
sub<G|x,y>);
```

```
A:=[Id(G1): i in [1..11]];

A[1]:=f(t);

A[2]:=f(t*x*t);

A[3]:=f(t * x^2 * t);

A[4]:=f(t * x * t * x * t);

A[5]:=f(t * x^2 * t * x * t);

A[6]:=f(t * x * t * y * x^-1 * t);

A[7]:=f(t * x * y * t * x^-1 * t);

A[8]:=f(t * x * t * x * y * t * x^-1 * t);

A[9]:=f(t * x * t * y * x^-1 * t * x^-1 * t);

A[10]:=f(t * x^2 * y * t * x^-1 * t * x^-1 * t);

prodim:=function(pt, Q, I)

v := pt;

for i in I do

v := v^(Q[i]);

end for;

return v;

end function;

1^(xx*yy*xx^3*yy*xx^3);

1^(xx*yy*xx^-1);

1^(xx^2*yy*xx);

1^(xx*yy*xx^-1*yy*xx*yy);
```

```
1^(xx);

1^(xx*yy*xx^-1*yy*xx);

1^(xx*yy);

1^(xx*yy*xx^-1*yy*xx*yy*xx);

1^(xx^2);

1^(xx*yy*xx^-1*yy*xx^2);

1^(xx*yy*xx);

1^(xx^3);

1^(xx*yy*xx^-1*yy*xx*yy*xx^2);

1^(xx*yy*xx^2);

1^(xx*yy*xx^2*yy);

1^(yy);

1^(xx*yy*xx^3*yy*xx^2);

1^(xx^2*yy);

1^(xx*yy*xx^-1*yy*xx*yy*xx*yy);

ts := [Id(G1): i in [1 .. 20] ];

ts[1]:=f(t);

ts[2]:=f(t^(x*y*x^3*y*x^3));

ts[3]:=f(t^(x*y*x^-1));

ts[4]:=f(t^(x^2*y*x));

ts[5]:=f(t^(x*y*x^-1*y*x*y));

ts[6]:=f(t^x);
```

```
ts[7]:=f(t^(x*y*x^-1*y*x));

ts[8]:=f(t^(x*y));

ts[9]:=f(t^(x*y*x^-1*y*x*y*x));

ts[10]:=f(t^(x^2));

ts[11]:=f(t^(x*y*x^-1*y*x^2));

ts[12]:=f(t^(x*y*x));

ts[13]:=f(t^(x^3));

ts[14]:=f(t^(x*y*x^-1*y*x*y*x^2));

ts[15]:=f(t^(x*y*x^2));

ts[16]:=f(t^(x*y*x^2*y));

ts[17]:=f(t^y);

ts[18]:=f(t^(x*y*x^3*y*x^2));

ts[19]:=f(t^(x^2*y));

ts[20]:=f(t^(x*y*x^-1*y*x*y*x*y));

cst := [null : i in [1 .. 144]] where null is [Integers() | ];

for i := 1 to 20 do

cst[prodim(1, ts, [i])] := [i];

end for;

m:=0;

for i in [1..144] do if cst[i] ne [] then m:=m+1; end if; end for; m;

N1:=Stabiliser (N,1);

#N;
```

```
SSS:={[1]};

SSS:=SSS^N;

SSS;

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do for n in IM do

if ts[1] eq n*ts[Rep(Seqq[i])[1]]

then print Rep(Seqq[i]);

end if;

end for;

end for;

N1s:=N1;

for n in N do if 1^n eq 2 then N1s:=sub<N|N1s,n>;
end if; end for;

Orbits(N1s);

T1:=Transversal(N,N1s);

T1;

for i in [1..#T1] do

ss:=[1]^T1[i];

cst[prodim(1, ts, ss)] := ss;end for;

m:=0; for i in [1..144] do if cst[i] ne []

then m:=m+1; end if; end for; m;
```

```
CompositionFactors(G1);

Orbits(N1s);

#k;

#G;

/* Break */

N13:=Stabiliser(N1,3);

SSS:={[1,3]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[3] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);

end if; end for; end for;
```

```
N13s:=N13;

N13; #N13;

for g in N do if [1,3]^g eq [7,9] then N13s:=sub<N|N13s,g>;
end if; end for;

T13:=Transversal(N,N13s);

#N13s;

for i in [1..#T13] do ss:=[1,3]^T13[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N13s);

#T13;

N15:=Stabiliser(N1,5);

SSS:={[1,5]}; SSS:=SSS^N;
```

```
#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[5] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);

end if; end for; end for;

N15s:=N15;

N15; #N15;

for g in N do if [1,5]^g eq [6,2] then N15s:=sub<N|N15s,g>;
end if; end for;

for g in N do if [1,5]^g eq [5,1] then N15s:=sub<N|N15s,g>;
end if; end for;
```

```
for g in N do if [1,5]^g eq [2,6] then N15s:=sub<N|N15s,g>;

end if; end for;

#N15s;

T15:=Transversal(N,N15s);

for i in [1..#T15] do ss:=[1,5]^T15[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N15s);

#T15;

N135:=Stabiliser(N13,5);

SSS:={[1,3,5]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);
```

```
Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[3]*ts[5] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);

end if; end for; end for;

N135s:=N135;

N135; #N135;

for g in N do if [1,3,5]^g eq [14,11,10] then N135s:=
sub<N|N135s,g>;
end if; end for;

T135:=Transversal(N,N135s);

#N135s;

for i in [1..#T135] do ss:=[1,3,5]^T135[i];

cst[prodim(1,ts,ss)]:=ss;
```

```
end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N135s);

#T135;

N1311:=Stabiliser(N13,11);

SSS:={[1,3,11]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[3]*ts[11] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
```

```
then print Rep(Seqq[i]);

end if; end for; end for;

N1311s:=N1311;

N1311; #N1311;

for g in N do if [1,3,11]^g eq [9,7,20] then N1311s:=
sub<N|N1311s,g>;
end if; end for;

T1311:=Transversal(N,N1311s);

#N1311s;

for i in [1..#T1311] do ss:=[1,3,11]^T1311[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N1311s);

#T1311;

N1313:=Stabiliser(N13,13);

SSS:={[1,3,13]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do
```

```
for n in IM do

if ts[1]*ts[3]*ts[13] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);

end if; end for; end for;

N1313s:=N1313;

N1313; #N1313;

/* Break */

for g in N do if [1,3,13]^g eq [12,10,6] then N1313s:=
sub<N|N1313s,g>;
 end if; end for;

for g in N do if [1,3,13]^g eq [17,5,16] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [8,14,9] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [19,15,4] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [20,4,15] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [11,6,10] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [7,9,14] then N1313s:=
sub<N|N1313s,g>;
end if; end for;
```

```
for g in N do if [1,3,13]^g eq [2,13,3] then N1313s:=
sub<N|N1313s,g>;
end if; end for;

for g in N do if [1,3,13]^g eq [18,16,5] then N1313s:=
sub<N|N1313s,g>;
end if; end for;
T1313:=Transversal(N,N1313s);

#N1313s;

for i in [1..#T1313] do ss:=[1,3,13]^T1313[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N1313s);

#T1313;

N157:=Stabiliser(N15,7);

SSS:={[1,5,7]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[5]*ts[7] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);
```

```
end if; end for; end for;

N157s:=N157;

N157; #N157;

for g in N do if [1,5,7]^g eq [3,15,9] then N157s:=
sub<N|N157s,g>;
 end if; end for;

for g in N do if [1,5,7]^g eq [6,2,8] then N157s:=
sub<N|N157s,g>;
end if; end for;

for g in N do if [1,5,7]^g eq [14,17,11] then N157s:=
sub<N|N157s,g>;
end if; end for;

for g in N do if [1,5,7]^g eq [16,4,10] then N157s:=
sub<N|N157s,g>;
end if; end for;

for g in N do if [1,5,7]^g eq [18,13,12] then N157s:=
sub<N|N157s,g>;
end if; end for;

T157:=Transversal(N,N157s);

#N157s;

for i in [1..#T157] do ss:=[1,5,7]^T157[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N157s);

#T157;
```

```
/* Break */

N1357:=Stabiliser(N135,7);

SSS:={[1,3,5,7]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[3]*ts[5]*ts[7] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]

then print Rep(Seqq[i]);

end if; end for; end for;

N1357s:=N1357;

N1357; #N1357;

for g in N do if [1,3,5,7]^g eq [17,4,7,5] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [4,17,16,20] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [3,1,15,9] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [12,18,20,16] then N1357s:=
sub<N|N1357s,g>;
```

```
end if; end for;

for g in N do if [1,3,5,7]^g eq [13,2,9,15] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [18,12,8,10] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [2,13,6,19] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [14,11,10,8] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

for g in N do if [1,3,5,7]^g eq [11,14,19,6] then N1357s:=
sub<N|N1357s,g>;
end if; end for;

T1357:=Transversal(N,N1357s);

#N1357s;

for i in [1..#T1357] do ss:=[1,3,5,7]^T1357[i];

cst[prodim(1,ts,ss)]:=ss;end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N1357s);

#T1357;

N13115:=Stabiliser(N1311,5);

SSS:={[1,3,11,5]}; SSS:=SSS^N;

#(SSS);
```

```
Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[3]*ts[11]*ts[5] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]

then print Rep(Seqq[i]);

end if; end for; end for;

N13115s:=N13115;

N13115; #N13115;

for g in N do if [1,3,11,5]^g eq [1,7,18,11] then N13115s:
=sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [1,9,5,18] then N13115s:
=sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [7,1,17,19] then N13115s:
=sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [3,1,12,15] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [9,1,6,13] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [3,7,15,14] then N13115s:=
sub<N|N13115s,g>;
end if; end for;
```

```
for g in N do if [1,3,11,5]^g eq [9,3,13,20] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [7,3,16,17] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [3,9,14,12] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [9,7,20,6] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

for g in N do if [1,3,11,5]^g eq [7,9,19,16] then N13115s:=
sub<N|N13115s,g>;
end if; end for;

T13115:=Transversal(N,N13115s);

#N13115s;

for i in [1..#T13115] do ss:=[1,3,11,5]^T13115[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N13115s);

#T13115;

/* Break */

N15719:=Stabiliser(N157,19);

SSS:={[1,5,7,19]}; SSS:=SSS^N;
```

```
#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[5]*ts[7]*ts[19] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]

then print Rep(Seqq[i]);

end if; end for; end for;

N15719s:=N15719;

N15719; #N15719;

for g in N do if [1,5,7,19]^g eq [6,9,11,4]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [17,7,5,10]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [10,14,16,7]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [8,20,14,3]

then N15719s:=sub<N|N15719s,g>;end if; end for;

for g in N do if [1,5,7,19]^g eq [1,11,9,13]

then N15719s:=sub<N|N15719s,g>; end if; end for;
```

```
for g in N do if [1,5,7,19]^g eq [13,18,19,11]
15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [19,11,15,5]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [12,3,18,8]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [6,16,14,17]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [17,14,20,12]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [17,2,4,16]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [12,2,10,14]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [4,16,20,9]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [15,8,2,12]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [13,4,2,6]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [10,19,18,1]
then N15719s:=sub<N|N15719s,g>; end if; end for;
```

```
for g in N do if [1,5,7,19]^g eq [8,15,11,1]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [1,18,3,15]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [15,5,13,18]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [7,19,3,14]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [3,15,9,20]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [20,12,5,15]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [16,6,18,13]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [19,10,2,17]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [12,20,16,6]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [6,2,8,20]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [20,9,17,2]
then N15719s:=sub<N|N15719s,g>; end if; end for;
```

```
for g in N do if [1,5,7,19]^g eq [16,7,12,2]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [11,4,8,18]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [5,10,4,11]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [9,13,7,16]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [4,13,5,1]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [15,3,19,10]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [13,9,15,8]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [10,5,12,3]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [8,18,6,9]

then N15719s:=sub<N|N15719s,g>; end if; end for;

for g in N do if [1,5,7,19]^g eq [3,14,1,5]
then N15719s:=sub<N|N15719s,g>;
end if; end for;

for g in N do if [1,5,7,19]^g eq [9,20,1,18]

then N15719s:=sub<N|N15719s,g>; end if; end for;
```

```
for g in N do if [1,5,7,19]^g eq [14,3,6,2]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [14,17,11,19]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [7,17,9,6]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [3,12,7,17]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [20,8,4,13]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [16,4,10,19]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [19,7,13,4]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [4,11,17,7]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [18,8,10,5]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [18,1,16,4]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [11,1,14,10]
then N15719s:=sub<N|N15719s,g>; end if; end for;
```

```
for g in N do if [1,5,7,19]^g eq [5,1,20,8]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [9,6,3,12]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [7,16,1,11]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [2,12,13,9]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [2,6,19,7]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [2,17,15,3]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [14,10,8,15]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [11,19,6,16]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [5,15,17,14]
then N15719s:=sub<N|N15719s,g>; end if; end for;
for g in N do if [1,5,7,19]^g eq [18,13,12,20]
then N15719s:=sub<N|N15719s,g>; end if; end for;
T15719:=Transversal(N,N15719s);
#N15719s;
```

```
for i in [1..#T15719] do ss:=[1,5,7,19]^T15719[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..144] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N15719s);

#T15719;
```

Appendix B

MAGMA Code for $M_{21} : (2 \times 2)$

```
N:=TransitiveGroup(10,30);
S:=Sym(10);

xx:=S!(1,2,10)(3,4,5)(6,7,8);

yy:=S!(1,7,3,4,2,5,6,8);

zz:=S!(1,2)(4,7)(5,8)(9,10);

N:=sub<S|xx,yy,zz>;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,

(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),

(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(z*x*y*t)^6>;

NN<a,b,c>:=Group<a,b,c|a^3,b^8,c^2,(b^-1*c)^2,

(c*a^-1)^3,(b^-1*a^-1*b^-1*a^-1*b^2*a),

(a*b^-2*a^-1*b*a^-1*b)>;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
```

```
ArrayP:=[Id(N): i in [1..#N]];

for i in [2..#N] do

P:=[Id(N): l in [1..#Sch[i]]];

for j in [1..#Sch[i]] do

if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;

if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;

if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;

if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;

if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;

end for;

PP:=Id(N);

for k in [1..#P] do

PP:=PP*P[k]; end for;

ArrayP[i]:=PP;

end for;

N1:=Stabiliser(N,1);

N1 eq sub<N|yy^-1*xx*zz*yy^2,yy^-2*xx*yy^-1>;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);

IN:=sub<G1|f(x),f(y), f(z)>;

sub<N|yy^-1*xx*zz*yy^2,yy^-2*xx*yy^-1> eq Stabiliser(N,1);

CompositionFactors(G1);

#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
```

```
DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);

A:=[Id(G1): i in [1..6]];

A[1]:=f(t);

A[2]:=f(t*x*t);

A[3]:=f(t * x * t * x * t);

A[4]:=f(t * x * t * x * t * y^-1 * t);
A[5]:=f(t * x * t * y * t * x^-1 * t * y * t);

prodim:=function(pt, Q, I)

v := pt;

for i in I do

v := v^(Q[i]);

end for;

return v;

end function;

1^xx;

1^(yy^2);

1^(yy^3);

1^(yy^5);

1^(yy^6);

1^yy;

1^(yy^7);

1^(xx^2*zz);

1^(xx^2);
```

```
ts := [Id(G1): i in [1 .. 10] ];

ts[1]:=f(t);

ts[2]:=f(t^x);

ts[3]:=f(t^(y^2));

ts[4]:=f(t^(y^3));

ts[5]:=f(t^(y^5));

ts[6]:=f(t^(y^6));

ts[7]:=f(t^y);

ts[8]:=f(t^(y^7));

ts[9]:=f(t^(x^2*z));

ts[10]:=f(t^(x^2));

cst := [null : i in [1 .. Index(G,sub<G|x,y>)]] where null is
[Integers() | ];

for i := 1 to 10 do

cst[prodim(1, ts, [i])] := [i];

end for;

m:=0;

m;

for i in [1..112] do if cst[i] ne [] then m:=m+1; end if; end for; m;

N1:=Stabiliser (N,1);

Orbits(N1);

N1s:=N1;
```

```
T1:=Transversal(N,N1s);

T1;

N12:=Stabiliser(N1,2);

SSS:={[1,2]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IN do

if ts[1]*ts[2] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);

end if; end for; end for;

N12s:=N12;

N12; #N12;

for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
end if; end for;

T12:=Transversal(N,N12s);

#N12s;

for i in [1..#T12] do ss:=[1,2]^T12[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
```

```
end for; m;

Orbits(N12s);

#T12;

N123:=Stabiliser(N12,3);

SSS:={[1,2,3]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IN do

if ts[1]*ts[2]*ts[3] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);

end if; end for; end for;

N123s:=N123;

N123; #N123;

for g in N do if [1,2,3]^g eq [7,5,4] then N123s:=sub<N|N123s,g>;
end if; end for;

for g in N do if [1,2,3]^g eq [2,1,3] then N123s:=sub<N|N123s,g>;
end if; end for;

T123:=Transversal(N,N123s);

#N123s;

for i in [1..#T123] do ss:=[1,2,3]^T123[i];
```

```
cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N123s);

#T123;

N1239:=Stabiliser(N123,9);

SSS:={[1,2,3,9]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IN do

if ts[1]*ts[2]*ts[3]*ts[9] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]

then print Rep(Seqq[i]);

end if; end for; end for;


N1239s:=N1239;

N1239; #N1239;

for g in N do if [1,2,3,9]^g eq [7,5,4,9] then N1239s:=sub<N|N1239s,g>;
end if; end for;
```

```
for g in N do if [1,2,3,9]^g eq [3,6,2,9] then N1239s:=sub<N|N1239s,g>;
end if; end for;

for g in N do if [1,2,3,9]^g eq [2,9,8,1] then N1239s:=sub<N|N1239s,g>;
end if; end for;

T1239:=Transversal(N,N1239s);

#N1239s;

for i in [1..#T1239] do ss:=[1,2,3,9]^T1239[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N1239s);

#T1239;

N123910:=Stabiliser(N1239,10);

SSS:={[1,2,3,9,10]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IN do

if ts[1]*ts[2]*ts[3]*ts[9]*ts[10] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]

then print Rep(Seqq[i]);
```

```
end if; end for; end for;

N123910s:=N123910;

N123910; #N123910;

for g in N do if [1,2,3,9,10]^g eq [2,10,4,9,1] then N123910s:=
sub<N|N123910s,g>;
end if; end for;

for g in N do if [1,2,3,9,10]^g eq [7,5,4,9,10] then N123910s:=
sub<N|N123910s,g>;
end if; end for;

for g in N do if [1,2,3,9,10]^g eq [2,1,3,10,9] then N123910s:=
sub<N|N123910s,g>;
end if; end for;

T123910:=Transversal(N,N123910s);

#N1239s;

for i in [1..#T123910] do ss:=[1,2,3,9,10]^T123910[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N123910s);
```

Appendix C

MAGMA Code for $L_3(4):2^2$

```
N:=TransitiveGroup(10,30);
S:=Sym(10);
xx:=S!(1,2,10)(3,4,5)(6,7,8);
yy:=S!(1,7,3,4,2,5,6,8);
zz:=S!(1,2)(4,7)(5,8)(9,10);
N:=sub<S|xx,yy,zz>;
#N;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(z*x*y*t)^6,(z*t)^8,(t*t^x)^2=y^4*z>;



NN<a,b,c>:=Group<a,b,c|a^3,b^8,c^2,(b^-1*c)^2,
(c*a^-1)^3,(b^-1*a^-1*b^-1*a^-1*b^2*a),
(a*b^-2*a^-1*b*a^-1*b)>;
#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
```

```
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq N!(3, 6)(4, 5)(7, 8)(9, 10)
 then Sch[i];

 end if; end for;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
CompositionFactors(G1);


#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
 Index(G,sub<G|x,y,z>);
IN:=sub<G1|f(x),f(y),f(z)>;
DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);

A:=[Id(G1): i in [1..11]];

A[1]:=f(t);
A[2]:=f(t * x * t);
A[3]:=f(t * x * t* x * t);
A[4]:=f( t * x * t * x * t * y^-1 * t);
A[5]:=f(t * x * t * y * t * x^-1 * t * y * t);

prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;

1^xx;
1^(yy^2);
1^(yy^3);
1^(yy^5);
1^(yy^6);
1^yy;
```

```
1^(yy^7);
1^(xx^2*zz);
1^(xx^2);



ts := [Id(G1): i in [1 .. 10] ];
ts[1]:=f(t);
ts[2]:=f(t^x);
ts[3]:=f(t^(y^2));
ts[4]:=f(t^(y^3));
ts[5]:=f(t^(y^5));
ts[6]:=f(t^(y^6));
ts[7]:=f(t^y);
ts[8]:=f(t^(y^7));
ts[9]:=f(t^(x^2*z));
ts[10]:=f(t^(x^2));

f(y^4*z)*ts[1]*ts[2]*ts[1]*ts[2];


cst := [null : i in [1 .. 112]] where null is [Integers() | ];
N1:=Stabiliser (N,1);
#N1;
SSS:={[1]};

SSS:=SSS^N;

SSS;

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do for n in IN do

if ts[1] eq n*ts[Rep(Seqq[i])[1]]

then print Rep(Seqq[i]);

end if;

end for;
```

```
end for;


N1s:=N1;

Orbits(N1s);

T1:=Transversal(N,N1s);

T1;

for i in [1..#T1] do

ss:=[1]^T1[i];

cst[prodim(1, ts, ss)] := ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne []

then m:=m+1; end if; end for; m;

Orbits(N1s);


N12:=Stabiliser(N1,2);

 SSS:={[1,2]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IN do
  if ts[1]*ts[2] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N12s:=N12;
  N12; #N12;


for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
```

```
end if; end for;

  T12:=Transversal(N,N12s);
#N12s;
  for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N12s);
#T12;
  T12:=Transversal(N,N12s);
#N12s;
  for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N12s);
#T12;


N123:=Stabiliser(N12,3);

 SSS:={[1,2,3]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IN do
  if ts[1]*ts[2]*ts[3] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N123s:=N123;
  N123; #N123;


for g in N do if [1,2,3]^g eq [3,6,2] then N123s:=sub<N|N123s,g>;
end if; end for;
```

```
for g in N do if [1,2,3]^g eq [2,1,3] then N123s:=sub<N|N123s,g>;
end if; end for;

  T123:=Transversal(N,N123s);
#N123s;
  for i in [1..#T123] do ss:=[1,2,3]^T123[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N123s);
#T123;

N1239:=Stabiliser(N123,9);

 SSS:={[1,2,3,9]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IN do
  if ts[1]*ts[2]*ts[3]*ts[9] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N1239s:=N1239;
  N1239; #N1239;

for g in N do if [1,2,3,9]^g eq [3,6,2,9] then N1239s:=sub<N|N1239s,g>;
 end if; end for;
for g in N do if [1,2,3,9]^g eq [2,9,8,1] then N1239s:=sub<N|N1239s,g>;
 end if; end for;
 T1239:=Transversal(N,N1239s);
#N1239s;
  for i in [1..#T1239] do ss:=[1,2,3,9]^T1239[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N1239s);
```

```
#T1239;


N123910:=Stabiliser(N1239,10);

 SSS:={[1,2,3,9,10]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IN do
  if ts[1]*ts[2]*ts[3]*ts[9]*ts[10]  eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N123910s:=N123910;
  N123910; #N123910;

for g in N do if [1,2,3,9,10]^g eq [2,10,4,9,1] then N123910s:=
sub<N|N123910s,g>; end if; end for;
for g in N do if [1,2,3,9,10]^g eq [3,6,2,9,10] then N123910s:=
sub<N|N123910s,g>; end if; end for;

for g in N do if [1,2,3,9,10]^g eq [1,2,6,10,9] then N123910s:=
sub<N|N123910s,g>; end if; end for;

T123910:=Transversal(N,N123910s);
#N1239s;
  for i in [1..#T123910] do ss:=[1,2,3,9,10]^T123910[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
Orbits(N123910s);
```

Appendix D

MAGMA Code for $L_3(4):2^2$ over $Aut(A_6)$

```
N:=TransitiveGroup(10,35);

#N;

Generators(N);

S:=Sym(10);

xx:=S!(1,2,10)(3,4,5)(6,7,8);

yy:=S!(1,7,3,4,2,5,6,8);

zz:=S!(1,2)(4,7)(5,8)(9,10);

ww:=S!(3,6)(4,7)(5,8);

N:=sub<S|xx,yy,zz,ww>;

#N;

G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^8,z^2,w^2,(y^-1*z)^2,

(x^-1*w*x*w),

(z*w)^2,(w*y^3*w*y^-1),(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x), t^2,
```

```
(t,x^(-1)*y^(-1)*x), (t,y*z*w*y^-1),(t*t^x)^2=y^4*z,
x * y * z * w * t * y^-1 * t * x * t * y^-1 * t * z * t>;
NN<a,b,c,d>:=Group<a,b,c,d|a^3,b^8,c^2,d^2,(b^-1*c)^2,(a^-1*d*a*d),
(c*d)^2,(d*b^3*d*b^-1),(c*a^-1)^3,(b^-1*a^-1*b^-1*a^-1*b^2*a)>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
```

```
N1:=Stabiliser(N,1);

N1 eq sub<N|xx^(-1)*yy^(-1)*xx,yy*zz*ww*yy^-1>;

#sub<G|x,y,z,w>;


f,G1,k:=CosetAction(G,sub<G|x,y,z,w,x*y*z*w*t*y^-1*t*x*t*y^-1*t*z*t>);

Index(G,sub<G|x,y,z,w>);

IN:=sub<G1|f(x),f(y),f(z),f(w)>;

IM:=sub<G1|f(x),f(y),f(z),f(w),f(x*y*z*w*t*y^-1*t*x*t*y^-1*t*z*t)>;

#IM;

CompositionFactors(IM);

#DoubleCosets(G,sub<G|x,y,z,w,x*y*z*w*t*y^-1*t*x*t*y^-1*t*z*t>,
sub<G|x,y,z,w>);
DoubleCosets(G,sub<G|x,y,x^2 * t * x * y * t * x^2 * y * t * x^2 * t>,
sub<G|x,y,z,w>);

A:=[Id(G1): i in [1..11]];

A[1]:=f(t);

A[2]:=f(z*x*t);

prodim:=function(pt, Q, I)

v := pt;

for i in I do

v := v^(Q[i]);

end for;

return v;

end function;
```

```
1^(zz);

1^(yy^2);

1^(yy^3);

1^(yy^2*xx^2);

1^(yy^2*ww);

1^(yy);

1^(yy^7);

1^(xx^2*zz);

1^(xx^2);

ts := [Id(G1): i in [1 .. 10] ];

ts[1]:=f(t);

ts[2]:=f(t^(z));

ts[3]:=f(t^(y^2));

ts[4]:=f(t^(y^3));

ts[5]:=f(t^(y^2*x^2));

ts[6]:=f(t^(y^2*w));

ts[7]:=f(t^y);

ts[8]:=f(t^(y^7));

ts[9]:=f(t^(x^2*z));

ts[10]:=f(t^(x^2));

cst := [null : i in [1 .. 56]] where null is [Integers() | ];

for i := 1 to 10 do
```

```
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..56] do if cst[i] ne [] then m:=m+1; end if; end for; m;
N1:=Stabiliser (N,1);
#N;
SSS:={[1]};
SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do for n in IM do
if ts[1] eq n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if;
end for;
end for;
N1s:=N1;

Orbits(N1s);
T1:=Transversal(N,N1s);
T1;
```

```
for i in [1..#T1] do

ss:=[1]^T1[i];

cst[prodim(1, ts, ss)] := ss;end for;

m:=0; for i in [1..56] do if cst[i] ne []

then m:=m+1; end if; end for; m;


N12:=Stabiliser(N1,2);

SSS:={[1,2]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[2] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);

end if; end for; end for;

N12s:=N12;

N12; #N12;

for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
end if; end for;

T12:=Transversal(N,N12s);

#N12s;
```

```
for i in [1..#T12] do ss:=[1,2]^T12[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..56] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N12s);

#T12;
```

Appendix E

MAGMA Code for $2^{\cdot}L_3(4){:}2$

```
TransitiveGroup(10,26);
S:=Sym(10);
xx:=S!(1, 2, 10)(3, 4, 5)(6, 7, 8);
yy:=S!(1, 3, 2, 6)(4, 5, 8, 7);
zz:=S!(1, 2)(4, 7)(5, 8)(9, 10);
N:=sub<S|xx,yy,zz>;
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^4,z^2,(y^-1*z)^2,y^-2*x^-1*y^2*x^-1,
(z*x^-1)^3,x^-1*y^-1*x^-1*y^-1*x^-1*y*x*y,(x^-1*y^-1*x*z)^3,t^2,
(t,y*x^-1*z*x*y),(t,z*x*y*x^-1*z),(t,y^x),(t,x*y^2),
((t*t^x)^2=y^2*z),
(z*x*y*x*t^(y*x))^6,
((z*x*y)^2*t^x)^8>;

NN<a,b,c>:=Group<a,b,c|a^3,b^4,c^2,(b^-1*c)^2,b^-2*a^-1*b^2*a^-1,
(c*a^-1)^3,a^-1*b^-1*a^-1*b^-1*a^-1*b*a*b,(a^-1*b^-1*a*c)^3>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..#N]];
 for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
  if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
  if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
  if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
  end for;
  PP:=Id(N);
```

```
  for k in [1..#P] do
  PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
f,G1,k:=CosetAction(G,sub<G|x,y,z>);
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
/* 6 */
M:=sub<G|x,y,z * t * y * x * t * x * t * y^-1 * t * x * t>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,z*t*y*x*t*x*t*y^-1*t*x*t>);
Index(G,sub<G|x,y,z>);
IN:=sub<G1|f(x),f(y),f(z)>;
IM:=sub<G1|f(x),f(y),f(z),f(z*t*y*x*t*x*t*y^-1*t*x*t)>;
#DoubleCosets(G,M,sub<G|x,y,z>);
DoubleCosets(G,M,sub<G|x,y,z>);

A:=[Id(G1): i in [1..11]];

A[1]:=f(t);
A[2]:=f(t * x * t);
A[3]:=f(t * x * t* x * t);
A[4]:=f(t * x * t * y * t * x * t);
A[5]:=f(t * x * t * x * t * x^-1 * y * t * z * t);


prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;

1^xx;
1^yy;
1^(yy*xx);
1^(yy*xx^2);
1^(yy^-1);
1^(yy*xx*zz);
1^(xx*yy*xx^2);
1^(xx^2*zz);
1^(xx^2);

ts := [Id(G1): i in [1 .. 10] ];
```

```
ts[1]:=f(t);
ts[2]:=f(t^x);
ts[3]:=f(t^y);
ts[4]:=f(t^(y*x));
ts[5]:=f(t^(y*x^2));
ts[6]:=f(t^(y^-1));
ts[7]:=f(t^(y*x*z));
ts[8]:=f(t^(x*y*x^2));
ts[9]:=f(t^(x^2*z));
ts[10]:=f(t^(x^2));
cst := [null : i in [1 .. 112]] where null is [Integers() | ];
N1:=Stabiliser (N,1);
#N1;
SSS:={[1]};

SSS:=SSS^N;

SSS;

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do for n in IM do

if ts[1] eq n*ts[Rep(Seqq[i])[1]]

then print Rep(Seqq[i]);

end if;

end for;

end for;

N1s:=N1;

Orbits(N1s);

T1:=Transversal(N,N1s);

T1;

for i in [1..#T1] do
```

```
ss:=[1]^T1[i];

cst[prodim(1, ts, ss)] := ss;

end for;

m:=0; for i in [1..112] do if cst[i] ne []

then m:=m+1; end if; end for; m;

Orbits(N1s);

N12:=Stabiliser(N1,2);

 SSS:={[1,2]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IM do
  if ts[1]*ts[2] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N12s:=N12;
  N12; #N12;

for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
end if; end for;

  T12:=Transversal(N,N12s);
#N12s;
  for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N12s);
#T12;
  T12:=Transversal(N,N12s);
#N12s;
```

```
  for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N12s);
#T12;

N123:=Stabiliser(N12,3);

 SSS:={[1,2,3]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IM do
  if ts[1]*ts[2]*ts[3] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N123s:=N123;
  N123; #N123;

for g in N do if [1,2,3]^g eq [3,6,2] then N123s:=sub<N|N123s,g>;
end if; end for;

for g in N do if [1,2,3]^g eq [2,1,3] then N123s:=sub<N|N123s,g>;
end if; end for;

  T123:=Transversal(N,N123s);
#N123s;
  for i in [1..#T123] do ss:=[1,2,3]^T123[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N123s);
#T123;

N1239:=Stabiliser(N123,9);
```

```
 SSS:={[1,2,3,9]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IM do
  if ts[1]*ts[2]*ts[3]*ts[9] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N1239s:=N1239;
  N1239; #N1239;

for g in N do if [1,2,3,9]^g eq [3,6,2,9] then N1239s:=
sub<N|N1239s,g>; end if; end for;
for g in N do if [1,2,3,9]^g eq [2,9,8,1] then N1239s:=
sub<N|N1239s,g>; end if; end for;
 T1239:=Transversal(N,N1239s);
#N1239s;
  for i in [1..#T1239] do ss:=[1,2,3,9]^T1239[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N1239s);
#T1239;

N123910:=Stabiliser(N1239,10);

 SSS:={[1,2,3,9,10]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IM do
  if ts[1]*ts[2]*ts[3]*ts[9]*ts[10]  eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
```

```
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N123910s:=N123910;
  N123910; #N123910;
for g in N do if [1,2,3,9,10]^g eq [2,10,4,9,1] then N123910s:=
sub<N|N123910s,g>;
end if; end for;
for g in N do if [1,2,3,9,10]^g eq [3,6,2,9,10] then N123910s:=
sub<N|N123910
s,g>; end if; end for;

for g in N do if [1,2,3,9,10]^g eq [1,2,6,10,9] then N123910s:=
sub<N|N123910s,g>;
end if; end for;
T123910:=Transversal(N,N123910s);
#N1239s;
  for i in [1..#T123910] do ss:=[1,2,3,9,10]^T123910[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..112] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N123910s);
```

Appendix F

MAGMA Code for $S(4, 3)$

```
N:=TransitiveGroup(10,7);
#N;
Generators(N);
S:=Sym(10);
xx:=S!(1, 9)(3, 4)(5, 10)(6, 7);
yy:=S!(1, 3, 5, 7, 9)(2, 4, 6, 8, 10);

G<x,y,t>:=Group<x,y,t|x^2,y^5,(x*y^-1)^3,t^2,(t,y^-1*x),
(t,y^2*x*y^-2*x*y^2),(t,t^(y*x*y^4)),

(t*t^(y*x*y^-1))^2,
(y*x*y^-1*t)^4,

(y*x*t^(y^2))^6,(y*t^y)^8>;


NN<a,b>:=Group<a,b|a^2,b^5,(a*b^-1)^3>;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..#N]];
 for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
  if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
```

```
  end for;
  PP:=Id(N);
  for k in [1..#P] do
  PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
M:=sub<G|x,y,(y * x * t * y * x * t * y)^2,
y * t * x * y^-1 * t * y^-2 * t * x * y^-1 * t * y^-2 * t>;

f,G1,k:=CosetAction(G,sub<G|x,y,(y * x * t * y * x * t * y)^2,
y * t * x * y^-1 * t * y^-2 * t * x * y^-1 * t * y^-2 * t>);

Index(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;

IM:=sub<G1|f(x),f(y),
f((y * x * t * y * x * t * y)^2),
f(y * t * x * y^-1 * t * y^-2 * t * x * y^-1 * t * y^-2 * t)>;
#DoubleCosets(G,M,sub<G|x,y>);
DoubleCosets(G,M,sub<G|x,y>);



A:=[Id(G1): i in [1..4]];

A[1]:=f(t);
A[2]:=f(t * x * y^-1 * t);
A[3]:=f(t * x * y^-1 * t * x * t);

prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;

1^(yy*xx*yy^-1);
1^(yy);
1^(yy*xx);
```

```
1^(yy^2);
1^(yy^3*xx);
1^(yy^3);
1^(yy*xx*yy^2);
1^(xx);
1^(yy^2*xx);

ts := [Id(G1): i in [1 .. 10] ];
ts[1]:=f(t);
ts[2]:=f(t^(y*x*y^-1));
ts[3]:=f(t^y);
ts[4]:=f(t^(y*x));
ts[5]:=f(t^(y^2));
ts[6]:=f(t^(y^3*x));
ts[7]:=f(t^(y^3));
ts[8]:=f(t^(y*x*y^2));
ts[9]:=f(t^x);
ts[10]:=f(t^(y^2*x));

cst := [null : i in [1 .. 36]] where null is [Integers() | ];
N1:=Stabiliser (N,1);
#N1;
SSS:={[1]};

SSS:=SSS^N;

SSS;

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do for n in IM do

if ts[1] eq n*ts[Rep(Seqq[i])[1]]

then print Rep(Seqq[i]);

end if;

end for;

end for;
```

```
N1s:=N1;

Orbits(N1s);

T1:=Transversal(N,N1s);

T1;

for i in [1..#T1] do

ss:=[1]^T1[i];

cst[prodim(1, ts, ss)] := ss;

end for;

m:=0; for i in [1..36] do if cst[i] ne []

then m:=m+1; end if; end for; m;

Orbits(N1s);


N12:=Stabiliser(N1,2);

SSS:={[1,2]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do

if ts[1]*ts[2] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);
```

```
end if; end for; end for;

N12s:=N12;

N12; #N12;

for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
end if; end for;

for g in N do if [1,2]^g eq [4,5] then N12s:=sub<N|N12s,g>;
end if; end for;

T12:=Transversal(N,N12s);

#N12s;

for i in [1..#T12] do ss:=[1,2]^T12[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N12s);

#T12;


N126:=Stabiliser(N12,6);

SSS:={[1,2,6]}; SSS:=SSS^N;

#(SSS);

Seqq:=Setseq(SSS);

Seqq;

for i in [1..#SSS] do

for n in IM do
```

```
if ts[1]*ts[2]*ts[6] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);

end if; end for; end for;

N126s:=N126;

N126; #N126;

for g in N do if [1,2,6]^g eq [2,1,6] then N126s:=sub<N|N126s,g>;
 end if; end for;

for g in N do if [1,2,6]^g eq [3,7,5] then N126s:=sub<N|N126s,g>;
 end if; end for;

for g in N do if [1,2,6]^g eq [8,9,4] then N126s:=sub<N|N126s,g>;
 end if; end for;

T126:=Transversal(N,N126s);

#N126s;

for i in [1..#T126] do ss:=[1,2,6]^T126[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1; end if;

end for; m;

Orbits(N126s);

#T126;

for m in IM do for n in IN do if ts[1]*ts[3] eq m*(ts[1])^n then "true";
break;end if; end for; end for;

for m in IM do for n in IN do if ts[1]*ts[2]*ts[3] eq m*(ts[1]*ts[2])^n
```

```
then "true";break;end if; end for; end for;

for m in IM do for n in IN do if ts[1]*ts[2]*ts[7] eq m*(ts[1]*ts[2])^n
then "true";break;end if; end for; end for;

for m in IM do for n in IN do if ts[1]*ts[2]*ts[6]*ts[10]
eq m*(ts[1]*ts[2]*ts[6])^n then "true";break;
end if; end for;end for;

for m in IM do for n in IN do if ts[1]*ts[2]*ts[6]*ts[1]
eq m*(ts[1]*ts[2]*ts[6])^n then "true";break;
end if; end for;end for;
```

Appendix G

MAGMA Code for S_7 over S_5

```
N:=TransitiveGroup(24,202);
#N;
Generators(N);
S:=Sym(24);

xx:=S!(3,17,11,7,5)(4,18,12,8,6)(9,14,22,20,15)(10,13,21,19,16);
yy:=S!(1,3)(2,4)(5,9)(6,10)(7,13)(8,14)(11,19)(12,20)(15,17)(16,18)
(21,23)(22,24);
N:=sub<S|xx,yy>;
#N;

FPGroup(N);

 G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^2,(t,x),

(y*x^2*t)^7,

((y*x)^2*t^(y*x^4))^2,

(y*x^-1*y*x*y*x^2*t*t^(y*x^3)*t^(y*x^4)*t^y*t),

(x^-1*y*t*t^(y*x)*t^(y*x^3)*t^y*t),

(x^2*y*x^-1*y*x^2*t^(y*x^2)*t^(y*x^2*y*x^2)*t^(y*x^2)*t*t^y*t),

(t*t^y)^3,

(t*t^(y*x^2*y*x^-1*y)),
```

```
(y*x*t)^5,(x*t^(y*x))^6>;

 NN<a,b>:=Group<a,b|a^5,b^2,(a^-1*b)^4,(a*b*a^-2*b*a)^2>;

#NN;

 Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..#N]];
 for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
 for j in [1..#Sch[i]] do
  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
  if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;

  end for;
  PP:=Id(N);
  for k in [1..#P] do
  PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
  end for;

for i in [1..#Sch] do if ArrayP[i] eq N!(3,17,11,7,5)(4,18,12,8,6)
(9,14,22,20,15)(10,13,21,19,16)
 then Sch[i];
end if; end for;

N1:=Stabiliser(N,1);
 N1 eq sub<N|xx>;

 DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
f,G1,k:=CosetAction(G,sub<G|x,y>);
 CompositionFactors(G1);

  IN:=sub<G1|f(x),f(y)>;
  sub<N|xx> eq Stabiliser(N,1);
  f,G1,k:=CosetAction(G,sub<G|x,y>);

A:=[Id(G1): i in [1..4]];
  A[1]:=f(t);   /* 1  */
  A[2]:=f(t*y*t);  /*2*/
  A[3]:=f(t*y*t*y*t);  /*3*/
```

```
prodim:=function(pt, Q, I)
  v := pt;
    for i in I do
        v := v^(Q[i]);
    end for;
  return v;
  end function;

1^(yy*xx*yy*xx^2*yy*xx^2*yy);

1^yy;

1^(yy*xx*yy*xx^2*yy*xx^2);

1^(yy*xx^4);

1^(yy*xx*yy*xx^2*yy*xx);

1^(yy*xx^3);

1^(yy*xx*yy*xx^2*yy);

1^(yy*xx*yy*xx);

1^(yy*xx^2*yy*xx^2);

1^(yy*xx^2);

1^(yy*xx*yy*xx^2*yy*xx^-1);

1^(yy*xx^3*yy);

1^(yy*xx*yy*xx^2);

1^(yy*xx*yy);

1^(yy*xx^2*yy*xx);

1^(yy*xx);

1^(yy*xx*yy*xx^2*yy*xx^3);

1^(yy*xx^2*yy);
```

```
1^(yy*xx*yy*xx^-1);

1^(yy*xx^2*yy*xx^-1);

1^(yy*xx*yy*xx^3);

1^(yy*xx^2*yy*xx^-1*yy);

1^(yy*xx*yy*xx^3*yy);

 ts:=[Id(G1): i in [1..24]];

 ts[1]:=f(t); ts[2]:=f(t^(y*x*y*x^2*y*x^2*y));

 ts[3]:=f(t^y); ts[4]:=f(t^(y*x*y*x^2*y*x^2));

 ts[5]:=f(t^(y*x^4)); ts[6]:=f(t^(y*x*y*x^2*y*x));

 ts[7]:=f(t^(y*x^3)); ts[8]:=f(t^(y*x*y*x^2*y));

 ts[9]:=f(t^(y*x*y*x)); ts[10]:=f(t^(y*x^2*y*x^2));

 ts[11]:=f(t^(y*x^2)); ts[12]:=f(t^(y*x*y*x^2*y*x^-1));

 ts[13]:=f(t^(y*x^3*y)); ts[14]:=f(t^(y*x*y*x^2));

 ts[15]:=f(t^(y*x*y)); ts[16]:=f(t^(y*x^2*y*x));

 ts[17]:=f(t^(y*x)); ts[18]:=f(t^(y*x*y*x^2*y*x^3));

 ts[19]:=f(t^(y*x^2*y)); ts[20]:=f(t^(y*x*y*x^-1));

 ts[21]:=f(t^(y*x^2*y*x^-1)); ts[22]:=f(t^(y*x*y*x^3));

 ts[23]:=f(t^(y*x^2*y*x^-1*y)); ts[24]:=f(t^(y*x*y*x^3*y));


cst := [null : i in [1 .. 42]] where null is [Integers() | ];
N1:=Stabiliser (N,1);
#N1;
SSS:={[1]};

SSS:=SSS^N;
```

```
SSS;
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do for n in IN do
if ts[1] eq n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if;
end for;
end for;
N1s:=N1;
for n in N do if 1^n eq 2 then N1s:=sub<N|N1s,n>;
end if; end for;
Orbits(N1s);
T1:=Transversal(N,N1s);
T1;
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..42] do if cst[i] ne []
then m:=m+1; end if; end for; m;
```

```
Orbits(N1s);


#G;

#k;

N13:=Stabiliser(N1,3);

 SSS:={[1,3]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
 for n in IN do
  if ts[1]*ts[3] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N13s:=N13;
  N13; #N13;

for g in N do if [1,3]^g eq [23,22] then N13s:=sub<N|N13s,g>;
end if; end for;
  T13:=Transversal(N,N13s);
#N13s;
  for i in [1..#T13] do ss:=[1,3]^T13[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..42] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N13s);
#T13;

N131:=Stabiliser(N13,1);

 SSS:={[1,3,1]}; SSS:=SSS^N;
 #(SSS);

  Seqq:=Setseq(SSS);
  Seqq;
 for i in [1..#SSS] do
```

```
 for n in IN do
  if ts[1]*ts[3]*ts[1] eq
  n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
  N131s:=N131;
  N131; #N131;

for g in N do if [1,3,1]^g eq [3,1,3] then N131s:=sub<N|N131s,g>;
end if; end for;

for g in N do if [1,3,1]^g eq [19,6,19] then N131s:=sub<N|N131s,g>;
end if; end for;

  T131:=Transversal(N,N131s);
#N131s;
  for i in [1..#T131] do ss:=[1,3,1]^T131[i];
 cst[prodim(1,ts,ss)]:=ss;

  end for;
 m:=0; for i in [1..42] do if cst[i] ne [] then m:=m+1; end if;
  end for; m;
  Orbits(N131s);
#T131;
```

Appendix H

MAGMA Codes ≅ Types Transitive Groups (10,3)

```
TransitiveGroup(10,3);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=5;
r11:=0; r12:=0;


G<x,y,t>:=Group<x,y,t|x^2,(y^-1*x)^2,y^10,t^2,(t,x*y^3),(t,t^(y^5)),
(t,t^y),(t,t^(y^2)),(t,t^(y^3)),(t,t^(y^4)),
(y^5*t^y)^r1,
(x*t^(y^2))^r2,
(x*t^y)^r3,
(x*t^(y^4))^r4,
(x*y*t^(y^5))^r5,
(x*y*t^y)^r6,
(x*y*t^(y^2))^r7,
(y^2*t^y)^r8,
(y^-4*t^y)^r9,
(y^-2*t^y)^r10,
(y*t^y)^r11,
(y^3*t^y)^r12>;
```

```
#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[4];

IsIsomorphic(NL[4],AbelianGroup (GrpPerm,[2,2,2,2,2,2]));
q,ff:=quo<G1|NL[4]>;
q;

IsIsomorphic(q,DihedralGroup(10));



TransitiveGroup(10,3);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=8; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=10;


G<x,y,t>:=Group<x,y,t|x^2,(y^-1*x)^2,y^10,t^2,(t,x*y^3),(t,t^(y^5)),
(t,t^y),(t,t^(y^2)),(t,t^(y^3)),(t,t^(y^4)),
(y^5*t^y)^r1,
(x*t^(y^2))^r2,
(x*t^y)^r3,
```

```
(x*t^(y^4))^r4,
(x*y*t^(y^5))^r5,
(x*y*t^y)^r6,
(x*y*t^(y^2))^r7,
(y^2*t^y)^r8,
(y^-4*t^y)^r9,
(y^-2*t^y)^r10,
(y*t^y)^r11,
(y^3*t^y)^r12>;


#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[9];

IsIsomorphic(NL[9],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[9]>;
q;

IsIsomorphic(q,DihedralGroup(10));



TransitiveGroup(10,3);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=8; r8:=10;
```

```
r9:=0; r10:=10;
r11:=0; r12:=0;


G<x,y,t>:=Group<x,y,t|x^2,(y^-1*x)^2,y^10,t^2,(t,x*y^3),(t,t^(y^5)),
(t,t^y),(t,t^(y^2)),(t,t^(y^3)),(t,t^(y^4)),
(y^5*t^y)^r1,
(x*t^(y^2))^r2,
(x*t^y)^r3,
(x*t^(y^4))^r4,
(x*y*t^(y^5))^r5,
(x*y*t^y)^r6,
(x*y*t^(y^2))^r7,
(y^2*t^y)^r8,
(y^-4*t^y)^r9,
(y^-2*t^y)^r10,
(y*t^y)^r11,
(y^3*t^y)^r12>;


#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[13];

IsIsomorphic(NL[13],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[13]>;
q;

IsIsomorphic(q,DihedralGroup(10));
```

Appendix I

MAGMA Codes $\cong$ Types Transitive Groups (10,7)

```
TransitiveGroup(10,7);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=5;
r9:=0; r10:=0;
r11:=0; r12:=5;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y^-1)^3,t^2,
(t,x^2 * y * x^2 * y * x^-1),(t,x * y * x^-2 * y * x^2 * y),
(y*x*y^-1*t^(y^2))^r12,
(y*x*y^-1*t^(y*x*y))^r11,
(y*x*y^-1*t^(y^3))^r10,
(y*x*y^-1*t)^r9,
(y*x*t^x)^r8,
(y*x*t)^r7,
(y*x*t^(y^2*x*y))^r6,
(y*x*t^(y^2))^r5,
(y*t)^r4,
(y*t^y)^r3,
(y^2*t)^r2,
(y^2*t^y)^r1>;
```

```
#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[1];

IsIsomorphic(NL[1],AbelianGroup (GrpPerm,[1]));
q,ff:=quo<G1|NL[1]>;
q;

IsIsomorphic(q,PSL(2,11));



TransitiveGroup(10,7);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=3;
r9:=0; r10:=0;
r11:=0; r12:=0;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y^-1)^3,t^2,
(t,x^2 * y * x^2 * y * x^-1),(t,x * y * x^-2 * y * x^2 * y),
(y*x*y^-1*t^(y^2))^r12,
(y*x*y^-1*t^(y*x*y))^r11,
(y*x*y^-1*t^(y^3))^r10,
```

```
(y*x*y^-1*t)^r9,
(y*x*t^x)^r8,
(y*x*t)^r7,
(y*x*t^(y^2*x*y))^r6,
(y*x*t^(y^2))^r5,
(y*t)^r4,
(y*t^y)^r3,
(y^2*t)^r2,
(y^2*t^y)^r1>;

#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[2,2,2,2]));
q,ff:=quo<G1|NL[2]>;
q;

IsIsomorphic(q,Alt(5));


TransitiveGroup(10,7);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
```

```
r9:=0; r10:=0;
r11:=0; r12:=4;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y^-1)^3,t^2,
(t,x^2 * y * x^2 * y * x^-1),(t,x * y * x^-2 * y * x^2 * y),
(y*x*y^-1*t^(y^2))^r12,
(y*x*y^-1*t^(y*x*y))^r11,
(y*x*y^-1*t^(y^3))^r10,
(y*x*y^-1*t)^r9,
(y*x*t^x)^r8,
(y*x*t)^r7,
(y*x*t^(y^2*x*y))^r6,
(y*x*t^(y^2))^r5,
(y*t)^r4,
(y*t^y)^r3,
(y^2*t)^r2,
(y^2*t^y)^r1>;

#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[1];


IsIsomorphic(NL[1],AbelianGroup (GrpPerm,[1]));
q,ff:=quo<G1|NL[1]>;
q;

IsIsomorphic(q,PGL(2,19));
```

```
TransitiveGroup(10,7);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=4;
r9:=0; r10:=0;
r11:=0; r12:=6;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y^-1)^3,t^2,
(t,x^2 * y * x^2 * y * x^-1),(t,x * y * x^-2 * y * x^2 * y),
(y*x*y^-1*t^(y^2))^r12,
(y*x*y^-1*t^(y*x*y))^r11,
(y*x*y^-1*t^(y^3))^r10,
(y*x*y^-1*t)^r9,
(y*x*t^x)^r8,
(y*x*t)^r7,
(y*x*t^(y^2*x*y))^r6,
(y*x*t^(y^2))^r5,
(y*t)^r4,
(y*t^y)^r3,
(y^2*t)^r2,
(y^2*t^y)^r1>;

#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;
```

```
NL[3];


IsIsomorphic(NL[3],AbelianGroup (GrpPerm,[4]));
q,ff:=quo<G1|NL[3]>;
q;

IsIsomorphic(q,Alt(5));



TransitiveGroup(10,7);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=5;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y^-1)^3,t^2,
(t,x^2 * y * x^2 * y * x^-1),(t,x * y * x^-2 * y * x^2 * y),
(y*x*y^-1*t^(y^2))^r12,
(y*x*y^-1*t^(y*x*y))^r11,
(y*x*y^-1*t^(y^3))^r10,
(y*x*y^-1*t)^r9,
(y*x*t^x)^r8,
(y*x*t)^r7,
(y*x*t^(y^2*x*y))^r6,
(y*x*t^(y^2))^r5,
(y*t)^r4,
(y*t^y)^r3,
(y^2*t)^r2,
(y^2*t^y)^r1>;

#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[2]));
q,ff:=quo<G1|NL[2]>;
q;

IsIsomorphic(q,PSL(2,11));
```

Appendix J

MAGMA Codes $\cong$ Types Transitive Groups (10,26)

```
TransitiveGroup(10,26);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=4; r14:=0;
r15:=0;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^4,z^2,(y^-1*z)^2,y^-2*x^-1*y^2*x^-1,
(z*x^-1)^3,x^-1*y^-1*x^-1*y^-1*x^-1*y*x*y,(x^-1*y^-1*x*z)^3,t^2,
(t,y*x^-1*z*x*y),(t,z*x*y*x^-1*z),(t,y^x),(t,x*y^2),
(z*x*y*x^-1*t^(y*x*z))^r1,
(z*x*y*x*t^(y*x))^r2,
(z*x*y*x*t^(y*x))^r3,
((z*x*y)^2*t^x)^r4,
(z*x*y*x*t^x)^r5,
(z*x*y*x*t^y)^r6,
(z*x*y*x^-1*t)^r7,
((z*x*y)^2*t)^r8,
(z*x*y*x*t)^r9,
(z*t^(y*x))^r10,
```

```
(z*x*y*t)^r11,
(z*x*t^x)^r12,
(z*t^x)^r13,
(z*t^y)^r14,
(z*x*t)^r15>;


#G;



f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[6];


IsIsomorphic(NL[6],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[6]>;
q;

IsIsomorphic(q,Alt(6));


TransitiveGroup(10,26);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=6; r14:=2;
```

```
r15:=6;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^4,z^2,(y^-1*z)^2,y^-2*x^-1*y^2*x^-1,
(z*x^-1)^3,x^-1*y^-1*x^-1*y^-1*x^-1*y*x*y,(x^-1*y^-1*x*z)^3,t^2,
(t,y*x^-1*z*x*y),(t,z*x*y*x^-1*z),(t,y^x),(t,x*y^2),
(z*x*y*x^-1*t^(y*x*z))^r1,
(z*x*y*x*t^(y*x))^r2,
(z*x*y*x*t^(y*x))^r3,
((z*x*y)^2*t^x)^r4,
(z*x*y*x*t^x)^r5,
(z*x*y*x*t^y)^r6,
(z*x*y*x^-1*t)^r7,
((z*x*y)^2*t)^r8,
(z*x*y*x*t)^r9,
(z*t^(y*x))^r10,
(z*x*y*t)^r11,
(z*x*t^x)^r12,
(z*t^x)^r13,
(z*t^y)^r14,
(z*x*t)^r15>;


#G;



f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[3,3,3,3,3,3,3,3,3]));
```

```
q,ff:=quo<G1|NL[2]>;
q;

IsIsomorphic(q,PGL(2,9));
```

Appendix K

MAGMA Codes ≅ Types Transitive Groups (10,30)

```
TransitiveGroup(10,30);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=6;
r13:=0; r14:=0;
r15:=0;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(y^2*x^-1*y*z*x^-1*t^(y^2))^r1,
(y^2*x^-1*y*z*x^-1*t^(y^5))^r2,
(x^-1*y*z*x*y*t^(y^2))^r3,
(x^-1*y*z*x*y*t^(y^5))^r4,
(y^2*x^-1*y*z*x^-1*t)^r5,
(y*x*z*x^-1*y^-2*t)^r6,
(x^-1*y*z*x*y*t)^r7,
```

```
((z*x*y)^2*t)^r8,
(z*x*y^-1*t)^r9,
(x*z*x*y*x*z*t)^r10,
(z*t^(y^2))^r11,
(z*x*y*t)^r12,
(z*x*t^x)^r13,
(z*x*t)^r14,
(z*t)^r15>;

#G;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;


NL[1];


IsIsomorphic(NL[1],AbelianGroup (GrpPerm,[1]));
q,ff:=quo<G1|NL[1]>;
q;


TransitiveGroup(10,30);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=10;
r13:=0; r14:=0;
r15:=4;
```

```
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(y^2*x^-1*y*z*x^-1*t^(y^2))^r1,
(y^2*x^-1*y*z*x^-1*t^(y^5))^r2,
(x^-1*y*z*x*y*t^(y^2))^r3,
(x^-1*y*z*x*y*t^(y^5))^r4,
(y^2*x^-1*y*z*x^-1*t)^r5,
(y*x*z*x^-1*y^-2*t)^r6,
(x^-1*y*z*x*y*t)^r7,
((z*x*y)^2*t)^r8,
(z*x*y^-1*t)^r9,
(x*z*x*y*x*z*t)^r10,
(z*t^(y^2))^r11,
(z*x*y*t)^r12,
(z*x*t^x)^r13,
(z*x*t)^r14,
(z*t)^r15>;

#G;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;


NL[3];


IsIsomorphic(NL[3],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[3]>;
q;
```

```
TransitiveGroup(10,30);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=4;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(y^2*x^-1*y*z*x^-1*t^(y^2))^r1,
(y^2*x^-1*y*z*x^-1*t^(y^5))^r2,
(x^-1*y*z*x*y*t^(y^2))^r3,
(x^-1*y*z*x*y*t^(y^5))^r4,
(y^2*x^-1*y*z*x^-1*t)^r5,
(y*x*z*x^-1*y^-2*t)^r6,
(x^-1*y*z*x*y*t)^r7,
((z*x*y)^2*t)^r8,
(z*x*y^-1*t)^r9,
(x*z*x*y*x*z*t)^r10,
(z*t^(y^2))^r11,
(z*x*y*t)^r12,
(z*x*t^x)^r13,
(z*x*t)^r14,
(z*t)^r15>;

#G;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;
```

```
 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[4];


IsIsomorphic(NL[4],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[4]>;
q;


TransitiveGroup(10,30);


r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=2; r12:=0;
r13:=0; r14:=6;
r15:=6;

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(y^2*x^-1*y*z*x^-1*t^(y^2))^r1,
(y^2*x^-1*y*z*x^-1*t^(y^5))^r2,
(x^-1*y*z*x*y*t^(y^2))^r3,
(x^-1*y*z*x*y*t^(y^5))^r4,
(y^2*x^-1*y*z*x^-1*t)^r5,
(y*x*z*x^-1*y^-2*t)^r6,
(x^-1*y*z*x*y*t)^r7,
((z*x*y)^2*t)^r8,
(z*x*y^-1*t)^r9,
(x*z*x*y*x*z*t)^r10,
(z*t^(y^2))^r11,
(z*x*y*t)^r12,
(z*x*t^x)^r13,
(z*x*t)^r14,
(z*t)^r15>;

#G;
```

```
f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[3,3,3,3,3,3,3,3,3]));
q,ff:=quo<G1|NL[2]>;
q;
```

Appendix L

MAGMA Codes ≅ Types Transitive Groups (10,35)

```
TransitiveGroup(10,35);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=7;
r21:=0; r22:=0;
r23:=0;


G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^8,z^2,w^2,(y^-1*z)^2,
(x^-1*w*x*w),
(z*w)^2,(w*y^3*w*y^-1),(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x), t^2,
(t,x^(-1)*y^(-1)*x), (t,y*z*w*y^(-1)),

(x*z*y*x*y*z*w*x^(-1)*t)^r1,
(x*z*y*x*y*z*w*x^(-1)*t^(y^2))^r2,
```

```
(x^(-1)*y*z*x*y*t^(y^2))^r3,
(x^(-1)*y*z*x*y*t)^r4,
(y*x*z*w*y^(-1)*x*t^(y^6))^r5,
(y*x*z*w*y^(-1)*x*t)^r6,
(x*z*x*y*x*z*t)^r7,
(((z*x*y)^2)*t)^r8,
(y*z*x*w*y*t)^r9,
(y*z*x*w*y*t^x)^r10,
(y*z*x*w*y*t^(y^2))^r11,
(z*x*y^(-1)*t^(y^2))^r12,
(z*x*y^(-1)*t^(y^5))^r13,
(z*x*y^(-1)*t)^r14,
(y*z*w*t^(x^2))^r15,
(y*z*w*t)^r16,
(z*x*y*t)^r17,
(y*w*t^(x^2))^r18,
(z*x*t^x)^r19,
(y*w*t)^r20,
(z*x*t)^r21,
(z*t^(y^2))^r22,
(z*t)^r23>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[1];
```

```
IsIsomorphic(NL[1],AbelianGroup (GrpPerm,[1]));
q,ff:=quo<G1|NL[1]>;
q;



TransitiveGroup(10,35);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=0;
r21:=0; r22:=0;
r23:=4;



G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^8,z^2,w^2,(y^-1*z)^2,
(x^-1*w*x*w),
(z*w)^2,(w*y^3*w*y^-1),(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x), t^2,
(t,x^(-1)*y^(-1)*x), (t,y*z*w*y^(-1)),

(x*z*y*x*y*z*w*x^(-1)*t)^r1,
(x*z*y*x*y*z*w*x^(-1)*t^(y^2))^r2,
(x^(-1)*y*z*x*y*t^(y^2))^r3,
(x^(-1)*y*z*x*y*t)^r4,
(y*x*z*w*y^(-1)*x*t^(y^6))^r5,
(y*x*z*w*y^(-1)*x*t)^r6,
(x*z*x*y*x*z*t)^r7,
(((z*x*y)^2)*t)^r8,
(y*z*x*w*y*t)^r9,
(y*z*x*w*y*t^x)^r10,
(y*z*x*w*y*t^(y^2))^r11,
(z*x*y^(-1)*t^(y^2))^r12,
(z*x*y^(-1)*t^(y^5))^r13,
(z*x*y^(-1)*t)^r14,
(y*z*w*t^(x^2))^r15,
(y*z*w*t)^r16,
```

```
(z*x*y*t)^r17,
(y*w*t^(x^2))^r18,
(z*x*t^x)^r19,
(y*w*t)^r20,
(z*x*t)^r21,
(z*t^(y^2))^r22,
(z*t)^r23>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[4];


IsIsomorphic(NL[4],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[4]>;
q;


TransitiveGroup(10,35);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
```

```
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=8;
r21:=0; r22:=0;
r23:=0;



G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^8,z^2,w^2,(y^-1*z)^2,
(x^-1*w*x*w),
(z*w)^2,(w*y^3*w*y^-1),(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x), t^2,
(t,x^(-1)*y^(-1)*x), (t,y*z*w*y^(-1)),

(x*z*y*x*y*z*w*x^(-1)*t)^r1,
(x*z*y*x*y*z*w*x^(-1)*t^(y^2))^r2,
(x^(-1)*y*z*x*y*t^(y^2))^r3,
(x^(-1)*y*z*x*y*t)^r4,
(y*x*z*w*y^(-1)*x*t^(y^6))^r5,
(y*x*z*w*y^(-1)*x*t)^r6,
(x*z*x*y*x*z*t)^r7,
(((z*x*y)^2)*t)^r8,
(y*z*x*w*y*t)^r9,
(y*z*x*w*y*t^x)^r10,
(y*z*x*w*y*t^(y^2))^r11,
(z*x*y^(-1)*t^(y^2))^r12,
(z*x*y^(-1)*t^(y^5))^r13,
(z*x*y^(-1)*t)^r14,
(y*z*w*t^(x^2))^r15,
(y*z*w*t)^r16,
(z*x*y*t)^r17,
(y*w*t^(x^2))^r18,
(z*x*t^x)^r19,
(y*w*t)^r20,
(z*x*t)^r21,
(z*t^(y^2))^r22,
(z*t)^r23>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y,z>);

#k;
```

```
CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[3];


IsIsomorphic(NL[3],AbelianGroup (GrpPerm,[4]));
q,ff:=quo<G1|NL[3]>;
q;
```

Appendix M

MAGMA Codes ≅ Types Transitive Groups (15,5)

```
r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=3;
r17:=5;
```

```
 G<x,y,t>:=Group<x,y,t|x^5,y^3,
(x*y*x)^2,t^2,(t,x*y^-1*x^-1*y*x),
(t,x^2*y),
 (x*y*x*t^(y*x*y^2))^r1,
 (x*y*x*t^y)^r2,
 (x*y*x*t^x)^r3,
 (x*y*x*t^(y^2))^r4,
 (x*y*x*t^(x^3))^r5,
 (x*y*x*t^(x*y))^r6,
 (x*t^y)^r7,
 (x*t^(y*x*y))^r8,
 (x*t^(x*y*x*y))^r9,
 (x*t^x)^r10,
```

```
 (x*t^(x^4))^r11,
 (x*t^x)^r12,
 (x*t^(x*y))^r13,
 (x*t^y)^r14,
 (x^-1*t^(x^2))^r15,
 (x^-1*t^(y*x*y^2))^r16,
 (x^-1*t^y)^r17>;


 #G;


 f,G1,k:=CosetAction(G,sub<G|x,y>);

 #k;


 CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;



 NL[3];
 IsIsomorphic(NL[3],AbelianGroup
(GrpPerm,[2,2,2,2,2]));
q,ff:=quo<G1|NL[3]>;
q;
IsIsomorphic(q,Alt(5));



r1:=0; r2:=0;
 r3:=0; r4:=0;
 r5:=0; r6:=0;
 r7:=0; r8:=0;
 r9:=0; r10:=0;
 r11:=0; r12:=0;
```

```
 r13:=0; r14:=0;
 r15:=0; r16:=4;
 r17:=3;



 G<x,y,t>:=Group<x,y,t|x^5,y^3,
(x*y*x)^2,t^2,(t,x*y^-1*x^-1*y*x),
(t,x^2*y),
 (x*y*x*t^(y*x*y^2))^r1,
 (x*y*x*t^y)^r2,
 (x*y*x*t^x)^r3,
 (x*y*x*t^(y^2))^r4,
 (x*y*x*t^(x^3))^r5,
 (x*y*x*t^(x*y))^r6,
 (x*t^y)^r7,
 (x*t^(y*x*y))^r8,
 (x*t^(x*y*x*y))^r9,
 (x*t^x)^r10,
 (x*t^(x^4))^r11,
 (x*t^x)^r12,
 (x*t^(x*y))^r13,
 (x*t^y)^r14,
 (x^-1*t^(x^2))^r15,
 (x^-1*t^(y*x*y^2))^r16,
 (x^-1*t^y)^r17>;


 #G;


 f,G1,k:=CosetAction(G,sub<G|x,y>);

 #k;


 CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
```

```
(NL[i]) then i; end if; end for;



 NL[6];
 IsIsomorphic(NL[6],AbelianGroup
(GrpPerm,[2,2,2,2,2,2]));
q,ff:=quo<G1|NL[6]>;
q;
IsIsomorphic(q,Alt(5));



r1:=0; r2:=0;
 r3:=0; r4:=0;
 r5:=0; r6:=0;
 r7:=0; r8:=0;
 r9:=0; r10:=0;
 r11:=0; r12:=0;
 r13:=0; r14:=0;
 r15:=0; r16:=8;
 r17:=3;



 G<x,y,t>:=Group<x,y,t|x^5,y^3,
(x*y*x)^2,t^2,(t,x*y^-1*x^-1*y*x),
(t,x^2*y),
 (x*y*x*t^(y*x*y^2))^r1,
 (x*y*x*t^y)^r2,
 (x*y*x*t^x)^r3,
 (x*y*x*t^(y^2))^r4,
 (x*y*x*t^(x^3))^r5,
 (x*y*x*t^(x*y))^r6,
 (x*t^y)^r7,
 (x*t^(y*x*y))^r8,
 (x*t^(x*y*x*y))^r9,
 (x*t^x)^r10,
 (x*t^(x^4))^r11,
 (x*t^x)^r12,
 (x*t^(x*y))^r13,
 (x*t^y)^r14,
 (x^-1*t^(x^2))^r15,
 (x^-1*t^(y*x*y^2))^r16,
```

```
 (x^-1*t^y)^r17>;


 #G;


 f,G1,k:=CosetAction(G,sub<G|x,y>);

 #k;

 CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[9];
IsIsomorphic(NL[9],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[9]>;
q;
IsIsomorphic(q,Alt(5));
```

Appendix N

MAGMA Codes $\cong$ Types Transitive Groups (20,15)

```
TransitiveGroup(20,15);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=2; r20:=4;
r21:=0;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
(x*t^x)^r8,
```

```
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[4];


IsIsomorphic(NL[4],AbelianGroup (GrpPerm,[2,2,2,2,2]));
q,ff:=quo<G1|NL[4]>;
q;

IsIsomorphic(q,Alt(5));


TransitiveGroup(20,15);
```

```
r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=3; r18:=0;
r19:=0; r20:=0;
r21:=3;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
(x*t^x)^r8,
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;
```

```
CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[3]));
q,ff:=quo<G1|NL[2]>;
q;

IsIsomorphic(q,Alt(7));


TransitiveGroup(20,15);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=4; r20:=2;
r21:=4;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
```

```
(x*t^x)^r8,
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[8];


IsIsomorphic(NL[8],AbelianGroup (GrpPerm,[2,2,2,2,2,2,2,2,2,2]));
q,ff:=quo<G1|NL[8]>;
q;

IsIsomorphic(q,Alt(5));
```

```
TransitiveGroup(20,15);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=5; r20:=2;
r21:=4;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
(x*t^x)^r8,
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


TransitiveGroup(20,15);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=5; r20:=0;
r21:=3;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
(x*t^x)^r8,
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
```

```
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[3,3,3,3,3]));
q,ff:=quo<G1|NL[2]>;
q;

IsIsomorphic(q,PSL(2,11));



TransitiveGroup(20,15);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
```

```
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=6; r20:=0;
r21:=3;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,t^2,(t,y*x),
(x^2*t^(x*y*x^-1*y))^r1,
(x^2*t^(x*y))^r2,
(x^2*t^(x^2*y*x^2*y))^r3,
(x^2*t)^r4,
(x*t^(x^2*y))^r5,
(x*t^(x^3*y))^r6,
(x*t^(x*y*x^3*y))^r7,
(x*t^x)^r8,
(x*y*t^(x^2))^r9,
(x*y*t^(x*y*x^2))^r10,
(x*y*t^(x^3*y*x^2))^r11,
(x*y*t^x)^r12,
(x*y*t^(x^2*y*x^2))^r13,
(x*y*t)^r14,
(x*y*t^(x^2*y*x^2*y*x^-1))^r15,
(x*y*t^(x^4))^r16,
(y*t^(x^2*y))^r17,
(y*t^x)^r18,
(y*t^(x^3*y))^r19,
(y*t^(x*y*x^2))^r20,
(y*t)^r21>;


#G;


f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
```

```
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[8];


IsIsomorphic(NL[8],AbelianGroup (GrpPerm,[2,2,2,2,2,3]));
q,ff:=quo<G1|NL[8]>;
q;
```

Appendix O

MAGMA Codes $\cong$ Types Transitive Groups (24,202)

```
TransitiveGroup(24,202);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=0;
r21:=0; r22:=0;
r23:=0; r24:=7;
r25:=5; r26:=6;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^2,(t,x),

(y*x*y*t^(y*x^2*y*x^2*y*x))^r1,

((y*x^2)^2*t^(y*x*y*x^4))^r2,

((y*x)^2*t^(y*x^4))^r3,
```

```
((y*x)^2*t^(y*x^3))^r4,

((y*x^2)^2*t^(y*x))^r5,

((y*x^2)^2*t^(y*x^3))^r6,

(y*x*t^(y*x^2*y*x*y))^r7,

(y*x*t^(y*x^4*y*x^2))^r8,

(y*x*y*t^(y*x^3*y*x))^r9,

(y*x*y*t^(y*x*y*x^3))^r10,

(y*x*y*t^(y*x^4*y*x))^r11,

(y*x^2*t^(y*x*y*x^4))^r12,

(y*x*y*t^(y*x^4))^r13,

(y*x*y*t^(y*x^3))^r14,

(y*x^2*t^(y*x))^r15,

(y*x^2*t^(y*x^4))^r16,

((y*x^2)^2*t)^r17,

(y*x*t^(y*x^4))^r18,

((y*x)^2*t)^r19,

(y*x*t^(y*x^2*y))^r20,

(y*x*t^(y*x^2))^r21,

(y*x*y*t^y)^r22,

(y*x*y*t)^r23,

(y*x^2*t)^r24,

(y*x*t)^r25,
```

```
(x*t^(y*x))^r26>;



#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[1];


IsIsomorphic(NL[1],AbelianGroup (GrpPerm,[1]));
q,ff:=quo<G1|NL[1]>;
q;


TransitiveGroup(24,202);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=0;
r21:=0; r22:=0;
r23:=0; r24:=6;
```

```
r25:=4; r26:=0;

G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^2,(t,x),

(y*x*y*t^(y*x^2*y*x^2*y*x))^r1,

((y*x^2)^2*t^(y*x*y*x^4))^r2,

((y*x)^2*t^(y*x^4))^r3,

((y*x)^2*t^(y*x^3))^r4,

((y*x^2)^2*t^(y*x))^r5,

((y*x^2)^2*t^(y*x^3))^r6,

(y*x*t^(y*x^2*y*x*y))^r7,

(y*x*t^(y*x^4*y*x^2))^r8,

(y*x*y*t^(y*x^3*y*x))^r9,

(y*x*y*t^(y*x*y*x^3))^r10,

(y*x*y*t^(y*x^4*y*x))^r11,

(y*x^2*t^(y*x*y*x^4))^r12,

(y*x*y*t^(y*x^4))^r13,

(y*x*y*t^(y*x^3))^r14,

(y*x^2*t^(y*x))^r15,

(y*x^2*t^(y*x^4))^r16,

((y*x^2)^2*t)^r17,

(y*x*t^(y*x^4))^r18,

((y*x)^2*t)^r19,

(y*x*t^(y*x^2*y))^r20,
```

```
(y*x*t^(y*x^2))^r21,

(y*x*y*t^y)^r22,

(y*x*y*t)^r23,

(y*x^2*t)^r24,

(y*x*t)^r25,

(x*t^(y*x))^r26>;



#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);



 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];


IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[5,5,5]));
q,ff:=quo<G1|NL[2]>;
q;



TransitiveGroup(24,202);

r1:=0; r2:=0;
```

```
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=0;
r21:=0; r22:=0;
r23:=0; r24:=4;
r25:=6; r26:=0;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^2,(t,x),

(y*x*y*t^(y*x^2*y*x^2*y*x))^r1,

((y*x^2)^2*t^(y*x*y*x^4))^r2,

((y*x)^2*t^(y*x^4))^r3,

((y*x)^2*t^(y*x^3))^r4,

((y*x^2)^2*t^(y*x))^r5,

((y*x^2)^2*t^(y*x^3))^r6,

(y*x*t^(y*x^2*y*x*y))^r7,

(y*x*t^(y*x^4*y*x^2))^r8,

(y*x*y*t^(y*x^3*y*x))^r9,

(y*x*y*t^(y*x*y*x^3))^r10,

(y*x*y*t^(y*x^4*y*x))^r11,

(y*x^2*t^(y*x*y*x^4))^r12,

(y*x*y*t^(y*x^4))^r13,

(y*x*y*t^(y*x^3))^r14,
```

```
(y*x^2*t^(y*x))^r15,

(y*x^2*t^(y*x^4))^r16,

((y*x^2)^2*t)^r17,

(y*x*t^(y*x^4))^r18,

((y*x)^2*t)^r19,

(y*x*t^(y*x^2*y))^r20,

(y*x*t^(y*x^2))^r21,

(y*x*y*t^y)^r22,

(y*x*y*t)^r23,

(y*x^2*t)^r24,

(y*x*t)^r25,

(x*t^(y*x))^r26>;


#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);


 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[2];
```

```
IsIsomorphic(NL[2],AbelianGroup (GrpPerm,[2]));
q,ff:=quo<G1|NL[2]>;
q;


TransitiveGroup(24,202);

r1:=0; r2:=0;
r3:=0; r4:=0;
r5:=0; r6:=0;
r7:=0; r8:=0;
r9:=0; r10:=0;
r11:=0; r12:=0;
r13:=0; r14:=0;
r15:=0; r16:=0;
r17:=0; r18:=0;
r19:=0; r20:=0;
r21:=0; r22:=0;
r23:=0; r24:=4;
r25:=0; r26:=4;


G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^2,(t,x),

(y*x*y*t^(y*x^2*y*x^2*y*x))^r1,

((y*x^2)^2*t^(y*x*y*x^4))^r2,

((y*x)^2*t^(y*x^4))^r3,

((y*x)^2*t^(y*x^3))^r4,

((y*x^2)^2*t^(y*x))^r5,

((y*x^2)^2*t^(y*x^3))^r6,

(y*x*t^(y*x^2*y*x*y))^r7,

(y*x*t^(y*x^4*y*x^2))^r8,

(y*x*y*t^(y*x^3*y*x))^r9,
```

```
(y*x*y*t^(y*x*y*x^3))^r10,

(y*x*y*t^(y*x^4*y*x))^r11,

(y*x^2*t^(y*x*y*x^4))^r12,

(y*x*y*t^(y*x^4))^r13,

(y*x*y*t^(y*x^3))^r14,

(y*x^2*t^(y*x))^r15,

(y*x^2*t^(y*x^4))^r16,

((y*x^2)^2*t)^r17,

(y*x*t^(y*x^4))^r18,

((y*x)^2*t)^r19,

(y*x*t^(y*x^2*y))^r20,

(y*x*t^(y*x^2))^r21,

(y*x*y*t^y)^r22,

(y*x*y*t)^r23,

(y*x^2*t)^r24,

(y*x*t)^r25,

(x*t^(y*x))^r26>;


#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

#k;

CompositionFactors(G1);
```

```
 NL:=NormalLattice(G1);
 NL;

 for i in [1..#NL] do if IsAbelian
(NL[i]) then i; end if; end for;


NL[3];


IsIsomorphic(NL[3],AbelianGroup (GrpPerm,[2,2]));
q,ff:=quo<G1|NL[3]>;
q;
```

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