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Exploring Flag Matroids and Duality

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Exploring Flag Matroids and Duality

A Thesis

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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Abstract

Matroids capture an abstraction of independence in mathematics, and in doing so, connect discrete mathematical structures that arise in a variety of contexts. A matroid can be defined in several cryptomorphic ways depending on which perspective of a matroid is most applicable to the given context. Among the many important concepts in matroid theory, the concept of matroid duality provides a powerful tool when addressing difficult problems. The usefulness of matroid duality stems from the fact that the dual of a matroid is itself a matroid. In this thesis, we explore a matroid-like object called a flag matroid. In particular, we suggest a notion of duality for flag matroids and we investigate the implications of this notion.

Acknowledgements

Thank you to the Math Department at California State University of San Bernardino. Their devotion to their students and their mathematical growth is nothing short of inspiring.

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Thank you to my family: my mother who instilled in me a love for mathematics from an early age, my father who taught me to take pride in my work, and my brothers who remind me the power of humility and a sense of humor for oneself.

Thank you to Dr. James Oxley of Louisiana State University, whose tremendous work in the field of Matroid Theory laid the foundations for this and countless other forays into this area of mathematics.

Finally, an enormous thank you to my academic advisor, Dr. Jeremy Aikin. Without his guidance, instruction, patience, and dedication to this research, this study would not have come to be. This paper is a testament to his investment in his students, and his passion for Matroid Theory is infectious.

Thank you.

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Chapter 1

Introduction

Matroid Theory is a relatively new field of mathematical study, first introduced in 1935 by Hassler Whitney. He developed the concept of a matroid and a few of its defining characteristics initially in an attempt to "capture abstractly the essence of dependence" [Oxl03]. The resultant field has provided links between group theory, graph theory, linear algebra, abstract algebra, combinatorics, and finite geometry. His work in matroids won Whitney the 1983 Wolf Prize, and in the subsequent years the field has garnered a substantial following [Gor12].

In this study, we begin with an introduction to matroids. We do so by describing and depicting matroids in a variety of ways and through a number of examples. With a foundation laid, focus moves to the idea of matroid duality - a crucial concept in the study of matroids. From this, we then introduce less central concepts including quotients, concordancy, Gale orderings, maximality and the increasing exchange property.

Having fleshed out the essential qualities of a matroid, attention shifts to flag matroids. Misnomers in and of themselves, flag matroids are not matroids, but rather collections of sets constructed using matroids, and subject to some similar properties. We define flags and flag matroids, and ultimately suggest a definition for the dual of a flag matroid. Once this definition is established, we prove that the dual of a flag matroid, as defined herein, its itself a flag matroid.

This study concludes by examining flag matroids and their duals on sets from one to four elements: a small sample that examines what further study of flag matroid duality may hold.

Chapter 2

What is a Matroid?

A matroid captures an abstraction of the notion of independence and, as such, can be defined and represented in a variety of fashions. In this chapter, we will introduce a number of these, focusing predominantly on those prominent in later portions of this study: the basis axioms by which a matroid may be defined, and matroids' geometric representations.

2.1 Defining a Matroid: Basis Axioms

A matroid consists of a finite set E together with a collection $\mathcal{B}$ of "basis" subsets adhering to a series of basis axioms. We define a matroid as follows:

Given a finite set E, called the ground set, let $\mathcal{B}$ be a collection of subsets of E satisfying the following three axioms:

(B1) $\mathcal{B} \neq \emptyset$

(B2) If B_1, $B_2 \in \mathcal{B}$, then $|B_1| = |B_2|$

(B3) If B_1, $B_2 \in \mathcal{B}$ and $x \in (B_1 - B_2)$, then there is an element $y \in (B_2 - B_1)$ so that $(B_1 - x \cup y) \in \mathcal{B}$ and $(B_2 - y \cup x) \in \mathcal{B}$. (The Strong Basis Exchange Property).

Then $\mathcal{B}$ is called the collection of bases of a *matroid* on the ground set E.

While there are many approaches to representing a matroid, this study will begin from a linear algebraic approach. Consider the following matrix over the field $GF(2)$:

$$
\begin{array}{c}
\quad\quad\;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \\
M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
\end{array}
$$

This matrix is one representation of a matroid we will, for now, call M. We will refer to the columns of this (and subsequently any) matrix as the elements of a matroid. Herein, they will be labelled as elements $1, 2, 3, 4$, and 5 progressing from left to right, resulting in the set $E = \{1, 2, 3, 4, 5\}$.

A subset of a matroid that can be repersented by a matrix is considered a *basis* in the matroid if the corresponding columns of the corresponding matrix are maximally linearly independent. That is, the addition of any other column of the matrix would result in a linearly dependent set. They thus form a *basis set* in the matroid.

Subsets that are linearly independent but not maximally linearly independent are referred to as *independent sets* in the matroid. Thus a set is independent if it is a subset of some basis.

By this definition, we see that the subsets, $\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}$ are all basis subsets as there is no linear combination of the elements in each that would result in the zero vector, and no other element may be added to any of the subsets without resulting in some linear combination yielding the zero vector. (Note: In future listing of collections of subsets, we omit the numerous braces and commas in the interest of brevity, instead writing individual subsets as if they were a single term, and the full collection within a single set of braces. The collections of basis sets seen above would be written $\{124, 125, 134, 135, 145, 234, 235, 245\}$.) At this point, the order of the elements in each subset does not matter. For example, $\{124\}$ and $\{241\}$ denote the same basis subset. This, however, will not be the case once attention turns to flags.

Sets of elements that are not contatined entirely in some basis are *dependent* in the matroid. In the example M having ground set E as described above, the collection of dependent subsets of the matroid would be $\{123, 345, 1234, 1235, 1245, 1345, 2345, 12345\}$ as each is linearly dependent in the corresponding matrix.

We denote the collection of bases of a matroid as $\mathcal{B}$. Again, let us turn to the example above. Note that here, every independent subset with three elements is also a basis, as every subset with four elements is dependent, thus the addition of another element to any independent subset described would result in a dependent set. The bases of our example matroid may be written: $\mathcal{B} = \{124, 125, 134, 135, 145, 234, 235, 245\}$. Notice that these bases are all of the same size.

We define the *rank* of an arbitrary subset X of E as the cardinality of the maximal independent subset of X. From (B2) of the definition of a matroid, all bases of a matroid are of equal cardinality. Thus we define the *rank of a matroid* as the rank of any of its bases. In our example, for any $B \in \mathcal{B}, |B| = 3$, therefore our matroid has rank 3.

There is one more term worth defining regarding the elements of a matroid. While it is not the focus of this study, its establishment may aide in the understanding of and communication regarding matroids. A subset is a *circuit* if it is minimally dependent. That is, the elimination of any element from the set results in an independent set.

Once more, consider the example above. $\{1245\}$ and $\{1345\}$ are circuits as eliminating any single element from either results in an independent subset. However, $\{1234\}$, $\{1235\}$, and $\{2345\}$ are not circuits as the elimination of 4 and 5 in the first two respectively result in $\{123\}$, while the elimination of 2 in the final yields $\{345\}$, both of which are still dependent sets. Moreover, subsets $\{123\}$ and $\{345\}$ are circuits, as they are both dependent subsets of three elements, but all two element subsets in this example are linearly independent.

Recall the basis axioms established at the beginning of this chapter:

(B1) $\mathcal{B} \neq \emptyset$

(B2) If B_1, $B_2 \in \mathcal{B}$, then $|B_1| = |B_2|$

(B3) If B_1, $B_2 \in \mathcal{B}$ and $x \in (B_1 - B_2)$, then there is an element $y \in (B_2 - B_1)$ so that $(B_1 - x \cup y) \in \mathcal{B}$ and $(B_2 - y \cup x) \in \mathcal{B}$.

By applying these axioms to our prior example, we confirm that the matrix M, with elements $E = \{1, 2, 3, 4, 5\}$ and their subsequent bases, form a matroid. Again, we say that this matroid has a rank of 3 as its bases each have a cardinality of 3.

2.2 Visualizing a Matroid

Any matroid that may be represented by a matrix is called a *representable matroid*. However, not all matroids are representable, and it becomes important that we establish a means of depicting matroids in a more general sense. Thus we establish the idea of a matroid geometry – the most general context by which we may view a matroid. We will now explore how a matrix and its vectors may result in the geometric representation of a matroid. Following this, we will present how some matroids may be presented as graphs and some as bipartite graphs, and how these too may result in matroid geometries. We conclude this introductory chapter by exploring how to determine if a collection of subsets of elements is a matroid without first visualizing it in one of these contexts.

2.2.1 From Matrix to Geometry

In this section we will develop the geometry of matroid from an initial matrix. Recall the representable matroid introduced at the beginning of this chapter.

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

In order to generate the geometry of a matroid, consider the vectors of each column in the matrix M. These columns of M drawn in $\mathbb{R}^3$ produce:

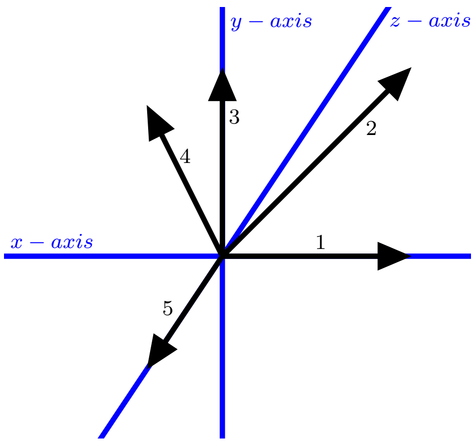

Figure 2.1: Matroid M as vectors in $\mathbb{R}^3$.

Having drawn the matroid in vector form, we place a hyperplane in general position observing where it intersects with a scalar multiple of each vector. Place points at these intersections.

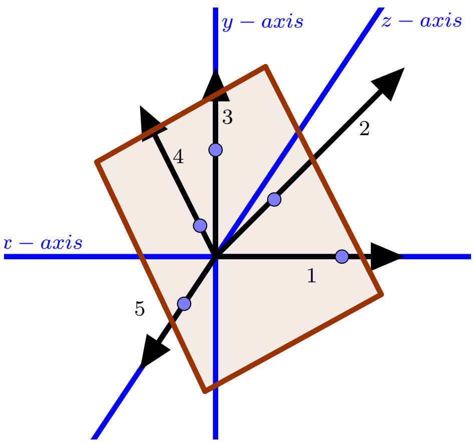

Figure 2.2: Hyperplane and vectors of M.

We label the points where the vectors intersect with the hyperplane by the same name as that of the column vector of the matrix M. These points will ultimately represent the elements of the matroid in the geometry. With these points in place, we make note of collinear points, remove the vectors, and observe the points on the hyperplane. In the case of our example, this is now in $\mathbb{R}^2$. The result is the geometric representation of matroid M.

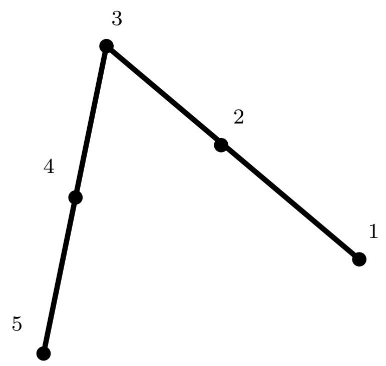

Figure 2.3: The geometry of matroid M.

Note that the three element circuits, like $\{123\}$, are collinear, and the four element circuits and dependent sets, like $\{1245\}$, are coplanar. Meanwhile, the bases, such as $\{135\}$, are not collinear.

To further illustrate this process, we consider another example. Let L be the matroid represented by the following matrix:

Example 2.1.

$$L = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

First, consider the corresponding vectors in $\mathbb{R}^2$.

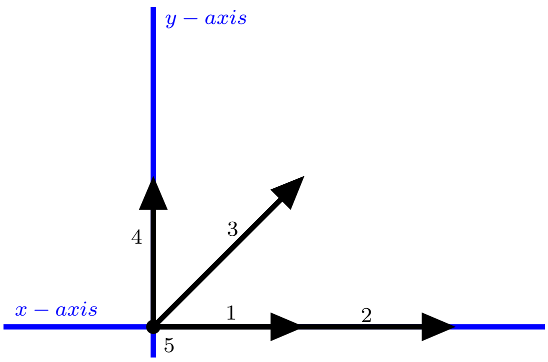

Figure 2.4: Matroid L as vectors in $\mathbb{R}^2$.

Note that Vectors 1 and 2 lie atop one another and form a linearly dependent set of size two. Meanwhile, Vector 5 has a magnitude of zero. We once again place a hyperplane in general position and observe the intersections.

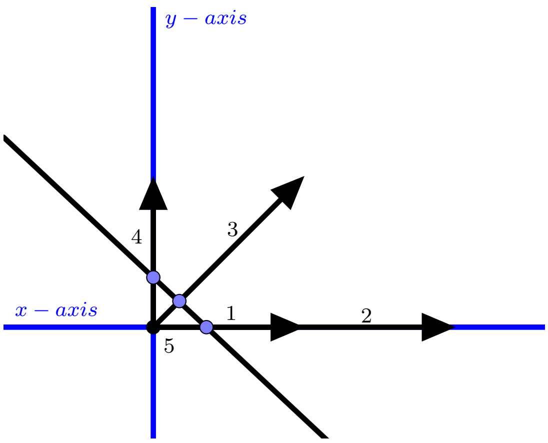

Figure 2.5: Hyperplane and vectors of the matroid L.

The result is the following matroid geometry:

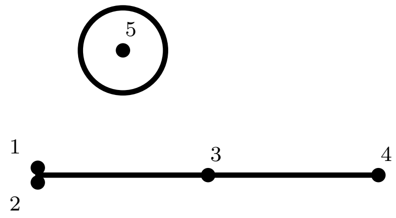

Figure 2.6: The geometry of the matroid L.

The vectors of 1 and 2 are multiples of one another and share direction. Therefore they extend to intersect with the hyperplane at the same point. As a result, the two elements lie in the same position in the geometry, forming a two element dependent set, stemming from the linearly dependent set of size two they formed in the matrix. Element 5 forms a linearly dependent set of size one in the initial matrix, effectively forming a single element dependent set in the matroid. As such, it does not intersect with a hyperplane in general position, and is depicted in the geometry within a cloud to denote it as a dependent singleton.

To summarize, given the matrix of a representable matroid, we may find its geometry by first designating the columns of the matrix as elements. We then graph the vectors in $\mathbb{R}^n$ where n is the number of non-zero rows of the matrix. Once the vectors are graphed, we place a hyperplane in general position and place points where some scalar multiple of each vector intersects the hyperplane. We label each of these points the same as the vector upon which they lie, before subsequently removing the vectors, hyperplane, and axes, and making note of which points are colinear, coplanar, etc.

2.2.2 Identifying Bases of Matroids in Graphs

Matrices and geometries are not the only means by which a matroid may be presented. Sometimes a matroid will take the form of a graph as seen in matroid K below. Matroids that can be represented by a graph are called *graphic matroids*, but like representable matroids, not all matroids are graphic.

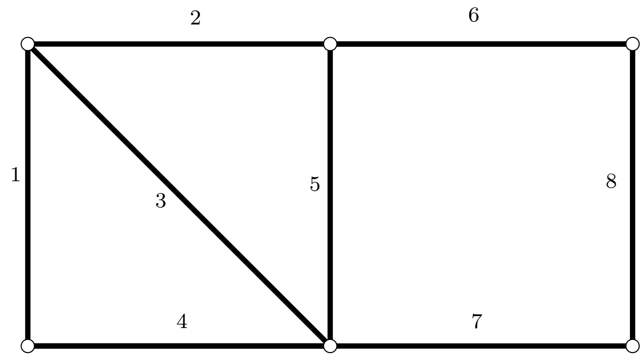

Figure 2.7: The graph of matroid K.

Although visually similar, graphs of matroids differ from geometries of matroids in a key regard: the elements of the matroid are depicted by the edges of the graph, not the points as in a geometry. As such, the vertices of a graph are herein depicted as open circles to help distinguish the two.

The subsets of the matroid are therefore depicted by collections of edges in the graph. A *path* from vertex u to vertex v in a graph G is a sequence of alternating vertices and edges $v_1 = u, e_1, v_2, e_2, v_3, e_3, ..., v_{k-1}, e_{k-1}, v_k = v$ such that, for $i = 1, 2, ..., k-1$, edge e_i is incident with vertices v_i and v_{i+1} and all edges and vertices in the sequence are distinct. For example, the collection of edges $\{1268\}$ in the graph K form a path by which one may "move" from the lower-leftmost vertex to the lower-rightmost. A collection of edges is said to be *connected* if there exists a path between any two vertices in the collection. Finally, a path where $v_1 = v_k$, for example $\{1254\}$, is called a *cycle*.

Meanwhile, a connected graph that does not contain any cycles is referred to as a *tree*, and the collections of edges in these trees form the independent sets of elements of the matroid depicted by the graph. More importantly, a tree that includes all vertices of a graph is called a *spanning tree*, and the collection of edges forming the spanning tree form a basis in the matroid. In matroid K these bases are:

$\mathcal{B} = \{12367, 12368, 12378, 12467, 12468, 12478, 12567, 12568, 12578, 12678, 13567, 13568, 13578, 14567, 14568, 14578, 23467, 23468, 23478, 24567, 24568, 24578, 34567, 34568, 34578\}$.

The rank of matroid K can therefore be determined to be five. A geometry depicting it would require four dimensions. Herein we see the utility of matroid graphs. If a matroid is graphic and of rank greater than four, the graph of the matroid provides a means of visualizing it in a simpler fashion than the geometry, which is limited by dimension.

Sometimes, a matroid may also be modelled by a bipartite graph. Such matroids are called *transversal matroids*. However, in visualizing a matroid in this manner, we introduce another means of conceptualizing the matroid. In a transversal matroid, the corresponding bipartite graph has as its two vertex sets (E, R) where E is the ground set of the matroid, and $|R|$ is the rank of the matroid. Consider the bipartite graph of matroid J below, where $E = \{1, 2, 3, 4, 5\}$ and $R : \{x, y, z\}$:

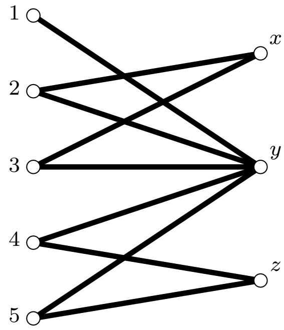

Figure 2.8: The bipartite graph of matroid J.

In a bipartite graph, a collection of edges wherein no two edges share a vertex is called a *matching*. Each of a transversal matroid is a subset of E corresponding to a matching in the bipartite graph. In matroid J:
$\mathcal{B} = \{124, 125, 134, 135, 234, 235, 245, 345\}$.

The vertices on the right do not depict elements of the matroid but determine the rank of the matroid. In matroid J above, the 3 vertices on the right indicate the matroid is of rank 3.

A collection of elements is thereby dependent if two or more elements are forced to be adjacent to the same vertex on the right side. For example, subset $\{123\}$ is dependent, as $\{1\}$ is adjacent to $\{y\}$ leaving $\{2\}$ and $\{3\}$ to either both be adjacent to $\{x\}$, or for one of the two to share vertex $\{y\}$ with $\{1\}$. A similar argument may be made for the subset $\{145\}$.

Given these bases and dependent sets, one may notice matroid J is structurally identical (isomorphic) to matroid M, although the elements have been relabelled.

2.2.3 Examples Given Sets of Bases

Not all matroids are representable, graphic, or transversal. However, one need not be given a matrix, graph, bipartite graph, or any other visual in order to determine if a collection of subsets of elements of E is indeed a matroid. Under the Basis Axioms previously established, one only needs to know the maximally independent subsets of the ground set, and check them against the axioms.

Consider the following matroid N. Let $E = \{1, 2, 3, 4, 5, 6, 7\}$, and $\mathcal{B} = \{125, 126, 135, 136, 245, 246, 345, 346\}$. A check of the basis axioms confirms that this is indeed a matroid. However, should we want to represent it in another fashion, we might construct the geometry of the matroid. In attempting to draw the geometry of a matroid without first being given its matrix, graph, or bipartite graph, one does the following:

Each single independent element is represented by a point. Any two elements that are independent span a line, while a dependent pair share a point. A three element independent set spans a plane, whereas the elements of a three element dependent set are collinear. It follows that if four elements are independent they are not coplanar. [Gor12]

By these criteria, and given the bases of N, we may construct the following geometry:

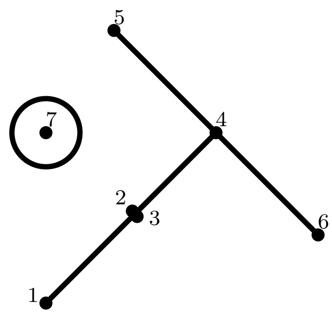

Figure 2.9: Geometry of matriod N.

Likewise, given a geometry, one may determine the associated bases and use them to determine if the geometry represents a matroid. Consider the following geometry of a rank three matroid P. As the matroid is rank three (and thereby 2-dimensional), we know the bases must be any collection of three elements that span the plane.

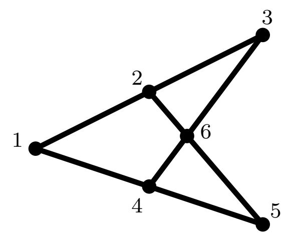

Figure 2.10: Geometry of matriod P.

Such collections of elements form the basis subsets and so $\mathcal{B} = \{124, 125, 126, 134, 135, 136, 146, 156, 234, 235, 236, 245, 246, 256, 345, 356, 456\}$. Basis axioms (B1) and (B2) are easily observed to be true, and an exhaustive comparison of each pair of bases would ultimately confirm (B3).

However, one must not confuse any collection of subsets declared as bases a matroid. Consider the following "matroid" Q: $\mathcal{B} = \{1235, 1236, 1245, 1246, 1345, 1346, 1356, 1357, 1367, 1456, 1457, 1467, 2345, 2346, 2356, 2357, 2367, 2456, 2457, 2467, 3567, 4567\}$. While this collection of subsets appears to correspond to the geometry shown below, where points $5, 6$, and 7 are not collinear, it is not a matroid.

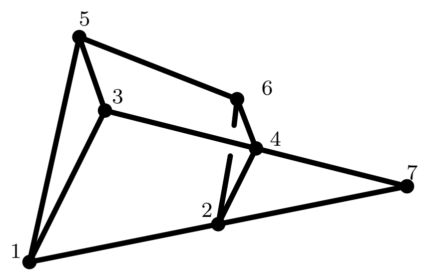

Figure 2.11: Geometry of above "matroid".

This collection of maximally independent sets fails to satisfy axiom (B3). Consider "basis" subsets $B_1 = \{1246\}$ and $B_2 = \{1567\}$. Recall from the definition of a matroid, (B3), if B_1, $B_2 \in \mathcal{B}$ and $x \in (B_1 - B_2)$, then there is an element $y \in (B_2 - B_1)$ so that $(B_1 - x \cup y) \in \mathcal{B}$ and $(B_2 - y \cup x) \in \mathcal{B}$. Let x be the single element 4 in B_1. There does not exist any element y in B_2 such that $(B_1 - x \cup y) \in \mathcal{B}$. If we replace element 4 in B_1 with element 5 from B_2, the set $\{1256\}$ forms a coplanar collection and therefore, a dependent set. Meanwhile, if we replace the element 4 in B_1 with 7 from B_2, the set $\{1267\}$ contains the collinear subset $\{127\}$ which is dependent. Neither of these resultant sets is a basis, and thus axiom (B3) is not satisfied. This geometry, which is not in fact a matroid, is referred to as the Escher Matroid.

2.3 Duality

Having established some of the ways in which matroids arise and can be viewed, we shift to one of the most useful of a matroid's aspects: duality. Notions of duality are prevelent in a number of fields of mathematics, but within the scope of matroid theory, duality is defined as follows:

Let M be a matroid on a collection of elements E, with bases
$\mathcal{B} = \{B_1, B_2, ..., B_n\}$. The *dual* of the matroid, denoted M^*, has the following collection

as its set of bases: $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$. This definition leads us to the following theorem.

Theorem 2.2. The dual of a matroid is also a matroid.

Proof. Let M be a matroid with bases $\mathcal{B} = \{B_1, B_2, ..., B_n\}$. Consider the set of all basis complements $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$.
(B1) M is matroid, therefore $\mathcal{B} \neq \emptyset$. As $B \subset E$ exists, for all $B \in \mathcal{B}$, the set $E - B$ exists, therefore $\mathcal{B}^* \neq \emptyset$.
(B2) We defined the rank r of a matroid to be the number of elements found within each of its bases. Thus for every basis $B_i \in \mathcal{B}$, we know $|B_i| = r$. It follows that the complement of every basis has size $|E| - |B_i|$. Each basis complement is of the same size and thus condition (B_2) is fulfilled.
(B3) Let $B_1^* = E - B_1$ and $B_2^* = E - B_2$ where $B_1, B_2 \in \mathcal{B}$ such that there exists element $x \in B_2$ but $x \notin B_1$. (Should no such element exist, B_1 and B_2 are the same basis.) Therefore, $x \in B_1^*$ but $x \notin B_2^*$. By the basis exchange axiom, there exists $y \in B_1 - B_2$ such that $(B_1 - y) \cup x$ and $(B_2 - x) \cup y$ are also bases of M. If we take the complement of these new bases, the results are $E - ((B_1 - y) \cup x)$ and $E - ((B_2 - x) \cup y)$, or rather $(B_1^* - x) \cup y$ and $(B_2^*) - y \cup x \in \mathcal{B}^*$. $\square$

Consider the following examples exhibiting the process of constructing the dual of a matroid.

Example 2.3. Consider a rank 2 matroid A on the set $E = \{1, 2, 3, 4\}$ with bases $\mathcal{B} = \{13, 14, 23, 24, 34\}$. The geometry of this matoid is seen below.

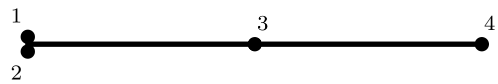

Figure 2.12: Geometry of matroid A.

We find the bases of the dual A^* of A by taking the complement of each basis of A. Therefore the bases of A^* are $\mathcal{B}^* = \{E-\{13\}, E-\{14\}, E-\{23\}, E-\{24\}, E-\{34\}\}$. This results in $\mathcal{B}^* = \{24, 23, 14, 13, 12\}$. Thus A^* has the following geometry:

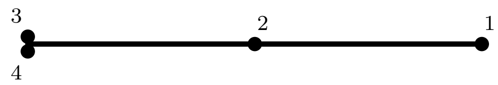

Figure 2.13: Geometry of matroid A^*.

A quick check of the basis axioms against this new set of bases confirms that A^* is a matroid as well. Moreover, the geometries of the original matroid and its dual are structurally the same, but their elements (namely the two-element dependent set) have been labelled differently. This result is not always the case, as we will see in the next example.

Example 2.4. Let D be a rank 3 matroid on five elements $E = \{1, 2, 3, 4, 5\}$ with the following set of bases: $\mathcal{B} = \{124, 125, 134, 135, 234, 235\}$. This matroid has the following geometry.

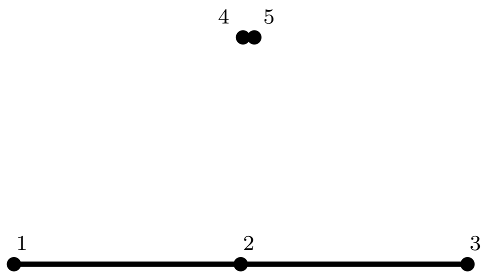

Figure 2.14: Geometry of matroid D.

Again, a check of the basis axioms confirms that this is a matroid. We calculate the bases of the dual D^* of D by once again taking the complement of each basis of D. The result is the collection of subsets $\mathcal{B}^* = \{35, 34, 25, 24, 15, 14\}$. This collection passes the basis axioms as well, and therefore D^* is also a matroid.

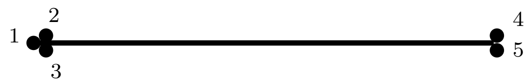

Figure 2.15: Geometry of matroid D^*.

Note that the resultant matroid D^* is no longer of rank 3, but now of rank 2. This difference in rank can be explained by the following lemma.

Lemma 2.5. Given a matroid M of rank r on n elements, the rank of the dual matroid M^* is $n - r$.

The proof of this lemma comes about as a result of the proof of Theorem 2.4. For clarity, we prove it here separately.

Proof. Let M be a matroid of rank r on n elements, whose dual is also a matroid M^*. By definition, the rank of a matroid is equal to the size of its bases. That is, $|B| = r : B \in \mathcal{B}$. The bases of the dual of a matroid, M^*, are defined as the complements of the bases of M. That is $B^* = \{E - B : B \in \mathcal{B}\}$. Every element in any basis B must also exist in E. Therefore $|E| - |B| = n - r$, and so $|B^*| = n - r$ $\square$

We have proven that the dual of a matroid is itself a matroid, and that the rank of this new matroid is equal to difference of the cardinality of the ground set E and rank of the original matroid.

Matroids M_1 and M_2 are *isomorphic* if there exists a bijective function $\phi : M_1 \to M_2$ such that $\{x_1, x_2, ...x_r\}$ is a basis in $\mathcal{B}(M_1)$ if and only if $\{\phi(x_1), \phi(x_2), ..., \phi(x_r)\}$ is a basis in $\mathcal{B}(M_2)$. We have seen in Example 2.5, an instance in which the dual of a matroid is isomorphic to the original matroid. In later sections we will refer to matroids who are isomorphic to their dual as being *self-dual*. Matroids that are isomorphic and whose elements are labelled the same within the geometry are considered *identically self-dual*.

In the final lines of this section we introduce another minor, though important, quality of matroid duality.

Lemma 2.6. The dual of the dual of a matroid is the original matroid. That is, if $M^* = N$, then $N^* = M$.

Proof. Let M be a matroid and the dual of M be the matroid N. The bases of the dual matroid N are found by taking the complement of those of M. Thus the dual of N, N^*, would require taking the complement of its bases, or rather the complement of the complement of those of M, which results in the bases of M. □

2.4 Some Matroid Constructions

In this final section of our introductory chapter on matroids, we explore a few matroid constructions and the results of these constructions. Often, one may wish to construct new matroids from existing ones by methods other than duality. While there are a number of ways to do so, herein we examine three means by which a matroid may be constructed from another: contraction, truncation, and Higgs lift.

Given a matroid M on ground set E, and let e be an element of E that is not a loop. The matroid constructed by the *contraction* of $e \in E$,written (M/e), has as its bases $\mathcal{B}(M/e)$, the set of all rank $r-1$ subsets $B - e$, for each $B \in \mathcal{B}(M)$. In the contraction of e, the element e is removed from the ground set.

Example 2.7. Let M be a rank 2 matroid on $E = \{1, 2, 3, 4\}$ with bases $\mathcal{B} = \{13, 14, 23, 24\}$.

Figure 2.16: Geometry of matroid M.

We contract M about element 2. As a result, the bases of $M/2$ are the bases of M that contained 2, that is $\{23\}$ and $\{24\}$ with 2 now deleted, $\mathcal{B}_{M/2} = \{34\}$.

Figure 2.17: Geometry of matroid M/2.

Contraction will become an important procedure during the final chapters of this work, as it is essential to proving our suggested notion of flag matroid duality.

Given a matroid M of rank r, the *truncation* of M to rank k, where $k < r$, is a matroid with all independent sets of size k in M as its bases.

Example 2.8. Consider the matroid N from Section 2.3, of rank 3, on the ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$, with bases $\mathcal{B} = \{125, 126, 135, 136, 245, 246, 345, 346\}$, and with the following geometry.

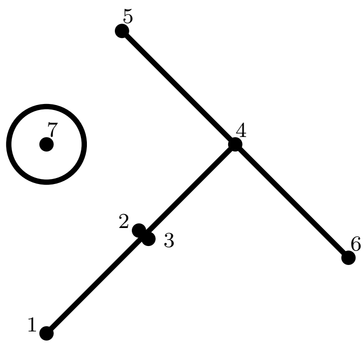

Figure 2.18: Geometry of matriod N.

Should we hope to truncate this matroid to rank 2, we must identify the independent sets of cardinality 2 in matroid N. These sets form the bases of a rank 2 matroid, N_2: $\mathcal{B}_2 = \{12, 13, 14, 15, 16, 24, 25, 26, 34, 35, 36, 45, 46, 56\}$. The element 7 does not appear in any independent set in N and therfore remains a dependent singleton in this truncation. Similarly, elements 2 and 3 form a two element dependent set in N and are therefore not a basis in N_2. The collection of these rank 2 bases satisfy the basis axioms, and thus N_2 is a matroid with geometry:

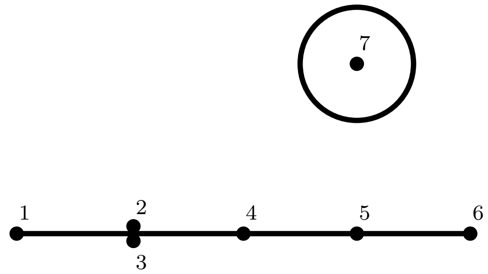

Figure 2.19: Geometry of matriod N_2.

By the same truncation process, we may construct a matroid of rank 1, N_1. We identify the single element independent sets in N, and declare them the bases of N_1. $\mathcal{B}_1 = \{1, 2, 3, 4, 5, 6\}$ which form a matroid, albeit an uninteresting one.

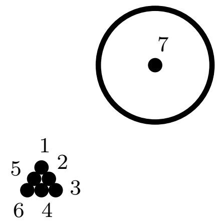

Figure 2.20: Geometry of matriod N_1.

Before we present a third method of matroid construction, we introduce terminology that describes the relationship between matroids and their truncations, which will be essential in defining the Higgs lift.

Given a matroid M of rank r and a matroid N of rank s where $s < r$, N is *quotient* to M if every basis of N is contained within a basis of M and every basis of M contains a basis of N. When one matroid is quotient to another we say that the two are *concordant*.

Due to the manner in which they were constructed, it is evident that matroids produced by truncation are quotient to their original matroid. By definition, truncation constructs matroids of smaller rank than that of the original matroid. Our third construction, the Higgs lift constructs new matroids given concordant matroids.

Given matroids L of rank l and M of rank m, on n elements, where L is quotient to M, the *Higgs lift* of L towards M, written $HL_M(L)$ is the matroid of rank $l+1$ whose bases are the $l+1$ sized subsets which are independent in M and have rank l in L.

Thus, while truncation resulted in matroids of smaller rank, Higgs lifts construct matroids of larger rank, provided we are given a matroid of even larger rank to lift towards that is concordant with the matroid we are lifting from. Again, in the interest of clarity, we examine another example.

Example 2.9. Consider matroid M of rank 4 on ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$ with bases $\mathcal{B}_M = \{1346, 1347, 1356, 1357, 1367, 1456, 1457, 1467, 2346, 2347, 2356, 2357, 2367, 2456, 2457, 2467, 3467, 3567, 4567\}$.

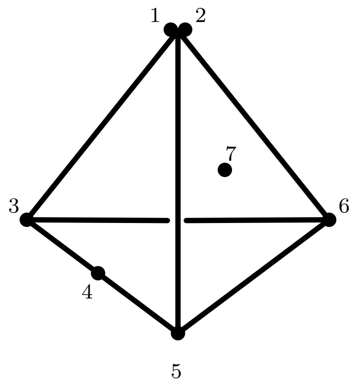

Figure 2.21: Geometry of matriod M.

Consider as well, matroid L, quotient to M, of rank 2 on the same elements, with bases $\mathcal{B}_L = \{13, 14, 15, 16, 17, 23, 24, 25, 26, 27, 34, 35, 36, 37, 45, 46, 47, 56, 57, 67\}$.

Figure 2.22: Geometry of matriod L.

As the rank of matroid L is 2, we can construct the Higgs lift matroid $HL_M(L)$ by identifying the independent sets of size 3 in M that are also of rank 2 in L and then by using these sets as the bases for our new matroid. Doing so results in the set of bases $\mathcal{B}_{HL_m(L)} = \{134, 135, 136, 137, 145, 146, 147, 156, 157, 167, 234, 235, 236, 237, 245, 246, 247, 256, 257, 267, 346, 347, 356, 357, 367, 456, 457, 467, 567\}$.

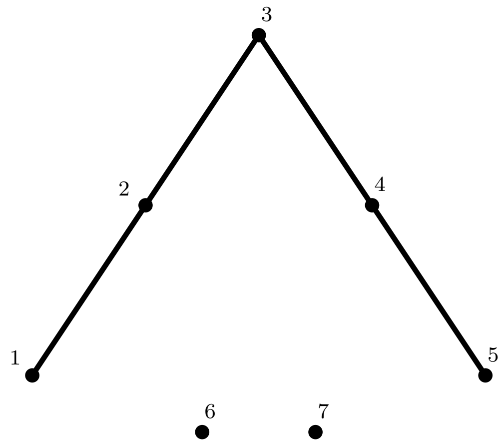

Figure 2.23: Geometry of matriod $HL_M(L)$.

We have now established three means by which matroids may be constructed from other matroids. Also in this section, we have established the ideas of quotient and concordancy. These two terms will be recurring themes throughout the later chapters of this work, and hold a central role in defining flag matroids and their duals.

Chapter 3

Gale Orderings

It is at this point, having established the fundamental aspects of a matroid, we can begin to examine relationships between the elements of a matroid and those of the symmetric group. We begin by establishing the concept of a Gale Ordering.

Consider the symmetric group on n elements, Sym_n, where n is finite. We are able to induce an ordering (denoted $\leq^w$) on these elements, effectively establishing which elements are greater than which. For example:

$$1 <^w 2 <^w 3 <^w ... <^w n-1 <^w n$$

We refer to this particular example as the identity ordering. However, we can consider any rearrangement of the n elements. For example:

$$4 <^w 9 <^w 2 <^w 1 <^w n <^w ...$$

When establishing a Gale ordering on a collection of n elements, we will use the symbol $<^w$ as no two elements can be weighted the same. However, when comparing the weights of subsets against one another, we will use the symbol $\leq^w$ as some individual elements within a subset may be compared against themselves in another subset, necessitating a need to represent potential equality.

As this is a concept we will make frequent use of, there is an established shorthand for Gale orderings written in two rows. The upper row depicts the identity ordering, while the lower rearranges the n elements, assigning each the weight of the element directly above it in the identity. For example, let $n = 4$ and consider the ordering $<^w$ shown:

$$\begin{pmatrix} 1234 \\ 4213 \end{pmatrix}$$

This corresponds to the ordering,

$$4 <^w 2 <^w 1 <^w 3$$

(Note: Despite the visual similarity to a matrix, and by extension the matrix representation of a matroid, the array representation of a Gale ordering is not a matrix.)

This concept can be be extrapolated to apply to subsets of $[n]$, where $[n] = \{1, 2, 3, 4, ..., n\}$, and ultimately to the bases of a matroid. Let $P_{n,k}$ be the collection of all k-element subsets of $[n]$. By inducing the ordering $\leq^w$ on $[n]$, subsets within $P_{n,k}$ are subject to the same ordering. If the ordered elements in one set A are greater than the corresponding ordered elements of another set B, the A is said to be greater than B.

Example 3.1. Let $A = \{134\}, B = \{123\} \in P_{4,3}$ and induce the ordering described above:

$$\begin{pmatrix} 1234 \\ 4213 \end{pmatrix}$$

Under this ordering we rewrite A and B as $A = (314)$ and $B = (312)$. The subsets have not changed, but we have rewritten them in descending order of weight under the given ordering $\leq^w$. As,

$$2 \leq^w 4,\ 1 \leq^w 1, \text{ and } 3 \leq^w 3$$

We can compare the full subsets, claiming $A \leq^w B$.

Example 3.2. To further illustrate the application of a Gale ordering, let us apply one within the contexts of a matroid. Recall matroid P from Section 2.2. Let P be a matroid of rank 3 on $E = \{1, 2, 3, 4, 5, 6\}$ with bases $\mathcal{B} = \{124, 125, 126, 134, 135, 136, 146, 156, 234, 235, 236, 245, 246, 256, 345, 356, 456\}$.

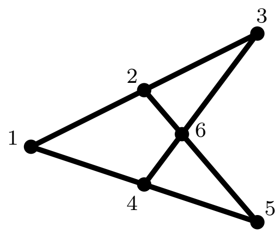

Figure 3.1: Geometry of matriod P.

Consider the following two Gale orderings $\leq^{w}{}_1$ and $\leq^{w}{}_2$

$$\leq^{w}{}_1 = \begin{pmatrix} 123456 \\ 526341 \end{pmatrix} \qquad\qquad \leq^{w}{}_2 = \begin{pmatrix} 123456 \\ 134265 \end{pmatrix}$$

If we induce the first Gale ordering $\leq^{w}{}_1$ on the matroid, we may rewrite the bases as follows: $\mathcal{B} = \{142, 125, 162, 143, 135, 136, 146, 165, 432, 325, 362, 425, 462, 625, 435, 365, 465\}$. Notice that the basis $\{143\}$ is greater than any other basis under this Gale ordering. In subsequent sections, we will refer to this as the *maximal basis* under $\leq^{w}{}_1$.

If we instead induce the second Gale ordering $\leq^{w}{}_2$ on the matroid, we may rewrite the bases as $\mathcal{B} = \{241, 521, 621, 431, 531, 631, 641, 561, 243, 523, 623, 524, 624, 562, 542, 563, 564\}$. Once again, a maximal basis arises, however, under this ordering this basis is $\{562\}$.

Having established the concept of Gale orderings, we turn our attention to their more general implications in terms of matroids.

Chapter 4

Maximality and Increasing Exchange Properties

In this chapter we introduce two new means by which a matroid may be defined using Gale orderings.

4.1 The Maximality Property

A collection of subsets $\mathcal{B}$ of n is said to satisfy the *Maximality Property* if for every $w \in Sym_n$, the collection $\mathcal{B}$ contains a unique member A that is maximal in $\mathcal{B}$ with respect to $\leq^w$; that is, $B \leq^w A$ for all $B \in \mathcal{B}$. We call A the w-maximal or Gale-maximal basis in $\mathcal{B}$. [Bor03]

Theorem 4.1. Let $\mathcal{B} \subseteq P_{n,k}$. Then $\mathcal{B}$ is a matroid if and only if $\mathcal{B}$ satisfies the Maximality Property.

In order to prove this theorem we must first show that a collection $\mathcal{B} \subseteq P_{n,k}$ that satisfies the Maximality Property also satisfies the basis axioms that define a matroid. We must then show that the bases of a matroid satisfy the Maximality Property.

Proof. Let a collection of subsets $\mathcal{B}$ of $P_{n,k}$ satisfy the Maximality Property. For (B1), by the Maximality Property, for every Gale ordering $\leq^w$ on n elements there exists a unique member $A \in \mathcal{B}$ that is maximal in $\mathcal{B}$ with respect to $\leq^w$. If such an element exists in $\mathcal{B}$ for every Gale ordering on the n elements, then $\mathcal{B}$ cannot be empty. Thus the collection of bases of the potential matroid M with bases $\mathcal{B}$ is a nonempty set, and (B1) is satisfied.

For (B2), by definition, $\mathcal{B} \subseteq P_{n,k}$ where $P_{n,k}$ is the collection of all subsets of cardinality k of n elements. Every $B \in \mathcal{B}$ has cardinality k, and thus $|B_1| = |B_2|$ for any $B_1, B_2 \in \mathcal{B}$. Therefore, (B2) is satisfied.

For (B3), let A and B be members of $\mathcal{B} \subseteq P_{n,k}$. Let $A - B = \{x_1, x_2, ..., x_k\}$ be the collection of all elements in A but not in B. Likewise, let $B - A = \{y_1, y_2, ..., y_l\}$ be all elements in B but not in A. Consider the Gale ordering $\leq^w$ shown below:

$$\text{(elements not in A or B)} <^w x_1 <^w y_1 <^w y_2 <^w ... <^w y_l <^w x_2 <^w ... <^w x_k.$$

By the Maximality Property, there must exist a maximal subset for any ordering induced on the n elements. A quick observation of our ordering above shows that neither A nor B is the maximal subset for this ordering, as x_1 is less than all elements in $B - A$, while all other elements of $A - B$ are greater than those of $B - A$. Let $C \in \mathcal{B}$ be the maximal subset with respect to this ordering. We know that, $A \leq^w C$, but A also contains the $k-1$ greatest elements under this Gale ordering. Therefore, C must also contain these elements. In order to exist in $P_{n,k}$, set C must contain a "k^{th}" element. In order to ensure this element is greater than x_1, it must come from $\{y_1, y_2, ..., y_l\} = B - A$, lets say y_c. Thus the maximal element under this Gale ordering $C = A - \{x_1\} \cup \{y_c\}$. This is, in effect, an application of the Basis Exchange Property, and can be performed using any two subsets $A, B \in \mathcal{B} \subseteq P_{n,k}$. Thus (B3) is satisfied.

Now let M be a matroid on n elements and therefore satisfy the Basis Exchange Property. Assume that a Gale ordering $\leq^w$ on these n elements does not result in a unique maximal basis. Instead, let bases A and B be unique bases in $\mathcal{B}$ such that they are simultaneously maximal on the ordering. Let $x \in A$ be the minimal element among all elements in A but not in B, with respect to $\leq^w$.

From the Basis Exchange Property, there exists an element $y \in B$ such that $A - \{x\} \cup \{y\}$ and $B - \{y\} \cup \{x\}$ are also bases of M. One of the following must be true: $x <^w y$ or $y <^w x$. If $x <^w y$, then $A <^w A - \{x\} \cup \{y\}$ which contradicts the initial claim that A is a maximal basis under $\leq^w$. If $y <^w x$, then $B <^w B - \{y\} \cup \{x\}$ which contradicts the initial claim that B is a maximal basis under $\leq^w$. Therefore, a matroid cannot have more than one maximal basis on any given Gale ordering. $\square$

To summarize, the Maximality Property states that if a collection of bases $\mathcal{B}$ is a matroid on n elements, then for any ordering $\leq^w$ on those n elements there exists a single subset $A \in \mathcal{B}$ such that A is weighted greater/heavier/etc. than any other basis of the matroid.

4.2 The Increasing Exchange Property

Let $\mathcal{B} \subseteq P_{n,k}$, let A_1, A_2 be distinct members of $\mathcal{B}$, and let $\leq^w$ be an arbitrary permutation. Then $\mathcal{B}$ is said to satisfy the *Increasing Exchange Property* if there is a transposition $t = (a, b)$, with $a <^w b$, such that for either A_1 or A_2, say A_i, containing a, but not containing b, $tA_i = A_i - a \cup b$ also belongs to $\mathcal{B}$. [Bor03]

More simply, through a transposition, a non-maximal basis may increase in weight on a given Gale ordering. This implies that, given a matroid with bases $\mathcal{B}$ and a Gale ordering, if a basis $B \in \mathcal{B}$ is not already maximal, there exists a series of transpositions by which it may become the Gale-maximal basis.

This property is an application of axiom $(B3)$ under a Gale ordering. With this in mind, we aim to show that if a collection of subsets satisfies the Increasing Exchange Property, then those subsets form the bases of a matroid.

Theorem 4.2. Let $\mathcal{B} \subseteq P_{n,k}$. $\mathcal{B}$ satisfies the Increasing Exchange Property if and only if $\mathcal{B}$ satisfies the Maximality Property.

Proof. Let $\mathcal{B} \subset P_{n,k}$ be a subset that satisfies the Maximality Property. As proven in the last section, the subsets of $\mathcal{B}$ are therefore the bases of some matroid M on n elements and therefore satisfy the Basis Exchange Property. Let $B_1, B_2 \in \mathcal{B}$, where B_1 and B_2 are distinct bases of M. By $(B3)$ there must exists elements $x \in B_1 - B_2$ and $y \in B_2 - B_1$ such that $B_1 - x \cup \{y\}$ and $B_2 - y \cup \{x\} \in \mathcal{B}$. Induce an arbitrary Gale ordering $\leq^w$ on the n elements. By definition of such an ordering, either $y <^w x$ or $x <^w y$. Let t be the transposition $t = (x, y)$. By $(B3)$ we know that tB_1, $tB_2 \in \mathcal{B}$. If $y <^w x$, then under this transposition B_1 increases in weight, replacing one of its elements, x, with a heavier one, y, thus $B_1 <^w tB_1$. Similarly, if $x <^w y$, $B_2 <^w tB_2$

Now let $\mathcal{B} \subset P_{n,k}$ satsify the Increasing Exchange Property. Assume $\mathcal{B}$ does not satisfy the Maximality Property. Therefore there exists some Gale ordering $\leq^w$ such that two distinct subsets in $\mathcal{B}$, say B_1, B_2, are both maximal under the given ordering. Let $x \in B_1 - B_2$ and $y \in B_2 - B_1$, and let transposition $t = (x, y)$. By the Increasing Exchange Property, either $B_1 <^w B_1 - x \cup y$ or $B_2 <^w B_2 - y \cup x$. That is, one of the subsets must increase in weight under the transposition. Thus, either B_1 or B_2 is not maximal under the Gale ordering, and we have a contradiction. $\square$

The Increasing Exchange Property on a collection of subsets $\mathcal{B} \subset P_{n,k}$ implies the Maximality Property on those subsets. As we have already proven that satisfaction of the Maximality Property makes a collection of such subsets a matroid, if $\mathcal{B}$ satisfies the Increasing Exchange Property on those elements, then $\mathcal{B}$ forms the bases of a matroid.

Chapter 5

Flags and Flag Matroids

We define a *flag* F to be a collection of finite sets

$$B^1 \subset B^2 \subset B^3 \subset ... \subset B^k$$

where the cardinality of each set is denoted $1, 2, 3, ..., k$ respectively (i.e. $\mid B^k \mid = k$).

For example, given the sets $\{1\}$, $\{14\}$, $\{124\}$, and $\{1234\}$, the collection $\{1 \subset 14 \subset 124 \subset 1234\}$ is a flag. In the interest of brevity, we write this flag $\{1423\}$ placing elements in the order they were included as we increased in set size. Note, that whereas the order of the elements in a basis did not matter (i.e. if $B_1 = \{123\}$, $B_2 = \{132\}$ then $B_1 = B_2$,) the same cannot be said for flags. $F_1 = \{123\} = \{1 \subset 12 \subset 123\}$ is not the same as $F_2 = \{132\} = \{1 \subset 13 \subset 123\}$.

We now define what constitutes a flag matroid.

A collection of $\mathcal{F}_n^{k_1,k_2,...,k_m}$ of flags of rank $k_1, k_2, ..., k_m$ is a *flag matroid* on n elements if the the following are true:

(F1) Each constituent M_i of $\mathcal{F}$ is a matroid.

(F2) The constituents $M_1, ..., M_m$ are concordant

(F3) Every flag $B_1 \subset ... \subset B_m$ such that B_i is a basis of M_i for $i = 1, ..., m$ belongs to $\mathcal{F}_n^{1,2,..,m}$.

Example 5.1. Consider the example from Section 2.1. Matroid M_3 has the following geometry:

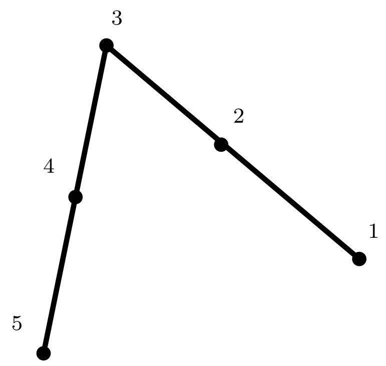

Figure 5.1: The Geometry of constituent matroid M_3.

This matroid has bases $\mathcal{B} = \{124, 125, 134, 135, 145, 234, 235, 245\}$. Consider as well matroids M_2 and M_1 concordant to M_3 with bases $\mathcal{B}_2 = \{12, 14, 23, 25, 35\}$ and $\mathcal{B}_1 = \{1, 2, 5\}$, respectively.

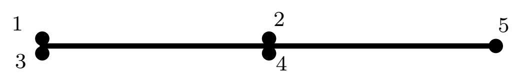

Figure 5.2: The Geometry of constituent matroid M_2.

Figure 5.3: Geometry of constituent matroid M_1.

As a result, the bases of these three matroids form a flag matroid, with flags $\mathcal{F}_5^{1,2,3} = \{124, 125, 142, 143, 145, 214, 215, 234, 235, 251, 253, 521, 523, 531, 532\}$.

At this point, it is important to acknowledge that a flag matroid is not a matroid, and is not defined by the basis axioms. Its name derives from a flag matroid's reliance on matroids for its construction, but we must always remember that it is not a matroid itself.

Recall that the Maximality Property (and subsequently the Increasing Exchange Property) from the previous sections did not apply exclusively to matroids, but rather to subsets of n elements. As such, we can apply their principles to the various flags in $\mathcal{F}_n^{k_1,k_2,\ldots,k_m}$ found within a flag matroid. With this in mind, we introduce the next theorem regarding flag matroids.

Theorem 5.2. A collection of flags $\mathcal{F}_n^{k_1,k_2,\ldots k_m}$ is a flag matroid if and only if it satisfies the Maximality Property.

Proof. Let $\mathcal{F}_n^{k_1,k_2,\ldots,k_m}$ be a flag matroid. Assume this collection of flags does not satisfy the Maximality Property. Instead, assume that for a Gale ordering $\leq^w$ there exist two distinct maximal flags F_1 and F_2. Let M_r denote the constituent matroid of lowest rank at which $B_1 \in F_1 \neq B_2 \in F_2$ are bases of M_r. If F_1 and F_2 are both maximal flags, then B_1 and B_2 must both be maximal bases in M_r under $\leq^w$. However, as was proven in Chapter 4 of this work, a collection $\mathcal{B}$ forms a matroid if and only if $\mathcal{B}$ satisfies the Maximality Property. As there is no unique maximal basis of M_r, "matroid" M_r is not a matroid which then violates condition (F1) of a flag matroid. We have a contradiction and $\mathcal{F}_n^{k_1,k_2,...,k_m}$ is therefore not a flag matroid.

Now assume a collection of flags $\mathcal{F}_n^{k_1,k_2,...,k_m} \subset P_{n,k}$ satisfy the Maximality Property. First, rewrite each flag in its elongated form, as a collection of subsets (i.e. $\{132\} = \{1 \subset 13 \subset 123\}$.) Create subsets of $P_{n,k}$ for $k = 1$, $k = 2$, ..., $k = m$ such that $\mathcal{B}_1$ contains all cardinality 1 subsets found in any flag, B_2 contains all cardinality 2 subsets found in any flag, and so on. As the Maximality Property held on the collection of flags, it continues to hold on these new subsets. Thus as each subset $\mathcal{B}_i$ satisfies the Maximality Property, each such collection also defines a matroid. Thus $\mathcal{F}$ is made up entirely of constituent matroids, satisfying (F1). As the elements of $\mathcal{B}_i$ came directly from flags, we know that for any $B_i \in \mathcal{B}_i$, there exist $B_j \in \mathcal{B}_j$ and $B_k \in \mathcal{B}_k$ for any ranks $j < i < k$ such that $B_j \subset B_i \subset B_k$. Therefore the constituent matroids are concordant, fulfilling condition (F2). Finally, as the basis elements were taken directly from a collection of flags satisfying the Maximality Property, no flag $F = \{B_1 \subset ... \subset X \subset ... \subset B_m\}$, where X is not a basis element of the corresponding constituent matroid of rank $|X|$, may exist within the collection of flags. Condition (F3) is satisfied, and $\mathcal{F}$ is a flag matroid. □

Consider again the flag matroid with constituent matroids M_3, M_2, and M_1: $\mathcal{F}_5^{1,2,3} = \{124, 125, 142, 143, 145, 214, 215, 234, 235, 251, 253, 521, 523, 531, 532\}$. By definition, this collection of flags must satisfy the Maximality Property for any Gale ordering. To illustrate this, let us induce the following Gale ordering on the elements $E = \{1, 2, 3, 4, 5\}$:

$$\leq^w : \begin{pmatrix} 12345 \\ 32415 \end{pmatrix}$$

Here, $\{4\}$, $\{1\}$, and $\{5\}$ are the three "heaviest" elements under this ordering and $\{415\} \in \mathcal{F}_5^{1,2,3}$. Therefore, $\{415\}$ is the maximal flag under this ordering. Consider another such ordering.

$$\leq^w : \begin{pmatrix} 12345 \\ 45312 \end{pmatrix}$$

Here, $\{3\}$, $\{1\}$, $\{2\}$ are the heaviest elements under the ordering. However, $\{123\}$ is not in our collection of flags. Therefore, we look for the next heaviest, $\{215\}$, which is in our collection. this process can be repearted for all 120 possible Gale orderings on 5 elements.

Chapter 6

Duals of Flag Matroids

In this chapter, we suggest a definition for the dual of a flag matroid, how these duals may be found, and prove that the dual of a flag matroid is itself a flag matroid.

6.1 Definition

Let $\mathcal{F}_n^{k_1,k_2,\ldots,k_m}$ denote a flag matroid with constituent matroids $M_1, M_2, ..., M_m$ where the superscript denotes the ranks of each constituent matroid. By definition of a flag matroid, each constituent matroid M_i is quotient to M_j for any $i < j \leq k$. We define the dual of $\mathcal{F}_n^{k_1,k_2,\ldots,k_m}$ as follows:

We begin by first taking the dual of each constituent matroid whose bases are used to form the flags of the original flag matroid $\mathcal{F}$. Whereas constituent matroids of the initial flag matroid were concordant, with M_1 quotient to M_2, quotient to, ..., quotient to M_m, the collection of the constituent duals of the matroids also are concordant, with M_m^* quotient to, ..., quotient to M_2^*, quotient to M_1^*. Thus there exist collections of bases of increasing rank in the dual of the flag matroid such that $B_n^* \subset ... \subset B_2^* \subset B_1^*$. These collections of nested bases form the flags for the dual of the initial flag matroid.

We present a simple example of this method of flag matroid dual construction to clarify the process. More complex flag matroid duals will be taken in the next chapter.

Consider matroids M_4, M_3, M_2 and M_1 on 5 elements $E = \{1, 2, 3, 4, 5\}$ of rank 4, 3, 2, and 1 respectively, with basis sets $\mathcal{B}_4 = \{1234\}$, $\mathcal{B}_3 = \{124, 134\}$, $\mathcal{B}_2 = \{12, 13\}$, and $\mathcal{B}_1 = \{1\}$.

A check of the basis axioms on each of these these sets confirms that these are matroids. Having done so, we now check if these matroids are concordant. Again, they are, as every basis in each matroid contains basis subsets of those of lower rank, and are each contained in basis subsets of higher rank. This relationship can be visualized in the following diagram:

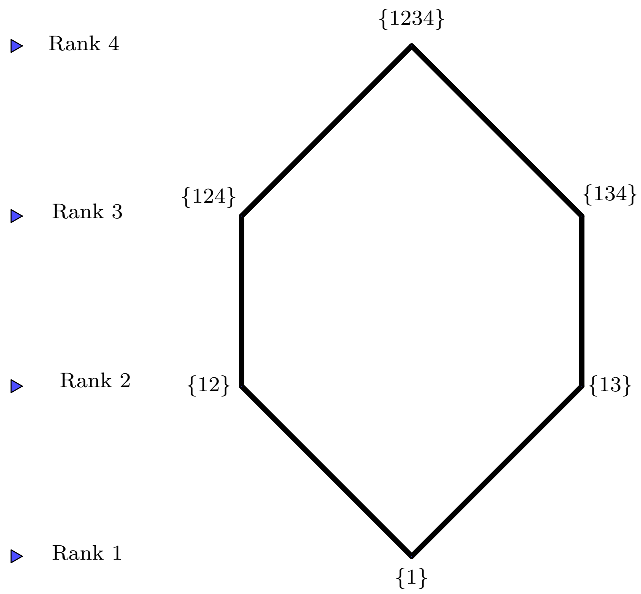

Figure 6.1: Diagram of bases as subsets of one another.

Therefore, M_1 is quotient to M_2 is quotient to M_3 is quotient to M_4. Having confirmed that the constituents are concordant matroids, we can then determine the flags of this collection of matroids by identifying which bases are contained within one another. This collection results in two flags, $\mathcal{F}_5^{1,2,3,4} = \{\{1 \subset 12 \subset 124 \subset 1234\}, \{1 \subset 13 \subset 134 \subset 1234\}\}$, or more simply: $\mathcal{F}_5^{1,2,3,4} = \{1243, 1342\}$. A check of the Maximality Property on flags confirms that these two flags form a flag matroid. Having established that our initial collection of matroids and bases form a flag matroid, we can now proceed to construct the dual.

We begin by taking the dual of each constituent matroid. The results are matroids M_4^*, M_3^*, M_2^*, and M_1^* of ranks 1, 2, 3, and 4 respectively, with bases $\mathcal{B}_4^* = \{5\}$,

$\mathcal{B}_4^* = \{25, 35\}$, $\mathcal{B}_2^* = \{245, 345\}$, $\mathcal{B}_1^* = \{2345\}$. By definition, each of these matroid duals is itself a matroid, and a check of the bases confirms that they are also concordant, with M_4^* quotient to M_3^* quotient to M_2^* quotient to M_1^*. This leads to a dual collection of flags, $\mathcal{F}_5^{1,2,3,4*} = \{\{5 \subset 25 \subset 245 \subset 2345\}, \{5 \subset 35 \subset 345 \subset 2345\}\}$, or more simply: $\mathcal{F}_5^{1,2,3,4*} = \{5243, 5342\}$. Maximality holds on these two flags, thus we know the dual of the original flag matroid is also a flag matroid. In the following section, we will prove that this is always the case, and that the dual of any flag matroid is itself a flag matroid.

6.2 A Theorem of Flag Matroid Duality

Before suggesting our theorem for matroid duality, we present a lemma that will be crucial to its proof.

Lemma 6.1. Let M_1 and M_2 be matroids. Then M_1 is a quotient to M_2 if and only if M_2^* is a quotient to M_1^*. [Oxl11]

Proof. Let M_1 be quotient to M_2 on ground set $E = \{1, 2, 3, ...n\}$. Then there exists a basis B_1 of M_1 and B_2 of M_2 such that $B_1 \subset B_2$. Because M_1 is quotient to M_2, the rank of M_2 is greater than the rank of M_1, and there exists some element(s) $x \in B_2 - B_1$. Now consider the bases of M_2^* and M_1^*. $B_2^* = E - B_2$ and $B_1^* = E - B_1$. But $B_1 \cup \{x : x \in B_2 - B_1\} = B_2$, therefore, $B_2^* = E - \{B_1 \cup x : x \in B_2 - B_1\}$ which can be rewritten as $B_2^* = B_1^* - \{x : x \in B_2 - B_1\}$. In short, $B_2^* \subset B_1^*$. This is true of any such bases in the original matroids and thus the duals of these matroids are also concordant. $\square$

With all the necessary foundations having been laid, we are now prepared to present the consequence of our notion of flag duality.

Theorem 6.2. The dual of a flag matroid is a flag matroid.

Proof. A collection F of flags of rank $(k_1, k_2, ...k_m)$ is a flag matroid if the following three conditions are true:

(F1) Each constituent matroid M_i of F is a matroid

(F2) The constituents $M_1, ..., M_m$ are concordant.

(F3) Every flag $B_1 \subset ... \subset B_m$ such that B_i is a basis of M_i for $i = 1, ..., m$ belongs to F.

In proving (F1), recall that each constituent of the initial flag matroid is a matroid, and that the dual of a matroid is a matroid. As the constituents of the dual of the flag matroid are duals of the original constituent matroid, every constituent of the dual flag matroid is a matroid.

In order for the (F2) quotient condition to hold, M_2^* must be quotient to M_1^* given M_1 is quotient to M_2. From Lemma A, we know this to be true, and thus (F2) is satisfied, We can extend this argument to any two matroids M_i quotient to M_j in the flag matroid F resulting in M_j^* quotient to M_i^* in the dual of the flag F^*.

Condition (F3) is fulfilled directly by our definition of the dual of a flag matroid. We have declared that every collection $B_n^* \subset ... \subset B_2^* \subset B_1^*$ forms a flag in the dual of the flag matroid. Thus (F3) is fulfilled $\square$

Therefore, the dual of a flag matroid as defined above is itself a flag matroid. While flag matroids are not matroids themselves, the method by which this dual of flag matroids may be taken parallels the notion of duality on matroids. Thus future work will involve determining which, if any, properties of matroid duality will apply to flag matroid duality.

Chapter 7

Flag Matroid Duals on $\mathbf{n} \leq 4$

In this chapter we consider all possible flag matroids on $1, 2, 3$, and 4 elements. Then, using our suggestion of flag matroid duality, we determine the dual of each flag matroid. Once we have done so, we determine which flag matroids are self-dual as defined below.

Flag matroids $\mathcal{F}_1^{k_1,k_2,\ldots,k_n}$ and $\mathcal{F}_2^{k_1,k_2,\ldots,k_n}$ are *isomorphic* if, for $i = 1, 2, \ldots, n$, each rank k_i constituent matroid in $\mathcal{F}_1$ is isomorphic to the rank k_i constituent matroid in $\mathcal{F}_2$.

A flag matroid is *self-dual* if the dual of the flag matroid, under the process we have defined, is isomorphic to the initial flag matroid. That is, the constituent matroids of the flag matroid are isomorphic to the constituent matroids of the dual. A flag matroid is *identically self-dual* if it is self-dual and if the labelling on the elements within the geometries of the constituent matroids are identical to those found in the initial flag matroid.

We construct the flag matroids by first considering all possible constituent matroids on n elements. Thus all the bases being used to build the flags of the flag matroid are coming from structures known to be matroids, fulfilling (F1). We then determine which collections of these constituent matroids are concordant, satisfying (F2). We then determine the flags of the flag matroid by calling every collection of bases of concordant constituent matroids contained within one another the flags of the flag matroid, as per (F3).

7.1 Flag Matroids on 1 Element

Let $E = \{1\}$. The case of $n = 1$ is simple, as there only exist two possible constituent matroids. The first is of rank 0:

Matriod M: $\mathcal{B} = \{\emptyset\}$

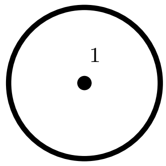

Figure 7.1: Rank 0 matroid on 1 element.

The second is of rank 1:

Matriod N: $\mathcal{B} = \{1\}$

Figure 7.2: Rank 1 matroid on 1 element.

Thus, the only viable flag matroid, is $\mathcal{F}_1^1 = \{\emptyset \subset 1\}$, or rather $\{1\}$. If we take the dual of this flag matroid as describe above, we see that the dual matroid M^* consists of basis $\{1\}$, while the dual matroid N^* has basis $\{\emptyset\}$, and together the two form a flag matroid of a single flag, $\mathcal{F}_1^{1^*} = \{\emptyset \subset 1\}$, or rather $\{1\}$. Thus, this single flag matroid on $n = 1$ element is identically self-dual.

Table 7.1: Duality of the flag matroid on one element.

Flag Matroid	Flags	Flag Duals	Dual Flag Matroid	Duality
$\mathcal{F}$	$\{1\}$	$\{1\}$	$\mathcal{F}$	Identically Self-Dual

7.2 Flag Matroids on 2 Elements

Let $E = \{1, 2\}$. The case on two elements is slightly more intricate, and suggests an important concept in constructing flag matroids: the labelling of the elements on the matroid structure matters. When dealing with two elements, there exists one possible matroid of rank 0:

Matriod M: $\mathcal{B} = \{\emptyset\}$

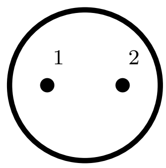

Figure 7.3: Rank 0 matroid on 2 elements.

There are three possible matroids of rank 1

Matriod N_1: $\mathcal{B} = \{1\}$, Matriod N_2: $\mathcal{B} = \{2\}$, and Matriod P: $\mathcal{B} = \{1, 2\}$

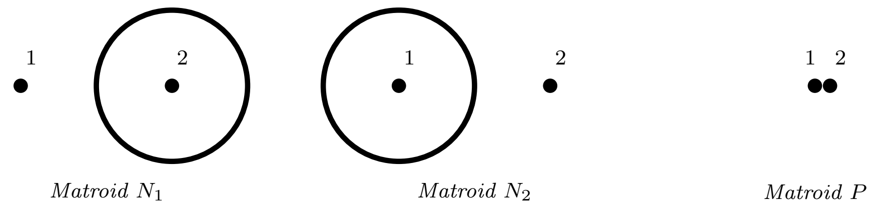

Figure 7.4: Rank 1 matroids on 2 elements.

There is only one rank 2 matroid on these two elements:

Matriod Q: $\mathcal{B} = \{1, 2\}$

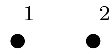

Figure 7.5: Rank 2 matroid on 2 elements.

If we consider the collection of all possible concordant matroids the results are three flag matroids. The first is built on matroids M, N_1, and Q: $\mathcal{F}_2^{0,1,2} = \{\emptyset \subset 1 \subset 12\} = \{12\}$. The second is built on matroids M, N_2, and Q: $\mathcal{G}_2^{0,1,2} = \{\emptyset \subset 2 \subset 12\} = \{21\}$. The third flag matroid is built on matroids M, P, and Q resulting in two flags: $\mathcal{H}_2^{0,1,2} = \{\emptyset \subset 1 \subset 12, \{\emptyset \subset 2 \subset 12\}\}, = \{12, 21\}$.

By taking the dual of the flag matroid as described above, we see that the dual of $\mathcal{F}_2^{0,1,2=}\{\emptyset \subset 1 \subset 12\}$ is $\mathcal{G}_2^{0,1,2} = \{\emptyset \subset 2 \subset 12\}$. Note, that while the flags within the flag matroids are different, the constituent matroids of $\mathcal{F}_2^{0,1,2}$ and $\mathcal{G}_2^{0,1,2}$ are isomorphic, thus we say $\mathcal{F}_2^{0,1,2}$ and $\mathcal{G}_2^{0,1,2}$ are self-dual, although not identically.

The results can be represented in the following table:

Table 7.2: Duality of flag matroids on two elements.

Flag Name	Flags	Flag Duals	Dual Flag Matroid	Duality
$\mathcal{F}$	$\{1 \subset 12\}$	$\{2 \subset 12\}$	$\mathcal{G}$	Self-Dual
$\mathcal{G}$	$\{2 \subset 12\}$	$\{1 \subset 12\}$	$\mathcal{F}$	Self-Dual
$\mathcal{H}$	$\{12, 21\}$	$\{12, 21\}$	$\mathcal{H}$	Identically Self-Dual

7.3 Flag Matroids on 3 Elements

Let $E = \{1, 2, 3\}$. As we shift focus to the case of three elements, the total number of constituent matroids, including each possible labelling of the elements, becomes lengthy, while the number of structurally unique matroids remains manageable. As such, in cases where multiple potential labellings exist, we will present the possible constituent matroid structures unlabelled, followed by a list of the possible sets of bases on each:

We first consider rank 0:

Matriod M:

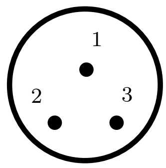

Figure 7.6: Rank 0 matroid on 3 elements.

This matroid M has the basis set $\mathcal{B}_M = \{\emptyset\}$

Next, rank 1.

Matroids N, P, Q

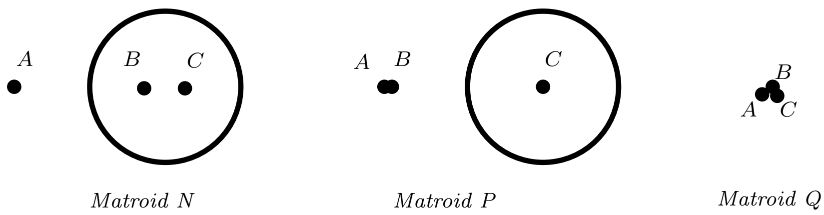

Figure 7.7: Rank 1 matroid structures on 3 elements.

Matroid N may be built upon bases $\mathcal{B}_{N_1} = \{1\}$, $\mathcal{B}_{N_2} = \{2\}$, or $\mathcal{B}_{N_3} = \{3\}$. Matroid P may be built upon bases $\mathcal{B}_{P_1} = \{1, 2\}$, $\mathcal{B}_{P_2} = \{13\}$, $\mathcal{B}_{P_3} = \{2, 3\}$. Matroid Q may be built on the set of bases $\mathcal{B}_Q = \{1, 2, 3\}$.

Next we consider rank 2

Matroids R, S, T

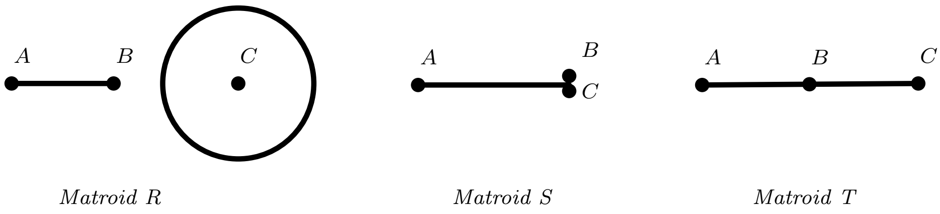

Figure 7.8: Rank 2 matroid structures on 3 elements.

Matroid R may be built upon bases $\mathcal{B}_{R_1} = \{12\}$, $\mathcal{B}_{R_2} = \{13\}$, or $\mathcal{B}_{R_3} = \{23\}$. Matroid S may be built upon bases $\mathcal{B}_{S_1} = \{12, 13\}$, $\mathcal{B}_{S_2} = \{12, 23\}$, $\mathcal{B}_{S_3} = \{13, 23\}$. Matroid T may be built on the set of bases $\mathcal{B}_T = \{12, 13, 23\}$.

Finally, consider rank 3.

Matroid U

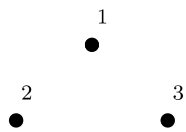

Figure 7.9: Rank 3 matroid structure on 3 elements.

This matroid U has the basis set $\mathcal{B}_U = \{123\}$.

We can now determine the collections of concordant matroids and construct the flag matroids accordingly. In the interest of brevity, we introduce a slight breach in notation. We will denote two flag matroids on isomorphic constituent matroids as $\mathcal{F}_1$ and $\mathcal{F}_2$. Here the subscript no longer indicates the number of elements, but denotes which isomorphic flag matroid we are referring to. (The difference between the two notations may be distinguished due to the lack of superscript denoting rank.)

There exist 28 potential flag matroids built on three elements. They are as follows:

Let $\mathcal{F}_1$ be the flag matroid built on constituent matroids M, N_1, R_1, and U. $\mathcal{F}_1 = \{123\}$.

Let $\mathcal{F}_2$ be the flag matroid built on constituent matroids M, N_1, R_2, and U. $\mathcal{F}_2 = \{132\}$.

Let $\mathcal{F}_3$ be the flag matroid built on constituent matroids M, N_2, R_1, and U. $\mathcal{F}_3 = \{213\}$.

Let $\mathcal{F}_4$ be the flag matroid built on constituent matroids M, N_2, R_3, and U. $\mathcal{F}_4 = \{231\}$.

Let $\mathcal{F}_5$ be the flag matroid built on constituent matroids M, N_3, R_2, and U. $\mathcal{F}_5 = \{312\}$.

Let $\mathcal{F}_6$ be the flag matroid built on constituent matroids M, N_3, R_3, and U. $\mathcal{F}_6 = \{321\}$.

Let $\mathcal{G}_1$ be the flag matroid built on constituent matroids M, N_1, S_1, and U. $\mathcal{G}_1 = \{123, 132\}$.

Let $\mathcal{G}_2$ be the flag matroid built on constituent matroids M, N_2, S_2, and U. $\mathcal{G}_2 = \{213, 231\}$.

Let $\mathcal{G}_3$ be the flag matroid built on constituent matroids M, N_3, S_3, and U. $\mathcal{G}_3 = \{312, 321\}$.

Let $\mathcal{H}_1$ be the flag matroid built on constituent matroids M, P_1, R_1, and U. $\mathcal{H}_1 = \{123, 213\}$.

Let $\mathcal{H}_2$ be the flag matroid built on constituent matroids M, P_2, R_2, and U. $\mathcal{H}_2 = \{132, 312\}$.

Let $\mathcal{H}_3$ be the flag matroid built on constituent matroids M, P_3, R_3, and U. $\mathcal{H}_3 = \{231, 321\}$.

Let $\mathcal{I}_1$ be the flag matroid built on constituent matroids M, P_1, S_1, and U. $\mathcal{I}_1 = \{123, 132, 213\}$.

Let $\mathcal{I}_2$ be the flag matroid built on constituent matroids M, P_1, S_2, and U. $\mathcal{I}_2 = \{123, 213, 231\}$.

Let $\mathcal{I}_3$ be the flag matroid built on constituent matroids M, P_1, S_3, and U. $\mathcal{I}_3 = \{132, 231\}$.

Let $\mathcal{I}_4$ be the flag matroid built on constituent matroids M, P_2, S_1, and U. $\mathcal{I}_4 = \{123, 132, 312\}$.

Let $\mathcal{I}_5$ be the flag matroid built on constituent matroids M, P_2, S_2, and U. $\mathcal{I}_5 = \{123, 321\}$.

Let $\mathcal{I}_6$ be the flag matroid built on constituent matroids M, P_2, S_3, and U. $\mathcal{I}_6 = \{132, 312, 321\}$.

Let $\mathcal{I}_7$ be the flag matroid built on constituent matroids M, P_3, S_1, and U. $\mathcal{I}_7 = \{213, 312\}$.

Let $\mathcal{I}_8$ be the flag matroid built on constituent matroids M, P_3, S_2, and U. $\mathcal{I}_8 = \{213, 231, 321\}$.

Let $\mathcal{I}_9$ be the flag matroid built on constituent matroids M, P_3, S_3, and U. $\mathcal{I}_9 = \{231, 312, 321\}$.

Let $\mathcal{J}_1$ be the flag matroid built on constituent matroids M, P_1, T, and U. $\mathcal{J}_1 = \{123, 132, 213, 231\}$.

Let $\mathcal{J}_2$ be the flag matroid built on constituent matroids M, P_2, T, and U. $\mathcal{J}_2 = \{123, 132, 312, 321\}$.

Let $\mathcal{J}_3$ be the flag matroid built on constituent matroids M, P_3, T, and U. $\mathcal{J}_3 = \{213, 231, 312, 321\}$.

Let $\mathcal{K}_1$ be the flag matroid built on constituent matroids M, Q, S_1, and U. $\mathcal{K}_1 = \{123, 132, 213, 312\}$. U Let $\mathcal{K}_2$ be the flag matroid built on constituent matroids M, Q, S_2, and U. $\mathcal{K}_2 = \{123, 213, 231, 321\}$.

Let $\mathcal{K}_3$ be the flag matroid built on constituent matroids M, Q, S_3, and U. $\mathcal{K}_3 = \{132, 231, 312, 321\}$.

Let $\mathcal{L}$ be the flag matroid built on constituent matroids M, Q, T, and U. $\mathcal{L} = \{123, 132, 213, 231, 312, 321\}$.

Having built the flag matroids, we can now consider the dual of each and its relationship to the original matroid. The results of this are found in the table on the next page.

Table 7.3: Duality of flag matroids on three elements.

Flag Matroid	Flags	Flag Duals	Dual Flag Matroid	Duality
$\mathcal{F}_1$	$\{123\}$	$\{321\}$	$\mathcal{F}_6$	Self-Dual
$\mathcal{F}_2$	$\{132\}$	$\{231\}$	$\mathcal{F}_4$	Self-Dual
$\mathcal{F}_3$	$\{213\}$	$\{312\}$	$\mathcal{F}_5$	Self-Dual
$\mathcal{F}_4$	$\{231\}$	$\{132\}$	$\mathcal{F}_2$	Self-Dual
$\mathcal{F}_5$	$\{312\}$	$\{213\}$	$\mathcal{F}_3$	Self-Dual
$\mathcal{F}_6$	$\{321\}$	$\{123\}$	$\mathcal{F}_1$	Self-Dual
$\mathcal{G}_1$	$\{123, 132\}$	$\{321, 231\}$	$\mathcal{H}_3$	
$\mathcal{G}_2$	$\{213, 231\}$	$\{312, 132\}$	$\mathcal{H}_2$	
$\mathcal{G}_3$	$\{312, 321\}$	$\{213, 123\}$	$\mathcal{H}_1$	
$\mathcal{H}_1$	$\{123, 213\}$	$\{321, 312\}$	$\mathcal{G}_3$	
$\mathcal{H}_2$	$\{312, 132\}$	$\{213, 231\}$	$\mathcal{G}_2$	
$\mathcal{H}_3$	$\{231, 321\}$	$\{132, 123\}$	$\mathcal{G}_1$	
$\mathcal{I}_1$	$\{123, 132, 213\}$	$\{231, 312, 321\}$	$\mathcal{I}_9$	Self-Dual
$\mathcal{I}_2$	$\{123, 213, 231\}$	$\{132, 312, 321\}$	$\mathcal{I}_6$	Self-Dual
$\mathcal{I}_3$	$\{132, 231\}$	$\{231, 132\}$	$\mathcal{I}_3$	Identically Self-Dual
$\mathcal{I}_4$	$\{123, 132, 312\}$	$\{213, 231, 321\}$	$\mathcal{I}_8$	Self-Dual
$\mathcal{I}_5$	$\{123, 321\}$	$\{123, 321\}$	$\mathcal{I}_5$	Identically Self-Dual
$\mathcal{I}_6$	$\{132, 312, 321\}$	$\{123, 213, 231\}$	$\mathcal{I}_2$	Self-Dual
$\mathcal{I}_7$	$\{213, 312\}$	$\{213, 312\}$	$\mathcal{I}_7$	Identically Self-Dual
$\mathcal{I}_8$	$\{213, 231, 321\}$	$\{312, 132, 123\}$	$\mathcal{I}_4$	Self-Dual
$\mathcal{I}_9$	$\{231, 312, 321\}$	$\{123, 132, 213\}$	$\mathcal{I}_1$	Self-Dual
$\mathcal{J}_1$	$\{123, 132, 213, 231\}$	$\{132, 231, 312, 321\}$	$\mathcal{K}_3$	
$\mathcal{J}_2$	$\{123, 132, 312, 321\}$	$\{321, 231, 213, 123\}$	$\mathcal{K}_2$	
$\mathcal{J}_3$	$\{213, 231, 312, 321\}$	$\{123, 132, 213, 312\}$	$\mathcal{K}_1$	
$\mathcal{K}_1$	$\{123, 132, 213, 312\}$	$\{213, 231, 312, 321\}$	$\mathcal{J}_3$	
$\mathcal{K}_2$	$\{321, 231, 213, 123\}$	$\{123, 132, 312, 321\}$	$\mathcal{J}_2$	
$\mathcal{K}_3$	$\{132, 231, 312, 321\}$	$\{123, 132, 213, 231\}$	$\mathcal{J}_1$	
$\mathcal{L}$	$\{123, 132, 213, 231, 312, 321\}$	$\{123, 132, 213, 231, 312, 321\}$	$\mathcal{L}$	Identically Self-Dual

7.4 Flag Matroids on 4 Elements

Due to our definition of the dual of a flag matroid, there exists an easier means by which to determine which flag matroids on a given set are self-dual. This method is confirmed by our $n = 1$, $n = 2$, and $n = 3$ cases, but will be employed more explicitely here where $n = 4$. Let $E = \{1, 2, 3, 4\}$

Once again we begin by considering the possible constituent matroids whose bases may form the flags of a flag matroid. Once more, when there exist multiple unique labellings on a matroid structure, we will instead label the matroid structure's elements with letters rather than numbers, and then make note of all possible collections of bases.

Consider rank 0:

Matroid M

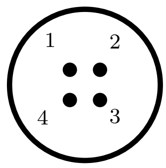

Figure 7.10: Rank 0 matroid structure on 4 elements.

This matroid M has the basis set $\mathcal{B}_M = \{\emptyset\}$

Now consider rank 1. There are four matroid geometries on four elements, each with a number of labellings.

Matroids P, Q, R, and S

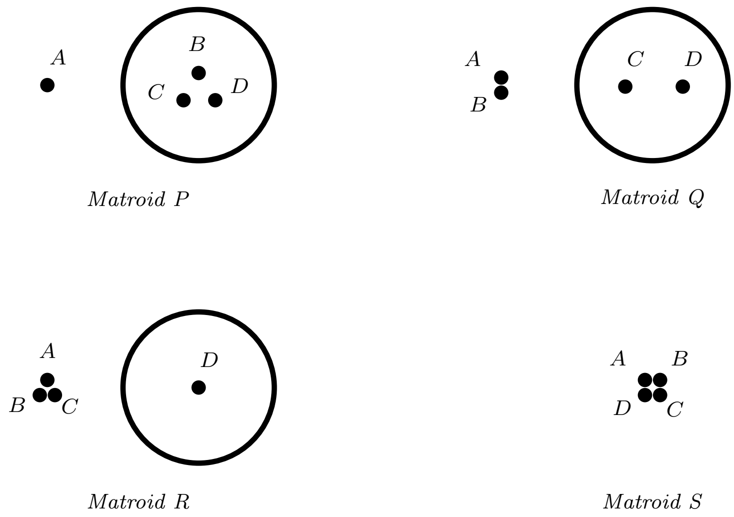

Figure 7.11: Rank 1 matroid structure on 4 elements.

Matroid P may be built upon bases $\mathcal{B}_{P_1} = \{1\}$, $\mathcal{B}_{P_2} = \{2\}$, $\mathcal{B}_{P_3} = \{3\}$, or $\mathcal{B}_{P_4} = \{4\}$. Matroid Q may be built upon bases $\mathcal{B}_{Q_1} = \{1,2\}$, $\mathcal{B}_{Q_2} = \{1,3\}$, $\mathcal{B}_{Q_3} = \{1,4\}$, $\mathcal{B}_{Q_4} = \{2,3\}$, $\mathcal{B}_{Q_5} = \{2,4\}$, or $\mathcal{B}_{Q_6} = \{3,4\}$. Matroid R may be built upon bases $\mathcal{B}_{R_1} = \{1,2,3\}$, $\mathcal{B}_{R_2} = \{1,2,4\}$, $\mathcal{B}_{R_3} = \{1,3,4\}$, or $\mathcal{B}_{R_4} = \{2,3,4\}$. Matroid S may be built upon bases $\mathcal{B}_S = \{1,2,3,4\}$

Next consider rank 2:

Matroids T, U, V, W, X, and Y

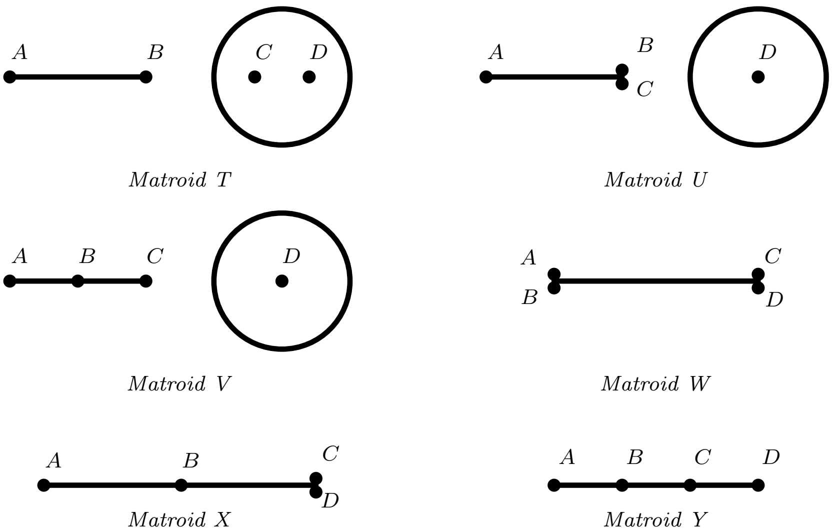

Figure 7.12: Rank 2 matroid structure on 4 elements.

It follows that matroid T may be built upon bases $\mathcal{B}_{T_1} = \{12\}$, $\mathcal{B}_{T_2} = \{13\}$, $\mathcal{B}_{T_3} = \{14\}$, $\mathcal{B}_{T_4} = \{23\}$, $\mathcal{B}_{T_5} = \{24\}$, or $\mathcal{B}_{T_6} = \{34\}$. Matroid U may be built upon bases $\mathcal{B}_{U_1} = \{12, 13\}$, $\mathcal{B}_{U_2} = \{12, 14\}$, $\mathcal{B}_{U_3} = \{12, 23\}$, $\mathcal{B}_{U_4} = \{12, 24\}$, $\mathcal{B}_{U_5} = \{12, 34\}$, $\mathcal{B}_{U_6} = \{13, 14\}$, $\mathcal{B}_{U_7} = \{13, 23\}$, $\mathcal{B}_{U_8} = \{13, 24\}$, $\mathcal{B}_{U_9} = \{13, 34\}$, $\mathcal{B}_{U_{10}} = \{14, 23\}$, $\mathcal{B}_{U_{11}} = \{14, 24\}$, or $\mathcal{B}_{U_{12}} = \{14, 34\}$, $\mathcal{B}_{U_{13}} = \{23, 24\}$, $\mathcal{B}_{U_{14}} = \{23, 34\}$, $\mathcal{B}_{U_{15}} = \{24, 34\}$. Matroid V may be built upon bases $\mathcal{B}_{V_1} = \{12, 13, 23\}$, $\mathcal{B}_{V_2} = \{12, 14, 24\}$, $\mathcal{B}_{V_3} = \{13, 14, 34\}$, or $\mathcal{B}_{V_4} = \{23, 24, 34\}$. Matroid W may be built upon bases $\mathcal{B}_{W_1} = \{13, 14, 23, 24\}$, $\mathcal{B}_{W_2} = \{12, 14, 23, 34\}$, or $\mathcal{B}_{W_3} = \{12, 13, 24, 34\}$. Matroid X may be built upon bases $\mathcal{B}_{X_1} = \{13, 14, 23, 24, 34\}$, $\mathcal{B}_{X_2} = \{12, 14, 23, 24, 34\}$, $\mathcal{B}_{X_3} = \{12, 13, 23, 24, 34\}$, $\mathcal{B}_{X_4} = \{12, 13, 14, 24, 34\}$, $\mathcal{B}_{X_5} = \{12, 13, 14, 23, 34\}$, or $\mathcal{B}_{X_6} = \{12, 13, 14, 23, 24\}$. Finally, matroid Y may be built upon the bases $\mathcal{B}_Y = \{12, 13, 14, 23, 24, 34\}$.

Next we consider rank 3

Matroids Z, H, J, and K

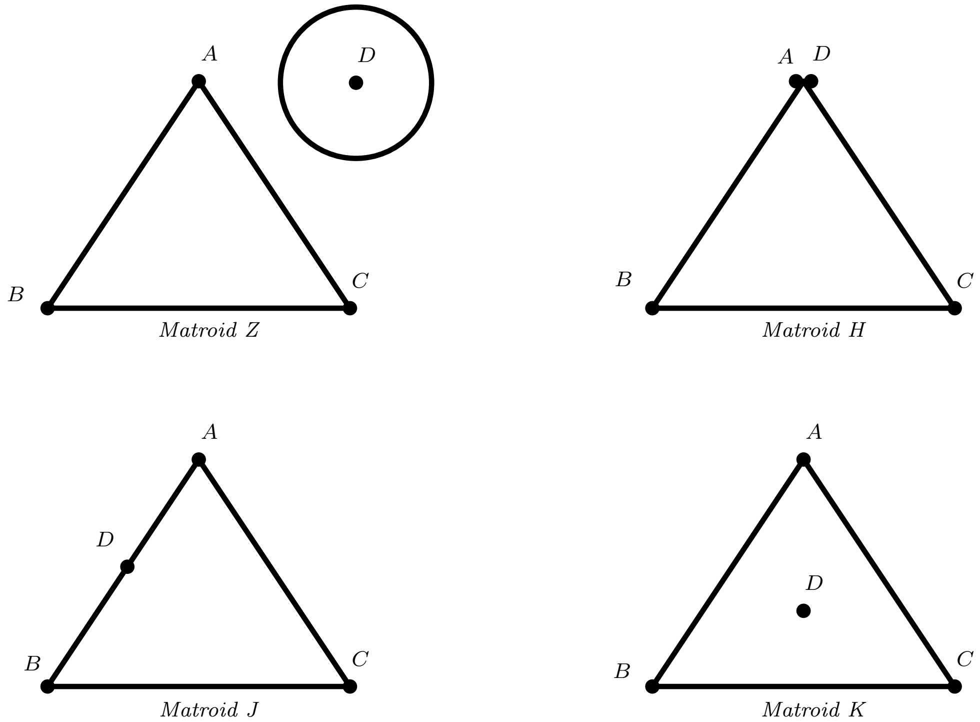

Figure 7.13: Rank 3 matroid structures on 4 elements.

The bases of matroid Z may be built upon the collections of bases $\mathcal{B}_{Z_1} = \{123\}$, $\mathcal{B}_{Z_2} = \{124\}$, $\mathcal{B}_{Z_3} = \{134\}$, $\mathcal{B}_{Z_4} = \{234\}$. Matroid H may be built upon bases $\mathcal{B}_{H_1} = \{123, 234\}$, $\mathcal{B}_{H_2} = \{124, 234\}$, $\mathcal{B}_{H_3} = \{134, 234\}$, $\mathcal{B}_{H_4} = \{123, 134\}$, $\mathcal{B}_{H_5} = \{124, 134\}$, and $\mathcal{B}_{H_6} = \{123, 124\}$. Matroid J may be built on bases $\mathcal{B}_{J_1} = \{123, 124, 134\}$, $\mathcal{B}_{J_2} = \{123, 124, 234\}$, $\mathcal{B}_{J_3} = \{123, 134, 234\}$, and $\mathcal{B}_{J_4} = \{124, 134, 234\}$. Matroid K may then be built on the collections of bases $\mathcal{B}_K = \{123, 124, 134, 234\}$.

Finally, we consider rank 4

Matroid L

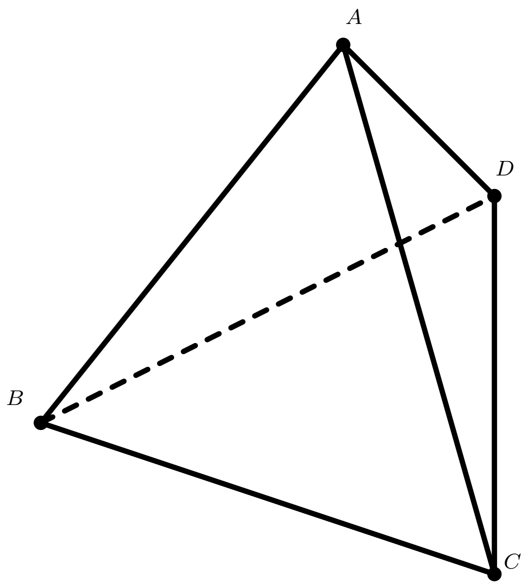

Figure 7.14: Rank 4 matroid structure on 4 elements.

Matroid DD may only be built on one collection of bases, $\mathcal{B}_L = \{1234\}$.

In total, there are nearly fifteen thousand possible combinations of constituent matroids we must check for concordancy, in order to determine which collections form flag matroids. Rather than an exhaustive search, we employ a strategy to determine which collections of constituent matroids form identically self-dual flag matroids.

We begin by first identifying a trend found among the identically self-dual flag matroids on three elements. In each of these four flag matroids, the rank 1 and rank 2 constituent matroids were duals of one another. This must be the case, as the dual of the rank 1 constituent matroid of the initial flag matroid becomes the rank 2 constituent matroid of the dual flag matroid, while the dual of the rank 2 constituent matroid of the original flag matroid becomes the rank 1 constituent matroid of the dual flag matroid. Similarily, the rank 0 and rank 3 constituent matroids are duals of one another.

In searching for a similar trend in the self-dual flag matroid on two elements, we observe the rank 2 constituent matroid is the dual of the rank 0 constituent matroid, while the rank 1 constituent matroid is self-dual. Thus, we suggest the following conjecture:

Conjecture 7.1. A flag matroid is identically self-dual if each of its constituent matroids of rank r is the dual of its constituent matroid of rank $n - r$.

Note that, under this definition, flag matroids with an odd number of constituent matroids, as seen in the cases of two and four elements, there must exist a self-dual constituent matroid in every self-dual flag matroid. When the number of constituent matroids is even, this does not occur.

With this conjecture in mind, we now have a means of finding all possible identically self-dual flag matroids on four elements. First, we consider collections of constituent matroids such that those of rank r and $n - r$ are duals of one another, and then check for concordancy. Should the collection of matroids be concordant, the result is a flag matroid.

Considering matroids of rank 0 and rank 4, all flag matroids must contain matroids M and K. These two are duals of one another, and thus they may be constituents in a self-dual flag matroid. Next we consider matroids of rank 1 and rank 3. The following pairs of matroids are duals of these ranks. They are as follows:

Table 7.4: Dual rank 1 and rank 3 matroids on four elements.

Rank 1 Constituent	Bases	Dual Bases	Dual Constituent of Rank 3
$\mathcal{K}$	$\{123, 124, 134, 234\}$	$\{4, 3, 2, 1\}$	$\mathcal{S}$
$\mathcal{J}_1$	$\{123, 124, 134\}$	$\{4, 3, 2\}$	$\mathcal{R}_4$
$\mathcal{J}_2$	$\{123, 124, 234\}$	$\{4, 3, 1\}$	$\mathcal{R}_3$
$\mathcal{J}_3$	$\{123, 134, 234\}$	$\{4, 2, 1\}$	$\mathcal{R}_2$
$\mathcal{J}_4$	$\{124, 134, 234\}$	$\{3, 2, 1\}$	$\mathcal{R}_1$
$\mathcal{H}_1$	$\{123, 234\}$	$\{4, 1\}$	$\mathcal{Q}_3$
$\mathcal{H}_2$	$\{124, 234\}$	$\{3, 1\}$	$\mathcal{Q}_2$
$\mathcal{H}_3$	$\{134, 234\}$	$\{2, 1\}$	$\mathcal{Q}_1$
$\mathcal{H}_4$	$\{123, 134\}$	$\{4, 2\}$	$\mathcal{Q}_5$
$\mathcal{H}_5$	$\{124, 134\}$	$\{3, 2\}$	$\mathcal{Q}_4$
$\mathcal{H}_6$	$\{123, 124\}$	$\{4, 3\}$	$\mathcal{Q}_6$
$\mathcal{Z}_1$	$\{123\}$	$\{4\}$	$\mathcal{P}_4$
$\mathcal{Z}_2$	$\{124\}$	$\{3\}$	$\mathcal{P}_3$
$\mathcal{Z}_3$	$\{134\}$	$\{2\}$	$\mathcal{P}_2$
$\mathcal{Z}_4$	$\{234\}$	$\{1\}$	$\mathcal{P}_1$

Finally, we consider which rank 2 constituent matroids are also self-dual:

Table 7.5: Self-dual rank 2 matroids on four elements.

Rank 2 Constituent	Bases	Dual Bases
$\mathcal{Y}$	$\{12, 13, 14, 23, 24, 34\}$	$\{34, 24, 23, 14, 13, 12\}$
$\mathcal{W}_3$	$\{12, 13, 24, 34\}$	$\{34, 24, 13, 12\}$
$\mathcal{W}_2$	$\{12, 14, 23, 34\}$	$\{34, 23, 14, 12\}$
$\mathcal{W}_1$	$\{13, 14, 23, 24\}$	$\{24, 23, 14, 13\}$
$\mathcal{U}_5$	$\{12, 34\}$	$\{34, 12\}$
$\mathcal{U}_8$	$\{13, 24\}$	$\{24, 13\}$
$\mathcal{U}_{10}$	$\{14, 23\}$	$\{23, 14\}$

With a collection of potential constituent matroids determined, we now check for concordancy between them and construct the following identically self-dual flag matroids.

Table 7.6: Self-dual flag matroids on four elements.

Flag Matroid	Constituents	Flag Matroid	Constituents
$\mathcal{F}_1$	$\mathcal{M}, \mathcal{S}, \mathcal{Y}, \mathcal{K}, \mathcal{L}$	$\mathcal{K}_1$	$\mathcal{M}, \mathcal{Q}_6, \mathcal{W}_1, \mathcal{H}_6, \mathcal{L}$
$\mathcal{F}_2$	$\mathcal{M}, \mathcal{S}, \mathcal{W}_3, \mathcal{K}, \mathcal{L}$	$\mathcal{K}_2$	$\mathcal{M}, \mathcal{Q}_6, \mathcal{U}_8, \mathcal{H}_6, \mathcal{L}$
$\mathcal{F}_3$	$\mathcal{M}, \mathcal{S}, \mathcal{W}_2, \mathcal{K}, \mathcal{L}$	$\mathcal{K}_3$	$\mathcal{M}, \mathcal{Q}_6, \mathcal{U}_{10}, \mathcal{H}_6, \mathcal{L}$
$\mathcal{F}_4$	$\mathcal{M}, \mathcal{S}, \mathcal{W}_1, \mathcal{K}, \mathcal{L}$	$\mathcal{L}_1$	$\mathcal{M}, \mathcal{Q}_5, \mathcal{W}_2, \mathcal{H}_4, \mathcal{L}$
$\mathcal{F}_5$	$\mathcal{M}, \mathcal{S}, \mathcal{U}_5, \mathcal{K}, \mathcal{L}$	$\mathcal{L}_2$	$\mathcal{M}, \mathcal{Q}_5, \mathcal{U}_5, \mathcal{H}_4, \mathcal{L}$
$\mathcal{F}_6$	$\mathcal{M}, \mathcal{S}, \mathcal{U}_8, \mathcal{K}, \mathcal{L}$	$\mathcal{L}_3$	$\mathcal{M}, \mathcal{Q}_5, \mathcal{U}_{10}, \mathcal{H}_4, \mathcal{L}$
$\mathcal{F}_7$	$\mathcal{M}, \mathcal{S}, \mathcal{U}_{10}, \mathcal{K}, \mathcal{L}$	$\mathcal{M}_1$	$\mathcal{M}, \mathcal{Q}_3, \mathcal{W}_3, \mathcal{H}_1, \mathcal{L}$
$\mathcal{G}_1$	$\mathcal{M}, \mathcal{J}_1, \mathcal{Y}, \mathcal{R}_4, \mathcal{L}$	$\mathcal{M}_2$	$\mathcal{M}, \mathcal{Q}_3, \mathcal{U}_5, \mathcal{H}_1, \mathcal{L}$
$\mathcal{G}_2$	$\mathcal{M}, \mathcal{J}_1, \mathcal{W}_3, \mathcal{R}_4, \mathcal{L}$	$\mathcal{M}_3$	$\mathcal{M}, \mathcal{Q}_3, \mathcal{U}_8, \mathcal{H}_1, \mathcal{L}$
$\mathcal{G}_3$	$\mathcal{M}, \mathcal{J}_1, \mathcal{W}_2, \mathcal{R}_4, \mathcal{L}$	$\mathcal{N}_1$	$\mathcal{M}, \mathcal{Q}_4, \mathcal{W}_3, \mathcal{H}_5, \mathcal{L}$
$\mathcal{G}_4$	$\mathcal{M}, \mathcal{J}_1, \mathcal{W}_1, \mathcal{R}_4, \mathcal{L}$	$\mathcal{N}_2$	$\mathcal{M}, \mathcal{Q}_4, \mathcal{U}_5, \mathcal{H}_5, \mathcal{L}$
$\mathcal{G}_5$	$\mathcal{M}, \mathcal{J}_1, \mathcal{U}_5, \mathcal{R}_4, \mathcal{L}$	$\mathcal{N}_3$	$\mathcal{M}, \mathcal{Q}_4, \mathcal{U}_8, \mathcal{H}_5, \mathcal{L}$
$\mathcal{G}_6$	$\mathcal{M}, \mathcal{J}_1, \mathcal{U}_8, \mathcal{R}_4, \mathcal{L}$	$\mathcal{M}_1$	$\mathcal{O}, \mathcal{Q}_2, \mathcal{W}_2, \mathcal{H}_2, \mathcal{L}$
$\mathcal{G}_7$	$\mathcal{M}, \mathcal{J}_1, \mathcal{U}_{10}, \mathcal{R}_4, \mathcal{L}$	$\mathcal{O}_2$	$\mathcal{M}, \mathcal{Q}_2, \mathcal{U}_5, \mathcal{H}_2, \mathcal{L}$
$\mathcal{H}_1$	$\mathcal{M}, \mathcal{J}_2, \mathcal{Y}, \mathcal{R}_3, \mathcal{L}$	$\mathcal{M}_3$	$\mathcal{O}, \mathcal{Q}_2, \mathcal{U}_{10}, \mathcal{H}_2, \mathcal{L}$
$\mathcal{H}_2$	$\mathcal{M}, \mathcal{J}_2, \mathcal{W}_3, \mathcal{R}_3, \mathcal{L}$	$\mathcal{P}_1$	$\mathcal{M}, \mathcal{Q}_1, \mathcal{W}_1, \mathcal{H}_3, \mathcal{L}$
$\mathcal{H}_3$	$\mathcal{M}, \mathcal{J}_2, \mathcal{W}_2, \mathcal{R}_3, \mathcal{L}$	$\mathcal{P}_2$	$\mathcal{M}, \mathcal{Q}_1, \mathcal{U}_8, \mathcal{H}_3, \mathcal{L}$
$\mathcal{H}_4$	$\mathcal{M}, \mathcal{J}_2, \mathcal{W}_1, \mathcal{R}_3, \mathcal{L}$	$\mathcal{P}_3$	$\mathcal{M}, \mathcal{Q}_1, \mathcal{U}_{10}, \mathcal{H}_3, \mathcal{L}$
$\mathcal{H}_5$	$\mathcal{M}, \mathcal{J}_2, \mathcal{U}_5, \mathcal{R}_3, \mathcal{L}$		
$\mathcal{H}_6$	$\mathcal{M}, \mathcal{J}_2, \mathcal{U}_8, \mathcal{R}_3, \mathcal{L}$		
$\mathcal{H}_7$	$\mathcal{M}, \mathcal{J}_2, \mathcal{U}_{10}, \mathcal{R}_3, \mathcal{L}$		
$\mathcal{I}_1$	$\mathcal{M}, \mathcal{J}_3, \mathcal{Y}, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_2$	$\mathcal{M}, \mathcal{J}_3, \mathcal{W}_3, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_3$	$\mathcal{M}, \mathcal{J}_3, \mathcal{W}_2, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_4$	$\mathcal{M}, \mathcal{J}_3, \mathcal{W}_1, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_5$	$\mathcal{M}, \mathcal{J}_3, \mathcal{U}_5, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_6$	$\mathcal{M}, \mathcal{J}_3, \mathcal{U}_8, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{I}_7$	$\mathcal{M}, \mathcal{J}_3, \mathcal{U}_{10}, \mathcal{R}_2, \mathcal{L}$		
$\mathcal{J}_1$	$\mathcal{M}, \mathcal{J}_4, \mathcal{Y}, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_2$	$\mathcal{M}, \mathcal{J}_4, \mathcal{W}_3, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_3$	$\mathcal{M}, \mathcal{J}_4, \mathcal{W}_2, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_4$	$\mathcal{M}, \mathcal{J}_4, \mathcal{W}_1, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_5$	$\mathcal{M}, \mathcal{J}_4, \mathcal{U}_5, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_6$	$\mathcal{M}, \mathcal{J}_4, \mathcal{U}_8, \mathcal{R}_1, \mathcal{L}$		
$\mathcal{J}_7$	$\mathcal{M}, \mathcal{J}_4, \mathcal{U}_{10}, \mathcal{R}_1, \mathcal{L}$		

Chapter 8

Conclusion

Over the course of this study we have defined a matroid in terms of its bases, then introduced a few of the fundamental ideas of matroids, most importantly the notion of duality. Having established what constitutes a matroid, we explored the implications of inducing a Gale ordering upon the elements of a matroid in the forms of the Maximality and Increasing Exchange Properties. We then went on to present the idea of a flag matroid. Although these are not matroids themselves, they are subject to variations of the Maximality and Increasing Exchange Properties of matroids. Ultimately, we suggest a new notion of duality on flag matroids, resulting in a theorem similar to that of duality on matroids.

The field of flag matroids is one that warrants substantial future attention. The notion of duality of flag matroids suggested herein stems from the fact that each flag matroid owes its construction to constituent matroids. Further study will determine what, if any, other characteristics of a matroid may apply in a modified fashion to flag matroids as well.

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