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Chaos and the stock market

Brent M. Monte

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CHAOS AND THE STOCK MARKET

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Art
in
Mathematics

by
Brent M. Monte
June 1994
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ABSTRACT

It has long been felt that the stock market behaved in a completely random manner. However, a relatively new branch of mathematics called "Chaos Theory" purports that there are systems which may appear random, but are in fact highly structured. These systems are deemed "chaotic."

Based on the work done by Edgar Peters, it will be shown that the Dow Jones Industrial Average, one of the primary indices of the New York Stock Exchange, exhibits the characteristics of a chaotic system.

The fact that the Dow Jones Industrial Average appears to be chaotic, questions the validity of any method of predicting stock market movements that is based on the random walk theory. A new way of trying to predict the movements of the stock market must now be developed.
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INTRODUCTION

For many years it has been assumed that the stock market and its related indices behaved in a random manner with stock price changes being normally distributed. In fact, the stock market has commonly been referred to as having a "random walk" in regard to its prices. The change in prices has been believed to be completely independent of what had happened the day before.

The Efficient Market Hypothesis (EMH) has now been a staple of academia for many years, and business professors have been teaching the EMH as a law of the stock market even though it has never been conclusively proven. Basically, what the EMH states is that current stock prices reflect all known information, and any change in stock prices comes about only when new information becomes available. This new information is then rapidly digested by the investment community, and the stock price reflects this new information rapidly, usually the same day that the news becomes public information. Since there is no way to tell if the next piece of information is going to be good or bad for a particular stock, or the market as a whole, there is no way to tell which way the price of stocks will go next. Thus, the market should behave randomly, depending on the latest news.
One nice result that manifests itself if the stock market does behave in a random manner with price changes being normally distributed is that a plot of the stock price changes would result in the normal distribution represented by a bell-shaped curve. This opens up a wide range of statistical procedures that may be used on the stock market data to aid in understanding the movements of the stock prices. However, most of the statistical procedures that are currently being used to analyze the stock market fail to work consistently. Although, if the stock market price changes are not random, then they should not be expected to be normally distributed. Thus, any statistical procedures based upon the normal distribution would have no basis for working anyway.

There are a number of various theories in addition to the EMH that are used to try to explain the movements of the stock market, but all with limited success. Since the EMH is probably the most common stock market theory being currently taught, the fundamental basis for the EMH is what this paper is questioning. Since none of the current methods of predicting the movements of the stock market have proven to be consistently reliable, a new method is then needed to try to understand how these price changes occur.

As far back as 1960, Benoit Mandelbrot was working with another type of economic data - income distribution.
Mandelbrot noted that although income distribution was assumed to be random, the distribution of income did not fit the normal distribution which he had expected. There were too many large changes in relation to the small changes; the bell-shaped curve did not fall off quickly enough. Coincidentally, this graph of income distribution matched a graph of cotton prices created by Hendrik Houthakker, so Mandelbrot recreated Houthakker's graph of cotton prices, this time using data back to 1900, and again obtained a graph matching the income distribution graph. Not only did the graph again have too many big changes, but when Mandelbrot changed the scale from daily to monthly to yearly, he again obtained the same graph with more large changes than the normal distribution would produce for "random" price changes [1].

Mandelbrot's work with income distribution and with cotton prices led him into a new field which was just about to emerge - chaos theory. Chaos in this context is not used as a synonym for randomness, but rather is used in relation to systems that behave in a nonlinear fashion. These systems may appear random to look at, but they are composed of a complex yet highly structured set of rules and bounds. Chaos theory deals with trying to understand which systems are chaotic, as opposed to being random or linear systems. One of the primary reasons to determine if a system is
indeed chaotic, is that if so, it is then composed of a rich
structure which could lead to understanding and predicting
how the system will behave in the short-term.

More recently, much more research is being done between
chaos theory and its relation to economic data. Chaos
theory has been related to such things as U.S. monthly
unemployment [2], capital goods and consumption goods [3],
and U.S. monthly pig iron production [4]. Edgar E. Peters
then considered the relationship between chaos theory and
the stock market.

The work of Peters in examining whether or not the
stock market behaves chaotically seems to be most
intriguing, especially in light of the thousands of people
around the world who watch the various stock market indices
and try to predict the direction of the stock market's next
move. Peters focused on the Standard and Poor's 500 company
index, hereafter referred to as the S&P 500.

In this paper, after explaining the method that Peters
invoked in evaluating whether or not the S&P 500 is chaotic,
an evaluation of another stock market index, the Dow Jones
Industrial Average (DJIA), will be performed to determine if
it may also possess the characteristics of a chaotic system.
EXPLANATION OF PETERS' PROCEDURE

Outline

Due to various anomalies present in the stock market, Peters decided to test the S&P 500 for evidence of the existence of chaos in the stock market. When Peters graphed the frequency distribution of the S&P 500 five-day returns from January 1928 through December 1989 he obtained a graph much like that obtained by Mandelbrot for cotton prices. Peters' graph had many more large changes in stock prices than the normal distribution, and also many more small changes than the normal distribution [5]. The graph was taller in the middle, and had longer, thicker tails than the normal distribution. Something seemed to be askew in the traditional thinking about the behavior of the stock market, and checking the stock market for evidence of chaos was the direction Peters decided to explore.

The typical way in which one determines if a particular system is chaotic is if the system exhibits two certain characteristics. The system must have a fractal dimension, and the system must exhibit sensitive dependence on initial conditions [6].

Peters' method to determine whether or not the S&P 500 behaved chaotically (i.e. had a chaotic attractor) was to see if the S&P 500 possessed both of these characteristics.
He first had to decide what variables he was going to use to model the S&P 500, and then he had to prepare the data for testing. Once this had been completed, Peters evaluated the dimensionality of the data stream, and tested the data stream to see if it exhibited sensitive dependence on initial conditions.

Preparing the Data

Traditionally, when an analysis of the stock market has been done, the values used have been based on the percentage change in prices. In an article published in "System Dynamics Review," the researcher Ping Chen found that based on experiments of various detrending methods applied to economic time series, the percentage rate of change method was a whitening process which was based on short-term scaling. Unfortunately, this process may remove any correlations which may exist, and these may in fact be the correlations which show whether or not a system behaves chaotically. However, by using a method called log linear detrending, any long-term correlations in economic data are retained since the time scale of the detrending process represents the entire time series used [7].

In log linear detrending, the basis for the data stream is the actual observed variable - the stock market price in this case. This seems reasonable as it is the same basis
used in the physical sciences. For example, in constructing the highly celebrated Lorenz attractor in chaos theory, the actual value of the variables are used, rather than the rate of change.

Log linear detrending of the data stream does two things. First, it uses the natural log of the variables rather than the variable values themselves. This results in a much smoother stream of data without losing any of the long-term correlations.

Secondly, log linear detrending is used in economic data to remove the effects of inflation on the data stream. If inflation was not removed from the data stream, the values would continue to spiral upward and the results would be intolerably skewed. Removing inflation is the economists method of placing a control on the collection of the data; all data is gathered from an equal starting point.

Peters incorporated this log linear detrending in the following method:

$$ S_i = \ln(P_i) - (a \cdot \ln(CPI_i) + b) $$

where

- $S_i$ = the detrended S&P 500 on month $i$,
- $P_i$ = the S&P price on month $i$, and
- $CPI_i$ = the Consumer Price Index (CPI) on month $i$.

The values of $a$ and $b$ are constants obtained by regressing the log of the S&P 500 against the log of the CPI over the period covered [8].

The purpose of the constants $a$ and $b$, is that they act
in canceling out the correlation between the S&P 500 and the CPI. Thus, the resulting detrended time series consists of only the value of the stock market less inflation.

It should be noted that removing inflation via the CPI index introduces the possibility that any chaos that might be observed may indicate chaotic properties of the CPI rather than the stock market. As this would be the case with any inflation index, there seems to be no way around admitting this possibility at the present time. However, it may prove beneficial in the future to test the CPI by itself for indications of chaotic behavior.

The Fractal Dimension

Once the data stream has been created, the next step is to determine the dimension of the S&P 500. The dimension of the S&P 500 is a measure of how complex the system is. The minimum number of equations needed to model a system is the next higher integer over the dimension. For example, if a system is determined to have a dimension of 3.5, the minimum number of equations needed to model the system would be four. If the S&P 500 can be shown to have a small dimension, then the possibility that the stock market may some day be able to be modeled would be far more likely than if the dimension was found to be high.

Conversely, if the S&P 500 is found to have a dimension
that is an integer, that would mean that there is no real underlying structure to the market, and it is essentially random. The limit of the graph of a system with dimension two, for example, would consist of an entire two dimensional area if an infinite number of points could be plotted. Obviously, this sort of structureless system could never be accurately modeled.

When speaking of dimension in this way we are referring to the Euclidean dimension in the standard way. That is, a line has dimension one, a circle has dimension two, and so on. In the case of an object with a fractal dimension however, the object would have a more complex structure than a similar object with an integer dimension. For example, an object with a dimension of 3.5 would have a structure far more complex than a three dimensional object, and in fact could not be accurately depicted in three dimensions. However, that same object of dimension 3.5 could easily be depicted in four dimensions, and would appear to be an object of clearly less than four dimensions. In four dimensions it would probably appear as a complex three dimensional object, which it of course is not.

In a paper published by The American Physical Society, Peter Grassberger and Itamar Procaccia have shown a method which can be used to determine the dimension of a system using only the time series of a single observable [9]. This
method seems ideally suited for the stream of data provided from the stock market, and it is the method which Peters used to determine the dimension of the S&P 500.

In order to find the dimension of the system represented by the time series, the correlation integral must be found. The correlation integral is the probability that a pair of points in the attractor are within a distance $R$ of one another. The correlation integral as used by Peters is as follows:

$$C_m(R) = \frac{1}{N^2} \sum_{i,j=1}^{N} z(R - |X_i - X_j|)$$

where

- $z(x) = 1$ if $x > 0$, 0 otherwise,
- $N$ = number of observations,
- $R$ = distance between the individual points,
- $X_k = (S_k, S_{k+t}, S_{k+2t}, \ldots, S_{k+(m-1)t})$,
- $S_k$ = the detrended S&P price on month $k$, and
- $C_m$ = correlation integral for dimension $m$.

In the calculation of $X_k$, $m$ stands for the dimension of the space being created, and $t$ stands for the time increment between coordinates. Wolf et al. have shown that the relation $m \cdot t = Q$, where $m$ is the embedding dimension, $t$ is the time lag, and $Q$ is the mean orbital period of the system is a good relationship for these three quantities [10]. The product $m \cdot t$ will be set at 48 months, as this is the period of the S&P 500 as determined by Peters through use of rescaled range analysis [5].

Grassberger and Procaccia have shown that as the value of $R$ is increased, $C_m$ approaches $R^D$ for small values of $m$. 


Then, $C_m = R^D$ as $m$ is increased. In this equation, $D$ is the dimension of the system that produced the time series [9]. Taking the natural log of both sides of the equation yields $\ln(C_m) = D \cdot \ln(R)$, or further still $D = \ln(C_m)/\ln(R)$.

The dimension $D$ referred to here is the Euclidean dimension, while the embedding dimension $m$ is the dimension of the multi-dimensional data stream being created. If the dimension $D$ turns out to be say 3.5, then the minimum embedding dimension $m$ needed to properly embed the system will be the next integer higher than 3.5, that is, $m = 4$.

The multi-dimensional data stream of the various $X_k$ created as above is called the phase space of the system. David Ruelle has proven mathematically that a phase space created in this way has the same fractal dimension and spectrum of Lyapunov exponents as the "real" phase space of the system [5]. The "real" phase space of the system would be the multi-dimensional space of the system of equations needed to accurately model the stock market. Since the equations of motion are not known, the phase space must be recreated from only the data stream that is available.

The method for determining the dimension $D$ of the phase space proceeds as follows:

1. Begin with a value of $m = 2$, $t$ set accordingly to keep the equation $m \cdot t = Q$, and an arbitrarily small $R$ value.
2. Incrementally increase R (this will increase $C_m$ at a rate of $R^D$).

3. Graph $\ln(C_m)/\ln(R)$ for the increasing R values and find the slope of the graph. This is the dimension $D$ for that particular $m$ value.

4. Increase the value of $m$ by one, adjusting $t$ accordingly, and repeat steps 1-3.

5. The dimension $D$ will eventually converge to its actual value as $m$ is increased.

Peters used a computer program written in Basic to perform this operation for various values of $m$. A copy of this program, converted into QuickBASIC can be found in appendix A.

After using this method to evaluate the dimension of the S&P 500, Peters arrived at an estimate of the dimension of 2.33. Not only did this show that the S&P 500 did indeed have a fractal dimension, but it also showed that if the S&P 500 was a chaotic system, then the minimum number of equations needed to model the system is only three - the next integer above the fractal dimension. Thus, it seems more realistic that the S&P 500 may eventually be modeled than if the dimension had been three or greater.

**Sensitive Dependence on Initial Conditions**

The second item which must be shown in order support that a system is indeed chaotic is the existence of sensitive dependence on initial conditions. Basically what
this means is that errors are not compounded linearly as in a system that behaves in a linear manner, but rather errors are compounded at an exponential rate. In this type of system, small errors in evaluating a system at any given time will turn into large errors in a relatively short period of time. Thus, a system exhibiting sensitive dependence on initial conditions may be modeled in the short term, but long term forecasting based on current conditions is meaningless.

The method that is most commonly used to show this sensitive dependence on initial conditions is the same method used by Peters in his work. The method entails finding the largest Lyapunov exponent of the phase space. If the largest Lyapunov exponent is positive, the system possesses sensitive dependence on initial conditions; if the largest Lyapunov exponent is zero or negative, no such dependence exists.

The Lyapunov exponent measures how quickly nearby points diverge in the phase space. There is one Lyapunov exponent for each dimension in the phase space. Thus, if the system can be modeled in a minimum of three dimensions, the dimension of the phase space is three, and there are three Lyapunov exponents.

A negative Lyapunov exponent would indicate contraction in that dimension; points would be all converging to a
common point. A zero Lyapunov exponent would indicate that the system is in a form of equilibrium for that dimension, the points are neither contracting nor diverging from each other on average in that dimension. A positive Lyapunov exponent, however, would indicate that the points are in fact diverging from one another.

The magnitude of the exponent relates the rate at which the points are either diverging from one another (positive exponent), or converging into a singular point (negative exponent). For example, if the largest Lyapunov exponent were 0.42 and current conditions could be measured to two bits of accuracy, all predictive power would be lost 4.8 iterations \(2.0^\frac{0.42}{0.42} = 4.8\) into the future.

In a three-dimensional system that possesses an attractor, the only possibilities for the spectra of Lyapunov exponents would be \((-,-,-)\), \((0,-,-)\), \((0,0,-)\), or \((+,0,-)\). Three negative exponents \((-,-,-)\) would mean that the points are converging to one point in all three dimensions, the system converges to a fixed point. The exponent system \((0,-,-)\) would indicate that two dimensions are contracting while one dimension is relatively stable, a limit cycle. For a system with exponents \((0,0,-)\), two dimensions are stable while one is in contraction, a two-dimensional limiting structure such as a two-torus. Finally, if the largest Lyapunov exponent is positive, the
For a system with Lyapunov exponents (+,0,-), the positive exponent shows that sensitive dependence on initial conditions exists. That is, small errors in evaluating initial conditions will result large errors in a short period of time due to the points diverging from each other. However, the existence of the negative exponent will keep the diverging points within the range of the attractor. Thus, although a system may behave chaotically making long term forecasts worthless based on current data, the system will be kept within a certain range of expected values.

The procedure followed by Peters is then to determine the largest Lyapunov exponent of the system. If the largest exponent is positive, sensitive dependence on initial conditions will exist, and the system will be determined to have a chaotic attractor.

Peters used a computer algorithm modified from one developed by Wolf et al. to determine the largest Lyapunov exponent [10]. The algorithm measures the divergence of nearby points in the reconstructed phase space over a fixed interval of time. If the distance is too large, the computer searches for a replacement point. This ensures that the points will not grow too far apart and fold into each other. Additionally, the angle between the points is
measured by the algorithm to try to keep the location of the points in the phase space as close as possible to that of the original set of points.

In addition to the data stream being input into the algorithm, an embedding dimension, a time lag, the time between data samples, an evolution time, and the maximum and minimum allowable distance between points, must all be chosen and input into the computer as well.

The embedding dimension should be larger than the phase space of the underlying attractor since a surface usually appears smoother in a higher dimension. However, the embedding dimension should not be too large or the data points will be too sparse for the algorithm to run efficiently.

The time lag represents how much time between data points should pass prior to choosing the next coordinate for each multi-dimensional point as the algorithm creates the phase space of the system.

The time between data samples refers to how much time passes between first coordinates of the points in the phase space.

The evolution time is the time the system is allowed to run before the new distance between the points is checked. If the evolution time is too large, the distance between the points may become too large, and the points can actually
begin to fold into one another. However, a shorter evolution time increases the likelihood of introducing errors regarding the orientation of the points in the system due to more replacement points being selected.

The maximum and minimum allowable distance between points represents the range of distances allowable between the two points being measured before the program would throw out one of the points and replace it with a more meaningful point.

**Selection of the Embedding Dimension m**

According to Wolf, an embedding of the phase space should occur if the embedding dimension (m) is selected to be greater than twice the dimension of the phase space. However, if the value of m is too large, noise in the data will tend to overwhelm the structure present in the data, and the points of the phase space may become too sparse in the higher dimension. Experimentation has shown that reliable results may be achieved for a value of m as low as the next integer higher than the dimension of the phase space. Peters used a value of \( m = 4 \) in determining the largest Lyapunov exponent of the S&P 500.

**Selection of the Time Lag t**

Peters used the mean orbital period of the stock market
of 48 months, and the embedding dimension of four, to obtain a time lag of 12 months, from the equation \( m \cdot t = Q \).

**Selection of the Time Between Data Samples**

The time between data samples refers to how much time passes between first coordinates of the points in the phase space. Wolf et al. merely say that this value is used to normalize the exponent, but gives no guidance as to what values have been found to work in experimentation [10]. Peters gives no indication of what values he used in running the Wolf algorithm on the S&P 500 data.

**Selection of the Evolution Time**

The evolution time should be long enough to measure stretching without measuring folds. A short evolution period results in more calculations, but requires fewer replacements and results in a more stable convergence. Peters feels that the shorter the evolution time the better, and he obtained convergence using an evolution time of six months [5].

**Selection of the Maximum and Minimum Distance Between Data Points**

The next thing that must be determined in order to calculate the Lyapunov exponent is the maximum and minimum
distance between data points before a replacement point is selected. Wolf et al. suggests using a number of no more than 10 percent of the length of the attractor in phase space [10]. Peters states that essentially this means that the maximum distance between data points should be no more than 10 percent of the difference between the maximum and minimum values of the time series. The minimum distance between data points that Peters uses is then calculated as 10 percent of the maximum distance between data points. These are the values that Peters uses when he runs the algorithm [5].

Wolf et al. arrived at this 10 percent number by experimentation, but Peters has found success in arriving at stable convergence of the largest Lyapunov exponent by using this guideline. However, Wolf further states in his paper that if the mechanism for chaos is not known (as is the case for the stock market), a wide range of evolution times should be used in order to check for exponent stability [10].

Calculation of the Lyapunov Exponent

Once the input parameters that have just been mentioned have been set, the largest Lyapunov exponent is found using the Wolf algorithm. A copy of the algorithm, modified into QuickBASIC can be found in appendix B of this paper.
What the Calculated Lyapunov Exponent Means

Using this algorithm, Peters found the largest Lyapunov exponent \( L_1 \) to be \( L_1 = 0.0241 \) bit. This means two things.

First, if one could model the S&P 500 and know initial conditions exactly, that person would know the S&P 500 in that month to one bit of precision. Even at that impossible amount of precision, all predictive power would be lost in about 42 months. This number is obtained by dividing the amount of precision \( 1 \) by \( L_1 (0.0241) \). Another way of looking at this result is that if one is trying to determine what the S&P 500 is going to do next month, looking back more than 42 months is entirely useless as the memory effect of the S&P 500 for those older months will have been completely eroded.

Secondly, the fact that the Lyapunov exponent is positive shows that the S&P 500 does exhibit sensitive dependence on initial conditions. Coupling this with the fractal dimension shows that according to the prevailing definitions being used at this time, the S&P 500 qualifies as a chaotic system. Thus, all of the work done on the stock market under the assumption that the stock market is random becomes very suspect. Chaos theory purports that there is an underlying structure to the S&P 500, and that in theory the S&P 500 can be modeled for at least short term
forecasting, although long term forecasting of the S&P 500 would be impossible.

One possible shortcoming of Peters' work, however, is that when Peters detrends the S&P 500 data using the CPI numbers, any characteristics of a chaotic system demonstrated after that point may be due to the CPI rather than the S&P 500. I will speak more about this issue later.
EXPLANATION OF MY PROCEDURES

Outline
In order to look further at the stock market in general, and Peters' procedure in particular, I decided to look at another index of the New York Stock Exchange. The index I chose to study was the Dow Jones Industrial Average (DJIA). As the S&P 500 which Peters examined is also an index of the New York Stock Exchange, I felt that it would be interesting to examine the DJIA in order to see whether or not this index also supported the contention made by Peters that the stock market was a chaotic system.

Following Peters' procedures, I tested the DJIA to determine if it had a fractal dimension, and whether or not it exhibited sensitive dependence on initial conditions.

Preparing the Data
The data that I used to perform my analysis was the month-end closing prices of the Dow Jones Industrial Average from January 1950 through December 1990. These 41 years of monthly data provided me with 492 observations with which to work. I also used the method employed by Peters of log linear detrending on this data to further prepare the data for testing.

The log linear detrending both smoothed the data stream
without losing any of the correlations which may have been present in the original data stream, and it enabled me to remove the effects of inflation from the original data stream.

I incorporated the log linear detrending in the following manner:

\[ D_i = \ln(P_i) - (a \cdot \ln(CPI_i)) \]

where

- \( D_i \) = the detrended DJIA on month \( i \),
- \( P_i \) = the DJIA month-end closing price on month \( i \), and
- \( CPI_i \) = the Consumer Price Index (CPI) on month \( i \).

The value of \( a \) is a constant obtained by regressing the log of the DJIA against the log of the CPI over the period covered. After regression, the constant \( a \) was selected to be 0.5 since that was the slope of the regression line. What this means is that for every one point that the log of the CPI increased, the log of the DJIA increased two points on average. This effectively removed the effects of inflation from the data stream. See page 24 for the graph of the detrended DJIA from January 1950 - December 1990.

The DJIA month-end closing values were obtained through America Online [11], while the CPI inflation numbers were obtained through the U.S. Bureau of the Census [12]. I adjusted the CPI numbers to reflect a constant dollar amount based on the value of one 1950 dollar.
The Fractal Dimension

Once the data stream had been created, I began to determine the dimension of the DJIA. Using the method outlined by Grassberger and Procaccia [9], I used the correlation integral in order to determine the dimension of the DJIA. This is also the same method that Peters used to determine the dimension of the S&P 500.

Using the $D_1$ data stream that I calculated, I created an $m$-dimensional phase space of $X_k$ data points where each $X_k = (D_k, D_{k+t}, D_{k+2t}, \ldots, D_{k+(m-1)t})$. In the calculation of the $X_k$, $m$ stands for the dimension of the space being created, and $t$ stands for the time increment between coordinates. I used the same relation as Peters that $m \cdot t = Q$, where $Q$ is the mean orbital period of the system. Peters set the product $m \cdot t$ at 48 months, as this is the period of the S&P 500 that he determined through the use of rescaled range analysis. Since the graph of my detrended DJIA data is nearly identical to Peters' graph of his detrended S&P 500 data, I assumed that the mean orbital period of the DJIA was also 48 months.

It should be noted that this mean orbital period value is only used as a benchmark in selecting input parameters for the programs determining the dimension and the largest Lyapunov exponent of the input data stream. If the mean orbital period value is different for the DJIA than it was
for the S&P 500, this could cause problems in obtaining convergence to a fixed value in either or both of the programs. However, once convergence to a fixed value is obtained in each of the programs, the input parameters are essentially irrelevant. Once each program converges to a fixed value, that is the correct value per the program, regardless of the input parameters. Thus, there is no risk foreseen in assuming a mean orbital period of 48 months for the DJIA.

I then used Peters' program to determine the correlation integral of the detrended DJIA data stream for various values of m. An example of the graph created from using Peters' program with $m = 3$ and $t = 16$ can be seen on page 27. The correlation integral for this $m$ value was obtained by determining the slope of the linear portion of the graph.

The method for determining the dimension $D$ of the phase space proceeded as follows:

1. Begin with a value of $m = 2$, $t$ set accordingly to keep the equation $m \cdot t = Q$, and an arbitrarily small $R$ value.

2. Incrementally increase $R$ (this will increase $C_m$ at a rate of $R^D$).

3. Graph $\ln(C_m)/\ln(R)$ for the increasing $R$ values and find the slope of the graph. This is the dimension $D$ for that particular $m$ value.
CORRELATION INTEGRAL

$M = 3, \; T = 16$
4. Increase the value of \( m \) by one, adjusting \( t \) accordingly, and repeat steps 1-3.

5. The dimension \( D \) will eventually converge to its actual value as \( m \) is increased.

After using this procedure, I obtained the following correlation integrals (CI) for each different embedding dimension (\( m \)) used:

<table>
<thead>
<tr>
<th>( m )</th>
<th>CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.69</td>
</tr>
<tr>
<td>3</td>
<td>2.07</td>
</tr>
<tr>
<td>4</td>
<td>2.15</td>
</tr>
<tr>
<td>5</td>
<td>2.16</td>
</tr>
<tr>
<td>6</td>
<td>2.17</td>
</tr>
<tr>
<td>7</td>
<td>2.17</td>
</tr>
</tbody>
</table>

Per the work done by Grassberger and Procaccia [9], this shows that the dimension of the DJIA is approximately 2.17. See page 29 for a graph of the correlation integral converging to the dimension as the embedding dimension is increased.

Similarly to Peters' work with the S&P 500 which exhibited a fractal dimension of the S&P 500 of 2.33, the fractal dimension of the DJIA at 2.17 shows that if the DJIA is indeed a chaotic system, then the minimum number of equations needed to model the system is only three - the next integer above the fractal dimension. Thus, modeling of the DJIA may someday become a reality.
DIMENSION = 2.17
DJIA 1950 - 1990
Sensitive Dependence on Initial Conditions

The second item which must be shown in order support that a system is indeed chaotic is the existence of sensitive dependence on initial conditions. The method I employed was to determine the largest Lyapunov exponent of the phase space by use of the Wolf algorithm. If the largest Lyapunov exponent is positive, the system possesses sensitive dependence on initial conditions; if the largest Lyapunov exponent is zero or negative, no such dependence exists.

In the Wolf algorithm, the data stream, an embedding dimension, a time lag, the time between data samples, an evolution time, and the maximum and minimum allowable distance between points all had to be chosen and input into the computer. A combination of these values which would result in convergence to a fixed value was needed. If convergence to a fixed value could be achieved, that fixed value would be the largest Lyapunov exponent.

Wolf et al. give many guidelines as to how to pick values for these different parameters [10], and it was these guidelines that I used. Peters also used these same guidelines in his work, and an explanation of each parameter can be found in the previous section of this paper under the explanation of Peters' procedure.
Selection of the Embedding Dimension $m$

I began attempting to find the largest Lyapunov exponent using an embedding dimension of four, which is the same value used by Peters. However, after many failed attempts with this dimension, I tried an embedding dimension of five. According to Wolf et al., an embedding of the phase space should occur if the embedding dimension is selected to be greater than twice the dimension of the phase space, and a surface looks smoother in a larger dimension [10], so the increase in dimension seemed reasonable. In fact, it was this value of $m = 5$ which I was using when I did achieve convergence to a fixed value.

Selection of the Time Lag $t$

To select the time lag, I used the Wolf et al. notion that at a maximum, the embedding dimension times the time lag should not be much greater than the mean orbital period [10]. With my value of $m = 5$, I chose a time lag of $t = 10$ since I was assuming a mean orbital period of 48 months.

Selection of the Time Between Data Samples

The time between data samples refers to how much time passes between first coordinates of the points in the phase space. Wolf et al. merely say that this value is used to normalize the exponent, but gives no guidance as to what
values have been found to work in experimentation [10]. Peters also gives no indication of what values he used in running the Wolf algorithm on the S&P 500 data and I settled on a value of eight through trial and error. In the algorithm, this parameter is labeled DT, so I attained convergence with a value of DT = 8.

**Selection of the Evolution Time**

The evolution time should be long enough to measure stretching without measuring folds. A short evolution period results in more calculations, but requires fewer replacements and results in a more stable convergence. Peters feels that the shorter the evolution time the better, and he obtained convergence using an evolution time of six months [5]. I also used an evolution time of six months which appears in the algorithm as EVOLV = 6.

**Selection of the Maximum and Minimum Distance Between Data Points**

The next thing that must be determined in order to calculate the Lyapunov exponent is the maximum and minimum distance between data points before a replacement point is selected. Wolf et al. suggests using a number of no more than 10 percent of the length of the attractor in phase space [10]. Peters uses 10 percent of the difference
between the minimum and maximum values of the time series for his maximum distance, and 10 percent of the maximum distance for his minimum distance [5]. Following those guidelines, since the maximum and minimum time series numbers in the detrended DJIA were 7.13 and 5.31 respectively, my maximum allowable distance was set at 0.182, and my minimum allowable distance was set at 0.018. In the algorithm these numbers appear respectively as SCALMX = 0.182 and SCALMN = 0.018.

Selection of the Minimum Time Between Pairs
Peters has one additional input parameter that is not in the Wolf algorithm. Peters denotes this parameter LAG, and it represents the minimum time between pairs. The Wolf algorithm has this fixed at 10, but I obtained convergence with LAG = 9.

Calculation of the Lyapunov Exponent
Once the input parameters that have just been mentioned were set as above the largest Lyapunov exponent was found using the Wolf algorithm.

What the Calculated Lyapunov Exponent Means
Using this algorithm, the largest Lyapunov exponent \( L_1 \) of the DJIA was found to be \( L_1 = 0.0209 \) bit. See page
LYAPUNOV EXPONENT

\[ L = 0.0209 \]
34 for the graph of the convergence of $L_1$.

First, this means that if one could model the DJIA and know initial conditions exactly, all predictive power would be lost in about 48 months. This number is obtained by dividing the amount of precision (1) by $L_1 (0.0209)$. Another way of looking at this result is that if one is trying to determine what the DJIA is going to do next month, looking back more than 48 months is entirely useless as the memory effect of the DJIA for those older months will have been completely eroded.

Secondly, the fact that the Lyapunov exponent is positive shows that the DJIA does exhibit sensitive dependence on initial conditions. Coupling this with the fractal dimension shows that according to the prevailing definitions being used at this time, the DJIA qualifies as a chaotic system. Thus, all of the work done on the stock market under the assumption that the stock market is random becomes very suspect. Chaos theory purports that there is an underlying structure to the DJIA, and that in theory the DJIA can be modeled for at least short term forecasting, although long term forecasting of the DJIA would be impossible.
CONCLUSIONS

It should be noted at this point that although the stock market has exhibited the characteristics of a chaotic system, the actual system of equations needed to model the movements of the stock market are far from becoming a reality. The existence of chaos merely states that such a system of equations exists, but chaos theory does not aid in the discovery of the equations themselves.

As referred to earlier, the next logical step would be to check the CPI for the possible existence of chaos. If no chaos is detected in the CPI, one can feel more certain that the stock market itself is chaotic. However, if the CPI is chaotic, it does not mean that the stock market is not chaotic, but it does show that the stock market would need to be tested again for chaos using some other means of removing inflation. Of course, the new index used to remove inflation should be shown to not be chaotic or the same problem will once again surface.

Under the assumption that the stock market is indeed chaotic, the quest for the system of three nonlinear differential equations which can model the stock market in the short-term should now follow. Assuredly, many people have been attempting to model the stock market for many years. Chaos theory has shown what to look for, now one
must determine what to base the system of equations on, and then define the equations which will actually work.
APPENDIX A: CORRELATION INTEGRAL PROGRAM

DIM X(2000)
DIM Z(1000, 10) 'EMBEDDING DIMENSIONS OF UP TO 10 ALLOWED
PRINT "INPUT NPT, DIMEN, TAU, DT, R:
INPUT NPT 'NUMBER OF OBSERVATIONS
INPUT DIMEN 'EMBEDDING DIMENSION
INPUT TAU 'TIME LAG FOR RECONSTRUCTING PHASE SPACE
INPUT DT 'INCREMENTS TO DISTANCE
INPUT R 'INITIAL DISTANCE
THETA = 0: THETA2 = 0: CR = 0: IND = 1:
K = 1: LAG = 0: SUM = 0: ITS = 0
OPEN "DELAY.PRN" FOR INPUT AS 1 LEN = 2000 'INPUT FILE
OPEN "CORDIM.PRN" FOR OUTPUT AS 2 LEN = 2000
VT$ = "####.####
FOR I = 1 TO NPT 'READ INPUT FILE
FOR J = 1 TO DIMEN
Z(I, J) = X(I + (J - 1)) * TAU) 'RECONSTRUCT THE PHASE SPACE
NEXT J
NEXT I
NPT = NPT - DIMEN * TAU 'MAXIMUM LENGTH OF PHASE SPACE
320 FOR K = 1 TO NPT
FOR I = 1 TO NPT
D = 0
FOR J = 1 TO DIMEN
D = D + (Z(LAG J) - Z(I, J)^2 'SQUARE OF DISTANCE
NEXT J
D = SQR(D) 'CALCULATION OF DISTANCE
IF D > R THEN THETA2 = 0 ELSE THETA2 = 1 'DISTANCE > R?
THETA = THETA + THETA2 'COUNTING POINTS
NEXT I
LAG = LAG + 1
NEXT K
CR = (1 / (NPT^2)) * THETA 'CALCULATING CORRELATION INTEGRAL
LPRINT USING VT$; CR; R 'PRINT FILE
L = L+1: IF L > 12 THEN END
R = R + DT
CR = 0: THETA = 0: THETA2 = 0: LAG = 0
GOTO 320
500 END
APPENDIX B: LARGEST LYAPUNOV EXPONENT PROGRAM

DIM X(1000), PT1(12), PT2(12)
DIM Z(1000, 5) 'ACCEPTS UP TO 5 DIMENSIONS
OPEN "LYAP.PRN" FOR OUTPUT AS 2 LEN = 500
VT$ = "###.##### ###.##### ###.##### ###.##### "
PRINT "NPT, DIM, TAU, DT, SCALMX, SCALMN, EVOLV, LAG?"
INPUT NPT 'NUMBER OF OBSERVATIONS
INPUT DIMEN 'EMBEDDING DIMENSION
INPUT TAU 'LAG TIME FOR PHASE SPACE
INPUT DT 'TIME BETWEEN DATA SAMPLES
INPUT SCALMX 'MAXIMUM DIVERGENCE
INPUT SCALMN 'MINIMUM DISTANCE
INPUT EVOLV 'EVOLUTION TIME
IND = 1 'POINTS TO FIDUCIAL TRAJECTORY
PRINT "NPT, DIM, TAU, DT, SCALMX, SCALMN, EVOLV, LAG?"
INPUT LAG 'TIME BETWEEN PAIRS
SUM = 0 'HOLDS RUNNING EXPONENT MINUS ONE DIVIDED BY TIME
ITS = 0 'TOTAL NUMBER OF PROPAGATION STEPS
OPEN "DELAY.PRN" FOR INPUT AS 1 LEN = 2500 'INPUT FILE
PRINT "READING DATA"
FOR I = 1 TO NPT
    INPUT #1, X(I) NEXT I
PRINT "DATA READ"
FOR I = 1 TO NPT - (DIMEN - 1) * TAU
    FOR J = 1 TO DIMEN
        Z(I, J) = X(I + (J - 1) * TAU) 'RECONSTRUCT PHASE SPACE
    NEXT J
    NEXT I
PRINT "DATA FORMATTED"
NPT = NPT - DIMEN * TAU - EVOLV 'MAX LENGTH OF PHASE SPACE
DI = 1000000000
FOR I = (LAG + 1) TO NPT 'FIND INITIAL PAIR
    D = 0
    FOR J = 1 TO DIMEN
        D = D + (Z(IND, J) - Z(I, J)) ^ 2 'CALCULATE DISTANCE
    NEXT J
    D = SQR(D)
    IF (D > DI) OR (D < SCALMN) GOTO 390 'STORE BEST POINT
    DI = D
    IND2 = I 'POINTS TO SECONDARY TRAJECTORY
    NEXT I
40
390 NEXT I
400 FOR J = 1 TO DIMEN 'COORDINATES OF EVOLVED POINTS
   PT1(J) = Z(IND + EVOLV, J)
   PT2(J) = Z(IND2 + EVOLV, J)
NEXT J
DF = 0
FOR J = 1 TO DIMEN 'COMPUTE FINAL DIVERGENCE
   DF = DF + (PT2(J) - PT1(J))^2
NEXT J
DF = SQR(DF)
ITS = ITS + 1
SUM = SUM + (LOG(DF / DI) / (EVOLV * DT * LOG(2)))
ZLYAP = SUM / ITS
LPRINT USING VT$; ZLYAP; EVOLV * ITS; DI; DF
INDOLD = IND2
ZMULT = 1
ANGLMX = .3
570 THMIN = 3.14
'LOOK FOR REPLACEMENT POINTS
FOR I = 1 TO NPT
   III = ABS(INT(I - (IND + EVOLV)))
   IF III < LAG GOTO 780 'REJECT IF REPLACEMENT POINT IS TOO
   CLOSE TO ORIGINAL
   DNEW = 0
   FOR J = 1 TO DIMEN
      DNEW = DNEW + (PT1(J) - Z(I, J))^2
   NEXT J
   DNEW = SQR(DNEW)
   IF (DNEW > ZMULT * SCALMX) OR (DNEW < SCALMN) GOTO 780
   DOT = 0
   FOR J = 1 TO DIMEN
      DOT = DOT + (PT1(J) - Z(I, J)) * (PT1(J) - PT2(J))
   NEXT J
   CTH = ABS(DOT / (DNEW * DF))
   IF (CTH > 1) THEN CTH = 1
   TH = COS(CTH)
   IF (TH > THMIN) GOTO 780
   THMIN = TH
   DII = DNEW
   IND2 = I
780 NEXT I
IF (THMIN < ANGLMX) GOTO 870
ZMULT = ZMULT + 1
IF (ZMULT < 5) GOTO 570
ZMULT = 1
ANGLMX = 2 * ANGLMX
IF (ANGLMX < 3.14) GOTO 570
IND2 = INDOLD + EVOLV
DII = DF
870 IND = IND + EVOLV
IF (IND >= NPT) GOTO 910
DI = DII
GOTO 400
910 END
REFERENCES


