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Tutte-Equivalent Matroids

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Tutte-Equivalent Matroids

A Thesis

Presented to the

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In Partial Fulfillment

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in

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Abstract

Matroid theory was introduced by Hassler Whitney in 1935. Whitney strived to capture an abstract notion of independence. A matroid is composed of a finite set E, called the ground set, together with a rule that tells us what it means for subsets of E to be independent. We begin by introducing matroids in the context of finite collections of vectors from a vector space over a specified field, where the notion of independence is linear independence. Then we will introduce the concept of a matroid invariant. Specifically, we will look at the Tutte polynomial, which is a well-defined two-variable invariant that can be used to determine differences and similarities between a collection of given matroids. The Tutte polynomial can tell us certain properties of a given matroid (such as the number of bases, independent sets, etc.) without the need to manually solve for them. Although the Tutte polynomial gives us significant information about a matroid, it does not uniquely determine a matroid. This thesis will focus on non-isomorphic matroids that have the same Tutte polynomial. We call such matroids Tutte-equivalent, and we will study the characteristics needed for two matroids to be Tutte-equivalent. Finally, we will demonstrate methods to construct families of Tutte-equivalent matroids.

Acknowledgements

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Chapter 1

Introduction to Matroid Theory

Matroid theory was first introduced by Hassler Whitney in 1935, when Whitney noticed properties of dependence common to graphs and matrices [Wel10]. Overall matroid theory combines important ideas from linear algebra, graph theory, and finite geometry [GM12]. Since its debut, matroid theory has grown significantly and remains an important research area in mathematics.

1.1 Definitions

Although a matroid can be defined in many cryptomorophic ways, we will use Whitney's initial definition in terms of a generalized notion of independence. A matroid M is built on a finite set of elements, E, called the *ground set*. Together with a family, $\mathcal{I}$, of subsets of E called *independent sets*.

Definition 1.1.1. A matroid M consists of a finite set E, together with a family of *independent subsets* $\mathcal{I}$ of E, such that $\mathcal{I}$ satisfies the following conditions:
(I1) Non triviality: $\mathcal{I} \neq \emptyset$;
(I2) Closed under subsets: If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;
(I3) Augmentation: If $I, J \in \mathcal{I}$ with $|I| < |J|$, then there exists an element $x \in J - I$ such that $(I \cup x) \in \mathcal{I}$.

If M is a matroid on the ground set E and $X \subseteq E$, then the rank of X, denoted

$r(X)$, is the cardinality of the largest independent set contained in X.

Definition 1.1.2. The *rank* of a matroid M is a non-negative increasing sub-modular function on subsets of E. That is, if $A, B \subseteq E$, then:

(r1) $0 \leq r(A) \leq |A|$;

(r2) If $A \subseteq B$, then $r(A) \leq r(B)$;

(r3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

The rank also tells us the dimension of the geometry associated with a matroid, according to the equation *rank = dimension + 1.* The maximal independent sets of a matroid are called *bases* and the collection of all bases is denoted $\mathcal{B}(M)$. An element that is common to every basis is called a *coloop*, and an element that is in no basis is called a *loop.* Sets of a matroid that are not independent are said to be *dependent.* A minimal dependent set is called a *circuit*, and the collection of all circuits is denoted $\mathcal{C}(M)$.

Example 1.1.1. Consider the matroid M on the ground set $E = \{a, b, c, d\}$ and with $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, abd, acd, bcd\}$. Note that $\mathcal{I}$ satisfies axioms $(I1)$, $(I2)$, and $(I3)$.

- The cardinality of a maximal independent set (basis) (e.g, {abd}, {acd}), is three. Therefore, the rank of M is three.
- Notice that the element d is contained in every basis. Therefore, d is a coloop.

When a student first hears the word matroid, they might initially think of a matrix. Indeed, a matrix over a given field is and example of a matroid. More precisely, *representable matroids* are those whose ground set can be represented by the column vectors of a matrix over a field. We declare a subset of column vectors to be independent whenever the vectors are linearly independent. A matroid that is representable over the two element field $\mathbb{F}_2$ is said to be *binary*, and matroids that are representable over $\mathbb{F}_3$ are called *ternary* matroids. A matroid that can be represented over all fields is called a *regular matroid.* One shall keep in mind, however, that not all matroids are representable. However, having a finite set of vectors as the ground set when working with representable matroids allows us to see the independence axioms that define matroids in a more familiar setting.

Example 1.1.2.

$$\mathbf{A} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Let $E = \{a, b, c, d, e, f\}$ be the set of column vectors of matrix A. Note that A is a rank 3 matrix over the field $\mathbb{F}_2$. Now consider the column vectors having nontrivial linear combinations that result in a zero vector. These will be the dependent sets of the associated matroid. All other sets of column vectors will be independent.
Notice the following:

- Since f is the zero vector, f is a loop. However, any single element from $E - \{f\}$ is independent.
- There are no dependent sets of column vectors of size 2 in $E - \{f\}$. So, any combination of two vectors from $E - \{f\}$ is independent.
- Since, $(1, 0, 0) + (0, 1, 0) - (1, 1, 0) = (0, 0, 0)$ and $(0, 1, 0) + (0, 0, 1) - (0, 1, 1) = (0, 0, 0)$, we see that {a,b,d} and {b,c,e} are dependent sets. All other combinations of three vectors taken from $E - \{f\}$ will be independent.
- Any combination of four vectors is dependent.
- Thus, $\mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, de, abc, abe, ace, acd, ade, bde, bcd, cde\}$. One can check that $\mathcal{I}$ satisfies the independence axioms $(I1)$, $(I2)$, and $(I3)$.

Spanning sets are subsets $A \subseteq E$ such that $r(A) = r(E)$, and the collection of all spanning sets of a matroid M is denoted $\mathcal{S}(M)$. A subset $F \subseteq E$ is a *flat* if F is *rank maximal*, meaning that for all $e \in E - F$, we have $r(F \cup e) > r(F)$. It is important to note that loops will be found in every flat (since loops have rank 0 and never increase the rank when added to a set). A *hyperplane* is a flat of rank $r(M) - 1$, and the collection of all hyperplanes of a matroid is denoted $\mathcal{H}(M)$.

Definition 1.1.3. If M is a matroid with independent sets, $\mathcal{I}(M)$, then the maximal independent sets are the *bases* $\mathcal{B}(\mathcal{M})$ of our matroid M if and only if they satisfy the following:

- (B1) $\mathcal{B}(\mathcal{M}) \neq \emptyset$.
- (B2) If $B_1, B_2 \in \mathcal{B}(M)$ and $B_1 \subseteq B_2$, then $B_1 = B_2$.
- (B3) If $B_1, B_2 \in \mathcal{B}(M)$ and $x \in B_1 - B_2$, then there exist $y \in B_2 - B_1$ such that $(B_1 - \{x\} \cup \{y\}) \in \mathcal{B}(M)$.

The properties that the collection of bases, $\mathcal{B}$, will be used to define a matroid in terms of bases in future lemmas.

1.2 Geometries

There are many perspectives from which to view matroids. One of the most advantageous perspectives is that of finite geometry, which is the perspective we will emphasize in this thesis.

Example 1.2.1. Consider the following matrix:

$$\begin{array}{c} \\ \mathbf{A} = \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Although one can easily find all the dependent and independent sets from A, this is not an ideal approach since not all matroids can be represented by a matrix. As the number of elements and rank of a matroid increases, creating a geometry from the given matrix will facilitate our research. The rank of A is two, and $\mathcal{I}(A) = \{\emptyset, a, b, c, d, ab, ac, bc, bd, cd\}$. Note the rank of a matroid also tells us the dimension of the associated geometry. So A will be a one-dimensional geometry. Next, we will need to follow some simple guidelines to create our geometry. First, we plot the vectors in a coordinate plane. Second, we place a line in "free" position, by making sure it is not parallel to any of the vectors. Last, we extend or reduce the magnitude of each vector, and/or reverse the direction of each vector so all vectors intersect the free line. Our matroid geometry will consist of the free line and the points on the line that result from the projected vectors (see Fig 1.1). Note, that any scalar multiple of a given vector will result in the same projected point in the geometry.

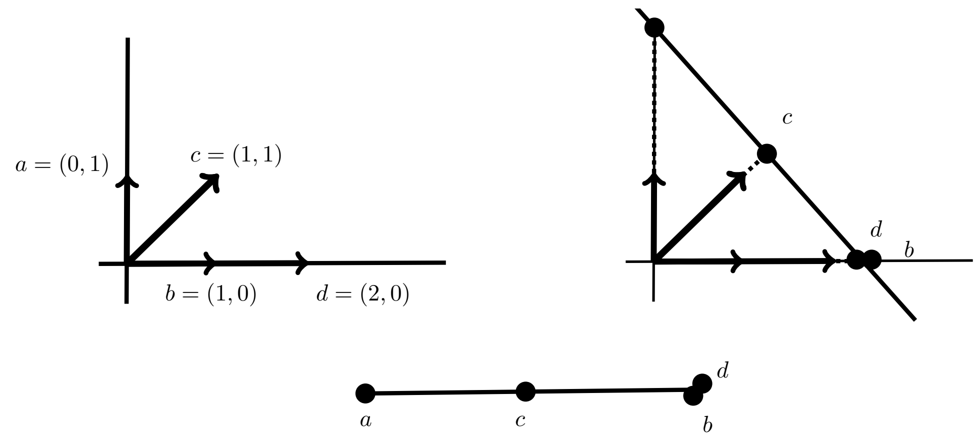

Figure 1.1: The geometric representation of the matroid derived from the matrix A.

1.3 Binary Matroids

A *binary matroid* is one that can be represented by a matrix over the field $\mathbb{F}_2$. Binary matroids play an important role since they contain unimodular matroids and graphic matroids, two fundamental classes in matroid theory [Fou87]. Additionally, binary matroids have many attractive properties that make them an interesting class of matroids for research.

Proposition 1.3.1. *If a matroid M is binary, then each of its minors are also binary.*

Matroid duality will be discussed in Section 1.4, but the following result speaks to the importance of matroid duality in that it behaves well with respect to matroid represent-ability over a given field.

Proposition 1.3.2. *If a matroid M is binary, its dual matroid M^* is also binary.*

Example 1.3.1. A rank r matroid on a ground set with n elements is called a *uniform matroid* if all of its circuits are of size $r+1$. We denote such a matroids by $U_{r,n}$. Uniform matroids play a key role in matroid represent-ability. If we are trying to determine whether a given matroid can be represented over a certain field, we look for the existence of smaller matroids "contained within" the given matroid. This notion of "contained within" will be made more precise in Section 1.4.1, where we will explore the concept of matroid minors. For now, an important illustration of this idea is the following theorem.

Theorem 1.3.1. *A matroid is binary if and only if it does not contain the uniform matroid* $U_{2,4}$ *as a minor.*

Binary matroids can be characterized in many ways aside from not having a $U_{2,4}$ minor. Below are some useful characterizations of binary matroids. Note, if X and Y are sets in M, the *symmetric difference*, denoted $X\Delta Y$, is equal to $(X - Y) \cup (Y - X)$.

Theorem 1.3.2. *Given a matroid* M*, the following are equivalent:*

(i) M *is binary.*

(ii) If C_1 *and* C_2 *are distinct circuits, then* $C_1\Delta C_2$ *contains a circuit.*

(iii) If C_1 *and* C_2 *are circuits, then* $C_1\Delta C_2$ *is a disjoint union of circuits.*

(iv) M *has no minor isomorphic to* $U_{2,4}$.

(v) Every rank $r - 2$ *flat of* M *is contained in at most three hyperplanes.*

1.4 Duality

An important tool in matroid theory is the ability to construct new matroids from those we already know about. One of the ways to create a new matroid is through the concept of matroid *duality*. The theory of duality is one of the most important tools at our disposal when attempting to solve many matroid theory problems. This theory was first introduced by Whitney in 1935 [Oxl11]. The dual of a matroid M is another matroid, written M^*, and it is defined on the same ground set E as follows:

Definition 1.4.1. Let M be a matroid on the ground set E. The *dual matroid* M^* will have the same ground set E such that

$$\mathcal{B}(M^*) = \{E - B : B \in \mathcal{B}(M)\}.$$

The rank of a subset A in M^* is denoted $r^*(A)$, and can be found by the following relationship:

Theorem 1.4.1. *Let M be a matroid with ground set E and rank function r. If $A \subseteq E$, then*

$$r^*(A) = r(E - A) + |A| - r(M).$$

Proof. Proof of this theorem can be found in [Oxl11]. □

Since the rank of a matroid is determined by the cardinality of its bases, and since the bases of the dual matroid are the complements of the bases of the original matroid, we get the following:

Proposition 1.4.1. $|E| - r(M) = r(M^*)$

The compliments of spanning sets, hyperplanes, and circuits also play an important role in matroid duality.

Proposition 1.4.2. *Let M be a matroid on the ground set E and suppose $S \subseteq E$. Then S is spanning if and only if $E - S$ is an independent set in the dual M^*.*

Proposition 1.4.3. *Let M be a matroid on the ground set E. Then circuits and hyperplanes of the dual matroid M^* are determined as follows:*

(i) $\mathcal{C}(M^*) = \{E - H : H \in \mathcal{H}(M)\}$

(ii) $\mathcal{H}(M^*) = \{E - C : C \in \mathcal{C}(M)\}$

Proof. (i) Let H be a hyperplane in M. If H is non-spanning, then there exists an $x_i \in E - H$ such that $H \cup x_i$ is spanning. Then $E - H$ is a dependent set in M^*, and $(E - H) - x_i$ is independent in M^*. Therefore, $E - H$ is a circuit in M^*.
(ii) Let $C \in \mathcal{C}(M)$, so C is a minimal dependent set and is therefore not contained in any basis of M. This implies that $E - C$ is maximal with respect to not containing a basis complement. Thus, $E - C \notin B^*$ and $r^*(E - C) < r^*(B^*)$. Therefore $E - C$ is a hyperplane in M^*. □

At times a matroid will contain a set that is both a circuit and a hyperplane. Let the set of circuit-hyperplanes of a matroid M be denoted $\mathcal{U}(M)$.

Corollary 1.4.1.1. *Let M be a matroid with the ground set E. If U is a circuit-hyperplane of M then its compliment will also be a circuit-hyperplane in M^*. That is, $\mathcal{U}(M^*) = \{E - U : U \in \mathcal{U}\}$.*

Example 1.4.1. Consider the matroid M with ground set $E = \{a, b, c, d, e\}$ in Figure 1.2.

- By Proposition 1.4.1, $r(M^*) = 2$.
- The basis of M are $\mathcal{B} = \{abd, abe, acd, ace, bdc, bce, bde, ade\}$, therefore the bases of M^* will be $\mathcal{B}^* = \{ce, cd, bd, ae, ad, ac, bc, be\}$
- Since $\mathcal{C}(M) = \{abc, cde, bdae\}$, we see that $\mathcal{H}(M^*) = \{de, ab, c\}$.
- By Corollary 1.4.1.1, the circuit-hyperplanes in M are $\mathcal{U}(M) = \{abc, cde\}$, therefore $\mathcal{U}(M^*) = \{de, ab\}$.

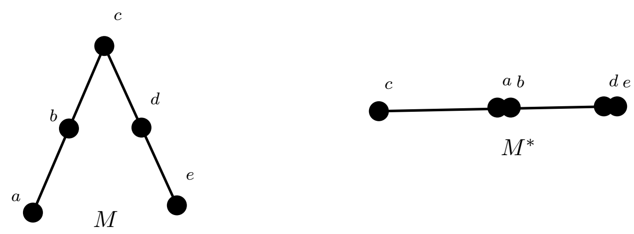

Figure 1.2: A rank 3 matroid M and its dual M^*.

As you may have predicted, loops and coloops also have an interesting relationship in duality. We already know the bases of the dual matroid M^* are the complements of the bases of M. Since a coloop e is in every basis of M, we can conclude that e will be in no basis of M^*. Therefore, we have the following:

Proposition 1.4.4. *Given a matroid M, e is a loop of M if and only if e is a coloop of M^*.*

For our research. matroid duality will allow us to work with some matroids of higher ranks that might otherwise be difficult to work with (matroids of rank 5, 6, 7...). If the dual of the matroid in question lives in rank 0, 1, 2, 3, or 4 it sometimes is easier for the researcher to analyze the dual of the matroid in question. We can do this because the dual of the dual gives us to the original matroid.

Proposition 1.4.5. *Given a matroid M, we have $(M^*)^* = M$.*

1.4.1 Deletion and Contraction

Another great tool in matroid theory is the *deletion* and *contraction* operations.Construction new matroids from a given matroid through a sequence of these operations produces something called a *matroid minor*. Both operations reduce the size of the ground set E by removing a chosen element.

Definition 1.4.2. Let M be a matroid with ground set E and independent sets $\mathcal{I}$.

1. ***Deletion***: For $e \in E$, where e is not an coloop. The matroid $M - e$ has ground set $E - \{e\}$. The independent sets of $M - e$ are the sets of $\mathcal{I}$ that do not contain e.

2. ***Contraction***: For any element $e \in E$, where e is not a loop, the matroid M/e has ground set $E - \{e\}$ and it's independent sets are formed by selecting all the members of $\mathcal{I}$ that contain e, and then removing e from such sets.

From Definition 1.4.2, we can see why it would be inconsistent to delete an coloop e. Recall, that coloops are in every basis of M. Our matroid $M - e$ would have no bases, which implies that $\mathcal{I} = \emptyset$, violating $(I1)$. Now lets see why we do not contract loops. Assume e is a loop, this implies that $e \notin \mathcal{I}(M)$. We would have no independent sets to remove e from when creating $\mathcal{I}(M/e)$. As we begin to practice these operations it would be easier for us to start by obtaining a list of all the independent sets of the original matroid M. We then separate the independent sets into two lists. The first those containing our element e, and our second, those that do not contain e. When applying the contraction operation, we will use our first list and delete e from any independent set. When using our deletion operation, we will simply use our second list without any modifications. Now, since contraction instructs us to delete e from our list of independent sets, we are also decreasing the cardinality of our bases by one. This leads us to the following proposition.

Proposition 1.4.6. *Let M be a matroid.*

1. *If e is not an coloop, then $r(M - e) = r(M)$*

2. *If e is not a loop, then $r(M/e) = r(M) - 1$*

Example 1.4.2. Consider Figure 1.3. The matrid M has ground set $E = \{a, b, c, d\}$ and independent sets $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ad, bc, cd\}$.

- For $M - a$ we adopt all the independent sets of M that do not contain a. So $\mathcal{I}(M-a) = \{\emptyset, b, c, d, bc, cd\}$. Also, our new matroid $M - a$ has remained in rank 2.

- For M/a we adopt all the independent sets of M that contain a, which are $\{a, ab, ad\}$, and delete a from each set. Therefore $\mathcal{I}(M/a) = \{\emptyset, b, d\}$. Note that c is still in the ground set of M/a, and since c is in no independent set, then c is a loop. Also note, by Proposition 1.4.6, our new matroid M/a has decreased to rank 1.

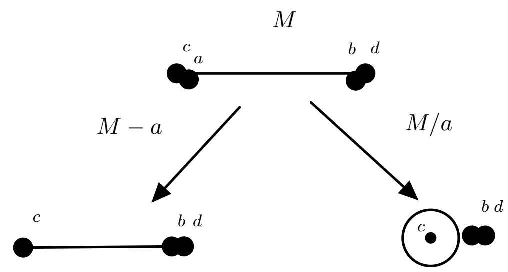

Figure 1.3: An example of a rank 2 matroid M, and the two matroids, $M - a$ and M/a, after deleting an contracting a from M.

Although, the conversions of our independent sets give us a lot of information about our new minor, we also need to give some attention to what occurs to the circuits and hyperplanes of M.

Proposition 1.4.7. *Let M be a matroid and suppose e is neither a coloop nor a loop. Then*

1. ***Deletion:*** *C is a circuit of $M - e$ if and only if $e \notin C$ and C is a circuit of M.*

2. ***Contraction:*** *C is a circuit of M/e if and only if*

   - *$C \cup e$ is a circuit of M, or*
   - *C is a circuit of M and $C \cup e$ contains no circuits except C.*

Proposition 1.4.8. *Let M be a matroid and suppose e is not a loop. Then $\mathcal{H}(M/e) = \{T - \{e\} : T \in \mathcal{H}(M)\}$.*

The process of deletion and contraction can be completed numerous times on the same matroid. We can delete elements of a matroid until one element remains in the ground set. Or, we can contract elements of a matroid until we've arrived at a rank 0 matroid. It is also important to note that the order of operations does not matter. If we delete a and contract b from matroid M the resulting matroid will be the same as if we first contracted b and then deleted a. This gives us the following proposition:

Proposition 1.4.9. *Let M be a matroid with ground set E, and $a, b \in E$. Assume the elements being contracted are not loops, and the elements being deleted are not coloops. Then,*

- $(M - a) - b = (M - b) - a$*;*
- $(M/a)/b = (M/b)/a$*;*
- $(M/a) - b = (M - b)/a$*.*

When we use the operations of deletion and contraction to create new matroids, we are producing a *minor* of the original matroid. Recall from Proposition 1.4.9 that the order in which we delete and contract the elements of a and b is not important. Our only restriction is to not delete and coloop or contract a loop.

Our next goal is to draw our new matroid minors. The deletion operation maintains the rank of M and only deletes e from the ground set E. The geometry of $M - e$ will look exactly the same as M, except of course, without the element e. The geometry for M/e will require more attention. We know that the rank of M/e will be one less than $r(M)$, which means that our dimension will also decrease. The geometry of M/e is viewed by projecting the elements of $E - \{e\}$ through the point e onto a space of dimension one less than that of the geometry of M.

Chapter 2

Matroid Invariants and Tutte-Equivalent Matroids

2.1 Matroid Invariants

In mathematics a quantity is said to be *invariant* if it remains unchanged under transformations. For example, geometries in a Euclidean space are invariant under isometric transformations. We can find invariants in every field of mathematics, and they are extremely useful in classifying mathematical objects. The determinant, eigenvectors, and eigenvalues of a square matrix are invariant under a change of basis. In graph theory, the *chromatic polynomial* is an invariant. Evaluating the chromatic polynomial yields the number of k-colorings of any graph G, and this is invariant under graph isomorphism. Matroid theory has several invariants. The most notable are the *corank-nullity polynomial* and *Tutte polynomial.* The latter will be the focus of our research. The Tutte polynomial captures a considerable amount of information about the matroid in question. However, not all invariants, including the Tutte polynomial are *complete invariants.*

Definition 2.1.1. A matroid isomorphism invariant is a function f on the class of all matroids such that $f(M) = f(N)$, when $M \cong N$.

The theory of invariants in matroid theory originated in graph theory, in the context of graph flows, and vertex colorings [BO92].

2.2 Corank-Nullity Polynomial

In 1932, Whitney introduced a polynomial that can be computed for any matroid with rank function r [Wel10]. The polynomial is a two variable polynomial, having independent variables x and y.

Definition 2.2.1. The *corank-nullity polynomial* of a matroid M with a ground set E and rank r is defined as:

$$s(M; x, y) = \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-r(A)},$$

where the *corank* of $A \subseteq E$ is $r(E) - r(A)$ and *nullity* is $|A| - r(A)$.

Notice that the corank-nullity polynomial is a matroid invariant. If we are given two isomorphic matroids M and N containing the same ground set, then every subset $A \subseteq E(M)$ will have a corresponding subset in N, $A' \subseteq E(N)$. Thus every term of the corank-nullity polynomial of $s(M; x, y)$ will have a corresponding term in the corank-nullity polynomial of $s(N; x, y)$.

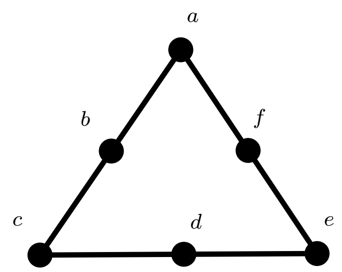

Figure 2.1: The $\mathcal{W}^3$ matroid.

Example 2.2.1. Refer to the matroid $\mathcal{W}^3$ in Figure 2.1. The table below describes the computation of $s(\mathcal{W}^3; x, y)$ by looking at all subsets A of $E(\mathcal{W}^3)$

Polynomial of $\mathcal{W}^3$					
A	$r(A)$	$\lvert A \rvert$	$r(E) - r(A)$	$\lvert A \rvert - r(A)$	$x^{r(E)-r(A)}y^{\lvert A \rvert - r(A)}$
$\emptyset$	0	0	3	0	x^3
a	1	1	2	0	x^2
ab	2	2	1	0	x
abc	2	3	1	1	xy
abd	3	3	0	0	1
$abcd$	3	4	0	1	y^1
$abcde$	3	5	0	2	y^2
E	3	6	0	3	y^3

Adding up the terms in the last column along with counting the possible ways of obtaining our sets gives us:

$$s(M; x, y) = x^3 + 6x^2 + 15x + 3xy + 17 + 15y + 6y^2 + y^3$$

2.3 The Tutte Polynomial

Tutte's motivation for his polynomial was inspired by the *chromatic polynomial*, $\chi(G, \lambda)$. Initially called the *dichromatic polynomial*, Henry Crapo later generalized Tutte's work to matroids, and Brylawski proved many further results concerning the Tutte polynomial [GM12].

Definition 2.3.1. Given a matroid M on ground set E, the *Tutte polynomial*, $t(M; x, y)$, can be computed recursively as follows:
For any element e of M, if e is neither a loop nor an coloop, then

$$t(M; x, y) = t(M - e; x, y) + t(M/e; x, y). \tag{2.1}$$

If e is a loop, then

$$t(M; x, y) = y \cdot t(M - e; x, y). \tag{2.2}$$

If e is an coloop, then

$$t(M; x, y) = x \cdot t(M/e; x, y). \tag{2.3}$$

If $E = \emptyset$, we define

$$t(M; x, y) = 1. \tag{2.4}$$

The Tutte polynomial is a special evaluation of the corank-nullity polynomial. Notice that $r(M)$ is the highest power of x in $t(M;x,y)$ while the nullity, $n(M)$, is the highest power of y in $t(M;x,y)$. Therefore rank and nullity are invariants that are computed by the Tutte polynomial. Additionally, since $|E| = r(M)+n(M)$ the cardinality of the ground set is also a Tutte polynomial invariant [BO92].

Theorem 2.3.1. *The Tutte polynomial for a given matroid M is an evaluation of the corank-nullity polynomial, and a well defined matroid invariant.*

$$t(M;x,y) = s(M;x-1,y-1)$$

Proof. Assume $|E| = 1$. Then e is either a loop or a coloop.
If e is a loop then $r(E) = 0$, and we get

$$\begin{aligned} s(M;x,y) &= x^{r(E)-r(E)}y^{|E|-r(E)} + x^{r(E)-r(\emptyset)}y^{|\emptyset|-r(\emptyset)} \\ &= x^{0-0}y^{1-0} + x^{0-0}y^{0-0} \\ &= y+1. \end{aligned}$$

Therefore $t(M;x,y) = y$, and it follows that $t(M;x,y) = s(M;x-1,y-1)$.
If e is a coloop, then $r(E) = 1$, so

$$\begin{aligned} s(M;x,y) &= x^{1-1}y^{1-1} + x^{1-0}y^{0-0} \\ &= x+1. \end{aligned}$$

Therefore $t(M;x,y) = x$, and it follows that $s(M;x-1,y-1) = t(M;x,y)$.
Now let $|E| \geq 2$, and assume our theorem holds for all matroids on $n-1$ elements. We need to show it holds for matroids on n elements. Let $e \in E$. **Case 1:** Assume e is neither a loop nor a coloop.We will denote r for the rank of M, r' for the rank function of $M-e$, and r'' for the rank function of M/e. Thus, $r'(A) = r(A)$ for all $A \subseteq E - \{e\}$, and $r''(A-e) = r(A) - 1$, when $e \in A$.
The ground set E will be separated into two classes. The class S_1 will be the collection of subsets containing e. The class S_2 is the collection of subsets that do not contain e. Then,

$$s(M;u,v) = \sum_{A \in S_1} u^{r(E)-r(A)} v^{|A|-r(A)} + \sum_{A \in S_2} u^{r(E)-r(A)} v^{|A|-r(A)}.$$

(1) Suppose $A \in S$, where $e \in A$. Since the rank is lowered by one in M/e, then for each of the subsets $r''(A-e) = r(A) - 1$. Also, the corank of A equals the corank of $A - e$ computed in M/e. Thus, we have the following equation:

$$r(E) - r(A) = r''(E-e) - r''(A-e).$$

For the nullity, the rank and cardinality will both decrease by one when we remove e. This implies

$$|A| - r(A) = |A - e| - r''(A-e).$$

Therefore,

$$\begin{aligned}
\sum_{A \in S_1} u^{r(E)-r(A)} v^{|A|-r(A)} &= \sum_{A \in S_1} u^{r''(E-e)-r''(A-e)} v^{|A-e|-r''(A-e)} \\
&= \sum_{B \subseteq E-e} u^{r''(E-e)-r''(B)} v^{|B|-r''(B)} \\
&= s(M/e;u,v)
\end{aligned}$$

(2) Now suppose $A \in S_2$. Since $e \notin A$, then $r(E) = r'(E-e)$ and $r(A) = r'(A)$. Therefore,

$$\begin{aligned}
\sum_{A \in S_2} u^{r(E)-r(A)} v^{|A|-r(A)} &= \sum_{A \in S_2} u^{r'(E-e)-r'(A)} v^{|A|-r'(A)} \\
&= s(M-e;u,v).
\end{aligned}$$

Upon combining these equations, we obtain

$$s(M;u,v) = s(M/e;u,v) + s(M-e;u,v).$$

But, by definition, $t(M;x,y) = t(M/e;x,y) + t(M-e;x,y)$. Applying our induction hypothesis to this equation we have:

$$\begin{aligned}
t(M;x,y) &= s(M/e;x-1,y-1) + s(M-e;x-1,y-1) \\
&= s(M;x-1,y-1).
\end{aligned}$$

Case 2: Assume e is a coloop. We will again partition our subsets into two classes. The class S_1 where $e \in A$ for all $A \in S_1$ and, S_2, where $e \notin A$ for all $A \in S_2$.

(1) Consider the sets A in S_1. The corank and nullity of A computed in M are the corank and nullity of $A-e$ computed in M/e. Therefore, we have

$$\sum_{A\in S_1} u^{r(E)-r(A)}v^{|A|-r(A)} = s(M/e;u,v).$$

Now, if A is a set in S_2, because e is a coloop, we will compare $r(A)$ and $r''(A)$ in M/e. When $e \notin A$, we have $r(A) = r''(A)$ in M/e, since contracting the coloop does not affect the rank of the sets not containing e. So our corank will change in M/e, where $r(E)-r(A) = (r''(E)+1)-r''(A)$, but our nullity will remain unchanged, $|A|-r(A) = |A|-r''(A)$. Then,

$$\begin{aligned}\sum_{A\in S_2} u^{r(E)-r(A)}v^{|A|-r(A)} &= \sum_{A\subseteq E-\{e\}} u^{(r''(E-e)+1)-r''(A)}v^{|A|-r''(A)}\\ &= u\cdot \sum_{A\subseteq E-\{e\}} u^{r''(E-e)-r''(A)}v^{|A|-r''(A)}\\ &= u\cdot s(M/e;u,v).\end{aligned}$$

Adding the results from the sets in S_1, and the sets in S_2, we get

$$\begin{aligned}s(M;u,v) &= s(M/e;u,v) + u\cdot s(M/e;u,v)\\ &= (1+u)\cdot s(M/e;u,v)\end{aligned}$$

But $u = x-1$ and $v = y-1$, so

$$\begin{aligned}s(M;u,v) &= (1+(x-1))s(M/e;x-1,y-1)\\ &= x\cdot t(M/e;x,y).\end{aligned}$$

Case 3: Assume e is a loop. We again partition our subsets in the same manner as in previous cases. For S_1, since e is a loop and $e\in A$, then $r(A) = r'(A)$ in $M-e$. Our corank will remain the same in $M-e$, $r(E)-r(A) = r'(E-e)-r'(A)$, and our nullity will change, $|A|-r(A) = |A|-(r'(A)-1)$. Then,

$$\begin{aligned}\sum_{A\in S_1} u^{r(E)-r(A)}v^{|A|-r(A)} &= \sum_{A\subseteq E-\{e\}} u^{r'(E-e)-r'(A)}v^{|A|-(r'(A)-1)}\\ &= v\cdot \sum_{A\subseteq E-\{e\}} u^{r'(E-e)-r'(A)}v^{|A|-r'(A)}\\ &= v\cdot s(M-e;u,v).\end{aligned}$$

For S_2, since $e \notin A$, we have $r(E) - r'(E-e)$ and $r(A) = r'(A)$. Therefore,

$$\begin{aligned}\sum_{A \in S_2} u^{r(E)-r(A)} v^{|A|-r(A)} &= \sum_{A \subseteq E-\{e\}} u^{r'(E-e)-r'(A)} v^{|A|-r'(A)} \\ &= s(M-e; u, v)\end{aligned}$$

Adding the results from the sets in S_1, and the sets in S_2, we get,

$$\begin{aligned}s(M; u, v) &= v \cdot (M-e; u, v) + s(M/e; u, v) \\ &= (v+1) \cdot s(M-e; u, v).\end{aligned}$$

But $u = x-1$ and $v = y-1$, so

$$\begin{aligned}s(M; u, v) &= (y-1+1)s(M-e; x-1, y-1) \\ &= y \cdot t(M-e; x, y).\end{aligned}$$

□

Although the Tutte polynomial is simply a special evaluation of the corank-nullity polynomial, the Tutte polynomial allows us to recursively compute the polynomial of a geometry or graph by using deletion and contraction. Ultimately, we will arrive at a collection of matroid minors whose Tutte polynomials we know. The sum of these polynomials gives us the Tutte polynomial for the given matroid.

Example 2.3.1. Consider the matroid Q_6 in Figure 2.2. The last row of the figure contains matroids $U_{2,3}$, $U_{1,3}$, $U_{2,2}$, $U_{1,1} \oplus U_{0,1}$, and $U_{0,3}$. The Tutte polynomials of these matroids are:

- $t(U_{2,3}; x, y) = x^2 + x + y$
- $t(U_{1,3}; x, y) = x + y + y^2$
- $t(U_{2,2}; x, y) = x^2$
- $t(U_{1,1}; x, y) \cdot t(U_{0,1}; x, y) = xy$
- $t(U_{0,3}; x, y) = y^3$

Putting these pieces together, we see that $t(Q_6; x, y) = x^3 + 3x^2 + 4x + 2xy + 4y + 3y^2 + y^3$.

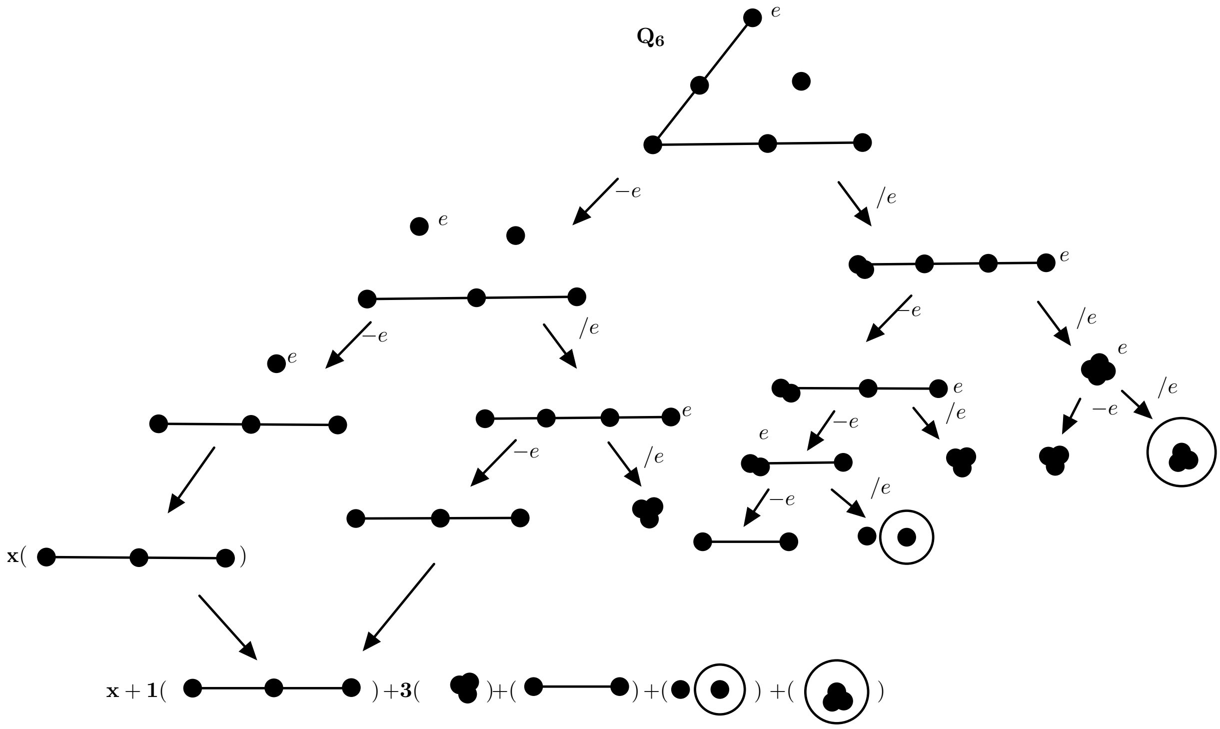

Figure 2.2: Computing the Tutte polynomial of a rank 3 matroid

One can find the rank and cardinality of the ground set visually. In Example 2.3.1 the rank of Q_6 is three and the ground set contains six elements. However, this is only a small amount of information that the Tutte polynomial tells us.

Proposition 2.3.1. *For any matroid M, the Tutte polynomial captures the following information:*

- $t(M;1,1)$ *gives the number of bases of M.*
- $t(M;2,1)$ *gives the number of independent sets of M.*
- $t(M;1,2)$ *gives the number of spanning sets of M.*
- $t(M;2,2) = 2^{|E|}$*, the number of subsets of M.*

The convenient relationship that a matroid has with its dual also applies to the

Tutte polynomial. Basically, if we know the Tutte polynomial of any given matroid, we also know the Tutte polynomial of its dual.

Theorem 2.3.2. *Given a matroid M, the Tutte polynomial of the dual matroid M^* is found by*

$$t(M^*; x, y) = t(M; y, x). \tag{2.5}$$

Proof. We will use the corank-nullity polynomial for this proof. Additionally, recall that that the Tutte polynomial is an evaluation of the corank-nullity polynomial. Therefore,

$$s(M; u, v) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Before we begin, recall the following relationships:
Using Theorem1.4.1, the rank of the set A in the dual matroid is given by

$$r^*(A) = r(E-A) + |A| - r(E).$$

The corank is

$$\begin{aligned} r^*(E) - r^*(A) &= |E| - r(E) - [r(E-A) + |A| - r(E)] \\ &= |E-A| - r(E-A). \end{aligned}$$

Also, the nullity is

$$\begin{aligned} |A| - r^*(A) &= |A| - [r(E-A) + |A| - r(E)] \\ &= r(E) - r(E-A). \end{aligned}$$

Notice the corank of A in M^* is equal to the nullity of $E-A$ in M, and the nullity of A in M^* is equal to the corank of $E-A$ in M.

From the observations, it follows that

$$\begin{aligned}
s(M^*; u, v) &= \sum_{A \subseteq E} (x-1)^{r^*(E)-r^*(A)}(y-1)^{|A|-r^*(A)} \\
&= \sum_{A \subseteq E} (x-1)^{|E-A|-r(E-A)}(y-1)^{r(E)-r(E-A)} \\
&= \sum_{A \subseteq E} (x-1)^{r^*(E)-r^*(A)}(y-1)^{|A|-r^*(A)} \\
&= \sum_{A \subseteq E} (x-1)^{|A'|-r(A')}(y-1)^{r(E)-r(A')}(\because A' = E - A) \\
&= s(M; y-1, x-1).
\end{aligned}$$

□

Our question now is to what extent does this information determine a unique matroid. In other words, given two matroids that have the same Tutte polynomial, is it possible that they will always be isomorphic? The answer is no. If we observe the two matroids in Figure 2.3 we can see that although both matroids having the same Tutte polynomial, the matroids are non-isomorphic. The remainder of this thesis is dedicated to studying such matroids.

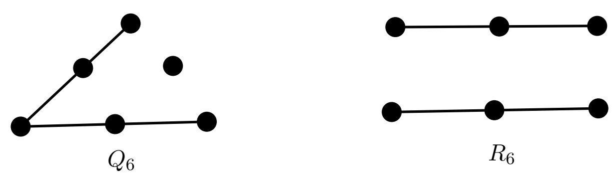

Figure 2.3: The matroids Q_6 and R_6 are the smallest examples of non-isomorphic matroids that have the same Tutte polynomial.

Chapter 3

Matroid Relaxation

3.1 Matroid Relaxation and the Tutte Polynomail

A matroid operation that has allowed one to find other matroids is called *matroid relaxation*. Several well known matroids have been created through this operation, the Fano and non-Fano matroids, Pappus and non-Pappus matroids are just two examples of this operation. Both, matroids have the same ground sets and almost the same bases. This operation in particular will be very fruitful in finding Tutte-equivalent matroids.

Definition 3.1.1. If M_1 and M_2 are two non-isomorphic matroids that have the same Tutte polynomial such that $t(M_1; x, y) = t(M_2; x, y)$, then we say that M_1 and M_2 are *Tutte-equivalent*.

Definition 3.1.2. Let M be a matroid having a subset $A \subset E$ such that, A is both a circuit and a hyperplane. *Relaxing* A will result in a new matroid M' with a new set of bases on the same ground set E such that $\mathcal{B}' = \mathcal{B}(M) \cup A$. Additionally, $\mathcal{C}(M') = \mathcal{C}(M) - A$.

When a circuit hyperplane A is relaxed in matroid M, our new matroid will be denoted M'_A. The pressing issue now, is to prove that relaxing a circuit hyperplane does result in a new matroid.

Theorem 3.1.1. *If A is a circuit hyperplane of a matroid M, the relaxation of A will result in a new matroid M'.*

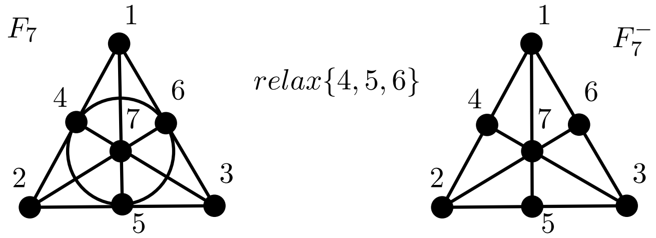

Figure 3.1: The non-Fano matroid F_7^- is a relaxation of the Fano matroid F_7

Proof. Assume M contains a circuit-hyperplane $A \in \mathcal{C}(M)$ and a collection of basis $B_1, B_2, B_3, \ldots B_n \in \mathcal{B}(M)$. By definition, relaxing A will result in a new matroid M'_A where $\mathcal{B}(M') = A \cup \mathcal{B}(M)$. Therefore M' will satisfy our $B3$ axiom in two cases.

Case 1: For all $x \in A$, there exist $y \in B_i$ (for any i) such that $A - \{x\} \cup \{y\} \in \mathcal{B}(M')$. Let $A \in \mathcal{C}(M)$, so $A - \{x\} \in \mathcal{I}(M)$, now assume with out loss of generality, there exist $B_2 \in \mathcal{I}(M)$. So in M we have two independent sets $I_1 = A - \{x\}$ and $I_2 = B_2$, where $|I_1| < |I_2|$. Then by our independent axiom $I3$, there exist an element $y \in I_2 - I_1$ such that $I_1 \cup \{y\} \in \mathcal{I}(M)$. Additionally $I_1 \cup \{y\} \in \mathcal{B}(M)$.

But $I_1 \cup \{y\} = (A - \{x\}) \cup \{y\}$. Since $(A - \{x\}) \cup \{y\} \in \mathcal{B}(M)$ and $\mathcal{B}(M') = \mathcal{B}(M) \cup A$, then we know $(A - \{x\}) \cup \{y\} \in \mathcal{B}(M')$.

Case 2: For all $x \in B_i$ (for any i), there exist $y \in A$ such that $B_i - \{x\} \cup \{y\} \in \mathcal{B}(M')$. Assume $A \in \mathcal{B}(M')$ and $B_2 \in \mathcal{B}(M)$. We need to show that there exist an element $y \in A$ such that $B_2 - \{x\} \cup \{y\} \in \mathcal{B}(M')$. If no such y exist, then $(B_2 - \{x\}) \cup \{y\} = r(M) - 1$ for all $y \in A$. Then we can assume that y is in the closure of $B_2 - \{x\}$ and $r(B_2 - \{x\}) \cup A = r(M) - 1$. A contradiction. □

Being able to create an additional matroid M' from the original M is a powerful tool when we are trying to find matroids having the same rank and ground set. Additionally, if a matroid contains n circuit hyperplanes we can relax said matroid to create at most n new matroids. The next question in order is, once we have relaxed a circuit hyperplane and created a new matroid M' how does it affect the Tutte polynomial. Luckily, one does not need to find the Tutte polynomial from scratch.

Theorem 3.1.2. *If M' is a relaxation of a matroid M, then the Tutte-polynomial of M'*

can be computed as follows,

$$t(M'; x, y) = t(M; x, y) - xy + x + y \tag{3.1}$$

Proof. We will prove this theorem using the corank-nullity polynomial. Let M be a matroid with $B_n \in \mathcal{B}(M)$, and circuit-hyperplanes $A_1, A_2, A_3...A_n \in \mathcal{A}(M)$. The set of circuit-hyperplanes are computed in the corank-nullity polynomial as follows:
Recall the rank and cardinality of A are, $r(\mathcal{A}) = r(E) - 1$ and $|\mathcal{A}| = r(E)$,

$$\sum_{\mathcal{A} \subseteq E} u^{r(E)-(r(E)-1)} v^{r(E)-(r(E)-1)}$$

$$\sum_{\mathcal{A} \subseteq E} u^1 v^1$$

.

The sets of bases, $\mathcal{B}(M)$, will also be computed at follows:

$$\sum_{\mathcal{B} \subseteq E} u^{r(E)-r(E)} v^{r(E)-r(E)}$$

$$\sum_{\mathcal{B} \subseteq E} u^0 v^0$$

The circuit-hyperplanes and basis sections for our corank-nullity polynomial will be described as

$$s(M; u, v) = \ldots + i(uv) + j(1) + \ldots$$

where i represents the number of circuit hyperplanes and j the number of basis found in M.
Without loss of generality we'll now relax the circuit hyperplane A_1 of M. Doing so, we will obtain a new matroid M' on the same ground set E whose basis will become $\mathcal{B}(M') = \mathcal{B}(M) \cup A_1$ and $\mathcal{C}(M') = \mathcal{C}(M) - A_1$.
Hence, the sections of the corank-nullity polynomial of M will change as follows:

$$s(M;u,v) = \ldots + (i-1)(uv) + (j+1)(1) + \ldots$$

Recall that the Tutte polynomial is an evaluation of the corank-nullity polynomial, $t(M;x,y) = s(M;x-1,y-1)$. Then, we can redefine the corank-nullity polynomial of M as:

$$t(M;x,y) = \ldots + i(x-1)(y-1) + j(1) + \ldots$$
$$t(M;x,y) = \ldots + ixy - ix - iy + (i+j) + \ldots$$

Matroid M' will be defined as

$$t(M';x,y) = \ldots + (i-1)(x-1)(y-1) + (j+1)(1) + \ldots$$
$$t(M';x,y) = \ldots + ixy - ix - iy + (i+j) - xy + x + y + \ldots$$

We can see that the difference between the Tutte polynomial of M and M' is $-xy+x+y$. Therefore, when a circuit hyperplane of a matroid M is relaxed, we obtain a new matroid M' whose Tutte polynomial can be derived by

$$t(M';x,y) = t(M;x,y) - xy + x + y$$

□

3.2 Parent and Descendant Matroids

The matroid relaxation tool has given us the ability to find at least one other matroid with the same ground set and rank as our original. However, if a matroid contains at least two or more circuit-hyperplanes it can produce two non-isomorphic matroids as well.

Definition 3.2.1. Assume we have two non-isomorphic matroids M_1 and M_2 where each one contains at least one circuit hyperplane. If we relax each matroids respective circuit-hyperplane so that M_1' and M_2' are isomorphic, then we can define M_1 and M_2 as having a *common relaxation descendant.* See Figure 3.3

Definition 3.2.2. Assume we have a matroid M containing at least two circuit-hyperplanes, A_1 and A_2. If we can relax each circuit-hyperplane individually such that we get two new non-isomorphic matroids M'_{A_1} and M'_{A_2} we can define M'_{A_1} and M'_{A_2} as having a *common relaxation parent.* See Figure 3.2.

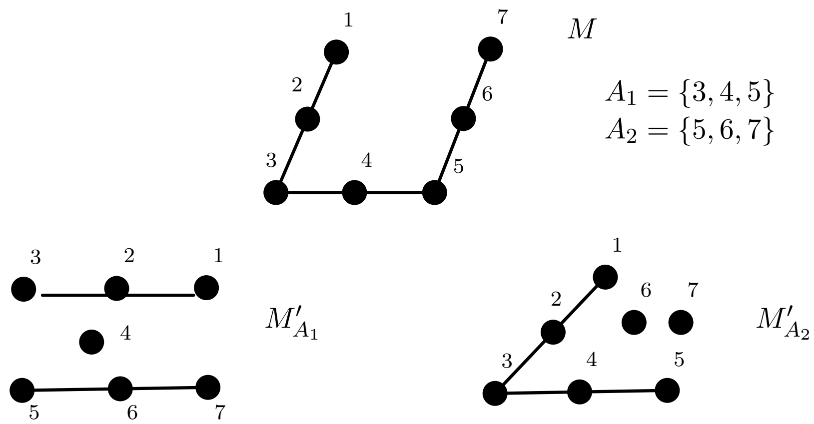

Figure 3.2: Relaxing two different circuit-hyperplanes in M results in two Tutte-equivalent matroids M'_{A_1} and M'_{A_2}

.

Approaching matroids by finding a common relaxation descendant or common relaxation parent allows us to find non-isomorphic matroids that will aways have the same Tutte polynomial.

Corollary 3.2.0.1. *If two matroids, M_1 and M_2 have a common relaxation descendant, such that $M_1' \cong M_2'$ then $t(M_1; x, y) = t(M_2; x, y)$.*

Proof. Assume we have two matroids M_1 and M_2 and we relax each matroid to discover that M_1' and M_2' have isomorphic relaxations. By Theorem 3.1.2, we know the Tutte polynomial of M_1' and M_2' will be $t(M_1'; x, y) = t(M_1; x, y) - xy + x + y$ and $t(M_2'; x, y) = t(M_2; x, y) - xy + x + y$ respectively. Since the relaxations of M_1 and M_2 are isomorphic their polynomials will be equal. That is $t(M_1; x, y) - xy + x + y = t(M_2; x, y) - xy + x + y$ If we wish to obtain the Tutte polynomial of the original matroid, we have to perform some algebraic manipulation to the polynomial of the relaxed matroid. Therefore $t(M_1; x, y) = t(M_2; x, y)$. $\square$

The Tutte polynomial of matroid that is a common relaxation parent can be found as follows.

Corollary 3.2.0.2. *Suppose a matroid M contains at least two circuit-hyperplanes $A_1, andA_2$, such that the respective relaxations result in two non-isomorphic matroids, $M'_{A_1} \ncong M'_{A_2}$. Then, Tutte-polynomials $t(M'_{A_1}; x, y) = t(M'_{A_2}; x, y)$.*

Proof. This proof follows from Theorem 3.1.2. $\square$

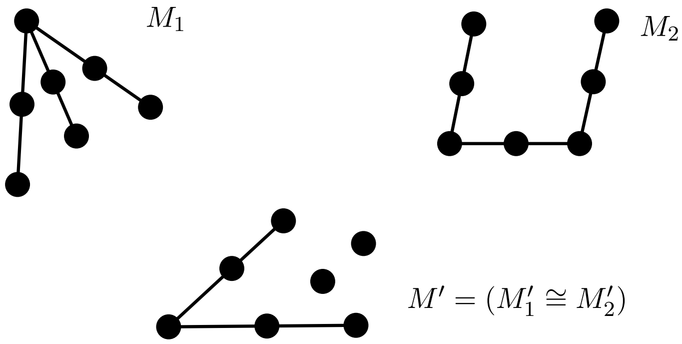

Figure 3.3: Relaxing a circuit-hyperplane from each matroid M_1 and M_2 results in isomorphic relaxations $M'_1 \cong M'_2$.

Chapter 4

Classifying Tutte-Equivalent Matroids

We have seen that the Tutte polynomial encodes a significant amount of information about a matroid. Additionally, we have found a method to find Tutte-equivalent matroids through matroid relaxation. The task now is to classify Tutte-equivalent matroids. In what follows, we will develop additional tools to construct Tutte-equivalent matroids within certain matroid classes. We begin by determining restrictions on the rank and the cardinality of the ground set for Tutte-equivalent matroids.

Lemma 4.1. *Any two rank 0 Tutte-equivalent matroids are isomorphic.*

Proof. Assume M_1 and M_2 are two matroids in rank 0 each on the same n-element ground set. All elements in a rank 0 matriod are loops, hence there is only one way to build such a matroid. Therefore, all such matroids will be isomorphic. □

Additionally, we can easily compute the Tutte polynomial for rank 0 matroids on n elements:

$$t(U_{0,n}; x, y) = y^n \tag{4.1}$$

Lemma 4.2. *Any two rank* 1 *Tutte-equivalent matroids are isomorphic.*

Proof. Suppose we have two matroids M_1 and M_2, such that $t(M_1; x, y) = t(M_2; x, y)$. Necessarily, both matroids must contain the same number of loops and parallel elements $|\ell_1| = |\ell_2|$ and $|p_1| = |p_2|$, where ℓ_i and p_i denotes the number of loops and parallel elements, respectively, in matroid M_i. However, since parallel elements in rank 1 can only occur in a single parallel class, then the two matroids must be isomorphic. $\square$

For a rank 1 matroid M on n elements, containing $\ell \geq 0$ loops and $n - \ell \geq 1$ parallel elements, it is not difficult to compute the Tutte polynomial of M:

$$t(M; x, y) = y^{\ell}(x + y + y^2 + ... + y^{n-\ell}) \tag{4.2}$$

Computing the Tutte polynomials for rank 2 matroids becomes more complicated, since such matroids can contain multiple parallel classes of elements. We first compute the Tutte polynomial for $U_{2,n}$, where $n \geq 2$. This can be seen by noticing that any single element contraction in $U_{2,n}$ produces the matroid $U_{1,n-1}$, while any single element deletion in $U_{2,n}$ produces the matroid $U_{2,n-1}$ (if $n \geq 3$).

$t(U_{2,n}; x, y) =$

$$x^2 + (n-2)x + (n-2)y + (n-3)y^2 + ... + (n-k-1)y^k + ...y^{n-2} \tag{4.3}$$

The Tutte polynomials for rank 2 matroids that contain $\ell \geq 1$ loops, but no parallel elements have the following formula:

$t(M_{2,n}; x, y) =$

$$y^{\ell}(x^2 + (n-\ell-2)x + (n-\ell-2)y + (n-\ell-3)y^2 + ...$$

$$+ (n-\ell-k-1)y^k + ... + y^{n-\ell-2}) \tag{4.4}$$

If a rank 2 matroid contains $\ell \geq 0$ loops and k parallel classes of size $m_1, m_2, ...m_k$, where $m_i \geq 2$, then the Tutte polynomial for M is:

$t(M_{2,n}; x, y) =$

$$y^l \Big[\textstyle\sum_{j=1}^{k} \sum_{i=1}^{m_j - 1} \big(x + y + y^2 + ...y^{n - \sum_{r=1}^{j} m_r + (j-2)}\big)$$

$$+ \big(x^2 + (N-2)x + (N-2)y + (N-3)y^2 + ... + y^{N-2}\big)\Big], \tag{4.5}$$

where $N = n - \sum_{r=1}^{k} m_r + k$.

Lemma 4.3. *Any two rank 2 Tutte-equivalent matroids are isomorphic.*

Proof. Assume we have two matroids M_1 and M_2 in rank 2 that have the same Tutte polynomial. Then, both matroids will have the same number of loops, single elements and the same number of parallel elements in each parallel class. Each parallel class of elements, will correspond to a unique place on an affine line. Reordering these parallel classes on this line simply creates isomorphic copies of M_1 and M_2, thus resulting in isomorphic matroids. $\square$

The implications of these lemmas is that our search of Tutte-equivalent matroids must begin in rank 3 and higher. However, as we see in the next results, the ground sets of two Tutte-equivalent matroids must also not be too small with respect to the ranks of the matroids. To prove this we recall Theorem 2.3.2, which implies the following:

Corollary 4.0.0.1. *If M_1 and M_2 are Tutte equivalent matroids, $t(M_1;x,y) = t(M_2;x,y)$, then $t(M_1^*;x,y) = t(M_2^*;x,y)$*

Proof. Since M_1 and M_2 are Tutte-equivalent, then $t(M_1;x,y) = t(M_2;x,y)$. By Theorem 2.3.2 $t(M_1^*;x,y) = t(M_1;y,x)$ and $t(M_2^*;x,y) = t(M_2;y,x)$. Therefore $t(M_1^*;x,y) = t(M_2^*;x,y)$. $\square$

Corollary 4.0.0.2. *If M_1 and M_2 are Tutte-equivalent matroids, each having rank $r \geq 3$, then $|E(M_i)| \geq r+3$, for $i = 1,2$.*

Proof. Suppose, to the contrary, that $r \leq |E(M_i)| < r+3$, for some i. Since M_1 and M_2 are Tutte-equivalent, $|E(M_1)| = |E(M_2)|$. Therefore, $r \leq |E(M_1)| = |E(M_2)| < r+3$. If $|E(M_1)| = |E(M_2)| = r$, then $r(M_1^*) = r(M_2^*) = 0$, and by Lemma 4.1, $M_1^* \cong M_2^*$, which implies $M_1 \cong M_2$. This contradicts the hypothesis that M_1 and M_2 are Tutte-equivalent. If $|E(M_1)| = |E(M_2)| = r+1$, then $r(M_1^*) = r(M_2^*) = 1$, and by Lemma 4.2, $M_1^* \cong M_2^*$, which implies $M_1 \cong M_2$. This again contradicts the hypothesis that M_1 and M_2 are Tutte-equivalent. Finally, if $|E(M_1)| = |E(M_2)| = r+2$, then $r(M_1^*) = r(M_2^*) = 2$, and by Lemma 4.3, $M_1^* \cong M_2^*$, which implies $M_1 \cong M_2$. This once again contradicts the hypothesis that M_1 and M_2 are Tutte-equivalent. Therefore, it follows that $|E(M_i)| \geq r+3$, for $i = 1,2$. $\square$

Thus, our search for Tutte-equivalent matroids must begin with rank 3 matroids having at least six elements. Figure 2.3 illustrates the smallest example of Tutte-equivalent matroids.

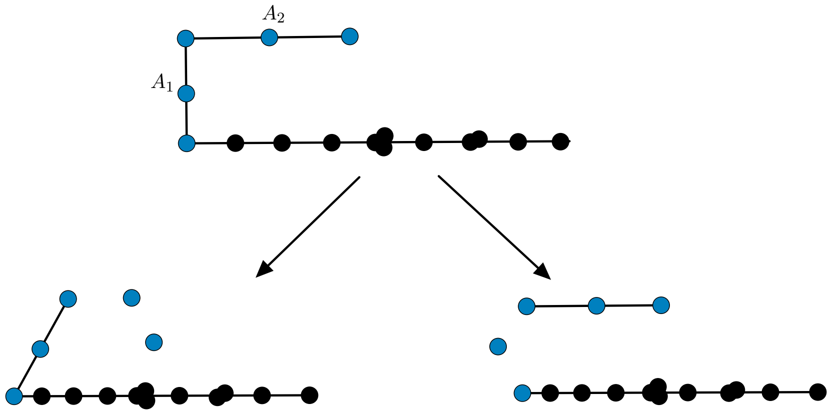

Figure 4.1: Adding additional elements to M to create new Tutte-equivalent matroids.

Once we have found a matroid M that is a common relaxation parent to at least two matroids M_1' and M_2' we can continue to relax circuit hyperplanes in M_1' and M_2' in hopes of finding additional Tutte-equivalent matroids, See Figure 5.1. Additionally, we can also use our parent matroid M as the foundation to construct new matroids by adding new elements to our ground set. The only restrictions are:

1. An element e added to the ground set cannot be a loop.
2. Elements cannot be added to the circuit-hyperplanes A_1 and A_2.

Example 4.0.1. The matroid in Figure 3.2 is in rank 3 with $|E| = 7$. We can construct another matroid in the same rank that is also a common relaxation parent by adding 10 additional elements as in Figure 4.1.

A similar approach is applied to matroids M_1 and M_2 that share a common relaxation descendant with one additional restriction. See Figure 4.2.

1. Any element e added to the ground set cannot be a loop.
2. Elements cannot be added to circuit hyperplane A.
3. The same amount of elements or parallel classes must be added to both M_1 and M_2.

Our initial research began by finding ways to construct and classify Tutte-equivalent matroids. Although matroid relaxation was an operation to obtain a new matroid, it quickly became a very useful tool to find Tutte-equivalent matroids. Matroid relaxation along with the correct rank and ground set gave way to a plethora of Tutte-equivalent matroids. However, our success in constructing Tutte-equivalent matroids has made it very difficult to classify them in any order. Therefore, we will turn our attention to explore the existence of Tutte-equivalent matroids in classes of matroids, particularly the binary field. We will try to find additional restrictions, if any, that Tutte-equivalent matroids need to hold to be in the binary field. Additionally, attempting to determine if common relaxation parents, or common relaxation descendants, will also produce binary matroids and other questions.

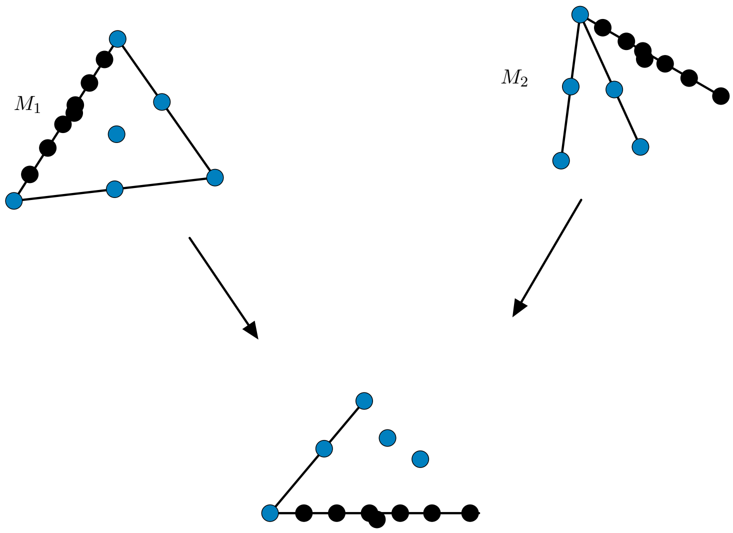

Figure 4.2: Constructing additional matroids that have a common relaxation descendant, by adding new elements to our ground set.

4.1 Binary Descendant and Parent Matroids

Binary matroids follow the same restrictions we have established for all matroids. To find Tutte-equivalent binary matroids they must must be in rank 3 or higher and contain $r+3$ elements. However, binary matroids can only contain at most $2^r - 1$ elements. So for example, a matroid M with $r(M) = 4$ can have a ground set that only contains from $7 \leq |E| \leq 15$ elements.

Now that we've restricted ourselves to binary matroids, we can now begin to classify the

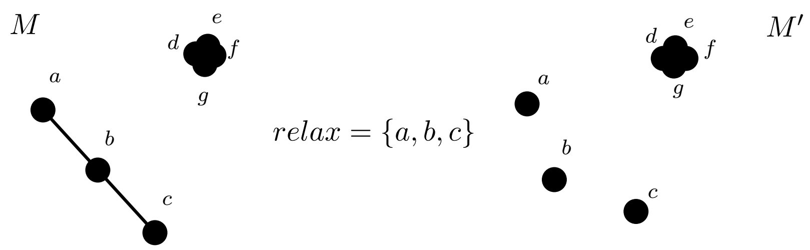

Figure 4.3: The matroid $M = U_{2,3} \oplus U_{1,4}$ is in the only structure where a binary matroid can have a binary relaxation M'.

results we've found from this class.

Definition 4.1.1. Let M_1 and M_2 be two matroids on disjoint ground sets E_1 and E_2 respectively. The *direct sum* is a new matroid $M_1 \oplus M_2$ with ground set $E = E_1 \cup E_2$ and independent sets $I_1 \cup I_2$.

Lemma 4.4. *A binary matroid in rank r with ground set $|E| = n$ that has a binary relaxation can only be in the structure of $U_{r-1,r} \oplus U_{1,n-r}$.*

Proof. Given a matroid M, its circuit-hyperplane is in the structure of $U_{r-1,r}$ containing $\binom{r}{r-2}$ flats. Any $r-2$ flat will be contained in two hyperplanes. The first, $r-2 \in A$ and the second is composed of $H_i = (r-2) \cup U_{1,n-r}$.Upon relaxing the circuit-hyperplane A we turned our dependent set independent, and increased the number of hyperplanes by one as well. Thus evey $r-2$ flat will be contained in at most three hyperplanes, and will continue to satisfy Theorem 1.3.2. $\square$

If M is a binary matroid with two Tutte-equivalent relaxations M_1' and M_2', then M must contain at least two circuit hyperplanes and a ground set of at least $r+3$ elements, where every $r-2$ flat is contained in at most 3 hyperplanes.

Lemma 4.5. *If M is a binary matroid, and a common relaxation parent to M_1' and M_2', then M_1' and M_2' are not binary matroids.*

Proof. If M is a binary matroid where every $r-2$ flat is contained in at most 2 hyperplanes, then $M = U_{r-1,r} \oplus U_{1,n-r}$. Thus, M is not a common relaxation parent. Now assume M is a binary matroid, and every $r-2$ flat is contained in three hyperplanes. If we relax A_1, then there will exist a $r-2$ flat in A_1 contained in more than three hyperplanes. The same result will follow for A_2. Therefore, M'_1 and M'_2 are not binary matroids. □

Lemma 4.6. *Common relaxation descendant binary matroids do not exist.*

Proof. Suppose, to the contrary. Assume M_1 and M_2 are two non-isomorphic binary matroid containing circuit-hyperplanes A_1 and A_2 respectively. Assume relaxing each matroids' circuit-hyperplane gives a binary relaxation where, $M'_{A_1} \cong M'_{A_2}$. From Lemma 4.4, the only binary matroids with a binary relaxation have structure of $U_{r-1,r} \oplus U_{1,n-r}$. Therefore, if M'_{A_1} and M'_{A_2} are binary, then $M_1 \cong M_2$. □

Although it is possible to find a binary matroid that is a common relaxation parent to two new matroids, once we relax a circuit hyperplane we've left our binary class. For example the Fano matroid F_7 can produce thirteen new matroids, none of which will be binary. See Fig 5.1.

Chapter 5

Conclusion

The goal of this thesis was to identify and study matroids that have the property of being Tutte-equivalent. We began learning that every matroid can be represented by a Tutte polynomial, and said polynomial encodes several pieces of information about our matroid. Such as:

- $t(M;1,1)$ gives the number of basis of M.
- $t(M;2,1)$ gives the number of independent sets of M.
- $t(M;1,2)$ gives the number of spanning sets of M.
- $t(M;2,2) = 2^{|E|}$ the number of subsets of M.

However, this also implies that there are pieces of information about our matroid that the Tutte polynomial can not capture. Therefore, it is possible that two geometrically different matroids can have the same Tutte polynomial. The next step was to find methods that will help us find non-isomorphic matroids that have the same Tutte polynomial, or Tutte-equivalent matroids.

We studied the Tutte-polynomial and the relation of the polynomial with the dual matroid, and circuit-hyperplane relaxation. From here we noticed that matroid relaxation is a very effective tool to help one find Tutte-equivalent matroids. We were able to classify Tutte-equivalent matroids in two ways: Common relaxation descendants, which consist of two matroids which have isomorphic relaxations. The other is a common relaxation parent, consisting of one matroid with two or more circuit-hyperplanes. When we relax each circuit in two or more separate instances it creates at least two Tutte-equivalent matroids.

The search for Tutte-equivalent matroids will begin in rank 3 and contain at least 3 more elements than the cardinality of the rank. Once two Tutte-equivalent matroids are found, they can be used as a foundation for Tutte-equivalent matroids with bigger ground sets. However, these methods gave way to an unlimited amount of Tutte-equivalent matroids making our attempt of classification very difficult. Instead, we turned our attention to Tutte-equivalent matroids in the binary field.

Finding Tutte-equivalent matroids in the binary field turned out to be unfeasible. Our research attempted to find binary matroids that would produce binary matroids through matroid relaxation. However, we soon discovered that only one type of binary matroid can have a binary relaxation. Therefore, one will not find common relaxation descendant matroids in the binary field. This means that when a binary matroid M exists that is a common relaxation parent, its descendants M_1 and M_2 will not be contained in the binary field.

Although the results of Tutte-equivalent matroids in the binary field seemed somewhat restrictive, it became a good starting point for future research. Ideally, future research will begin to look into *ternary matroids*, geometries representable in the field $\mathbb{F}_3$, and then *regular matroids*, those representable over all fields.

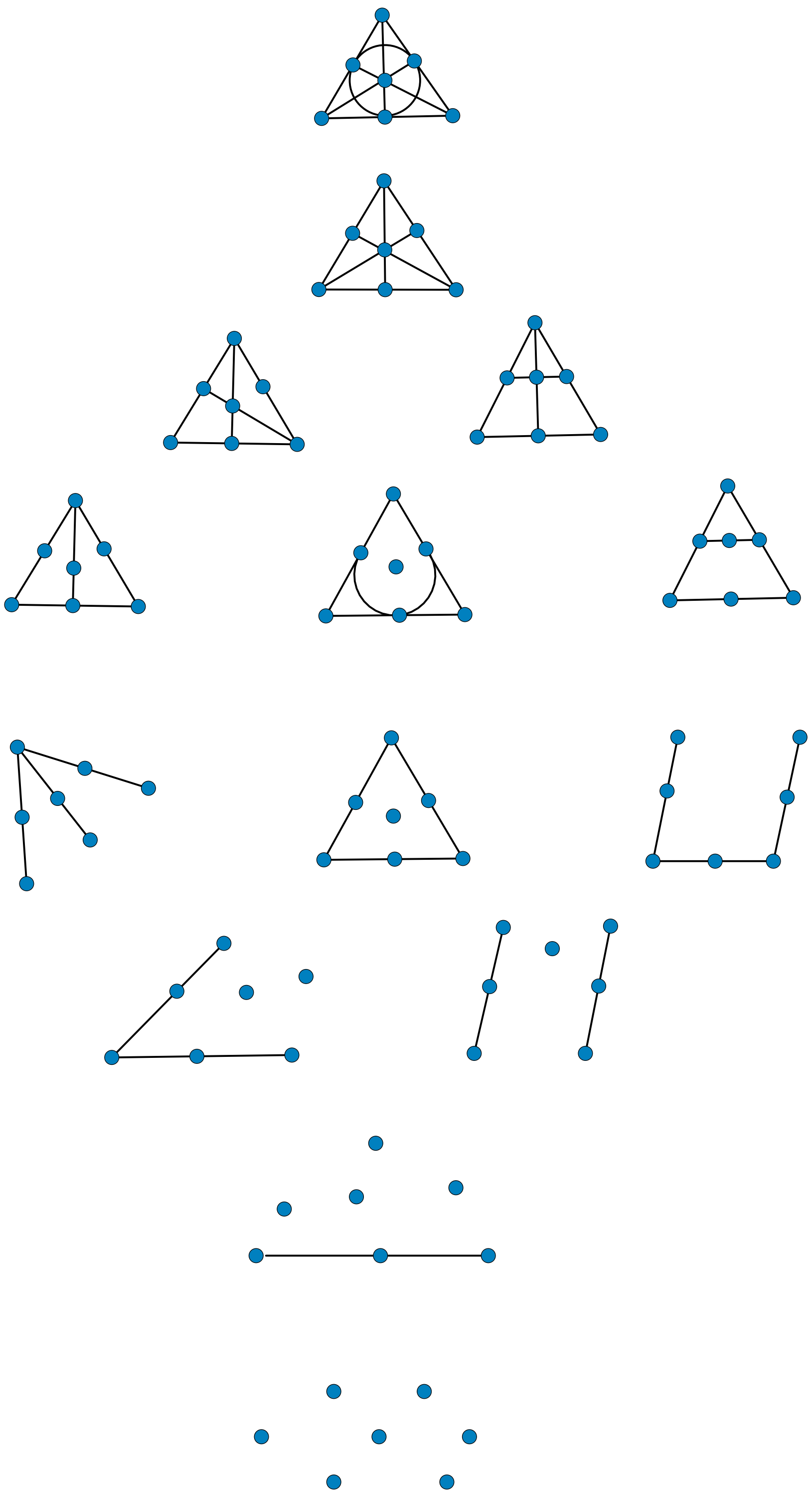

Figure 5.1: The stages of relaxing a circuit-hyperplane of the Fano matriod, F_7, and the Tutte-equivalent matroids

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