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Progenitors Involving Simple Groups

Nicholas R. Andujo

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Nicholas Raymond Andujo

September 2018
Progenitors Involving Simple Groups

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September 2018

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Abstract

I will be going over writing representations of both permutation and monomial progenitors, which include $2^4 : D_4$, $2(7) : L_2(7)$ as permutation progenitors, and monomial progenitors $7(2) : S_3 \times 2$, $11^2 : (5 : 2) \cdot 5$, $11^3 : (25 : 3)$, $11^4 : (4 : 5) \cdot 5$. Also, the images of these different progenitors at both lower and higher fields and orders.

We will also do the double coset enumeration of $S_5$ over $D_5$, $S_6$ over $5 : 4$, $A_5 \times A_5$ over $(5 : 2) \cdot 5$, and go on to do the double coset enumeration over maximal subgroups for larger constructions. We will also do the construction of sporatic group $M_{22}$ over maximal subgroup $A_7$, and also $J1$ with the monomial representation $7(2) : S_3 \times 2$ over maximal subgroup $PSL(2, 11)$.

We will also look at different extension problems of composition factors of different groups, and determine the isomorphism types of each extension.
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Introduction

We are interested in finite groups. Since every finite groups is composed of simple groups and we are able to solve extension problems, we are interested in finite non-abelian simple groups. Now it has been shown that progenitors factored by appropriate relations give finite non-abelian simple groups including Sporadic simple groups. So I am interested in finding homomorphic images of progenitors.

The automorphisms of $M^{*n}$ are permutations of the free generators $t_i$. If $M = 2$ then $N$ will simply act by conjugation as permutations of the $n$ involutary symmetric generators. Thus, elements of $N$ can be gathered on the left, every element of the progenitor can be represented as $Pw$, where $P \in N$ and $w$ is a word in the symmetric generators. Indeed this representation is unique provided $w$ is simplified so that adjacent symmetric generators are distinct. Thus any additional relation by which we must factor the progenitor must have the form $Pw(t_0, t_1, ..., t_{n-1})$. Let $N$ be a group of permutations on $n$ letters. $2^{*n} : N$ means $2^{*n}$ extended by $N$ acting as automorphisms (by conjugations). The objective here is to factor the progenitor by relations, that equate elements of $N$ to the product of $t_i$s, that give finite homomorphic images.

We will illustrate the method of construction through the following example. The progenitor $2^{*7} : D_{14}$ is the free product of seven copies of the cyclic group $C_2$ of order 2 extended by $D_{14}$, the dihedral group of degree 14. In order to find finite homomorphic images of the infinite progenitor $2^{*7} : D_{14}$, we must factor by suitable relations.
Chapter 1

Writing Monomial Progenitors

1.1 Progenitor Preliminaries

Definition 2.1.1 (Permutation). If $X$ is a nonempty set, a permutation is the bijective mapping $f : X \to X$.

Definition 2.1.2. (Semigroup) A semigroup $(G, \ast)$ is a nonempty set $G$ equipped with an associative operation $\ast$. [Rot95]

Definition 2.1.3 (Group). A group is a semigroup $G$ containing an element $e$ such that

(i) $e \ast a = a = a \ast e$ for all $a \in G$

(ii) for every $a \in G$, there is an element $b \in G$ with $a \ast b = e = b \ast a$. [Rot95]

Definition 2.1.4. (Free Group) If $X$ is a nonempty subset of a group $F$, then $F$ is a free group with basis $X$ if, for every group $G$ and every function $f : X \to G$, there exists a unique homomorphism $\varphi : F \to G$ extending $f$. Moreover, $X$ generates $F$. [Rot95]

Definition 2.1.5. (Presentation) Let $X$ be a set and let $\Delta$ be a family of words on $X$. A group $G$ has generators $X$ and relations $\Delta$ if $G \cong F/R$, where $F$ is the free group with basis $X$ and $R$ is the normal subgroup of $F$ generated by $\Delta$. The ordered pair $(X|\Delta)$ is called a presentation of $G$. [Rot95]
Definition 2.1.6. (Symmetric Group) The symmetric group, denoted \( S_n \), is the set of all permutations of the nonempty set \( X = \{1, 2, ..., n\} \). \( S_n \) is a group of order \( n! \) on \( n \) letters.

Definition 2.1.7. (Disjoint) Two permutations \( \alpha, \beta \in S_X \) are disjoint if every \( x \) moved by one is fixed by the other. In symbols, if \( \alpha(a) \neq a \), then \( \beta(a) = a \), and if \( \alpha(b) = b \), then \( \beta(b) \neq b \).

Theorem 2.1.8. Every permutation of \( S_n \) for \( n \geq 2 \) is a product of disjoint cycles.

Theorem 2.1.9. Let \( \alpha \in S_X \), \( \alpha \) is either a cycle or a product of disjoint cycles. [Rot95]

Definition 2.1.10. (Order) Let \( G \) be a group. The order of \( G \) is the number of elements in \( G \), denoted \( |G| \) (#\( G \) in Magma).

Definition 2.1.11. (Progenitor) A member of a family of infinite groups, the members of which include among their homomorphic images all the non-abelian simple groups, is called a progenitor.

Definition 2.1.12. (Character) Let \( A(x) = (a_{ij}(x)) \) be a matrix representation of \( G \) of degree \( m \). We consider the characteristic polynomial of \( A(x) \), namely
\[
\det(\lambda I - A(x)) = \begin{vmatrix}
\lambda - a_{11}(x) & \lambda - a_{12}(x) & \ldots & \lambda - a_{1m}(x) \\
\lambda - a_{21}(x) & \lambda - a_{22}(x) & \ldots & \lambda - a_{2m}(x) \\
\ldots & \ldots & \ldots & \ldots \\
\lambda - a_{m1}(x) & \lambda - a_{m2}(x) & \ldots & \lambda - a_{mm}(x)
\end{vmatrix}
\]

This is a polynomial of degree \( m \) in \( \lambda \), and inspection shows that the coefficient of \( -\lambda^{m-1} \) is
\[
\varphi(x) = a_{11}(x) + a_{22}(x) + \ldots + a_{mm}(x)
\]

It is customary to call the right-hand side of this equation the trace of \( A(x) \), abbreviated to \( \text{tr} A(x) \), so that \( \varphi(x) = \text{tr} A(x) \)
We regard $\varphi(x)$ as a function on $G$ with values in $K$, and we call it the character of $A(x)$.[Led87]

**Theorem 2.1.13.** The number of irreducible character of $G$ is equal to the number of conjugacy classes of $G$.[Led87]

**Definition 2.1.14.** (Degree of a Character) The sum of squares of the degrees of the distinct irreducible characters of $G$ is equal to $|G|$. The degree of a character $\chi$ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character.[Led87]

**Definition 2.1.15.** (Lifting Process) Let $N$ be a normal subgroup of $G$ and suppose that $A_0(Nx)$ is a representation of degree $m$ of the group $G/N$. Then $A(x) = A_0(Nx)$ defines a representation of $G/N$ lifted from $G/N$. If $\varphi_0(Nx)$ is a character of $A_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \varphi(u) = m = \varphi(1)$. The lifting process preserves irreducibility.[Led87]

**Definition 2.1.16.** (Induced Character) Let $H \leq G$ and $\varphi(u)$ be a character of $H$ and define $\varphi(x) = 0$ if $x \in H$, then

$$\varphi^G(x) = \begin{cases} 
\varphi(x), & x \in H \\
0 & x \notin H
\end{cases}$$

is an induced character of $G$.[Led87]

**Definition 2.1.17.** (Formula for Induced Character) Let $G$ be a finite group and $H$ be a subgroup such that $[G : H] = n$. Let $C_\alpha, \alpha = 1, 2, \ldots m$ be the conjugacy classes of $G$ with $|C_\alpha| = h_\alpha, \alpha = 1, 2, \ldots m$. Let $\varphi$ be a character of $H$ and $\varphi^G$ be the character of $G$ induced from the character $\alpha$ of $H$ up to $G$. The values of $\varphi^G$ on the $m$ classes of $G$ are given by:

$$\varphi^G_\alpha(x) = \frac{n}{h_\alpha} \sum_{\omega \in C_\alpha \cap H} \varphi(\omega), \alpha = 1, 2, 3, \ldots, m$$
1.2 Permutation Representations

1.2.1 \(2^*4 : D_4\)

We want to write the progenitor \(2^*4 : D_4\). Our control group with this progenitor is \(N = D_4\). We want to first write a presentation for \(D_4\) given by,
\[G < x, y | x^4 = y^2 = (x^{-1}y)^2 = 1 >\]
We check in Magma if the above presentation gives \(D_4\).

```magma
> G<x,y>:=Group<x,y|x^4,y^2,(x^-1*y)^2>;
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> s,t:=IsIsomorphic(G1,DihedralGroup(4));
> s;
true
```

We see that the corresponding permutation representation is \(N = < x, y >\), where in this case, \(x = (1, 2, 3, 4)\) and \(x = (1, 3)\). Now we add the free product \(2^*4\) to this group to form our progenitor. Since we now have \(2^*4\) in our progenitor, that would mean that we have 5 \(t\)'s of order 2. We will add a \(t\) of order 2 and let \(t\) commute with the point stabilizer of 1 in \(N\). Therefore, a presentation for the progenitor \(2^*4 : N\) is given by
\[G < x, y, t | x^4, y^2, (x^{-1}y)^2, t^2, (t, N^1) >\], where \(t \sim t_1\). \((t, N^1)\) means that \(1^g = 1\) for every \(g \in N^1\). Using Magma, we can see that the point stabilizer of 1 in \(N\) is equal to \(< (2, 4) >\). Now we use the Schreier System to convert the permutations into words. Thus, we have \(N^1 = < x^2 y >\) (see below).

```magma
> S:=Sym(4);
> xx:=S!(1, 2, 3, 4);
> yy:=S!(1, 3);
> N:=sub<S|xx,yy>;
> NN<x,y>:=Group<x,y|y^2,x^4,(x^-1*y)^2>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..8]];
> for i in [2..8] do
> for j in [1..#Sch[i]] do
> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
> if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
> end for;
> PP:=Id(N);
> for k in [1..#P] do
> PP:=PP*P[k]; end for;
> end for;
```
> Stabiliser(N,1);
Permutation group acting on a set of cardinality 4
Order = 2
   (2, 4)
> for i in [1..8] do if ArrayP[i] eq N!(2, 4) then Sch[i];
for|if> end if;
for> end for;
x^2 * y

So, a representation of the progenitor \( G = 2^*4 : D_4 \) is given by
\[
G < x, y, t > =< x, y, t | x^4, y^2, (x^{-1}y)^2, t^2, t^2y = t >
\]
We now want to write the progenitor $2^7 : L_2(7)$. Our control group with this progenitor is $N = L_2(7)$. The presentation for the control group $L_2(7)$, which is of order 168, is the following,
\[ G < x, y > := \text{Group} < x, y | x^7, y^2, (yx^{-1})^3, (xyx)^4 > . \]
We check in Magma if the above presentation gives $L_2(7)$.

\begin{verbatim}
> G<x,y>:=Group<x,y|x^7,y^2,(y*x^-1)^3,(x*y*x^-2)^4>;
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> s,t:=IsIsomorphic(G1,PSL(2,7));
> s;
true
\end{verbatim}

We see that our presentation is isomorphic to $\text{PSL}(2,7)$, which is also isomorphic to $L_2(7)$. The corresponding permutation representation is $N = \langle x, y \rangle$, where in this case, $x = (1, 2, 3, 4, 5, 6, 7)$ and $x = (1, 2)(3, 6)$. Now we add the free product $2^7$ to this group to form our progenitor, which would mean that we have $7^t$'s of order $2$ ($|t_i|$). Letting $t$ commute with the one point stabilizer $N^1$, we now have a presentation for the progenitor $2^7 : N$ is given by
\[ G < x, y, t > := \text{Group} < x, y, t | x^7, y^2, (yx^{-1})^3, (xyx)^4, t^2, (t, N^1) > , \]
where $t \sim t_1$. Using Magma, we will now see what the point stabilizer comes out to be and also convert it into words for our presentation.

\begin{verbatim}
> S:=Sym(7);
> xx:=S!(1, 2, 3, 4, 5, 6, 7);
> yy:=S!(1, 2)(3, 6);
> N:=sub<S|xx,yy>;
> NN<x,y>:=Group<x,y|x^7,y^2,(y*x^-1)^3,(x*y*x^-2)^4>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..168]];
> for i in [2..168] do
| for P:=[Id(N): l in [1..#Sch[i]]] do
| for j in [1..#Sch[i]] do
| if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
| if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
| if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
| end for;
end for;
> PP:=Id(N);
> for k in [1..#P] do
| PP:=PP*P[k]; end for;
> ArrayP[i]:=PP;
> Stabiliser(N,1);
Permutation group acting on a set of cardinality 7
\end{verbatim}
Order $= 24 = 2^3 \cdot 3$

$(3, 7)(5, 6)$
$(2, 5)(4, 6)$
$(2, 3, 5)(4, 7, 6)$
$(3, 5)(6, 7)$

> for $i$ in [1..168] do if ArrayP[i] eq N!(3, 7)(5, 6) then
  Sch[i];
for|if> end if; end for;

$x^3 \cdot y \cdot x^{-3}$

> for $i$ in [1..168] do if ArrayP[i] eq N!(2, 5)(4, 6) then
  Sch[i];
for|if> end if; end for;
y * $x^{-2} \cdot y \cdot x^{-2} \cdot y$

> for $i$ in [1..168] do if ArrayP[i] eq N!(2, 3, 5)(4, 7, 6) then
  Sch[i];
for|if> end if; end for;
x * $y \cdot x^{-3} \cdot y \cdot x^{-2} \cdot y$

> for $i$ in [1..168] do if ArrayP[i] eq N!(3, 5)(6, 7) then
  Sch[i];
for|if> end if; end for;
x * y * $x^{-3} \cdot y \cdot x^{-3}$

We see now that point stabilizer $N^1$ contains 4 permutations, $N^1 = < (3, 7)(5, 6), (2, 5)(4, 6), (2, 3, 5)(4, 7, 6), (3, 5)(6, 7) >$. We have 3 of order 2, and 1 of order 3. Thus $2^3 \cdot 3 = 24$ and the order of $N^1 = 24$. Converting these to words, have have,

$N^1 = < x^3yx^{-3}, yx^{-2}yx^2y, xyx^3yx^{-2}y, xyx^{-3}yx^3 >$. We must now have all the permutations in the point stabilizer $N^1$ commute with $t$ to complete our representation. Therefore we have:

$G < x, y, t > = Group < x, y, t | x^7, y^2, (yx^{-1})^3, (xyx^2)^4, t^2, t^3yx^{-3} = t, t^yx^{-2}yx^2y = t, t^yx^{-3}yx^3 = t >$
1.3 Monomial Representations

1.3.1 $7^2 : m \, D_{12}$

We will now move on to writing monomial progenitor representations. In this process, we will take a subgroup $H$ of a group $G$, and induce characters of the rows of the character table of $H$ using the lifting process. Once we find the right character row to induce in $H$, we will write matrix representations $A(x)$ and $A(y)$. Once we have the matrix representations reduced in the field of the free group, we will use the characters in the matrix to generate the new permutations that will be in our monomial progenitor.

In this example, we have $G = 7^2 : m \, D_{12}$, which represents 2 $t$’s of order 7 with the control group $N = D_{12}$. The control group will be having a presentation of $D =< x, y | x^5 = y^2 = (xy)^2 = 1 >$, and $x = (1, 2, 3, 4, 5, 6)$, $y = (1, 4)(2, 3)(5, 6)$.

Let us now look at the character table of $G$.

Table 1.1: Character Table of $G$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$\chi.1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi.2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi.3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi.4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi.5$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi.6$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

As you see, there are 2 rows that have an index of 2. Let $H$ be a subgroup of $G$ of index 2 such that $\frac{|G|}{|H|} = 2$. That is $\frac{12}{6} = 2$, $|H| = 6$. So, we have a subgroup $H \leq G$ of order 6, with a presentation $H =< a, b | (a^4b^2) = (a^2b^2a^2) = 1 >$, and $a = (1, 2, 3, 4, 5, 6)$, $b = (1, 5, 3)(2, 6, 4)$. Let’s now take a look at the character table of $H$.

Table 1.2: Character Table of $H$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$\varphi.1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\varphi.2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\varphi.3$</td>
<td>1</td>
<td>1</td>
<td>$\alpha$</td>
<td>$\alpha^2$</td>
<td>$\alpha^2$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\varphi.4$</td>
<td>1</td>
<td>-1</td>
<td>$\alpha$</td>
<td>$\alpha^2$</td>
<td>$-\alpha^2$</td>
<td>$-\alpha$</td>
</tr>
<tr>
<td>$\varphi.5$</td>
<td>1</td>
<td>1</td>
<td>$\alpha^2$</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\alpha^2$</td>
</tr>
<tr>
<td>$\varphi.6$</td>
<td>1</td>
<td>-1</td>
<td>$\alpha^2$</td>
<td>$\alpha$</td>
<td>$-\alpha$</td>
<td>$-\alpha^2$</td>
</tr>
</tbody>
</table>
(α is the 3rd root of unity)

We will need to see which character row \( \varphi.n \) of \( H \) equals either \( X.5 \) or \( X.6 \) of \( G \) since both \( X.5 \) and \( X.6 \) have index of 2). To do this, we use the formula to induce characters of \( H \) up to \( D_{12} \).

\[
\varphi^G_\alpha(x) = \frac{n}{h_\alpha} \sum_{\omega \in \mathcal{C}_\alpha \cap H} \varphi(\omega), \alpha = 1, 2, 3, ..., m
\]

First, we find the smallest field that defines \( \alpha \), in which we get \( \mathbb{Z}_7 \{0\} \). So, this implies that \( \alpha = 2, \alpha^2 = 4 \) and \( \alpha^3 = 8 \equiv 1 \).

For \( \varphi^G_1 \)

\[
\varphi^G_1(1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_1 = 2 \varphi(1)_1 = 2
\]

\[
\varphi^G_1((14)(25)(36)) = 2 \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_1 = 2 \varphi(1)_1 = 2
\]

\[
\varphi^G_1((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in 1 \cap H} = \varphi(1)_1 = \frac{2}{3} \varphi(0)_1 = 0
\]

\[
\varphi^G_1((15)(24)) = \frac{2}{3} \sum_{\omega \in (5)(24) \cap H} = \varphi(1)_1 = \frac{2}{3} \varphi(0)_1 = 0
\]

\[
\varphi^G_1((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (53)(264))_1 = (1) \varphi(1 + 1)_1 = 2
\]

\[
\varphi^G_1((12456)) = \frac{2}{2} \sum_{\omega \in (12456) \cap H} = \varphi((12456) + (65432))_1 = (1) \varphi(1 + 1)_1 = 2
\]

For \( \varphi^G_2 \)

\[
\varphi^G_2(1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_2 = 2 \varphi(1)_2 = 2
\]

\[
\varphi^G_2((14)(25)(36)) = \frac{2}{1} \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_2 = 2 \varphi(-1)_2 = -2
\]

\[
\varphi^G_2((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in (14)(23)(56) \cap H} = \varphi(1)_2 = \frac{2}{3} \varphi(0)_2 = 0
\]

\[
\varphi^G_2((15)(24)) = \frac{2}{3} \sum_{\omega \in (15)(24) \cap H} = \varphi(1)_2 = \frac{2}{3} \varphi(0)_2 = 0
\]

\[
\varphi^G_2((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (153)(264))_2 = (1) \varphi(1 + 1)_2
\]
\[ \varphi_2^G((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_2 = (1)\varphi(-1 + (-1))_2 = -2 \]

For \( \varphi_3^G \)
\[ \varphi_3^G(1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_3 = 2\varphi(1)_3 = 2 \]
\[ \varphi_3^G((14)(25)(36)) = \frac{2}{3} \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_3 = 2\varphi(1)_3 = 2 \]
\[ \varphi_3^G((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in (14)(23)(56) \cap H} = \varphi(\emptyset)_3 = \frac{2}{3}\varphi(0)_3 = 0 \]
\[ \varphi_3^G((15)(24)) = \frac{2}{3} \sum_{\omega \in (15)(24) \cap H} = \varphi(\emptyset)_3 = \frac{2}{3}\varphi(0)_3 = 0 \]
\[ \varphi_3^G((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (153)(264))_3 = (1)\varphi(\omega + \omega^2)_3 = 2 + 4 = 6 \equiv 7 - 1 \]
\[ \varphi_3^G((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_3 = (1)\varphi(\omega^2 + \omega)_3 = 4 + 2 = 6 \equiv 7 - 1 \]

For \( \varphi_4^G \)
\[ \varphi_4^G(1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_4 = 2\varphi(1)_4 = 2 \]
\[ \varphi_4^G((14)(25)(36)) = \frac{2}{3} \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_4 = 2\varphi(-1)_4 = -2 \]
\[ \varphi_4^G((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in (14)(23)(56) \cap H} = \varphi(\emptyset)_4 = \frac{2}{3}\varphi(0)_4 = 0 \]
\[ \varphi_4^G((15)(24)) = \frac{2}{3} \sum_{\omega \in (15)(24) \cap H} = \varphi(\emptyset)_4 = \frac{2}{3}\varphi(0)_4 = 0 \]
\[ \varphi_4^G((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (153)(264))_4 = (1)\varphi(\omega + \omega^2)_4 = 2 + 4 = 6 \equiv 7 - 1 \]

\[ \varphi_4^G((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_4 = (1)\varphi(-1 + (-1))_4 = -2 \]
\[ 2 + 4 = 6 \equiv \bar{7} - 1 \]

\[ \varphi^G_4((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_4 = (1)\varphi(-\omega^2 + -\omega)_4 = -4 + (-2) = -6 \equiv \bar{7} 1 \]

For \( \varphi^G_5 \)

\[ \varphi^G_5 (1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_5 = 2\varphi(1)_5 = 2 \]

\[ \varphi^G_5((14)(25)(36)) = \frac{2}{3} \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_5 = 2\varphi(1)_5 = 2 \]

\[ \varphi^G_5((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in (14)(23)(56) \cap H} = \varphi(\emptyset)_5 = \frac{2}{3}\varphi(0)_5 = 0 \]

\[ \varphi^G_5((15)(24)) = \frac{2}{3} \sum_{\omega \in (15)(24) \cap H} = \varphi(\emptyset)_5 = \frac{2}{3}\varphi(0)_5 = 0 \]

\[ \varphi^G_5((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (153)(264))_5 = (1)\varphi(\omega^2 + \omega)_5 = 4 + 2 = 6 \equiv \bar{7} - 1 \]

\[ \varphi^G_5((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_5 = (1)\varphi(\omega + \omega^2)_5 = 2 + 4 = 6 \equiv \bar{7} - 1 \]

For \( \varphi^G_6 \)

\[ \varphi^G_6 (1) = \frac{2}{1} \sum_{\omega \in 1 \cap H} = \varphi(1)_6 = 2\varphi(1)_6 = 2 \]

\[ \varphi^G_6((14)(25)(36)) = \frac{2}{1} \sum_{\omega \in (14)(25)(36) \cap H} = \varphi((14)(25)(36))_6 = 2\varphi(-1)_6 = -2 \]

\[ \varphi^G_6((14)(23)(56)) = \frac{2}{3} \sum_{\omega \in (14)(23)(56) \cap H} = \varphi(\emptyset)_6 = \frac{2}{3}\varphi(0)_6 = 0 \]

\[ \varphi^G_6((15)(24)) = \frac{2}{3} \sum_{\omega \in (15)(24) \cap H} = \varphi(\emptyset)_6 = \frac{2}{3}\varphi(0)_6 = 0 \]
\[ \varphi_6^G((135)(246)) = \frac{2}{2} \sum_{\omega \in (135)(246) \cap H} = \varphi((135)(246) + (153)(264))_6 = (1)\varphi(\omega^2 + \omega)_6 = 4 + 2 = 6 \equiv 7 - 1 \]
\[ \varphi_6^G((123456)) = \frac{2}{2} \sum_{\omega \in (123456) \cap H} = \varphi((123456) + (165432))_6 = (1)\varphi(-\omega + -\omega^2)_6 = -2 + (-4) = -6 \equiv 7 1 \]

We therefore see that \( \varphi_6 = \chi.6 \). Next, we put together matrix representations \( A(x) \) and \( A(y) \) using the following formula.

\[ A(n) = \begin{bmatrix} B(T_1 n T_1^{-1}) & B(T_1 n T_2^{-1}) & \ldots & B(T_1 n T_m^{-1}) \\ B(T_2 n T_1^{-1}) & B(T_2 n T_2^{-1}) & \ldots & B(T_2 n T_m^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ B(T_m n T_1^{-1}) & B(T_m n T_2^{-1}) & \ldots & B(T_m n T_m^{-1}) \end{bmatrix} \]

Where we would have \( B((14)(25)(36)) = -1, B((135)(246)) = 4, B((153)(264)) = 2, B((123456)) = 5, B((165432)) = 3, \) and \( B(g) = 0, g \not\in G \). Also, with \( H \leq G \), we would have the right transversals be \( T_1 = e \) and \( T_2 = (1, 4)(2, 3)(5, 6) \) since we have an index of order 2. Having \( x = (1, 2, 3, 4, 5, 6) \) and \( y = (1, 4)(2, 3)(5, 6) \), we then have

\[ A(x) = \begin{bmatrix} B(T_1(1, 2, 3, 4, 5, 6)T_1^{-1}) & B(T_1(1, 2, 3, 4, 5, 6)T_2^{-1}) \\ B(T_2(1, 2, 3, 4, 5, 6)T_1^{-1}) & B(T_2(1, 2, 3, 4, 5, 6)T_2^{-1}) \end{bmatrix} \]
\[ A(x) = \begin{bmatrix} B((1, 2, 3, 4, 5, 6)) & B((1, 3)(4, 6)) \\ B((1, 5)(2, 4)) & B((1, 6, 5, 4, 3, 2)) \end{bmatrix} \]
\[ A(x) = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \]

\[ A(y) = \begin{bmatrix} B(T_1(1, 4)(2, 3)(5, 6)T_1^{-1}) & B(T_1(1, 4)(2, 3)(5, 6)T_2^{-1}) \\ B(T_2(1, 4)(2, 3)(5, 6)T_1^{-1}) & B(T_2(1, 4)(2, 3)(5, 6)T_2^{-1}) \end{bmatrix} \]
\[ A(y) = \begin{bmatrix} B((1, 4)(2, 3)(5, 6)) & B(e) \\ B(e) & B((1, 4)(2, 3)(5, 6)) \end{bmatrix} \]
\[ A(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

Verifying the the orders of the matrices. \( < A(x), A(y) > \) and \( |A(x)| = 6, |A(y)| = 2 \) can generate \( |A(x)A(y)| = 6 + 2 = 12 \).

Thus, we can write permutation representations of the above matrices using the following formula. For each matrix entry, \( a_{ij} = n \) implies \( t_i \) goes to \( t_j^n \). The labeling of \( t' s \) is given as follows from a field of order \( \mathbb{Z}_7 \setminus \{0\} \).
We will start by looking at the matrix $A$. Similarly, for all powers of $t$, this gives the permutation $\text{xx}$. This implies that we have the presentation for $G$.

Therefore, we have the presentation as follows:

$$G = \langle x, y, t | x^2 = (x^{-1}y)^2 = x^6 = t^{11} = (t, \text{Normalizer}(N, < t_1 >) > \rangle$$

Where the $\text{Normalizer}(N, < t_1 >)$ is the stabilizer of all powers of $t_1$ in $N$, such as $t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6$. Thus, we want the permutations in $N$ that stabilize $\{1, 3, 5, 7, 9, 11\}$ set wise. We get the following permutation group that stabilizes $N$.

Converting the permutation group into words in terms of $x$ and $y$, and also letting $t$ commute with the generators of the stabilizer. We have $x^{-1}$ being in the stabilizer. Therefore, we have the presentation as follows:

$$7^2 : m \; D_{12} = G = \langle x, y, t | y^2 = (x^{-1}y)^2 = x^6 = t^7 = (t^{x^{-1}} = t) >$$

<table>
<thead>
<tr>
<th>1. $t_1$</th>
<th>2. $t_2$</th>
<th>3. $t_1^2$</th>
<th>4. $t_2^2$</th>
<th>5. $t_1^3$</th>
<th>6. $t_2^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. $t_1^4$</td>
<td>8. $t_2^4$</td>
<td>9. $t_1^5$</td>
<td>10. $t_2^5$</td>
<td>11. $t_1^6$</td>
<td>12. $t_2^6$</td>
</tr>
</tbody>
</table>

Table 1.3: Labeling of $t_i'$s

<table>
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<tr>
<th>1. $t_1$</th>
<th>2. $t_2$</th>
<th>3. $t_1^2$</th>
<th>4. $t_2^2$</th>
<th>5. $t_1^3$</th>
<th>6. $t_2^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. $t_1^4$</td>
<td>8. $t_2^4$</td>
<td>9. $t_1^5$</td>
<td>10. $t_2^5$</td>
<td>11. $t_1^6$</td>
<td>12. $t_2^6$</td>
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</tbody>
</table>

Table 1.4: Permutations of $t_i'$s of $A(x)$

<table>
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<th>3. $t_1^2$</th>
<th>4. $t_2^2$</th>
<th>5. $t_1^3$</th>
<th>6. $t_2^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. $t_1^4$</td>
<td>8. $t_2^4$</td>
<td>9. $t_1^5$</td>
<td>10. $t_2^5$</td>
<td>11. $t_1^6$</td>
<td>12. $t_2^6$</td>
</tr>
</tbody>
</table>

Table 1.5: Permutations of $t_i'$s of $A(x)$

We will start by looking at the matrix $A(x)$. Since $a_{11} = 5$, therefore $t_1$ goes to $t_1^5$ and similarly for all powers of $t_1$. Also, since $a_{22} = 3$, $t_2$ is going to $t_2^3$. Therefore, we have the following:

This gives the permutation $xx = (1, 9, 7, 11, 3, 5)(2, 6, 4, 12, 8, 10)$ for $A(x)$. Now, for $A(y)$, since $a_{12} = 1$, then $t_1$ goes to $t_2$. Also, since $a_{21} = 1$, thus $t_2$ goes to $t_1$. We then have the following:

This gives the permutation $yy = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$ for $A(y)$.
1.3.2 $11^2 : m (5 : 2) \cdot 5$

We will write a presentation for the monomial progenitor $11^2 : m (5 : 2) \cdot 5$. $11^2$ represents 2 $t$'s of order 11.

Let $G = (5 : 2) \cdot 5 = \langle (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15), (1, 7, 5, 6, 4, 10, 3, 9, 2, 8)(11, 24, 15, 23, 14, 22, 13, 21, 12, 25)(16, 18, 20, 17, 19) \rangle$


Table 1.6: Character Table of $G$

| Classes | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | ...
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------
| Size    | 1     | 5     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | ...
| Order   | 1     | 2     | 5     | 5     | 5     | 5     | 5     | 5     | 5     | ...
| $X.1$   | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | ...
| $X.2$   | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | ...
| $X.3$   | 1     | -1    | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.4$   | 1     | 1     | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.5$   | 1     | -1    | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.6$   | 1     | 1     | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.7$   | 1     | -1    | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.8$   | 1     | -1    | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
| $X.9$   | 1     | 1     | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | $Z_1$ | ...
|        |       |       |       |       |       |       |       |       |       | ...
| ...    | ...   | ...   | ...   | ...   | ...   | ...   | ...   | ...   | ...   | ...

Now, we must induce a non-trivial linear character of $H$, we will use $X.9$ up to $G$ in this case. Next, we find the right transversals of $H$ in $G$. Using MAGMA, these are $\{ e, (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15) \}$

Which we label as:

$T_1 = e$


Now, the matrices $A(x)$ and $A(y)$ will be a representation of $G$ induced from the character in $H$, $X.9$. The general forms for $A(x)$ and $A(y)$ are

$$A(x) = \begin{bmatrix} B(T_1 x T_1^{-1}) & B(T_1 x T_2^{-1}) \\ B(T_2 x T_1^{-1}) & B(T_2 x T_2^{-1}) \end{bmatrix}$$

$$A(y) = \begin{bmatrix} B(T_1 y T_1^{-1}) & B(T_1 y T_2^{-1}) \\ B(T_2 y T_1^{-1}) & B(T_2 y T_2^{-1}) \end{bmatrix}$$
Table 1.7: Character Table of $H$

<table>
<thead>
<tr>
<th>Classes</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
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<th>$C_9$</th>
<th>$C_{10}$</th>
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</tr>
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<td>1</td>
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<td>$Z_1_3$</td>
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<td>$Z_1_3$</td>
<td>$Z_1_4$</td>
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<td>$Z_1_4$</td>
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</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Such that $B(x)$ is the value of the induced character for the class of $H$ containing $x$ (recall if $x \notin H$, then $B(x) = 0$).

also $y = (1, 7, 5, 6, 4, 10, 3, 9, 2, 8)(11, 24, 15, 23, 14, 22, 13, 21, 12, 25)(16, 18, 20, 17, 19)$

We also have classes of $H$
Class 1 = e
Class 2 = (1, 5, 4, 3, 2)(6, 10, 9, 8, 7)(11, 15, 14, 13, 12)(16, 20, 19, 18, 17)(21, 25, 24, 23, 22)
Class 3 = (1, 4, 2, 5, 3)(6, 9, 7, 10, 8)(11, 14, 12, 15, 13)(16, 19, 17, 20, 18)(21, 24, 22, 25, 23)

Class 23 = (1, 16, 7, 24, 14)(2, 17, 8, 25, 15)(3, 18, 9, 21, 11)(4, 19, 10, 22, 12)(5, 20, 6, 23, 13)
Class 24 = (1, 14, 24, 7, 16)(2, 15, 25, 8, 17)(3, 11, 21, 9, 18)(4, 12, 22, 10, 19)(5, 13, 23, 6, 20)
Class 25 = (1, 7, 14, 16, 24)(2, 8, 15, 17, 25)(3, 9, 11, 18, 21)(4, 10, 12, 19, 22)(5, 6, 13, 20, 23)

We then have
$B(T_1 x T_1^{-1}) = 0$
$B(T_1 x T_2^{-1}) = 1$
$B(T_2 x T_1^{-1}) = 1$
\[
B(T_2 x T_2^{-1}) = 0 \\
B(T_1 y T_1^{-1}) = 0 \\
B(T_1 y T_2^{-1}) = 4 \\
B(T_2 y T_1^{-1}) = 5 \\
B(T_2 y T_2^{-1}) = 0
\]

Substituting into \( A(x) = \) and \( A(y) = \) we have

\[
A(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A(y) = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}
\]

(With roots of unity represented as 4 and 5 from a field order of 11)

So, with \(< A(x), A(y) >\) and \(|A(x)| = 2, |A(y)| = 10\) can generate \(|A(x)A(y)| = 2 \cdot 5^2 = 50\)

Now, we can write permutation representations of the above matrices using the following formula. For each matrix entry, \( a_{ij} = n \) implies \( t_i \) goes to \( t_j^n \) The labeling of \( t_i \)'s is given as follows from a field of order 11.

**Table 1.8: Labeling of \( t_i \)'s**

| \( 1.t_1 \) | \( 2.t_2 \) | \( 3.t_1^2 \) | \( 4.t_2^2 \) | \( 5.t_1^3 \) | \( 6.t_2^3 \) | \( 7.t_1^4 \) | \( 8.t_2^4 \) | \( 9.t_1^5 \) | \( 10.t_2^5 \) | \( 11.t_1^6 \) | \( 12.t_2^6 \) | \( 13.t_1^7 \) | \( 14.t_2^7 \) | \( 15.t_1^8 \) | \( 16.t_2^8 \) | \( 17.t_1^9 \) | \( 18.t_2^9 \) | \( 19.t_1^{10} \) | \( 20.t_2^{10} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |

We will start by looking at the matrix \( A(x) \). Since \( a_{12} = 1 \) therefore \( t_1 \) goes to \( t_2 \) and similarly for all powers of \( t_1 \) such as \( t_1^2 \) going to \( t_2^2 \). Therefore, we have the following

**Table 1.9: Permutations of \( t_i \)'s of \( A(x) \)**

| \( 1.t_1 \) | \( 2.t_2 \) | \( 3.t_1^2 \) | \( 4.t_2^2 \) | \( 5.t_1^3 \) | \( 6.t_2^3 \) | \( 7.t_1^4 \) | \( 8.t_2^4 \) | \( 9.t_1^5 \) | \( 10.t_2^5 \) | \( 11.t_1^6 \) | \( 12.t_2^6 \) | \( 13.t_1^7 \) | \( 14.t_2^7 \) | \( 15.t_1^8 \) | \( 16.t_2^8 \) | \( 17.t_1^9 \) | \( 18.t_2^9 \) | \( 19.t_1^{10} \) | \( 20.t_2^{10} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |
| \( 2.t_2 \) | \( 1.t_1 \) | \( 4.t_2^2 \) | \( 3.t_1^2 \) | \( 6.t_2^3 \) | \( 5.t_1^3 \) | \( 8.t_2^4 \) | \( 7.t_1^4 \) | \( 10.t_2^5 \) | \( 9.t_1^5 \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |
| \( 11.t_1^6 \) | \( 12.t_2^6 \) | \( 13.t_1^7 \) | \( 14.t_2^7 \) | \( 15.t_1^8 \) | \( 16.t_2^8 \) | \( 17.t_1^9 \) | \( 18.t_2^9 \) | \( 19.t_1^{10} \) | \( 20.t_2^{10} \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |

This gives the permutation, \( xx = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20) \) for \( A(x) \). Now, for \( A(y) \), since \( a_{12} = 4 \), then \( t_1 \) goes to \( t_2^4 \). Also, since \( a_{21} = 5 \), thus \( t_2 \) goes to \( t_1^5 \). We
Table 1.10: Permutations of $t_i's$ of $A(y)$

<table>
<thead>
<tr>
<th></th>
<th>1. $t_1$</th>
<th>2. $t_2$</th>
<th>3. $t_1^2$</th>
<th>4. $t_2^2$</th>
<th>5. $t_1^5$</th>
<th>6. $t_2^5$</th>
<th>7. $t_1^4$</th>
<th>8. $t_2^4$</th>
<th>9. $t_1^3$</th>
<th>10. $t_2^3$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
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<td>↓</td>
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<tr>
<td>8.</td>
<td>$t_2^4$</td>
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<td>$t_1^4$</td>
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<td>$t_2^9$</td>
<td>$t_1^3$</td>
</tr>
<tr>
<td>11.</td>
<td>$t_1^6$</td>
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<td>$t_1^7$</td>
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<td>$t_2^7$</td>
<td>$t_1^6$</td>
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<td></td>
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<tr>
<td></td>
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<td>↓</td>
<td>↓</td>
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<td>↓</td>
<td>↓</td>
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</tr>
<tr>
<td>4.</td>
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<td>$t_1^2$</td>
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<td>$t_1^1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

then have the following

This gives the permutation,

$yy = (1, 8, 17, 6, 7, 10, 5, 2, 9, 18)(3, 16, 13, 12, 15, 20, 11, 4, 19, 14)$ for $A(y)$.

So, we have the presentation for $G$ being

$< x, y, t | x^2, y^{10}, (x * y)^5, (y^{-1} * x * y^2 * x * y^{-1}), t^{11}, (t, Normalizer(N, < t_1 >)) >$

Where the $Normalizer(N, < t_1 >)$ is the stabilizer of all powers of $t_1$ in $N$, such as $t_1, t_1^2, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}$. Thus, we want the permutations in N that stabilize $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ set wise. We get the following permutation that stabilizes $N$.

$(2, 8, 10, 18, 6)(4, 16, 20, 14, 12)$.

We now change the permutation into words in terms of $x$ and $y$, and letting $t$ commute with the generators of the stabilizer. Using MAGMA, we have $y^{-1}xy^{-2}$ being in the stabilizer. Therefore, the presentation of the progenitor $11^{*2} : (5 : 2)^*5$

$< x, y, t | x^2, y^{10}, (xy)^5, (y^{-1}xy^2xy^{-1}), t^{11}, (y^{-1}xy^{-2} = t) >$

$11^{*2} : (5 : 2)^*5 N$
1.3.3 11^3 :_{m} (25 : 3)

We will write a presentation for the monomial progenitor 11^3 :_{m} (25 : 3). 11^3 represents 3 t’s of order 11.

Let \( G = \langle (2, 6, 25)(3, 11, 19)(4, 16, 13)(5, 21, 7)(8, 10, 20)(9, 15, 14)(12, 24, 22)(17, 18, 23), (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25) \rangle \)

First we need to induce a non-trivial linear character from a subgroup \( H \) of \( G \) such that \( \frac{|G|}{|H|} = 3 \) (the number of t’s in our presentation). \( |G| = 75 \), therefore we must have \( |H| = 25 \).


Table 1.11: Character Table of \( G \)

<table>
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<th>Classes</th>
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<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_7 )</th>
<th>( C_8 )</th>
<th>( C_9 )</th>
<th>( C_{10} )</th>
<th>( C_{11} )</th>
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<td>Z1</td>
<td>Z1</td>
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<td>0</td>
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<td>Z2</td>
<td>Z2</td>
<td>Z1</td>
<td>Z1_4</td>
<td>Z1_3</td>
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<td>Z1</td>
<td>Z1_2</td>
<td>Z1_3</td>
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</tr>
</tbody>
</table>

(note: \( J \) is the 3rd root of unity)

Now, we must induce a non-trivial linear character of \( H \), we will use \( X.5 \) up to \( G \) in this case. Next, we find the right transversals of \( H \) in \( G \). Using MAGMA, these are


Which we label as:
\( T_1 = e \)
\( T_2 = (2, 6, 25)(3, 11, 19)(4, 16, 13)(5, 21, 7)(8, 10, 20)(9, 15, 14)(12, 24, 22)(17, 18, 23) \)
\( T_3 = (2, 25, 6)(3, 19, 11)(4, 13, 16)(5, 7, 21)(8, 20, 10)(9, 14, 15)(12, 22, 24)(17, 23, 18) \)
Table 1.12: Character Table of $H$

<table>
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<tr>
<th>Classes</th>
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<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
<th>$C_8$</th>
<th>$C_9$</th>
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</tr>
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<td>1</td>
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<td>...</td>
</tr>
<tr>
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<td>$Z_1$</td>
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<td>$Z_{14}$</td>
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<td>$Z_{13}$</td>
<td>$Z_{14}$</td>
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</tr>
<tr>
<td>$X.10$</td>
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<td>$Z_{14}$</td>
<td>$Z_{13}$</td>
<td>$Z_{12}$</td>
<td>$Z_1$</td>
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<td>$Z_{13}$</td>
<td>$Z_{14}$</td>
<td>$Z_1$</td>
<td>1 ...</td>
</tr>
</tbody>
</table>

Now, the matrices $A(x)$ and $A(y)$ will be a representation of $G$ induced from the character in $H$, $X.5$. The general forms for $A(x)$ and $A(y)$ are

$$A(x) = \begin{bmatrix}
B(T_1 x T_1^{-1}) & B(T_1 x T_2^{-1}) & B(T_1 x T_3^{-1}) \\
B(T_2 x T_1^{-1}) & B(T_2 x T_2^{-1}) & B(T_2 x T_3^{-1}) \\
B(T_3 x T_1^{-1}) & B(T_3 x T_2^{-1}) & B(T_3 x T_3^{-1})
\end{bmatrix}$$

$$A(y) = \begin{bmatrix}
B(T_1 y T_1^{-1}) & B(T_1 y T_2^{-1}) & B(T_1 y T_3^{-1}) \\
B(T_2 y T_1^{-1}) & B(T_2 y T_2^{-1}) & B(T_2 y T_3^{-1}) \\
B(T_3 y T_1^{-1}) & B(T_3 y T_2^{-1}) & B(T_3 y T_3^{-1})
\end{bmatrix}$$

Such that $B(x)$ is the value of the induced character for the class of $H$ containing $x$ (recall if $x \notin H$, then $B(x) = 0$).

Let $x = (2, 6, 25)(3, 11, 19)(4, 16, 13)(5, 21, 7)(8, 10, 20)(9, 15, 14)(12, 24, 22)(17, 18, 23)$

and $y = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)$

We also have classes of $H$

Class 1 = $e$

Class 2 = $(1, 22, 18, 14, 10)(2, 23, 19, 15, 6)(3, 24, 20, 11, 7)(4, 25, 16, 12, 8)$

$(5, 21, 17, 13, 9)$

Class 3 = $(1, 18, 10, 22, 14)(2, 19, 6, 23, 15)(3, 20, 7, 24, 11)(4, 16, 8, 25, 12)$

$(5, 17, 9, 21, 13)$
Substituting into formula. For each matrix entry, a |.

Now, (1, 13, 25, 7, 19)(2, 14, 21, 8, 20)(3, 15, 22, 9, 16)(4, 11, 23, 10, 17)
(5, 12, 24, 6, 18)

Class 23 = (1, 13, 25, 7, 19)(2, 14, 21, 8, 20)(3, 15, 22, 9, 16)(4, 11, 23, 10, 17)
(5, 12, 24, 6, 18)

Class 24 = (1, 19, 7, 25, 13)(2, 20, 8, 21, 14)(3, 16, 9, 22, 15)(4, 17, 10, 23, 11)
(5, 18, 6, 24, 12)

Class 25 = (1, 25, 19, 13, 7)(2, 21, 20, 14, 8)(3, 22, 16, 15, 9)(4, 23, 17, 11, 10)
(5, 24, 18, 12, 6)

We then have:

\[
B(T_1xT_1^{-1}) = 0 \quad B(T_1yT_1^{-1}) = 4
\]
\[
B(T_1xT_2^{-1}) = 1 \quad B(T_1yT_2^{-1}) = 0
\]
\[
B(T_1xT_3^{-1}) = 0 \quad B(T_1yT_3^{-1}) = 0
\]
\[
B(T_2xT_1^{-1}) = 0 \quad B(T_2yT_1^{-1}) = 0
\]
\[
B(T_2xT_2^{-1}) = 0 \quad B(T_2yT_2^{-1}) = 5
\]
\[
B(T_2xT_3^{-1}) = 1 \quad B(T_2yT_3^{-1}) = 0
\]
\[
B(T_3xT_1^{-1}) = 1 \quad B(T_3yT_1^{-1}) = 0
\]
\[
B(T_3xT_2^{-1}) = 0 \quad B(T_3yT_2^{-1}) = 0
\]
\[
B(T_3xT_3^{-1}) = 0 \quad B(T_3yT_3^{-1}) = 5
\]

Substituting into \(A(x) = \) and \(A(y) = \) we have

\[
A(x) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \quad A(y) = \begin{bmatrix}
4 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{bmatrix}
\]

(With roots of unity represented as 4 and 5 from a field of order 11)

Now, \(< A(x), A(y) > \) with \(|A(x)| = 3, |A(y)| = 5 \) can generate
\(|A(x), A(y)| = 3 * 5^2 = 75

Now, we can write permutation representations of the above matrices using the following

Table 1.13: Labeling of \(t'_j\)s

\begin{tabular}{cccccccccccc}
1.t1 & 2.t2 & 3.t3 & 4.t4 & 5.t5 & 6.t6 & 7.t7 & 8.t8 & 9.t9 & 10.t10 & 11.t11 & 12.t12 \\
21.t25 & 22.t26 & 23.t27 & 24.t28 & 25.t29 & 26.t30 & 27.t31 & 28.t32 & 29.t33 & 30.t34 & 31.t35 & 32.t36 \\
\end{tabular}

We will start by looking at the matrix \(A(x)\). Since \(a_{12} = 1\), therefore \(t_1\) goes
to \(t_2\). Also, \(a_{23} = 1, a_{31} = 1\) indicates \(t_2\) goes to \(t_3\), \(t_3\) goes to back to \(t_1\) and similarly
for all powers of \(t_1, t_2, t_3\). Therefore, we have the following
This gives the permutation $xx = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)(19, 20, 21)(22, 23, 24)(25, 26, 27)(28, 29, 30)$ for $A(x)$. Now, for $A(y)$, since $a_{11} = 4$, then $t_1$ goes to $t_1^4$. Also, $a_{22} = 5$ implies that $t_2$ goes to $t_2^5$, and $a_{33} = 5$ also implies that $t_3$ goes to $t_3^5$. We then have the following

This gives the permutation $yy = (1, 10, 13, 25, 7)(2, 14, 8, 11, 26)(3, 15, 9, 12, 21)(4, 22, 28, 19, 16)(5, 29, 17, 23, 20)(6, 30, 18, 24, 21)$ for $A(y)$. This gives us the presentation for $G$ being

\[
\langle x, y, t | x^3, y^5, (x \ast y)^3, (y^{-1} \ast x)^3, (y^{-1} \ast x^{-1})^3, y^{11}, (t, Normalizer(N, < t_1 >)) \rangle
\]

Where the $Normalizer(N, < t_1 >)$ is the stabilizer of all powers of $t_1$ in $N$, such as $t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}$. Thus, we want the permutations in $N$ that stabilize $\{1, 4, 7, 10, 13, 16, 19, 22, 25, 28\}$ set wise.

We get the following permutations that stabilizes $N$.

- $(2, 26, 11, 8, 14)(3, 15, 9, 12, 27)(5, 20, 23, 17, 29)(6, 30, 18, 24, 21)$
- $(1, 7, 25, 13, 10)(3, 12, 15, 27, 9)(4, 16, 19, 28, 22)(6, 24, 30, 21, 18)$

We now change the permutation into words in terms of $x$ and $y$, and letting $t$ commute with the generators of the stabilizer. Using MAGMA, we have $xya^{-1}ya^{-2}$, and $(x^{-1}, y)$ being all in the stabilizer. Therefore, we have the presentation of the progenitor
to be

\[
<x, y, t | x^3, y^5, (x \cdot y)^3, (y^{-1} \cdot x)^3, (y^{-1} \cdot x^{-1})^3, t^{11}, (t^{xy^{-1}y^{-2}} = t), (t^{(x^{-1}, y)} = t) >
\]

\[11^{x^3 : m (25 : 3)}\]
1.3.4 \(11^4 \cdot_m (4 : 5)^5\)

We will write a presentation for the monomial progenitor \(11^4 \cdot_m (4 : 5)^5\). \(11^4\) represents 4 t’s of order 11.

Let \(G = \langle (1, 20, 21, 8)(2, 16, 22, 9)(3, 17, 23, 10)(4, 18, 24, 6)(5, 19, 25, 7),
(1, 2, 3, 4, 5)(6, 19, 21, 15, 10, 18, 25, 14, 9, 17, 24, 13, 8, 16, 23, 12, 7, 20, 22, 11)\rangle\)

First we need to induce a non-trivial linear character from a subgroup \(H\) of \(G\) such that \(\frac{|G|}{|H|} = 4\) (the number of t’s in our presentation).

\(|G| = 100\), therefore we must have \(|H| = 25\). So, let \(H = S[9]\) subgroup.
Then \(S[9]\) subgroup = \(|H| = 25\)

Table 1.16: Character Table of \(G\)

<table>
<thead>
<tr>
<th>Classes</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_4)</th>
<th>(C_5)</th>
<th>(C_6)</th>
<th>(C_7)</th>
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<th>(C_9)</th>
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<td>...</td>
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<td>...</td>
</tr>
</tbody>
</table>

(note: \(I\) is the 4th root of unity)

Now, we must induce a non-trivial linear character of \(H\), we will use \(X.9\) up to \(G\) in this case. Next, we find the right transversals of \(H\) in \(G\). Using MAGMA, these are

\[
\{e, (1, 20, 21, 8)(2, 16, 22, 9)(3, 17, 23, 10)(4, 18, 24, 6)(5, 19, 25, 7),
(1, 8, 21, 20)(2, 9, 22, 16)(3, 10, 23, 17)(4, 6, 24, 18)(5, 7, 25, 19),
\]

Which we label as:
\(\text{T}_1 = e\)
\(\text{T}_2 = (1, 20, 21, 8)(2, 16, 22, 9)(3, 17, 23, 10)(4, 18, 24, 6)(5, 19, 25, 7)\)
\(\text{T}_3 = (1, 8, 21, 20)(2, 9, 22, 16)(3, 10, 23, 17)(4, 6, 24, 18)(5, 7, 25, 19)\)
Now, the matrices $A(x)$ and $A(y)$ will be a representation of $G$ induced from the character in $H$, $X.9$. The general forms for $A(x)$ and $A(y)$ are

$$A(x) = \begin{bmatrix}
B(T_1 x T_1^{-1}) & B(T_1 x T_2^{-1}) & B(T_1 x T_3^{-1}) & B(T_1 x T_4^{-1}) \\
B(T_2 x T_1^{-1}) & B(T_2 x T_2^{-1}) & B(T_2 x T_3^{-1}) & B(T_2 x T_4^{-1}) \\
B(T_3 x T_1^{-1}) & B(T_3 x T_2^{-1}) & B(T_3 x T_3^{-1}) & B(T_3 x T_4^{-1}) \\
B(T_4 x T_1^{-1}) & B(T_4 x T_2^{-1}) & B(T_4 x T_3^{-1}) & B(T_4 x T_4^{-1})
\end{bmatrix}$$

$$A(y) = \begin{bmatrix}
B(T_1 y T_1^{-1}) & B(T_1 y T_2^{-1}) & B(T_1 y T_3^{-1}) & B(T_1 y T_4^{-1}) \\
B(T_2 y T_1^{-1}) & B(T_2 y T_2^{-1}) & B(T_2 y T_3^{-1}) & B(T_2 y T_4^{-1}) \\
B(T_3 y T_1^{-1}) & B(T_3 y T_2^{-1}) & B(T_3 y T_3^{-1}) & B(T_3 y T_4^{-1}) \\
B(T_4 y T_1^{-1}) & B(T_4 y T_2^{-1}) & B(T_4 y T_3^{-1}) & B(T_4 y T_4^{-1})
\end{bmatrix}$$

Such that $B(x)$ is the value of the induced character for the class of $H$ containing $x$ (recall if $x \notin H$, then $B(x) = 0$).

Let $x = (1, 20, 21, 8)(2, 16, 22, 9)(3, 17, 23, 10)(4, 18, 24, 6)(5, 19, 25, 7)$

and $y = (1, 2, 3, 4, 5)(6, 19, 21, 15, 10, 18, 25, 14, 9, 17, 24, 13, 8, 16, 23, 12, 7, 20, 22, 11)$

We also have classes of $H$

Class 1 = $e$

Class 2 = $(1, 5, 4, 3, 2)(6, 10, 9, 8, 7)(11, 15, 14, 13, 12)(16, 20, 19, 18, 17)$

$(21, 25, 24, 23, 22)$

Class 3 = $(1, 4, 2, 5, 3)(6, 9, 7, 10, 8)(11, 14, 12, 15, 13)(16, 19, 17, 20, 18)$

$(21, 24, 22, 25, 23)$
We then have:

\[
B(T_1 T_1^{-1}) = 0 \quad B(T_3 T_1^{-1}) = 1 \quad B(T_1 y T_1^{-1}) = 0 \quad B(T_3 y T_1^{-1}) = 0 \\
B(T_1 T_2^{-1}) = 1 \quad B(T_3 T_2^{-1}) = 0 \quad B(T_1 y T_2^{-1}) = 0 \quad B(T_3 y T_2^{-1}) = 0 \\
B(T_1 T_3^{-1}) = 0 \quad B(T_3 T_3^{-1}) = 0 \quad B(T_1 y T_3^{-1}) = 1 \quad B(T_3 y T_3^{-1}) = 0 \\
B(T_1 T_4^{-1}) = 0 \quad B(T_3 T_4^{-1}) = 0 \quad B(T_1 y T_4^{-1}) = 0 \quad B(T_3 y T_4^{-1}) = 9 \\
B(T_2 T_1^{-1}) = 0 \quad B(T_4 T_1^{-1}) = 0 \quad B(T_2 y T_1^{-1}) = 4 \quad B(T_4 y T_1^{-1}) = 0 \\
B(T_2 T_2^{-1}) = 0 \quad B(T_4 T_2^{-1}) = 0 \quad B(T_2 y T_2^{-1}) = 0 \quad B(T_4 y T_2^{-1}) = 3 \\
B(T_2 T_3^{-1}) = 0 \quad B(T_4 T_3^{-1}) = 1 \quad B(T_2 y T_3^{-1}) = 0 \quad B(T_4 y T_3^{-1}) = 0 \\
B(T_2 T_4^{-1}) = 1 \quad B(T_4 T_4^{-1}) = 0 \quad B(T_2 y T_4^{-1}) = 0 \quad B(T_4 y T_4^{-1}) = 0 \\
\]

Substituting into \( A(x) = \) and \( A(y) = \) we have

\[
A(x) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \quad A(y) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 \\
0 & 3 & 0 & 0
\end{bmatrix}
\]

(With roots of unity represented as 4, 5, 9 and 3 from a field order of 11)

Now, < \(A(x), A(y)\) > with \(|A(x)| = 4, |A(y)| = 20\) can generate

\(|A(x)A(y)| = 2^2 \cdot (5 \cdot 2^2) = 100\)

Now, we can write permutation representations of the above matrices using the following formula. For each matrix entry, \(a_{ij} = n\) implies \(t_i\) goes to \(t_j^n\) The labeling of \(t_j^n\)’s is given as follows from a field of order 11.

<table>
<thead>
<tr>
<th>1.t₁</th>
<th>2.t₂</th>
<th>3.t₃</th>
<th>4.t₄</th>
<th>5.t₁²</th>
<th>6.t₂²</th>
<th>7.t₃²</th>
<th>8.t₄²</th>
<th>9.t₁³</th>
<th>10.t₂³</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.t₃³</td>
<td>12.t₄³</td>
<td>13.t₁⁴</td>
<td>14.t₂⁴</td>
<td>15.t₃⁴</td>
<td>16.t₄⁴</td>
<td>17.t₁⁵</td>
<td>18.t₂⁵</td>
<td>19.t₃⁵</td>
<td>20.t₄⁵</td>
</tr>
<tr>
<td>21.t₁⁶</td>
<td>22.t₂⁶</td>
<td>23.t₃⁶</td>
<td>24.t₄⁶</td>
<td>25.t₁⁷</td>
<td>26.t₂⁷</td>
<td>27.t₃⁷</td>
<td>28.t₄⁸</td>
<td>29.t₁⁹</td>
<td>30.t₄⁹</td>
</tr>
<tr>
<td>31.t₃⁸</td>
<td>32.t₄⁸</td>
<td>33.t₁⁹</td>
<td>34.t₂⁹</td>
<td>35.t₃⁹</td>
<td>36.t₄⁹</td>
<td>37.t₁¹⁰</td>
<td>38.t₂¹⁰</td>
<td>39.t₃¹⁰</td>
<td>40.t₄¹⁰</td>
</tr>
</tbody>
</table>

We will start by looking at the matrix \(A(x)\). Since \(a_{12} = 1\), therefore \(t₁\) goes to \(t₂\). Also, \(a_{24} = 1, a_{43} = 1, a_{3₁} = 1\) indicates \(t₂\) goes to \(t₄\), \(t₄\) goes to \(t₃\), \(t₃\) goes to back to \(t₁\), and similarly for all powers of \(t₁, t₂, t₃, t₄\). Therefore, we have the following
Table 1.19: Permutations of $t'_i$s of $A(x)$

<table>
<thead>
<tr>
<th>$t'_1$</th>
<th>$t'_2$</th>
<th>$t'_3$</th>
<th>$t'_4$</th>
<th>$t'_5$</th>
<th>$t'_6$</th>
<th>$t'_7$</th>
<th>$t'_8$</th>
<th>$t'_9$</th>
<th>$t'_10$</th>
</tr>
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<tbody>
<tr>
<td>\downarrow</td>
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</tr>
<tr>
<td>11.t^2_3</td>
<td>12.t^4_3</td>
<td>13.t^1_3</td>
<td>14.t^2_4</td>
<td>15.t^3_3</td>
<td>16.t^4_3</td>
<td>17.t^1_4</td>
<td>18.t^2_5</td>
<td>19.t^3_5</td>
<td>20.t^4_5</td>
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</tr>
<tr>
<td>21.t^6_1</td>
<td>22.t^6_2</td>
<td>23.t^6_3</td>
<td>24.t^6_4</td>
<td>25.t^6_5</td>
<td>26.t^6_6</td>
<td>27.t^6_7</td>
<td>28.t^6_8</td>
<td>29.t^6_9</td>
<td>30.t^6_10</td>
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<tr>
<td>31.t^8_3</td>
<td>32.t^8_4</td>
<td>33.t^8_5</td>
<td>34.t^8_6</td>
<td>35.t^8_7</td>
<td>36.t^8_8</td>
<td>37.t^8_9</td>
<td>38.t^8_10</td>
<td>39.t^8_11</td>
<td>40.t^8_12</td>
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</tr>
<tr>
<td>21.t^6_1</td>
<td>22.t^6_2</td>
<td>23.t^6_3</td>
<td>24.t^6_4</td>
<td>25.t^6_5</td>
<td>26.t^6_6</td>
<td>27.t^6_7</td>
<td>28.t^6_8</td>
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<td>30.t^6_10</td>
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<tr>
<td>31.t^8_3</td>
<td>32.t^8_4</td>
<td>33.t^8_5</td>
<td>34.t^8_6</td>
<td>35.t^8_7</td>
<td>36.t^8_8</td>
<td>37.t^8_9</td>
<td>38.t^8_10</td>
<td>39.t^8_11</td>
<td>40.t^8_12</td>
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</tbody>
</table>

Table 1.20: Permutations of $t'_i$s of $A(y)$

<table>
<thead>
<tr>
<th>$t'_1$</th>
<th>$t'_2$</th>
<th>$t'_3$</th>
<th>$t'_4$</th>
<th>$t'_5$</th>
<th>$t'_6$</th>
<th>$t'_7$</th>
<th>$t'_8$</th>
<th>$t'_9$</th>
<th>$t'_10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>\downarrow</td>
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<td>\downarrow</td>
</tr>
<tr>
<td>35.t^9_3</td>
<td>1.t^9_1</td>
<td>12.t^9_4</td>
<td>14.t^9_2</td>
<td>27.t^9_3</td>
<td>5.t^9_1</td>
<td>24.t^9_6</td>
<td>30.t^9_8</td>
<td>19.t^9_5</td>
<td>9.t^9_3</td>
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</tr>
<tr>
<td>21.t^10_1</td>
<td>22.t^10_2</td>
<td>23.t^10_3</td>
<td>24.t^10_4</td>
<td>25.t^10_5</td>
<td>26.t^10_6</td>
<td>27.t^10_7</td>
<td>28.t^10_8</td>
<td>29.t^10_9</td>
<td>30.t^10_10</td>
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</tr>
<tr>
<td>31.t^10_8</td>
<td>32.t^10_9</td>
<td>33.t^10_10</td>
<td>34.t^10_11</td>
<td>35.t^10_12</td>
<td>36.t^10_13</td>
<td>37.t^10_14</td>
<td>38.t^10_15</td>
<td>39.t^10_16</td>
<td>40.t^10_17</td>
</tr>
<tr>
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</tbody>
</table>

This gives the permutation $xx = (1, 2, 4, 3)(5, 6, 8, 7)(9, 10, 12, 11)(13, 14, 16, 15)(17, 18, 20, 19)(21, 22, 24, 23)(25, 26, 28, 27)(29, 30, 32, 31)(33, 34, 36, 35)(37, 38, 40, 39)$ for $A(x)$. Now, for $A(y)$ with $a_{13} = 1$, would indicate $t_1$ goes to $t^9_3$ since $a_{34} = 9$. Also, $a_{42} = 3$ implies that $t^9_3$ goes to $t^3_{4} \Rightarrow t^5_{4}$, and $a_{21} = 4$ also implies that $t^5_{4}$ goes to $t^5_{4} \Rightarrow t^9_{3}$. We then have the following

This gives the permutation $yy = (1, 35, 20, 34, 33, 15, 4, 14, 13, 11, 36, 10, 9, 19, 16, 18, 17, 3, 12, 2)(5, 27, 40, 26, 25, 31, 8, 30, 29, 23, 28, 22, 21, 39, 32, 38, 37, 7, 24, 6)$ for $A(y)$.

This gives us the presentation for $G$ being
<x, y, t | x^4, y^{20}, (x * y)^5, (x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^2 * y), (x^{-1} * y^{-2} * x^{-1} * y * x^2 * y), (y^{-3} * x^2 * y^{-2} * x), t^{11}, (t, Normalizer(N, < t_1 >)) >

Where the Normalizer(N, < t_1 >) is the stabilizer of all powers of $t_1$ in $N$, such as $t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}$. Thus, we want the permutations in $N$ that stabilize \{1, 5, 9, 13, 17, 21, 25, 29, 33, 37\} set wise.

We get the following permutations that stabilizes $N$.

(2, 14, 18, 34, 10)(3, 35, 15, 11, 19)(4, 12, 36, 20, 16)(6, 30, 38, 26, 22)  
(7, 27, 31, 23, 39)(8, 24, 28, 40, 32),  
(1, 13, 17, 33, 9)(3, 35, 15, 11, 19)(4, 20, 12, 16, 36)(5, 29, 37, 25, 21)  
(7, 27, 31, 23, 39)(8, 40, 24, 32, 28)

We now change the permutation into words in terms of $x$ and $y$, and letting $t$ commute with the generators of the stabilizer. Using MAGMA, we have $yx$, and $xy^2x$ being all in the stabilizer. Therefore, the presentation of the progenitor $11^*^4 : _m (4 : 5)^5 \Rightarrow$

< x, y, t | x^4, y^{20}, (xy)^5, (x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^2 * y), (x^{-1} * y^{-2} * x^{-1} * y * x^2 * y), (y^{-3} * x^2 * y^{-2} * x), t^{11}, (tyx = t), (t^{xy} = t) >

11^*^4 : _m (4 : 5)^5
Chapter 2

Double Coset Enumeration

2.1 Preliminaries

Definition 4.1.1 (Right Coset). Let \( K \) be a subgroup of \( G \) and \( h \in G \). A right coset of \( K \) in \( G \) is the subset of \( G \) dened as

\[ Kh = \{kh | k \in K \}, \]

where \( h \) is a representative of \( Kh \).

Definition 4.1.2 (Double Coset). Let \( H \) and \( K \) be subgroups of \( G \). Then we define a double coset of \( G \) as

\[ HgK = \{Hgk | k \in K, g \in G\}. \]

We note that every double coset is composed of single cosets and \( G \) is the union of all double cosets of \( G \).

Definition 4.1.3 (Stabiliser). The stabiliser of \( w \) in \( N \), where \( w \) in a word of \( tis \) is given by

\[ Nw = \{n \in N | w^n = w\}. \]

Definition 4.1.4 (Coset Stabiliser). The coset stabiliser of \( w \) in \( Nw \), where \( w \) is a word of \( tis \) is given by

\[ N(w) = \{n \in N | Nw^n = Nw\}. \]

We note that \( Nw \leq N(w) \).

Definition 4.1.5 (G-Orbit). Let \( x \in X \). The set \( XG = \{xg | g \in G\} \) is the \( G \)-orbit.

Definition 4.1.6 (Maximal Subgroup). A subgroup \( M \neq 1 \leq G \) is a maximal normal subgroup of \( G \) if there is no normal subgroup \( N \) of \( G \) with \( M < N < G \). [Rot12]
2.2 Permutation Progenitor Double Coset Enumeration

2.2.1 Construction of $S_5$ over $D_6$

Consider the group $N = 2^{*6} \cdot D_6$ factored by the relations $(yxt)^3, (xt)^5, (yt^x)^4 = 1$

That is, $G = S_5 \cong 2^{*6} \cdot D_6$/\langle (yxt)^3, (xt)^5, (yt^x)^4 = 1 \rangle$

With $N$ being transitive on 6 letters of order 2.

$2^{*6} = (t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 > * < t_6)$

$N = D_6$

$x \sim (1, 2, 3, 4, 5, 6)$

$y \sim (1, 4)(2, 3)(5, 6)$

$t \sim t_1$

We will first expand the relations:

\begin{align*}
(1) \; (yxt)^3 &= 1 \\
\Rightarrow (yxt)(yxt)(yxt) &= 1 \\
\Rightarrow (yx^3t_1^2)(yt_1)(xt_1) &= 1 \\
\Rightarrow yxt_1t_2 &= 1 \\
\Rightarrow yxt_2t_3 &= 1 \\
\Rightarrow yxt_3t_4 &= t_1 \\
\Rightarrow Nt_1t_5 &= Nt_1 \\
\Rightarrow (yt^x)^4 &= 1 \\
\Rightarrow (yt_2)(yt_2)(yt_2)(yt_2) &= 1 \\
\Rightarrow (yx^3t_1^2)(yt_1)(xt_1) &= 1 \\
\Rightarrow y^{-1}t_2yt_2y^{-1}t_2yt_2 &= 1 \\
\Rightarrow t_3^2t_2t_3 &= 1 \\
\Rightarrow t_3t_2t_3 &= 1 \\
\Rightarrow t_3t_2 &= t_3 \\
\Rightarrow Nt_2t_6 &= Nt_2 \\
\Rightarrow Nt_2t_6 &= Nt_2 \\
\end{align*}

We can also conjugate all the previous relation by all elements of $x$ and $y$ from $N = D_6$. An example of this would be $(Nt_1t_5)^x = (N_1)^x$. Since, $x = (1, 2, 3, 4, 5, 6)$, we have $Nt_2t_6 = Nt_2$. Taking the elements

$D_6 = \{e, (1, 2, 3, 4, 5, 6), (1, 4)(2, 3)(5, 6), (1, 6, 5, 4, 3, 2), (1, 3, 5)(2, 4, 6), (1, 3)(4, 6), (1, 5)(2, 4), (1, 5, 3)(2, 6, 4), (1, 4)(2, 5)(3, 6), (1, 2)(3, 6)(4, 5), (1, 6)(2, 5)(3, 4), (2, 6)(3, 5)\}$

So, we then obtain the following additional relations:

\begin{align*}
\text{For } Nt_1t_5 &= Nt_1, \text{ we have:} \\
(yxt_1t_5 = t_1)^e &\iff yxt_1t_5 = t_1 \\
(yxt_1t_5 = t_1)^x &\iff x^3yt_2t_6 = t_2 \\
(yxt_1t_5 = t_1)^y &\iff xy_4t_6 = t_4 \\
(yxt_1t_5 = t_1)^{-1} &\iff xy_6t_4 = t_6 \\
(yxt_1t_5 = t_1)^{-2} &\iff xy_3t_1 = t_3 \\
(yxt_1t_5 = t_1)^{-3} &\iff x^3yt_6t_2 = t_6 \\
(yxt_1t_5 = t_1) &\iff x^3yt_3t_5 = t_3 \\
\end{align*}
For $Nt_5 t_4 = Nt_1 t_2 t_3$, we have:

\[
\begin{align*}
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^x & \iff x^{-1} t_5 t_4 t_3 = t_1 t_2 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{yx} & \iff x t_1 t_2 t_3 = t_5 t_4 \\
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^x & \iff x^{-1} t_6 t_5 t_4 = t_2 t_3 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{x^2} & \iff x^{-1} t_3 t_2 t_1 = t_5 t_6 \\
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^y & \iff x t_6 t_1 t_2 = t_4 t_3 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{x^3} & \iff x^{-1} t_2 t_1 t_6 = t_4 t_5 \\
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^{-1} & \iff x^{-1} t_4 t_3 t_2 = t_6 t_1 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{x^2 y} & \iff x t_4 t_5 t_6 = t_2 t_1 \\
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^{x^2} & \iff x^{-1} t_1 t_6 t_5 = t_3 t_4 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{y^2} & \iff x t_2 t_3 t_4 = t_6 t_5 \\
(x^{-1} t_5 t_4 t_3 = t_1 t_2)^{xy} & \iff x t_5 t_6 t_1 = t_3 t_2 & (x^{-1} t_5 t_4 t_3 = t_1 t_2)^{x^3 y} & \iff x t_3 t_4 t_5 = t_1 t_6 \\
\end{align*}
\]

For $t_3 t_2 = t_2 t_3$, we have:

\[
\begin{align*}
(t_3 t_2 = t_2 t_3)^x & \iff t_3 t_2 = t_2 t_3 & (t_3 t_2 = t_2 t_3)^{yx} & \iff t_3 t_4 = t_4 t_3 \\
(t_3 t_2 = t_2 t_3)^x & \iff t_4 t_3 = t_3 t_4 & (t_3 t_2 = t_2 t_3)^{x^2} & \iff t_1 t_6 = t_6 t_1 \\
(t_3 t_2 = t_2 t_3)^y & \iff t_2 t_3 = t_3 t_2 & (t_3 t_2 = t_2 t_3)^{x^3} & \iff t_5 t_6 = t_6 t_5 \\
(t_3 t_2 = t_2 t_3)^{-1} & \iff t_2 t_1 = t_1 t_2 & (t_3 t_2 = t_2 t_3)^{x^2 y} & \iff t_6 t_1 = t_1 t_6 \\
(t_3 t_2 = t_2 t_3)^{x^2} & \iff t_5 t_4 = t_4 t_5 & (t_3 t_2 = t_2 t_3)^{y^2} & \iff t_4 t_5 = t_5 t_4 \\
(t_3 t_2 = t_2 t_3)^{xy} & \iff t_1 t_2 = t_2 t_1 & (t_3 t_2 = t_2 t_3)^{x^3 y} & \iff t_6 t_5 = t_5 t_6 \\
\end{align*}
\]

We can also solve and use these relations to show the relations between cosets. Beginning the double coset enumeration, we first need to calculate the total number of unique cosets of $N$ in $G$. This is the index of $G$ in $N$. The index will be the order of $G$ divided by the order of $N$.

$$\frac{G}{N} = \frac{120}{12} = 10.$$ 

Now we know that we will have 10 unique single cosets.

Also, using the magma code "#DoubleCosets(G,sub < G | x, y >, sub < G | x, y >);", you can calculate the number of double cosets in your double coset enumeration.

```magma
> #DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
3
```

This lets us know that there will be 3 unique double cosets. We will then use this information as we do the double coset enumeration of $G$ over $N$.

**1st Double Coset [*]**

We have that $N e N = \{N e^n | n \in \mathbb{N}\} = \{N\}$.

$N$ is transitive on $\{1, 2, 3, 4, 5, 6\}$; moreover, $\frac{N}{N} = \frac{12}{12} = 1$.

Thus, this indicates that there is 1 single coset in the double coset $N e N$. Therefore, orbit of $N$ on $\{1, 2, 3, 4, 5, 6\}$ is the single orbit $\{1, 2, 3, 4, 5, 6\}$. Now, select a representative $t_i$ from orbit. We will select 1 from the orbit in this case.

$1 \in \{1, 2, 3, 4, 5, 6\} \Rightarrow N t_1 \in \{1\}$

Now, we shall extend $N t_1$ to a new double coset $[1]$. 


2nd Double Coset [1]

We now have that \( Nt_1N = \{Nt_1^n | n \in \mathbb{N}\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\} \).

The coset stabilizer \( N^{(1)} \) are elements of \( D_6 \) that fix 1. So, we then have \( N^{(1)} \geq N^1 = <(2,6)(3,5)> \Rightarrow \{e,(2,6)(3,5)\} \) which shows us that there are two elements in the coset stabilizer. This would then imply that the number of single cosets in \( Nt_1N \) are \( \frac{N}{N^{(1)}} = \frac{12}{2} = 6 \). Thus, The orbits of \( N^{(1)} \) on \( \{1,2,3,4,5,6\} \) would be \( \{1\}, \{2,6\}, \{3,5\}, \{4\} \).

From this, we could see that 2 and 6, and also 3 and 5 share the same orbit due to the coset stabilizer containing \((2,6)(3,5)\). We will now select a representative \( t_i \) from each orbit.

1 \( \in \{1\} \Rightarrow Nt_1t_1 \)

Since each \( t_i \) is of order 2, \( t_1 * t_1 = e \), which implies that \( Nt_1t_1 = Ne \in [\ast] \)

2 \( \in \{2,6\} \Rightarrow Nt_1t_2 \)

This takes both \( Nt_1t_2 \) and \( Nt_1t_6 \) to a new double coset [12]

5 \( \in \{3,5\} \Rightarrow Nt_1t_5 \)

From the expansion of the relation \((yxt)^3\), we have that \( Nt_1t_5 = Nt_1 \). This would indicate that both \( Nt_1t_5 \) and \( Nt_1t_3 \) loop back to [1]

4 \( \in \{4\} \Rightarrow Nt_1t_4 \)

Which we will investigate later on.

3rd Double Coset [12]

\( Nt_1t_2N = \{Nt_1t_2^n | n \in \mathbb{N}\} = \{Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_5, Nt_1t_6\} \).

The coset stabilizer \( N^{(12)} \) are elements of \( D_6 \) that fix both 1 and 2. Which implies that \( N^{(12)} \geq N^{12} = <e> \) which happens to only be the identity \( e \).

Let us further investigate to see if there are any other elements in the stabilizing group \( N^{(12)} \). Looking at relation (1) \( yxt_1t_5 = t_1 \), and from earlier seeing that, \((yxt_1t_5 = t_1)^x \Rightarrow x^3yt_1t_6 = t_2 \), which then implies,

\( t_1x^3yt_6 = t_1t_2 \) (multiplying by \( t_1 \) on the right hand side).

\( \Rightarrow x^3yt_1t_6 = t_1t_2 \).

From the relation (2) \( x^{-1}t_3t_4t_3 = t_1t_2 \), and also from earlier seeing that,

\( (x^{-1}t_3t_4t_3 = t_1t_2)^x \Rightarrow x^{-1}t_2t_1t_6 = t_4t_5 \).

Finally, from relation (3) \( t_3t_2 = t_2t_3 \), and also from earlier seeing that,

\( (t_3t_2 = t_2t_3)^x \Rightarrow t_3t_1 = t_1t_2 \).

We have, \( x^{-1}t_1t_2t_6 = t_4t_5 \) (substituting (3) \( t_2t_1 = t_1t_2 \) into (2) \( x^{-1}t_2t_1t_6 = t_4t_5 \)).

From this, we do see that both \( Nt_1t_2t_6 = Nt_1t_2 \), and \( Nt_1t_2t_6 = Nt_1t_5 \). Which then implies that \( Nt_1t_2 = Nt_4t_5 \).

Also, having \( (t_3t_2 = t_2t_3)^x \Rightarrow t_5t_4 = t_4t_5 \). We have,

\( \Rightarrow Nt_1t_2 = Nt_5t_4 \).

From these two equalities, we would have that \((Nt_1t_2)^x = Nt_4t_5 = Nt_1t_2 \), and that \((Nt_1t_2)^yx = Nt_5t_4 = Nt_1t_2 \). Which indicates that both elements \( x^3 \) and \( yx \) are in the coset stabilizing group. Thus, The coset stabilizer becomes \( N^{(12)} \geq N^{12} = <(1,5)(2,4),(1,4)(2,5)(3,6)> \).

So, with the both \( yx \) and \( x^3 \) being of order 2, this implies that the coset stabilizer is of
order 4, and the number of single cosets in $N_{t_1}t_2N$ are $\frac{N}{N(12)} = \frac{12}{4} = 3$. Therefore, The orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 4, 5\}$, and $\{3, 6\}$. We will now select a representative $t_i$ from each orbit.

$2 \in \{1, 2, 4, 5\} \Rightarrow N_{t_1}t_2t_2$. Since each $t_i$ is of order 2, $t_2 \ast t_2 = e$, which implies that $N_{t_1}t_2t_2 = N_{t_1}$. This indicates that cosets $N_{t_1}t_2t_1$, $N_{t_1}t_2t_2$, $N_{t_1}t_2t_4$, and $N_{t_1}t_2t_5$ go back to the double coset. [1]

$3 \in \{3, 6\} \Rightarrow N_{t_1}t_2t_3$. From the relation $N_{t_5}t_4 = N_{t_1}t_2t_3$, if we substitute $t_5t_4$ with $t_1t_2$ by the earlier expansion of the relations, we then have that $N_{t_1}t_2 = N_{t_1}t_2t_3$. Which then implies that both $N_{t_1}t_2t_3$ and $N_{t_1}t_2t_6$ loop back to the double coset [12]. Finally, going back to $4 \in \{4\} \Rightarrow N_{t_1}t_4$ in the double coset $N_{t_1}N = [1]$. From the relation (1) $N_{t_1}t_5 = N_{t_1}$. We have $yxt_1t_5 = t_1$

$\Rightarrow yxt_4t_2 = t_4$ (Conjugating by $x^3$)

$\Rightarrow t_1yxt_4t_2 = t_1t_4$ (multiplying by $t_1$ on the left hand side).

$\Rightarrow yxt_5t_2 = t_1t_4$

$\Rightarrow N_{t_1}t_2t_2 = N_{t_1}t_4$ (from $N_{t_1}t_2 = N_{t_5}t_4$)

Simplifying, we get that, $N_{t_1} = N_{t_1}t_4$.

Thus, we have that $N_{t_1} = N_{t_1}t_4$ also loops back to [1].

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:

$G = N \cup N_{t_1}N \cup N_{t_1}t_2N$, and that

$|G| \leq (|N| + \frac{|N|}{|N(1)|} + \frac{|N|}{|N(12)|}) \times |N|$

$|G| \leq (1 + 6 + 3) \times 12$

$|G| \leq 10 \times 12$

$|G| \leq 120$
2.2.2 Construction of $S_6$ over $D_6$

Consider the group $N = 2^{*5} : (5 : 4)$ factored by the relation $(yx^{-1}t^x)^5 = 1$

That is, $G = S_6 \cong \frac{2^{*5} : (5 : 4)}{(yx^{-1}t^x)^5 = 1}$

With $N$ being transitive on 5 letters of order 2.

$2^{*5} = < t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 >$

$N = (5 : 4)$

$x \sim (1, 2, 3, 4, 5)$

$y \sim (1, 2, 4, 3)$

$t \sim t_1$

Expanding the relation we have:

1) $(yx^{-1}t^x)^5 = 1$

$\Rightarrow (yx^{-1}t^x)(yx^{-1}t^x)(yx^{-1}t^x)(yx^{-1}t^x)(yx^{-1}t^x) = 1$

$\Rightarrow (yx^{-1}t_2)(yx^{-1}t_2)(yx^{-1}t_2)(yx^{-1}t_2)(yx^{-1}t_2) = 1$

$\Rightarrow yx^{-1}((yx^{-4})^{-3}(yx^{-1})^4)((yx^{-1})^{-3}(yx^{-1})^3)((yx^{-1})^{-2}t_2(yx^{-1})^2)$
\[(yx^{-1})^{-1}t_2yx^{-1})t_2 = 1\]
\[\Rightarrow yx^{-1}t_2(xyx^{-1})t_2(yx^{-1})t_2 = 1\]
\[\Rightarrow yx^{-1}t_2t_4t_5t_3t_2 = 1\]
\[\Rightarrow Nt_2t_4 = Nt_2t_3t_5\]

Conjugating all the previous relation by all elements of \(x\) and \(y\) from the group \(5 : 4\).

We then obtain the following additional relations:

\[
\begin{align*}
(yx^{-1}t_2t_4 = t_2t_3t_5)^c &\iff yx^{-1}t_2t_4 = t_2t_3t_5 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^yx^{-1} &\iff yx^{-1}t_3t_2 = t_3t_5t_4 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^x &\iff x^{-1}yt_3t_5 = t_3t_4t_1 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}y &\iff yt_1 = t_2t_4t_3 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^y &\iff x^{-1}yt_4t_3 = t_4t_1t_5 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-2} &\iff yxt_5t_2 = t_5t_1t_2 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}y^{-1} &\iff yt_3t_4 = t_3t_1t_2 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^y^{-1} &\iff xy_1t_2 = t_1t_4t_5 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}x &\iff yxt_2t_3 = t_2t_5t_1 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1} &\iff yxt_4t_1 = t_4t_5t_2 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}x^{-1} &\iff x^{-1}yt_5t_1 = t_5t_3t_4 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)x^y &\iff yxt_1t_5 = t_1t_3t_2 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)y^2 &\iff xy_2t_5 = t_2t_1t_4 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}x &\iff xy^{-1}t_4t_5 = t_4t_2t_3 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^xy &\iff yx^{-1}t_5t_3 = t_5t_4t_2 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)x^y &\iff yx_5t_4 = t_5t_2t_1 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{2x} &\iff yt_4t_2 = t_4t_3t_1 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^2 &\iff yxt_3t_1 = t_3t_2t_5 \\
(yx^{-1}t_2t_4 = t_2t_3t_5)^{-1}y &\iff x^{-1}yt_1t_4 = t_1t_3t_5 \end{align*}
\]

Calculating the total number of unique cosets of \(N\) in \(G\). We have:

\[
\frac{|G|}{|N|} = \frac{720}{20} = 36.
\]

Now we know that we will have 36 unique single cosets.

1st Double Coset [⋆]

\(NeN = \{Ne^n | n \in \mathbb{N}\} = \{N\}\). \(N\) is transitive on \(\{1, 2, 3, 4, 5\}\); moreover, \(\frac{|N|}{|N|} = \frac{20}{20} = 1\) So, \(NeN\) has one single orbit \(\{t_1, t_2, t_3, t_4, t_5\}\). Thus, orbit of \(N\) on \(\{1, 2, 3, 4, 5\}\) is \(\{1, 2, 3, 4, 5\}\). Now, select a representative \(t_i\) from orbit. We will select 1 from the orbit in this case.
that the number of single cosets in $N_t \in N$ takes 

For the coset stabilizer $N(1)$ we have, 

This would then imply that the number of single cosets in $N_t N$ are 

Thus, orbits of $N(1)$ on $\{1, 2, 3, 4, 5\}$ are $\{1\}$, and $\{2, 4, 5, 3\}$. Now, select a representative $t_i$ from orbit. 

1 $\in$ $\{1\} \Rightarrow N_{t_1}t_1 \Rightarrow Ne \in \star$ 

2 $\in$ $\{2, 4, 5, 3\} \Rightarrow N_{t_1}t_2 \in [12]$, which goes to a new double coset $[12]$ 

3rd Double Coset $[12]$ 

$N_{t_1}t_2 N = \{N_{t_1}t_2^n|n \in N\}$ 

Now, conjugating the single coset representative $N_{t_1}t_2$ by the transversals, we have our distinct double cosets being, 

Finding these distinct single cosets aids us later on when finding which cosets are equal
by the relations.
We will now select a representative \( t_i \) from each orbit.

1 \( \in \{1\} \Rightarrow Nt_1t_2t_1 \in [121] \), which goes to a new double coset [121].
2 \( \in \{2\} \Rightarrow Nt_1t_2t_2 \in [1], \) which goes back to [1].
3 \( \in \{3\} \Rightarrow Nt_1t_3t_3 \in [123] \), which goes to a new double coset [123].
4 \( \in \{4\} \Rightarrow Nt_1t_4t_4 \). Let’s now look at the relation \( Nt_4t_4 = t_2t_3t_5 \).

Since, \( (Nt_4t_4)^{-1} = (Nt_2t_3t_5)^{-1} ) \Rightarrow Nt_1t_3 = Nt_1t_2, \) and we also have that \( Nt_1t_3 \in [12] \). This indicates that \( Nt_1t_2t_4 \in [12] \).

5 \( \in \{5\} \Rightarrow Nt_1t_5t_5 \). Let’s now look again at the relation \( Nt_2t_4 = Nt_2t_3t_5 \).

Since, \( (Nt_2t_4)^{-1} = (Nt_2t_3t_5)^{-1} \Rightarrow Nt_1t_2 = Nt_1t_4t_5 \),
\( \Rightarrow Nt_1t_2t_5 = Nt_1t_4t_5 \) (right multiplication by \( t_5 \))
\( \Rightarrow Nt_1t_2t_5 = Nt_1t_4t_5 \) and we also have that \( Nt_1t_4 \in [12] \).

This indicates that \( Nt_1t_2t_5 \in [12] \).

4th Double Coset [121]

\( Nt_1t_2N = \{ Nt_1t_2t_1^n | n \in \mathbb{N} \} \)

For the coset stabilizer \( N^{(121)} \) we have, \( N^{(121)} \geq N^{121} = < e > \). Which indicates that only the identity \( e \) fixes 1, 2, and 1. However, we have from our relation that,
\( xyt_1t_2 = t_1t_4t_5, \Rightarrow t_5xyt_1t_2 = t_5t_1t_4t_5 \) (multiplying \( t_5 \) on the left hand side).
\( \Rightarrow xyt_2t_1t_2 = t_5t_1t_4t_5 \)
(also, since \( x^{-1}yt_1t_1 = t_5t_3t_4 \))
\( \Rightarrow xyt_2t_1t_2 = x^{-1}yt_3t_4t_5t_1 \)
\( \Rightarrow (xyt_2t_1t_2)^{y^{-1}}y = (x^{-1}yt_3t_4t_5)^{y^{-1}}y \) (conjugating by \( yx^{-1}y \))
\( \Rightarrow xyt_1t_2t_1 = yx^{-1}t_3t_5t_3 \)
\( \Rightarrow Nt_1t_2t_1 = Nt_3t_5t_3 \)

So, \( Nt_1t_2t_1 = (Nt_3t_5t_3)^{yx} = Nt_1t_2t_1 \) implies that \( yx \) is in the stabilizing group \( N^{(121)} \).

Which implies \( N^{(121)} \geq N^{121} = < yx > = < (1, 3, 2, 5) > \). Since there are 4 elements in
\( N^{121} = \{ e, (1, 3, 2, 5), (1, 2)(3, 5), (1, 5, 2, 3) \} \), this would then imply that the number of single cosets in \( Nt_1t_2t_1N \) are \( \frac{|N|}{|N^{121}|} = \frac{20}{4} = 5 \). Thus, the orbits of \( N^{(121)} \) on \( \{1, 2, 3, 4, 5\} \) are \( \{1, 2, 3, 5\} \), and \( \{4\} \). Since we have 5 distinct single cosets in [121], and using the transversals
\( T_{121} = \{ e, (1, 2, 3, 4, 5), (1, 4, 2, 5, 3, (1, 3, 5, 2, 4) \} \). We have that,
\( Nt_1t_2t_1 = Nt_3t_5t_3 = Nt_2t_1t_2 = Nt_5t_3t_5, \)
\( Nt_2t_3t_2 = Nt_4t_1t_4 = Nt_3t_2t_3 = Nt_1t_4t_1, \)
\( Nt_4t_5t_4 = Nt_5t_1t_5 = Nt_2t_4t_2 = Nt_1t_5t_1, \)
\( Nt_5t_4t_5 = Nt_1t_3t_1 = Nt_5t_4t_5 = Nt_3t_1t_3, \)
\( Nt_3t_5t_3 = Nt_5t_2t_5 = Nt_4t_3t_4 = Nt_2t_5t_2. \)

We will now select a representative \( t_i \) from each orbit we have for [121],
1 \( \in \{1, 2, 3, 5\} \Rightarrow Nt_1t_2t_1t_1 \Rightarrow Nt_1t_2 \in [12] \)
4 \( \in \{4\} \Rightarrow Nt_1t_2t_1t_4 \). Let’s now look at the relation \( xyt_2t_5 = t_2t_1t_4, \)
$\Rightarrow t_1yt_2t_5 = t_1t_2t_1t_4$ (multiplying $t_1$ on the left hand side).
$\Rightarrow xyt_4t_5 = t_1t_2t_1t_4$.
$\Rightarrow N_t t_2t_1t_4 = N_t t_2t_5$, it is proved in the investigation of the next coset that $N_t t_2t_5 \in \{123\}$
$\Rightarrow N_t t_2t_1t_4 \in \{123\}$.

5th Double Coset [123]

$N_{t_1t_2t_3}N = \{N_{t_1t_2t_3}^n | n \in \mathbb{N}\}$

For the coset stabilizer $N^{(123)}$ we have, $N^{(123)} \supseteq N^{123} = \langle e \rangle$. Which indicates that only the identity $e$ fixes 1, 2 and 3. However, we have from our relation that, $yt_1t_3 = t_1t_2t_4$, (also, since $yx^{-1}t_2t_4 = t_2t_3t_5$).
$\Rightarrow yt_1t_3 = t_1x^{-1}t_2t_3t_5$.
$\Rightarrow yt_1t_3 = xy^{-1}t_1t_2t_3t_5$.
$\Rightarrow yt_1t_3t_5 = xy^{-1}t_1t_2t_3t_5$ (multiplying $t_5$ on the right hand side).
$\Rightarrow yt_1t_5 = xy^{-1}t_1t_2t_3$.
$\Rightarrow N_{t_1t_5} = N_{t_1t_2t_3}$

So, $N_{t_1t_2t_3} = (N_{t_1t_2t_3})^{-1} = N_{t_1t_2t_3}$ implies that $yx^{-1}$ is in the stabilizing group $N^{(123)}$. Which implies $N^{(123)} \supseteq N^{123} = \langle xy^{-1} \rangle = \langle (2, 5, 3, 4) \rangle$. Since there are 4 elements in $N^{121} = \{e, (2, 5, 3, 4), (2, 3)(4, 5), (2, 4, 3, 5)\}$, this would then imply that the number of single cosets in $N_{t_1t_2t_3}N$ are $\frac{|N|}{|N^{(123)}|} = \frac{20}{4} = 5$. Thus, the orbits of $N^{(123)}$ on $\{1, 2, 3, 4, 5\}$ are $\{2, 5, 3, 4\}$, and $\{1\}$. Since we have 5 distinct single cosets in [123], and using the transversals

$T_{123} = \{e, (1, 2, 3, 4, 5), (1, 5, 4, 3, 2), (1, 3, 4, 2), (1, 4, 2, 5, 3)\}$. We have that,

$N_{t_1t_2t_3} = N_{t_1t_3t_5} = N_{t_1t_5t_4} = N_{t_1t_4t_2}$,

$N_{t_2t_3t_4} = N_{t_2t_4t_3} = N_{t_2t_1t_5} = N_{t_2t_5t_3}$,

$N_{t_5t_1t_2} = N_{t_5t_2t_4} = N_{t_5t_4t_3} = N_{t_5t_3t_1}$,

$N_{t_3t_4t_1} = N_{t_3t_4t_5} = N_{t_3t_5t_2} = N_{t_3t_2t_1}$,

$N_{t_4t_5t_1} = N_{t_4t_3t_5} = N_{t_4t_3t_2} = N_{t_4t_2t_5}$.

Taking a look at the relations $N_{t_2t_5t_1} = N_{t_4t_1t_3} = N_{t_4t_3t_2} = N_{t_4t_2t_5}$ we got from one of the transversals, we now see that $(N_{t_1t_2t_3})^{x^2} \Rightarrow N_{t_4t_5t_1}$, and $N_{t_4t_5t_1} = N_{t_4t_2t_5}$.

This fully proves that $N_{t_4t_2t_5} = N_{t_1t_2t_1t_4} \in [123]$ from the fourth coset. We will now select a representative $t_i$ from each orbit we have for [123],

$3 \in \{2, 5, 3, 4\} \Rightarrow N_{t_2t_3t_5} \Rightarrow N_{t_1t_2} \in [12]$.

$1 \in \{1\} \Rightarrow N_{t_1t_2t_3}$.

$\Rightarrow t_5xyt_2t_3 = t_2t_5t_1$ (multiplying $t_5$ on the left hand side).
$\Rightarrow xyt_1t_2t_3 = t_5t_2t_5$.
$\Rightarrow xyt_1t_2t_3 = t_5t_2t_5$ (multiplying $t_1$ on the right hand side).
$\Rightarrow xyt_1t_2t_3t_1 = t_5t_2t_5t_1$.
$\Rightarrow xyt_1t_2t_3t_1 = t_5t_2t_5$.
$\Rightarrow N_{t_1t_2t_3} = N_{t_5t_2t_5}$.

Looking at the relations $N_{t_3t_4t_5} = N_{t_5t_2t_5} = N_{t_4t_3t_4} = N_{t_2t_5t_2}$ from the fourth coset, we now see that $(N_{t_1t_2t_1})^{x^2} \Rightarrow N_{t_3t_4t_3}$, and $N_{t_3t_4t_3} = N_{t_5t_2t_5}$. We have that
\(N_{t5t2t5} = N_{t1t2t3t1} \in [121].\)

We then conclude:

\[G = N_eN \cup N_{t1N} \cup N_{t1t2N} \cup N_{t1t2t1N} \cup N_{t1t2t3N},\]

and that

\[|G| \leq (\frac{|N|}{|N^2|} + \frac{|N|}{|N(1)|} + \frac{|N|}{|N(12)|} + \frac{|N|}{|N(121)|} + \frac{|N|}{|N(123)|}) \times |N|\]

\[|G| \leq (1 + 5 + 20 + 5 + 5) \times 20\]

\[|G| \leq 36 \times 20\]

\[|G| \leq 720\]

Figure 2.2: Construction of \(S_6\) over \(5 : 4\)
2.2.3 Construction of $A_5 \times A_5$ over $[(5 : 2)\cdot 5]$

Consider the group $N = 2^{*25} : [(5 : 2)\cdot 5]$ factored by the relation $(y^{-1}t^{yx})^3 = 1$

That is, $G = A_5 \times A_5 \cong \frac{2^{*25} : [(5 : 2)\cdot 5]}{(y^{-1}t^{yx})^3 = 1}$

With $N$ being transitive on 25 letters of order 2.

$2^{*25} = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25} \rangle$

$N = [(5 : 2)\cdot 5]$

$x \sim (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15)$

$y \sim (1, 7, 5, 6, 4, 10, 3, 9, 2, 8)(11, 24, 15, 23, 14, 22, 13, 21, 12, 25)$

$t \sim t_1$

We will first expand the relation:

1) $(y^{-1}t_1^{yx})^3 = 1,$

$\Rightarrow (y^{-1}t_1^{yx})(y^{-1}t_1^{yx})(y^{-1}t_1^{yx}) = 1$

$\Rightarrow (y^{-1}t_{12})(y^{-1}t_{12})(y^{-1}t_{12}) = 1$

$\Rightarrow y^{-3}(y^{-1}t_{12}y^{-2})(yt_{12}y^{-1})t_{12} = 1$

$\Rightarrow y^{-3}t_{12}t_{12}^{-1}t_{12}^{-1} = 1$

$\Rightarrow y^{-3}t_{13}t_{21}t_{12} = 1$

$\Rightarrow Nt_{13}t_{21} = Nt_{12}$

We can also conjugate the previous relation by all elements of $x$ and $y$ from $N = T_{25}$. An example of this would be $(Nt_{13}t_{21})^x = (Nt_{12})^x$. Since, $x = (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15)$, we have $Nt_{8}t_{21} = Nt_{7}$. So, we then obtain the following additional relations:

$(13, 21 \sim 12) \iff (8, 21 \sim 7)$

$(13, 21 \sim 12) \iff (21, 12 \sim 25)$

$(13, 21 \sim 12) \iff (22, 13 \sim 21)$

$(13, 21 \sim 12) \iff (1, 12 \sim 5)$

$(13, 21 \sim 12) \iff (6, 24 \sim 10)$

$(13, 21 \sim 12) \iff (21, 7 \sim 25)$

$(13, 21 \sim 12) \iff (12, 25 \sim 11)$

$(13, 21 \sim 12) \iff (22, 8 \sim 21)$

$(13, 21 \sim 12) \iff (14, 22 \sim 13)$

$(13, 21 \sim 12) \iff (15, 23 \sim 14)$

$(13, 21 \sim 12) \iff (7, 25 \sim 6)$

$(13, 21 \sim 12) \iff (11, 24 \sim 15)$

$(13, 21 \sim 12) \iff (5, 11 \sim 4)$

$(13, 21 \sim 12) \iff (12, 5 \sim 11)$

$(13, 21 \sim 12) \iff (13, 1 \sim 12)$

$(13, 21 \sim 12) \iff (25, 11 \sim 24)$
(13 21 ∼ 12) ⇔ (24 15 ∼ 23)
(13 21 ∼ 12) ⇔ (23 14 ∼ 22)
(13 21 ∼ 12) ⇔ (24 15 ∼ 23)
(13 21 ∼ 12) ⇔ (25 6 ∼ 24)
(13 21 ∼ 12) ⇔ (23 9 ∼ 22)
(13 21 ∼ 12) ⇔ (24 10 ∼ 23)
(13 21 ∼ 12) ⇔ (11 4 ∼ 15)
(13 21 ∼ 12) ⇔ (15 3 ∼ 4)

Then, by comparing these relations, we also get the following equivalencies.

(1 ∼ 21)
(6 ∼ 11)
(2 ∼ 22)
(7 ∼ 12)
(3 ∼ 23)
(8 ∼ 13)
(4 ∼ 24)
(9 ∼ 14)
(5 ∼ 25)
(10 ∼ 15)

We first need to calculate the total number of unique cosets of N in G. This is the index of G in N. The index will be the order of G divided by the order of N.

\[ \frac{G}{N} = \frac{3600}{50} = 72. \]

Now we know that we will have 72 unique single cosets.

1st Double Coset [⋆]

\[ NeN = \{ Ne^n | n \in \mathbb{N} \} = \{ N \}. \]

N is transitive on \{1, 2, ..., 24, 25\}; moreover, \( \frac{N}{N} = \frac{50}{50} = 1. \) So, NeN has one single orbit \{t_1, t_2, ..., t_{24}, t_{25}\}. Thus, orbit of N on \{1, 2, ..., 24, 25\} is \{1, 2, ..., 24, 25\}. Now, select a representative \( t_i \) from orbit. We will select 1 from the orbit in this case.

\[ 1 \in \{1, 2, ..., 24, 25\} \Rightarrow Nt_1 \in [1] \]

Now, we shall extend \( Nt_1 \) to a new double coset [1].

2nd Double Coset [1]

\[ Nt_1N = \{ Nt_1^n | n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(1)} \) are elements of \((5 : 2)^*5\) that fix 1. So, we then have \( N^{(1)} \geq N^1 = < (6, 22)(7, 23)(8, 24)(9, 25)(10, 21)(11, 16)(12, 17)(13, 18)(14, 19)(15, 20) > \)
Since, the coset stabilizer is of order 2, this would imply that the number of single cosets in $N_{t1}N$ are $\frac{N}{N_{t1}N} = \frac{50}{2} = 25$ Thus, The orbits of $N^{(1)}$ on $\{1,2,\ldots,24,25\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6,22\}, \{7,23\},\{8,24\},\{9,25\},\{10,21\},\{11,16\},\{12,17\},\{13,18\},\{14,19\},\{15,20\}$ From this, we could see which $t_i^s$ are sharing orbits just by looking at the permutations in the stabilizer $N^{(1)}$. We will now select a representative $t_i$ from each orbit.

1 $\in \{1\} \Rightarrow N_{t1}t_1 = N \in [\ast]$
2 $\in \{2\} \Rightarrow N_{t1}t_2 \sim N_{t1}t_5 \in [12]$
3 $\in \{3\} \Rightarrow N_{t1}t_3 \sim N_{t1}t_4 \in [13]$
4 $\in \{4\} \Rightarrow N_{t1}t_4 \sim N_{t1}t_3 \in [13]$
5 $\in \{5\} \Rightarrow N_{t1}t_5 \sim N_{t1}t_2 \in [12]$
6 $\in \{6,22\} \Rightarrow N_{t1}t_6 \sim N_{t1}t_9 \sim N_{t1}t_{11} \sim N_{t1}t_{14} \in [16]$
7 $\in \{7,23\} \Rightarrow N_{t1}t_7 \sim N_{t1}t_8 \sim N_{t1}t_{12} \sim N_{t1}t_{13} \in [1]$
8 $\in \{8,24\} \Rightarrow N_{t1}t_8 \sim N_{t1}t_7 \sim N_{t1}t_{12} \sim N_{t1}t_{13} \in [1]$
9 $\in \{9,25\} \Rightarrow N_{t1}t_9 \sim N_{t1}t_6 \sim N_{t1}t_{11} \sim N_{t1}t_{14} \in [16]$
10 $\in \{10,21\} \Rightarrow N_{t1}t_{10} \sim N_{t1}t_{15} \in [110]$
11 $\in \{11,16\} \Rightarrow N_{t1}t_{11} \sim N_{t1}t_6 \sim N_{t1}t_9 \sim N_{t1}t_{14} \in [16]$
12 $\in \{12,17\} \Rightarrow N_{t1}t_{12} \sim N_{t1}t_7 \sim N_{t1}t_8 \sim N_{t1}t_{13} \in [1]$
13 $\in \{13,18\} \Rightarrow N_{t1}t_{13} \sim N_{t1}t_7 \sim N_{t1}t_8 \sim N_{t1}t_{12} \in [1]$
14 $\in \{14,19\} \Rightarrow N_{t1}t_{14} \sim N_{t1}t_6 \sim N_{t1}t_9 \sim N_{t1}t_{14} \sim N_{t1}t_{14} \in [16]$
15 $\in \{15,20\} \Rightarrow N_{t1}t_{15} \sim N_{t1}t_{10} \in [110]$

We will now find all the single cosets in the double coset enumeration with words of length two.

3rd Double Coset [12]

$N_{t1}t_2N = \{N_{t1}t_2^n | n \in N\}$


This now implies the coset stabilizer is now of order 10, and the number of single cosets in $N_{t1}t_2N$ are $\frac{N}{N^{(12)}} = \frac{50}{10} = 5$. Now, The orbits of $N^{(12)}$ on $\{1,2,\ldots,24,25\}$ are $\{1,20,15,10,21\},\{2,16,11,6,22\},\{3,17,12,7,23\},\{4,18,13,8,24\},\{5,19,14,9,25\}.$ We will now select a representative $t_i$ from each orbit.

1 $\in \{1,20,15,10,21\} \Rightarrow N_{t1}t_1t_1 \in [121]$
2 $\in \{2,16,11,6,22\} \Rightarrow N_{t1}t_2t_2 \sim N_{t1}t_2t_5 \in [1]$
Now, for $N_t 1_3$, we have that $N_t 1_3 N \sim N_t 1_2 t_3 N \sim N_t 1_2 t_2 t_3$ by earlier relations, and also conjugating the permutations $x$ and $xy^5$ such that $(N_t 1_2 t_3 N)(xy^5)^2 = N_t 1_3 N$, and $(N_t 20 t_1 17 N)^x = N_t 1_3 N$, $x$ and $xy^5$ are also in the stabilizer $N^{(13)}$. Thus, The coset stabilizer $N^{(13)} \geq N^{(13)} = < (6, 22)(7, 23)(8, 24)(9, 25)(10, 21)(11, 16)(12, 17)(13, 18)(14, 19)(15, 20), (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15), (1, 20, 10, 21, 15)(2, 16, 6, 22, 11)(3, 17, 7, 23, 12)(4, 18, 8, 24, 13)(5, 19, 9, 25, 14) >$

This now implies the coset stabilizer is also of order 10, and the number of single cosets in $N_t 1_3 N$ are $\frac{50}{10} = 5$. Now, The orbits of $N^{(13)}$ on $\{1, 2, ..., 24, 25\}$ are $\{1, 2, 10, 15, 21\}$, $\{2, 16, 11, 6, 22\}$, $\{3, 17, 12, 7, 23\}$, $\{4, 18, 13, 8, 24\}$, $\{5, 19, 14, 9, 25\}$

We will now select a representative $t_i$ from each orbit.

$1 \in \{1, 2, 10, 15, 21\} \Rightarrow N_t 1_3 t_1 \sim N_t 1_3 t_4 \in [16]$

$2 \in \{2, 16, 11, 6, 22\} \Rightarrow N_t 1_3 t_2 \sim N_t 1_3 t_3 \in [1]$

$3 \in \{3, 17, 12, 7, 23\} \Rightarrow N_t 1_3 t_3 \sim N_t 1_3 t_1 \in [16]$

$4 \in \{4, 18, 13, 8, 24\} \Rightarrow N_t 1_3 t_4 \sim N_t 1_3 t_1 \in [16]$

$5 \in \{5, 19, 14, 9, 25\} \Rightarrow N_t 1_3 t_5 \in [110]$

We will now select a representative $t_i$ from each orbit.

$1 \in \{1, 2, 10, 15, 21\} \Rightarrow N_t 1_6 t_1 \sim N_t 1_6 t_10 \in [121]$

$2 \in \{2, 22\} \Rightarrow N_t 1_6 t_2 \sim N_t 1_6 t_5 \sim N_t 1_6 t_6 \sim N_t 1_6 t_9 \in [1]$
\[ 3 \in \{3, 23\} \Rightarrow N_1t_0t_3 \sim N_1t_0t_4 \sim N_1t_0t_7 \sim N_1t_0t_8 \in [16] \]
\[ 4 \in \{4, 24\} \Rightarrow N_1t_0t_4 \sim N_1t_0t_3 \sim N_1t_0t_7 \sim N_1t_0t_8 \in [16] \]
\[ 5 \in \{5, 25\} \Rightarrow N_1t_0t_5 \sim N_1t_0t_2 \sim N_1t_0t_6 \sim N_1t_0t_9 \in [1] \]
\[ 6 \in \{6, 16\} \Rightarrow N_1t_0t_6 \sim N_1t_0t_2 \sim N_1t_0t_5 \sim N_1t_0t_9 \in [1] \]
\[ 7 \in \{7, 17\} \Rightarrow N_1t_0t_7 \sim N_1t_0t_3 \sim N_1t_0t_4 \sim N_1t_0t_8 \in [16] \]
\[ 8 \in \{8, 18\} \Rightarrow N_1t_0t_8 \sim N_1t_0t_3 \sim N_1t_0t_4 \sim N_1t_0t_7 \in [16] \]
\[ 9 \in \{9, 19\} \Rightarrow N_1t_0t_9 \sim N_1t_0t_2 \sim N_1t_0t_5 \sim N_1t_0t_6 \in [1] \]
\[ 10 \in \{10, 20\} \Rightarrow N_1t_0t_{10} \sim N_1t_0t_1 \in [121] \]
\[ 11 \in \{11\} \Rightarrow N_1t_0t_{11} \sim N_1t_0t_{14} \in [13] \]
\[ 12 \in \{12\} \Rightarrow N_1t_0t_{12} \sim N_1t_0t_{13} \in [12] \]
\[ 13 \in \{13\} \Rightarrow N_1t_0t_{13} \sim N_1t_0t_{12} \in [12] \]
\[ 14 \in \{14\} \Rightarrow N_1t_0t_{14} \sim N_1t_0t_{11} \in [13] \]
\[ 15 \in \{15\} \Rightarrow N_1t_0t_{15} \in [1615] \]

6th Double Coset [110]

\[ N_1t_{10}N = \{N_1t_{10}^n | n \in \mathbb{N}\} \]

The coset stabilizer \(N^{(110)}\) are elements of \((5 : 2)^5\) that fix both 1 and 10. Which implies that \(N^{(110)} \geq N^{110} = <e>\), which is also only the identity.

For \(N_1t_{10}\) however, we have that \(N_1t_{10}N \sim N_5t_9N \sim N_{25}t_{19}N\) by earlier relations, and also conjugating the permutations \(y^2\) and \(xy\) such that \((N_5t_9N)^y = N_1t_{10}N\), and \((N_{25}t_{19}N)^{(xy)} = N_1t_{10}N\). Hence \(N_5t_9N\) and \(N_{25}t_{19}N\) are also in the stabilizer \(N^{(110)}\). Thus, the coset stabilizer \(N^{(110)} \geq N^{110} = <(1, 5, 4, 3, 2)(6, 10, 9, 8, 7)(11, 15, 14, 13, 12)\)
\(\in 16, 20, 19, 18, 17) \in 21, 25, 24, 23, 22), (1, 25, 4, 23, 2, 21, 5, 24, 3, 22)\)
\((6, 20, 9, 18, 16, 10, 19, 8, 17, 11, 15, 14, 13, 12) > \]

This now implies the coset stabilizer is also of order 10, and the number of single cosets in \(N_1t_{10}N\) are \(\frac{N}{N_1t_{10}} = \frac{50}{10} = 5\). Now, the orbits of \(N^{(110)}\) on \(\{1, 2, \ldots, 24, 25\}\) are \(\{11, 15, 14, 13, 12\}, \{1, 5, 25, 4, 24, 3, 23, 2, 22, 21\}, \{6, 10, 20, 9, 18, 17, 16\}\)

We will now select a representative \(t_i\) from each orbit.
\[ 11 \in \{11, 15, 14, 13, 12\} \Rightarrow N_1t_{10}t_{11} \in [13] \]
\[ 1 \in \{1, 5, 25, 4, 24, 3, 23, 2, 22, 21\} \Rightarrow N_1t_{10}t_1 \sim N_1t_{10}t_{10} \in [1] \]
\[ 10 \in \{6, 10, 20, 9, 18, 17, 16\} \Rightarrow N_1t_{10}t_{10} \sim N_1t_{10}t_1 \in [1] \]

We will now find the remaining single cosets with the double coset enumeration of words length three.

7th Double Coset [121]

\[ N_1N_2N_1N = \{N_1N_2N_1t_2^n | n \in \mathbb{N}\} \]

The coset stabilizer \(N^{(121)}\) are elements of \((5 : 2)^5\) that fix all of 1,2 and 1. Which implies that \(N^{(121)} \geq N^{121} = <(6, 22)(7, 23)(8, 24)(9, 25)(10, 21)(11, 16)(12, 17)(13, 18)(14, 19)(15, 20) >.\)
We have that \( N_t1N_t2N_t1 \sim N_t5N_t1N_t5 \) by earlier relations, and also conjugating the permutations \( y^2 \) and \( xy^{-1}xy^{-2} \) such that \( (N_t5N_t1N_t5)(y^2)^{-1} = N_t1N_t2N_t1 \), and \( (N_t5N_t1N_t5)(xy^{-1}xy^{-2})^{-1} = N_t1N_t2N_t1 \), \( y^2 \) and \( xy^{-1}xy^{-2} \) are also in the stabilizer \( N^{(121)} \). Thus, The coset stabilizer \( N^{(121)} \geq N^{121} = \langle 6, 22, 7, 23, 8, 24, 9, 25, 10, 21 \rangle(11, 16)(12, 17)(13, 18)(14, 19)(15, 20), (1, 5, 4, 3, 2)(6, 10, 9, 8, 7)(11, 15, 14, 13, 12)(16, 20, 19, 18, 17)(21, 25, 24, 23, 22), (1, 5, 4, 3, 2)(6, 21, 9, 24, 7, 22, 10, 25, 8, 23)(11, 20, 14, 18, 12, 16, 15, 19, 13, 17) > .

This now implies the coset stabilizer is also of order 10, and the number of single cosets in \( N_t1N_t2N_t1N \) are \( \frac{N}{N_{(110)}} = \frac{50}{10} = 5 \). Now, The orbits of \( N^{(121)} \) on \( \{1, 2, ..., 24, 25\} \) are \( \{11, 15, 14, 13, 12\}, \{1, 5, 24, 23, 2, 3, 22, 21\}, \{6, 10, 9, 18, 17, 16\} \).

We will now select a representative \( t_i \) from each orbit.

1 \( \in \{1, 5, 4, 3, 2\} \Rightarrow N_t1N_t2N_t1N_t1 \in [12]
6 \( \in \{6, 22, 10, 21, 9, 25, 8, 24, 7, 23\} \Rightarrow N_t1N_t2N_t1N_t6 \sim N_t1N_t2N_t1t_{11} \in [16]
11 \( \in \{11, 16, 15, 20, 14, 19, 13, 18, 12, 17\} \Rightarrow N_t1N_t2N_t1t_{11} \sim N_t1N_t2N_t1N_t6 \in [16]

**8th Double Coset** [1615]

\[ N_t1N_t6N_{t15}N = \{N_t1N_t6N_{t15}^n | n \in \mathbb{N}\} \]

The coset stabilizer \( N^{(1615)} \) are elements of \( (5 : 2)^5 \) that fix all of 1, 6 and 15. Which implies that \( N^{(1615)} \geq N^{1615} = \langle e \rangle \), which is also only the identity.

We finally have that \( N_t1N_t6N_{t15} \sim N_t20N_t11N_t10 \sim N_t7N_t7N_{t23} \) by earlier relations, and also conjugating the permutations \( x \) and \( y \) such that \( (N_t20N_t11N_t10)x = N_t1N_t6N_{t15} \), and \( (N_t7N_t7N_{t23})^y = N_t1N_t6N_{t15} \), \( x \) and \( y \) are also in the stabilizer \( N^{(1615)} \). Thus, The coset stabilizer \( N^{(1615)} \geq N^{1615} = \langle (1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15), (1, 7, 5, 6, 4, 10, 3, 9, 2, 8)(11, 24, 15, 23, 14, 22, 13, 21, 12, 25)(16, 18, 20, 17, 19) \rangle > .

This implies the coset stabilizer is of order 50, and the number of single cosets in \( N_t1N_t6N_{t15}N \) are \( \frac{N}{N^{1615}} = \frac{50}{50} = 1 \). Now, The orbits of \( N^{(1615)} \) on \( \{1, 2, ..., 24, 25\} \) are \( \{1, 7, 12, 17, 2, 5, 3, 19, 25, 6, 9, 16, 11, 4, 14, 2, 18, 24, 10, 22, 8, 15, 13, 23, 21\} \).

We will now select one last representative \( t_i \) from the orbit.

15 \( \in \{1, 20, 7, 17, 12, 5, 3, 19, 25, 6, 9, 16, 11, 4, 14, 2, 18, 24, 10, 22, 8, 15, 13, 23, 21\} \)
\Rightarrow N_t1N_t6N_{t15}t_{15} \in [16]

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:

\[ G = N \cup N_t1N \cup N_t1t_2N \cup N_t1t_3N \cup N_t1t_6N \cup N_t1t_{10}N \cup N_t1t_{21}N \cup N_t1t_{6}N_{t15}N \text{, and that} \]

\[ |G| \leq (|N^*| + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(1615)}|}) \times |N| \]

\[ |G| \leq (1 + 25 + 5 + 5 + 25 + 5 + 5 + 1) \times 50 \]
\[ |G| \leq 72 \times 50 \]

\[ |G| \leq 3600 \]

Figure 2.3: Construction of \( A_5 \times A_5 \) over \((5 : 2)^\ast 5\)
2.2.4 Construction of $M_{22}$ over Maximal Subgroup $A_7$

We have found that we can construct a larger group over a maximal subgroup and still retain the same information. As mentioned earlier, we would let $M \neq 1$ be a subgroup of $G$ such that $N \leq M \leq G$. We can write $G$ as set of single cosets in $M$. Letting $M = N\omega N$ and $G = M\omega N$, where $\omega$ is a word of the symmetric generators.

In this case, we have the sporadic simple group $M_{22}$. Let us look at the classification theorem for finite simple groups.

**Finite simple groups** are to be found among:
- The cyclic groups of prime order.
- The alternating groups of degree at least 5.
- The Chevalley and twisted Chevalley groups, and the Tits group.
- The 26 sporadic simple groups.

We this time, consider $\frac{2^{*7} \cdot (7.3)}{(yx^{-1}t^{x^{-1}})^{5} \cdot (xt)^{8} \cdot (yt)^{11} - 1} \cong A_{7} = M \leq G$

With the maximal subgroup $M$ being the size of $A_{7}$ as well as being transitive on 7 letters of order 2. That is, $M = \langle N, yx^{-1}y^{-1}txty^{-1}tx^{-1}tyx \rangle$.

$2^{*7} = \langle t_1 \rangle \ast \langle t_2 \rangle \ast \langle t_3 \rangle \ast \langle t_4 \rangle \ast \langle t_5 \rangle \ast \langle t_6 \rangle \ast \langle t_7 \rangle$.

$x \sim (1, 2, 3, 4, 5, 6, 7)$

$y \sim (1, 2, 4)(3, 6, 5)$

$t \sim t_1$

Overall, the presentation for our group we have,

$G<x, y, t> := \text{Group}<x, y, t | y^3, y*x^-2*y^-1*x, t^2, t*(y*x^-1) = t, (y*x^-1*t*(x^-3))^{-5}, (x*t)^8, (y*t)^{11}>$

With the maximal subgroup such that,

$> M := \text{sub}<G | x, y, y*x^-1*y^-1*t*x*y*t*y^-1*t*x^-1*t*y*x>$

$> f, G1, k := \text{CosetAction}(G, M)$

$> \text{CompositionFactors}(G1)$

$G$

| M22
| 1

Let us expand $yx^{-1}y^{-1}txty^{-1}tx^{-1}tyx$ being in $M$.

$yx^{-1}y^{-1}txty^{-1}tx^{-1}tyx \in M$

$y((yx)^{-1}t(yx))((yx)^{-1}t(yx)^{-1})((x^{-1})^{-1}t(x^{-1}))((yx^{-1})^{-1}t(yx^{-1}))$

$((yx)^{-1}t(yx)) \in M$
\[
\Rightarrow y t^x t^y t^{-1} t x^{-1} t y x^{-1} t y x \in M
\]
\[
\Rightarrow y t_3 t_1 t_7 t_1 t_3 \in M
\]

Which, then implies \( M t_3 t_1 t_7 t_1 t_3 = M \Leftrightarrow M t_3 t_1 = M t_3 t_1 \)

We next expand the relations.

\[(1) \ (y x^{-1} t^x)^5 = 1 \quad \Rightarrow (y x^{-1} t_4) \ldots (y x^{-1} t_4) = 1 \quad \Rightarrow (x t_1) \ldots (x t_1) = 1
\]
\[(2) \ (x t)^8 = 1
\]
\[
\Rightarrow (y x^{-1})^2 t^p x_{\frac{t^4}{t_1}} t_1^2 (y x^{-1})^2 t^p x_{\frac{t^4}{t_1}} = 1
\]
\[
\Rightarrow x y^{-1} t_7 t_4 t_6 t_7 t_4 = 1
\]
\[
\Rightarrow x y^{-1} t_7 t_4 t_6 = t_4 t_7
\]
\[
\Rightarrow N t_7 t_4 t_6 = N t_4 t_7
\]
\[
(3) \ (y t)^{11} = 1
\]
\[
\Rightarrow (y t_1) \ldots (y t_1) = 1
\]
\[
\Rightarrow y^2 t_3 t_4 t_2 t_1 t_4 t_3 t_2 t_1 t_4 t_2 t_1 = 1
\]
\[
\Rightarrow y^2 t_2 t_1 t_4 t_2 t_1 t_4 t_3 t_2 t_1 t_4 = 1
\]
\[
\Rightarrow y^2 t_2 t_1 t_4 t_2 t_1 = t_1 t_2 t_1 t_2
\]
\[
\Rightarrow N t_2 t_1 t_4 t_2 t_1 t_4 = N t_1 t_2 t_4 t_1 t_2
\]

Beginning the double coset enumeration, we first need to calculate the total number of unique cosets of \( M \) of \( N \) in \( G \). This is the index of \( G \) in \( M \). The index will be the order of \( G \) divided by the order of \( M \).

\[
\frac{|G|}{|M|} = \frac{443520}{2520} = 176.
\]

Now we know that we will have 176 unique single cosets.

Calculating the number of double cosets in your double coset enumeration.

\[
> \text{#DoubleCosets}(G,H,\text{sub}<G|x,y>);
\]

12
This lets us know that there will be 12 unique double cosets. We will then use this information as we do the double coset enumeration of $G$ over $M$.

1st Double Coset $[*]$

$MeN = \{ Me^n | n \in \mathbb{N} \} = \{ N \}$. $N$ is transitive on $\{1, 2, 3, 4, 5, 6, 7\}$; moreover, $\frac{|N|}{21} = \frac{49}{21} = 1$. Thus, orbit of $N$ on $\{1, 2, 3, 4, 5, 6, 7\}$ is $\{1, 2, 3, 4, 5, 6, 7\}$. Now, select a representative $t_1$ from orbit. We will select 1 from the orbit in this case.

$1 \in \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow M_{t_1} \in [1]$

Now, we shall extend $M_{t_1}$ to a new double coset $[1]$.

2nd Double Coset $[1]$

$M_{t_1}N = \{ M_{t_1}^n | n \in \mathbb{N} \}$

The coset stabilizer $N^{(1)}$ are elements of $7 : 3$ that fix 1. So, we then have $N^{(1)} \geq N^1 = < (2, 3, 5)(4, 7, 6) >$, which indicates the order of the stabilizing group $N^{(1)}$ would be 4. Which would imply that the number of single cosets in $M_{t_1}N$ are $\frac{N}{N^{(1)}} = \frac{21}{3} = 7$. Thus, The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}, \{2, 3, 5\}$, and $\{4, 7, 6\}$.

We will now select a representative $t_2$ from each orbit.

$1 \in \{1\} \Rightarrow N_{t_1}t_1 = N \in [*]$

$2 \in \{2, 3, 5\} \Rightarrow N_{t_1}t_2 \in [12]$, which goes to a new double coset $[12]$

$4 \in \{4, 7, 6\} \Rightarrow N_{t_1}t_2 \in [14]$, which goes to a new double coset $[14]$

Words of length two.

3rd Double Coset $[12]$

$M_{t_1}t_2N = \{ M_{t_1}t_2^n | n \in \mathbb{N} \}$

The coset stabilizer $N^{(12)}$ fixes both 1 and 2. So, we then have $N^{(12)} \geq N^{12} = < e >$, which indicates the stabilizing group $N^{(12)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $M_{t_1}t_2N$ are $\frac{N}{N^{(12)}} = \frac{21}{1} = 21$. Thus, The orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\},$ and $\{7\}$.

We will now select a representative $t_3$ from each orbit.

$1 \in \{1\} \Rightarrow M_{t_1}t_2t_1$, Let's now look at the relation ****

$2 \in \{2\} \Rightarrow M_{t_1}t_2t_2 \Rightarrow M_{t_1} \in [1]$

$3 \in \{3\} \Rightarrow M_{t_1}t_2t_3 \in [123]$, which goes to a new double coset $[123]$

$4 \in \{4\} \Rightarrow M_{t_1}t_2t_4 \in [124]$, which goes to a new double coset $[124]$

$5 \in \{5\} \Rightarrow M_{t_1}t_2t_5 \in [125]$, which goes to a new double coset $[125]$

$6 \in \{6\} \Rightarrow M_{t_1}t_2t_6$, Let's now look at the relation ****

$7 \in \{7\} \Rightarrow M_{t_1}t_2t_7 \in [127]$, which goes to a new double coset $[125]$

4th Double Coset $[14]$

$M_{t_1}t_4N = \{ M_{t_1}t_4^n | n \in \mathbb{N} \}$
The coset stabilizer $N^{(14)}$ fixes both 1 and 4. So, we then have $N^{(14)} \geq N^{14} = \langle e \rangle$, which indicates the stabilizing group $N^{(14)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $Mt_4N$ are $\frac{N}{N^{(14)}} = \frac{21}{1} = 21$.

Thus, The orbits of $N^{(14)}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, and $\{7\}$.

We will now select a representative $t_i$ from each orbit.

1 $\in \{1\} \Rightarrow Mt_1t_4t_1$, Lets now look at the relation ****

2 $\in \{2\} \Rightarrow Mt_1t_4t_2$, Lets now look at the relation ****

3 $\in \{3\} \Rightarrow Mt_1t_4t_3$, Lets now look at the relation ****

4 $\in \{4\} \Rightarrow Mt_1t_4t_4 \Rightarrow Mt_1 \in [1]$ 

5 $\in \{5\} \Rightarrow Mt_1t_4t_5$, Lets now look at the relation ****

6 $\in \{6\} \Rightarrow Mt_1t_4t_6 \in [146]$, which goes to a new double coset $[146]$ 

7 $\in \{7\} \Rightarrow Mt_1t_4t_7 \in [147]$, which goes to a new double coset $[147]$ 

Words of length three.

5th Double Coset $[123]$

$Mt_1t_2t_3N = \{Mt_1t_2t_3^n|n \in \mathbb{N}\}$

The coset stabilizer $N^{(123)}$ fixes both 1, 2 and 3. So, we then have $N^{(123)} \geq N^{123} = \langle e \rangle$, which indicates the stabilizing group $N^{(123)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $Mt_1t_2t_3N$ are $\frac{N}{N^{(123)}} = \frac{21}{1} = 21$.

Thus, The orbits of $N^{(123)}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, and $\{7\}$.

We will now select a representative $t_i$ from each orbit.

1 $\in \{1\} \Rightarrow Mt_1t_2t_3t_1$, Lets now look at the relation ****

2 $\in \{2\} \Rightarrow Mt_1t_2t_3t_2$, Lets now look at the relation ****

3 $\in \{3\} \Rightarrow Mt_1t_2t_3t_3 \Rightarrow Mt_1t_2 \in [12]$

4 $\in \{4\} \Rightarrow Mt_1t_2t_3t_4$, Lets now look at the relation ****

5 $\in \{5\} \Rightarrow Mt_1t_2t_3t_5$, Lets now look at the relation ****

6 $\in \{6\} \Rightarrow Mt_1t_2t_3t_6 \in [1236]$, which goes to a new double coset $[1236]$ 

7 $\in \{7\} \Rightarrow Mt_1t_2t_3t_7$, Lets now look at the relation ****

6th Double Coset $[124]$

$Mt_1t_2t_4N = \{Mt_1t_2t_4^n|n \in \mathbb{N}\}$

The coset stabilizer $N^{(124)}$ fixes both 1, 2 and 4. So, we then have $N^{(124)} \geq N^{124} = \langle e \rangle$, which indicates the stabilizing group $N^{(124)}$ only contains the identity $e$.

****

Which would imply that the number of single cosets in $Mt_1t_2t_4N$ are $\frac{N}{N^{(124)}} = \frac{21}{3} = 7$.

Thus, The orbits of $N^{(124)}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1, 2, 4\}$, $\{3, 6, 5\}$, and $\{7\}$.

We will now select a representative $t_i$ from each orbit.

4 $\in \{1, 2, 4\} \Rightarrow Mt_1t_2t_4t_4 \Rightarrow Mt_1t_2 \in [12]$

3 $\in \{3, 6, 5\} \Rightarrow Mt_1t_2t_3$, Lets now look at the relation ****

7 $\in \{7\} \Rightarrow Mt_1t_2t_4t_7$, Lets now look at the relation ****
7th Double Coset \[125\]

\[Mt_1t_2t_5N = \{Mt_1t_2t_5^n | n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(125)}\) fixes both 1, 2 and 5. So, we then have \(N^{(125)} \geq N^{125} = \langle e \rangle\), which indicates the stabilizing group \(N^{(125)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2t_5N\) are \(\frac{N}{N^{(125)}} = \frac{21}{1} = 21\).

Thus, The orbits of \(N^{(125)}\) on \{1, 2, 3, 4, 5, 6, 7\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, and \{7\}. We will now select a representative \(t_i\) from each orbit.

1 \(\in\) \{1\} \(\Rightarrow\) \(Mt_1t_2t_5t_1\), Lets now look at the relation ****
2 \(\in\) \{2\} \(\Rightarrow\) \(Mt_1t_2t_5t_2\), Lets now look at the relation ****
3 \(\in\) \{3\} \(\Rightarrow\) \(Mt_1t_2t_5t_3\), Lets now look at the relation ****
4 \(\in\) \{4\} \(\Rightarrow\) \(Mt_1t_2t_5t_4\), Lets now look at the relation ****
5 \(\in\) \{5\} \(\Rightarrow\) \(Mt_1t_2t_5t_5 \Rightarrow Mt_1t_2 \in [12]\)
6 \(\in\) \{6\} \(\Rightarrow\) \(Mt_1t_2t_5t_6\), Lets now look at the relation ****
7 \(\in\) \{7\} \(\Rightarrow\) \(Mt_1t_2t_5t_7\), Lets now look at the relation ****

8th Double Coset \[127\]

\[Mt_1t_2t_7N = \{Mt_1t_2t_7^n | n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(127)}\) fixes both 1, 2 and 7. So, we then have \(N^{(127)} \geq N^{127} = \langle e \rangle\), which indicates the stabilizing group \(N^{(127)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2t_7N\) are \(\frac{N}{N^{(127)}} = \frac{21}{1} = 21\).

Thus, The orbits of \(N^{(127)}\) on \{1, 2, 3, 4, 5, 6, 7\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, and \{7\}. We will now select a representative \(t_i\) from each orbit.

1 \(\in\) \{1\} \(\Rightarrow\) \(Mt_1t_2t_7t_1 \in [1271]\), which goes to a new double coset \[1271\]
2 \(\in\) \{2\} \(\Rightarrow\) \(Mt_1t_2t_7t_2\), Lets now look at the relation ****
3 \(\in\) \{3\} \(\Rightarrow\) \(Mt_1t_2t_7t_3\), Lets now look at the relation ****
4 \(\in\) \{4\} \(\Rightarrow\) \(Mt_1t_2t_7t_4\), Lets now look at the relation ****
5 \(\in\) \{5\} \(\Rightarrow\) \(Mt_1t_2t_7t_5\), Lets now look at the relation ****
6 \(\in\) \{6\} \(\Rightarrow\) \(Mt_1t_2t_7t_6\), Lets now look at the relation ****
7 \(\in\) \{7\} \(\Rightarrow\) \(Mt_1t_2t_7t_7 \Rightarrow Mt_1t_2 \in [12]\)

9th Double Coset \[146\]

\[Mt_1t_4t_6N = \{Mt_1t_4t_6^n | n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(146)}\) fixes both 1, 4 and 6. So, we then have \(N^{(146)} \geq N^{146} = \langle e \rangle\), which indicates the stabilizing group \(N^{(146)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_4t_6N\) are \(\frac{N}{N^{(146)}} = \frac{21}{1} = 21\).

Thus, The orbits of \(N^{(146)}\) on \{1, 2, 3, 4, 5, 6, 7\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, and \{7\}. We will now select a representative \(t_i\) from each orbit.

1 \(\in\) \{1\} \(\Rightarrow\) \(Mt_1t_4t_6t_1\), Lets now look at the relation ****
2 \(\in\) \{2\} \(\Rightarrow\) \(Mt_1t_4t_6t_2\), Lets now look at the relation ****
3 \in \{3\} \Rightarrow M_{t1}t_4t_6t_3, \text{ Lets now look at the relation ****}
4 \in \{4\} \Rightarrow M_{t1}t_4t_6t_4, \text{ Lets now look at the relation ****}
5 \in \{5\} \Rightarrow M_{t1}t_4t_6t_5, \text{ Lets now look at the relation ****}
6 \in \{6\} \Rightarrow M_{t1}t_4t_6t_6 \Rightarrow M_{t1}t_4 \in [14]
7 \in \{7\} \Rightarrow M_{t1}t_4t_6t_7, \text{ Lets now look at the relation ****}

10th Double Coset \[147\]

\[M_{t1}t_4t_7N = \{M_{t1}t_4t_7^n \mid n \in \mathbb{N}\}\]
The coset stabilizer \(N^{(147)}\) fixes both 1, 4 and 7. So, we then have \(N^{(147)} \geq N^{147} = \langle e \rangle\), which indicates the stabilizing group \(N^{(147)}\) only contains the identity \(e\).
Which would imply that the number of single cosets in \(M_{t1}t_4t_7N\) are \(\frac{N}{N^{(147)}} = \frac{21}{3} = 7\). Thus, The orbits of \(N^{(147)}\) on \(\{1, 2, 3, 4, 5, 6, 7\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\},\) and \(\{7\}\). We will now select a representative \(t_i\) from each orbit.
1 \in \{1\} \Rightarrow M_{t1}t_4t_7t_1, \text{ Lets now look at the relation ****}
2 \in \{2\} \Rightarrow M_{t1}t_4t_7t_2, \text{ Lets now look at the relation ****}
3 \in \{3\} \Rightarrow M_{t1}t_4t_7t_3, \text{ Lets now look at the relation ****}
4 \in \{4\} \Rightarrow M_{t1}t_4t_7t_4, \text{ Lets now look at the relation ****}
5 \in \{5\} \Rightarrow M_{t1}t_4t_7t_5, \text{ Lets now look at the relation ****}
6 \in \{6\} \Rightarrow M_{t1}t_4t_7t_6, \text{ Lets now look at the relation ****}
7 \in \{7\} \Rightarrow M_{t1}t_4t_7t_7 \Rightarrow M_{t1}t_4 \in [14]

Words of length four

11th Double Coset \[1236\]

\[M_{t1}t_2t_3t_6N = \{M_{t1}t_2t_3t_6^n \mid n \in \mathbb{N}\}\]
The coset stabilizer \(N^{(1236)}\) fixes both 1, 2, 3 and 6. So, we then have \(N^{(1236)} \geq N^{1236} = \langle e \rangle\), which indicates the stabilizing group \(N^{(1236)}\) only contains the identity \(e\).
Which would imply that the number of single cosets in \(M_{t1}t_2t_3t_6N\) are \(\frac{N}{N^{(1236)}} = \frac{21}{3} = 7\). Thus, The orbits of \(N^{(1236)}\) on \(\{1, 2, 3, 4, 5, 6, 7\}\) are \(\{1, 5, 7\}, \{3, 4, 6\},\) and \(\{2\}\). We will now select a representative \(t_i\) from each orbit.
1 \in \{1, 5, 7\} \Rightarrow M_{t1}t_2t_3t_6t_1, \text{ Lets now look at the relation ****}
2 \in \{3, 4, 6\} \Rightarrow M_{t1}t_2t_3t_6t_6 \Rightarrow M_{t1}t_2t_3 \in [123]
3 \in \{2\} \Rightarrow M_{t1}t_2t_3t_6t_2, \text{ Lets now look at the relation ****}

12th Double Coset \[1271\]

\[M_{t1}t_2t_7t_1N = \{M_{t1}t_2t_7t_1^n \mid n \in \mathbb{N}\}\]
The coset stabilizer \(N^{(1271)}\) fixes both 1, 2, 7 and 1. So, we then have \(N^{(1271)} \geq N^{1271} = \langle e \rangle\), which indicates the stabilizing group \(N^{(1271)}\) only contains the identity \(e\).
Which would imply that the number of single cosets in \(M_{t1}t_2t_7t_1N\) are \(\frac{N}{N^{(1271)}} = \frac{21}{3} = 7\).
= 7. Thus, the orbits of $N^{(1236)}$ on \{1, 2, 3, 4, 5, 6, 7\} are \{1, 5, 7\}, \{3, 4, 6\}, and \{2\}. We will now select a representative $t_i$ from each orbit.

1 ∈ \{1, 5, 7\} ⇒ $Mt_1t_2t_7t_1t_1$, Let’s now look at the relation ****

3 ∈ \{3, 6, 4\} ⇒ $Mt_1t_2t_7t_1t_3 ⇒ Mt_1t_2t_3 \in [123]

2 ∈ \{2\} ⇒ $Mt_1t_2t_7t_1t_2$, Let’s now look at the relation ****

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:

$$G = N \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_2t_3N \cup Mt_1t_2t_4N \cup Mt_1t_2t_5N \cup Mt_1t_2t_7N \cup Mt_1t_4t_6N \cup Mt_1t_4t_7N \cup Mt_1t_2t_3t_7N \cup Mt_1t_2t_7t_1N,$$

and that

$$|G| ≤ (|N^*| + \frac{|N|}{|N^{(17)}|} + \frac{|N|}{|N^{(11)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(124)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(127)}|}) \times |N|$$

$$|G| ≤ (1 + 7 + 21 + 21 + 7 + 21 + 21 + 7 + 7) \times 2520$$

$$|G| ≤ 176 \times 2520$$

$$|G| ≤ 443520$$

Figure 2.4: Construction of $M_{22}$ over Maximal Subgroup $A_7$
2.2.5 Construction of $PSL(2, 17)$ over Maximal Subgroup $12 : 2$

Consider the group $N = 2^{12} : A_4$ factored by the relations $(x^2tx^{-1}y)^4, (x^2ty^2)^4, (x^2t)^3 = 1$

That is, $G = PSL(2, 17) \cong \frac{2^{12} : A_4}{(x^2tx^{-1}y)^4, (x^2ty^2)^4, (x^2t)^3} = 1$

With $N$ being the size of $A_4$ as well as being transitive on 12 letters of order 2.

$2^{12} = <t_1> \ast <t_2> \ast <t_3> \ast <t_4> \ast <t_5> \ast <t_6> \ast <t_7> \ast <t_8> \ast <t_9> \ast <t_{11}> \ast <t_{12}>$

$N = A_4(12)$

$x \sim (1, 9, 5)(2, 4, 3)(6, 8, 7)(10, 12, 11)$

$y \sim (1, 11, 6)(2, 9, 7)(3, 10, 5)(4, 8, 12)$

$t \sim t_1$

We will also let $H$ be the maximal subgroup generated by the control group $A_4$ and $xtx^{-1}ty^{-1}tx^{-1}t$. In which we have $H = \langle N, xtx^{-1}ty^{-1}tx^{-1}t \rangle = (12 : 2)$ where $|H| = 24$.

We will first expand the relations:

(1) $(x^2tx^{-1}y)^4 = 1$

$\Rightarrow (x^2t_2)(x^2t_2)(x^2t_2)(x^2t_2) = 1$

$\Rightarrow x^8x^{-6}t_2x^6x^{-4}t_2x^4x^{-2}t_2x^2t_2 = 1$

$\Rightarrow x^2t_6^4t_2t_2^4t_2 = 1$

$\Rightarrow x^2t_2t_4t_3t_2 = 1$

$\Rightarrow Nt_2t_4 = Nt_2t_3$

(2) $(x^2ty^2)^4 = 1$

$\Rightarrow (x^2t_6)(x^2t_6)(x^2t_6)(x^2t_6) = 1$

$\Rightarrow x^8x^{-6}t_6x^6x^{-4}t_6x^4x^{-2}t_6x^2t_6 = 1$

$\Rightarrow x^2t_6^4t_6^4t_6 = 1$

(3) $(x^2t)^3 = 1$

$\Rightarrow (x^2t_11)(x^2t_11)(x^2t_11) = 1$

$\Rightarrow x^6x^{-4}t_11x^4x^{-2}t_11x^2t_11 = 1$

$\Rightarrow t_11^2t_11^2t_11 = 1$

$\Rightarrow t_10t_12t_11 = 1$

$\Rightarrow Nt_{10} = Nt_{11}t_{12}$

Let us now expand $xtx^{-1}ty^{-1}tx^{-1}t$ being in $H$.

$yx^{-1}y^{-1}txy^{-1}tx^{-1}tyx \in M$
\[ y^{txy^{-1}}t^{tx^{-1}}t \in H \]

\[ x^{2yt^{-1}}t_{2t6t1t5t1} \in H \]

Which, then implies \( Ht_{2t6t1} = Ht_{1t5} \Leftrightarrow Ht_{1t5} = Ht_{2t6t1} \)

We can also conjugate all the previous relation by all elements of \( x \) and \( y \) from \( N = T_{12} \). Like the previous DCE, an example of this would be \((N_{t10})^x = (N_{t11t12})^x\). Since, \( x = (1, 9, 5)(2, 4, 3)(6, 8, 7)(10, 12, 11) \), we have \( N_{t12} = N_{t10t11} \). So, we then obtain the following additional relations:

\[
\begin{align*}
(2 & 4 \sim 2 3) \Leftrightarrow (1 6 \sim 1 11) & (6 8 \sim 6 7) \Leftrightarrow (1 12 \sim 1 2) \\
(2 & 4 \sim 2 3) \Leftrightarrow (3 2 \sim 3 3) & (6 8 \sim 6 7) \Leftrightarrow (2 1 \sim 2 12) \\
(2 & 4 \sim 2 3) \Leftrightarrow (4 3 \sim 4 2) & (6 8 \sim 6 7) \Leftrightarrow (3 5 \sim 3 10) \\
(2 & 4 \sim 2 3) \Leftrightarrow (5 7 \sim 5 12) & (6 8 \sim 6 7) \Leftrightarrow (4 9 \sim 4 11) \\
(2 & 4 \sim 2 3) \Leftrightarrow (6 11 \sim 6 1) & (6 8 \sim 6 7) \Leftrightarrow (5 10 \sim 5 3) \\
(2 & 4 \sim 2 3) \Leftrightarrow (7 12 \sim 7 5) & (6 8 \sim 6 7) \Leftrightarrow (7 6 \sim 7 8) \\
(2 & 4 \sim 2 3) \Leftrightarrow (8 10 \sim 8 9) & (6 8 \sim 6 7) \Leftrightarrow (8 7 \sim 8 6) \\
(2 & 4 \sim 2 3) \Leftrightarrow (9 8 \sim 9 10) & (6 8 \sim 6 7) \Leftrightarrow (9 11 \sim 9 4) \\
(2 & 4 \sim 2 3) \Leftrightarrow (10 9 \sim 10 8) & (6 8 \sim 6 7) \Leftrightarrow (10 3 \sim 10 5) \\
(2 & 4 \sim 2 3) \Leftrightarrow (11 4 \sim 11 6) & (6 8 \sim 6 7) \Leftrightarrow (11 4 \sim 11 9) \\
(2 & 4 \sim 2 3) \Leftrightarrow (12 2 \sim 12 7) & (6 8 \sim 6 7) \Leftrightarrow (12 2 \sim 12 1) \\
(10 \sim 11 12) \Leftrightarrow (1 \sim 8 3) & \\
(10 \sim 11 12) \Leftrightarrow (2 \sim 9 7) & \\
(10 \sim 11 12) \Leftrightarrow (3 \sim 1 8) & \\
(10 \sim 11 12) \Leftrightarrow (4 \sim 5 6) & \\
(10 \sim 11 12) \Leftrightarrow (5 \sim 6 4) & \\
(10 \sim 11 12) \Leftrightarrow (6 \sim 4 5) & \\
(10 \sim 11 12) \Leftrightarrow (7 \sim 2 9) & \\
(10 \sim 11 12) \Leftrightarrow (8 \sim 3 1) & \\
(10 \sim 11 12) \Leftrightarrow (9 \sim 7 2) & \\
(10 \sim 11 12) \Leftrightarrow (11 \sim 12 10) & \\
(10 \sim 11 12) \Leftrightarrow (12 \sim 10 11) & 
\end{align*}
\]

We can also solve and use these relations to show the relations between cosets. Beginning the double coset enumeration, we first need to calculate the total number of unique cosets of \( H \) of \( N \) in \( G \). This is the index of \( G \) in \( H \). The index will be the order of \( G \) divided by the order of \( H \).

\[ \frac{G}{H} = \frac{2448}{24} = 102. \]

Now we know that we will have 102 unique single cosets.
Also, using the magma code “#DoubleCosets(G,H,sub< G|x, y >);”, you can calculate the number of double cosets in your double coset enumeration.

\[
> \text{#DoubleCosets}(G,H,\text{sub} < G|x, y >); \\
12
\]

This lets us know that there will be 12 unique double cosets. We will then use this information as we do the double coset enumeration of \( G \) over \( H \).

1st Double Coset [*]

\( H eN = \{ H e^n | n \in \mathbb{N} \} = \{ N \} \). \( N \) is transitive on \( \{1, 2, ..., 11, 12\} \); moreover,

\[
\frac{N}{N} = \frac{24}{24} = 1.
\]

So, \( H eN \) has one single orbit \( \{ t_1, t_2, ..., t_{11}, t_{12} \} \). Thus, orbit of \( N \) on \( \{1, 2, ..., 11, 12\} \) is \( \{1, 2, ..., 11, 12\} \). Now, select a representative \( t_i \) from orbit. We will select 1 from the orbit in this case.

\( 1 \in \{1, 2, ..., 11, 12\} \Rightarrow Ht_1 \in [1] \)

Now, we shall extend \( Ht_1 \) to a new double coset [1].

2nd Double Coset [1]

\( Ht_1N = \{ Ht_1^n | n \in \mathbb{N} \} \)

The coset stabilizer \( N^{(1)} \) are elements of \( T_{12} \) that fix 1. So, we then have

\( N^{(1)} \geq N = \langle e \rangle \), which indicates that there are no permutations of \( A_4 \) that fix 1.

This would imply that the number of single cosets in \( Ht_1N \) are

\[
\frac{N}{N^{(1)}} = \frac{12}{1} = 12.
\]

Thus, the orbits of \( N^{(1)} \) on \( \{1, 2, ..., 11, 12\} \) are \( \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\} \). From this, we could see that each orbit doesn’t share any other \( t_i \)s due to the stabilizer only containing the identity. We will now select a representative \( t_i \) from each orbit.

\[
\begin{align*}
1 \in \{1\} & \Rightarrow Ht_1t_1 = He \in [\ast] \\
2 \in \{2\} & \Rightarrow Ht_1t_2 \sim Ht_1t_{12} \in [12] \\
3 \in \{3\} & \Rightarrow Ht_1t_3 \sim Ht_1t_8 \in [1] \\
4 \in \{4\} & \Rightarrow Ht_1t_4 \in [14] \\
5 \in \{5\} & \Rightarrow Ht_1t_5 \in [15] \\
6 \in \{6\} & \Rightarrow Ht_1t_6 \sim Ht_1t_{11} \in [16] \\
7 \in \{7\} & \Rightarrow Ht_1t_7 \in [17] \\
8 \in \{8\} & \Rightarrow Ht_1t_8 \sim Ht_1t_3 \in [1] \\
9 \in \{9\} & \Rightarrow Ht_1t_9 \in [19] \\
10 \in \{10\} & \Rightarrow Ht_1t_{10} \in [110] \\
11 \in \{11\} & \Rightarrow Ht_1t_{11} \sim Ht_1t_6 \in [16] \\
12 \in \{12\} & \Rightarrow Ht_1t_{12} \sim Ht_1t_2 \in [12]
\end{align*}
\]

We will now start to find all the single cosets in the double coset enumeration with words of length two.
3rd Double Coset \([12]\)

\[H_{t_1t_2}N = \{H_{t_1t_2}^n | n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(12)}\) are elements of \(A_4\) that fix both 1 and 2. Which implies that \(N^{(12)} \geq N^{12} = \langle e \rangle\) which also happens to be the identity.

There are also no permutations in this double coset that conjugate with any relations that are contained in the stabilizer. So, this still would imply that the coset stabilizer is only of order 1, and the number of single cosets in \(N_{t_1t_2}N\) are \(\frac{N}{N_{t_1t_2}} = \frac{12}{1} = 12\). Now, The orbits of \(N^{(12)}\) on \(\{1, 2, ..., 24, 25\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\).

We will now select a representative \(t_i\) from each orbit.

1 \(\in \{1\}\) \(\Rightarrow H_{t_1t_2}t_1 \sim H_{t_1t_2}t_3 \in [121]\)
2 \(\in \{2\}\) \(\Rightarrow H_{t_1t_2}t_2 \sim H_{t_1t_2}t_{12} \in [1]\)
3 \(\in \{3\}\) \(\Rightarrow H_{t_1t_2}t_3 \sim H_{t_1t_2}t_1 \in [121]\)
4 \(\in \{4\}\) \(\Rightarrow H_{t_1t_2}t_4 \sim H_{t_1t_2}t_6 \sim H_{t_1t_2}t_7 \in [19]\)
5 \(\in \{5\}\) \(\Rightarrow H_{t_1t_2}t_5 \in [14]\)
6 \(\in \{6\}\) \(\Rightarrow H_{t_1t_2}t_6 \sim H_{t_1t_2}t_4 \sim H_{t_1t_2}t_7 \in [19]\)
7 \(\in \{7\}\) \(\Rightarrow H_{t_1t_2}t_7 \sim H_{t_1t_2}t_4 \sim H_{t_1t_2}t_6 \in [19]\)
8 \(\in \{8\}\) \(\Rightarrow H_{t_1t_2}t_8 \in [12]\)
9 \(\in \{9\}\) \(\Rightarrow H_{t_1t_2}t_9 \in [17]\)
10 \(\in \{10\}\) \(\Rightarrow H_{t_1t_2}t_{10} \in [16]\)
11 \(\in \{11\}\) \(\Rightarrow H_{t_1t_2}t_{11} \in [110]\)
12 \(\in \{12\}\) \(\Rightarrow H_{t_1t_2}t_{12} \sim H_{t_1t_2}t_1 \in [1]\)

4th Double Coset \([14]\)

\[H_{t_1t_4}N = \{H_{t_1t_4}^n | n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(14)}\) are elements of \(A_4\) that fix both 1 and 4. Which, like \(N^{(12)}\), also implies that \(N^{(14)} \geq N^{14} = \langle e \rangle\).

Also similar to \(N^{(12)}\), are no permutations in this double coset that conjugate with any relations that are contained in the stabilizer as well.

This implies the coset stabilizer is of order 1, and the number of single cosets in \(N_{t_1t_4}N\) are \(\frac{N}{N_{t_1t_4}} = \frac{12}{1} = 12\). Now, The orbits of \(N^{(14)}\) on \(\{1, 2, ..., 12\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\).

We will now select a representative \(t_i\) from each orbit.

1 \(\in \{1\}\) \(\Rightarrow H_{t_1t_4}t_1 \sim H_{t_1t_4}t_2 \sim H_{t_1t_4}t_8 \in [14]\)
2 \(\in \{2\}\) \(\Rightarrow H_{t_1t_4}t_2 \sim H_{t_1t_4}t_1 \sim H_{t_1t_4}t_8 \in [14]\)
3 \(\in \{3\}\) \(\Rightarrow H_{t_1t_4}t_3 \in [121]\)
4 \(\in \{4\}\) \(\Rightarrow H_{t_1t_4}t_4 \in [1]\)
5 \(\in \{5\}\) \(\Rightarrow H_{t_1t_4}t_5 \in [16]\)
6 \(\in \{6\}\) \(\Rightarrow H_{t_1t_4}t_6 \in [15]\)
7 \(\in \{7\}\) \(\Rightarrow H_{t_1t_4}t_7 \in [147]\)
8 \(\in \{8\}\) \(\Rightarrow H_{t_1t_4}t_8 \sim H_{t_1t_4}t_1 \sim H_{t_1t_4}t_2 \in [14]\)
9 \(\in \{9\}\) \(\Rightarrow H_{t_1t_4}t_9 \in [17]\)
10 \in \{10\} \Rightarrow H_{t_1 t_4} t_{t_{10}} \in [12]
11 \in \{11\} \Rightarrow H_{t_1 t_4} t_{t_{11}} \sim H_{t_1 t_4} t_{t_{12}} \in [19]
12 \in \{12\} \Rightarrow H_{t_1 t_4} t_{t_{12}} \sim H_{t_1 t_4} t_{t_{11}} \in [19]

5th Double Coset [15]

\begin{align*}
H_{t_1 t_5} N &= \{ H_{t_1 t_5}^n | n \in \mathbb{N} \} \\
\text{The coset stabilizer } N^{(15)} \text{ are elements of } A_4 \text{ that fix both 1 and 5. Which implies that } \\
N^{(15)} &= N^{15} = < e > \\
\text{This implies the coset stabilizer is of order 1, and the number of single cosets in } N t_{t_5} N \text{ are } N^{\frac{N}{15}} = \frac{12}{1} = 12. \\
\text{The orbits of } N^{(15)} \text{ on } \{1, 2, ..., 11, 12\} \text{ are } \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}. \\
\text{We will now select a representative } t_i \text{ from each orbit.} \\
1 \in \{1\} \Rightarrow H_{t_1 t_5 t_1} \sim H_{t_1 t_5 t_2} \sim H_{t_1 t_5 t_{12}} \in [15] \\
2 \in \{2\} \Rightarrow H_{t_1 t_5 t_2} \sim H_{t_1 t_5 t_1} \sim H_{t_1 t_5 t_{12}} \in [15] \\
3 \in \{3\} \Rightarrow H_{t_1 t_5 t_3} \sim H_{t_1 t_5 t_4} \sim H_{t_1 t_5 t_{10}} \in [16] \\
4 \in \{4\} \Rightarrow H_{t_1 t_5 t_4} \sim H_{t_1 t_5 t_5} \sim H_{t_1 t_5 t_{10}} \in [16] \\
5 \in \{5\} \Rightarrow H_{t_1 t_5 t_5} \in [1] \\
6 \in \{6\} \Rightarrow H_{t_1 t_5 t_6} \in [14] \\
7 \in \{7\} \Rightarrow H_{t_1 t_5 t_7} \sim H_{t_1 t_5 t_9} \in [19] \\
8 \in \{8\} \Rightarrow H_{t_1 t_5 t_8} \sim H_{t_1 t_5 t_{11}} \in [17] \\
9 \in \{9\} \Rightarrow H_{t_1 t_5 t_9} \sim H_{t_1 t_5 t_7} \in [19] \\
10 \in \{10\} \Rightarrow H_{t_1 t_5 t_{10}} \sim H_{t_1 t_5 t_3} \sim H_{t_1 t_5 t_4} \in [16] \\
11 \in \{11\} \Rightarrow H_{t_1 t_5 t_{11}} \sim H_{t_1 t_5 t_8} \in [17] \\
12 \in \{12\} \Rightarrow H_{t_1 t_5 t_{12}} \sim H_{t_1 t_5 t_1} \sim H_{t_1 t_5 t_2} \in [15] \\
\end{align*}

6th Double Coset [16]

\begin{align*}
H_{t_1 t_6} N &= \{ H_{t_1 t_6}^n | n \in \mathbb{N} \} \\
\text{The coset stabilizer } N^{(16)} \text{ are elements of } A_4 \text{ that fix both 1 and 6. Which implies that } \\
N^{(16)} \geq N^{16} = < e > \\
\text{This implies the coset stabilizer is also of order 1, and the number of single cosets in } N t_{t_6} N \text{ are } N^{\frac{N}{16}} = \frac{12}{1} = 12. \\
\text{The orbits of } N^{(16)} \text{ on } \{1, 2, ..., 11, 12\} \text{ are } \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}. \\
\text{We will now select a representative } t_i \text{ from each orbit.} \\
1 \in \{1\} \Rightarrow H_{t_1 t_6} t_1 \sim H_{t_1 t_6} t_8 \in [161] \\
2 \in \{2\} \Rightarrow H_{t_1 t_6} t_2 \sim H_{t_1 t_6} t_4 \sim H_{t_1 t_6} t_7 \in [15] \\
3 \in \{3\} \Rightarrow H_{t_1 t_6} t_3 \in [16] \\
4 \in \{4\} \Rightarrow H_{t_1 t_6} t_4 \sim H_{t_1 t_6} t_2 \sim H_{t_1 t_6} t_7 \in [15] \\
5 \in \{5\} \Rightarrow H_{t_1 t_6} t_5 \in [14] \\
6 \in \{6\} \Rightarrow H_{t_1 t_6} t_6 \sim H_{t_1 t_6} t_{11} \in [1] \\
7 \in \{7\} \Rightarrow H_{t_1 t_6} t_7 \sim H_{t_1 t_6} t_2 \sim H_{t_1 t_6} t_4 \in [15] \\
8 \in \{8\} \Rightarrow H_{t_1 t_6} t_8 \sim H_{t_1 t_6} t_1 \in [161] \\
9 \in \{9\} \Rightarrow H_{t_1 t_6} t_9 \in [17]
10 ∈ \{10\} ⇒ H_{t_1 t_6 t_10} ∈ [12]
11 ∈ \{11\} ⇒ H_{t_1 t_6 t_{11}} ∼ H_{t_1 t_6 t_6} ∈ [1]
12 ∈ \{12\} ⇒ H_{t_1 t_6 t_{12}} ∈ [110]

7th Double Coset [17]

\[ H_{t_1 t_7} N = \{ H_{t_1 t_7^n} | n ∈ \mathbb{N} \} \]

The coset stabilizer \( N^{(17)} \) are elements of \( A_4 \) that fix both 1 and 7. Which implies that \( N^{(17)} ≥ N^{17} = ⟨ e ⟩ \).

This implies the coset stabilizer is also of order 1, and the number of single cosets in \( N t_{17} N \) are \( \frac{N}{N^{17}} = \frac{12}{1} = 12 \). The orbits of \( N^{(17)} \) on \{1, 2, ..., 11, 12\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}.

We will now select a representative \( t_i \) from each orbit.

1 ∈ \{1\} ⇒ H_{t_1 t_7 t_1} ∼ H_{t_1 t_7 t_3} ∈ [17]
2 ∈ \{2\} ⇒ H_{t_1 t_7 t_2} ∈ [19]
3 ∈ \{3\} ⇒ H_{t_1 t_7 t_3} ∼ H_{t_1 t_7 t_1} ∈ [17]
4 ∈ \{4\} ⇒ H_{t_1 t_7 t_4} ∈ [147]
5 ∈ \{5\} ⇒ H_{t_1 t_7 t_5} ∈ [14]
6 ∈ \{6\} ⇒ H_{t_1 t_7 t_6} ∈ [10]
7 ∈ \{7\} ⇒ H_{t_1 t_7 t_7} ∈ [161]
8 ∈ \{8\} ⇒ H_{t_1 t_7 t_8} ∈ [12]
9 ∈ \{9\} ⇒ H_{t_1 t_7 t_9} ∈ [110]
10 ∈ \{10\} ⇒ H_{t_1 t_7 t_{10}} ∈ [12]
11 ∈ \{11\} ⇒ H_{t_1 t_7 t_{11}} ∼ H_{t_1 t_7 t_{12}} ∈ [15]
12 ∈ \{12\} ⇒ H_{t_1 t_7 t_{12}} ∼ H_{t_1 t_7 t_{11}} ∈ [15]

8th Double Coset [19]

\[ H_{t_1 t_9} N = \{ H_{t_1 t_9^n} | n ∈ \mathbb{N} \} \]

The coset stabilizer \( N^{(19)} \) are elements of \( A_4 \) that fix both 1 and 9. Which implies that \( N^{(19)} ≥ N^{19} = ⟨ e ⟩ \).

This implies the coset stabilizer is also of order 1, and the number of single cosets in \( N t_{19} N \) are \( \frac{N}{N^{19}} = \frac{12}{1} = 12 \). The orbits of \( N^{(19)} \) on \{1, 2, ..., 11, 12\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}.

We will now select a representative \( t_i \) from each orbit.

1 ∈ \{1\} ⇒ H_{t_1 t_9 t_1} ∼ H_{t_1 t_9 t_6} ∼ H_{t_1 t_7 t_{11}} ∈ [19]
2 ∈ \{2\} ⇒ H_{t_1 t_9 t_2} ∈ [17]
3 ∈ \{3\} ⇒ H_{t_1 t_9 t_3} ∈ [14]
4 ∈ \{4\} ⇒ H_{t_1 t_9 t_4} ∼ H_{t_1 t_9 t_5} ∈ [15]
5 ∈ \{5\} ⇒ H_{t_1 t_9 t_5} ∼ H_{t_1 t_9 t_4} ∈ [15]
6 ∈ \{6\} ⇒ H_{t_1 t_9 t_6} ∼ H_{t_1 t_9 t_1} ∼ H_{t_1 t_9 t_{11}} ∈ [19]
7 ∈ \{7\} ⇒ H_{t_1 t_9 t_7} ∼ H_{t_1 t_9 t_8} ∼ H_{t_1 t_9 t_{10}} ∈ [12]
8 ∈ \{8\} ⇒ H_{t_1 t_9 t_8} ∼ H_{t_1 t_9 t_7} ∼ H_{t_1 t_9 t_{10}} ∈ [12]
9 ∈ \{9\} ⇒ H_{t_1 t_9 t_9} ∈ [1]
10 \in \{10\} \Rightarrow H_{t_1t_9t_{10}} \sim H_{t_1t_9t_7} \sim H_{t_1t_9t_8} \in [12]
11 \in \{11\} \Rightarrow H_{t_1t_9t_{11}} \sim H_{t_1t_9t_1} \sim H_{t_1t_9t_6} \in [19]
12 \in \{12\} \Rightarrow H_{t_1t_9t_{12}} \in [14]

**9th Double Coset [110]**

\[H_{t_1t_{10}}N = \{H_{t_1}t_1^n|n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(110)}\) are elements of \(A_4\) that fix both 1 and 9. Which implies that \(N^{(110)} \geq N^{110} = \langle e \rangle\).

This time however, since we now have that \((H_{t_1t_{10}})^y = H_{t_1t_{10}}\). The permutation \(yx\) is also in the stabilizer \(N^{(110)} \geq N^{110} = \langle (1,10)(2,5)(3,12)(4,7)(6,9)(8,11) \rangle\).

This implies the coset stabilizer is now of order 2, and the number of single cosets in \(N_{t_1t_{10}}N\) are

\(\frac{N^{(110)}}{\mathbb{N}^{110}} = \frac{12}{2} = 6\). The orbits of \(N^{(110)}\) on \(\{1,2,\ldots,11,12\}\) are \(\{1,10\},\{2,5\},\{3,12\},\{4,7\},\{6,9\},\{8,11\}\).

We will now select a representative \(t_i\) from each orbit.
1 \in \{1,10\} \Rightarrow H_{t_1t_{10}t_1} \in [1]
2 \in \{2,5\} \Rightarrow H_{t_1t_{10}t_2} \in [17]
3 \in \{3,12\} \Rightarrow H_{t_1t_{10}t_3} \in [16]
4 \in \{4,7\} \Rightarrow H_{t_1t_{10}t_4} \in [147]
6 \in \{6,9\} \Rightarrow H_{t_1t_{10}t_6} \in [14]
8 \in \{8,11\} \Rightarrow H_{t_1t_{10}t_8} \in [12]

We will now find the remaining single cosets with the double coset enumeration of words length three

**10th Double Coset [121]**

\[H_{t_1t_2t_1}N = \{H_{t_1}t_2^nt_1^n|n \in \mathbb{N}\}\]

The coset stabilizer \(N^{(121)}\) are elements of \(A_4\) that fix all of 1,2 and 1. Which implies that

\(N^{(121)} \geq N^{121} = \langle e \rangle\).

Since we also have that \((H_{t_1t_2t_2}t^{-1}) = H_{t_1t_2t_1}N\) The permutation \(xy^{-1}\) is also in the stabilizer \(N^{(121)} \geq N^{121} = \langle (1,2,12)(3,7,11)(4,5,6)(8,9,10) \rangle\). This now implies the coset stabilizer is of order 3, and the number of single cosets in \(H_{t_1t_2t_1}N\) are

\(\frac{N^{(121)}}{\mathbb{N}^{121}} = \frac{12}{3} = 4\). Now, The orbits of \(N^{(121)}\) on \(\{1,2,\ldots,11,12\}\) are \(\{1,2,12\},\{3,7,11\},\{4,5,6\},\{8,9,10\}\).

We will now select a representative \(t_i\) from each orbit.
1 \in \{1,2,12\} \Rightarrow H_{t_1t_2t_1t_1} \sim H_{t_1t_2t_1t_3} \in [12]
3 \in \{3,7,11\} \Rightarrow H_{t_1t_2t_1t_3} \sim H_{t_1t_2t_2t_1} \in [12]
4 \in \{4,5,6\} \Rightarrow H_{t_1t_2t_2t_4} \in [14]
8 \in \{8,9,10\} \Rightarrow H_{t_1t_2t_1t_8} \in [121]
11th Double Coset [147]

\[ Ht_1 t_4 t_7 N = \{ Ht_1 t_4 t_7^n \mid n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(147)} \) are elements of \( A_4 \) that fix all of 1,4 and 7. Which implies that
\[ N^{(147)} \geq N^{147} = \langle e \rangle. \]

Since we also have that \( (N t_7 t_1 t_1)^{x y} \Rightarrow Ht_1 t_4 t_7 N, \) and also \( (Ht_4 t_1 t_1 N)^{x y^{-1} x} \Rightarrow Ht_1 t_4 t_7 N. \) The permutations \( x y \) and \( x y^{-1} x \) are also in the stabilizer \( N^{(147)} \geq \)
\[ N^{147} = \langle (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12), (1, 4)(2, 11)(3, 6)(5, 8)(7, 10)(9, 12) \rangle. \]

This now implies the coset stabilizer is of order 3, and the number of single cosets in \( Ht_1 t_4 t_7 N \) are \( \frac{N}{N^{(147)}} = \frac{12}{4} = 3. \) Now, The orbits of \( N^{(147)} \) on \( \{1, 2, ..., 11, 12\} \) are \( \{1, 7, 4, 10\}, \{2, 8, 11, 5\}, \{3, 9, 6, 12\}. \)

We will now select a representative \( t_i \) from each orbit.
\[ 7 \in \{1, 7, 4, 10\} \Rightarrow Ht_1 t_4 t_7 N t_7 \in [14] \]
\[ 2 \in \{2, 8, 11, 5\} \Rightarrow Ht_1 t_4 t_7 N t_2 \in [17] \]
\[ 3 \in \{3, 9, 6, 12\} \Rightarrow Ht_1 t_4 t_7 N t_3 \in [10] \]

12th Double Coset [161]

\[ Ht_1 t_6 t_1 N = \{ Ht_1 t_6 t_1^n \mid n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(161)} \) are elements of \( A_4 \) that fix all of 1,6 and 1. Which implies that
\[ N^{(161)} \geq N^{161} = \langle e \rangle. \]

Since we also have that \( (Ht_1 t_1 t_1 t_1 N)^{x^{-1} y} \Rightarrow Ht_1 t_6 t_1 N. \) The permutation \( x^{-1} y \) is also in the stabilizer \( N^{(161)} \geq N^{161} = \langle (1, 11, 6)(2, 9, 7)(3, 10, 5)(4, 8, 12) \rangle. \) This now implies the coset stabilizer is of order 3, and the number of single cosets in \( Ht_1 t_6 t_1 N \) are \( \frac{N}{N^{(161)}} = \frac{12}{3} = 4. \) Now, The orbits of \( N^{(161)} \) on \( \{1, 2, ..., 11, 12\} \) are \( \{1, 11, 6\}, \{2, 9, 7\}, \)
\[ \{3, 10, 5\}, \{4, 8, 12\}. \]

We will now select a representative \( t_i \) from each orbit.
\[ 1 \in \{1, 11, 6\} \Rightarrow Ht_1 t_6 t_1 t_1 \sim Ht_1 t_6 t_1 t_4 \in [16] \]
\[ 2 \in \{2, 9, 7\} \Rightarrow Ht_1 t_6 t_1 t_2 \in [17] \]
\[ 3 \in \{3, 10, 5\} \Rightarrow Ht_1 t_6 t_1 t_3 \in [161] \]
\[ 4 \in \{4, 8, 12\} \Rightarrow Ht_1 t_6 t_1 t_4 \sim Ht_1 t_6 t_1 t_1 \in [16] \]

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:
\[
|G| \leq (|N^*| + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(147)}|} + \frac{|N|}{|N^{(161)}} + \frac{|N|}{|N^{(147)}} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(110)}} + \frac{|N|}{|N^{(110)}} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(110)}|} \times |H|
\]
$|G| \leq (1 + 12 + 12 + 12 + 12 + 12 + 12 + 6 + 4 + 3 + 4) \times 24$

$|G| \leq 102 \times 24$

$|G| \leq 2448$

Figure 2.5: Construction of $PSL(2, 17)$ over Maximal Subgroup $A_4$
2.3 Monomial Progenitor Double Coset Enumeration

2.3.1 Construction of $S_5$ over $D_8$

Consider $\frac{5^2 \cdot D_8}{(xyt)^3} \cong S_5 = G$

With $N$ being transitive on 2 letters of order 5 since we have a monomial progenitor.

$5^2 = \langle t_1 \rangle \triangleleft D_8$

$N = D_8$

$x \sim (1, 5, 7, 3)(2, 4, 8, 6)$

$y \sim (1, 2)(3, 4)(5, 6)(7, 8)$

$t \sim t_1$

We will first expand the relation:

$$(xyt)^3 = 1$$

$$(xyt)(xyt)(xyt) = 1$$

$$(xy)^3(xy)^{-2}t_1(xy)^2(xy)^{-1}t_1(xy)t_1 = 1$$

$$(xyt_1^2t_1^2t_1 = 1$$

$$(xyt_1^2t_1 = 1$$

$$(xyt_1^2t_1^3 = t_1^4$$

$$(Nt_1t_2^3 = Nt_1^4$$

Since, we have a monomial progenitor, we will have the labeling of $t'_i$s be as follows,

Table 2.1: Labeling of $5^2 : D_8$ $t'_i$s

<table>
<thead>
<tr>
<th>$t_1^n$</th>
<th>$t_2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.t_1$</td>
<td>$2.t_2$</td>
</tr>
<tr>
<td>$3.t_1^2$</td>
<td>$4.t_2^2$</td>
</tr>
<tr>
<td>$5.t_1^3$</td>
<td>$6.t_2^3$</td>
</tr>
<tr>
<td>$7.t_1^4$</td>
<td>$8.t_2^4$</td>
</tr>
</tbody>
</table>

We can also conjugate all the previous relation by all elements of $x$ and $y$ from $N = D_8$. Taking the elements of $5^2 : D_8$ We then obtain the following additional relations:

$$(xyt_1^2t_2^3 = t_1^4)c \Rightarrow xyt_1^2t_2^3 = t_1^4$$

$$(xyt_1^2t_2^3 = t_1^4)x \Rightarrow xyt_1^2t_2 = t_1^2$$

$$(xyt_1^2t_2^3 = t_1^4)y \Rightarrow yxt_2^3t_1^3 = t_2^4$$

$$(xyt_1^2t_2^3 = t_1^4)x^{-1} \Rightarrow yxt_1^2t_2^4 = t_1^3$$

$$(xyt_1^2t_2^3 = t_1^4)x^2 \Rightarrow yxt_1^4t_2^2 = t_1$$

$$(xyt_1^2t_2^3 = t_1^4)y \Rightarrow yxt_1^2t_1^3 = t_2^2$$

$$(xyt_1^2t_2^3 = t_1^4)x^x \Rightarrow yxt_2^3t_1^4 = t_2^3$$

$$(xyt_1^2t_2^3 = t_1^4)x^2y \Rightarrow yxt_2^4t_1^2 = t_2$$
Beginning the double coset enumeration, we first need to calculate the total number of unique cosets of $N$ in $G$.

$$\frac{G}{N} = \frac{120}{8} = 15.$$ 

Now we know that we will have 15 unique single cosets.

Calculating the number of double cosets in your double coset enumeration.

```plaintext
> #DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
4
```

This lets us know that there will be 4 unique double cosets. We will then use this information as we do the double coset enumeration of $G$ over $N$.

**1st Double Coset [∗]**

We have that $NeN = \{Ne^n | n ∈ \mathbb{N}\} = \{N\}$.

$N$ is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8\}$; moreover, $\frac{N}{N} = \frac{8}{8} = 1$.

Thus, this indicates that there is 1 single coset in the double coset $NeN$. Therefore, the orbit of $N$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is the single orbit $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Now, select a representative $t_1$ from orbit. We will select 1 from the orbit in this case.

$1 ∈ \{1, 2, 3, 4, 5, 6, 7, 8\} ⇒ Nt_1 ∈ [1]$

Now, we shall extend $Nt_1$ to a new double coset $[1]$.

**2nd Double Coset [1]**

$Nt_1N = \{Nt_1^n | n ∈ \mathbb{N}\}$

The coset stabilizer $N^{(1)}$ are elements of the monomial representation $5^{*2} : D_8$ that fix 1. So, we then have $N^{(1)} ≥ N^1 = < e >$, which indicates only the identity $e$ would be in the stabilizing group $N^{(1)}$. Which would imply that the number of single cosets in $Nt_1N$ are $\frac{N}{N^{(1)}} = \frac{8}{1} = 8$. Thus, The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}$, and $\{8\}$.

We will now select a representative $t_2$ from each orbit.

$1 ∈ \{1\} ⇒ Nt_1t_1 ⇒ Nt_1^2 ∈ [1]$

$2 ∈ \{2\} ⇒ Nt_1t_2 ∈ [12]$, which goes to a new double coset $[12]$

$3 ∈ \{3\} ⇒ Nt_1t_2^2 ⇒ Nt_1^3 ∈ [1]$

$4 ∈ \{4\} ⇒ Nt_1t_2^2$, Let us look at the relation $xyt_1t_2^3 = t_1^4$.

Multiplying by $t_2^3$ on the right hand side, we have, $xyt_1t_2^3t_2^4 = t_1^4t_2^4$

$⇒ xyt_1t_2^2 = t_1^4t_2^4$, and conjugating by the coset transversal $(t_1^4t_2^4)^2 = t_1t_2 ∈ [12]$

$5 ∈ \{5\} ⇒ Nt_1t_1^3 ⇒ Nt_1^4 ∈ [1]$

$6 ∈ \{6\} ⇒ Nt_1t_2^3 ∈ [16]$, by the relation $xyt_1t_2^3 = t_1^4$.

$Nt_1t_2^3 ⇒ t_1^4 ∈ [16]$

$7 ∈ \{7\} ⇒ Nt_1t_1^4 ⇒ Ne ∈ [∗]$
8 ∈ {8} ⇒ \(Nt_1t_2^4 \in [18]\), which goes to a new double coset [18]

3rd Double Coset [12]

\(Nt_1t_2N = \{Nt_1t_2^n | n ∈ \mathbb{N}\}\)

The coset stabilizer \(N^{(12)}\) are elements of the monomial representation \(5^{*2} : D_8\) that fix both 1 and 2. So, we then have \(N^{(12)} \supseteq N^{12} = < e >\), which indicates only the identity \(e\) would be in the stabilizing group \(N^{(12)}\). However, since we have that

\[
(xy_1t_2^3 = t_1^4)x^2y \Rightarrow yxt_2^4t_1^2 = t_2,
\]

\(⇒ t_1yxt_2^4t_1^2 = t_1t_2\), multiplying \(t_1\) on the left hand side.

\(⇒ yxt_2^4t_1^2 = t_1t_2\).

Also, since we have \(yxt_2t_1^3 = t_2^4\),

\(⇒ yxt_2t_1^3t_1^4 = t_2^4t_1^4\), multiplying \(t_1^4\) on the right hand side.

\(⇒ yxt_2t_1^2 = t_2^4t_1^4\).

Thus, \(xyt\) also stabilizes \(N\), so we have \(N\) are elements of the monomial representation \(5^{*2} : D_8\) that fix both 1 and 8. So, we then have \(N^{(12)} \supseteq N^{18} = < e >\), which indicates only the identity \(e\) would be in the stabilizing group \(N^{(12)}\). However, since we have that

\[
(xy_1t_2^3 = t_1^4)x^2y \Rightarrow yxt_2^4t_1^2 = t_2,
\]

\(⇒ t_1yxt_2^4t_1^2 = t_1t_2\), multiplying \(t_1\) on the left hand side.

\(⇒ yxt_2^4t_1^2 = t_1t_2\).

Also, since we have \(yxt_2t_1^3 = t_2^4\),

\(⇒ yxt_2t_1^3t_1^4 = t_2^4t_1^4\), multiplying \(t_1^4\) on the right hand side.

\(⇒ yxt_2t_1^2 = t_2^4t_1^4\).

Thus, the orbits of \(N^{(12)}\) on \{1, 2, 3, 4, 5, 6, 7, 8\} are \{1, 8\}, \{2, 7\}, \{3, 6\}, and \{4, 5\}.

We will now select a representative \(t_i\) from each orbit.

8 ∈ \{1, 8\} ⇒ \(Nt_1t_2t_2^4 \Rightarrow Nt_1 \in [1]\)

2 ∈ \{2, 7\} ⇒ \(Nt_1t_2t_2 \Rightarrow Nt_1t_2^2\). Let us look at the relation \(xyt_1t_2^3 = t_1^4\).

\(⇒ xyt_1t_2^3t_2^2 = t_1^4t_2^4\), multiplying \(t_2^4\) on the right hand side.

\(⇒ xyt_1t_2^2 = t_1^4t_2^4\), in which \(t_1^4t_2^4\) is in the double coset [12].

\(6 \in \{3, 6\} ⇒ Nt_1t_2t_2^3 \Rightarrow Nt_1t_2t_2^8 \in [18]\)

5 ∈ \{4, 5\} ⇒ \(Nt_1t_2t_2^3\). Let us look at the new relation \(t_1t_2 = t_2^4t_1^4\).

Substituting, we have that \(Nt_1t_2t_2^3 = Nt_2t_1t_2^4\), \(⇒ Nt_2t_1t_2^8\), and since \(yxt_2t_2^4t_1^2 = t_2\),

we have that \(Nt_2t_1t_2^8 = Nt_2 \in [1]\).

3rd Double Coset [18]

\(Nt_1t_2^4N = \{Nt_1t_2^4n | n ∈ \mathbb{N}\}\)

The coset stabilizer \(N^{(18)}\) are elements of the monomial representation \(5^{*2} : D_8\) that fix both 1 and 8. So, we then have \(N^{(18)} \supseteq N^{18} = < e >\), which indicates only the identity \(e\) would be in the stabilizing group \(N^{(18)}\). However, looking at the relation

\[
xyt_1t_2^3 = t_1^4,
\]

\(⇒ xyt_1t_2^3t_2 = t_1^4t_2^4\), multiplying \(t_2\) on the right hand side.

\(⇒ xyt_1t_2^3 = t_1^4t_2^4\).

Also, taking the relation \(yxt_2t_1^3 = t_4^4\),

\(⇒ t_1yxt_2t_1^3 = t_1t_4^4\), multiplying \(t_1\) on the left hand side.

\(⇒ yxt_2t_1^3 = t_1t_4^4\).

Finally, taking the relation \(yxt_2t_1^2 = t_2\),

\(⇒ t_1^4yxt_2t_1^2 = t_1^4t_2\), multiplying \(t_1^4\) on the left hand side.

\(⇒ yxt_2t_1^2 = t_1^4t_2\).
Therefore, altogether, we have that \( t_1 t_2^4 = t_1^4 t_2 = t_2^3 t_1^3 = t_2^2 t_1^2 \). Since,

\[(t_1^4 t_2)x^2 = (t_2^3 t_1^3)xy = (t_2^2 t_1^2)y^x = t_1 t_2^4,\]

we have that \( x^2, xy, \) and \( yx \) are all in the coset stabilizing group \( N^{(18)} = \langle x^2, xy, yx \rangle \).

Which would imply that the number of single cosets in \( N t_1 t_2^4 N \) are \( \frac{N}{N^{(18)}} = \frac{8}{4} = 2 \).

Thus, The orbits of \( N^{(18)} \) on \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) are \( \{1, 7, 6, 4\} \), and \( \{2, 8, 3, 5\} \).

We will now select a representative \( t_i \) from each orbit.

\( 4 \in \{1, 7, 6, 4\} \Rightarrow N t_1 t_2^4 t_2^2 \Rightarrow N t_1 t_2 \in [12] \)

\( 2 \in \{2, 8, 3, 5\} \Rightarrow N t_1 t_2^4 t_2 \Rightarrow N t_1 \in [12] \)

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:

\[ G = Ne \cup N t_1 N \cup N t_1 t_2 N \cup N t_1 t_8 N, \] and that

\[ |G| \leq (|N| + \frac{|N|}{|N^{(11)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(18)}|}) \times |N| \]

\[ |G| \leq (1 + 8 + 4 + 2) \times 8 \]

\[ |G| \leq 15 \times 8 \]

\[ |G| \leq 120 \]
2.3.2 Construction of $J_1$ over Maximal subgroup $PSL(2, 11)$

Consider \( (xyt)^2(yt)^m ytyt^2 ytyt^{-1} ytyt^{-1} = 1 \) \( \cong PSL(2, 11) = M \leq G \)

With the maximal subgroup $M$ being the size of $PSL(2, 11)$ as well as being transitive on 2 letters of order 7. That is, \( M = \langle N, x^{-1} y t^{-1} y t y t y t x t x \rangle \).

\( 7^{*2} = \langle t_1 \rangle \ast \langle t_2 \rangle . \)
\( x \sim (1, 9, 7, 11, 3, 5)(2, 6, 4, 12, 8, 10) \)
\( y \sim (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \)
\( t \sim t_1 \)

Overall, with the monomial representation. The final group presentation is,

\[
G<x, y, t> := \text{Group}<x, y, t | y^2, (x^{-1} y)^2, x^6, t^7, t(x^{-1}) = t^5,
\]
\[
y t y t^2 y t y t^2 y t^2 y t^{-1} y t^2 y t^2 y t^2 y t^2 = 1,
\]
\[
(x y t)^5, (y t)^{10}, (x t)^8, (y t)^{11} >;
\]

With the maximal subgroup such that,

\[
> M := \text{sub}<G|x, y, x^{-1} t^{-1} y t^{-1} y t y t y t x y t^x,x y t^x>;
\]
Recalling from earlier, we have the labeling of \( t'_i \)s to be,

\[
\begin{array}{|c|c|}
\hline
 t_1^n & t_2^n \\
\hline
 1.t_1 & 2.t_2 \\
 3.t_1^2 & 4.t_2^2 \\
 5.t_1^3 & 6.t_2^3 \\
 7.t_1^4 & 8.t_2^4 \\
 9.t_1^5 & 10.t_2^5 \\
11.t_1^6 & 12.t_2^6 \\
\hline
\end{array}
\]

Expanding the relations we have,

1. \((xyt)^5 \Rightarrow xyt_1t_2^5t_1 = t_1^6t_2^2 \)
2. \((yt)^{10} \Rightarrow t_1t_2t_1t_2 = t_2^6t_1^6t_2^6t_1^6t_2^6 \)
(Center) \( ytyt^2yt^2y^2t^2y^2t^2y^2 \Rightarrow ytt_1t_2^2t_1t_2^2 = t_1^5t_2t_1^5t_2^2t_1 \)
(Subgroup) \( x^{-1}t^6y^6ytytxyt \in M \Rightarrow Mt_2^6t_1^6t_2 = Mt_1^2t_1^6t_2^6t_1^6 \)

Beginning the double coset enumeration, we first need to calculate the total number of unique cosets of \( M \) of \( N \) in \( G \). This is the index of \( G \) in \( M \). The index will be the order of \( G \) divided by the order of \( M \).

\[
\frac{|G|}{|M|} = \frac{175560}{660} = 266.
\]

Now we know that we will have 266 unique single cosets.

Calculating the number of double cosets in your double coset enumeration.

\[
> \#\text{DoubleCosets}(G,H,\text{sub}<G|x,y>);
\]

29

This lets us know that there will be 29 unique double cosets. We will then use this information as we do the double coset enumeration of \( G \) over \( M \).

1st Double Coset \([*]\)

\( MeN = \{Me^n|n \in \mathbb{N}\} = \{N\} \). \( N \) is transitive on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \); moreover,
\[
\frac{|N|}{|N_e|} = \frac{12}{12} = 1. \text{ Thus, orbit of } N \text{ on } \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \text{ is } \{1, 2, 3, 4, 5, 6, 7\}. 
\]

Now, select a representative \( t_i \) from orbit. We will select 1 from the orbit in this case.

\[ 1 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \Rightarrow Mt_1 \in [1] \]

Now, we shall extend \( Mt_1 \) to a new double coset \([1]\).

**2nd Double Coset** \([1]\)

\( Mt_1 N = \{Mt_1^n | n \in \mathbb{N}\} \)

The coset stabilizer \( N^{(1)} \) are elements of the monomial representation \( 7^{*2} : S_3 \times 2 \) that fix 1. So, we then have

\[ N^{(1)} \cong N^1 = < e >, \text{ which indicates only the identity } e \text{ would be in the stabilizing group } N^{(1)}. \]

Which would imply that the number of single cosets in \( Mt_1 N \) are

\[ \frac{|N|}{|N_e|} = \frac{12}{1} = 12. \text{ Thus, The orbits of } N^{(1)} \text{ on } \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \text{ are } \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{ and } \{12\}. \]

We will now select a representative \( t_i \) from each orbit.

\[ 1 \in \{1\} \Rightarrow Mt_1 t_1 \Rightarrow Mt_1^2 \in [1] \]

\[ 2 \in \{2\} \Rightarrow Mt_1 t_2 \in [12], \text{ which goes to a new double coset } [12] \]

\[ 3 \in \{3\} \Rightarrow Mt_1 t_3^2 \Rightarrow Mt_1^3 \in [1] \]

\[ 4 \in \{4\} \Rightarrow Mt_1 t_4^2 \in [14], \text{ which goes to a new double coset } [14] \]

\[ 5 \in \{5\} \Rightarrow Mt_1 t_5^3 \Rightarrow Mt_1^4 \in [1] \]

\[ 6 \in \{6\} \Rightarrow Mt_1 t_6^3 \in [16], \text{ which goes to a new double coset } [16] \]

\[ 7 \in \{7\} \Rightarrow Mt_1 t_7^4 \Rightarrow Mt_1^5 \in [1] \]

\[ 8 \in \{8\} \Rightarrow Mt_1 t_8^4 \in [18], \text{ which goes to a new double coset } [18] \]

\[ 9 \in \{9\} \Rightarrow Mt_1 t_9^5 \Rightarrow Mt_1^6 \in [1] \]

\[ 10 \in \{10\} \Rightarrow Mt_1 t_{10}^5 \in [110], \text{ which goes to a new double coset } [110] \]

\[ 11 \in \{11\} \Rightarrow Mt_1 t_{11}^6 \Rightarrow Me \in [\ast] \]

\[ 12 \in \{12\} \Rightarrow Mt_1 t_{12}^6 \in [112], \text{ which goes to a new double coset } [112] \]

**Words of length two.**

**3rd Double Coset** \([12]\)

\( Mt_1 t_2 N = \{Mt_1 t_2^n | n \in \mathbb{N}\} \)

The coset stabilizer \( N^{(12)} \) are elements of the monomial representation \( 7^{*2} : S_3 \times 2 \) that fix both 1 and 2. So, we then have \( N^{(12)} \cong N^{12} = < e >, \text{ which indicates only the identity } e \text{ would be in the stabilizing group } N^{(12)}. \]

Which would imply that the number of single cosets in \( Mt_1 t_2 N \) are

\[ \frac{|N|}{|N_e|} = \frac{12}{1} = 12. \text{ Thus, The orbits of } N^{(12)} \text{ on } \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \text{ are } \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{ and } \{12\}. \]

We will now select a representative \( t_i \) from each orbit.

\[ 1 \in \{1\} \Rightarrow Mt_1 t_2 t_1 \in [121], \text{ which goes to a new double coset } [121] \]

\[ 2 \in \{2\} \Rightarrow Mt_1 t_2 t_2, \text{ which goes to } [18] \text{ by relation.} \]

\[ 3 \in \{3\} \Rightarrow Mt_1 t_2 t_1^2, \text{ which goes to } [121] \text{ by relation.} \]
4 \in \{4\} \Rightarrow M_{t_1}t_2^2t_2^3 \Rightarrow M_{t_1}t_2^3 \in [16]

5 \in \{5\} \Rightarrow M_{t_1}t_2^2t_2^1 \in [125], which goes to a new double coset [125]

6 \in \{6\} \Rightarrow M_{t_1}t_2^3, which goes to [112] by relation.

7 \in \{7\} \Rightarrow M_{t_1}t_2^4 \in [127], which goes to a new double coset [127]

8 \in \{8\} \Rightarrow M_{t_1}t_2^4 \Rightarrow M_{t_1}t_2^5 \in [110]

9 \in \{9\} \Rightarrow M_{t_1}t_2^5 \in [129], which goes to a new double coset [129]

10 \in \{10\} \Rightarrow M_{t_1}t_2^5, which goes to [14] by relation.

11 \in \{11\} \Rightarrow M_{t_1}t_2^6 \in [1211], which goes to a new double coset [1211]

12 \in \{12\} \Rightarrow M_{t_1}t_2^6 \Rightarrow M_{t_1} \in [1]

4th Double Coset [14]

\[ M_{t_1}t_2^2N = \{ M_{t_1}t_2^{2n} | n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(14)} \) are elements of the monomial representation \( 7^{*2} : S_3 \times 2 \) that fix both 1 and 4. So, we then have \( N^{(14)} \geq N^{14} = < e > \), which indicates only the identity \( e \) would be in the stabilizing group \( N^{(14)} \). Which would imply that the number of single cosets in \( M_{t_1}t_2^2N \) are \( \frac{N}{N^{(14)}} = \frac{12}{2} = 12 \). Thus, The orbits of \( N^{(14)} \) on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \) are \( \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \) and \( \{12\} \).

We will now select a representative \( t_i \) from each orbit.

1 \in \{1\} \Rightarrow M_{t_1}t_2^2t_1 \in [141], which goes to a new double coset [141]

2 \in \{2\} \Rightarrow M_{t_1}t_2^2t_2 \Rightarrow M_{t_1}t_2^3 \in [16]

3 \in \{3\} \Rightarrow M_{t_1}t_2^2t_1^2, which goes to [141] by relation.

4 \in \{4\} \Rightarrow M_{t_1}t_2^2t_2^2, which goes to [12] by relation.

5 \in \{5\} \Rightarrow M_{t_1}t_2^3t_1^3 \in [145], which goes to a new double coset [145]

6 \in \{6\} \Rightarrow M_{t_1}t_2^3t_2^3, which goes to [18] by relation.

7 \in \{7\} \Rightarrow M_{t_1}t_2^3t_1^4, which goes to [14] by relation.

8 \in \{8\} \Rightarrow M_{t_1}t_2^3t_2^4 \Rightarrow M_{t_1}t_2^6 \in [112]

9 \in \{9\} \Rightarrow M_{t_1}t_2^3t_2^5, which goes to [14] by relation.

10 \in \{10\} \Rightarrow M_{t_1}t_2^3t_2^5 \Rightarrow M_{t_1} \in [1]

11 \in \{11\} \Rightarrow M_{t_1}t_2^3t_1^6, which goes to [14] by relation.

12 \in \{12\} \Rightarrow M_{t_1}t_2^3t_2^6, which goes to [110] by relation.

5th Double Coset [16]

\[ M_{t_1}t_2^3N = \{ M_{t_1}t_2^{3n} | n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(16)} \) are elements of the monomial representation \( 7^{*2} : S_3 \times 2 \) that fix both 1 and 6. So, we then have \( N^{(16)} \geq N^{16} = < e > \), which indicates only the identity \( e \) would be in the stabilizing group \( N^{(16)} \). Which would imply that the number of single cosets in \( M_{t_1}t_2^3N \) are \( \frac{N}{N^{(16)}} = \frac{12}{2} = 12 \). Thus, The orbits of \( N^{(16)} \) on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \) are \( \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \) and \( \{12\} \).

We will now select a representative \( t_i \) from each orbit.
1 \in \{1\} \Rightarrow Mt_1t_2^3t_1 \in [161], \text{ which goes to a new double coset } [161]
2 \in \{2\} \Rightarrow Mt_1t_2^3t_2, \text{ which goes to } [112] \text{ by relation.}
3 \in \{3\} \Rightarrow Mt_1t_2^3t_1^2 \in [163], \text{ which goes to a new double coset } [163]
4 \in \{4\} \Rightarrow Mt_1t_2^3t_2^2, \text{ which goes to } [18] \text{ by relation.}
5 \in \{5\} \Rightarrow Mt_1t_2^3t_1^3 \in [165], \text{ which goes to a new double coset } [165]
6 \in \{6\} \Rightarrow Mt_1t_2^3t_2^3, \text{ which goes to } [110] \text{ by relation.}
7 \in \{7\} \Rightarrow Mt_1t_2^3t_1^4 \in [167], \text{ which goes to a new double coset } [167]
8 \in \{8\} \Rightarrow Mt_1t_2^3t_2^4 \Rightarrow Mt_1 \in [1]
9 \in \{9\} \Rightarrow Mt_1t_2^3t_1^5 \in [169], \text{ which goes to a new double coset } [169]
10 \in \{10\} \Rightarrow Mt_1t_2^3t_2^5 \Rightarrow Mt_1t_2 \in [12]
11 \in \{11\} \Rightarrow Mt_1t_2^3t_1^6 \in [1611], \text{ which goes to a new double coset } [1611]
12 \in \{12\} \Rightarrow Mt_1t_2^3t_2^6 \Rightarrow Mt_1t_2^2 \in [14]

6th Double Coset \ [18]

Mt_1t_2^4N = \{ Mt_1t_2^4n|n \in \mathbb{N} \}

The coset stabilizer \(N^{(18)}\) are elements of the monomial representation \(7^{\times 2} : S_3 \times 2\) that fix both 1 and 8. So, we then have \(N^{(18)} \geq N^{18} = \langle e \rangle\), which indicates only
the identity e would be in the stabilizing group \(N^{(18)}\). Which would imply that the
number of single cosets in \(Mt_1t_2^4N\) are \(\frac{N}{N^{(18)}} = \frac{12}{1} = 12\). Thus, The orbits of \(N^{(18)}\) on
\(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{ and } \{12\}\).

We will now select a representative \(t_i\) from each orbit.
1 \in \{1\} \Rightarrow Mt_1t_2^4t_1, \text{ which goes to } [169] \text{ by relation.}
2 \in \{2\} \Rightarrow Mt_1t_2^4t_2 \Rightarrow Mt_1t_2^5 \in [110]
3 \in \{3\} \Rightarrow Mt_1t_2^4t_1^2, \text{ which goes to } [125] \text{ by relation.}
4 \in \{4\} \Rightarrow Mt_1t_2^4t_2^2 \Rightarrow Mt_1t_2^6 \in [112]
5 \in \{5\} \Rightarrow Mt_1t_2^4t_1^3, \text{ which goes to } [129] \text{ by relation.}
6 \in \{6\} \Rightarrow Mt_1t_2^4t_2^3 \Rightarrow Mt_1 \in [1]
7 \in \{7\} \Rightarrow Mt_1t_2^4t_1^4 \in [187], \text{ which goes to a new double coset } [187]
8 \in \{8\} \Rightarrow Mt_1t_2^4t_2^4, \text{ which goes to } [14] \text{ by relation.}
9 \in \{9\} \Rightarrow Mt_1t_2^4t_1^5 \in [189], \text{ which goes to a new double coset } [189]
10 \in \{10\} \Rightarrow Mt_1t_2^4t_2^5, \text{ which goes to } [16] \text{ by relation.}
11 \in \{11\} \Rightarrow Mt_1t_2^4t_1^6, \text{ which goes to } [163] \text{ by relation.}
12 \in \{12\} \Rightarrow Mt_1t_2^4t_2^6, \text{ which goes to } [12] \text{ by relation.}

7th Double Coset \ [110]

Mt_1t_2^5N = \{ Mt_1t_2^5n|n \in \mathbb{N} \}

The coset stabilizer \(N^{(110)}\) are elements of the monomial representation \(7^{\times 2} : S_3 \times 2\) that fix both 1 and 10. So, we then have \(N^{(110)} \geq N^{110} = \langle e \rangle\), which indicates only
the identity e would be in the stabilizing group \(N^{(110)}\). Which would imply that the
number of single cosets in \(Mt_1t_2^5N\) are \(\frac{N}{N^{(110)}} = \frac{12}{1} = 12\). Thus, The orbits of \(N^{(110)}\) on
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, and \{12\}.

We will now select a representative \(t_i\) from each orbit.

1 \(\in\) \{1\} \(\Rightarrow\) \(Mt_1t_2^2t_1\), which goes to \([110]\) by relation.

2 \(\in\) \{2\} \(\Rightarrow\) \(Mt_1t_2^5t_2\), which goes to \([14]\) by relation.

3 \(\in\) \{3\} \(\Rightarrow\) \(Mt_1t_2^3t_1^2 \in [1103]\), which goes to a new double coset \([1103]\)

4 \(\in\) \{4\} \(\Rightarrow\) \(Mt_1t_2^5t_2^2 \Rightarrow Mt_1 \in [1]\)

5 \(\in\) \{5\} \(\Rightarrow\) \(Mt_1t_2^3t_1^3\), which goes to \([1211]\) by relation.

6 \(\in\) \{6\} \(\Rightarrow\) \(Mt_1t_2^5t_2^3 \Rightarrow Mt_1t_2 \in [12]\)

7 \(\in\) \{7\} \(\Rightarrow\) \(Mt_1t_2^5t_1^4\), which goes to \([1611]\) by relation.

8 \(\in\) \{8\} \(\Rightarrow\) \(Mt_1t_2^5t_2^4\), which goes to \([12]\) by relation.

9 \(\in\) \{9\} \(\Rightarrow\) \(Mt_1t_2^5t_1^5\), which goes to \([1211]\) by relation.

10 \(\in\) \{10\} \(\Rightarrow\) \(Mt_1t_2^5t_2^5\), which goes to \([16]\) by relation.

11 \(\in\) \{11\} \(\Rightarrow\) \(Mt_1t_2^5t_1^6\), which goes to \([1611]\) by relation.

12 \(\in\) \{12\} \(\Rightarrow\) \(Mt_1t_2^5t_2^6 \Rightarrow Mt_1t_2^2 \in [18]\)

8th Double Coset \([112]\)

\(Mt_1t_2^5N = \{Mt_1t_2^5n | n \in \mathbb{N}\}\)

The coset stabilizer \(N^{(112)}\) are elements of the monomial representation \(7^* : S_3 \times 2\) that fix both 1 and 12. So, we then have \(N^{(112)} \geq N^{112} = \langle e \rangle\), which indicates only the identity \(e\) would be in the stabilizing group \(N^{(112)}\).

Since we have that \((Mt_2^5t_1^3)^{xy} = Mt_1t_2^6\) by relation, this implies that \(Mt_1t_2^6 = (Mt_2^5t_1^3)^{xy} = Mt_1t_2^6\), so \(xy\) would be in the stabilizing group \(N^{(112)} = \langle xy \rangle\).

Which would imply that the number of single cosets in \(Mt_1t_2^5N\) are \(\frac{N}{N^{(112)}} = \frac{12}{2} = 6\).

Thus, the orbits of \(N^{(112)}\) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1, 10\}, \{2, 5\}, \{3, 6\}, \{4, 11\}, \{7, 12\}, and \{8, 9\}.

We will now select a representative \(t_i\) from each orbit.

10 \(\in\) \{1, 10\} \(\Rightarrow\) \(Mt_2^5t_2^5 \Rightarrow Mt_1t_2^4 \in [18]\)

2 \(\in\) \{2, 5\} \(\Rightarrow\) \(Mt_2^5t_2 \Rightarrow Mt_1 \in [1]\)

6 \(\in\) \{3, 6\} \(\Rightarrow\) \(Mt_2^5t_2^3 \Rightarrow Mt_1t_2^5 \in [14]\)

4 \(\in\) \{4, 11\} \(\Rightarrow\) \(Mt_1t_2^5t_2^2\), which goes to \([110]\) by relation.

12 \(\in\) \{7, 12\} \(\Rightarrow\) \(Mt_1t_2^5t_2^6\), which goes to \([16]\) by relation.

8 \(\in\) \{8, 9\} \(\Rightarrow\) \(Mt_1t_2^5t_2^4\), which goes to \([12]\) by relation.

9th Double Coset \([121]\)

\(Mt_1t_2t_1N = \{Mt_1t_2t_1^n | n \in \mathbb{N}\}\)

The coset stabilizer \(N^{(121)}\) fixes both 1, 4 and 7. So, we then have \(N^{(121)} \geq N^{121} = \langle e \rangle\), which indicates the stabilizing group \(N^{(121)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2t_1N\) are \(\frac{N}{N^{(121)}} = \frac{12}{1} = 12\).
Thus, the orbits of $N^{(121)}$ on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, and \{12\}.

We will now select a representative $t_i$ from each orbit.

1 ∈ \{1\} ⇒ $Mt_1 t_2 t_1 t_1$, which goes to [127] by relation.
2 ∈ \{2\} ⇒ $Mt_1 t_2 t_1 t_2$, which goes to [12] by relation.
3 ∈ \{3\} ⇒ $Mt_1 t_2 t_1 t_1^2$, which goes to [125] by relation.
4 ∈ \{4\} ⇒ $Mt_1 t_2 t_1 t_2^2$, which goes to [127] by relation.
5 ∈ \{5\} ⇒ $Mt_1 t_2 t_1 t_1^3$, which goes to [1211] by relation.
6 ∈ \{6\} ⇒ $Mt_1 t_2 t_1 t_2^3$, which goes to [125] by relation.
7 ∈ \{7\} ⇒ $Mt_1 t_2 t_1 t_2^4$, which goes to [129] by relation.
8 ∈ \{8\} ⇒ $Mt_1 t_2 t_1 t_2^5$, which goes to [129] by relation.
9 ∈ \{9\} ⇒ $Mt_1 t_2 t_1 t_1^5$, which goes to [121] by relation.
10 ∈ \{10\} ⇒ $Mt_1 t_2 t_1 t_2^5$, which goes to [1211] by relation.
11 ∈ \{11\} ⇒ $Mt_1 t_2 t_1 t_1^6$, which goes to [12] by relation.
12 ∈ \{12\} ⇒ $Mt_1 t_2 t_1 t_2^6$, which goes to [121] by relation.

**10th Double Coset [125]**

$Mt_1 t_2 t_1^3 N = \{ Mt_1 t_2 t_1^{3n} | n \in \mathbb{N} \}$

The coset stabilizer $N^{(125)}$ fixes both 1, 2 and 5. So, we then have $N^{(125)} \geq N^{125} = \langle e \rangle$, which indicates the stabilizing group $N^{(125)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $Mt_1 t_2 t_1^3 N$ are $\frac{N}{N^{(125)}} = \frac{12}{5} = 12$.

Thus, the orbits of $N^{(125)}$ on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, and \{12\}.

We will now select a representative $t_i$ from each orbit.

1 ∈ \{1\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_1$, which goes to [127] by relation.
2 ∈ \{2\} ⇒ $Mt_1 t_2 t_1 t_2 t_1^3$, which goes to [12] by relation.
3 ∈ \{3\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2$, which goes to [125] by relation.
4 ∈ \{4\} ⇒ $Mt_1 t_2 t_1 t_2^4 t_1^3 t_2$, which goes to [127] by relation.
5 ∈ \{5\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_1^3 t_2$, which goes to [1211] by relation.
6 ∈ \{6\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^3 t_1^3 t_2$, which goes to [125] by relation.
7 ∈ \{7\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^4 t_1^4$, which goes to [129] by relation.
8 ∈ \{8\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^5 t_1^4$, which goes to [129] by relation.
9 ∈ \{9\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^5 t_1^5$, which goes to [121] by relation.
10 ∈ \{10\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^5 t_1^5$, which goes to [1211] by relation.
11 ∈ \{11\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^5 t_1^6$, which goes to [12] by relation.
12 ∈ \{12\} ⇒ $Mt_1 t_2 t_1 t_1^3 t_2^6$, which goes to [121] by relation.

**11th Double Coset [127]**

$Mt_1 t_2 t_1^4 N = \{ Mt_1 t_2 t_1^{4n} | n \in \mathbb{N} \}$

The coset stabilizer $N^{(127)}$ fixes both 1, 2 and 7. So, we then have $N^{(127)} \geq N^{127} = \langle e \rangle$, which indicates the stabilizing group $N^{(127)}$ only contains the identity $e$. 
Which would imply that the number of single cosets in $Mt_1t_2t_1^4N$ are $\frac{N}{N^{(127)}} = \frac{12}{1} = 12$. Thus, The orbits of $N^{(127)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{and } \{12\}$.

We will now select a representative $t_i$ from each orbit.

1. $1 \in \{1\} \Rightarrow Mt_1t_2t_1^4t_1$, which goes to $[129]$ by relation.
2. $2 \in \{2\} \Rightarrow Mt_1t_2t_1^4t_2$, which goes to $[127]$ by relation.
3. $3 \in \{3\} \Rightarrow Mt_1t_2t_1^4t_1^2$, which goes to $[1211]$ by relation.
4. $4 \in \{4\} \Rightarrow Mt_1t_2t_1^4t_2^2$, which goes to $[1274]$ by relation.
5. $5 \in \{5\} \Rightarrow Mt_1t_2t_1^4t_1^3$, which goes to $[12]$ by relation.
6. $6 \in \{6\} \Rightarrow Mt_1t_2t_1^4t_2^3$, which goes to $[127]$ by relation.
7. $7 \in \{7\} \Rightarrow Mt_1t_2t_1^4t_1^4$, which goes to $[121]$ by relation.
8. $8 \in \{8\} \Rightarrow Mt_1t_2t_1^4t_2^4$, which goes to $[1278]$ by relation.
9. $9 \in \{9\} \Rightarrow Mt_1t_2t_1^4t_1^5$, which goes to $[125]$ by relation.
10. $10 \in \{10\} \Rightarrow Mt_1t_2t_1^4t_2^5$, which goes to $[1274]$ by relation.
11. $11 \in \{11\} \Rightarrow Mt_1t_2t_1^4t_1^6$, which goes to $[121]$ by relation.
12. $12 \in \{12\} \Rightarrow Mt_1t_2t_1^4t_2^6$, which goes to $[1274]$ by relation.

12th Double Coset $[129]$

$Mt_1t_2t_1^5N = \{ Mt_1t_2t_1^{5n} | n \in \mathbb{N} \}$

The coset stabilizer $N^{(129)}$ fixes both 1, 2 and 9. So, we then have $N^{(129)} \geq N^{129} = \langle e \rangle$, which indicates the stabilizing group $N^{(129)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $Mt_1t_2t_1^5N$ are $\frac{N}{N^{(129)}} = \frac{12}{1} = 12$.

Thus, The orbits of $N^{(129)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{and } \{12\}$.

We will now select a representative $t_i$ from each orbit.

1. $1 \in \{1\} \Rightarrow Mt_1t_2t_1^{5t_1}$, which goes to $[121]$ by relation.
2. $2 \in \{2\} \Rightarrow Mt_1t_2t_1^{5t_2}$, which goes to $[129]$ by relation.
3. $3 \in \{3\} \Rightarrow Mt_1t_2t_1^{5t_1^2}$, which goes to $[12]$ by relation.
4. $4 \in \{4\} \Rightarrow Mt_1t_2t_1^{5t_2^2}$, which goes to $[163]$ by relation.
5. $5 \in \{5\} \Rightarrow Mt_1t_2t_1^{5t_1^3}$, which goes to $[121]$ by relation.
6. $6 \in \{6\} \Rightarrow Mt_1t_2t_1^{5t_2^3}$, which goes to $[169]$ by relation.
7. $7 \in \{7\} \Rightarrow Mt_1t_2t_1^{5t_1^4}$, which goes to $[125]$ by relation.
8. $8 \in \{8\} \Rightarrow Mt_1t_2t_1^{5t_2^4}$, which goes to $[187]$ by relation.
9. $9 \in \{9\} \Rightarrow Mt_1t_2t_1^{5t_1^5}$, which goes to $[1211]$ by relation.
10. $10 \in \{10\} \Rightarrow Mt_1t_2t_1^{5t_2^5}$, which goes to $[125]$ by relation.
11. $11 \in \{11\} \Rightarrow Mt_1t_2t_1^{5t_1^6}$, which goes to $[127]$ by relation.
12. $12 \in \{12\} \Rightarrow Mt_1t_2t_1^{5t_2^6}$, which goes to $[189]$ by relation.

13th Double Coset $[1211]$

$Mt_1t_2t_1^{6N} = \{ Mt_1t_2t_1^{6n} | n \in \mathbb{N} \}$

The coset stabilizer $N^{(1211)}$ fixes both 1, 2 and 11. So, we then have $N^{(1211)} \geq N^{1211}$.
= <e>, which indicates the stabilizing group \(N^{(1211)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2t_1^6N\) are \(N_{N^{(1211)}} = \frac{12}{2} = 12\).

Thus, The orbits of \(N^{(1211)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \) and \(\{12\}\).

We will now select a representative \(t_i\) from each orbit.

1 \(\in\{1\} \Rightarrow Mt_1t_2t_1^6t_1\), which goes to \([12]\) by relation.
2 \(\in\{2\} \Rightarrow Mt_1t_2t_1^6t_2\), which goes to \([1211]\) by relation.
3 \(\in\{3\} \Rightarrow Mt_1t_2t_1^6t_1t_1^2\), which goes to \([129]\) by relation.
4 \(\in\{4\} \Rightarrow Mt_1t_2t_1^6t_2^2\), which goes to \([103]\) by relation.
5 \(\in\{5\} \Rightarrow Mt_1t_2t_1^6t_1^3\), which goes to \([121]\) by relation.
6 \(\in\{6\} \Rightarrow Mt_1t_2t_1^6t_2^3\), which goes to \([1611]\) by relation.
7 \(\in\{7\} \Rightarrow Mt_1t_2t_1^6t_1t_1^4\), which goes to \([121]\) by relation.
8 \(\in\{8\} \Rightarrow Mt_1t_2t_1^6t_2^4\), which goes to \([110]\) by relation.
9 \(\in\{9\} \Rightarrow Mt_1t_2t_1^6t_1^5\), which goes to \([127]\) by relation.
10 \(\in\{10\} \Rightarrow Mt_1t_2t_1^6t_2^5\), which goes to \([1611]\) by relation.
11 \(\in\{11\} \Rightarrow Mt_1t_2t_1^6t_1t_1^6\), which goes to \([125]\) by relation.
12 \(\in\{12\} \Rightarrow Mt_1t_2t_1^6t_2^6\), which goes to \([110]\) by relation.

14th Double Coset \([141]\)

\(Mt_1t_2t_1^2N = \{Mt_1t_2t_1^2t_1^n|n \in N\}\)

The coset stabilizer \(N^{(141)}\) fixes both 1, 4 and 1. So, we then have \(N^{(141)} \geq N^{141} = <e>\), which indicates the stabilizing group \(N^{(141)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2^2t_1N\) are \(N_{N^{(141)}} = \frac{12}{2} = 12\).

Thus, The orbits of \(N^{(141)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \) and \(\{12\}\).

We will now select a representative \(t_i\) from each orbit.

1 \(\in\{1\} \Rightarrow Mt_1t_2^2t_1t_1\), which goes to \([14]\) by relation.
2 \(\in\{2\} \Rightarrow Mt_1t_2^2t_1t_2\), which goes to \([187]\) by relation.
3 \(\in\{3\} \Rightarrow Mt_1t_2^2t_1t_1^2\), which goes to \([145]\) by relation.
4 \(\in\{4\} \Rightarrow Mt_1t_2^2t_1t_2^2\), which goes to \([1414]\) by relation.
5 \(\in\{5\} \Rightarrow Mt_1t_2^2t_1t_1^3\), which goes to \([14]\) by relation.
6 \(\in\{6\} \Rightarrow Mt_1t_2^2t_1t_2^3\), which goes to \([187]\) by relation.
7 \(\in\{7\} \Rightarrow Mt_1t_2^2t_1t_1^4\), which goes to \([141]\) by relation.
8 \(\in\{8\} \Rightarrow Mt_1t_2^2t_1t_2^4\), which goes to \([1414]\) by relation.
9 \(\in\{9\} \Rightarrow Mt_1t_2^2t_1t_1^5\), which goes to \([141]\) by relation.
10 \(\in\{10\} \Rightarrow Mt_1t_2^2t_1t_2^5\), which goes to \([1103]\) by relation.
11 \(\in\{11\} \Rightarrow Mt_1t_2^2t_1t_1^6\), which goes to \([14]\) by relation.
12 \(\in\{12\} \Rightarrow Mt_1t_2^2t_1t_2^6\), which goes to \([141]\) by relation.

15th Double Coset \([145]\)

\(Mt_1t_2^2t_1^3N = \{Mt_1t_2^2t_1^3t_1^n|n \in N\}\)
The coset stabilizer \(N^{(145)}\) fixes both 1, 4 and 5. So, we then have \(N^{(145)} \geq N^{145} = \langle e \rangle\), which indicates the stabilizing group \(N^{(145)}\) only contains the identity \(e\).

Since we have that \((Mt_1t_2t_1)^x = Mt_1t_2t_1^3\), \((Mt_2t_1^5t_2)^x = Mt_1t_2t_1^3\), \((Mt_2^3t_1^3t_2^2)^xy = Mt_1t_2t_2^3\), and \((Mt_2^6t_1^3t_2)^x = Mt_1t_2t_1^3\) all by relation, then we

\[
\begin{align*}
\text{which would imply that the number of single cosets in } \langle e \rangle, \text{ which indicates the stabilizing group } Mt_1t_2t_1^3, \text{ all by relation, then we}
\end{align*}
\]

Thus, The orbits of \(N^{(145)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1, 7, 10, 6, 12, 3\}\), and \(\{2, 4, 5, 9, 11, 8\}\).

We will now select a representative \(t_i\) from each orbit.

7 \(\in\) \{1, 7, 10, 6, 12, 3\} \(\Rightarrow\) \(Mt_1t_2^2t_1^3t_1^4\), which goes to \([14]\) by relation.

5 \(\in\) \{2, 4, 5, 9, 11, 8\} \(\Rightarrow\) \(Mt_1t_1^2t_1^3t_1^3\), which goes to \([141]\) by relation.

16th Double Coset \([161]\)

\(Mt_1t_2^3t_1^N = \{Mt_1t_2^3t_1^n | n \in \mathbb{N}\}\)

The coset stabilizer \(N^{(161)}\) fixes both 1, 6 and 1. So, we then have \(N^{(161)} \geq N^{161} = \langle e \rangle\), which indicates the stabilizing group \(N^{(161)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2t_3t_1^1\) are \(\frac{N}{N^{(161)}} = \frac{12}{6} = 2\).

Thus, The orbits of \(N^{(161)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\), and \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\).

We will now select a representative \(t_i\) from each orbit.

1 \(\in\) \{1\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_1\), which goes to \([167]\) by relation.

2 \(\in\) \{2\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_2\), which goes to \([168]\) by relation.

3 \(\in\) \{3\} \(\Rightarrow\) \(Mt_1t_2^3t_2^1t_1^2\), which goes to \([165]\) by relation.

4 \(\in\) \{4\} \(\Rightarrow\) \(Mt_1t_2^3t_2^1t_2\), which goes to \([164]\) by relation.

5 \(\in\) \{5\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_3\), which goes to \([1611]\) by relation.

6 \(\in\) \{6\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_3\), which goes to \([165]\) by relation.

7 \(\in\) \{7\} \(\Rightarrow\) \(Mt_1t_2^3t_2^1t_1^1\), which goes to \([169]\) by relation.

8 \(\in\) \{8\} \(\Rightarrow\) \(Mt_1t_2^3t_1^2t_1\), which goes to \([168]\) by relation.

9 \(\in\) \{9\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_5\), which goes to \([163]\) by relation.

10 \(\in\) \{10\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_5\), which goes to \([165]\) by relation.

11 \(\in\) \{11\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_6\), which goes to \([16]\) by relation.

12 \(\in\) \{12\} \(\Rightarrow\) \(Mt_1t_2^3t_1^1t_6\), which goes to \([1618]\) by relation.

17th Double Coset \([163]\)

\(Mt_1t_2^3t_1^2N = \{Mt_1t_2^3t_1^{2n} | n \in \mathbb{N}\}\)

The coset stabilizer \(N^{(163)}\) fixes both 1, 6 and 3. So, we then have \(N^{(163)} \geq N^{163} = \langle e \rangle\), which indicates the stabilizing group \(N^{(163)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(Mt_1t_2^3t_1^2\) are \(\frac{N}{N^{(163)}} = \frac{12}{1} = 12\).

Thus, The orbits of \(N^{(163)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\).

We will now select a representative \(t_i\) from each orbit.
19th Double Coset

$$Mt_1t_2^3t_1^2t_1, \text{ which goes to } [165] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2, \text{ which goes to } [169] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_1^2, \text{ which goes to } [161] \text{ by relation.}$$

$$Mt_1t_2^3t_2^2t_2^2, \text{ which goes to } [18] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_1^3, \text{ which goes to } [167] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2^3, \text{ which goes to } [187] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_1^4, \text{ which goes to } [1611] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2^4, \text{ which goes to } [189] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2^5, \text{ which goes to } [16] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2^6, \text{ which goes to } [129] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_1^6, \text{ which goes to } [169] \text{ by relation.}$$

$$Mt_1t_2^3t_1^2t_2^6, \text{ which goes to } [125] \text{ by relation.}$$

18th Double Coset [165]

$$Mt_1t_2^3t_1^3N = \{Mt_1t_2^3t_1^{3n} | n \in \mathbb{N}\}$$

The coset stabilizer $N^{(165)}$ fixes both 1, 6 and 5. So, we then have $N^{(165)} \geq N^{165} = \langle e \rangle$, which indicates the stabilizing group $N^{(165)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $Mt_1t_2^3t_1^3N$ are $N = \frac{12}{7} = 12$.

Thus, the orbits of $N^{(165)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \text{ and } \{12\}$.

We will now select a representative $t_i$ from each orbit.

1. $1 \in \{1\} \Rightarrow Mt_1t_2^3t_1^3t_1$, which goes to [1611] by relation.
2. $2 \in \{2\} \Rightarrow Mt_1t_2^3t_1^3t_2$, which goes to [1618] by relation.
3. $3 \in \{3\} \Rightarrow Mt_1t_2^3t_1^3t_1^2$, which goes to [167] by relation.
4. $4 \in \{4\} \Rightarrow Mt_1t_2^3t_1^3t_2^2$, which goes to [1618] by relation.
5. $5 \in \{5\} \Rightarrow Mt_1t_2^3t_1^3t_1^3$, which goes to [169] by relation.
6. $6 \in \{6\} \Rightarrow Mt_1t_2^3t_1^3t_2^3$, which goes to [165] by relation.
7. $7 \in \{7\} \Rightarrow Mt_1t_2^3t_1^3t_1^4$, which goes to [16] by relation.
8. $8 \in \{8\} \Rightarrow Mt_1t_2^3t_1^3t_2^4$, which goes to [161] by relation.
9. $9 \in \{9\} \Rightarrow Mt_1t_2^3t_1^3t_1^5$, which goes to [161] by relation.
10. $10 \in \{10\} \Rightarrow Mt_1t_2^3t_1^3t_2^5$, which goes to [161] by relation.
11. $11 \in \{11\} \Rightarrow Mt_1t_2^3t_1^3t_1^6$, which goes to [163] by relation.
12. $12 \in \{12\} \Rightarrow Mt_1t_2^3t_1^3t_2^6$, which goes to [1214] by relation.

19th Double Coset [167]

$$Mt_1t_2^3t_1^4N = \{Mt_1t_2^3t_1^{4n} | n \in \mathbb{N}\}$$

The coset stabilizer $N^{(167)}$ fixes both 1, 6 and 7. So, we then have $N^{(167)} \geq N^{167} = \langle y \rangle$, which indicates the stabilizing group $N^{(167)}$ only contains the identity $e$.

Since we have that $(Mt_2^6t_1^3t_2^3)^y = Mt_1t_2^3t_1^4$, by relation. We then have that $y$ would be in the stabilizing group $N^{(167)} = \langle y \rangle$.

Which would imply that the number of single cosets in $Mt_1t_2^3t_1^4N$ are $N = \frac{12}{2} = 6$. 

Thus, The orbits of $N^{(167)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1, 2\}, \{3, 4\}, \{5, 6\}$, $\{7, 8\}, \{9, 10\}$, and $\{11, 12\}$.

We will now select a representative $t_i$ from each orbit.

1 $\in \{1, 2\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1$, which goes to [169] by relation.
3 $\in \{3, 4\} \Rightarrow M_{t_2} t_2^3 t_1^5 t_1^2$, which goes to [161] by relation.
5 $\in \{5, 6\} \Rightarrow M_{t_2} t_2^3 t_1^5 t_1^3$, which goes to [16] by relation.
7 $\in \{7, 8\} \Rightarrow M_{t_2} t_2^3 t_1^5 t_2^4$, which goes to [163] by relation.
9 $\in \{9, 10\} \Rightarrow M_{t_2} t_2^3 t_1^5 t_1^5$, which goes to [165] by relation.
11 $\in \{11, 12\} \Rightarrow M_{t_2} t_2^3 t_1^5 t_2^6$, which goes to [161] by relation.

20th Double Coset [169]

$M_{t_1} t_2^3 t_1^5 N = \{M_{t_1} t_2^3 t_1^5 t_i | n \in \mathbb{N}\}$

The coset stabilizer $N^{(169)}$ fixes both 1, 6, and 9. So, we then have $N^{(169)} \geq N^{169} = < e >$, which indicates the stabilizing group $N^{(169)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $M_{t_1} t_2^3 t_1^5 N$ are $\frac{N}{N^{(169)}} = \frac{12}{2} = 12$.

Thus, The orbits of $N^{(169)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}$, and $\{12\}$.

We will now select a representative $t_i$ from each orbit.

1 $\in \{1\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1$, which goes to [169] by relation.
2 $\in \{2\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2$, which goes to [187] by relation.
3 $\in \{3\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1^2$, which goes to [16] by relation.
4 $\in \{4\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^2$, which goes to [129] by relation.
5 $\in \{5\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1^3$, which goes to [161] by relation.
6 $\in \{6\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^3$, which goes to [163] by relation.
7 $\in \{7\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1^4$, which goes to [165] by relation.
8 $\in \{8\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^4$, which goes to [189] by relation.
9 $\in \{9\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^5$, which goes to [161] by relation.
10 $\in \{10\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^5$, which goes to [125] by relation.
11 $\in \{11\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_1^6$, which goes to [167] by relation.
12 $\in \{12\} \Rightarrow M_{t_1} t_2^3 t_1^5 t_2^6$, which goes to [18] by relation.

21st Double Coset [1611]

$M_{t_1} t_2^3 t_1^6 N = \{M_{t_1} t_2^3 t_1^6 t_i | n \in \mathbb{N}\}$

The coset stabilizer $N^{(1611)}$ fixes both 1, 6, and 11. So, we then have $N^{(1611)} \geq N^{1611} = < e >$, which indicates the stabilizing group $N^{(1611)}$ only contains the identity $e$.

Which would imply that the number of single cosets in $M_{t_1} t_2^3 t_1^6 N$ are $\frac{N}{N^{(1611)}} = \frac{12}{2} = 12$.

Thus, The orbits of $N^{(1611)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}$, and $\{12\}$.

We will now select a representative $t_i$ from each orbit.

1 $\in \{1\} \Rightarrow M_{t_1} t_2^3 t_1^6 t_1$, which goes to [16] by relation.
2 $\in \{2\} \Rightarrow M_{t_1} t_2^3 t_1^6 t_2$, which goes to [110] by relation.
3 ∈ \{3\} \Rightarrow M_{t_1t_2^2}t_1^6t_1^2, which goes to [169] by relation.
4 ∈ \{4\} \Rightarrow M_{t_1t_2^2}t_1^6t_2^2, which goes to [1211] by relation.
5 ∈ \{5\} \Rightarrow M_{t_1t_2^2}t_1^6t_1^3, which goes to [166] by relation.
6 ∈ \{6\} \Rightarrow M_{t_1t_2^2}t_1^6t_2^3, which goes to [1103] by relation.
7 ∈ \{7\} \Rightarrow M_{t_1t_2^2}t_1^6t_1^4, which goes to [161] by relation.
8 ∈ \{8\} \Rightarrow M_{t_1t_2^2}t_1^6t_2^4, which goes to [110] by relation.
9 ∈ \{9\} \Rightarrow M_{t_1t_2^2}t_1^6t_1^5, which goes to [167] by relation.
10 ∈ \{10\} \Rightarrow M_{t_1t_2^2}t_1^6t_2^5, which goes to [1611] by relation.
11 ∈ \{11\} \Rightarrow M_{t_1t_2^2}t_1^6t_1^6, which goes to [169] by relation.
12 ∈ \{12\} \Rightarrow M_{t_1t_2^2}t_1^6t_2^6, which goes to [1211] by relation.

22nd Double Coset [187]

\[ M_{t_1t_2^4}t_1^4N = \{ M_{t_1t_2^4}t_1^4n | n \in \mathbb{N} \} \]

The coset stabilizer \(N^{(187)}\) fixes both 1, 8 and 7. So, we then have \(N^{(187)} \geq N^{187} = e\), which indicates the stabilizing group \(N^{(187)}\) only contains the identity \(e\).

Which would imply that the number of single cosets in \(M_{t_1t_2^4}t_1^4N\) are \(\frac{N}{N^{(187)}} = \frac{12}{2} = 12\).

Thus, The orbits of \(N^{(187)}\) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, and \{12\}.

We will now select a representative \(t_i\) from each orbit.
1 ∈ \{1\} \Rightarrow M_{t_1t_2^4}t_1^4t_1, which goes to [189] by relation.
2 ∈ \{2\} \Rightarrow M_{t_1t_2^4}t_1^4t_2, which goes to [187] by relation.
3 ∈ \{3\} \Rightarrow M_{t_1t_2^4}t_1^4t_1^2, which goes to [163] by relation.
4 ∈ \{4\} \Rightarrow M_{t_1t_2^4}t_1^4t_2^2, which goes to [1103] by relation.
5 ∈ \{5\} \Rightarrow M_{t_1t_2^4}t_1^4t_1^3, which goes to [18] by relation.
6 ∈ \{6\} \Rightarrow M_{t_1t_2^4}t_1^4t_2^3, which goes to [141] by relation.
7 ∈ \{7\} \Rightarrow M_{t_1t_2^4}t_1^4t_1^4, which goes to [125] by relation.
8 ∈ \{8\} \Rightarrow M_{t_1t_2^4}t_1^4t_2^4, which goes to [1414] by relation.
9 ∈ \{9\} \Rightarrow M_{t_1t_2^4}t_1^4t_1^5, which goes to [129] by relation.
10 ∈ \{10\} \Rightarrow M_{t_1t_2^4}t_1^4t_2^5, which goes to [141] by relation.
11 ∈ \{11\} \Rightarrow M_{t_1t_2^4}t_1^4t_1^6, which goes to [169] by relation.
12 ∈ \{12\} \Rightarrow M_{t_1t_2^4}t_1^4t_2^6, which goes to [1414] by relation.

23rd Double Coset [189]

\[ M_{t_1t_2^4}t_1^5N = \{ M_{t_1t_2^4}t_1^5n | n \in \mathbb{N} \} \]

The coset stabilizer \(N^{(189)}\) fixes both 1, 8 and 9. So, we then have \(N^{(189)} \geq N^{189} = e\), which indicates the stabilizing group \(N^{(189)}\) only contains the identity \(e\).

Since we have that \((M_{t_1t_2^5}t_1^5t_2^4)^y = M_{t_1t_2^4}t_1^5\), by relation. We then have that \(y\) would be in the stabilizing group \(N^{(145)} = y\).

Which would imply that the number of single cosets in \(M_{t_1t_2^4}t_1^5N\) are \(\frac{N}{N^{(189)}} = \frac{12}{2} = 6\).

Thus, The orbits of \(N^{(189)}\) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, and \{11, 12\}. 

We will now select a representative \( t_i \) from each orbit.

1 \( \in \{ 1, 2 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1 \), which goes to \([125]\) by relation.

3 \( \in \{ 3, 4 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1^2 \), which goes to \([18]\) by relation.

5 \( \in \{ 5, 6 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1^3 \), which goes to \([169]\) by relation.

7 \( \in \{ 7, 8 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1^4 \), which goes to \([129]\) by relation.

9 \( \in \{ 9, 10 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1^5 \), which goes to \([163]\) by relation.

11 \( \in \{ 11, 12 \} \Rightarrow M t_1 t_2^4 t_1^5 t_1^6 \), which goes to \([187]\) by relation.

**24th Double Coset** \([1103]\)

\[ M t_1 t_2^5 t_1^2 N = \{ M t_1 t_2^5 t_1^2 n | n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(1103)} \) fixes 1, 10 and 3. So, we then have \( N^{(1103)} \geq N^{1103} = \langle e \rangle \), which indicates the stabilizing group \( N^{(1103)} \) only contains the identity \( e \).

Since we have that \( (M t_1 t_2^5 t_1^2) x^3 = M t_1 t_2^4 t_1^5 \), by relation. We then have that \( x^3 \) would be in the stabilizing group \( N^{(1103)} = \langle x^3 \rangle \).

Which would imply that the number of single cosets in \( M t_1 t_2^5 t_1^2 N \) are \( \frac{N}{N^{(1103)}} = \frac{12}{2} = 6 \).

Thus, The orbits of \( N^{(1103)} \) on \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \} \) are \( \{ 1, 11 \}, \{ 2, 12 \}, \{ 3, 9 \}, \{ 4, 10 \}, \{ 5, 7 \}, \) and \( \{ 6, 8 \} \).

We will now select a representative \( t_i \) from each orbit.

1 \( \in \{ 1, 11 \} \Rightarrow M t_1 t_2^5 t_1^2 t_1 \), which goes to \([121]\) by relation.

2 \( \in \{ 2, 12 \} \Rightarrow M t_1 t_2^5 t_1^2 t_2 \), which goes to \([141]\) by relation.

9 \( \in \{ 3, 9 \} \Rightarrow M t_1 t_2^5 t_1^2 t_1^5 \), which goes to \([10]\) by relation.

4 \( \in \{ 4, 10 \} \Rightarrow M t_1 t_2^5 t_1^2 t_2^2 \), which goes to \([141]\) by relation.

5 \( \in \{ 5, 7 \} \Rightarrow M t_1 t_2^5 t_1^2 t_1^3 \), which goes to \([161]\) by relation.

6 \( \in \{ 6, 8 \} \Rightarrow M t_1 t_2^5 t_1^2 t_2^3 \), which goes to \([187]\) by relation.

**Words of length four**

**25th Double Coset** \([1274]\)

\[ M t_1 t_2 t_4 t_2^2 N = \{ M t_1 t_2 t_4 t_2^2 n | n \in \mathbb{N} \} \]

The coset stabilizer \( N^{(1274)} \) fixes 1, 2, 7, and 4. So, we then have \( N^{(1274)} \geq N^{1274} = \langle e \rangle \), which indicates the stabilizing group \( N^{(1274)} \) only contains the identity \( e \).

Since we have that \( (M t_1 t_2 t_4 t_2^2) x^3 = M t_1 t_2 t_1^4 t_2^2 \), by relation. We then have that \( x y \) would be in the stabilizing group \( N^{(1274)} = \langle x y \rangle \).

Which would imply that the number of single cosets in \( M t_1 t_2 t_4 t_2^2 N \) are \( \frac{N}{N^{(1274)}} = \frac{12}{2} = 6 \).

Thus, The orbits of \( N^{(1274)} \) on \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \} \) are \( \{ 1, 10 \}, \{ 2, 5 \}, \{ 3, 6 \}, \{ 4, 11 \}, \{ 7, 12 \}, \) and \( \{ 8, 9 \} \).

We will now select a representative \( t_i \) from each orbit.

10 \( \in \{ 1, 10 \} \Rightarrow M t_1 t_2 t_1^4 t_2^2 t_1 \), which goes to \([121]\) by relation.

2 \( \in \{ 2, 5 \} \Rightarrow M t_1 t_2 t_1^4 t_2^2 t_2 \), which goes to \([141]\) by relation.

3 \( \in \{ 3, 6 \} \Rightarrow M t_1 t_2 t_1^4 t_2^2 t_1^2 \), which goes to \([10]\) by relation.

4 \( \in \{ 4, 11 \} \Rightarrow M t_1 t_2 t_1^4 t_2^2 t_2^2 \), which goes to \([141]\) by relation.
7 ∈ {7, 12} ⇒ Mt_1t_2t_1t_2t_4t_1^4, which goes to [1611] by relation.
8 ∈ {8, 9} ⇒ Mt_1t_2t_1t_2^2t_4^2, which goes to [187] by relation.

26th Double Coset [1278]

Mt_1t_2t_1t_2t_4t_2N = \{Mt_1t_2t_1t_2t_4t_2^n | n ∈ \mathbb{N}\}
The coset stabilizer N\(^{(1278)}\) fixes 1, 2, 7, and 8. So, we then have N\(^{(1278)}\) ≥ N\(^{(1278)}\) = < e >, which indicates the stabilizing group N\(^{(1278)}\) only contains the identity e.
Since we have that (Mt_1^2t_2t_4t_1^2t_2^4)x = Mt_1t_2t_1t_2t_4t_2^4, (Mt_2^5t_1^3t_2t_4^6t_1^5xy = Mt_1t_2t_1t_2t_4^2, (Mt_2^3t_1^5t_2t_4^5t_1^6)y = Mt_1t_2t_1t_2t_4^4, and (Mt_2^6t_1^6t_2^3t_1^3)x = Mt_1t_2t_1t_2t_4^4 all by relation, then we have that x^2, y, x, y, x^3y would be in the stabilizing group N\(^{(1278)}\) = < x^2, y, x, y, x^3y >.
Which would imply that the number of single cosets in Mt_1t_2t_1t_2t_4t_2N are \(\frac{N}{N^{(1278)}} = \frac{12}{6}\) = 2. Thus, The orbits of N\(^{(1278)}\) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1, 7, 10, 6, 12, 3, 2, 4, 5, 9, 11, 8\}.
We will now select a representative t_i from each orbit.
6 ∈ \{1, 7, 10, 6, 12, 3\} ⇒ Mt_1t_2t_1t_2t_4t_2^3, which goes to [127] by relation.
2 ∈ \{2, 4, 5, 9, 11, 8\} ⇒ Mt_1t_2t_1t_2t_4^2t_2, which goes to [1274] by relation.

27th Double Coset [1414]

Mt_1t_2^2t_1t_2t_2^2N = \{Mt_1t_2^2t_1t_2t_2^2n | n ∈ \mathbb{N}\}
The coset stabilizer N\(^{(1414)}\) fixes 1, 4, 1, and 4. So, we then have N\(^{(1414)}\) ≥ N\(^{(1414)}\) = < e >, which indicates the stabilizing group N\(^{(1414)}\) only contains the identity e.
Since we have that (Mt_1^6t_2t_4^6t_1^2)t_2^2 = Mt_1t_2t_1t_2t_2^4, by relation. We then have that y would be in the stabilizing group N\(^{(1414)}\) = < y >.
Which would imply that the number of single cosets in Mt_1t_2^2t_1t_2^2N are \(\frac{N}{N^{(1414)}} = \frac{12}{2}\) = 6. Thus, The orbits of N\(^{(1414)}\) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} are \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, and \{11, 12\}.
We will now select a representative t_i from each orbit.
2 ∈ \{1, 2\} ⇒ Mt_1t_2^2t_1t_2^2t_2, which goes to [187] by relation.
4 ∈ \{3, 4\} ⇒ Mt_1t_2^2t_1t_2^2t_2^2, which goes to [187] by relation.
6 ∈ \{5, 6\} ⇒ Mt_1t_2^2t_1t_2^2t_2^3, which goes to [1414] by relation.
8 ∈ \{7, 8\} ⇒ Mt_1t_2^2t_1t_2^2t_2^4, which goes to [141] by relation.
10 ∈ \{9, 10\} ⇒ Mt_1t_2^2t_1t_2^2t_2^5, which goes to [141] by relation.
12 ∈ \{11, 12\} ⇒ Mt_1t_2^2t_1t_2^2t_2^6, which goes to [1103] by relation.

28th Double Coset [1614]

Mt_1t_2^3t_1t_2^2N = \{Mt_1t_2^3t_1t_2^2n | n ∈ \mathbb{N}\}
The coset stabilizer N\(^{(1614)}\) fixes 1, 6, 1, and 4. So, we then have N\(^{(1614)}\) ≥ N\(^{(1614)}\) = < e >, which indicates the stabilizing group N\(^{(1614)}\) only contains the identity e.
Since we have that (Mt_1^2t_4t_1t_2^4)x^3 = Mt_1t_2t_1t_2^4, (Mt_2^5t_1^3t_2t_4^6t_1^5)x = Mt_1t_2t_1t_2^4,
and \((Mt_2^3t_1^5t_2^5t_1^6)^gyx = Mt_1t_2t_1^4t_2^4\) all by relation, then we have that \(x^3.x^2.y, \text{ and } yx,\) would be in the stabilizing group \(N^{(1614)} = \langle x^3, x^2.y, yx \rangle\).

Which would imply that the number of single cosets in \(Mt_1t_2^3t_1t_2^2N\) are \(\frac{N}{N^{(1614)}} = \frac{123}{4}\)

Thus, The orbits of \(N^{(1614)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1, 6, 11, 8\},\)

\(\{2, 9, 12, 3\}, \text{ and } \{4, 5, 10, 7\}\).

\(1 \in \{1, 6, 11, 8\} \Rightarrow Mt_1t_2^3t_1t_2^2t_1, \text{ which goes to } [1618] \text{ by relation.}\)

\(2 \in \{2, 9, 12, 3\} \Rightarrow Mt_1t_2^3t_1t_2^2t_2, \text{ which goes to } [165] \text{ by relation.}\)

\(10 \in \{4, 5, 10, 7\} \Rightarrow Mt_1t_2^3t_1t_2^2t_2^5, \text{ which goes to } [161] \text{ by relation.}\)

29th Double Coset \([1618]\)

\(Mt_1t_2^3t_1t_2^4N = \{Mt_1t_2^3t_1t_2^4n | n \in \mathbb{N}\}\)

The coset stabilizer \(N^{(1618)}\) fixes 1, 6, 1, and 8. So, we then have \(N^{(1618)} = \langle e \rangle\), which indicates the stabilizing group \(N^{(1618)}\) only contains the identity \(e\).

Since we have that \((Mt_2^3t_1^5t_2^2t_1^2)^gyx = Mt_1t_2^3t_1t_2^4\), by relation. We then have that \(yx^2\) would be in the stabilizing group \(N^{(1618)} = \langle yx^2 \rangle\).

Which would imply that the number of single cosets in \(Mt_1t_2^3t_1t_2^4N\) are \(\frac{N}{N^{(1618)}} = \frac{123}{2}\)

Thus, The orbits of \(N^{(1618)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) are \(\{1, 4\}, \{2, 7\}, \{3, 8\},\)

\(\{5, 12\}, \{6, 9\}, \text{ and } \{10, 11\}\).

We will now select a representative \(t_i\) from each orbit.

\(1 \in \{1, 4\} \Rightarrow Mt_1t_2^3t_1t_2^3t_1, \text{ which goes to } [1618] \text{ by relation.}\)

\(2 \in \{2, 7\} \Rightarrow Mt_1t_2^3t_1t_2^2t_2, \text{ which goes to } [165] \text{ by relation.}\)

\(3 \in \{3, 8\} \Rightarrow Mt_1t_2^3t_1t_2^2t_1^2, \text{ which goes to } [1614] \text{ by relation.}\)

\(5 \in \{5, 12\} \Rightarrow Mt_1t_2^3t_1t_2^3t_1^3, \text{ which goes to } [161] \text{ by relation.}\)

\(6 \in \{6, 9\} \Rightarrow Mt_1t_2^3t_1t_2^2t_2^3, \text{ which goes to } [161] \text{ by relation.}\)

\(10 \in \{10, 11\} \Rightarrow Mt_1t_2^2t_1t_2^2t_2^5, \text{ which goes to } [161] \text{ by relation.}\)

Now, we have completed the double coset enumeration since the right coset is closed under multiplication. We then conclude:

\[G = N \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_2^2N \cup Mt_1t_2^3N \cup Mt_1t_2^4N \cup Mt_1t_2^5N \cup Mt_1t_2^6N \cup Mt_1t_2t_1N \cup Mt_1t_2^3t_1N \cup Mt_1t_2^4t_1N \cup Mt_1t_2^5t_1N \cup Mt_1t_2^6t_1N \cup Mt_1t_2^3t_1^2N \cup Mt_1t_2^3t_1^3N \cup Mt_1t_2^3t_1^4N \cup Mt_1t_2^3t_1^5N \cup Mt_1t_2^3t_1^6N \cup Mt_1t_2^3t_1^7N \cup Mt_1t_2^3t_1^8N \cup Mt_1t_2^4t_1^9N \cup Mt_1t_2^5t_1^{10}N \cup Mt_1t_2^6t_1^{11}N \cup Mt_1t_2^7t_1^{12}N, \text{ and that}\]

\[|G| \leq (|N^*| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|}) + (\frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(16)}|} + \frac{|N|}{|N^{(18)}|} + \frac{|N|}{|N^{(110)}|} + \frac{|N|}{|N^{(112)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(127)}|} + \frac{|N|}{|N^{(129)}|} + \frac{|N|}{|N^{(131)}|} + \frac{|N|}{|N^{(133)}|} + \frac{|N|}{|N^{(135)}|} + \frac{|N|}{|N^{(137)}|} + \frac{|N|}{|N^{(139)}|} + \frac{|N|}{|N^{(141)}|} + \frac{|N|}{|N^{(143)}|} + \frac{|N|}{|N^{(145)}|} + \frac{|N|}{|N^{(147)}|} + \frac{|N|}{|N^{(149)}|} + \frac{|N|}{|N^{(151)}|} + \frac{|N|}{|N^{(153)}|} + \frac{|N|}{|N^{(155)}|} + \frac{|N|}{|N^{(157)}|} + \frac{|N|}{|N^{(159)}|} + \frac{|N|}{|N^{(161)}|} + \frac{|N|}{|N^{(163)}|} + \frac{|N|}{|N^{(165)}|} + \frac{|N|}{|N^{(167)}|} + \frac{|N|}{|N^{(169)}|} + \frac{|N|}{|N^{(171)}|} + \frac{|N|}{|N^{(173)}|} + \frac{|N|}{|N^{(175)}|} + \frac{|N|}{|N^{(177)}|} + \frac{|N|}{|N^{(179)}|} + \frac{|N|}{|N^{(181)}|} + \frac{|N|}{|N^{(183)}|} + \frac{|N|}{|N^{(185)}|} + \frac{|N|}{|N^{(187)}|} + \frac{|N|}{|N^{(189)}|} + \frac{|N|}{|N^{(191)}|} + \frac{|N|}{|N^{(193)}|}) \times |N|\]

\[|G| \leq (1 + 12 + 12 + 12 + 12 + 12 + 6 + 12 + 12 + 12 + 12 + 12 + 2 + 12 + 12 + 12 + 6 + 12 + 12 + 12 + 6 + 6 + 2 + 6 + 3 + 6) \times 660\]
$|G| \leq 266 \times 660$

$|G| \leq 175560$

Figure 2.7: Construction of $J_1$ over Maximal Subgroup $PSL(2, 11)$
Chapter 3

Progenitor Images

3.1 Permutation Progenitors

3.1.1 2^6 : S₄ × 2

G<x, y, t> := Group<x, y, t | y^2, (x^−1*y)^2, x^6, t^2, t^∗(x^3*y)=t, (x^3*t)^a1, (y*t)^b1, (y*t^x)^c1, (y*x*t)^d1, (y*x*t^∗(x^2))^e1, (x^2*t)^f1, (x*t)^a2>

Conjugacy Classes of group N

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<th>Length</th>
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<td>(1, 2, 3, 4, 5, 6)</td>
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Table 3.1: \(2^6 : S_4 \times 2\) Progenitor Images

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<td>((A_5 \times PGL(2,11)) \times 2^4)</td>
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<td>((PSL(3,4) \times (2 \times 3)) : 2)</td>
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<td>8</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>((3^2 \times 2^{13}) : 2^2)</td>
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</table>

3.1.2 \(2^7 : (7 : 3)\)

\[ G<x,y,t> := \text{Group}<x,y,t | y^3, y*x^2*y^1*x^1, t^2, t^1(y*x^1) = t, \]
\[ (x*y^1*1*t)^1^a1, (x*y^1*1*t^1(x))^1^b1, (x*y^1*1*t^1(x^3))^1^c1, \]
\[ (y*x^1*1*t)^1^d1, (y*x^1*1*t^1(x))^1^e1, (y*x^1*1*t^1(x^3))^1^f1, \]
\[ (x^1*3*t)^1^a2, (x*t)^1^b2>; \]

Conjugacy Classes of group N

------------------------

[1] Order 1  Length 1
   Rep Id(N)

[2] Order 3  Length 7
   Rep (2, 5, 3) (4, 6, 7)

[3] Order 3  Length 7
   Rep (2, 3, 5) (4, 7, 6)

[4] Order 7  Length 3
   Rep (1, 2, 3, 4, 5, 6, 7)

[5] Order 7  Length 3
   Rep (1, 4, 7, 3, 6, 2, 5)
Table 3.2: $2^7 : (7 : 3)$ Progenitor Images

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3.1.3 $2^8 : D_{16}$

$G<x, y, t> = \text{Group}<x, y, t | y^2, (x-1*y)^2, x^8, t^2, t^y(x*x^3)=t, (x^4*t)^a1, (x^3*t)^b1, (y*t*(x^2))^c1, (x*y*t*(x^2))^d1, (x*y^t*(x))^e1, (x^2*t)^f1, (x*y^t)^a2, (y*t)^b2, (x*t)^c2>$

Conjugacy Classes of group N

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Rep (1, 5) (2, 6) (3, 7) (4, 8)

[3] Order 2  Length 4
Rep (1, 6) (2, 5) (3, 4) (7, 8)

[4] Order 2  Length 4
Rep (1, 5) (2, 4) (6, 8)

Rep (1, 3, 5, 7) (2, 4, 6, 8)

[6] Order 8  Length 2
Rep (1, 2, 3, 4, 5, 6, 7, 8)

[7] Order 8  Length 2
Rep (1, 4, 7, 2, 5, 8, 3, 6)
Table 3.3: $2^8 : D_{16}$ Progenitor Images

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<th>d1</th>
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3.1.4 $2^{11} : D_{22}$

\[ G < x, y, t > := \text{Group} < x, y, t \mid y^2, (x^{11} - 1 + y^2, x^2, t^2, t \cdot (y \cdot x^2) = t, \]
\[ (y \cdot t)^{-1 \cdot a1}, (y \cdot t \cdot x)^{-1 \cdot b1}, (y \cdot t \cdot (x^2)) \cdot c1, (y \cdot t \cdot (x^3)) \cdot d1, \]
\[ (y \cdot t \cdot (x^4)) \cdot e1, (x \cdot t) \cdot f1, (x^2 \cdot t) \cdot a2, (x^3 \cdot t) \cdot b2, (x^4 \cdot t) \cdot c2, \]
\[ (x^5 \cdot t) \cdot d2 >; \]

Conjugacy Classes of group N

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Table 3.4: $2^{11} : D_{22}$ Progenitor Images

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3.1.5 $2^{11} : L_2(11)$

$\begin{align*}
G<x,y,t>:=&\text{Group}<x,y,t | y^2, (y*x^3-1)^3, x^3-11, (x*y*x^3-3*y*x^2)^2, \\
&\quad t^2, (t^y)t, (t^y)^t, (t^y)^{(x^2*y*x^3-3)}=t, (x*y*x^3-1*t)^a_1, \\
&\quad (x*y*x^3-1*t^y)^b_1, (x*y*x^3-1*t^x)^c_1, (x*y*t)^d_1, \\
&\quad (x*y^3*t^y)^e_1, (x*y^3*t^x)^f_1, (y*x^3*t)^a_2, (y*x^3*t^y)^b_2, \\
&\quad (y*x^3*t^x)^c_2, (((y*x^3)^2)*t)^d_2, (((y*x^3)^2)*t^x)^e_2, \\
&\quad (((y*x^3)^2)*t^x)^f_2>;
\end{align*}$

Table 3.5: $2^{11} : L_2(11)$ Progenitor Images

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<tr>
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<th>b1</th>
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<th>d1</th>
<th>e1</th>
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3.1.6 \(2^{11} : M_{11}\)

\[G<x,y,z,t>:=\text{Group}\langle x, y, z, t \mid x^{11}, y^5, z^4, y*x^3*y^{-1}x^{-1}, z^{-1}y^2z^{-1}, x^{-1}z^{-2}x^2z^{-1}x^{-1}z^{-1}, z^{-1}x^{-1}z^{-1}x^{-2}y^{-1}z^{-1}x^5y, t^2, t^z = t, t^y = t\rangle.
\]

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3.1.7 $2^{12} : S_4$

$G\langle x, y, t \rangle := \text{Group}<x, y, t|x^3, y^3, (x^{-1}y^{-1})^2, t^2, (x*y*t)^a, (x*y*t \cdot x)^b1, (x*y*t \cdot y)^c1, (x*t)^d1, (x*t \cdot (x*y^{-1}))^e1, (x*t \cdot (y^2))^f1, (x*t \cdot (y))^a2, (x^2*t \cdot t)^b2, (x^2*t \cdot (x*y^{-1}))^c2, (x^2*t \cdot (y^2))^d2, (x^2*t \cdot (y))^e2>$

Conjugacy Classes of group N

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Table 3.7: $2^{12} : S_4$ Progenitor Images

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<td>PGL(2,25) × $S_5 \times A_5 \times 2$</td>
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<td>4</td>
<td>PGL(2,81) × 2</td>
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</table>
3.1.8 $2^{*15} : S_5 \times 3$

\[G < x, y, t> := \text{Group:<x,y,t|y^2,(x*y*x^-1*y)^2,x^-1*y*x^-5*y*x^-4,(x^-1*y)^6,y*x^2*y*x^-3*y*x^2*y*x^2,t^2,}
\]
\[t^((y*x^-1*y*x^-1*y*x^-2*y)=t, t^((x*y*x^-1*y*x^-3)=t, t^((x*y)^3)=t, t^((y*x^-2*y*x^2*y)=t, (x^-3*y*x^-4*t)^a_1,}
\]
\[(x^-3*y*x^-4*t*(x))^-b_1, (x^-2*t)^c_1, (x*t)^d_1,}
\[(y*x^-4*t*(x))^-e_1, (x^-5*y*x*t)^f_1, (x^-4*y*t^*(x))^-a_2,}
\[(x^-4*y*t)^b_2>;\]

<table>
<thead>
<tr>
<th>Index</th>
<th>a1</th>
<th>b1</th>
<th>c1</th>
<th>d1</th>
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<td>$S(6,2) \times 2$</td>
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<td>$(A_6 \times 64 \times 64) : 6$</td>
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3.1.9 \(2^{19}: D_{38}\)

\[G<x,y,t>:=\text{Group}<x,y,t|y^2, (x^{-1}y)^2, x^{-19}, t^2, t^y=t, (y^t^x)\text{^a1}, (x^5t)\text{^b1}, (y^t^x)\text{^c1}, (x^4t)\text{^d1}, (y^t^x)\text{^e1}, (x^3t)\text{^f1}, (y^t^x)^2, (x^2t)\text{^b2}, (y^t^x)\text{^c2}, (x^t)\text{^d2}>;\]

Conjugacy Classes of group N
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[1] Order 1 Length 1
   Rep Id(N)

[2] Order 2 Length 19
   (9, 12)(10, 11)

   Rep (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19)

   Rep (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 2, 4, 6, 8, 10, 12, 14, 16, 18)

   Rep (1, 4, 7, 10, 13, 16, 19, 3, 6, 9, 12, 15, 18, 2, 5, 8, 11, 14, 17)

   Rep (1, 5, 9, 13, 17, 2, 6, 10, 14, 18, 3, 7, 11, 15, 19, 4, 8, 12, 16)

[7] Order 19 Length 2
   Rep (1, 6, 11, 16, 2, 7, 12, 17, 3, 8, 13, 18, 4, 9, 14, 19, 5, 10, 15)

[8] Order 19 Length 2
   Rep (1, 7, 13, 19, 6, 12, 18, 5, 11, 17, 4, 10, 16, 3, 9, 15, 2, 8, 14)

[9] Order 19 Length 2
   Rep (1, 8, 15, 3, 10, 17, 5, 12, 19, 7, 14, 2, 9, 16, 4, 11, 18, 6, 13)
[10] Order 19  Length 2  
   Rep  (1, 9, 17, 6, 14, 3, 11, 19, 8, 16, 5, 13, 2, 10, 18, 7, 15, 4, 12)

   Rep  (1, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15, 5, 14, 4, 13, 3, 12, 2, 11)

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<th>c1</th>
<th>d1</th>
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<th>b2</th>
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3.1.10 $2^{+26} : D_{52}$

\[ G<x, y, t> := \text{Group}\langle x, y, t \mid y^2, (x^1+y)^2, x^{-13}, t^2, t^{(y*x^12)} = t, \\
(x^a*t^2*t^2)^a_1, (x^2*t^1)^b_1, (x^2*t^2)^c_1, (x^3)^d_1, \\
(x^4*t^e_1, (x^5*t)^f_1, (x^6*t)^a_2 >; \]

Conjugacy Classes of group N

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<th>Length</th>
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<td>(8, 9)(17, 25)(18, 26)(19, 24)(20, 23)(21, 22)</td>
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</table>
[10] Order 13  
    Length 2  
    Rep (1, 11, 21, 5, 15, 25, 9, 20, 3, 14, 24, 7, 18)  
    (2, 12, 22, 6, 16, 26, 10, 19, 4, 13, 23, 8, 17)  

    Length 2  
    Rep (1, 22, 15, 10, 3, 23, 18, 12, 5, 26, 20, 13, 7,  
    2, 21, 16, 9, 4, 24, 17, 11, 6, 25, 19, 14, 8)  

[12] Order 26  
    Length 2  
    Rep (1, 10, 18, 26, 7, 16, 24, 6, 14, 22, 3, 12, 20,  
    2, 9, 17, 25, 8, 15, 23, 5, 13, 21, 4, 11, 19)  

[13] Order 26  
    Length 2  
    Rep (1, 23, 20, 16, 11, 8, 3, 26, 21, 17, 14, 10,  
    5, 2, 24, 19, 15, 12, 7, 4, 25, 22, 18, 13, 9, 6)  

[14] Order 26  
    Length 2  
    Rep (1, 12, 21, 6, 15, 26, 9, 19, 3, 13, 24, 8, 18,  
    2, 11, 22, 5, 16, 25, 10, 20, 4, 14, 23, 7, 17)  

[15] Order 26  
    Length 2  
    Rep (1, 26, 24, 22, 20, 17, 15, 13, 11, 10, 7, 6, 3,  
    2, 25, 23, 21, 19, 18, 16, 14, 12, 9, 8, 5, 4)  

[16] Order 26  
    Length 2  
    Rep (1, 13, 25, 12, 24, 10, 21, 8, 20, 6, 18, 4, 15,  
    2, 14, 26, 11, 23, 9, 22, 7, 19, 5, 17, 3, 16)  

Table 3.10: $2^{*26} : D_{52}$ Progenitor Images

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<td>0</td>
<td>0</td>
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<td>3</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>$(2^{11} \times 3^4 \times 13)$</td>
</tr>
</tbody>
</table>
3.1.11 $2^{*28} : (7 : 2)^4$

$$G<x, y, z, t> := \text{Group}\{x, y, z, t \mid x^2, y^2, z^2, (x*y)^2, (y*z)^2, (x*z)^{14}, t^2, (z*t)^{(z*x)}^a1, (y*z*t)^{(z*x)^3}^b1, (y*z*t)^{(z*x)^2}^c1, (z*t)^d1, (y*z*x*z*t)^{(z*x)}^6)^e1, (y*z*x*z*t)^{(z*x)}^2)^f1, (y*z*t)^a2\};$$

<table>
<thead>
<tr>
<th>Index</th>
<th>a1</th>
<th>b1</th>
<th>c1</th>
<th>d1</th>
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<th>f1</th>
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<td>0</td>
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<td>3</td>
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<td>$A_6 \times 28 \times 49 \times 49$</td>
</tr>
<tr>
<td>870</td>
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<td>2</td>
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<td>0</td>
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<td>$PSL(2, 29) \times 2$</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>$(21 \times 128 \times 128)$</td>
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</table>
3.1.12 \(2^{*55} : (55 \times 10) : 2\)

\[G<x, y, t> = \text{Group}<x, y, t \mid y \cdot x^{-2} \cdot y^{-1} \cdot x, y \cdot x^{-1} \cdot y^{-1} \cdot x^{-3} \cdot y \cdot 4 \cdot x \cdot y^{-1}, y^{20}, t^{2}, t^{((x^{-1} \cdot y \cdot x^{-1}) \cdot 3)} = t, (y \cdot x^{-1} \cdot t \cdot (x^{5})) \cdot a_{1}, (y \cdot x^{-1} \cdot t \cdot (x^{3})) \cdot b_{1}, (y \cdot x^{-1} \cdot t \cdot (x^{1})) \cdot c_{1}, (y \cdot x^{-1} \cdot t \cdot (x^{3})) \cdot d_{1}, (y \cdot 5 \cdot x^{-1} \cdot y \cdot 5 \cdot t \cdot (x^{3})) \cdot e_{1}, (y \cdot 3 \cdot x^{-1} \cdot y \cdot 4 \cdot x^{-1} \cdot t \cdot (x^{11})) \cdot f_{1}, (y \cdot 5 \cdot x^{-1} \cdot y \cdot 5 \cdot t^{(x^{5})}) \cdot a_{2}, (y \cdot 3 \cdot x^{-1} \cdot y \cdot 4 \cdot x^{-1} \cdot t \cdot (x^{5})) \cdot b_{2}, (x \cdot t) \cdot c_{2}>;\]

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<th>(c_{1})</th>
<th>(d_{1})</th>
<th>(e_{1})</th>
<th>(f_{1})</th>
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<td>0</td>
<td>4</td>
<td>(64 \times 32 \times 55) : 2</td>
<td></td>
</tr>
<tr>
<td>36</td>
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<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>(S_{6})</td>
<td></td>
</tr>
<tr>
<td>38016</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>8</td>
<td>(M_{12} \times 4) : 2</td>
<td></td>
</tr>
</tbody>
</table>
3.1.13 $2^{*64} : D_{128}$

$G<x, y, t>:=\text{Group}<x, y, t | y^2, (x^1+y)^2, x^{64}, t^2, t^{(y*x)}=t, (x^{32}*t)^a1, (x^{11}*t)^b1, (x^{13}*t)^c1, (x*t)^d1, (y*t^{(x^{12})})^e1, (y*t^{(x^{11})})^f1, (y*t^{(x^{10})})^a2, (y*t^{(x^9)})^b2>;$

<table>
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<th>c1</th>
<th>d1</th>
<th>e1</th>
<th>f1</th>
<th>a2</th>
<th>b2</th>
<th>Image</th>
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<td>6</td>
<td>4</td>
<td>$(U(3, 7) \times 4) : 2$</td>
</tr>
<tr>
<td>42</td>
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<td>3</td>
<td>4</td>
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<td>0</td>
<td>0</td>
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<td>$PGL(2, 17) \times 2$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$S_6 \times 6$</td>
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</tbody>
</table>
3.1.14  $2^{*100} : [(25 \times 10) : 2]^2$

$G<x, y, t> := \text{Group}<x, y, t | y*x*y^-1*x^2*y^-1*x*y,$
$y^-1*x*y*x^2*y*x*y^-1, x^-1*y^-1*x^-1*y^-1*x^-1*y^-1*x^-1*y^-1,x^-1*y^-1*x^-1*y^-1,x^-1*y^-1*x^-1,$
$y^-1*x*y^-1*x^3*y^2*x^-2, y^10, x^15*y^-1*x^-1*y*x^4, t^2,$
$t^*(y*x*y*x^-1*y) = t, (x^-6*y^-1*x*y^-1*t)^a1,$
$(x^4*y*x^-1*y^3*t^*(x^3)) ^b1, (x^4*y*x^-1*y^3*t^*(x^2)) ^c1,$
$(x^4*y*x^-1*y^3*t^*(x)) ^d1, (x^7*y^2*t) ^e1, (y^-4*x*t) ^f1,$
$(x^9*y^4*t) ^a2, (x^2*y^-1*x^-1*y^-1*t) ^b2>$

<table>
<thead>
<tr>
<th>Index</th>
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<th>b1</th>
<th>c1</th>
<th>d1</th>
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<th>f1</th>
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<th>b2</th>
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<td>4</td>
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<td>4</td>
<td>$S_6 \times 128 \times 64 \times 25 \times 6$</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>$U(3, 9) \times 25 \times 4$</td>
</tr>
<tr>
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<td>0</td>
<td>4</td>
<td>3</td>
<td>$S_6 \times 6$</td>
</tr>
</tbody>
</table>
3.2 Monomial Progenitors

3.2.1 $3^3 : A_4$

$G<x, y, t> := \text{Group}<x, y, t | x^3, y^3, (x^{-1}y^{-1})^2, t^3, t^*(y*x) = t, t^*(x*y^{-1}x) = t^2, (x*y*t^*(y))^{-a_1}, (x*y*t^*(x))^{-b_1}, (x*y*t)^{-c_1}, (x*t^*(x*y))^d, (x*t)^{-e_1}, (x^{-1}*t^*(x*y))^f, (x^{-1}*t)^{-a_2}>$

Conjugacy Classes used for relations

<table>
<thead>
<tr>
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<th>Order</th>
<th>Length</th>
</tr>
</thead>
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<td>3</td>
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<tr>
<td></td>
<td>Rep (1, 4)(2, 5)</td>
<td></td>
</tr>
<tr>
<td>[2]</td>
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<td>4</td>
</tr>
<tr>
<td></td>
<td>Rep (1, 2, 3)(4, 5, 6)</td>
<td></td>
</tr>
<tr>
<td>[3]</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Rep (1, 3, 2)(4, 6, 5)</td>
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</table>

<p>| Table 3.15: $3^3 : A_4$ Progenitor Images |
|---|---|---|---|---|---|---|---|---|---|</p>
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<th>b1</th>
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<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$PSL(2,13) \times 12$</td>
</tr>
<tr>
<td>91</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>$PSL(2,13)$</td>
</tr>
<tr>
<td>210</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>$A_7$</td>
</tr>
</tbody>
</table>
3.2.2 $5^2 : D_8$

$G<x,y,t>:=\text{Group}<x,y,t|y^2,x^4,(x^{-1}*y)^2,t^5,t^5(x^{-1})=t^2,(x^2*t)^a1,(y*t^a(x))^b1,(y*t^c1,(x*y^t^a(y))^d1,(x*y*t)^e1,(x*t^c(y))^f1,(x*t)^a2>;$

Conjugacy Classes used for relations

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<tr>
<td>[2]</td>
<td>2</td>
<td>2</td>
<td>(1, 2)(3, 4)(5, 6)(7, 8)</td>
</tr>
<tr>
<td>[3]</td>
<td>2</td>
<td>2</td>
<td>(1, 6)(2, 3)(4, 7)(5, 8)</td>
</tr>
<tr>
<td>[4]</td>
<td>4</td>
<td>2</td>
<td>(1, 5, 7, 3)(2, 4, 8, 6)</td>
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</table>

Table 3.16: $5^2 : D_8$ Progenitor Images

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<th>f1</th>
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<td>0</td>
<td>0</td>
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<td>6</td>
<td>0</td>
<td>0</td>
<td>$S_5 \times 16$</td>
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<tr>
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<td>0</td>
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<td>$(S_6 \times 6) : 2$</td>
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<td>8</td>
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<td>8</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>$PSL(4,3) \times 8$</td>
</tr>
</tbody>
</table>
3.2.3 \( 5^2 : 16 \cdot 4 \)

\[ G<x, y, t> := \text{Group}<x, y, t| x^2, y^2, (x*y)^4, t^5, t^{(x*y*x)} = t^4, t^{(y)} = t, ((x*y)^2 + t)^a_1, (y*t)^b_1, (x*y*x*t)^c_1, (x*y*x*t^{(x)})^d_1, (x*y*t)^e_1 >; \]

Conjugacy Classes used for relations
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<td>4</td>
<td>2</td>
<td>(1, 8, 7, 2)(3, 4)(5, 6)</td>
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Table 3.17: \( 5^2 : 16 \cdot 4 \) Progenitor Images

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<td>25</td>
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<td>0</td>
<td>4</td>
<td>( 8 \times 25 )</td>
</tr>
<tr>
<td>90</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>( A_4 : 2 )</td>
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3.2.4  $5^4 : 16^4$

$G<x,y,t> := \text{Group}<x,y,t|x^2,y^4,(y^{1-x})^4,(y*x*y^{1-x})^2,t^5,$
$t^*(x*y)^2*x=t,t^*(y^{1})=t^3,(x*t^*(x*y))^a1,(y*t^*(y*x))^b1,$
$(x*y)^c1,(x*y*x*y^4-x)*t^*(x*y))\hat{d}1,(y*x*y*t^*(x*y))\hat{e}1,$
$((x*y)^2*t^*(y))^\hat{f}1,(y*x*y*t)^a2>;$

Conjugacy Classes used for relations
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<tbody>
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</table>

<table>
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<th>Length 4</th>
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Table 3.18: $5^4 : 16^4$ Progenitor Images

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<td>5</td>
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<td>0</td>
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<td>$(S_5 \times A_4 \times 4) : 4$</td>
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<tr>
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<td>0</td>
<td>0</td>
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<td>5</td>
<td>$PSL(4,3) \times 4$</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>$(M_{12} \times 2) : 2$</td>
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</table>
3.2.5 \(7^2 : S_3 \times 2\)

\(G<x,y,t> = \text{Group}<x,y,t | y^2, (x^{-1}y)^2, x^6, t^7, t(x^{-1}) = t^5, (x^3t)^a1, (x^2t^y)^b1, (x^2t)^c1, (y^t(x)^2)^d1, (x^yt(x))e1, (x^yt^x)^f1, (y^t(x))^a2, (x^yt)^b2, (x^yt^y)^c2, (y^t)^d2, (x^t)^e2>\)

Conjugacy Classes used for relations

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3.2.6 $13^{*4} : D_8 \times S_3$

$G<x,y,t> := \text{Group}<x,y,t| y^2, x^4, (x^{-1}y)^2, t^{13}, t^{(x^{-1})} = t^5, t^{(x^2)} = t^{12}, (yx^t)^a_1, (x^t(y^x))^b_1, (x^t(y))^c_1, (x^2t)^d_1, (yt)^e_1, (x^t(y)^f_1, (x^t)^a_2>$

Conjugacy Classes used for relations

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<td>(15, 35)(16, 36)(17, 29)(18, 30)(19, 31)(20, 32)</td>
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<tr>
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<td>(15, 24)(16, 27)(18, 45)(20, 47)(21, 34)(23, 36)</td>
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<td>(7, 11, 43, 39)(8, 40, 44, 12)(13, 21, 33, 25)</td>
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<tr>
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<td>(14, 26, 34, 22)(15, 23, 35, 27)</td>
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<td>(16, 28, 36, 24)</td>
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<p>| Table 3.20: $13^{*4} : D_8 \times S_3$ Progenitor Images |
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<th>d1</th>
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<th>f1</th>
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<td>6</td>
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<td>4</td>
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<td>$PGL(2,25) \times 4$</td>
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3.2.7 $17^*^2 : D_{16}$

$G<x, y, t> := \text{Group}<x, y, t|x^2, y^4, (y^{-1}x)^4, (yx^4y^{-1}x)^2, t^5, t^c(x^2y^2x)t, t^c(y^{-1})t^c3, (x^4t)^a1, (y^t(x^3))^{b1}, (yt^c(x^2))^c1, (yt^c(x))^d1, (yt)^e1, (x^4yt^c(x^3))^f1, (xy^t(x^2))^a2, (xy^t(x))^b2>;$

Conjugacy Classes used for relations
-------------------------------------
[1] Order 2 Length 4
  Rep (1, 30) (2, 15) (3, 26) (4, 31) (5, 22) (6, 13) (7, 18)
  (8, 29) (9, 14) (10, 11) (12, 27) (16, 25) (17, 32) (19, 28)
  (20, 23) (21, 24)

  Rep (1, 7, 31, 25) (2, 26, 32, 8) (3, 15, 29, 17)
  (4, 18, 30, 16) (5, 23, 27, 9) (6, 10, 28, 24)
  (11, 13, 21, 19) (12, 20, 22, 14)

[3] Order 8 Length 2
  Rep (1, 29, 7, 17, 31, 3, 25, 15)
  (2, 16, 26, 4, 32, 18, 8, 30)
  (5, 21, 23, 19, 27, 11, 9, 13)
  (6, 14, 10, 12, 28, 20, 24, 22)

[4] Order 8 Length 2
  Rep (1, 17, 25, 29, 31, 15, 7, 3)
  (2, 4, 8, 16, 32, 30, 26, 18)
  (5, 19, 9, 21, 27, 13, 23, 11)
  (6, 12, 24, 14, 28, 22, 10, 20)

Table 3.21: $17^*^2 : D_{16}$ Progenitor Images

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<th>d1</th>
<th>e1</th>
<th>f1</th>
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<th>b2</th>
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<td>$PSL(2,17)$</td>
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3.2.8 17^{2} : D_{96}

G<x, y, t> := Group<x, y, t | y^2, x^4, (x^-1*y)^2, t^17, (x^-1)*t = t^4, (x^2*t)^a1, (y*t^x)^b1, (y*t^c1, (x*y*t^y)^d1, (x*y*t)^e1, (x*t^y)^f1, (x*t)^a2 >;

Conjugacy Classes for relations
-------------------------------
[1] Order 2  Length 1
(8, 26) (9, 23) (10, 24) (11, 21) (12, 22) (13, 19)
(14, 20) (15, 17) (16, 18)

[2] Order 2  Length 2
  Rep (1, 2) (3, 4) (5, 6) (7, 8) (9, 10) (11, 12) (13, 14)
(15, 16) (17, 18) (19, 20) (21, 22) (23, 24) (25, 26)
(27, 28) (29, 30) (31, 32)

[3] Order 2  Length 2
  Rep (1, 26) (2, 7) (3, 18) (4, 15) (5, 10) (6, 23) (8, 31)
(9, 28) (11, 20) (12, 13) (14, 21) (16, 29) (17, 30)
(19, 22) (24, 27) (25, 32)

  Rep (1, 25, 31, 7) (2, 8, 32, 26) (3, 17, 29, 15)
(4, 16, 30, 18) (5, 9, 27, 23) (6, 24, 28, 10)
(11, 19, 21, 13) (12, 14, 22, 20)

Table 3.22: 17^{2} : D_{96} Progenitor Images

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3.2.9 19^2: D_{72}

\[ G<x, y, t> := \text{Group}<x, y, t | y^2, (x^\ast y)^2, x^\ast 18, t^\ast 19, t^\ast (x^\ast 1) = t^\ast 2, \\
(x^\ast y^\ast t^\ast (x^\ast 4))^\ast a_1, (x^\ast y^\ast t^\ast (x^\ast 3))^\ast b_1, (x^\ast y^\ast t^\ast (x^\ast 2))^\ast c_1, \\
(x^\ast y^\ast t^\ast (x))^\ast d_1, (x^\ast 6^\ast t^\ast e_1, (x^\ast 6^\ast t^\ast (y))^\ast f_1, (x^\ast y^\ast t)^\ast a_2>; \]

Conjugacy Classes used for relations
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3.2.10  \(23^2 : D_{22}\)

\[
G<x,y,t>:=\text{Group}<x,y,t|y^2,(x^2-1*y)^2, x^{11}, t^{23}, t^{(x^3-3)}=t^4, \\
(y*t)^a1,(y*t^((x^4)))^b1,(y*t^((x^3-1))^c1,(x^5*t)^d1, \\
(x^5*t^((y)))^e1,(x^4*t)^f1,(x^4*t^((y)))^a2>;
\]

Conjugacy Classes used for relations
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Table 3.24: \(23^2 : D_{22}\) Progenitor Images

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3.2.11 31*2: \( D_{10} \times 3 \)

\( G<x, y, t>:= \text{Group}\langle x, y, t | y^2, x^4*y*x*y, t^3, t^31, t^2 \rangle = t^{16}, t^{x^2} = t^9, (y*x^{-1}*t^x)\rangle^a, (y*x^{-1}*t^x)\rangle^b, \\
(y*x^{-1}*t^x)\rangle^c, (x*y*t^x)\rangle^d, (x*y*t)\rangle^e, \\
(y*x^{-1}*t)\rangle^f, (y*x^{-1}*t)\rangle^a >; \\
\]

Conjugacy Classes used for relations

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</tbody>
</table>
3.2.12 $67^2: D_{66}$

$G< x, y, t | y^2, (x^{-1}y)^2, x^{-33}, t^{67}, t^x = t^{56}, (y^x)^3, (x^16t^y)^b, (x^{14}t^y)^c, (x^{14}t^y)^d, (x^{14}t^y)^e, (y^x)^f, (y^x)^{a_2}, (y^x)^{b_2}>$

Conjugacy Classes used for relations

<table>
<thead>
<tr>
<th></th>
<th>Order</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1]</td>
<td>2</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Rep (1, 2) (3, 4) (5, 6) (7, 8) (9, 10) (11, 12) (13, 14) (15, 16) (17, 18) (19, 20) (21, 22) (23, 24) (25, 26) (27, 28) (29, 30) (31, 32) (33, 34) (35, 36) (37, 38) (39, 40) (41, 42) (43, 44) (45, 46) (47, 48) (49, 50) (51, 52) (53, 54) (55, 56) (57, 58) (59, 60) (61, 62) (63, 64) (65, 66) (67, 68) (69, 70) (71, 72) (73, 74) (75, 76) (77, 78) (79, 80) (81, 82) (83, 84) (85, 86) (87, 88) (89, 90) (91, 92) (93, 94) (95, 96) (97, 98) (99, 100) (101, 102) (103, 104) (105, 106) (107, 108) (109, 110) (111, 112) (113, 114) (115, 116) (117, 118) (119, 120) (121, 122) (123, 124) (125, 126) (127, 128) (129, 130) (131, 132)</td>
<td></td>
</tr>
<tr>
<td>[2]</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>[3]</td>
<td>11</td>
<td>2</td>
</tr>
</tbody>
</table>
116, 132, 104, 
86, 84, 54) (7, 71, 111, 69, 93, 41, 109, 51, 65, 57, 119) (8, 120, 58, 66, 52, 110, 42, 94, 70, 112, 72) (11, 107, 33, 37, 73, 129, 97, 77, 31, 19, 45) (12, 46, 20, 32, 78, 98, 130, 74, 38, 34, 108) (13, 125, 61, 21, 63, 39, 91, 23, 81, 67, 75) (14, 76, 68, 82, 24, 92, 40, 64, 22, 62, 126)


Table 3.26: $67^2: D_{66}$ Progenitor Images

<table>
<thead>
<tr>
<th>Index</th>
<th>a1</th>
<th>b1</th>
<th>c1</th>
<th>d1</th>
<th>e1</th>
<th>f1</th>
<th>a2</th>
<th>b2</th>
<th>Image</th>
</tr>
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<tr>
<td>2278</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>$PSL(2,67)$</td>
</tr>
<tr>
<td>150348</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>$PSL(2,67) \times 66$</td>
</tr>
</tbody>
</table>
\[ 3.2.13 \quad 97^{\ast 2} : D_{96} \]

\[ G \langle x, y, t \rangle := \text{Group} < x, y, t | y^2, x^48, (x^{-1}y)^2, t^97, t^x = t^2, (x*y*t^x)^{\ast 1}, (x*y*t^x)^{\ast 5} > \]

Conjugacy Classes used for relations

<table>
<thead>
<tr>
<th>Class</th>
<th>Order</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Rep \((1, 4)(2, 97)(3, 8)(5, 12)(6, 99)(7, 16)\)
Rep \((70, 131)(71, 144)(73, 148)(74, 133)(75, 152)(77, 156)(78, 135)\)
Rep \((89, 180)(90, 141)(91, 184)(93, 188)(94, 143)(95, 192)(98, 145)\)

[1] Order 2 Length 24
Rn \((1, 4)(2, 97)(3, 8)(5, 12)(6, 99)(7, 16)\)
Rn \((10, 101)(11, 24)(13, 28)(14, 103)(15, 32)(17, 36)(18, 105)\)
Rn \((30, 111)(31, 64)(33, 68)(34, 113)(35, 72)(37, 76)(38, 115)(39, 80)\)
Rn \((70, 131)(71, 144)(73, 148)(74, 133)(75, 152)(77, 156)(78, 135)\)
Rn \((79, 160)(81, 144)(82, 137)(83, 168)(85, 172)(86, 139)(87, 176)\)
Rn \((89, 180)(90, 141)(91, 184)(93, 188)(94, 143)(95, 192)(98, 145)\)
128) (19, 55, 117) (20, 118, 56) (21, 177, 187)
(22, 188, 178) (23, 105, 63) (24, 64, 106) (25, 33, 133) (26, 134, 34)
(29, 83, 79) (30, 80, 84) (35, 61, 95) (36, 96, 62) (37, 183, 165)
(38, 166, 184) (39, 111, 41) (40, 42, 112) (43, 161, 181) (44, 182, 162)
(45, 89, 57) (46, 58, 90) (51, 67, 73) (52, 74, 68) (53, 189, 143)
(54, 144, 190) (59, 167, 159) (60, 160, 168) (65, 145, 175)
(66, 176, 146) (71, 123, 191) (72, 192, 124) (75, 173, 137) (76, 138, 174)
(81, 151, 153) (82, 154, 152) (87, 129, 169) (88, 170, 130)
(91, 179, 115) (92, 116, 180) (93, 107, 185) (94, 186, 108) (97, 157, 131)
(98, 132, 158) (103, 135, 147) (104, 148, 136) (109, 113, 163)
(110, 164, 114) (119, 141, 125) (120, 126, 142)

68) (57, 111, 135, 81) (58, 82, 136, 112) (65, 93, 127, 99) (66, 100, 128, 94) (73, 75, 119, 117) (74, 118, 120, 76) (83, 101, 109, 91) (84, 92, 110, 102)


[5] Order 8 Length 2
Table 3.27: $97^{*2} : D_{96}$ Progenitor Images

<table>
<thead>
<tr>
<th>Index</th>
<th>a1</th>
<th>b1</th>
<th>c1</th>
<th>dl</th>
<th>el</th>
<th>fl</th>
<th>a2</th>
<th>b2</th>
<th>Image</th>
</tr>
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<tbody>
<tr>
<td>9506</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>$PSL(2,97) \times 2$</td>
</tr>
<tr>
<td>456288</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>$PSL(2,97) \times 98$</td>
</tr>
</tbody>
</table>
Chapter 4

Isomorphism Types

4.1 Isomorphism Group Preliminaries

Definition 4.1.1 (Isomorphism). Let $G$ and $H$ be groups. The mapping $\phi : G \to H$ is said to be an isomorphism if the following are true.
1. $\phi$ is a homomorphism
2. $\phi$ is one-to-one
3. $\phi$ is onto

Definition 4.1.2. (Normal Subgroup) Let $H$ be a subgroup of $G$. We say $H$ is a normal subgroup in $G$, denoted $H \Delta G$, if $\forall g \in G$,

$$g^{-1}Hg = H$$

Definition 4.1.3. (First Isomorphism Theorem) Let $\phi : G \to H$ be a homomorphism with kernel $K$. Then the following are true.
1. $K \Delta G$
2. $K/G \cong \text{im}\phi$

Definition 4.1.4. (Second Isomorphism Theorem) Let $H$ and $K$ be subgroups of $G$, where $H$ is normal. Then the following are true.
1. $H \cap K \Delta K$
2. $K/(H \cap K) \cong HK/H$

Definition 4.1.5. (Extension) If $K$ and $Q$ are groups, then an extension of $K$ by $Q$ is a group $G$ having a normal subgroup $K_1 \cong K$ with $G/K_1 \cong Q$. [Rot95]

Definition 4.1.6. (Symmetric Group) The symmetric group, denoted $S_n$, is the set of all permutations of the nonempty set $X = \{1, 2, \ldots, n\}$. $S_n$ is a group of order $n!$ on $n$ letters.
Definition 4.1.7. (Direct Product) If $H$ and $K$ are groups, then their direct product, denoted by $H \times K$, is the group with elements all ordered pairs $(h, k)$, where $h \in H$ and $k \in K$ and with operation

$$(h, k)(h', k') = (hh', kk').$$

Definition 4.1.8. (Semi-Direct Product) A group $G$ is a semi-direct product of the subgroups $K$ by the subgroups $Q$, denoted by $G = K : Q$, if $K$ is normal in $G$ and $K$ has a complement $Q_1 \cong Q$.\[Rot95\]

Definition 4.1.9. (Central Extension) A central extension of $K$ by $Q$ is an extension $G$ of $K$ by $Q$ with $K \leq Z(G)$.\[Rot95\]

Definition 4.1.10. (Mixed Extension) A mixed extension is a combination of a central extension with a semi-direct product, where the center of the group is not the largest abelian subgroup. We can then factor the group by the largest abelian subgroup and solve the extension problem with a mixture of semi-direct product and central extension properties.
4.2 Direct Product

Let $G = \frac{2^{21} \cdot D_{22}}{(x^3 t)^3, (x^4 t)^8, (x^4 t)^3 = 1}$.

This implies that $G$ has the following composition factors,

\[
\text{CompositionFactors}(G1); \\
\begin{array}{l}
A(1, 23) = L(2, 23) \\
\ast \\
Cyclic(2) \\
1
\end{array}
\]

As you could see, the composition factors contain both $PSL(2, 23)$ and $C_2$. By definition, we need to show that both of these groups are normal in $G$. To do this, we will need to see the lattice of the normal subgroups in $G$. We bring this up on Magma.

Normal subgroup lattice
-----------------------

---  
[3] Order 6072 Length 1 Maximal Subgroups: 1  
---  
[2] Order 2 Length 1 Maximal Subgroups: 1  
---  
[1] Order 1 Length 1 Maximal Subgroups:  

This shows us that there are normal subgroups of order 2 and 6072 that are both separate from each other in the lattice. Which could mean that have a direct product between both of these subgroups. Letting $A = NL[3]$, and $B = NL[2]$, (which represent the normal subgroups isomorphic to $PSL(2, 23)$ and $C_2$) we shall see if there is a direct product between both of these subgroups of $G$.

```plaintext
> A:= NL[3];
> B:= NL[2];
> D:=DirectProduct(A,B);
> s:=IsIsomorphic(G1,D);
> s;
true
```

Now, the best way to show $PSL(2, 23) \times C_2$ is isomorphic to $G$ is to first write a presentation for $PSL(2, 23)$.

```plaintext
> P:=PSL(2,23);
> FPGroup(P);
```
Finitely presented group on 2 generators

Relations

\[
\begin{align*}
.&1^{11} = \text{Id}(\$) \\
.&2^3 = \text{Id}(\$) \\
(&.1^{-1} \cdot .2^{-1})^4 = \text{Id}(\$) \\
(&.1 \cdot .2 \cdot .1 \cdot .2^{-1} \cdot .1^{-1} \cdot .2^{-1})^2 = \text{Id}(\$) \\
.&1^2 \cdot .2^{-1} \cdot .1^5 \cdot .2 \cdot .1 \cdot .2^{-1} \cdot .1^{-1} \\
\end{align*}
\]

\[
> \text{H<x,y>:=Group<x,y|x^{11},y^3,(x^{-1}y^{-1})^4,}
\]

\[
(x*y*x*y^-1*x^-1*y^-1)^2, \\
(x^2*y^-1*x^5*y*x*y^-1*x^-1*y*x); \\
> \text{f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);}
\]

\[
> \text{s:=IsIsomorphic(H1,PSL(2,23));}
\]

\[
> \text{s;}
\]

true

We will now add a generator of order 2 and make that generator commute with the generators of PSL(2,23).

\[
\begin{align*}
> \text{H<x,y,z>:=Group<x,y,z|x^{11},y^3,(x^{-1}y^{-1})^4,}
\]

\[
(x*y*x*y^-1*x^-1*y^-1)^2, \\
(x^2*y^-1*x^5*y*x*y^-1*x^-1*y*x,z^2,(x,z),(y,z)); \\
> \text{f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);}
\]

\[
> \text{s:=IsIsomorphic(H1,G1);}
\]

\[
> \text{s;}
\]

true

We therefore have that \( G \cong PSL(2,23) \times 2 \)
4.3 Semi-Direct Product

We will now look at the group $G = \frac{S_3 \times 2 S_3}{(yxt)^3, (yxt^2)^2, (xt)^5 = 1}$

This time, we have $G$ having the following composition factors,

```plaintext
CompositionFactors(G1);
  G
  | Cyclic(2)
  *
  | Alternating(5)
  1
```

Both $A_5$ and $C_2$ are part of the composition factors. We will now look at the normal lattice of $G$.

Normal subgroup lattice
------------------------

- [3] Order 120 Length 1 Maximal Subgroups: 2
- [2] Order 60 Length 1 Maximal Subgroups: 1
- [1] Order 1 Length 1 Maximal Subgroups:

This time, we see that the normal lattice contains a subgroup of order 60, but none of order 2. So, we do not have a direct product. However, we also will check to see the order of the center to determine if there is a central, or mixed extension.

```plaintext
> Center(G1);
Permutation group acting on a set of cardinality 10
Order = 1
```

This now implies that we do have a semi-direct product. Since have that $A_5$ is normal in $G$, we must find the element that extends $A_5$. Let us first write a presentation for $A_5$.

```plaintext
> H<a,b>:=Group<a,b|a^2,b^3,(a*b)^5>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,NL[2]);
> s;
true
> AA:=Alt(5);
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,AA);
> s;
true
```
We must now find the element of order 2 extending \( NL[2] \) of order 60 to \( G \). Once we find that element, we will use the SchreierSystem to find the action of the third generator on the first two.

\[
\text{for } i \text{ in } NL[3] \text{ do if Order(i) eq 2 and i notin NL[2] and NL[3] eq sub<NL[3]|NL[2],i> then c:=i; break; for|if> end if; end for; }
\]

\( c; (1, 2)(3, 7)(5, 9) \)

\[
\text{for } i \text{ in } [1..#Sch] \text{ do if ArrayP[i] eq G1!(2, 3, 6, 4, 8, 5) (7, 10, 9) then Sch[i]; for|if> end if; end for; x}
\]

\[
\text{for } i \text{ in } [1..#Sch] \text{ do if ArrayP[i] eq G1!(2, 4)(3, 6)(5, 8) (7, 9) then Sch[i]; for|if> end if; end for; y}
\]

After determining all the generators, we will see that conjugating by \( c \) will give us the action of the third generator. We then have the following,

\[
a^c = a
\]

\[
b^c = ab^{-1}aba^{-1}
\]

Finally, we add the third generator to the existing presentation we have that is isomorphic to \( NL[2] \). We thus have the following presentation,

\[
H\langle a, b, c \rangle := \text{Group} < a, b, c | a^2, b^3, (a^b)^5, c^2, a^c = a, b^c = a*b^{-1}*a*b*a*b^{-1} >;
\]

\[
f1, H1, k1 := \text{CosetAction} (H, \text{sub} < H | \text{Id}(H) >);
\]

\[
s := \text{IsIsomorphic} (H1, G1);
\]

\[
s;
\]

true

Therefore, we have that \( G \cong A_5 : 2 \)
4.4 Central Extension

To see if we have a central extension, we would need to compute the center of a group $G$. We would then factor by that center, and by definition, we would need to find which elements of the factor group can be written in terms of $\text{Center}(G)$. We first let $G = D_8$. with the following composition factors,

```plaintext
> G:=TransitiveGroup(8,4);
> GD:=TransitiveGroupDescription(8,4);
> GD;
D_8(8)
> CompositionFactors(G);
  G
   | Cyclic(2)
  * | Cyclic(2)
   | Cyclic(2)
   *
  1

We would then look at the normal lattice of $G$ to see where the center may be,

Normal subgroup lattice
-----------------------
[6] Order 8 Length 1 Maximal Subgroups: 3 4 5
---
---
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

We then calculate the center of $G$

```plaintext
> Center(G);
Permutation group acting on a set of cardinality 8
Order = 2
   (1, 3)(2, 8)(4, 6)(5, 7)
> Center(G) eq NL[2]
true
```

We will now need to factor $G$ by the center we just computed. We will call this quotient group $Q$, which has the following composition factors and normal lattice.
We now need to write a presentation for the quotient group $Q$. By looking at the composition factors, you see that there are two cyclic groups of order 2. Also, looking at the normal lattice of $Q$, we also notice that there are two separate normal subgroups of order 2. By definition, we could easily see that we have can write down a presentation of a direct product of $C_2$ and $C_2$.

```plaintext
> H<x,y>:=Group<x,y|x^2,y^2,(x,y)>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,Q);
> s;
true
```

As you see, $H$ is isomorphic to $Q$. We will also need to convert $x, y$ to elements in $G$ using the transversals of $NL[2]$ in $G$.

```plaintext
> A:=f1(x);
> B:=f1(y);
> T:=Transversal(G,NL[2]);
> ff(T[2]) eq A;
```
true
> ff(T[3]) eq B;
true

Since we now have that both $A$ and $B$ are equivalent to $T[2]$ and $T[3]$, we can assign them as follows, and also $C$ as the center we found earlier,

> A:=T[2];
> B:=T[3];
> C:=NL[2].1;

Before we begin to write our presentation, we have to see if any of the generators $A, B$ can be written in terms of the center $C$.

> for i in [1..2] do if A^2 eq C^i then i; end if; end for;
> for i in [1..2] do if B^2 eq C^i then i; end if; end for;
> for i in [1..2] do if (A,B) eq C^i then i; end if; end for;
> 1

As you see, $C$ can be written in terms of $(A,B)$. We will now add $C$ of order 2 to the presentation of $H$, and by definition, we will also have $C$ commute with both $A$ and $B$.

> H<a,b,c>:=Group<a,b,c|a^2,b^2,(a,b)=c,c^2,(a,c),(b,c)>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,G);
> s;
true

Therefore, we have $G \cong (2^*(2 \times 2))$. 

Appendix A: MAGMA Code for Monomial Presentation of $11^2 : (5 : 2) \cdot 5$

T := TransitiveGroups(25);
IsAbelian(T[3]);
G := T[3];
     (9, 14)(10, 15);
yy := G!(1, 7, 5, 6, 4, 10, 3, 9, 2, 8)
     (11, 24, 15, 23, 14, 22, 13, 21, 12, 25)
     (16, 18, 20, 17, 19);
CG := CharacterTable(G);
CG;

/* Displays Character Table of G */

S := Subgroups(G);
for i in [1..#S] do if Index(G, S[i]'subgroup) eq 2 then i;
    end if;
end for;
H := S[9]'subgroup;
CH := CharacterTable(H);
CH;

/* Displays Character Table of H */

I := Induction(CH[9], G);
I eq CG[12];
C := CyclotomicField(5);
A := [[C.1, 0]: i in [1..2]];
for i, j in [1..2] do A[i, j] := 0; end for;
T := Transversal(G, H);
T;
for i, j in [1..2] do if T[i] * xx * T[j]^-1 in H then
(T[i] * xx * T[j]^-1); end if; end for;
GG := GL(2, C);
B := [[C.1, 0] : i in [1..2]];
for i, j in [1..2] do B[i, j] := 0; end for;
for i, j in [1..2] do if T[i] * yy * T[j]^-1 in H then
B[i, j] := CH[9]
(T[i] * yy * T[j]^-1); end if; end for;
GG!B;
HH := sub<GG|A, B>;
IsIsomorphic(HH, G);
G2 := GL(2, 11);
N := H;
T := Transversal(G, H);

mat := function(n, p, D, k)
for i, j in [1..k] do if T[i] * p * T[j]^-1 in H then
if CH[n](T[i] * p * T[j]^-1) eq C.1 then D[i, j] := 4;
end if;
if CH[n](T[i] * p * T[j]^-1) eq C.1^2 then D[i, j] := 5;
end if;
if CH[n](T[i] * p * T[j]^-1) in {1, 0} then
D[i, j] := CH[n](T[i] * p * T[j]^-1);
end if; end if; end for;
return D;
end function;

G2 := GL(2, 11);
A2 := G2!mat(9, xx, A, 2);
A2;
B2 := G2!mat(9, yy, B, 2);
B2;
HHH := sub<G2|A2, B2>;
IsIsomorphic(G, HHH);
Order(A2);
Order(B2);
HHH;
#HHH;
C := CyclotomicField(11);
A := [[C.1, 0] : i in [1..2]];
for i, j in [1..2] do A[i, j] := 0; end for;
for i, j in [1..2] do
if $T[i]xxT[j]^{-1}$ in $H$ then
end if; end for;
for $i, j$ in $[1..2]$ do $A[i,j] := 0$;
end for;
for $i, j$ in $[1..2]$ do if $T[i]xxT[j]^{-1}$ in $H$ then
end if; end for;
$B := \begin{bmatrix} C.1, 0 \end{bmatrix}$ : $i$ in $[1..2]$;
for $i, j$ in $[1..2]$ do $B[i,j] := 0$;
end for;
for $i, j$ in $[1..2]$ do if $T[i]yyT[j]^{-1}$ in $H$ then
$B[i,j] := CH_9(T[i]yyT[j]^{-1})$;
end if; end for;
$GG := GL(2, C)$;
$T := Transversal(G, H)$;
for $i, j$ in $[1..2]$ do if $T[i]xxT[j]^{-1}$ in $H$ then
end if; end for;
$GG!A$;
Order($GG!A$);
Order($xx$);
$B := \begin{bmatrix} C.1, 0 \end{bmatrix}$ : $i$ in $[1..2]$;
for $i, j$ in $[1..2]$ do $B[i,j] := 0$;
end for;
for $i, j$ in $[1..2]$ do if $T[i]yyT[j]^{-1}$ in $H$ then
$B[i,j] := CH_9(T[i]yyT[j]^{-1})$;
end if; end for;
$GG!B$;
Order($GG!B$);

$S := Sym(20)$;
$xx := S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)$
$(15, 16)(17, 18)(19, 20)$;
$yy := S!(1, 8, 17, 6, 7, 10, 5, 2, 9, 18)$
$(3, 16, 13, 12, 15, 20, 11, 4, 19, 14)$;
$N := sub<S|xx, yy>$;
FPGroup($N$);
$NN^{<x,y>} := Group<x, y|x^2, y^{-1}xx^2, y^{-1}yy^{-1}, y^{10}, (x^y)^{-1}5>$;
Sch := SchreierSystem($NN$, sub<$NN|Id(NN)>$);
ArrayP := [Id($N$) : $i$ in $[1..50]$];
for $i$ in $[2..50]$ do
P := [Id($N$) : $l$ in $[1..\#Sch[i]]$];
for $j$ in $[1..\#Sch[i]]$ do
if Eltseq(Sch[i][j]) eq 1 then $P[j] := xx$; end if;
if Eltseq(Sch[i][j]) eq 2 then $P[j] := yy$; end if;
if Eltseq(Sch[i][j]) eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..50] do Sch[i], ArrayP[i]; end for;
Stabiliser(N,[1,3,5,7,9,11,13,15,17,19]);
Permutation group acting on a set of cardinality 25

G<x,y,t>:=Group<x,y,t|x^2,y^-1*x*y^-2*x*y^-1,y^10,t^11,
(x*y^-1)^5,t^(y^-1*x*y^-2)=t>;
Appendix B: MAGMA Code for Monomial Presentation of $11^3 : [(5 \times 5) : 3]$

T:=TransitiveGroups(25);
IsAbelian(T[6]);
G:=T[6];
xx:=G!(2, 6, 25)(3, 11, 19)(4, 16, 13)(5, 21, 7)(8, 10, 20)
  (9, 15, 14)(12, 24, 22)(17, 18, 23);
yy:=G!(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)
  (16, 17, 18, 19, 20)(21, 22, 23, 24, 25);
CG:=CharacterTable(G);
CG;

/*Displays Character Table of G*/

S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 3 then i;
end if; end for;
CH:=CharacterTable(H);
CH;

/*Displays Character Table of H*/

I:=Induction(CH[5],G);
I eq CG[7];
C:=CyclotomicField(5);
A:=[[C.1,0]: i in [1..3]];
for i,j in [1..3] do A[i,j]:=0; end for;
T:=Transversal(G,H);
T;
for i,j in [1..3] do if T[i]*xx*T[j]^(-1) in H then
A[i,j]:=CH[5](T[i]*xx*T[j]ˆ-1); end if; end for;
GG:=GL(3, C);
GG!A;
B:=[[C.1,0]: i in [1..3]];
for i, j in [1..3] do B[i,j]:=0; end for;
for i, j in [1..3] do if T[i]*yy*T[j]ˆ-1 in H then
B[i,j]:=CH[5](T[i]*yy*T[j]ˆ-1); end if; end for;
GG!B;
HH:=sub<GG|A,B>;
IsIsomorphic(HH, G);
G2:=GL(3, 11);
N:=H;
T:=Transversal(G,H);
mat := function(n,p,D,k)
for i, j in [1..k] do if T[i]*p*T[j]ˆ-1 in H then
if CH[n](T[i]*p*T[j]ˆ-1) eq C.1 then D[i,j]:=4; end if;
if CH[n](T[i]*p*T[j]ˆ-1) eq C.1^2 then D[i,j]:=5; end if;
if CH[n](T[i]*p*T[j]ˆ-1) in {1, 0} then
D[i,j]:=CH[n](T[i]*p*T[j]ˆ-1); end if;
end if; end for;
return D;
end function;
G2:=GL(3, 11);
A2:=G2!mat(5, xx, A, 3);
A2;
B2:=G2!mat(5, yy, B, 3);
B2;
HHH:=sub<G2|A2,B2>;
IsIsomorphic(G, HHH);
Order(A2);
Order(B2);
HHH;
#HHH;
S:=Sym(30);
xx:=S!(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)
(16, 17, 18)(19, 20, 21)(22, 23, 24)(25, 26, 27)
(28, 29, 30);
yy:=S!(1, 10, 13, 25, 7)(2, 14, 8, 11, 26)
(3, 15, 9, 12, 27)
(4, 22, 28, 19, 16)(5, 29, 17, 23, 20)
(6, 30, 18, 24, 21);
N:=sub<S|xx,yy>;
FPGroup(N);
NN<x,y>:=Group<x,y|xˆ3,yˆ5, (y^-1*x)^3, (y^-1*x^-1)^3>;
Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..75]];
for i in [2..75] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
  end for;
for i in [1..75] do Sch[i], ArrayP[i]; end for;
Stabiliser(N,{1,4,7,10,13,16,19,22,25,28});

G<x,y,t>:=Group<x,y,t|xˆ3,yˆ5, (y^-1*x)^3, (y^-1*x^-1)^3, t^11, t^((x*y*x^-1*y^-2)=t, t^((x^-1,y))=t^3>;
Appendix C: MAGMA Code for Monomial Presentation of $11^4 : [(5 : 4)^*5]$

```magma
T:=TransitiveGroups(25);
IsAbelian(T[7]);
G:=T[7];
xx:=G!(1, 20, 21, 8) (2, 16, 22, 9) (3, 17, 23, 10)
     (4, 18, 24, 6) (5, 19, 25, 7);
yy:=G!(1, 2, 3, 4, 5)
     (6, 19, 21, 15, 10, 18, 25, 14, 9, 17, 24, 13, 8, 16,
      23, 12, 7, 20, 22, 11);
CG:=CharacterTable(G);
CG;

/*Displays Character Table of G*/

S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 4 then i;
end if; end for;
H:=S[9]`subgroup;
CH:=CharacterTable(H);
CH;

/*Displays Character Table of H*/

I:=Induction(CH[9],G);
I eq CG[25];
true
C:=CyclotomicField(5);
A:=[[C.1,0]: i in [1..4]];
for i,j in [1..4] do A[i,j]:=0; end for;
T:=Transversal(G,H);
```
for i, j in [1..4] do if \( T[i] \cdot xx \cdot T[j]^{-1} \) in \( H \) then 
\[ A[i,j] := CH[9](T[i] \cdot xx \cdot T[j]^{-1}); \]
end if; end for;

\[ GG := GL(4, C); \]
\[ GG!A; \]

\[ B := [[C.1, 0] \colon i \in [1..4]]; \]
for i, j in [1..4] do \( B[i,j] := 0 \); end for;
for i, j in [1..4] do if \( T[i] \cdot yy \cdot T[j]^{-1} \) in \( H \) then 
\[ B[i,j] := CH[9](T[i] \cdot yy \cdot T[j]^{-1}); \]
end if; end for;

\[ GG!B; \]

\[ HH := sub<GG|A,B>; \]
\[ IsIsomorphic(HH, G); \]

\[ G2 := GL(4, 11); \]
\[ N := H; \]
\[ T := Transversal(G, H); \]

\[ mat := function(n, p, D, k) \]
for i, j in [1..k] do if \( T[i] \cdot p \cdot T[j]^{-1} \) in \( H \) then 
if \( CH[n](T[i] \cdot p \cdot T[j]^{-1}) \) eq C.1 then \( D[i,j] := 4; \) end if;
if \( CH[n](T[i] \cdot p \cdot T[j]^{-1}) \) eq C.1^2 then \( D[i,j] := 5; \) end if;
if \( CH[n](T[i] \cdot p \cdot T[j]^{-1}) \) eq C.1^3 then \( D[i,j] := 9; \) end if;
if \( CH[n](T[i] \cdot p \cdot T[j]^{-1}) \) eq C.1^4 then \( D[i,j] := 3; \) end if;
end if; end for;
return D;
end function;

\[ G2 := GL(4, 11); \]
\[ A2 := G2!mat(9, xx, A, 4); \]
\[ A2; \]
\[ B2 := G2!mat(9, yy, B, 4); \]
\[ B2; \]
\[ HHH := sub<G2|A2,B2>; \]
\[ IsIsomorphic(G, HHH); \]
\[ Order(A2); \]
\[ Order(B2); \]
\[ HHH; \]

\[ S := Sym(40); \]
\[ xx := S!(1, 2, 4, 3)(5, 6, 8, 7)(9, 10, 12, 11)(13, 14, 16, 15) \]
\[ (17, 18, 20, 19)(21, 22, 24, 23)(25, 26, 28, 27) \]
\[ (29, 30, 32, 31) \]
\[ (33, 34, 36, 35)(37, 38, 40, 39); \]
\[ yy := S!(1, 3, 36, 18, 33, 35, 16, 2, 13, 15, 12, 34, 9, 11, 20, \]
\(14, 17, 19, 4, 10\)
\((5, 7, 28, 38, 25, 27, 32, 6, 29, 31, 24, 26, 21, 23, 40, 30, 37, 8, 22)\);
N:=sub<\langle xx,yy \rangle>;
FPGroup(N);
NN<x,y>:=Group<x,y|x^4,y*x^-1*y^-1*x^-1*y^-1*x^2*y,
  x^-1*y^-2*x^-1*y*x^2*y,
y^-3*x^2*y^2*2*x>;
Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..100]];
for i in [2..100] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
for i in [1..100] do Sch[i], ArrayP[i]; end for;
Stabiliser(N,\{1,5,9,13,17,21,25,29,33,37\});
Permutation group acting on a set of cardinality 40
G<x,y,t>:=Group<x,y,t|x^4,y*x^-1*y^-1*x^-1*y^-1*x^2*y,
x^-1*y^-2*x^-1*y*x^2*y,
y^-3*x^2*y^2*2*x,t^11,t^(x^-1*y^-1)=t,t^(x*y^2*x)=t^4>;
Appendix D: MAGMA Code for the DCE of $S_5$ over $D_{12}$

S:=Sym(6);
xx:=S!(1, 2, 3, 4, 5, 6);
yy:=S!(1, 4)(2, 3)(5, 6);
N:=sub<S|xx,yy>;
a1:=0; b1:=0; c1:=0; d1:=3; e1:=0; f1:=0; a2:=5;
G<x,y,t>:=Group<x,y,t|y²,(x⁻¹*y)²,x⁶,t²,t^(x⁻³*y)=t,
(x²*t)⁻¹, (y*t⁻¹*x)⁻¹,c1,(y*x*t⁻¹(x²))⁻¹,e1,
(x*x*t⁻¹)⁻¹,f1,(x*t⁻¹)⁻¹*a2>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#sub<G|x,y>;
#k;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i][j]) eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i][j]) eq -1 then P[j]:=f(x)⁻¹; end if;
if Eltseq(Sch[i][j]) eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i][j]) eq 3 then P[j]:=f(t); end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
IN:=sub<G1|f(x),f(y)>;
NL:=NormalLattice(G1);
NL;
s:=IsIsomorphic(G1,Sym(5));
\texttt{prodim := function(pt, Q, I)}
\texttt{v := pt;}
\texttt{for i in I do}
\texttt{v := v^Q[i];}
\texttt{end for;}
\texttt{return v;}
\texttt{end function;}

\texttt{ts := [Id(G1): i in [1 .. 6];}
\texttt{ts[1]:=f(t);}
\texttt{ts[2]:=f(t^x);}
\texttt{ts[3]:=f(t^(x^2));}
\texttt{ts[4]:=f(t^y);}
\texttt{ts[5]:=f(t^(x^-2));}
\texttt{ts[6]:=f(t^(x^-1));}
\texttt{cst := [null : i in [1 .. 10]] where null is [Integers() | ];}
\texttt{for i := 1 to 6 do}
\texttt{cst[prodim(1, ts, [i])] := [i];}
\texttt{end for;}
\texttt{m:=0;}
\texttt{for i in [1..10] do if cst[i] ne [] then m:=m+1;}
\texttt{end if; end for; m;}
\texttt{Orbits(N);}

\texttt{N1:=Stabiliser(N,1);}
\texttt{#N/#N1;}
\texttt{Orbits(N1);}

\texttt{N12:=Stabiliser(N,[1,2]);}
\texttt{SSS:=[(1,2)];}
\texttt{SSS:=SSS\^{N};}
\texttt{#(SSS);}
\texttt{Seqq:=Setseq(SSS);}
\texttt{Seqq;}
\texttt{for i in [1..#SSS] do}
\texttt{for n in IN do}
\texttt{if ts[1]*ts[2] eq}
\texttt{n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]}
then print Rep(Seqq[i]);
end if; end for; end for;
N12; #N12;
N12s:=N12;
for g in N do if 1^g eq 5 and 2^g eq 4 then 
N12s:=sub<N|N12s,g>; end if;
end for;
for g in N do if 1^g eq 4 and 2^g eq 5 then 
N12s:=sub<N|N12s,g>; end if;
end for;
#N12s; N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do 
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne [] 
then m:=m+1; end if; end for; m;
Orbits(N12s);

/* Coset Equality Loop */

/*******************************************************************************/
for g in IN do for h in IN do if ts[n]*ts[n]*... eq 
g*(ts[n]*ts[n]*...)^h
then "true"; gg:=g; hh:=h; break; end if; end for; end for;
/******************************************************************************/

N16:=Stabiliser(N,[1,6]);
SSS:={[1,6]};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do 
for n in IN do
if ts[1]*ts[6] eq 
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
Appendix E: MAGMA Code for DCE of $S_6$ over 5:4

```
S:=Sym(5);
xx:=S!(1, 2, 3, 4, 5);
yy:=S!(1, 2, 4, 3);
N:=sub<S|xx,yy>;
a1:=0; b1:=0; c1:=0; d1:=5; e1:=0; f1:=0; a2:=0;
G<x,y,t>:=Group<x,y,t|y^4,x^-5,y^-1*x^-2*y*x^-1,t^2,
t^*(y*x*y)=t,t^*(y*x^-1)*t=a1,
(x*y^-1*t*(x))^-1, (y*x^-1*t)^c1, (y*x^-1*t*(x))^-d1,
(y*x*y*t*(x))^-e1, (y*x*y*t)^f1, (x*t)^a2>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#sub<G|x,y>;
#k;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
  P:=[Id(G1): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
    if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
  end for;
  PP:=Id(G1);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
  end for;
IN:=sub<G1|f(x),f(y)>;
s:=IsIsomorphic(G1,Sym(6));
```
s;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#N;
#G/#N;
prodim := function(pt, Q, I)
v := pt;
for i in I do
    v := vˆQ[i];
end for;
return v;
end function;
ts := [Id(G1): i in [1 .. 5] ];
ts[1] := f(t);
ts[2] := f(tˆ(x));
ts[3] := f(tˆ(x^2));
ts[4] := f(tˆ(x^3));
ts[5] := f(tˆ(x^4));
cst := [null : i in [1 .. 36]] where null is [Integers() | ];
for i := 1 to 5 do
    cst[prodim(1, ts, [i])] := [i];
end for;
m := 0;
for i in [1 .. 36] do if cst[i] ne [] then m := m + 1;
end if; end for; m;
Orbits(N);
N1 := Stabiliser(N, 1);
#N/#N1;
Orbits(N1);
N12 := Stabiliser(N, [1, 2]);
SSS := ([1, 2]);
SSS := SSS \ N;
#(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1 .. #SSS] do
    for n in IN do
            n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
N12; #N12;
N12s := N12;
\#N12s; N12s;
T12:=\text{Transversal}(N,N12s);
for i in [1..\#T12] do
    ss:=[1,2]^{T12[i]};
    \text{cst}[\text{prodim}(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..36] do if \text{cst}[i] \neq []
then m:=m+1; end if; end for; m;
\text{Orbits}(N12s);

N121:=\text{Stabiliser}(N,[1,2,1]);
SSS:=[(1,2,1)];
SSS=SSS^N;
\#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..\#SSS] do
    for n in IN do
        if ts[1]*ts[2]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
N121; \#N121;
N121s:=N121;
for g in N do if 1^g eq 3 and 2^g eq 5 and 1^g eq 3 then
N121s:=\text{sub}<N|N121s,g>; end if; end for;
\#N121s; N121s;
T121:=\text{Transversal}(N,N121s);
for i in [1..\#T121] do
    ss:=[1,2,1]^{T121[i]};
    \text{cst}[\text{prodim}(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..36] do if \text{cst}[i] \neq []
then m:=m+1; end if; end for; m;
\text{Orbits}(N121s);

N123:=\text{Stabiliser}(N,[1,2,3]);
SSS:=[(1,2,3)];
SSS=SSS^N;
\#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..\#SSS] do
    for n in IN do
n \cdot ts[Rep(Seqq[i])[1]] \cdot ts[Rep(Seqq[i])[2]] \cdot ts[Rep(Seqq[i])[3]]
then print \( \text{Rep(Seqq[i])} \);
end if; end for; end for;
N123; #N123;
N123s:=N123;
for \( g \) in \( N \) do if \( 1^g \) eq 1 and \( 2^g \) eq 3 and \( 3^g \) eq 5 then 
N123s:=sub<N|N123s,g>;
end if; end for;
#N123s; N123s;
T123:=Transversal(N,N123s);
for \( i \) in \([1..#T123]\) do
ss:=[1,2,3]^T123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for \( i \) in \([1..36]\) do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N123s);
Appendix F: MAGMA Code for DCE of $S_5$ over $D_8$

```magma
S:=Sym(8);
x:=S!(1, 5, 7, 3)(2, 4, 8, 6);
y:=S!(1, 2)(3, 4)(5, 6)(7, 8);
N:=sub<S|xx,yy>;
a1:=0; b1:=0; c1:=0; d1:=0; e1:=3; f1:=0; a2:=0;
G<x,y,t>:=Group<x,y,t|y^2,x^4,(x^-1*y)^2,t^5,t^2(x^-1)=t^2,
(x^2*t)^a1,(y*t^x)^b1,(y*t)^c1,(x*y*t^y)^d1,(x*y*t)^e1,
(x^t*y)^f1,(x^t)^a2>;
#G;
f,Gl,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#sub<G|x,y>;
#k;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
IN:=sub<G1|f(x),f(y)>;
s:=IsIsomorphic(G1, Sym(5));
s;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
```
#N;
#G/#N;
prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
  return v;
end function;

P := [Id(G) : i in [1 .. 12]]; 
P[1] := f(t);
P[2] := f(t^y);
P[3] := f(t^(x*y));
P[4] := f(t^y);
P[5] := f(t^(y*x));
P[6] := f(t^(-y+1));
P[7] := f(t^(-y+1));
P[8] := f(t^(-y+1));
cst := [null : i in [1 .. 15]] where null is [Integers() | ];
for i := 1 to 8 do
  cst[prodim(1, P, [i])] := [i];
end for;
m := 0;
for i in [1 .. 15] do
  if cst[i] ne [] then m := m+1;
end if;
end for;
Orbits(N);

N1 := Stabiliser(N, 1);
#N/#N1;
Orbits(N1);

N12 := Stabiliser(N, [1, 2]);
SSS := ([1, 2]);
N := SSS^N;
(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1 .. #SSS] do
  for n in IN do
    n * P[Rep(Seqq[i])[1]] * P[Rep(Seqq[i])[2]]
    then print Rep(Seqq[i]);
  end if;
end for;
end for;
N12; #N12;
\[ N_{12s} := N_{12}; \]
for \( g \) in \( N \) do if \( 1^g \) eq 8 and \( 2^g \) eq 7 then
\[ N_{12s} := \text{sub}<N|N_{12s},g>; \]
end if; end for;
\#\( N_{12s}; \) \( N_{12s}; \)
\[ T_{12} := \text{Transversal}(N,N_{12s}); \]
for \( i \) in \([1..\#T_{12}]\) do
\[ ss := [1,2]^{T_{12}[i]}; \]
cst[\text{prodim}(1, ts, ss)] := ss;
end for;
m := 0; for \( i \) in \([1..15]\) do if \( \text{cst}[i] \) ne \([\ ]\) then m := m+1; end if; end for; m;
\text{Orbits}(N_{12s});

\[ N_{16} := \text{Stabiliser}(N,[1,6]); \]
\[ SSS := \{[1,6]\}; \]
\[ SSS := SSS \cup N; \]
\#(SSS);
\text{Seqq} := \text{Setseq}(SSS);
\text{Seqq};
for \( i \) in \([1..\#SSS]\) do
for \( n \) in \( IN \) do
if \( ts[1]*ts[6] \) eq
\[ n*ts[\text{Rep}(	ext{Seqq}[i])[1]]*ts[\text{Rep}(	ext{Seqq}[i])[2]] \]
then print \text{Rep(Seqq}[i]);
end if; end for; end for;
\text{Orbits}(N_{12s});

\[ N_{16s} := N_{16}; \]
\[ N_{16s} := \text{Stabiliser}(N,[1,6]); \]
\[ SSS := \{[1,6]\}; \]
\[ SSS := SSS \cup N; \]
\#(SSS);
\text{Seqq} := \text{Setseq}(SSS);
\text{Seqq};
for \( i \) in \([1..\#SSS]\) do
for \( n \) in \( IN \) do
if \( ts[1]*ts[6] \) eq
\[ n*ts[\text{Rep}(	ext{Seqq}[i])[1]]*ts[\text{Rep}(	ext{Seqq}[i])[2]] \]
then print \text{Rep(Seqq}[i]);
end if; end for; end for;
\text{Orbits}(N_{16s});

\[ N_{18} := \text{Stabiliser}(N,[1,8]); \]
\[ SSS := \{[1,8]\}; \]
\[ SSS := SSS \cup N; \]
\#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if \(ts[1]*ts[8] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]\)
then print Rep(Seqq[i]);
end if; end for; end for;
N18; #N18;
N18s:=N18;
for g in N do if 1^g eq 7 and 8^g eq 2 then
N18s:=sub<N|N18s,g>;
end if; end for;
for g in N do if 1^g eq 6 and 8^g eq 5 then
N18s:=sub<N|N18s,g>;
end if; end for;
for g in N do if 1^g eq 4 and 8^g eq 3 then
N18s:=sub<N|N18s,g>;
end if; end for;
#N18s; N18s;
T18:=Transversal(N,N18s);
for i in [1..#T18] do
ss:=[1,8]^T18[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N18s);
Appendix G: MAGMA Code for DCE of $A_5 \times A_5$ over $(5 : 2)^5$

```magma
S:=Sym(25);
x:=S!(1, 20)(2, 16)(3, 17)(4, 18)(5, 19)(6, 11)(7, 12)(8, 13)(9, 14)(10, 15);
y:=S!(1, 7, 5, 6, 4, 10, 3, 9, 2, 8)(11, 24, 15, 23, 14, 22, 13, 21, 12, 25)(16, 18, 20, 17, 19);
N:=sub<S|xx, yy>;
a1:=0; b1:=3;
G<x,y,t>:=Group<x, y, t | x^2, y^10, (x*y)^5, (y^-1*x*y^2*x*y^-1), t^2, t^((x*y*x*y^4)^t), (y^-1*(y*x))^t*a1, (y^-1*(y*x))^t*b1>;
f, G1, k := CosetAction(G, sub<G|x, y>);
CompositionFactors(G1);
Sch:=SchreierSystem(G, sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
  P:=[Id(G1): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j] := f(x); end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j] := f(y); end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j] := f(y)^-1; end if;
    if Eltseq(Sch[i])[j] eq 3 then P[j] := f(t); end if;
  end for;
  PP := Id(G1);
  for k in [1..#P] do
    PP := PP*P[k];
  end for;
  ArrayP[i] := PP;
end for;
NL:=NormalLattice(G1);
G<x,y,t>:=Group<x, y, t | x^2, y^10, (x*y)^5, (y^-1*x*y^2*x*y^-1), t^2, t^((x*y*x*y^4)^t), (y^-1*(y*x))^3, x*y*x*y*x*t*y*x*t*y^-3*t*y*t>;
f, G1, k := CosetAction(G, sub<G|x, y>);
```

IN:=sub<\text{G}1|f(x), f(y)>;
CompositionFactors(G1);
s:=IsIsomorphic(G1, DirectProduct(Alt(5), Alt(5))); s;
#N;
#G/#N;
#DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^\langle Q[i]\rangle;
  end for;
  return v;
end function;
ts := [Id(G1): i in [1 .. 25]];
for i := 1 to 25 do
  ts[i] := prodim(1, ts, [i]);
end for;
cst := [null : i in [1 .. Index(G, sub<G|x,y>)]]
where null is [Integers() | ];
for i := 1 to 25 do
  cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..72] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N);

N1:=Stabiliser(N,1);
#N/#N1;
Orbits(N1);

N12:=Stabiliser(N,[1,2]);
SSS:={[1,2]};
SSS=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12; #N12;
N12s:=N12;
for g in N do if 1^g eq 20 and 2^g eq 16 then
N12s:=sub<N|N12s,g>;
end if; end for;
#N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12s);

/*****************************/
for g in IN do for h in IN do if ts[1]*ts[7] eq g*(ts[1])^h
then "true"; gg:=g; hh:=h; break; end if; end for; end for;
****************************************************************************/
N12s:=sub\langle N | N12s, g \rangle;
end if; end for
*****************************************************************************
for i in [1..#N] do
if ArrayP[i] eq gg then Sch[i]; end if; end for;
*****************************************************************************

N13:=Stabiliser(N,[1,3]);
SSS:={[1,3]};
SSS:=SSS\N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3] eq 
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N13; #N13;
N13s:=N13;
for g in N do if 1^g eq 20 and 3^g eq 17 then
N13s:=sub\langle N | N13s, g \rangle;
end if; end for;
#N13s;
T13:=Transversal(N,N13s);
for i in [1..#T13] do
ss:=[1,3]^T13[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N13s);

N16:=Stabiliser(N,[1,6]);
SSS:={[1,6]};
SSS:=SSS\N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[6] eq 
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N16; #N16;
then print Rep(Seqq[i]);
end if; end for; end for;
N16; #N16;
N16s:=N16;
for g in N do if $1^g \equiv 21$ and $6^g \equiv 16$ then
N16s:=sub<N\g>;<N|N16s,g>;
end if; end for;
#N16s;
T16:=Transversal(N,N16s);
for i in [1..#T16] do
ss:=[1,6]^T16[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N16s);

N17:=Stabiliser(N,[1,7]);
SSS:={(1,7)};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[7] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N17; #N17;
N17s:=N17;
for g in N do if $1^g \equiv 21$ and $7^g \equiv 17$ then
N17s:=sub<N\g>;<N|N17s,g>;
end if; end for;
#N17s;
T17:=Transversal(N,N17s);
for i in [1..#T17] do
ss:=[1,7]^T17[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N17s);
N110:=Stabiliser(N,[1,10]);
SSS:={[1,10]};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[10] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N110; #N110;
N110s:=N110;
for g in N do if 1^g eq 5 and 10^g eq 9 then
N110s:=sub<N|N110s,g>;
end if; end for;
for g in N do if 1^g eq 25 and 10^g eq 19 then
N110s:=sub<N|N110s,g>;
end if; end for;
N110s;
T110:=Transversal(N,N110s);
for i in [1..#T110] do
ss:=[1,10]^T110[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N110s);

/*Length 3 Words*/

N121:=Stabiliser(N,[1,2,1]);
SSS:={(1,2,1)};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N121; #N121;
N121s:=N121;
for g in N do if 1^g eq 5 and 2^g eq 1 and 
1^g eq 5 then
N121s:=sub<N|N121s,g>;
end if; end for;
#N121s;
T121:=Transversal(N,N121s);
for i in [1..#T121] do
ss:=[1,2,1]^T121[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N121s);

N1615:=Stabiliser(N,[1,6,15]);
SSS:={(1,6,15)};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[6]*ts[15] eq 
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1615; #N1615;
N1615s:=N1615;
for g in N do if 1^g eq 20 and 6^g eq 11 
and 15^g eq 10 then
N1615s:=sub<N|N1615s,g>;
for g in N do if 1^g eq 7 and 6^g eq 4 
and 15^g eq 23 then
N1615s:=sub<N|N1615s,g>;
end if; end for;
#N1615s;
T1615:=Transversal(N,N1615s);
for i in [1..#T1615] do
ss:=[1,6,15]^T1615[i];
cst[prodim(1, ts, ss)]:=ss;
m := 0; for i in [1..72] do if cst[i] ne [] then m := m + 1; end if; end for; m;

Orbits(N1615s);
Appendix H: MAGMA Code for the DCE of $PSL(2,17)$ over Maximal Subgroup $A_4$

S:=Sym(12);
xx:=S!(1, 9, 5)(2, 4, 3)(6, 8, 7)(10, 12, 11);
yy:=S!(1, 11, 6)(2, 9, 7)(3, 10, 5)(4, 8, 12);
N:=sub<S|xx,yy>;
a1:=0; b1:=0; c1:=0; d1:=0; e1:=0; f1:=0; a2:=0; b2:=0;
c2:=4; d2:=4; e2:=3;
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y^{-1})^2,t^2,
(x*y*t)^a1,(x*y*t*x)^b1,(x*y*t*y)^c1,(x*t)^d1,
(x*t*(x*y)^{-1})^e1,(x*t*(y^2))^f1,(x*t*(y))^a2,
(x^2*t)^b2,(x^2*t*(x*y)^{-1})^c2,(x^2*t*(y^2))^d2,
(x^2*t*(y))^e2>;
H:=sub<G|x,y,x*t*x*t*y^{-1}*t*x*t*x^{-1}*t>;
f,G1,k:=CosetAction(G,H);
CompositionFactors(G1);
#sub<G|x,y>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^{-1}; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^{-1}; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
IH:=sub<G1|f(x),f(y),f(x*t*x*t*y^-1+t*x*t*x^-1+t)>;
IN:=sub<G1|f(x),f(y)>;
NL:=NormalLattice(G1);
#sub<G1,x,y>;
#k;
#DoubleCosets(G,H,sub<G1,x,y>);
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
ts := [Id(G1): i in [1..12]];
ts[1]:=f(t);
ts[2]:=f(t^(x*y^2));
ts[3]:=f(t^((y^2*x)^2));
ts[4]:=f(t^(x*y^2*x));
ts[5]:=f(t^(x^2));
ts[6]:=f(t^(y^2));
ts[7]:=f(t^(x*y));
ts[8]:=f(t^((y^2*x)^2));
ts[9]:=f(t^((x*y)^2));
ts[10]:=f(t^(y*x));
ts[11]:=f(t^((y^2*x)^2));
ts[12]:=f(t^((y*x)^2));
cst := [null: i in [1..102]] where null is [Integers() | ];
for i := 1 to 12 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..102] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

/* Use this loop to determine equality between cosets */

/*****************************/
for g in IH do for h in IN do if ts[n]*ts[n]*... eq g*(ts[n]*ts[n]...)^h
then "true"; gg:=g; hh:=h; break; end if; end for; end for;

***********************************************************

N12:=Stabiliser(N,[1,2]);
SS:={[1,2]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12; #N12;
N12s:=N12;
for g in N do if 1^g eq 1 and 2^g eq 12 then
N12s:=sub<N|N12s,g>;
end if; end for;
#N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12);

N14:=Stabiliser(N,[1,4]);
SS:={[1,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
\begin{verbatim}
N14; #N14;
N14s:=N14;
#N14s;
T14:=Transversal(N,N14s);
for i in [1..#T14] do
   ss:=[1,4]^T14[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
   then m:=m+1; end if; end for; m;
Orbits(N14s);

N15:=Stabiliser(N,[1,5]);
SS:=[[1,5]];
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
   for n in IH do
      if ts[1]*ts[5] eq
         n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      then print Rep(Seqq[i]);
      end if; end for;
end for;
N15; #N15;
N15s:=N15;
#N15s;
T15:=Transversal(N,N15s);
for i in [1..#T15] do
   ss:=[1,5]^T15[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
   then m:=m+1; end if; end for; m;
Orbits(N15s);

N16:=Stabiliser(N,[1,6]);
SS:=[[1,6]];
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
\end{verbatim}
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[6] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N16; #N16;
N16s:=N16;
#N16s;
T16:=Transversal(N,N16s);
for i in [1..#T16] do
ss:=[1,6]^T16[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N16s);

N17:=Stabiliser(N,[1,7]);
SS:={[1,7]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[7] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N17; #N17;
N17s:=N17;
#N17s;
T17:=Transversal(N,N17s);
for i in [1..#T17] do
ss:=[1,7]^T17[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N17s);

N19:=Stabiliser(N,[1,9]);
SS:={[1,9]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[9] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N19; #N19;
N19s:=N19;
#N19s;
T19:=Transversal(N,N19s);
for i in [1..#T19] do
ss:=[1,9]^T19[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N19s);

N110:=Stabiliser(N,[1,10]);
SS:={{1,10}};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[10] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N110; #N110;
N110s:=N110;
for g in N do if 1^g eq 10 and 10^g eq 1 then
N110s:=sub<N|N110s,g>;
end if; end for;
N110s;
#N110s;
\[ T_{110} := \text{Transversal}(N, N_{110s}); \]
for \( i \) in \([1..\#T_{110}]\) do
\[ \text{ss} := [1,10]^T_{110[i]}; \]
\[ \text{cst}[\text{prodim}(1, \text{ts}, \text{ss})] := \text{ss}; \]
end for;
\[ m := 0; \text{for} \ i \ \text{in} \ [1..102] \ \text{do} \text{if} \ \text{cst}[i] \neq [] \ \text{then} \ m := m + 1; \ \text{end if}; \ \text{end for}; \ m; \]
\text{Orbits}(N_{110s});

\[ N_{121} := \text{Stabiliser}(N, [1,2,1]); \]
\[ \text{SS} := \{[1,2,1]\}; \]
\[ \text{SS} := \text{SS}^N; \]
\[ \#\text{SS}; \]
\[ \text{Seqq} := \text{Setseq}(\text{SS}); \]
\[ \text{Seqq}; \]
for \( i \) in \([1..\#\text{SS}]\) do
for \( n \) in \( \text{IH} \) do
if \( \text{ts}[1]\ast\text{ts}[2]\ast\text{ts}[1] \equiv \]
\[ n\ast\text{ts}[\text{Rep}(\text{Seqq}[i])[1]]\ast\text{ts}[\text{Rep}(\text{Seqq}[i])[2]] \]
\[ \ast\text{ts}[\text{Rep}(\text{Seqq}[i])[3]] \]
then print \( \text{Rep}(\text{Seqq}[i]) \);
end if; end for; end for;
\[ N_{121s} := N_{121}; \]
for \( g \) in \( N \) do if \( 1^g \equiv 2 \text{ and } 2^g \equiv 1 \text{ and } 1^g \equiv 2 \text{ then} \]
\[ N_{121s} := \text{sub}<N|N_{121s},g>; \]
end if; end for;
\[ N_{121s}; \ #N_{121s}; \]
\[ T_{121} := \text{Transversal}(N, N_{121s}); \]
for \( i \) in \([1..\#T_{121}]\) do
\[ \text{ss} := [1,2,1]^T_{121[i]}; \]
\[ \text{cst}[\text{prodim}(1, \text{ts}, \text{ss})] := \text{ss}; \]
end for;
\[ m := 0; \text{for} \ i \ \text{in} \ [1..102] \ \text{do} \text{if} \ \text{cst}[i] \neq [] \ \text{then} \ m := m + 1; \ \text{end if}; \ \text{end for}; \ m; \]
\text{Orbits}(N_{121s});

\[ N_{147} := \text{Stabiliser}(N, [1,4,7]); \]
\[ \text{SS} := \{[1,4,7]\}; \]
\[ \text{SS} := \text{SS}^N; \]
\[ \#\text{SS}; \]
\[ \text{Seqq} := \text{Setseq}(\text{SS}); \]
\[ \text{Seqq}; \]
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N147s:=N147;
for g in N do if 1^g eq 7 and 4^g eq 10 and 7^g eq 1
then N147s:=sub<N|N147s,g>;
end if; end for;
for g in N do if 1^g eq 4 and 4^g eq 1 and 7^g eq 10 then
N147s:=sub<N|N147s,g>;
end if; end for;
N147s; #N147s;
N147s:=Transversal(N,N147s);
for i in [1..#T147] do
ss:=[1,4,7]^T147[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N147s);

N161:=Stabiliser(N,[1,6,1]);
SS:={[1,6,1]};
SS:=SS^N;

#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[6]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N161s:=N161;
for g in N do if 1^g eq 11 and 6^g eq 1 and 1^g eq 11 then
N161s:=sub<N|N161s,g>;
end if; end for;
N161s; #N161s;
T161:=Transversal(N,N161s);
for i in [1..#T161] do
ss:=\[1,6,1\]^T161[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..102] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N161s);
Appendix I: MAGMA Code for the DCE of Sporadic Group $M_{22}$ over Maximal Subgroup $A_7$

```
S:=Sym(7);
xx:=S!(1, 2, 3, 4, 5, 6, 7);
yy:=S!(1, 2, 4)(3, 6, 5);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|y^3,y*x^-2*y^-1*x,t^2,t^(y*x^-1)=t,
(y*x^-1*t^(-(x^-3)))^5,(x*t)^8,(x^-3*t)^8,(y*t)^11>;
H:=sub<G|x,y,y*x^-1*y^-1*t*x*y*t*y^-1*t*x^-1*t*y*x>;
G1,k:=CosetAction(G,H);
CompositionFactors(G1);
#sub<G|x,y>;
#k;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
  P:=[Id(G1): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
    if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
  end for;
P:=Id(G1);
for k in [1..#P] do
  PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
IH:=sub<G1|f(x),f(y),f(y*x^-1*y^-1*t*x*y*t*y^-1*t*x^-1*t*y*x)>;
```
IN:=sub<Gl| f(x), f(y)>;
NL:=NormalLattice(G1);
#DoubleCosets(G,H,sub<G|x,y>);
#H;
#G/#H;
prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^Q[i];
  end for;
  return v;
end function;
ts := [Id(G1): i in [1 .. 7] ];
ts[1]:=f(t);
ts[2]:=f(t^x);
ts[3]:=f(t^x^2);
ts[4]:=f(t^x^3);
ts[5]:=f(t^x^4);
ts[6]:=f(t^x^5);
ts[7]:=f(t^x^6);
cst := [null : i in [1 .. 176]] where null is [Integers() | ];
for i := 1 to 7 do
  cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..176] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N1:=Stabiliser(N,1);
#N/#N1;
Orbits(N1);
N12:=Stabiliser(N,[1,2]);
SS:={[1,2]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12; #N12;
N12s:=N12;
#N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12s);
N14:=Stabiliser(N,[1,4]);
SS:={[1,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N14; #N14;
N14s:=N14;
#N14s;
T14:=Transversal(N,N14s);
for i in [1..#T14] do
ss:=[1,4]^T14[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N14s);
N123:=Stabiliser(N,[1,2,3]);
SS:={[1,2,3]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N123; #N123;
N123s:=N123;
#N123s;
T123:=Transversal(N,N123s);
for i in [1..#T123] do
ss:=[1,2,3]^T123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N123s);

N124:=Stabiliser(N,[1,2,4]);
SS:={[1,2,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N124; #N124;
N124s:=N124;
for g in N do if 1^g eq 2 and 2^g eq 4 and
4^g eq 1 then
N124s:=sub<N|N124s,g>;
for g in N do if 1^g eq 4 and 2^g eq 1 and
4^g eq 2 then
N124s:=sub<N|N124s,g>;
end if; end for; end if; end for;
N124s;  #N124s;
T124:=Transversal(N,N124s);
for i in [1..#T124] do
  ss:=[1,2,4]^T124[i];
  cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N124s);

N125:=Stabiliser(N,[1,2,5]);
SS:={[1,2,5]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    if ts[1]*ts[2]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N125;  #N125;
N125s:=N125;
N125s;  #N125s;
T125:=Transversal(N,N125s);
for i in [1..#T125] do
  ss:=[1,2,5]^T125[i];
  cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N125s);

N127:=Stabiliser(N,[1,2,7]);
SS:={[1,2,7]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2]*ts[7] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N127; #N127;
N127s:=N127;
N127s; #N127s;
T127:=Transversal(N,N127s);
for i in [1..#T127] do
ss:=[1,2,7]^T127[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N127s);

N146:=Stabiliser(N,[1,4,6]);
SS:={[1,4,6]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4]*ts[6] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N146; #N146;
N146s:=N146;
N146s; #N146s;
T146:=Transversal(N,N146s);
for i in [1..#T146] do
ss:=[1,4,6]^T146[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..176] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N146s);
N147 := Stabiliser(N, [1, 4, 7]);
SS := {[1, 4, 7]};
SS := SS^N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4]*ts[7] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N147; #N147;
N147s := N147;
N147s; #N147s;
T147 := Transversal(N, N147s);
for i in [1..#T147] do
ss := [1, 4, 7]^T147[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..176] do if cst[i] ne []
then m := m + 1; end if; end for; m;
Orbits(N147s);

N1236 := Stabiliser(N, [1, 2, 3, 6]);
SS := {[1, 2, 3, 6]};
SS := SS^N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2]*ts[3]*ts[6] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1236; #N1236;
N1236s := N1236;
for g in N do if 1^g eq 5 and 2^g eq 2
and $3^g = 6$
and $6^g = 4$ then
N1236s := sub<N|N1236s,g>;
end if; end for;
N1236s; #N1236s;
T1236 := Transversal(N,N1236s);
for i in [1..#T1236] do
ss := [1,2,3,6]^T1236[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..176] do if cst[i] ne []
then m := m+1; end if; end for; m;
Orbits(N1236s);

N1271 := Stabiliser(N,[1,2,7,1]);
SS := {[1,2,7,1]};
SS := SS^N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2]*ts[7]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1271; #N1271;
N1271s := N1271;
for g in N do if $1^g = 5$ and $2^g = 2$ and
$7^g = 1$
and $1^g = 5$ then
N1271s := sub<N|N1271s,g>;
end if; end for;
N1271s; #N1271s;
T1271 := Transversal(N,N1271s);
for i in [1..#T1271] do
ss := [1,2,7,1]^T1271[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..176] do if cst[i] ne []
then m := m+1; end if; end for; m;
Orbits(N1271s);
Appendix J: MAGMA Code for the DCE of Sporadic Group $J_1$ over Maximal Subgroup $PSL(2, 11)$

```magma
S := Sym(12);
x := S!(1, 9, 7, 11, 3, 5)(2, 6, 4, 12, 8, 10);
xx := S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12);
N := sub<S | xx, yy >;
G<x, y, t> := Group<x, y, t | y^2, (x^-1*y)^2, x^6*t^7, t*(x^-1) = t^5, y*t*y*t^2*y*t^2*y*t^-1*y*t^2, (x*y*t)^5, (y*t)^10>;
H := sub<G | x, y^-1*t^-1*y*t^-1*y*t*y*t*x*y*t*x >;
G1, k := CosetAction(G, H);
CompositionFactors(G1);
IH := sub<G1 | f(x), f(y), f(x^-1*t^-1*y*t^-1*y*t*y*t*x*y*t*x) >;
IN := sub<G1 | f(x), f(y) >;
NL := NormalLattice(G1);
#DoubleCosets(G, H, sub<G | x, y >);
#H;
#G/#H;
prodim := function(pt, Q, I)
v := pt;
for i in I do
  v := v^(Q[i]);
end for;
return v;
end function;
ts := [Id(G1) : i in [1 .. 12] ];
ts[1] := f(t);
ts[2] := f(t^y);```

\[ts[3] := f(t^{x^2})\]
\[ts[4] := f(t^{y^2})\]
\[ts[5] := f(t^{x^1})\]
\[ts[6] := f(t^{y^x})\]
\[ts[7] := f(t^{x^2})\]
\[ts[8] := f(t^{x^2y})\]
\[ts[9] := f(t^x)\]
\[ts[10] := f(t^{xy})\]
\[ts[11] := f(t^{x^3})\]
\[ts[12] := f(t^{x^3y})\]
\[cst := \{null : i in [1 .. 266]\ where null is [Integers() | \};\]

for i := 1 to 12 do
    cst[prodim(1, ts, [i])] := [i];
end for;

m := 0;
for i in [1..266] do if cst[i] ne [] then m := m + 1; end if; end for; m;

N1 := Stabiliser(N, 1);
\#N/#N1;
Orbits(N1);

N12 := Stabiliser(N, [1, 2]);
SS := {[1, 2]};
SS := SS^N;
SS;
\#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..\#SS] do
    for n in IH do
        if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]] then print Rep(Seqq[i]); end if; end for; end for;
N12; \#N12;
N12s := N12;
\#N12s;
T12 := Transversal(N, N12s);
for i in [1..\#T12] do
    ss := [1, 2]^T12[i];
    cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N12s);

N14:=Stabiliser(N,[1,4]);
SS:={[1,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N14; #N14;
N14s:=N14;
#N14s;
T14:=Transversal(N,N14s);
for i in [1..#T14] do
ss:=[1,4]^T14[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N14s);

N16:=Stabiliser(N,[1,6]);
SS:={[1,6]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[6] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N16; #N16;
N16s:=N16;
N16s;
T16:=Transversal(N,N16s);
for i in [1..#T16] do
ss:=[1,6]^T16[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N16s);
N18:=Stabiliser(N,[1,8]);
SS:={[1,8]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N18; #N18;
N18s:=N18;
#N18s;
T18:=Transversal(N,N18s);
for i in [1..#T18] do
ss:=[1,8]^T18[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N18s);
N110:=Stabiliser(N,[1,10]);
SS:={[1,10]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
 n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N110; #N110;
N110s := N110;
#N110s;
T110 := Transversal(N, N110s);
for i in [1..#T110] do
ss := [1, 10]^T110[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..266] do if cst[i] ne []
then m := m + 1; end if; end for; m;
Orbits(N110s);

N112 := Stabiliser(N, [1, 12]);
SS := {{1, 12}};
SS := SS^N;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
 n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N112; #N112;
N112s := N112;
for g in N do if 1^g eq 10 and 12^g eq 7
then N112s := sub<N|N112s, g>;
end if; end for;
#N112s;
T112 := Transversal(N, N112s);
for i in [1..#T112] do
ss := [1, 12]^T112[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..266] do if cst[i] ne []
then m := m + 1; end if; end for; m;
Orbits(N112s);
N121 := Stabiliser(N, [1, 2, 1]);
SS := {[1, 2, 1]};
SS := SS^N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if;
end for;
end for;
N121; #N121;
N121s := N121;
#N121s;
T121 := Transversal(N, N121s);
for i in [1..#T121] do
ss := [1, 2, 1]^T121[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..266] do if cst[i] ne []
then m := m+1;
end if;
end for;
m;
Orbits(N121s);

N125 := Stabiliser(N, [1, 2, 5]);
SS := {[1, 2, 5]};
SS := SS^N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if;
end for;
end for;
N125; #N125;
N125s := N125;
#N125s;
T125 := Transversal(N, N125s);
for i in [1..#T125] do
  ss:=[1,2,5]^T125[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N125s);

N127:=Stabiliser(N,[1,2,7]);
SS:={[1,2,7]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
      n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      *ts[Rep(Seqq[i])[3]]
    then print Rep(Seqq[i]);
      end if; end for; end for;
N127; #N127;
N127s:=N127;
#N127s;
T127:=Transversal(N,N127s);
for i in [1..#T127] do
  ss:=[1,2,7]^T127[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N127s);

N129:=Stabiliser(N,[1,2,9]);
SS:={[1,2,9]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N129; #N129;
N129s:=N129;
#N129s;
T129:=Transversal(N,N129s);
for i in [1..#T129] do
ss:=[1,2,9]^T129[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N129s);

N1211:=Stabiliser(N,[1,2,11]);
SS:={[1,2,11]};
SS:=SS^N;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1211; #N1211;
N1211s:=N1211;
#N1211s;
T1211:=Transversal(N,N1211s);
for i in [1..#T1211] do
ss:=[1,2,11]^T1211[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1211s);

N141:=Stabiliser(N,[1,4,1]);
SS:={[1,4,1]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N141; #N141;
N141s:=N141;
#N141s;
T141:=Transversal(N,N141s);
for i in [1..#T141] do
ss:=[1,4,1]^T141[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N141s);

N145:=Stabiliser(N,[1,4,5]);
SS:={[1,4,5]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[4]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
/*
[ 1, 4, 5 ]
[ 7, 8, 9 ]
[ 10, 11, 2 ]
[ 6, 5, 4 ]
[ 3, 2, 11 ]
N145;  #N145;
N145s:=N145;
for g in N do if 1^g eq 7 and 4^g eq 8 and 5^g eq 9 then
  N145s:=sub<N|N145s,g>;
end if; end for;
for g in N do if 1^g eq 10 and 4^g eq 11 and 5^g eq 2 then
  N145s:=sub<N|N145s,g>;
end if; end for;
for g in N do if 1^g eq 6 and 4^g eq 5 and 5^g eq 4 then
  N145s:=sub<N|N145s,g>;
end if; end for;
for g in N do if 1^g eq 12 and 4^g eq 9 and 5^g eq 8 then
  N145s:=sub<N|N145s,g>;
end if; end for;
#N145s;
T145:=Transversal(N,N145s);
for i in [1..#T145] do
  ss:=[1,4,5]^T145[i];
  cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N145s);

N161:=Stabiliser(N,[1,6,1]);
SS:={[1,6,1]};
SS:=SS`N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    if ts[1]*ts[6]*ts[1] eq
      n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      *ts[Rep(Seqq[i])[3]]
    then print Rep(Seqq[i]);
  end if; end for; end for;
N161; #N161;
N161s:=N161;
#N161s;
T161:=Transversal(N,N161s);
for i in [1..#T161] do
ss:=[1,6,1]^T161[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N161s);

N163:=Stabiliser(N,[1,6,3]);
SS:={[1,6,3]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N163; #N163;
N163s:=N163;
#N163s;
T163:=Transversal(N,N163s);
for i in [1..#T163] do
ss:=[1,6,3]^T163[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N163s);

N165:=Stabiliser(N,[1,6,5]);
SS:={[1,6,5]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
    if ts[1]*ts[6]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        *ts[Rep(Seqq[i])[3]]
    then print Rep(Seqq[i]);
    end if;
end for;
end for;
N165; #N165;
N165s:=N165;
#N165s;
T165:=Transversal(N,N165s);
for i in [1..#T165] do
    ss:=[1,6,5]^T165[i];
    cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N165s);

N167:=Stabiliser(N,[1,6,7]);
SS:=\{[1,6,7]\};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
    for n in IH do
        if ts[1]*ts[6]*ts[7] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
            *ts[Rep(Seqq[i])[3]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
/
[ 1, 6, 7 ]
[ 2, 5, 8 ]
*/
N167; #N167;
N167s:=N167;
for g in N do if 1^g eq 2 and 6^g eq 5 and 7^g eq 8 then
    N167s:=sub<N|N167s,g>;
end if; end for;
#N167s;
T167:=Transversal(N,N167s);
for i in [1..#T167] do
ss:=[1,6,7]ˆT167[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N167s);

N169:=Stabiliser(N,[1,6,9]);
SS:={{1,6,9}};
SS:=SSˆN;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[6]*ts[9] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N169; #N169;
N169s:=N169;
#N169s;
T169:=Transversal(N,N169s);
for i in [1..#T169] do
ss:=[1,6,9]ˆT169[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N169s);

N1611:=Stabiliser(N,[1,6,11]);
SS:={{1,6,11}};
SS:=SSˆN;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
  if ts[1]*ts[6]*ts[11] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    *ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if; end for; end for;
N1611; #N1611;
N1611s:=N1611;
#N1611s;
T1611:=Transversal(N,N1611s);
for i in [1..#T1611] do
  ss:=[1,6,11]^T1611[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N1611s);

N187:=Stabiliser(N,[1,8,7]);
SS:={[1,8,7]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    if ts[1]*ts[8]*ts[7] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      *ts[Rep(Seqq[i])[3]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N187; #N187;
N187s:=N187;
#N187s;
T187:=Transversal(N,N187s);
for i in [1..#T187] do
  ss:=[1,8,7]^T187[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N187s);
N1103 := Stabiliser(N, [1, 10, 3]);
SS := {[1, 10, 3]};
SS := SS \times N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    then print Rep(Seqq[i]);
  end if;
end for;
end for;
/*
  [ 1, 10, 3 ]
  [ 11, 4, 9 ]
*/
N1103; #N1103;
N1103s := N1103;
for g in N do if 1^g eq 11 and 10^g eq 4 and 3^g eq 9 then
  N1103s := sub<N|N1103s,g>;
end if;
end for;
#N1103s;
T1103 := Transversal(N,N1103s);
for i in [1..#T1103] do
  ss := [1, 10, 3]^T1103[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..266] do if cst[i] ne []
  then m := m+1; end if;
end for;
Orbits(N1103s);

N1274 := Stabiliser(N, [1, 2, 7, 4]);
SS := {[1, 2, 7, 4]};
SS := SS \times N;
SS;
#SS;
Seqq := Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
/*
[ 1, 2, 7, 4 ]
[ 10, 5, 12, 11 ]
*/
N1274; #N1274;
N1274s:=N1274;
for g in N do if 1^g eq 10 and 2^g eq 5 and 7^g eq 12 and
4^g eq 11 then N1274s:=sub<N|N1274s,g>;
end if; end for;
#N1274s;
T1274:=Transversal(N,N1274s);
for i in [1..#T1274] do
ss:=[1,2,7,4]^T1274[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1274s);
N1278:=Stabiliser(N,[1,2,7,8]);
SS:={1,2,7,8};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[2]*ts[7]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1278; #N1278;
N1278s:=N1278;
for g in N do if 1^g eq 7 and 2^g eq 4 and
7^g eq 3 and
8^g eq 2 then N1278s:=sub<N|N1278s,g>;
end if; end for;
for g in N do if 1^g eq 10 and 2^g eq 5 and 7^g eq 12 and 8^g eq 9 then N1278s:=sub<N|N1278s,g>;
end if; end for;
for g in N do if 1^g eq 6 and 2^g eq 9 and 7^g eq 10 and 8^g eq 11 then N1278s:=sub<N|N1278s,g>;
end if; end for;
for g in N do if 1^g eq 12 and 2^g eq 11 and 7^g eq 6 and 8^g eq 5 then N1278s:=sub<N|N1278s,g>;
end if; end for;
#N1278s;
T1278:=Transversal(N,N1278s);
for i in [1..#T1278] do
  ss:=[1,2,7,8]^T1278[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N1278s);

N1414:=Stabiliser(N,[1,4,1,4]);
SS:={[1,4,1,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
    if ts[1]*ts[4]*ts[1]*ts[4] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N1414; #N1414;
N1414s:=N1414;
for g in N do if 1^g eq 2 and 4^g eq 3 and 1^g eq 2 and 4^g eq 3 then N1414s:=sub<N|N1414s,g>;
end if; end for;

T1414:=Transversal(N, N1414s);
for i in [1..#T1414] do
  ss:=[1,4,1,4]ˆT1414[i];
  cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
  then m:=m+1; end if; end for; m;

Orbits(N1414s);

N1614:=Stabiliser(N, [1,6,1,4]);
SS:={[1,6,1,4]};
SS:=SS^N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
  for n in IH do
      n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      *ts[Rep(Seqq[i])[3]]
      *ts[Rep(Seqq[i])[4]]
    then print Rep(Seqq[i]);
  end if; end for; end for;
/*
[ 1, 6, 1, 4 ]
[ 6, 1, 6, 5 ]
[ 11, 8, 11, 10 ]
[ 8, 11, 8, 7 ]
*/
N1614; #N1614;
N1614s:=N1614;
for g in N do if 1^g eq 6 and 6^g eq 1 and 1^g eq 6
  and 4^g eq 5 then N1614s:=sub<N|N1614s,g>;
end if; end for;
for g in N do if 1^g eq 11 and 6^g eq 8 and 1^g eq 11
  and 4^g eq 10 then N1614s:=sub<N|N1614s,g>;
end if; end for;
for g in N do if 1^g eq 8 and 6^g eq 11 and 1^g eq 8
  and 4^g eq 7 then N1614s:=sub<N|N1614s,g>;
end if; end for;

#N1614s;
T1614:=Transversal(N,N1614s);
for i in [1..#T1614] do
ss:=[1,6,1,4]ˆT1614[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1614s);

N1618:=Stabiliser(N,[1,6,1,8]);
SS:={[1,6,1,8]};
SS:=SS\N;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
for n in IH do
if ts[1]*ts[6]*ts[1]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1618; #N1618;
N1618s:=N1618;
for g in N do if 1^g eq 4 and 6^g eq 9 and 1^g eq 4
and 8^g eq 3 then N1618s:=sub<N|N1618s,g>;
end if; end for;
#N1618s;
T1618:=Transversal(N,N1618s);
for i in [1..#T1618] do
ss:=[1,6,1,8]ˆT1618[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..266] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1618s);
Bibliography


