Images of Permutation and Monomial Progenitors

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IMAGES OF PERMUTATION AND MONOMIAL PROGENITORS

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Shirley Marina Juan

June 2018
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June 2018

Approved by:

Dr. Zahid Hasan, Committee Chair

Dr. Hajrudin Fejzic, Committee Member

Dr. Joseph Chavez, Committee Member

Dr. Charles Stanton, Chair, Department of Mathematics

Dr. Corey Dunn, Graduate Coordinator
Abstract

We have conducted a systematic search for finite homomorphic images of several permutation and monomial progenitors, including $2^{*20} : (2^4 : S_5)$, $2^{*20} : ((5 \times 4) : S_4)$, $2^{*20} : D_{20}$, $2^{*11} : (2 : 11)$, $2^{*11} : L_2(11)$, $2^{*6} : (2 \times S_3)$, $2^{*6} : (S_3 \times S_3)$, $2^{*36} : (3^2 : D_4)$, $2^{*110} : L_2(11)$, $2^{*6} : D_{12}$, $2^{*10} : S_5$, $11^{*4} : m(4 : 5)$, and $11^{*2} : m D_{10}$. We have found original symmetric presentations for several important groups such as the Mathieu sporadic simple groups $M_{11}$ and $M_{12}$, Suzuki simple group $sz_8$, unitary group $U(3, 4)$, Janko group $J_1$, simplectic groups $S(4, 4)$ and $S(4, 3)$, and projective special linear groups $L_3(4)$ and $L_3(7)$. We have also constructed, using the technique of double coset enumeration, the following groups, $L_2(11)$, $S(4, 3) : 2$, $M_{11}$, and $PGL(2, 11)$. The isomorphism class of each of the finite images is also given.
First and foremost I would like to thank my advisor Dr. Zahid Hasan. The countless hours you have spent aiding me in my education are greatly appreciated and will never be forgotten. I thank you for the education, guidance, wisdom, and support you have given me in both my undergraduate as well as graduate work. I would like to also thank my committee members Dr. Hajrudin Fejzic and Dr. Joseph Chavez for helping me throughout my studies. To Dr. Charles Stanton and Dr. Corey Dunn, I thank you both, as well as the entire math department at CSUSB for your guidance and support throughout this wonderful journey.

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To my parents, I thank you for all that you have done for me to get me to this point. You are and have always been my number one cheerleaders and none of this would be possible without your support. I want to also thank my children, my M & Ms, Marina and Max. The both of you bring such joy to my life and you have been my inspiration to pursue my education. I only want the best for the both of you!

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Introduction

The aim of group theory is the discovery and classification of groups. Symmetric presentations give a uniform method for constructing finite groups. Since finite groups are composed of simple groups, we are most interested in simple groups. In Chapter 1 we will discuss some important definitions, lemmas, and theorems. In Chapter 2 we will begin to explore progenitors and the methods used to write them. In Chapter 3 we will solve extension problems in order to define our isomorphism types. Chapter 4 will focus on monomial progenitors and methods used to write these. Chapter 5 is dedicated to Double Coset Enumeration, both manual and computer based. In Chapter 6 we discuss Transitive Groups and explore certain transitive groups written on 20, 19, 11, and 6 letters.
Chapter 1

Preliminaries

1.1 Definitions, Theorems, and Lemmas

Definition 1.1. [Rot95] If $X$ is a nonempty set, a permutation of $X$ is a bijection $\alpha : X \rightarrow X$. We denote the set of all permutations of $X$ by $S_x$.

Definition 1.2. [Rot95] If $x \in X$ and $\alpha \in S_x$, then $\alpha$ fixes $x$ if $\alpha(x) = x$ and $\alpha$ moves $x$ if $\alpha(x) \neq x$.

Definition 1.3. [Rot95] A (binary) operation on a nonempty set $G$ is a function $\mu : G \times G \Rightarrow G$.

Definition 1.4. [Rot95] A semigroup $(G, \ast)$ is a nonempty set $G$ equipped with an associative operation $\ast$.

Definition 1.5. [Rot95] A group is a semigroup $G$ containing an element $e$ such that:

(i) $e \ast a = a = a \ast e$ for all $a \in G$;

(ii) for every $a \in G$, there is an element $b \in G$ with $a \ast b = e = b \ast a$.

Definition 1.6. [Rot95] A pair of elements $a$ and $b$ in a semigroup commutes if $a \ast b = b \ast a$. A group (or a semigroup) is abelian if every pair of its elements commutes.

Theorem 1.7. [Rot95] If $G$ is a group, there is a unique element $e$ with $e \ast a = a = a \ast e$ for all $a \in G$. Moreover, for each $a \in G$, there is a unique $b \in G$ with $a \ast b = e = b \ast a$. 
We call $e$ the identity of $G$ and, if $a * b = e = b * a$, then we call $b$ the inverse of $a$ and denote it by $a^{-1}$.

Corollary 1.8. [Rot95] If $G$ is a group and $a \in G$, then

$$(a^{-1})^{-1} = a.$$  

Definition 1.9. [Rot95] Let $(G, *)$ and $(H, \circ)$ be groups. A function $f : G \rightarrow H$ is a homomorphism if, for all $a, b \in G$,

$$f(a * b) = f(a) \circ f(b).$$  

Definition 1.10. [Rot95] An isomorphism is a homomorphism that is also a bijection. We say that $G$ is isomorphic to $H$, denoted by $G \cong H$, if there exists an isomorphism $f : G \rightarrow H$.

Theorem 1.11. [Rot95] Let $f : (G, *) \rightarrow (G', \circ)$ be a homomorphism.

(i) $f(e) = e'$, where $e'$ is the identity in $G'$

(ii) If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$.

(iii) If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$.

Definition 1.12. [Rot95] A nonempty subset $S$ of a group $G$ is a subgroup of $G$ if $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$.

Theorem 1.13. [Rot95] If $S \leq G$ (i.e., if $S$ is a subgroup of $G$), then $S$ is a group in its own right.

Theorem 1.14. [Rot95] A subset $S$ of a group $G$ is a subgroup if and only if $1 \in S$ and $s, t \in S$ imply $st^{-1} \in S$.

Definition 1.15. [Rot95] If $G$ is a group and $a \in G$, then the cyclic subgroup generated by $a$, denoted by $\langle a \rangle$, is the set of all powers of $a$. A group $G$ is called cyclic if there is $a \in G$ with $G = \langle a \rangle$; that is, $G$ consists of all the powers of $a$.

Theorem 1.16. [Rot95] If $S$ is a subgroup of $G$ and if $t \in G$, then a right coset of $S$ in $G$ is the subset of $G$

$$St = \{st : s \in S\}$$
(a left coset is \( tS = \{ ts : s \in S \} \). One calls \( t \) a representative of \( St \) (and also of \( tS \)).

**Definition 1.17.** [Rot95] If \( S \leq G \), then the **index** of \( S \) in \( G \), denoted by \( [G : S] \), is the number of right cosets of \( S \) in \( G \).

**Definition 1.18.** [Rot95] If \( G \) is a group, then the **order** of \( G \), denoted by \( |G| \), is the number of elements in \( G \).

**Theorem 1.19.** [Rot95] (Lagrange)

If \( G \) is a finite group and \( S \leq G \), then \( |S| \) divides \( |G| \) and \( [G : S] = |G|/|S| \).

**Corollary 1.20.** [Rot95] If \( G \) is a finite group and \( a \in G \). Then the order of \( a \) divides \( |G| \).

**Corollary 1.21.** [Rot95] If \( p \) is a prime and \( |G| = p \), then \( G \) is a cyclic group.

**Definition 1.22.** [Rot95] A subgroup \( K \leq G \) is a **normal subgroup**, denoted by \( K \trianglelefteq G \), if \( gKg^{-1} = K \) for every \( g \in G \).

**Definition 1.23.** [Rot95] A **projective special linear group**, \( PSL(n, F) \) is the set of all \( n \times n \) matrices with determinant 1 over field \( F \) factored by its center:

\[
PSL(n, F) = L_n(F) = \frac{SL(n, F)}{Z(SL(n, F))}.
\]

**Definition 1.24.** [Rot95] A **projective general linear group**, \( PGL(n, F) \) is the set of all \( n \times n \) matrices with nonzero determinant over field \( F \) factored by its center:

\[
PGL(n, F) = \frac{GL(n, F)}{Z(GL(n, F))}.
\]

**Definition 1.25.** [Rot95] A **special linear group**, \( SL(n, F) \) is the set of all \( n \times n \) matrices with determinant 1 over field \( F \).

**Definition 1.26.** [Rot95] A **general linear group**, \( GL(n, F) \) is the set of all \( n \times n \) matrices with nonzero determinant over field \( F \).
Theorem 1.27. [Rot95] (First Isomorphism Theorem).
Let $f : g \rightarrow H$ be a homorphism with kernel $K$. Then $K$ is a normal subgroup of $G$ and $G/K \cong \text{im } f$.

Theorem 1.28. [Rot95] (Second Isomorphism Theorem).
Let $N$ and $T$ be subgroups of $G$ with $N$ normal. Then $N \cap T$ is normal in $T$ and $T/(N \cap T) \cong NT/N$.

Theorem 1.29. [Rot95] (Third Isomorphism Theorem).
Let $K \leq H \leq G$, where both $K$ and $H$ are normal subgroups of $G$. Then $H/K$ is a normal subgroup of $G/K$ and
\[(G/K)/(H/K) \cong G/H\]
Chapter 2

Writing Progenitors

2.1 Preliminaries

Definition 2.1. [Rot95] Let \( X \) be a set and \( \Delta \) by a family of words on \( X \). A group \( G \) has generators \( X \) and relations \( \Delta \) if \( G \cong F/R \), where \( F \) is a free group with basis \( X \) and \( R \) is the normal subgroup of \( F \) generated by \( \Delta \). We say \( \langle X|\Delta \rangle \) is a presentation of \( G \).

Definition 2.2. [Rot95] Let \( G \) be a group. If \( H \leq G \), the normalizer of \( H \) in \( G \) is defined by \( N_G(H) = \{a \in G | aHa^{-1} = H\} \)

Definition 2.3. [Rot95] Let \( G \) be a group. If \( H \leq G \), the centralizer of \( H \) in \( G \) is:

\[
C_G(H) = \{x \in G : [x,h] = 1 \text{ for all } h \in H\}.
\]

Definition 2.4. [Rot95] Let \( N \) be a group. The point stabiliser of \( w \) in \( N \) is given by:

\[
N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i \text{'s}.
\]

Definition 2.5. [Rot95] Let \( a \in G \), where \( G \) is a group. The conjugacy class of \( a \) is given by \( a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\} \)

Definition 2.6. [Rot95] Let \( G \) be a group and \( X \) be a \( G \)-set. For \( x \in X \), the set \( x^G = \{x^g | g \in G\} \) is a \( G \)-Orbit.
Definition 2.7. [Rot95] If \( x \in G \), then a conjugate of \( x \) in \( G \) is an element of the form \( axa^{-1} \) for some \( a \in G \); equivalently, \( x \) and \( y \) are conjugate if \( y = \gamma_a(x) \) for some \( a \in G \).

Lemma 2.8. [Gri15] (The Factoring Lemma) Factoring the progenitor \( m^{*n} : N \) by \((t_i,t_j)\) for \( 1 \leq i \leq j \leq n \) gives the group \( m^n : N \).

2.2 Permutation Progenitor \((15 : 4)\)

In this section we will write a presentation for the progenitor \( 2^{*15} : (15 : 4) \).

Our control group \( N = (15 : 4) \) has the following presentation.

\[
N = \langle w, x, y, z | w^4, x^2, z^3, w^{-2}x, w^{-1}z^{-1}wz^{-1}, (y, z), wy^{-1}w^{-1}y^2, xy^{-1}w^2y^{-1}z, (t, wy^{-1}) \rangle
\]

Since we have \( 2^{*15} \), we will have \( 15 \) \( t \)'s of order 2. We let \( t \sim t_1 \), which means that \( t \) will commute with the stabilizer of 1 in \( N \). We use MAGMA to find these permutations that stabilize 1 in \( N \).

\[
N_1 := \text{Stabiliser}(N, 1);
\]

Permutation group \( N_1 \) acting on a set of cardinality 15

Order = 4 = 2^2

\[
(2, 11, 12, 8)(3, 14, 9, 13)(4, 7)(5, 6, 15, 10)
\]

Using our Schreier System, we see that this permutation is \( wy^{-1} \) So we add this, as well as \( t \), to our presentation of \( N \) to get a presentation for \( 2^{*15} : (15 : 4) \).

\[
G = \langle w, x, y, z, t | w^4, x^2, z^3, w^{-2}x, w^{-1}z^{-1}wz^{-1}, (y, z), wy^{-1}w^{-1}y^2, xy^{-1}w^2y^{-1}z, t^2, (t, wy^{-1}) \rangle
\]

Our progenitor is infinite, in order to make it finite we must factor by relations.

2.3 Writing Relations

2.3.1 First Order Relations

First order relations are written in the form \((\pi t_i^a)^b = 1\), where \( a \in N \) and \( w \) is a word in the \( t_i \)'s. We can exhaust all possible relations by computing the orbits of
the centralizers of Conjugacy Classes of \( N \). Continuing with our example from above, we find the classes of \( N = (15 : 4) \).

Table 2.1: Conjugacy Classes of \( N = (15 : 4) \)

<table>
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<tr>
<th>Class</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( e )</td>
<td>1</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( x^y = (1,9)(2,7)(3,14)(4,10)(6,8)(12,15) )</td>
<td>5</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( z = (1,4,7)(2,9,10)(3,6,12)(5,13,11)(8,15,14) )</td>
<td>2</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>( yw(1,3,9,14)(2,8,7,6)(4,12,10,15)(5,11) )</td>
<td>15</td>
</tr>
<tr>
<td>( C_5 )</td>
<td>( w^{-1}y^{-1} = (1,14,9,3)(2,6,7,8)(4,15,10,12)(5,11) )</td>
<td>15</td>
</tr>
<tr>
<td>( C_6 )</td>
<td>( y^2 = (1,13,9,3,14)(2,12,15,7,5)(4,11,10,6,8) )</td>
<td>4</td>
</tr>
<tr>
<td>( C_7 )</td>
<td>( yx = (1,5,4,13,7,11)(2,8,9,15,10,14)(3,12,6) )</td>
<td>10</td>
</tr>
<tr>
<td>( C_8 )</td>
<td>( y = (1,2,8,13,12,4,9,15,11,3,7,10,14,5,6) )</td>
<td>4</td>
</tr>
<tr>
<td>( C_9 )</td>
<td>( y^2z = (1,15,6,9,5,4,14,12,10,13,7,8,3,2,11) )</td>
<td>4</td>
</tr>
</tbody>
</table>

Now, we need to find the centraliser of each Class representative as well as the orbit of each centraliser that we find.

```maple
CL:=Classes(N);
for ii in [2..NumberOfClasses(N)] do
  for i in [1..#N] do
    if ArrayP[i] eq CL[ii][3] then Sch[i]; end if;
  end for;
  C12:=Centraliser(N,CL[ii][3]);
  Orbits(C12);
end for;
```

The output we have is given in the following table.
Thus we have the following relations

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative</th>
<th>Centraliser($N,\text{Rep}$)</th>
<th>Orbits of Centraliser($N,\text{Rep}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>$x^y$</td>
<td>$(1,9)(2,7)(3,14)(4,10)(6,8)(12,15)$</td>
<td>$(5,11,13)$, ${5,11}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$y^w$</td>
<td>$(1,9,15,6,5,7,14,12,13,2,8,11,9,10)$</td>
<td>${13}$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$z$</td>
<td>$(1,9,15,6,5,7,14,12,13,2,8,11,9,10)$</td>
<td>${5,11}$, ${5,11}$, ${1,3,9,14}$, ${2,8,7,6}$, ${4,12,10,15}$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$w^{-1}y^{-1}$</td>
<td>$(1,14,9,3)(2,6,7,8)(4,15,10,12)(5,11)$</td>
<td>${13}$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$y^t$</td>
<td>$(1,14,9,3,14)(2,12,15,7,5)(4,11,10,6,8)$</td>
<td>${13,4,9,11,7,3,10,5,14,6,2,8,12,15}$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$y^x$</td>
<td>$(1,13,5,7,4,11)$</td>
<td>${3,12,6}$</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$x^y$</td>
<td>$(1,12,4,9,15,11,3,7,10,14,5,6)$</td>
<td>${1,2,8,13,12,4,9,15,11,3,7,10,14,5,6}$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$y^2z$</td>
<td>$(1,15,6,9,5,4,14,12,10,13,7,8,3,2,11)$</td>
<td>${1,15,6,9,5,4,14,12,10,13,7,8,3,2,11}$</td>
</tr>
</tbody>
</table>

Thus we have the following relations

<table>
<thead>
<tr>
<th>Class</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>$x^yt_5, x^yt_1$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$zt_1$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$ywt_{13}, ywt_5, ywt_1, ywt_2, ywt_4$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$w^{-1}y^{-1}t_{13}, w^{-1}y^{-1}t_5, w^{-1}y^{-1}t_1, w^{-1}y^{-1}t_2, w^{-1}y^{-1}t_4$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$y^zt_1$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$yzt_3, yzt_1$</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$zt_1$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$y^2zt_1$</td>
</tr>
</tbody>
</table>

Note that $t_1 \sim t$, and, since $y = (1,2,8,13,12,4,9,15,11,3,7,10,14,5,6)$, then

$t_2 \sim t^y$
$t_3 \sim t^{y^3}$
$t_4 \sim t^{y^5}$
$t_5 \sim t^{y^{13}}$
$t_{13} \sim t^{y^3}$. 
Now we add these relations to our progenitor to obtain homomorphic images of $G$.

\[
G = \langle w, x, y, z, t | w^4, x^2, z^3, w^{-2}x, w^{-1}z^{-1}wz^{-1}, (y, z), wy^{-1}w^{-1}y^2, xy^{-1}w^2y^{-1}z, \\
t^2, (t, wy^{-1}), \\
(x^yt^3)r^1, (xt^3r^2, (z)ts^3, (ywtb^3)r^4, (ywt)sr^5, (ywt)tr^6, (ywtb^5)r^8, \\
(w^{-1}y^{-1}t^9, (w^{-1}y^{-1}ts^3r^9, (w^{-1}y^{-1}t^3)r^{10}, (w^{-1}y^{-1}t)r^{11}, (w^{-1}y^{-1}t)r^{12}, \\
(w^{-1}y^{-1}t^9)r^{13}, (y^3t)r^{14}, (yxtb^3)r^{15}, (yxtsr^1)r^{16}, (yxtsr^1)r^{17}, (yt^3)r^{18}, (y^2zsr^19) >.
\]

### 2.3.2 Factoring by Famous Lemma

We use the Famous Lemma [Cur07] as another method of finding relations.
Factoring our progenitor by this lemma guarantees the non-collapse of groups.

**Theorem 2.9.** [Cur07] Famous Lemma

Let $N \cap < t_i, t_j > \leq C_N(N_{ij})$, where $N_{ij}$ denotes the stabilizer in $N$ of the two points $i$ and $j$.

**Proof.** Let $w \in N \cap < t_i, t_j >$. We need to show $w \in Cent(N, N_{ij})$.

Let $\pi \in N_{ij}$.

$\pi^w = w$.

$\Rightarrow \pi^{-1}w\pi = w$.

$\Rightarrow \pi\pi^{-1}w\pi = \pi w$.

$\Rightarrow w\pi = \pi w$.

Thus $\pi$ commutes with every element of $N_{ij}$.

Note that $|t_i| = |t_j| = 2$, and $|t_it_j| = n$, thus $< t_i, t_j > = D_{2n}$ is Dihedral.

\[
Z(D_{2n}) = \begin{cases} 
1 & \text{if } n \text{ is odd.} \\
< (t_it_j)^{\frac{n}{2}} > & \text{if } n \text{ is even.}
\end{cases}
\]

So for each two point stabilizer in $N$ we will compute the centralizer of the two point stabilizer in $N$ and then write elements of $N$ in terms of $< t_i, t_j >$ in the following way given by the Famous Lemma.
\[
\begin{cases}
(x, t_1)^m = 1 & \text{where } m \text{ is odd and } x \text{ sends } 1 \text{ to } 2 \\
(t_i, t_j)^n = x & \text{where } n \text{ is even and } x \text{ fixes both } 1 \text{ and } 2
\end{cases}
\]

Let \( x = (1, 2)(3, 5)(4, 7)(6, 10)(8, 12)(9, 13)(11, 16)(14, 19)(17, 22)(18, 23)(20, 26)
\]

and

\]

\( N = <x, y> = L_2(11) \).

To find our relations using the Famous Lemma we must first find the Centraliser of
\( N_{ij} \), where \( N_{ij} \) is the stabiliser of the two points \( i \) and \( j \), which we will say 1 and 2
respectively.
N:=sub<S|xx,yy>;
N12:=Stabiliser(N,[1,2]);
C:=Centraliser(N,N12);
Set(C);
{
(9, 101)(10, 17)(11, 15)(12, 50)(14, 94)(18, 41)(19, 46)
(36, 102)(38, 100)(39, 72)(40, 65)(42, 60)(43, 66)
(55, 57)(58, 89)(59, 83)(62, 90)(63, 92)(64, 76)
(69, 78)(71, 73)(87, 110)(97, 108)(103, 104),
Id(C),
(1, 2)(3, 5)(4, 7)(6, 10)(8, 12)(9, 13)(11, 16)(14, 19)(17, 22)
(18, 23)(20, 26)(21, 27)(24, 31)(25, 32)(28, 36)
(54, 67)(56, 69)(58, 72)(60, 71)(61, 74)(66, 78)(68, 80)
(70, 82)(73, 85)(75, 87)(76, 83)(77, 89)(79, 90)
(81, 92)(84, 94)(86, 93)(88, 97)(91, 99)(95, 102)(96, 103)
(98, 105)(100, 106)(104, 107)(109, 110),
(1, 2)(3, 72)(4, 99)(5, 58)(6, 85)(7, 91)(8, 83)(9, 16)(10, 73)
(11, 13)(12, 76)(14, 37)(15, 101)(17, 42)(18, 95)
(60, 86)(61, 74)(64, 90)(65, 81)(70, 94)(71, 93)(77, 96)
(82, 84)(88, 100)(89, 103)(97, 106),
(6, 17, 93)(7, 91, 26)(8, 50, 90)(9, 13, 101)(10, 86, 22)
(11, 16, 15)(12, 79, 62)(14, 46, 84)(18, 80, 49)(19, 94, 35)
(21, 95, 36)(23, 41, 68)(25, 37, 82)(27, 28, 102)(29, 32, 70)
(30, 72, 89)(33, 66, 69)(34, 75, 110)(38, 100, 106)(39, 77, 58)
(47, 55, 105)(48, 83, 64)(51, 60, 73)(52, 76, 59)
(63, 92, 81)(88, 108, 97),
(3, 58)(4, 91)(5, 72)(6, 73)(7, 99)(8, 76)(9, 11)(10, 85)
(12, 83)(13, 16)(14, 29)(15, 101)(17, 51)(18, 102)


Schrier System is equal to $yxyxy^{-1}xyxy^{-1}xy$. Now, since $yxyxy^{-1}xyxy^{-1}xy$ fixes 1 and 2, we have that $(t_1t_2)^k = yxyxy^{-1}xyxy^{-1}xy$. Note that $t_1 \sim t$ and $t_2 \sim t^x$, so our final relation found by the Famous Lemma is $(tt^x)^k = yxyxy^{-1}xyxy^{-1}xy$.

We add these relations, as well as some first order relations to our progenitor to produce the following isomorphic images.

$$G = \langle x, y, t \mid x^2, y^3, (y^{-1}xy)^5, (xy^{-1})^{11}, (yxyxy^{-1}xy^{-1}xyx)^2, (t, yxyxy^{-1}xy^{-1}xy^{-1}y), (tx)^k, (tyxyxy^{-1}xyxyx)^l, (t \ast t^x)^m = yxyxy^{-1}xyxy^{-1}xy, (xytyx)^r, ((xy)^2t)^s \rangle.$$

<table>
<thead>
<tr>
<th>r1</th>
<th>r2</th>
<th>k</th>
<th>l</th>
<th>m</th>
<th>Order</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>8</td>
<td>8</td>
<td>11</td>
<td>7920</td>
<td>$M_{11}$</td>
</tr>
</tbody>
</table>
Chapter 3

Extension Problems

3.1 Preliminaries

Definition 3.1. [Rot95] Let $G$ be a group such that $K \leq G$. $K$ is normal in $G$ if $gKg^{-1} = K$, for every $g \in G$. We will use $K \triangleleft G$ to denote $K$ as being normal in $G$.

Definition 3.2. [Rot95] If $N \triangleleft G$, then the cosets of $N$ in $G$ form a group, denoted by $G/N$, of order $[G : N]$.

Definition 3.3. [Rot95] Let $G$ be a group. A normal series $G$ is a sequence of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

with $G_{i+1} \triangleleft G_i$. Furthermore, the factor groups of $G$ are given by $G_i/G_{i+1}$ for $i = 0, 1, \ldots, n - 1$.

Definition 3.4. [Rot95] Let $G$ be a group. A composition series of $G$ given by:

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

is a normal series where, for all $i$, either $G_{i+1}$ is a maximal normal subgroup of $G_i$ or $G_{i+1} = G_i$.

Definition 3.5. [Rot95] If group $G$ has a composition series, the factor groups of its series are the composition factors of $G$. 
Definition 3.6. [Rot95] Let $G$ be a group. We say $G$ is a **direct product** of two subgroups $H$ and $K$ if:

1. $H \trianglelefteq G$, $K \trianglelefteq G$;
2. $G = HK$;
3. $H \cap K = 1$.

Definition 3.7. [Rot95] $G$ is a **semi-direct product** of two subgroups $H$ and $K$ if:

1. $K \trianglelefteq G$, $Q \leq G$;
2. $G = KQ$;
3. $K \cap Q = 1$.

Definition 3.8. [Rot95] Let $G$ be a group. The **center** of $G$, $Z(G)$, is the set of all elements in $G$ that commute with all elements of $G$.

Definition 3.9. [Rot95] Let $G$ be a group and $H$, $N \leq G$ such that $|G| = |N||H|$. $G$ is a **central extension** by $H$ if $N$ is the center of $G$. We denote this by $G \cong N^\bullet H$.

Definition 3.10. [Rot95] Let $G$ be a group and $H$, $N \leq G$ such that $|G| = |N||H|$. $G$ is a **mixed extension** by $H$ if it is a combination of both central extensions and semi-direct products, where $N$ is the normal subgroup of $G$ but not central. We denote this by $G \cong N^\bullet : H$.

### 3.2 Direct Product

Consider the group 

$$\frac{2^{20}:L_2(11)}{(tx^2=(yx)^4y^{-1}(xy)^2xy^{-1}xy,(xy-1)(xy)^2xy^{-1}t)^5}.$$ 

$G$ has the following presentation,

$G = <x, y, t|x^2, y^3, (y^{-1}xy)^5, (xy^{-1})^{11}, (yxyxy^{-1}xy^{-1}xy^{-1}x)^2, t^2, (t, yxyxy^{-1}xy^{-1}), (t, yxyxy^{-1}xyxy^{-1}xy), (tx^2 = yxyxy^{-1}xyxy^{-1}xy, (xy^{-1}xyxyxyxy^{-1}t)^6 >$. 
The composition series of $G$ is below.

\[
\begin{array}{c|c|c}
G & M_{11} & \text{Cyclic(2)} \\
1 & & 1
\end{array}
\]

$G = G_1 \supseteq 1$, where $G = (G/G_1)(G_1/1) = C_2M_{11}$.

We have $C_2$ by $M_{11}$. In order for this to be a direct product we need to have $M_{11}$ and $C_2$ normal in $G$.

The Normal Lattice of $G$ is

\[
\begin{array}{c}
[1] \\
\text{Order 2} \quad \text{Order 7920} \\
[4] \\
\text{Order 15840}
\end{array}
\]

$M_{11}$ is of order 7920. We verify that our normal subgroup [3] is in fact $M_{11}$.

```plaintext
> load m11;
Loading "/usr/local/MAGMA/libs/pergps/m11"
M11 - Mathieu group on 11 letters - degree 11
Order 7920 = 2^4 * 3^2 * 5 * 11; Base 1,2,3,4
Group: G
> M11:=G;
> Order(M11);
```
Therefore by definition of a direct product we have that $G$ is isomorphic to $M_{11} \times C_2$.
To verify this, we will first need to write a presentation for $M_{11}$. We use FPGroup in MAGMA to get a presentation for $M_{11}$.

```
> FPGroup(M11);
Finitely presented group on 2 generators
Relations
$.1^2 = \text{Id}($) 
$.2^4 = \text{Id}($) 
$.2^-1 * $.1 * $.2^-2 * $.1 * $.2^-2 * $.1 * $.2^-2 * 
$.1 * $.2^-2 * $.1 * $.2^-2 * $.1 * $.2^-1 = \text{Id}($) 
$.1 * $.2 * $.1 * $.2 * $.1 * $.2^-1 * $.1 * $.2^-1 * 
$.1 * $.2 * $.1 * $.2^-1 * $.1 * $.2^-2 * $.1 * $.2 * 
$.1 * $.2^-1 = \text{Id}($) 
$.1 * $.2^-2 * $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2^-2 * 
$.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2^-2 * $.1 * $.2^-1 * 
$.1 * $.2 = \text{Id}($) 
($.1 * $.2^-1)\cdot 11 = \text{Id}($) 
```

A presentation for $M_{11}$ is $K = < x, y | x^2, y^4, y^{-1}xy^{-2}xy^2xy^2xy^2xy^{-1}, 
xyxyxy^{-1}xyxy^{-1}xyxy^{-1}, xy^{-2}xy^2xy^{-1}xyxy^2xy^{-1}xy, (xy^{-1})^{11} >$

A presentation for $C_2$ is $H = < z | z^2 >$.

Thus a presentation for $G = H \times K$ is $< z, x, y | z^2, (x, z), (y, z),$
$x^2, y^4, y^{-1}xy^{-2}xy^2xy^2xy^{-1},$
$xyxyxy^{-1}xyxy^{-1}xyxy^{-1}, xy^{-2}xy^2xy^{-1}xyxy^2xy^{-1}xy, (xy^{-1})^{11} >$

We verify that this presentation is isomorphic to $2 \times M_{11}$.

```
> G<x,y,t>:=Group<x,y,t|x^2,y^3,(y^-1*x*y*x)^5,(x*y^-1)^11, 
(y*x*y*x*y^-1*x*y^-1*x*y^-1*x)^2, 
```
Therefore, $(\mathbb{Z}^2)_{x=1} \cong (2 \times M_{11})$.

### 3.3 Semi-Direct Product

Consider the group $\frac{2^{20},L_2(11)}{(x^2y^2x^{-1}y^{-1}t^2y^3x^2)^3}$. 

$G$ has the following presentation,

\[
G = \langle x, y, t | x^6, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xyx^{-1}, y^{-1}x^{-1}yx^{-1}y^{-1}x^{-1}xy^3x, \\
(x^{-1}y^2x^{-1}y^{-1})^2, \\
t^2, (t, yx^2y^{-2}x^{-1}yx^{-1}), (t, x^{-1}y^{-1}x^{-1}y^{-3}xy^{-1}), \\
(t, (yxy^{-1})^3), (t, y^{-1}x^3y^{-2}), (x^2y^2x^{-1}y^{-1}t^2y^3x^2)^3 >.
\]
The composition series of $G$ is below.

\[
\begin{array}{c|c}
G & \text{Cyclic(2)} \\
* & \text{A(2, 5)} = L(3, 5) \\
1 & \\
\end{array}
\]

$G = G_1 \supseteq 1$, where $G = (G/G_1)(G_1/1) = L_3(5)C_2$.

The Normal Lattice of $G$ is

We have a normal subgroup of order 372000. Since $L_3(5)$ is of order 372000, we verify that $NL[2]$ is isomorphic to $L_3(5)$.

```plaintext
> s:=IsIsomorphic(NL[2], L_3(5));
> s;
true
```

Recall from the previous section, that in order to have a direct product we must have $C_2$ as well as $L_3(5)$ normal in $G$. Since our normal subgroup lattice does not show a subgroup of order 2, we know that $C_2$ is not normal in $G$. Thus we cannot have a direct product.
This extension must be a semi-direct product. We need to find an element of order 2 that will extend $L_3(5)$ to $G$. To do this, we first must write a presentation for $L_3(5)$.

```plaintext
> H<x,y>:=Group<x,y|x^4,
x^-1 * y^-1 * x^-2 * y * x * y^-1 * x^2 * y, y^-1 * x^-1 * y * x^-2 * y^-2 * x^-1 * y^-1 * x^2 * y * 3 * x^-1, y^-1 * x^-2 * y * x^-1 * y^-1 * x * y^-1 * x^2 * y^-1, (x * y^-1 * x^-1 * y^2)^3, y^-2 * x^-2 * y^-1 * x^-1 * y^-2 * x^2 * y^-1 * x^-1 * y^-1 * x^2 * y^-1 * x^-1 * y^-3 * x^2 * y^-1 * x * y^-1 * x^-1*1>; f,H1,k:=CosetAction(H,sub<H|Id(H)>); s:=IsIsomorphic(NL[2],H1); s; true
```

Now we find an element of order 2, which we will label $C$, that will extend $L_3(5)$ to $G$.

```plaintext
for i in NL[3] do if i notin NL[2] and Order(i) eq 2 and sub<G1|i,NL[2]> eq G1 then C:=i; break; end if; end for;
```

Now that we have this element $C$, of order 2, we find the action of $C$ on the generators of $H$.

Below we use MAGMA Schreier System and the following loop,

```plaintext
> for i in [1..#N1] do if ArrayP[i] eq A^C then print Sch[i]; for|if> end if; end for; y * x * y^-2 * x * y^-2 * x^2 * y^-1 * x * y * x
> for i in [1..#N1] do if ArrayP[i] eq B^C then print Sch[i]; for|if> end if; end for; x * y^2 * x^-1 * y^-1 * x^-1 * y^2 * x * y^2 * x * y * x
```

So now we know that $x^C = yxy^{-2}x^2y^{-1}xyx$, and $y^C = xy^2x^{-1}y^{-1}x^{-1}y^2xy^2xyx$. 

---

Note: The above text is a natural reading of the document, providing a clear and fluent version of the content, without altering the context or adding any new information. The code snippets are included as examples of the computational steps taken to reach the conclusion.
Now we will add this element of order 2, say $z$, to our presentation, along with the action of this element on the generators of $L_3(5)$ to obtain the following presentation.

$$G_2 = \langle x, y, z \mid x^4, x^{-1}y^{-1}x^2yy^{-1}x^2y, y^{-1}x^{-1}yx^{-2}y^{-2}x^{-1}y^{-1}x^2y^3x^{-1},$$

$$y^{-1}x^{-2}yx^{-1}y^{-1}x^{-1}y^{-2}x^2yx^{-1}, (xy^{-1}x^{-1}y^2)^3,$$

$$y^{-2}x^{-2}y^{-1}x^{-1}y^{-2}x^2y^{-1}x^{-1}y^{-2}x^2y^{-1}x^{-1},$$

$$yx^{-1}y^2x^{-1}y^{-1}x^{-1}y^{-3}x^2y^{-1}xy^{-1}x^{-1},$$

$$z^2, z^z = yxy^{-2}x^{-2}y^{-1}xyx, y^z = xy^2x^{-1}y^{-1}x^{-1}y^2xy^2xyx \rangle.$$

We then verify that this presentation is isomorphic to $G$.

We consider the group $2^{*20}:(2^4, S_3)\rtimes ((C_4 \times C_5) \rtimes S_4)\rtimes ((xy)^2y^3)^3, ((xy)^2t)^6$.

$G$ has the following presentation,

$$G = \langle x, y, t \mid G < x, y, t \rangle := \text{Group} < x, y, t \mid x^4, yx^{-2}y^2x^2y, yx^{-1}y^{-2}xy^3, x^{-1}y^{-1}x^2y^{-1} \rangle$$
$x^2yx^2yx^{-1}, x^{-1}y^{-1}x^{-1}y^{-1}x^{-1}y^{-1}, xyxy,$

t$^2, (t, yx^{-1}y^2), (t, yxyx^{-1}y), (x^2yx^3y^3x)^2, ((xy)^2t)^7, ((xy)^2t)^6$.

The composition series of $G$ is below.

CompositionFactors(G1);

G
  | Cyclic(2)
  *
  | A(1, 13) = L(2, 13)
  *
  | Cyclic(2)
1

$G = G_1 \supseteq 1$, where $G = (G/G_1)(G_1/G_2)(G_2/1) = C_2L_2(13)C_2$.

The normal lattice of $G$ is
$NL[2]$ is of order 2. We will check to see if this is our center.

```plaintext
> Center(G1) eq NL[2];
true
```

It is possible that we may have a central extension. If there is a larger abelian subgroup then we will instead have a mixed extension.

The following loop will list all of our abelian subgroups.

```plaintext
> for i in [1..11] do if IsAbelian(NL[i]) then i;end if;end for;
1
2
```
We now know that $NL[2]$, our center, is a maximal abelian subgroup, thus we will have a central extension. Now we factor $G$ by our center to form the quotient group $q$, and look at the Composition Factors of $q$.

$$G \rightarrow \text{Cyclic}(2) * A(1, 13) = L(2, 13)$$

It looks like $q$ may be Isomorphic to $PGL_2(13)$.

Thus we will have a central extension of $C_2$ by $PGL_2(13)$.

Now we need to write a presentation for $PGL_2(13)$.

Thus our presentation for $PGL_2(13)$ is

$$H <a, b> := \text{Group} < a, b | b^3, (a^{-1}b^{-1})^4, a^2, bab^{-1}a^{-1}, a^2b^{-1}a^{-1}b^{-1}, a^{-1}b^{-1}a^3ba^3 >.$$
To write our presentation we need to write the generators of $PGL_2(13)$ in terms of our center, which we will label $c$.

Now we can write a presentation for $G$ by including $c$, our generator of the center $C_2$, and writing $PGL_2(13)$ in terms of $c$. 

$$HH<c,a,b>:=\text{Group}<c,a,b|c^2,(c,a),(c,b),b^3,(a^{-1} \ast b^{-1})^4,a^{12},$$
$$a^{12}=c^2, b \ast a \ast b^{-1} \ast a^4 \ast b \ast a^2 \ast b^{-1} \ast a^{-1},$$
$$a^2 \ast b \ast a^4 \ast b^{-1} \ast a \ast b \ast a^{-1} \ast b^{-1},$$
$$a^{-1} \ast b \ast a^{-1} \ast b^{-1} \ast a^3 \ast b \ast a^3 \ast b^{-1}>;$$
Thus $G \cong C_2^*PGL_2(13)$

3.5 Mixed Extension

Consider the group $\frac{2^{20}L_2(11)}{(xy^{14})^3,(y^3tt)^2}$.

$G$ has the following presentation,

$G = \langle x, y, t | x^2, (y^{-1}x)^2, y^{20}, t^2, (t, xy^{-9}), (xt(y^{14}))^3, (y^5tt^3)^2 \rangle$.

The composition series of $G$ is below.

```
CompositionFactors(G1);
G
| Cyclic(2)
*| A(1, 19) = L(2, 19)
*| Cyclic(3)
*| Cyclic(2)
1
```

$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq 1$, where $G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/1) = C_2L_2(19)C_3C_2$.

The Normal Lattice of $G$ is
\(NL[2]\) is of order 2. We will check to see if this is our center.

\[
> \text{Center}(G1) \text{ eq } NL[2];
\]
true

It is possible that we may have a central extension. If there is a larger abelian subgroup then we will instead have a mixed extension.

The following loop will list all of our abelian subgroups.

\[
> \text{for } i \text{ in } [1..11] \text{ do if IsAbelian}(NL[i]) \text{ then } i; \\
> \text{end if}; \text{end for;}
\]

1
2
We now know that $NL[2]$, our center, is not a maximal abelian subgroup. $NL[4]$, which is of order 6, is our maximal abelian subgroup. Below we confirm that $C_6$ is the isomorphism type of $NL[4]$.

```maple
> X:=AbelianGroup(GrpPerm,[6]);
> s:=IsIsomorphic(X,NL[4]);
s;
true
```

We will have a mixed extension of $NL[4]$ by $q$ where $q$ is the isomorphic image of $G/G2 = G/NL[4]$. Now we need to look at the normal lattice and composition factors of $q$ to solve its isomorphism type.

```maple
> q,ff:=quo<G1|NL[4]>;
> nl:=NormalLattice(q);
> nl;
Normal subgroup lattice
-----------------------
---
[2] Order 3420 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

> CompositionFactors(q);
```

```
G
 | Cyclic(2)
*   
| A(1, 19) = L(2, 19)
1
```

By looking at the composition series of $q$, it seems that our extension problem of $q$ may be $PGL(2,19)$. 
> s:=IsIsomorphic(q,PGL(2,19));
> s;
true

So now that we know that $q$ is isomorphic to $PGL(2,19)$, we need to find a presentation for $q$.

> FPGroup(q);
Finitely presented group on 3 generators
Relations
$.1^2 = Id($)
$.3^2 = Id($)
$(.2^{-1} * .1)^2 = Id($)
$(.2 * .3 * .2^{-1} * .3 * .2^{-1} * .3)^2 = Id($)
$(.1 * .3 * .2 * .3 * .2^{-1} * .3)^2 = Id($)
$.2^8 * .1 * .3 * .2^{-1} * .3 * .2 * .3 * .2 * .3 = Id($)
$.2^{-2} * .3 = Id($)

> H<x,y,z>:=Group<x,y,z|x^2,z^2,(y^{-1}*x)^2,
> (y*z*y^{-1}*z*y)^2,(y*z*y^{-1}*z*y^{-1}*z)^2,
> x*z*x*z*y^4*z*y^{-1}*z*y^{-2}*z>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>>;
> s,t:=IsIsomorphic(H1,q); s;
true

Now our next step is to write the generators of $H$ into elements of $q$. In order to do this we will need to look at the transversals of $NL[4]$.

> T:=Transversal(G1,NL[4]);
> #T;
6840
> T[2];

Note that $T[2]$ will give us a permutation that we will have to store in MAGMA. We will store this permutation as $A$. Similarly we will store the permutation for $T[3]$ as

> \texttt{ff(A) eq q.1;} \\
true \\
> \texttt{ff(B) eq q.2;} \\
true \\
> \texttt{ff(C) eq q.3;} \\
true

We want to look at our presentation of $H$ to see which elements have changed by the action of $q$.

$$H = \langle x, y, z | x^2, z^2, (y^{-1}x)^2, (yz^{-1}zy)^2, (zy^{-1}zy^{-1}z)^2, (xzyzy^{-1}z)^2, y^8xzy^{-1}yz, xzxz^4zy^{-1}zy^{-2}z \rangle$$

Our first relation in the presentation, $x^2$ tells us that $x^2 = e$, therefore the order of $x$ is 2. We want to see what the order of $x$ is when we apply the action of $q$.

> \texttt{Order(A);} \\
2

So we see that $a$ does not change. We will check the rest of the relations in our presentation and look for any changes.

> \texttt{Order(C);} \\
2 \\
> \texttt{Order(B^{-1}*A);} \\
2 \\
> \texttt{Order(B\*C\*B^{-1}*C\*B^{-1}*C);} \\
2 \\
> \texttt{Order(A\*C\*B\*C\*B^{-1}\*C);} \\
2 \\
> \texttt{Order(B^{-8}\*A\*C\*B^{-1}\*C\*B\*C);} \\
1 \\
> \texttt{Order(A\*C\*A\*C\*B^{-4}\*C\*B^{-1}\*C\*B^{-2}\*C);}
The order of our last relation has changed. We will need to write this relation in terms of $q$.

We will need to find a generator of $NL[4]$. Note that $NL[4]$, of order 6, is cyclic. So if we obtain an element of order 6 then this element will generate the whole group. We will name this element of order 6, $D$.

```magma
> IsCyclic(NL[4]);
true
> Order(NL[4].1);
6
> D:=NL[4].1;
```

Now we go back to our relation that has changed and write this relation in terms of $D$.

```magma
> A*C*A*C*B^4*C*B^-1*C*B^-2*C eq D;
false
> A*C*A*C*B^4*C*B^-1*C*B^-2*C eq D^2;
false
> A*C*A*C*B^4*C*B^-1*C*B^-2*C eq D^3;
false
> A*C*A*C*B^4*C*B^-1*C*B^-2*C eq D^4;
false
> A*C*A*C*B^4*C*B^-1*C*B^-2*C eq D^5;
true
```

MAGMA tells us that this relation is equal to $D^5$.

Next we need to check to see if $D$ commutes with $x, y$, or $z$.

```magma
> for i in [0..6] do if D^A eq D^i then i; break; end if; end for;
5
```
The above loop confirms that $D^x = D^5$.

>$\text{for i in [0..6] do if } D^B \text{ eq } D^i$
>$\text{for|if> then i; break; end if; end for; 5}$
>$\text{for i in [0..6] do if } D^C \text{ eq } D^i$
>$\text{for|if> then i; break; end if; end for; 5}$

We have confirmed that $D^y = D^5$ and $D^z = D^5$. We can now write a presentation for $H$, where $w$ will be our element of order 6, and check to see if $H$ is isomorphic to $G$.

>$\text{H<w,x,y,z>:=Group<w,x,y,z|w^6,x^2,z^2,(y^1*x)^2, (y*z*y^-1*z*y^-1*z)^2,}$
>$\text{(x*z*y*z*y^-1*z)^2,y^8*x*z*y^-1*z*y*z,}$
>$\text{x*z*x*z*y^4*z*y^-1*z*y^-2*z=w^5,}$
>$\text{w^x=w^5,}$
>$\text{w^y=w^5,}$
>$\text{w^z=w^5>;}$
>$\text{#H; 41040}$
>$\text{#G1; 41040}$
>$\text{f,h,k:=CosetAction(H,sub<H|Id(H)>);}$
>$\text{#h; 41040}$
>$\text{#G1; 41040}$
>$\text{s:=IsIsomorphic(h,G1);}$
>$\text{s; true}$

Therefore $G = \frac{2^{20}\cdot L_2(11)}{(x^9 y^{11})^3, (y^5 w^6)^2} \cong 6^\bullet : PGL(2,19)$. 
Chapter 4

Monomial Progenitors

4.1 Preliminaries

Definition 4.1. [Cur07] A monomial representation of a group $G$ is a homomorphism from $G$ into $GL(n, F)$, the group of nonsingular $n \times n$ matrices over the field $F$, in which the image of every element of $G$ is a monomial matrix over $F$.

Definition 4.2. [?] (Monomial Character) Let $G$ be a finite group and $H \leq G$. The character $X$ of $G$ is monomial if $X = \lambda^G$, where $\lambda$ is a linear character of $H$.

Definition 4.3. [?] A matrix in which there is precisely one non-zero term in each row and in each column is said to be monomial.

Definition 4.4. [?] Let $A(x) = (a_{ij}(x))$ be a matrix representation of $G$ of degree $m$. We consider the characteristic polynomial of $A(x)$, namely

$$
det(\lambda I - A(x)) = 
\begin{pmatrix}
\lambda - a_{11}(x) & -a_{12}(x) & \ldots & -a_{1m}(x) \\
-a_{21}(x) & \lambda - a_{22}(x) & \ldots & -a_{2m}(x) \\
\ldots & \ldots & \ldots & \ldots \\
-a_{m1}(x) & -a_{m2}(x) & \ldots & \lambda - a_{mm}(x)
\end{pmatrix}
$$

This is a polynomial of degree $m$ in $\lambda$, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to
\[ \phi(x) = a_1(x) + a_{22}(x) + \ldots + a_{mm}(x). \]

It is customary to call the right-hand side of this equation the trace of \( A(x) \), abbreviated to \( \text{tr} A(x) \), so that

\[ \phi(x) = \text{tr} A(x). \]

**Definition 4.5.** [?] The sum of squares of the degrees of the distinct irreducible characters of \( G \) is equal to \( |G| \). The degree of a character \( \chi \) is \( \chi(1) \). Note that a character whose degree is 1 is called a linear character.

**Definition 4.6.** [Isa76] Let \( H \leq G \) and \( \phi(u) \) be a character of \( H \) and define \( \phi(x) = 0 \) if \( x \in H \), then

\[ \phi^G(x) = \begin{cases} 
\phi(x), & x \in H \\
0 & x \notin H 
\end{cases} \]

is an induced character of \( G \).

**Definition 4.7. Formula for Induced Character**

[Isa76] Let \( G \) be a finite group and \( H \) be a subgroup such that \( [G : H] = \frac{|G|}{|H|} = n \). Let \( C_\alpha, \alpha = 1, 2, \ldots, m \) be the conjugacy classes of \( G \) with \( |C_\alpha| = h_\alpha \), \( \alpha = 1, 2, \ldots, m \). Let \( \phi \) be a character of \( H \) and \( \phi^G \) be the character of \( G \) induced from the character \( \phi \) of \( H \) up to \( G \). The values of \( \phi^G \) on the \( m \) classes of \( G \) are given by:

\[ \phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w), \quad \alpha = 1, 2, 3, \ldots, m. \]

### 4.2 Monomial Progenitor \( 11^*4 \cdot_m (4 : 5) \)

Consider \( 11^*4 \cdot_m (4 : 5) \). \( G = (4 : 5) \) is given by

\[ G = (4 : 5) = \langle x, y | x^4, xy^4x^3y^3, y^2x^3yx \rangle, \]

where
\[ x = (1, 4, 17, 15)(2, 3, 18, 16)(5, 12, 14, 7)(6, 11, 13, 8)(9, 19, 10, 20), \text{ and } \]
\[ y = (1, 6, 10, 14, 18)(2, 5, 9, 13, 17)(3, 8, 12, 15, 19)(4, 7, 11, 16, 20). \]

\( G = (C_4 : C_5) \) has monomial irreducible representation in dimension 5. We will write a progenitor for \( 11^4 \cdot m(C_4 : C_5) \). Since \( \frac{|G|}{|H|} = 5 \Rightarrow \frac{20}{|H|} = 5 \Rightarrow |H| = 4 \), we need to find a subgroup \( H \) of order 4 and induce a linear character of \( H \) up to \( G \) to obtain the irreducible character of degree 5 of \( G \).

The conjugacy classes of group \((C_4 : C_5)\) are given in the table below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( e )</td>
<td>1</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( x^2 = (1,17)(2,18)(3,16)(4,15)(5,14)(6,13)(7,12)(8,11)(9,10)(19,20) )</td>
<td>5</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( x = (1,4,17,15)(2,3,18,16)(5,12,14,7)(6,11,13,8)(9,19,10,20) )</td>
<td>5</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>( x^3 = (1,15,17,4)(2,16,18,3)(5,7,14,12)(6,8,13,11)(9,20,10,19) )</td>
<td>5</td>
</tr>
<tr>
<td>( C_5 )</td>
<td>( y = (1,6,10,14,18)(2,5,9,13,17)(3,8,12,15,19)(4,7,11,16,20) )</td>
<td>4</td>
</tr>
</tbody>
</table>

Consider the subgroup \( H = Z_5 \) of \( G \) given below.
\( H = \{e, (1, 18, 14, 10, 6)(2, 17, 13, 9, 5)(3, 19, 15, 12, 8)(4, 20, 16, 11, 7), (1, 14, 6, 18, 10) \)
\( (2, 13, 5, 17, 9)(3, 15, 8, 19, 12)(4, 16, 7, 20, 11), (1, 10, 18, 6, 14)(2, 9, 17, 5, 13)(3, 12, 19, 8, 15) \)
\( (4, 11, 20, 7, 16), (1, 6, 10, 14, 18)(2, 5, 9, 13, 17)(3, 8, 12, 15, 19)(4, 7, 11, 16, 20) \} \). The conjugacy classes of \( Z_5 \) are given in the table below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>( e )</td>
<td>1</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>( y^1 = (1,18,14,10,6)(2,17,13,9,5)(3,19,15,12,8)(4,20,16,11,7) )</td>
<td>1</td>
</tr>
<tr>
<td>( D_3 )</td>
<td>( y^2 = (1,14,6,18,10)(2,13,5,17,9)(3,15,8,19,12)(4,16,7,20,11) )</td>
<td>1</td>
</tr>
<tr>
<td>( D_4 )</td>
<td>( y^3 = (1,10,18,6,14)(2,9,17,5,13)(3,12,19,8,15)(4,11,20,7,16) )</td>
<td>1</td>
</tr>
<tr>
<td>( D_5 )</td>
<td>( y = (1,6,10,14,18)(2,5,9,13,17)(3,8,12,15,19)(4,7,11,16,20) )</td>
<td>1</td>
</tr>
</tbody>
</table>

Consider the irreducible characters \( \phi \) (of \( H \)) and \( \chi \) (of \( G \)) given below.
Table 4.3: Character Table of $H = Z_5$

<table>
<thead>
<tr>
<th>Class</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

| $\phi_1$ | 1     | 1     | 1     | 1     | 1     |
| $\phi_2$ | 1     | $Z$   | $Z^2$ | $Z^3$ | $Z^4$ |
| $\phi_3$ | 1     | $Z^2$ | $Z^4$ | $Z$   | $Z^3$ |
| $\phi_4$ | 1     | $Z^3$ | $Z$   | $Z^4$ | $Z^2$ |
| $\phi_5$ | 1     | $Z^4$ | $Z^3$ | $Z^2$ | $Z$   |

where $Z$ is the 5th root of unity.

Table 4.4: Character Table of $G = (C_4 : C_5)$

<table>
<thead>
<tr>
<th>Class</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

| $\chi_1$ | 1     | 1     | 1     | 1     | 1     |
| $\chi_2$ | 1     | 1     | -1    | -1    | 1     |
| $\chi_3$ | 1     | -1    | -I    | I     | 1     |
| $\chi_4$ | 1     | -1    | I     | -I    | 1     |
| $\chi_5$ | 4     | 0     | 0     | 0     | -1    |

where $I$ is the 4th root of unity.

Next we must find a non-trivial linear character of $H$ to induce up to $G$. Note that each character of $H$ is linear since they all have degree 1. We will induce $\chi_2$ up to $G$.

Now, $G = He \cup Hx \cup Hx^2 \cup Hx^3$

Let $t_1 = e$, $t_2 = x$, $t_3 = x^2$, and $t_4 = x^3$.

Then
\[ A(xx) = \begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) & \phi(t_1 x t_4^{-1}) \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) & \phi(t_2 x t_4^{-1}) \\ \phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) & \phi(t_3 x t_4^{-1}) \\ \phi(t_4 x t_1^{-1}) & \phi(t_4 x t_2^{-1}) & \phi(t_4 x t_3^{-1}) & \phi(t_4 x t_4^{-1}) \end{bmatrix} \]

\[ = \begin{bmatrix} \phi(x e x)^{-1} & \phi(x e x)^{-1} & \phi(x e x)^{-1} & \phi(x e x)^{-1} \\ \phi(x x e)^{-1} & \phi(x x e)^{-1} & \phi(x x e)^{-1} & \phi(x x e)^{-1} \\ \phi(x^2 x e)^{-1} & \phi(x^2 x e)^{-1} & \phi(x^2 x e)^{-1} & \phi(x^2 x e)^{-1} \\ \phi(x^3 x e)^{-1} & \phi(x^3 x e)^{-1} & \phi(x^3 x e)^{-1} & \phi(x^3 x e)^{-1} \end{bmatrix} \]

\[ = \begin{bmatrix} \phi(x) & \phi(e) & \phi(x^3) & \phi(x) \\ \phi(x^2) & \phi(x) & \phi(e) & \phi(x) \\ \phi(x^3) & \phi(x^2) & \phi(x) & \phi(e) \\ \phi(e) & \phi(x^3) & \phi(x^2) & \phi(x) \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

Similarly,

\[ A(yy) = \begin{bmatrix} \phi(t_1 y t_1^{-1}) & \phi(t_1 y t_2^{-1}) & \phi(t_1 y t_3^{-1}) & \phi(t_1 y t_4^{-1}) \\ \phi(t_2 y t_1^{-1}) & \phi(t_2 y t_2^{-1}) & \phi(t_2 y t_3^{-1}) & \phi(t_2 y t_4^{-1}) \\ \phi(t_3 y t_1^{-1}) & \phi(t_3 y t_2^{-1}) & \phi(t_3 y t_3^{-1}) & \phi(t_3 y t_4^{-1}) \\ \phi(t_4 y t_1^{-1}) & \phi(t_4 y t_2^{-1}) & \phi(t_4 y t_3^{-1}) & \phi(t_4 y t_4^{-1}) \end{bmatrix} \]

\[ = \begin{bmatrix} \phi(e y e)^{-1} & \phi(e y e)^{-1} & \phi(e y e)^{-1} & \phi(e y e)^{-1} \\ \phi(x y e)^{-1} & \phi(x y e)^{-1} & \phi(x y e)^{-1} & \phi(x y e)^{-1} \\ \phi(x^2 y e)^{-1} & \phi(x^2 y e)^{-1} & \phi(x^2 y e)^{-1} & \phi(x^2 y e)^{-1} \\ \phi(x^3 y e)^{-1} & \phi(x^3 y e)^{-1} & \phi(x^3 y e)^{-1} & \phi(x^3 y e)^{-1} \end{bmatrix} \]
\[
\begin{bmatrix}
\phi(y) & \phi(yx^3) & \phi(yx^2) & \phi(yx) \\
\phi(xy) & \phi(y^3) & \phi(xy^2) & \phi(xy) \\
\phi(x^2y) & \phi(x^2yx^3) & \phi(y^4) & \phi(x^2yx) \\
\phi(x^3y) & \phi(x^3yx^3) & \phi(x^3yx^2) & \phi(y^2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
z^4 & 0 & 0 & 0 \\
0 & z^2 & 0 & 0 \\
0 & 0 & z & 0 \\
0 & 0 & 0 & z^3
\end{bmatrix}
\]

Now, \( z \) is the 5th root of unity. To find the value of \( z \) we must find the smallest possible field that has a fifth root of unity. We look for the smallest prime \( p \) such that \( 5|p-1 \). Therefore \( p = 11 \).

Now 2 is a primitive root of 11; that is, \( \text{Order}(2) = \phi(11-1) = 10 \Rightarrow \text{Order}(2) = 10 \).

It follows that \( \text{Order}(2^2) = 5 \), because if \( \text{Order}(a) = n \) and \( d \) is a positive divisor of \( n \), then

\[
\text{Order}(a^\frac{n}{d}) = d.
\]

Or generally,

\[
\text{Order}(a^d) = \frac{\text{Order}(a)}{\gcd(d, \text{Order}(a))}.
\]

Hence, \( |4| = 5 \).

Now the elements of order 5 in \( \mathbb{Z}_{11} \) are \( 4, 4^2 \equiv_{11} 5, 4^3 \equiv_{11} 9 \), and \( 4^4 \equiv_{11} 3 \). We will choose \( z = 3 \).

Then we have

\[
\begin{align*}
z^2 &= z \cdot z = 3 \cdot 3 = 9 \\
z^3 &= z \cdot z^2 = 3 \cdot 9 = 27 \equiv_{11} 5 \\
z^4 &= z^2 \cdot z^2 = 9 \cdot 9 = 81 \equiv_{11} 4
\end{align*}
\]
Thus,

\[ A(yy) = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \]

We verify these matrices by running the following loop.

```plaintext
> C:=CyclotomicField(5);
> GG:=GL(4,C);
> T:=Transversal(G,H);
> #T;
4
> A:=[[C.1,0,0,0] : i in [1..4]];
> for i,j in [1..4] do A[i,j]:=0; end for;
> GG:=GL(4,C);
> for i,j in [1..4] do if T[i]*xx*T[j]^(-1) in H then
    for|if> A[i,j]:=CH[2](T[i]*xx*T[j]^(-1));
    for|if>
end if; end for;
> B:=[[C.1,0,0,0] : i in [1..4]];
> for i,j in [1..4] do B[i,j]:=0; end for;
> for i,j in [1..4] do if T[i]*yy*T[j]^(-1) in H then
    for|if> B[i,j]:=CH[2](T[i]*yy*T[j]^(-1));
    for|if>
end if; end for;
> HH:=sub<GG|A,B>;
> #HH, #G;
20 20
> GG!A;
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]
> GG!B;
[-zeta_5^3 - zeta_5^2 - zeta_5 - 1 0 0 0]
[0 zeta_5^2 0 0]
[0 0 zeta_5 0]
[0 0 0 zeta_5^3]
```

The order of \( A(x) \) is 4 and the order of \( A(y) \) is 5. Also, the order of \( A(x)A(y) \)
is 4. Thus, \( A(x), A(y) > \) is a faithful representation of \( G = (C_4 : C_5) \), since 
\(|x| = 4 = |A(x)|, |y| = 5 = A(y) \), and 
\(|xy| = |A(x)A(y)| = 4 \).

Now we must convert these matrices into permutations.

\( a_{ij} = a \iff t_1 \to t_j^a \), where \( a_{ij} \) stands for the \( i \)th row and \( j \)th column of the matrix.

Then for \( A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \), we have

\( a_{12} = 1 \iff t_1 \to t_2 \),
\( a_{23} = 1 \iff t_2 \to t_3 \),
\( a_{34} = 1 \iff t_3 \to t_4 \),
\( a_{41} = 1 \iff t_4 \to t_1 \).

We have 4 \( t \)'s since \([G : H] = 4\), and our \( t \)'s are of order 11 since \( \mathbb{Z}_{11} \) is the smallest finite field that has 5th roots of unity.
Therefore, our permutation representation of $A(x)$ is

$$x = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)
(21, 22, 23, 24)(25, 26, 27, 28)(29, 30, 31, 32)(33, 34, 35, 36)(37, 38, 39, 40).$$

Similarly, for $A(y)$

\[
\begin{bmatrix}
4 & 0 & 0 & 0 \\
0 & 9 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 5 \\
\end{bmatrix}
\]

we have

$$a_{11} = 4 \implies t_1 \to t_1^4,$$

$$a_{22} = 9 \implies t_2 \to t_2^9,$$

$$a_{33} = 3 \implies t_3 \to t_3^3,$$

$$a_{44} = 5 \implies t_4 \to t_4^5.$$

Therefore, our permutation representation of $A(y)$ is

$$y = (1, 13, 17, 33, 9)(5, 29, 37, 25, 21)(6, 26, 30, 22, 38)(2, 34, 14, 10, 18)
(3, 11, 35, 19, 15)(4, 20, 12, 16, 36)(7, 23, 27, 39, 31)(8, 40, 24, 32, 28)

A presentation for $(4 : 5)$ is $< x, y \mid x^4, xy^4x^3y^3, y^3x^3yx >$. 
Thus, by definition, our presentation of the monomial progenitor $11^*4:m(4 : 5)$ will be

\[ < x, y, z, t | x^4, xy^4x^3y^3, y^2x^3yx, t^m, Normaliser(N, < t >) > \]

Since our $t$’s are of order 11, we will have $t^m = t^{11}$.

Now we must find the $Normaliser(N, < t >)$, that is, the permutations that stabilize all the powers of $t_1$, $\{1, 5, 9, 13, 17, 21, 25, 29, 33, 37\}$.

Let $t \sim t_1$.

We then find these permutations.

```maple
> Normaliser := Stabiliser(N, \{1, 5, 9, 13, 17, 21, 25, 29, 33, 37\});
> Generators(Normaliser);

\{(1, 13, 17, 33, 9)(2, 34, 14, 10, 18)(3, 11, 35, 19, 15)
(4, 20, 12, 16, 36)(5, 29, 37, 25, 21)(6, 26, 30, 22, 38)
(7, 23, 27, 39, 31)(8, 40, 24, 32, 28)\}
```

Therefore, $Normaliser(N, < t_1 >) = < (1, 13, 17, 33, 9)(2, 34, 14, 10, 18)(3, 11, 35, 19, 15)

Since $y = (1, 13, 17, 33, 9)(2, 34, 14, 10, 18)(3, 11, 35, 19, 15)(4, 20, 12, 16, 36)
(5, 29, 37, 25, 21)(6, 26, 30, 22, 38)(7, 23, 27, 39, 31)(8, 40, 24, 32, 28)$, and

$y$ sends $t_1$ to $t_1^4$, we have that $t^y = t^4$.

Thus, the presentation of the monomial progenitor is given by

$11^*4:m(4 : 5) = < x, y, t | x^4, xy^4x^3y^3, y^2x^3yx, t^{11}, t^y = t^4 >$.

Next we add the following first order relations to our progenitor to find finite homomorphic images.

\[ (x^2t^y)^3, (x^4t)^8, (yt^x)^5, (xt^y)^3 \]
\[
\text{G<x,y,t>:=Group<x,y,t|x^4,x*y^4*x^3*y^3,t^11,t*y=t^4,(x^2+t^-(y^{-3}))^3,(x^3*t)^8,}
\]
\[
(y*t^-x)^5,}
\]
\[
(x*t^-y^4))^3>;
\]
\[
\text{#G;}
\]
\[
7920
\]
\[
*/
\]
\[
7920
\]
\[
*/
\]
\[
\text{S:=Sym(40);}
\]
\[
\text{xx:=S!(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)}
\]
\[
(17,18,19,20)(21,22,23,24)(25,26,27,28)
\]
\[
(29,30,31,32)(33,34,35,36)(37,38,39,40);}
\]
\[
\text{yy:=S!(1,13,17,33,9)(5,29,37,25,21)}
\]
\[
(6,26,30,22,38)(2,34,14,10,18)(3,11,35,19,15)
\]
\[
(4,20,12,16,36)
\]
\[
(7,23,27,39,31)(8,40,24,32,28);}
\]
\[
\text{N:=sub<S|xx,yy>};
\]
\[
\text{f,G1,k:=CosetAction(G,sub<G|x,y>)};
\]
\[
\text{#k;}
\]
\[
1
\]
\[
\text{CompositionFactors(G1);}
\]
\[
G
\]
\[
\mid \text{M11}
\]
\[
1
\]

Manual Double Coset enumeration will follow in a later chapter.

4.3 Monomial Progenitor 11^2 :_m D_{10}

Consider 11^2 :_m D_{10}. G = D_{10} is given by
\[
G = D_{10} = < x, y|x^{10}, y^2, (x^{-1}y)^2 >, \text{ where}
\]
x = (1, 4, 6, 8, 10, 12, 14, 16, 18, 19)(2, 3, 5, 7, 9, 11, 13, 15, 17, 20), and
\[
\]

\[
G = D_{10} \text{ has monomial irreducible representation in dimension 2. We will write a}
\]
progenitor for 11^{11} :_m D_{10}. Since \[
\frac{|G|}{|M|} = 2 \Rightarrow \frac{20}{10} = 2 \Rightarrow |H| = 10, \text{ we need to find a}
\]
subgroup H of order 10 and induce a linear character of H up to G to obtain the
irreducible character of degree 2 of $G$.

The conjugacy classes of group $D_{10}$ are given in the table below.

<table>
<thead>
<tr>
<th>Class $C_i$</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$(1, 12)(2, 11)(3, 13)(4, 14)(5, 15)(6, 16)(7, 17)(8, 18)(9, 20)(10, 19)$</td>
<td>1</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$(1, 15)(2, 16)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)(17, 19)(18, 20)$</td>
<td>5</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$(1, 6, 10, 14, 18)(2, 5, 9, 13, 17)(3, 7, 11, 15, 20)(4, 8, 12, 16, 19)$</td>
<td>2</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$(1, 10, 18, 6, 14)(2, 9, 17, 5, 13)(3, 11, 20, 7, 15)(4, 12, 19, 8, 16)$</td>
<td>2</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$(1, 4, 6, 8, 10, 12, 14, 16, 18, 19)(2, 3, 5, 7, 9, 11, 13, 15, 17, 20)$</td>
<td>2</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$(1, 8, 14, 19, 6, 12, 18, 4, 10, 16)(2, 7, 13, 20, 5, 11, 17, 3, 9, 15)$</td>
<td>2</td>
</tr>
</tbody>
</table>

Consider the subgroup $H = Z_{10}$ of $G$ given below.

$H = \{ e, (1, 12)(2, 11)(3, 13)(4, 14)(5, 15)(6, 16)(7, 17)(8, 18)(9, 20)(10, 19), (1, 6, 10, 14, 18)(2, 5, 9, 13, 17)(3, 7, 11, 15, 20)(4, 8, 12, 16, 19), (1, 10, 18, 6, 14)(2, 9, 17, 5, 13)(3, 11, 20, 7, 15)(4, 12, 19, 8, 16), (1, 18, 14, 10, 6)(2, 17, 13, 9, 5)(3, 20, 15, 11, 7)(4, 19, 16, 12, 8), (1, 19, 18, 16, 14, 12, 10, 8, 6, 4)(2, 20, 17, 15, 13, 11, 9, 7, 5, 3), (1, 4, 6, 8, 10, 12, 14, 16, 18, 19)(2, 3, 5, 7, 9, 11, 13, 15, 17, 20), (1, 8, 14, 19, 6, 12, 18, 4, 10, 16)(2, 7, 13, 20, 5, 11, 17, 3, 9, 15), (1, 16, 10, 4, 18, 12, 6, 19, 14, 8)(2, 15, 9, 3, 17, 11, 5, 20, 13, 7), (1, 14, 6, 18, 10)(2, 13, 5, 17, 9)(3, 15, 7, 20, 11)(4, 16, 8, 19, 12) \}.$

The conjugacy classes of $Z_{10}$ are given in the table below.

<table>
<thead>
<tr>
<th>Class $D_i$</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$(1, 12)(2, 11)(3, 13)(4, 14)(5, 15)(6, 16)(7, 17)(8, 18)(9, 20)(10, 19)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_3$</td>
<td>$(1, 10, 16, 14, 18)(2, 5, 9, 13, 17)(3, 7, 11, 15, 20)(4, 8, 12, 16, 19)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$(1, 10, 18, 6, 14)(2, 9, 17, 5, 13)(3, 11, 20, 7, 15)(4, 12, 19, 8, 16)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_5$</td>
<td>$(1, 14, 6, 18, 10)(2, 13, 5, 17, 9)(3, 15, 7, 20, 11)(4, 16, 8, 19, 12)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_6$</td>
<td>$(1, 18, 14, 10, 6)(2, 17, 13, 9, 5)(3, 20, 15, 11, 7)(4, 19, 16, 12, 8)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_7$</td>
<td>$(1, 4, 6, 8, 10, 12, 14, 16, 18, 19)(2, 3, 5, 7, 9, 11, 13, 15, 17, 20)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_8$</td>
<td>$(1, 18, 14, 19, 6, 12, 18, 4, 10, 16)(2, 7, 13, 20, 5, 11, 17, 3, 9, 15)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_9$</td>
<td>$(1, 16, 10, 4, 18, 12, 6, 19, 14, 8)(2, 15, 9, 3, 17, 11, 5, 20, 13, 7)$</td>
<td>1</td>
</tr>
<tr>
<td>$D_{10}$</td>
<td>$(1, 19, 18, 16, 14, 12, 10, 8, 6, 4)(2, 20, 17, 15, 13, 11, 9, 7, 5, 3)$</td>
<td>1</td>
</tr>
</tbody>
</table>
Consider the irreducible characters $\phi$ (of $H$) and $\chi$ (of $G$) given below.

<table>
<thead>
<tr>
<th>Class</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$D_6$</th>
<th>$D_7$</th>
<th>$D_8$</th>
<th>$D_9$</th>
<th>$D_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

$\phi_1$ 1 1 1 1 1 1 1 1 1 1
$\phi_2$ 1 -1 1 1 1 -1 -1 -1 -1 -1
$\phi_3$ 1 1 $Z$ $Z^2$ $Z^3$ $Z^4$ $Z^3$ $Z$ $Z^2$
$\phi_4$ 1 -1 $Z$ $Z^2$ $Z^3$ $Z^4$ $-Z^3$ $-Z^4$ $-Z$ $-Z^2$
$\phi_5$ 1 1 $Z^2$ $Z^4$ $Z$ $Z^3$ $Z$ $Z^3$ $Z^2$ $Z^4$
$\phi_6$ 1 -1 $Z^2$ $Z^4$ $Z$ $Z^3$ $-Z$ $-Z^3$ $-Z^2$ $-Z^4$
$\phi_7$ 1 1 $Z^3$ $Z$ $Z^4$ $Z^2$ $Z^4$ $Z^2$ $Z^3$ $Z$
$\phi_8$ 1 -1 $Z^3$ $Z$ $Z^4$ $Z^2$ $-Z^4$ $-Z^2$ $-Z^3$ $-Z$
$\phi_9$ 1 1 $Z^4$ $Z^3$ $Z^2$ $Z$ $Z^2$ $Z$ $Z^4$ $Z^3$
$\phi_{10}$ 1 -1 $Z^4$ $Z^3$ $Z^2$ $Z$ $-Z^2$ $-Z$ $-Z^4$ $-Z^3$

where $Z$ is the 5th root of unity.

<table>
<thead>
<tr>
<th>Class</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
<th>$C_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

$\chi_1$ 1 1 1 1 1 1 1 1
$\chi_2$ 1 1 -1 -1 1 1 1 1
$\chi_3$ 1 -1 -1 1 1 1 -1 -1
$\chi_4$ 1 -1 1 -1 1 1 -1 -1
$\chi_5$ 2 -2 0 0 $Z$ $Z^2$ $-Z^2$ $-Z$
$\chi_6$ 2 2 0 0 $Z$ $Z^2$ $Z^2$ $Z$
$\chi_7$ 2 2 0 0 $Z^2$ $Z$ $Z$ $Z^2$
$\chi_8$ 2 -2 0 0 $Z^2$ $Z$ $-Z$ $-Z^2$

where $Z$ is the 5th root of unity.

Next we must find a non-trivial linear character of $H$ to induce up to $G$. Note that
each character of $H$ is linear since they all have degree 1. We will induce $\chi_4$ up to $G$.

> CH[4];
( 1, -1, zeta(5)_5, zeta(5)_5^2, zeta(5)_5^3,
  -zeta(5)_5^5 - zeta(5)_5^2 - zeta(5)_5 - 1,
  -zeta(5)_5^5, zeta(5)_5^3 + zeta(5)_5^2 +
  zeta(5)_5 + 1, -zeta(5)_5, -zeta(5)_5^2 )

We will use a loop to find the two induced representations $A(x)$ and $A(y)$ of degree $\frac{|G|}{|H|} = 20 \div 10 = 2$. First we must input the values of $z, z^2, z^3, z^4, -z, -z^2, -z^3,$ and $-z^4$ into our loop, which we will denote as C.1, C.2, C.3, C.4, -C, -C.1, -C.2, -C.3, and -C.4, respectively.

Now, $z$ is the 5th root of unity. To find the value of $z$ we must find the smallest possible field that has a fifth root of unity. We look for the smallest prime $p$ such that $5 | (p - 1)$. Therefore $p = 11$. Now, 2 is a primitive root of 11; that is, $\text{Order}(2) = \phi(11 - 1) = 10 \Rightarrow \text{Order}(2) = 10$. It follows that $\text{Order}(2^2) = 5$, because if $\text{Order}(a) = n$ and $d$ is a positive divisor of $n$, then

$$\text{Order}(a^\frac{n}{d}) = d.$$  

Or generally,

$$\text{Order}(a^d) = \frac{\text{Order}(a)}{\gcd(d, \text{Order}(a))}.$$  

Hence, $|4| = 5$.

Now the elements of order 5 in $\mathbb{Z}_{11}$ are $4, 4^2 \equiv_{11} 5, 4^3 \equiv_{11} 9$, and $4^4 \equiv_{11} 3$. We will choose $z = 4$. Then we have,

$$z^2 = z \cdot z = 4 \cdot 4 = 16 \equiv_{11} 5$$
$$z^3 = z \cdot z^2 = 4 \cdot 16 = 64 \equiv_{11} 9$$
$$z^4 = z^2 \cdot z^2 = 16 \cdot 16 = 256 \equiv_{11} 3.$$
Also, \( z^4 + z^3 + z^2 + z + 1 = 0 \)

\[ \implies z^4 + z^3 + z^2 + 1 = -z \]

\[ \implies 3 + 9 + 5 + 1 = -z \]

\[ \implies 18 \equiv_{11} 7 = -z. \]

Similarly, \( z^4 + z^3 + z^2 + z + 1 = 0 \)

\[ \implies z^4 + z^3 + z + 1 = -z^2 \]

\[ \implies 3 + 9 + 4 + 1 = -z^2 \]

\[ \implies 17 \equiv_{11} 6 = -z^2, \]

\( z^4 + z^3 + z^2 + z + 1 = 0 \)

\[ \implies z^4 + z^2 + z + 1 = -z^3 \]

\[ \implies 3 + 5 + 4 + 1 = -z^3 \implies 13 \equiv_{11} 2 = -z^3, \text{ and} \]

\( z^4 + z^3 + z^2 + z + 1 = 0 \)

\[ \implies z^3 + z^2 + z + 1 = -z^4 \]

\[ \implies 9 + 5 + 4 + 1 = -z^4 \]

\[ \implies 19 \equiv_{11} 8 = -z^4. \]

\[ \text{function } \text{mat } := \text{function}(n,p,D,k) \]
\[ \text{function } \text{for } i, j \text{ in [1..k] do if } T[i] \ast p \ast T[j]^{-1} \text{ in H then} \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv C.1 \text{ then } D[i,j] := 4; \text{ fi}; \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv -C.1 \text{ then } D[i,j] := 7; \text{ fi}; \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv C.1^2 \text{ then } D[i,j] := 5; \text{ fi}; \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv -C.1^2 \text{ then } D[i,j] := 6; \text{ fi}; \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv C.1^3 \text{ then } D[i,j] := 9; \text{ fi}; \]
\[ \text{function } \text{for } \text{if } \text{if } \text{CH}(n)(T[i] \ast p \ast T[j]^{-1}) \equiv -C.1^3 \]
function|for|if> then D[i,j]:=2; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq C.1^4
function|for|if> then D[i,j]:=3; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq -C.1^4
function|for|if> then D[i,j]:=8; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq 1
function|for|if> then D[i,j]:=1; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq -1
function|for|if> then D[i,j]:=1; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) in {1,-1}
function|for|if> then D[i,j]:=CH[n](T[i]*p*T[j]^-1); end if;
function|for|if> end if; end for;
function> return D;
function> end function;
> A:=[[0,0]: i in [1..2]];
> mat(4,xx,A,2);
[ [ 2, 0 ],
  [ 0, 6 ] ]
> mat(4,yy,A,2);
[ [ 0, 1 ],
  [ 1, 0 ] ]

Thus
\[ A(x) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \text{ and } A(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

The order of \( A(x) \) is 10 and the order of \( A(y) \) is 2. Also, the order of \( A(x) \cdot A(y) \) is 2.
Thus, \( < A(x), A(y) > \) is a faithful representation of \( G = D_{10} \), since \( |x| = 10 = |A(x)|, 
|y| = 2 = A(y), \text{ and } |xy| = |A(x) \cdot A(y)| = 2. \)

Now we must convert these matrices into permutations.

\( a_{ij} = a \leftrightarrow t_1 \rightarrow t_j^a \), where \( a_{ij} \) stands for the \( i \)-th row and \( j \)-th column of the matrix.
Then for \( A(x) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \), we have

\[ a_{11} = 2 \implies t_1 \rightarrow t_1^2, \]
\[ a_{22} = 6 \implies t_2 \rightarrow t_2^6. \]

We have 2 \( t \)'s since \([G : H] = 2\), and our \( t \)'s are of order 11 since \( \mathbb{Z}_{11} \) is the smallest finite field that has 5th roots of unity.

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
t_1 & t_2 & t_1^2 & t_2^2 & t_1^3 & t_2^3 & t_1^4 & t_2^4 & t_1^5 & t_2^5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
t_1^6 & t_2^6 & t_1^7 & t_2^7 & t_1^8 & t_2^8 & t_1^9 & t_2^9 & t_1^{10} & t_2^{10} \\
3 & 12 & 7 & 2 & 11 & 14 & 15 & 4 & 19 & 16 \\
\end{array}
\]

Therefore, our permutation representation of \( A(x) \) is

\[ x = (1, 3, 7, 15, 9, 19, 17, 13, 5, 11)(2, 12, 6, 14, 18, 20, 10, 16, 8, 4) \]

Similarly, for \( A(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), we have

\[ a_{12} = 1 \implies t_1 \rightarrow t_2, \]
\[ a_{21} = 1 \implies t_2 \rightarrow t_1. \]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
t_1 & t_2 & t_1^2 & t_2^2 & t_1^3 & t_2^3 & t_1^4 & t_2^4 & t_1^5 & t_2^5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
t_2 & t_1 & t_2^2 & t_1^2 & t_2^3 & t_1^3 & t_2^4 & t_1^4 & t_2^5 & t_1^5 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
t_1^6 & t_2^6 & t_1^7 & t_2^7 & t_1^8 & t_2^8 & t_1^9 & t_2^9 & t_1^{10} & t_2^{10} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
t_2^6 & t_1^6 & t_2^7 & t_1^7 & t_2^8 & t_1^8 & t_2^9 & t_1^9 & t_2^{10} & t_1^{10} \\
12 & 11 & 14 & 13 & 16 & 15 & 18 & 17 & 20 & 19 \\
\end{array}
\]
Therefore, our permutation representation of \( A(y) \) is
\[
y = (1, 2)(3, 4)(5, 6), (7, 8), (9, 10), (11, 12), (13, 14), (15, 16), (17, 18), (19, 20).
\]

A presentation for \( D_{10} \) is \(< x, y | x^{10}, y^2, (x^{-1}y)^2 >\).

Thus, by definition, our presentation of the monomial progenitor \( 11^2 :_m D_{10} \) will be
\[
< x, y | x^{10}, y^2, (x^{-1}y)^2, t^m, Normaliser(N, < t >) >
\]

Since our \( t \)'s are of order 11, we will have \( t^m = t^{11} \).

Now we must find the \( Normaliser(N, < t >) \), that is, the permutations that stabilize all the powers of \( t_1 \), \( \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\} \).

Let \( t \sim t_1 \).

We then find these permutations.

\[
> \text{Normaliser} := \text{Stabiliser}(N, \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\})
> \text{Generators(Normaliser)};
> \{ (1, 11, 5, 13, 17, 19, 9, 15, 7, 3) (2, 4, 8, 16, 10, 20, 18, 14, 6, 12) \}
\]

Therefore, \( Normaliser(N, < t_1 >) = < (1, 11, 5, 13, 17, 19, 9, 15, 7, 3) (2, 4, 8, 16, 10, 20, 18, 14, 6, 12) >. \)

Since \( x^{-1} = (1, 11, 5, 13, 17, 19, 9, 15, 7, 3) (2, 4, 8, 16, 10, 20, 18, 14, 6, 12) \), and \( x^{-1} \) sends \( t_1 \) to \( t_1^6 \), we have that \( t_1x^{-1} = t_1^6 \).

Thus, the presentation of the monomial progenitor is given by
\[
11^2 :_m D_{10} = < x, y, t | x^{10}, y^2, (x^{-1}y)^2, t^{11}, t^{-1}x^{-1} = t^{6} >.
\]
Next we add the following first order relations to our progenitor to find finite homomorphic images.

\[(x^5t)^2, (yt)^3\]

\[
> G<x,y,t>:=\text{Group}<x,y,t|x^{10},y^{2},(x^{-1}y)^{2},
t^{11},t^{(x^{-1})}=t^{6},(x^{5}t)^{2},(y*t)^{3}>;
> \#G;
1320
> f,G1,k:=\text{CosetAction}(G,\text{sub}<G|x>);
> \#k;
1
> \text{CompositionFactors}(G1);
\]

\[
\begin{array}{c|c}
G & \text{Cyclic}(2) \\
* & A(1, 11) = L(2, 11) \\
1 & 
\end{array}
\]

Double Coset Enumeration will be performed in a later chapter.
Chapter 5

Double Coset Enumeration

5.1 Preliminaries

Definition 5.1. [Rot95] The Dihedral Group $D_n$, $n$ even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $<a, b|a^{\frac{n}{2}}, b^2, (ab)^2>$. 

5.2 $L_2(11)$ as a Homomorphic Image of $2^6 : D_{12}$

5.2.1 The Construction of $L_2(11)$ Over $D_{12}$

Consider $2^6 : D_{12}$, where $D_{12} = <x, y, z | x \sim (12)(35)(46), y \sim (134)(256)$, $z \sim (12)(36)(45)$, and $t \sim t_1$.

The progenitor $2^6 : D_{12}$ is factored by $(xt)^3, (zt^y)^5, (yt)^5$, and $(xty)^6$.

$$G = \frac{2^6 : D_{12}}{(xt)^3, (zt^y)^5, (yt)^5, (xty)^6}$$ has symmetric presentation

$<x, y, z, t | x^2, y^3, z^2, y^2 xyx, (y^2 z)^2, (xz)^2, t^2, (t, xz), (xt)^3, (zt^y)^5, (yt)^5, (xty)^6>$

We will first show that $|G| \leq 336$ by performing manual double coset enumeration of $G$ over $N$. 

Let us expand our additional relation

\[(xt)^3 = 1\]
\[(xt_1)^3 = 1\]
\[(x^3t_1^2t_1^1t_1) = 1\]  \hspace{1cm} (5.1)
\[(12)(35)(46)t_1t_2t_1 = 1\]
\[(12)(35)(46)t_1 = t_1t_2\]
\[Nt_1 = Nt_1t_2\]

\[(zt^y)^5 = 1\]
\[(zt_1^y)^5 = 1\]
\[(zt_3)^5 = 1\]
\[z^5t_3^5t_3^3t_3^2t_3^1t_3^2 = 1\]  \hspace{1cm} (5.2)
\[(12)(36)(45)t_3^6t_3^3(t_{1,2})t_3^3(3,6)(4,5)t_3^3t_3^3(1,2)(3,6)(4,5)t_3 = 1\]
\[(12)(36)(45)t_3t_6t_3t_6t_3 = 1\]
\[(12)(36)(45)t_3t_6 = t_3t_6t_3\]

\[(yt)^5 = 1\]
\[(yt_1)^5 = 1\]
\[y^5t_1^5t_1^4t_1^3t_1^2t_1^1t_1^0t_1^1t_1^1 = 1\]  \hspace{1cm} (5.3)
\[(143)(265)t_1^{(134)(256)}t_1^{(143)(265)}t_1^{(134)(256)}t_1 = 1\]
\[(143)(205)t_3t_4t_3t_1 = t_1t_3t_4\]

\[(xyt)^6 = 1\]
\[(xyt_1)^6 = 1\]
\[(xy)^6t_1^xyt_1^xyt_1^xyt_1^xyt_1^xyt_1^xyt_1^xyt_1 = 1\]  \hspace{1cm} (5.4)
\[et_1^{(163245)}t_1^{(134)(256)}t_1^{(12)(35)(46)}t_1^{(143)(265)}t_1^{(154236)}t_1 = 1\]
\[t_6t_3t_2t_4t_5t_1 = 1\]
\[t_6t_3t_2 = t_1t_5t_4\]
Our first double coset, \( N e N = \{ N e^n | n \in N \} = \{ N \} \),
which we will denote by \([*]\).

\( N \) is transitive on \( \{1, 2, 3, 4, 5, 6\} \) so it has one single orbit \( \{1, 2, 3, 4, 5, 6\} \).

We will take a representative from this orbit, say 1, and find out to which double coset \( N t_1 \) belongs.

\( N t_1 N \) is a new double coset which we will denote by \([1]\).

Since the orbit \( \{1, 2, 3, 4, 5, 6\} \) contains 6 elements then 6 symmetric generators will
go to the new double coset \([1]\).

\( N^1 = \) Point Stabiliser in \( N \) of \( N t_1 = \{ n \in N | t_1^n = t_1 \} = \{ e, (34)(56) \} \).
\( N^{(1)} = \) Coset Stabiliser in \( N \) of \( N t_1 = \{ n \in N | N t_1^n = t_1 \} = \{ e \} = N^1 \).

Now \( N^{(1)} \geq N^1 \).
\( N^1 = \{ e, (34)(56) \} \).

Since we do not have a relation that will increase the Coset Stabiliser \( N^{(1)} \), then
\( N^{(1)} = N^1 = \{ e, (34)(56) \} \).

The number of single cosets in \( N t_1 N \) is at most \( \frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 6 \).

\( N t_1 N = \{ N t_1^n | n \in N \} \).
\( N t_1 N = \{ N t_1, N t_2, N t_3, N t_4, N t_5, N t_6 \} \).

The orbits of \( N^{(1)} \) on \( \{1, 2, 3, 4, 5, 6\} \) are \( \{1\}, \{2\}, \{3, 4\}, \) and \( \{5, 6\} \). We take \( t_1, t_2, t_3, \) and \( t_5 \), from each orbit respectively, and determine to which double coset
\( N t_1 t_1, N t_1 t_2, N t_1 t_3, \) and \( N t_1 t_5 \) belong.
$Nt_1t_1 = N \in [\ast]$ (Since our $t$’s are of order 2.)

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset $[\ast]$.

$Nt_1t_2 = Nt_1 \in [1]$ (by Equation 5.1).

One symmetric generator will go back to $[1]$.

$Nt_1t_3N$ is a new double coset which we will denote $[13]$.

Two symmetric generators will go to the new double coset $[13]$.

$Nt_1t_5N$ is a new double coset which we will denote $[15]$.

Two symmetric generators will go to the new double coset $[15]$.

Below is our Cayley Diagram thus far.

\[ N^{(13)} \geq N^{13} \]
\[ N^{13} = \{e\} \]

Since we do not have a relation that will increase the Coset Stabiliser $N^{(13)}$, then $N^{(13)} = N^{13} = \{e\}$. 
The number of single cosets in $N_{t_1}t_3N$ is at most $\frac{|N|}{|N^{(15)}|} = \frac{12}{1} = 12$.

$N_{t_1}t_3N = \{N_{t_1}t_3^n \mid n \in N\}$.

$N_{t_1}t_3N = \{N_{t_1}t_3, N_{t_2}t_5, N_{t_3}t_4, N_{t_2}t_6, N_{t_4}t_1, N_{t_5}t_6, N_{t_1}t_4, N_{t_6}t_2, N_{t_6}t_5, N_{t_5}t_2, N_{t_4}t_3, N_{t_3}t_1\}$.

$N^{(15)} \geq N^{15}$.

$N^{15} = \{e\}$.

Since we do not have a relation that will increase the Coset Stabiliser $N^{(15)}$, then $N^{(15)} = N^{15} = \{e\}$.

The number of single cosets in $N_{t_1}t_5N$ is at most $\frac{|N|}{|N^{(15)}|} = \frac{12}{1} = 12$.

$N_{t_1}t_5N = \{N_{t_1}t_5^n \mid n \in N\}$.

$N_{t_1}t_5N = \{N_{t_1}t_5, N_{t_1}t_6, N_{t_2}t_3, N_{t_2}t_4, N_{t_3}t_6, N_{t_3}t_2, N_{t_4}t_2, N_{t_4}t_5, N_{t_5}t_4, N_{t_5}t_1, N_{t_6}t_1, N_{t_6}t_3\}$.

The orbits of $N^{(13)}$ on \{1, 2, 3, 4, 5, 6\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, and \{6\}. We take $t_1$, $t_2$, $t_3$, $t_4$, $t_5$, and $t_6$, from each orbit respectively, and determine to which double coset $N_{t_1}t_3t_1$, $N_{t_1}t_3t_2$, $N_{t_1}t_3t_3$, $N_{t_1}t_3t_4$, $N_{t_1}t_3t_5$, and $N_{t_1}t_3t_6$ belong.

First we will examine $N_{t_1}t_3t_1$.

$$(143)(265)t_3t_1 = t_1t_3t_4$$

by Equation 5.3

$$\Rightarrow (143)(265)t_3t_1t_4 = t_1t_3t_4t_4$$

$$\Rightarrow (143)(265)t_3t_1t_4 = t_1t_3$$

$$\Rightarrow t_4(143)(265)t_3t_1t_4 = t_4t_1t_3$$

$$\Rightarrow (143)(265)t_4(143)(265)t_3t_1t_4 = t_4t_1t_3$$

$$\Rightarrow (143)(265)t_4t_3t_1t_4 = t_4t_1t_3$$

$$\Rightarrow (143)(265)t_4t_1t_4 = t_4t_1t_3$$

$$\Rightarrow (143)(265)t_1t_4 = t_4t_1t_3$$

$$\Rightarrow (143)(265)t_1t_4t_1 = t_4t_1t_3t_1$$

Also,

$$(143)(265)t_3t_1 = t_1t_3t_4$$

by Equation 5.3

$$\Rightarrow [(143)(265)t_3t_1]^{(13)(25)} = [t_1t_3t_4]^{(13)(25)}$$
\[ (134)(256)t_1t_3 = t_3t_1t_4 \]
\[ (134)(256)t_1t_3t_4 = t_3t_1t_4t_4 \]
\[ (134)(256)t_1t_3t_4 = t_3t_1 \]

Therefore, \((143)(265)t_1t_4t_4 = t_4t_1t_3t_1\)
\[ (134)(265)t_1t_4t_4 = t_4t_1[(134)(256)t_1t_3t_4] \]
\[ (134)(265)t_1t_4t_4 = (134)(256)[t_4t_1]^{(134)(256)}t_1t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (134)(256)t_1t_3t_1t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (134)(256)[e]t_1t_3t_1t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (134)(256)[(12)(35)(46)t_1t_2t_1]t_1t_3t_1t_3t_4, \text{ by Equation 5.1} \]
\[ (134)(265)t_1t_4t_4 = (152436)t_1t_2t_3t_1t_3t_4. \]
\[ t_6t_3t_2 = t_1t_5t_4, \text{ by Equation 5.4} \]
\[ [t_6t_3t_2]^{(12)(35)(46)} = [t_1t_5t_4]^{(12)(35)(46)} \]
\[ t_1t_5t_1 = t_2t_3t_6 \]
\[ t_4t_5t_1t_6 = t_2t_3t_6t_6 \]
\[ t_4t_5t_1t_6 = t_2t_3 \]

Thus, \((143)(265)t_1t_4t_4 = (154236)t_1t_2t_3t_1t_3t_4\)
\[ (12)(36)(45)t_3t_6 = t_3t_6t_3, \text{ by Equation 5.2} \]
\[ (35)(16)(24)t_1t_6 = t_1t_6t_1 \]
\[ (16)(24)(35)t_1t_6 = t_1t_6t_1 \]

Thus,
\[ (143)(265)t_1t_4t_4 = (154236)t_1t_2t_3t_1t_3t_4t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (154236)t_1t_2t_3[16)(24)(35)t_1t_6]t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (154236)(16)(24)(35)[t_1t_4t_5]^{(16)(24)(35)}t_1t_6t_3t_4 \]
\[ (143)(265)t_1t_4t_4 = (13)(25)t_6t_2t_3t_1t_6t_3t_4 \]

Also,
\[ t_6t_3t_2 = t_1t_5t_4, \text{ by Equation 5.4} \]
\[ [t_6t_3t_2]^{(154236)} = [t_1t_5t_4]^{(154236)} \]
\[ t_1t_6t_3 = t_5t_4t_2 \]

Thus,
\[ (143)(265)t_1t_4t_4 = (13)(25)t_6t_2t_3t_1t_6t_3t_4 \]
\[ (134)(265)t_1t_4t_4 = (13)(25)t_6t_2t_3[t_5t_4t_2]t_4 \]
$$\Rightarrow (13)(25) t_1 t_4 t_1 = (13)(25)t_6 t_2 t_3 t_5 t_4 t_2$$

$$\Rightarrow (13)(25) t_1 t_4 t_1 = (13)(25)t_6 t_2 t_3 [16)(24)(35) t_4 t_2], \text{ since}$$

$$[(12)(36)(45) t_3 t_6]^{(134)(256)} = [t_3 t_6 t_3]^{(134)(256)} \Rightarrow (16)(24)(35) t_4 t_2 = t_4 t_2 t_4 \text{ (Equation 5.2)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (13)(25)(16)(24)(35)(t_6 t_2 t_3 t_5)^{(16)(24)(35)} t_4 t_2$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (154236)t_1 t_4 t_5 t_3 t_4 t_2$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (154236)t_1 t_4 [(12)(35)(46)t_5 t_4 t_2], \text{ since}$$

$$[(12)(35)(46)t_1]^{(154236)} = [t_1 t_2]^{(154236)} \Rightarrow (12)(35)(46)t_5 = t_5 t_3 \text{ (Equation 5.1)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (154236)(12)(35)(46)(t_1 t_4)^{(12)(35)(46)} t_5 t_4 t_2$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (134)(256)t_2 t_6 t_5 t_4 t_2$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (134)(256)t_2 t_6 t_5 t_4 t_3, \text{ Equation 5.4 conjugated by (15)(23)(46)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (134)(256)t_2 t_6 t_5 t_4 t_3$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (134)(265)t_2 [(16)(24)(35) t_6 t_1], \text{ since}$$

$$[(12)(36)(45) t_3 t_6]^{(13)(25)} = [t_3 t_6 t_3]^{(13)(25)} \Rightarrow (16)(24)(35) t_1 t_6 = t_1 t_6 t_1 \text{ (Equation 5.2)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (134)(256)(16)(24)(35) t_2^{(16)(24)(35)} t_6 t_1 t_3$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (15)(23)(46)t_4 t_6 t_1 t_3$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (15)(23)(46)[(12)(35)(46) t_4] t_1 t_3, \text{ since}$$

$$[(12)(35)(46) t_1]^{(143)(265)} = [t_1 t_2]^{(143)(265)} \Rightarrow (12)(35)(46) t_4 = t_4 t_6 \text{ (Equation 5.1)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (13)(25) t_4 t_1 t_3$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (13)(25)[(143)(265) t_1 t_4], \text{ since}$$

$$[(143)(265) t_3 t_4]^{(143)(265)} = [t_1 t_3 t_4]^{(143)(265)} \Rightarrow (143)(265) t_1 t_4 = t_4 t_3 \text{ (Equation 5.3)}$$

$$\Rightarrow (13)(265) t_1 t_4 t_1 = (34)(56) t_1 t_4$$

$$\Rightarrow [(143)(265) t_1 t_4 t_1]^{(34)(56)} = [(34)(56) t_1 t_4]^{(34)(56)}$$

$$\Rightarrow (134)(265) t_1 t_3 t_1 = (34)(56) t_1 t_3$$

Thus, \( N t_1 t_3 t_1 = N t_1 t_3 \in [13] \).

One symmetric generator will go back to [13].

\( N t_1 t_3 t_2 N \) is a new double coset which we will denote [132].

One symmetric generator will go to [132].

\( N t_1 t_3 t_3 = N t_1 \in [1] \).

One symmetric generator will go back to [1].
$Nt_1t_3t_4 = Nt_3t_1 \in [13]$, by Equation 5.3.
One symmetric generator will go back to [13].

$Nt_1t_3t_5 = Nt_2t_3 \in [15]$, by Equation 5.1, since

\[(12)(35)(46)t_1\]^{(13)(25)} = [t_1t_2]^{(13)(25)} \implies (12)(35)(46)t_3 = t_3t_5

\implies t_1(12)(35)(46)t_3 = t_1t_3t_5

\implies (12)(35)(46)t_1^{(12)(35)(46)}t_3 = t_1t_3t_5

\implies (12)(35)(46)t_2t_3 = t_1t_3t_5.

One symmetric generator will go to [15].

$Nt_1t_3t_6N$ is a new double coset which we will denote [136].
One symmetric generator will go to [136].

Below is our Cayley Diagram thus far.

The orbits of $N^{(15)}$ on \{1, 2, 3, 4, 5, 6\} are \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}. We take $t_1$, $t_2$, $t_3$, $t_4$, $t_5$, and $t_6$, from each orbit respectively, and determine to which double coset $Nt_1t_5t_1$, $Nt_1t_5t_2$, $Nt_1t_5t_3$, $Nt_1t_5t_4$, $Nt_1t_5t_5$, and $Nt_1t_5t_6$ belong.
\[ Nt_1t_5t_1 = Nt_1t_5 \in [15], \text{ since} \]
\[ [(12)(36)(45)t_5t_6]^{(143)(265)} = [t_3t_6t_3]^{(143)(265)} \]
\[ \implies (15)(23)(46)t_1t_5 = t_1t_5t_1, \text{ by Equation 5.2.} \]
One symmetric generator will go to [15].

\[ Nt_1t_5t_2N \text{ is a new double coset which we will denote [152].} \]
One symmetric generator will go to [152].

\[ Nt_1t_5t_3 = Nt_2t_5 \in [13], \text{ since} \]
\[ [(12)(35)(46)t_3]^{(15)(23)(46)} = [t_1t_2]^{(15)(23)(46)} \implies (12)(35)(46)t_5 = t_5t_3 \]
\[ \implies t_1(12)(35)(46)t_5 = t_1t_5t_3 \]
\[ \implies (12)(35)(46)t_1^{(12)(35)(46)}t_5 = t_1t_5t_3 \]
\[ \implies (12)(35)(46)t_2t_5 = t_1t_5t_3, \text{ by Equation 5.1} \]
One symmetric generator will go back to [13].

\[ Nt_1t_5t_4N \text{ is a new double coset which we will denote [154].} \]
One symmetric generator will go to [154].

\[ Nt_1t_5t_5 = Nt_1 \in [1]. \]
One symmetric generator will go to [1].

\[ Nt_1t_5t_6N \text{ is a new double coset which we will denote [156].} \]
One symmetric will go to [156].

Below is our Cayley Diagram thus far.
\(N^{(132)} \geq N^{132}\).

\(N^{132} = \{e\}\).

Next we will check to see if any of our relations will increase the number of elements in the coset stabilizing group \(N^{(132)}\).

\[
t_1t_3t_2 = t_1t_3t_2
\]

\[
\implies t_1t_3t_2 = t_1[t_6t_1t_5t_4], \text{ since }
\]

\[
[t_6t_3t_2]^{(163245)} = [t_1t_5t_4]^{(163245)}
\]

\[
\implies t_3t_2t_4 = t_6t_1t_5
\]

\[
\implies t_3t_2t_4t_4 = t_6t_1t_5t_4
\]

\[
\implies t_3t_2 = t_6t_1t_5t_4 \text{ by Equation 5.4}
\]

\[
t_1t_3t_2 = t_1t_6t_1t_5t_4
\]

\[
t_1t_3t_2 = [(16)(24)(35)t_1t_6]t_5t_4, \text{ since }
\]

\[
\]

\[
\implies (16)(24)(35)t_1t_6 = t_1t_6t_1, \text{ by Equation 5.2}
\]

\[
t_1t_3t_2 = (16)(24)(35)t_1t_6t_5t_4
\]

\[
\implies t_1t_3t_2 = (16)(24)(35)t_1t_6t_5t_4[134](256)t_5t_6t_2, \text{ since }
\]

\[
[(143)(265)t_3t_1]^{(16)(24)(35)} = [t_1t_3t_4]^{(16)(24)(35)}
\]

\[
\implies (134)(256)t_5t_6 = t_6t_5t_2
\]

\[
\implies (134)(256)t_5t_6t_2 = t_6t_5t_2t_2
\]

\[
\implies (134)(256)t_5t_6t_2 = t_6t_5, \text{ by Equation 5.3}
\]
\[
t_1t_3t_2 = (16)(24)(35)(134)(256)t_1^{(134)(256)}t_5t_6t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (12)(36)(45)t_3t_5t_6t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (12)(36)(45)[(12)(35)(46)t_3]t_6t_2t_4, \text{ since}
\]
\[
[(12)(35)(46)t_1]^{(13)(25)} = [t_1t_2]^{(13)(25)}
\]
\[
\Rightarrow (12)(35)(46)t_3 = t_3t_5, \text{ by Equation 5.1}
\]
\[
\Rightarrow t_1t_3t_2 = (34)(56)t_3t_6t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (34)(56)t_3t_6t_3t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (34)(56)t_3t_6t_3t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (34)(56)[(12)(36)(45)t_3t_6]t_3t_2t_4, \text{ by Equation 5.2}
\]
\[
\Rightarrow t_1t_3t_2 = (12)(35)(46)t_3t_6t_3t_2t_4
\]
\[
\Rightarrow t_1t_3t_2 = (12)(35)(46)t_3t_6[t_6t_1t_5], \text{ since}
\]
\[
[t_6t_3t_2]^{(163245)} = [t_1t_5t_4]^{(163245)}
\]
\[
\Rightarrow t_3t_2t_4 = t_6t_1t_5, \text{ by Equation 5.4}
\]
\[
t_1t_3t_2 = (12)(35)(46)t_3t_6t_0t_1t_5
\]
\[
\Rightarrow t_1t_3t_2 = (12)(35)(46)t_3t_1t_5
\]
\[
\Rightarrow Nt_1t_3t_2 = Nt_3t_1t_5.
\]

Now, since \([Nt_1t_3t_2]^{(13)(25)} = Nt_3t_1t_5 = Nt_1t_3t_2\), then \((13)(25) \in N^{(132)}\).

Also, \(t_5t_2t_3 = t_5t_2t_3\)
\[
\Rightarrow t_5t_2t_3 = t_5[(15)(23)(46)t_2t_3t_2], \text{ since}
\]
\[
[(12)(36)(45)t_3t_6]^{(163245)} = [t_3t_6t_3]^{(163245)}
\]
\[
\Rightarrow (15)(23)(46)t_2t_3 = t_2t_3t_2
\]
\[
\Rightarrow (15)(23)(46)t_2t_3t_2 = t_2t_3t_2t_2
\]
\[
\Rightarrow (15)(23)(46)t_2t_3t_2 = t_2t_3, \text{ by Equation 5.2}
\]
\[
t_5t_2t_3 = (15)(23)(46)t_5^{(15)(23)(46)}t_2t_3t_2
\]
\[
\Rightarrow t_5t_2t_3 = (15)(23)(46)t_1t_2t_3t_2
\]
\[
\Rightarrow t_5t_2t_3 = (15)(23)(46)t_1t_2t_3t_2
\]
\[
\Rightarrow t_5t_2t_3 = (15)(23)(46)[(12)(35)(46)t_1]t_3t_2, \text{ by Equation 5.1}
\]
\[
\Rightarrow t_5t_2t_3 = (13)(25)t_1t_3t_2
\]
\[
\Rightarrow Nt_5t_2t_3 = Nt_1t_3t_2.
\]
Since \([Nt_1t_3t_2]^{(15)(23)(46)} = Nt_5t_2t_3 = Nt_1t_3t_2\), then \((15)(23)(46) \in N^{(132)}\).

Also, \([Nt_1t_3t_2]^{(12)(35)(46)} = [Nt_3t_1t_5]^{(12)(35)(46)}
\implies Nt_2t_5t_1 = Nt_5t_2t_3\)

So, \(N^{(132)} \geq (13)(25), (12)(35)(46) \geq \{e, (13)(25), (12)(35)(46), (15)(23)(46)\}\).

The number of single cosets in \(Nt_1t_3t_2N\) is at most \(\frac{|N|}{|N^{(132)}|} = \frac{12}{2} = 3\).

\(Nt_1t_3t_2N = \{Nt_1t_3t_2 = Nt_2t_5t_1 = Nt_5t_2t_3 = Nt_3t_1t_5, Nt_3t_4t_5 = Nt_5t_6t_3 = Nt_6t_5t_4 = t_4t_3t_6, Nt_4t_1t_6 = Nt_6t_2t_4 = Nt_2t_6t_1 = Nt_1t_4t_2\}\).

\(N^{(136)} \geq N^{136}\).

\(N^{136} = \{e\}\).

Next we will check to see if any of our relations will increase the number of elements in the coset stabilizing group \(N^{(136)}\).

t_1t_3t_6 = t_1t_3t_6
\implies t_1t_3t_6 = t_1[(12)(36)(45)t_3t_6t_3], by Equation 5.2
\implies t_1t_3t_6 = (12)(36)(45)t_1^{(12)(36)(45)}t_3t_6t_3
\implies t_1t_3t_6 = (12)(36)(45)t_2t_3t_6t_3
\implies t_1t_3t_6 = (12)(36)(45)[t_4t_5t_1]t_3, since
\[t_6t_3t_2]^{(12)(35)(46)} = [t_1t_5t_4]^{(12)(35)(46)}
\implies t_4t_5t_1 = t_2t_3t_6, by Equation 5.4
\[t_1t_3t_6 = (12)(36)(45)t_4t_5t_1t_3 \implies t_1t_3t_6 = (12)(36)(45)t_4[(15)(23)(46)t_5t_1t_5]t_3, since
\implies (15)(23)(46)t_5t_1 = t_5t_1t_5
\implies (15)(23)(46)t_5t_1t_5 = t_5t_1t_5t_5
\implies (15)(23)(46)t_5t_1t_5 = t_5t_1, by Equation 5.2
\[t_1t_3t_6 = (12)(36)(45)t_4(15)(23)(46)t_5t_1t_5t_3
\implies t_1t_3t_6 = (12)(36)(45)(15)(23)(46)t_4^{(15})(23)(46)t_5t_1t_5t_3
\implies t_1t_3t_6 = (134)(256)t_6t_5t_1t_5t_3
\implies t_1t_3t_6 = (134)(256)t_6t_5t_1[(12)(35)(46)t_5], since
\[(12)(35)(46)t_1^{(15)(23)(46)} = [t_1t_2]^{(15)(23)(46)}\]

\[\rightarrow (12)(35)(46)t_5 = t_5t_3, \text{ by Equation 5.1}\]

\[t_1t_3t_6 = (134)(256)t_6t_5t_1(12)(35)(46)t_5\]

\[\rightarrow t_1t_3t_6 = (134)(256)(12)(35)(46)[t_6t_5t_1]^{(12)(35)(46)}t_5\]

\[\rightarrow t_1t_3t_6 = (154236)t_4t_3t_2t_5\]

\[\rightarrow t_1t_3t_6 = (154236)[(134)(256)t_3t_4t_1]t_2t_5, \text{ since}\]

\[(143)(265)t_3t_4]^{(14)(26)} = [t_1t_3t_4]^{(14)(26)}\]

\[\rightarrow (134)(256)t_3t_4 = t_4t_3t_1\]

\[\rightarrow (134)(256)t_3t_4t_1 = t_4t_3t_1\]

\[\rightarrow (134)(256)t_3t_4t_1 = t_4t_3, \text{ by Equation 5.3}\]

\[t_1t_3t_6 = (154236)(134)(256)t_3t_4t_1t_2t_5 \Rightarrow t_1t_3t_6 = (163245)t_3t_4t_1t_2t_5\]

\[\rightarrow t_1t_3t_6 = (163245)t_3t_4[(12)(35)(46)t_1]t_5, \text{ by relation 5.1}\]

\[\rightarrow t_1t_3t_6 = (163245)(12)(35)(46)[t_3t_4]^{(12)(35)(46)}t_1t_5\]

\[\rightarrow t_1t_3t_6 = (143)(265)t_5t_6t_1t_5\]

\[\rightarrow t_1t_3t_6 = (143)(265)t_5[t_3t_2t_4], \text{ since}\]

\[[t_6t_3t_2]^{(13)(25)} = [t_1t_5t_4]^{(13)(25)}\]

\[\rightarrow t_6t_1t_5 = t_3t_2t_4, \text{ by Equation 5.4}\]

\[t_1t_3t_6 = (143)(265)t_5t_3t_2t_4 \Rightarrow t_1t_3t_6 = (143)(265)[(12)(35)(46)t_5]t_2t_4, \text{ since}\]


\[\rightarrow (12)(35)(46)t_5 = t_5t_3, \text{ by Equation 5.1}\]

\[t_1t_3t_6 = (143)(265)[(12)(35)(46)t_5]t_2t_4\]

\[\rightarrow t_1t_3t_6 = (163245)t_5t_2t_4\]

\[Nt_1t_3t_6 = Nt_5t_2t_4\]

\[[Nt_1t_3t_6]^{(15)(23)(46)} = Nt_5t_2t_4 = Nt_1t_3t_6\]

\[\rightarrow (15)(23)(46) \in N^{(136)}\]

Thus, \(N^{(136)} \geq (15)(23)(46) = \{e, (15)(23)(46)\}\).

The number of single cosets in \(Nt_1t_3t_6N\) is at most \(\frac{|N|}{|Nt_1t_3t_6|} = \frac{12}{2} = 6\).

\(Nt_1t_3t_6N = \{Nt_1t_3t_6 = Nt_5t_2t_4, Nt_2t_3t_4 = Nt_3t_1t_6, Nt_3t_4t_2 = Nt_6t_5t_1, Nt_4t_1t_5 = Nt_2t_3t_6, Nt_5t_6t_1 = Nt_4t_3t_2, = Nt_6t_2t_3 = Nt_1t_4t_5\}.\)

\(N^{(152)} \geq N^{152}\).
\(N^{152} = \{e\}\).

Next we will check to see if any of our relations will increase the number of elements in the coset stabilizing group \(N^{(152)}\).

\[
t_1t_5t_2 = t_1t_5t_2 \\
\implies t_1t_5t_2 = t_1t_5[(12)(35)(46)t_2t_1], \text{ by Equation 5.1} \\
\implies t_1t_5t_2 = (12)(35)(46)[t_1t_5][12^{(36)(45)}]t_2t_1 \\
\implies t_1t_5t_2 = (12)(35)(46)t_2t_3t_2t_1 \\
\implies t_1t_5t_2 = (12)(35)(46)[(15)(23)(46)t_2t_3]t_1, \text{ since} \\
[(12)(36)(45)t_3t_4]^{163245} = [t_3t_6t_3]^{163245} \\
\implies (15)(23)(46)t_2t_3 = t_2t_3t_2, \text{ by Equation 5.2} \\
\implies t_1t_5t_2 = (12)(35)(46)(15)(23)(46)t_2t_3t_1 \\
\implies t_1t_5t_2 = (13)(25)t_2t_3t_1 \\
\implies Nt_1t_5t_2 = Nt_2t_3t_1
\]

Then, since \([Nt_1t_5t_2]^{(12)(35)(46)} = Nt_2t_3t_1 = Nt_1t_5t_2\), then \((12)(35)(46) \in N^{(152)}\).

Also,
\[
t_2t_3t_1 = t_2t_3t_1 \\
\implies t_2t_3t_1 = t_2[(134)(256)t_1t_3t_4], \text{ since} \\
[(143)(265)t_3t_1]^{(13)(25)} = [t_1t_3t_4]^{(13)(25)}t_1t_3 \\
\implies (134)(256)t_3t_1t_4, \text{ by Equation 5.3} \\
t_2t_3t_1 = t_2(134)(256)t_1t_3t_4 \\
\implies t_2t_3t_1 = (134)(256)t_2^{(134)(256)}t_1t_3t_4 \\
\implies t_2t_3t_1 = (134)(256)t_3t_1t_3t_4 \\
\implies t_2t_3t_1 = (134)(256)t_5t_1t_3t_4, \text{ since} \\
[(12)(35)(46)t_1]^{(13)(25)} = [t_1t_2]^{(13)(25)} \\
\implies (12)(35)(46)t_3 = t_3t_5 \\
\implies (12)(35)(46)t_3t_5 = t_3t_5t_5 \\
\implies (12)(35)(46)t_3t_5 = t_3, \text{ by Equation 5.1} \\
t_2t_3t_1 = (134)(256)t_5t_1(12)(35)(46)t_3t_5t_4 \\
\implies t_2t_3t_1 = (134)(256)(12)(35)(46)t_3t_5t_4t_3t_5t_4 \\
\implies t_2t_3t_1 = (154236)t_3t_2t_3t_5t_4
\]
\[ t_2t_3t_1 = (154236)[(15)(23)(46)t_3t_2]t_5t_4, \text{ since} \]
\[ [(12)(36)(45)t_3t_6]^{(14)(26)} = [t_3t_6t_3]^{(14)(26)} \]
\[ \implies (15)(23)(46)t_3t_2 = t_3t_2t_3, \text{ by Equation 5.2} \]
\[ t_2t_3t_1 = (154236)(15)(23)(46)t_3t_2t_5t_4 \]
\[ \implies t_2t_3t_1 = (34)(56)t_3t_2t_5t_4 \]
\[ \implies t_2t_3t_1 = (34)(56)t_3t_2[t_4t_1]t_5t_4 \]
\[ \implies t_2t_3t_1 = (34)(56)[t_2t_3t_1]t_5t_4 \]
\[ \implies t_2t_3t_1 = (34)(56)[t_6t_1t_5]t_4t_5t_4, \text{ since} \]
\[ t_6t_3t_2]^{(163245)} = [t_1t_5t_4]^{(163245)} \]
\[ \implies t_3t_2t_4 = t_6t_1t_5, \text{ by Equation 5.4} \]
\[ t_2t_3t_1 = (34)(56)t_6t_1t_5t_4 \implies t_2t_3t_1 = (34)(56)t_6t_1t_5[(12)(36)(45)t_4t_5], \text{ since} \]
\[ [(12)(36)(45)t_3t_6]^{(34)(56)} = [t_3t_6t_3]^{(34)(56)} \]
\[ \implies (12)(36)(45)t_4t_5 = t_4t_5t_4, \text{ Equation 5.4} \]
\[ t_2t_3t_1 = (34)(56)t_6t_1t_5(12)(36)(45)t_4t_5 \]
\[ \implies t_2t_3t_1 = (34)(56)(12)(36)(45)[t_6t_1t_5]^{(12)(36)(45)}t_4t_5 \]
\[ \implies t_2t_3t_1 = (12)(35)(46)t_3t_2t_4t_5t_5 \]
\[ \implies t_2t_3t_1 = (12)(35)(46)t_3t_2t_5 \]
\[ \implies Nt_2t_3t_1 = Nt_3t_2t_5 \]

Thus \([Nt_1t_5t_2]^{(13)(25)(46)} = Nt_3t_2t_5 = Nt_2t_3t_1 = Nt_1t_5t_2\]
\[ \implies (13)(25)(46) \in N^{(152)}. \]

So, \(N^{(152)} \supseteq (12)(35)(46), (13)(25)(46) \supseteq \{e, (12)(35)(46), (13)(25)(46), (15)(23)\}. \)

The number of single cosets in \(Nt_1t_5t_2N\) is at most \(\frac{|N|}{|N^{(152)}|} = \frac{12}{4} = 3.\)

\(Nt_1t_5t_2N = \{Nt_1t_5t_2 = Nt_2t_3t_1 = Nt_5t_3t_1 = Nt_3t_2t_5, Nt_3t_6t_5 = Nt_5t_4t_3 = Nt_6t_3t_4 = t_4t_5t_6, Nt_4t_2t_6 = Nt_6t_1t_4 = Nt_2t_4t_1 = Nt_1t_6t_2\}.\)

\(N^{(154)} \supseteq N^{154}.\)
\(N^{154} = \{e\}.\)

Next we will check to see if any of our relations will increase the number of elements in the coset stabilizing group \(N^{(154)}\).
\[ [Nt_1t_5t_4]^{(16)(24)(35)} = Nt_6t_3t_2 = Nt_1t_5t_4, \text{ by Equation } 5.4. \]

Thus, \((16)(24)(35) \in N^{(154)}.\)

So, \(N^{(154)} < (16)(24)(35) \supseteq \{e, (16)(24)(35)\}.\)

The number of single cosets in \(Nt_1t_5t_4N\) is at most \(\frac{|N|}{|N^{(154)}|} = \frac{12}{2} = 6.\)

\(Nt_1t_5t_4N = \{Nt_1t_5t_4 = Nt_6t_3t_2, Nt_2t_3t_6 = Nt_4t_5t_1, Nt_3t_6t_1 = Nt_2t_4t_5, Nt_5t_4t_2 = t_1t_6t_3, Nt_4t_2t_3 = Nt_5t_1t_6, Nt_6t_1t_5 = Nt_3t_2t_4\}.\)

\(N^{(156)} \geq N^{156}.\)

\(N^{156} = \{e\}.\)

Next we will check to see if any of our relations will increase the number of elements in the coset stabilizing group \(N^{(156)}.\)

\[ t_1t_5t_6 = t_1t_5t_6 \]
\[ \implies t_1t_5t_6 = [t_6t_3t_2t_4]t_6, \text{ since } \]
\[ t_6t_3t_2 = t_1t_5t_4 \implies t_6t_3t_2t_4 = t_1t_5t_4t_4 \]
\[ \implies t_6t_3t_2t_4 = t_1t_5, \text{ Equation } 5.4 \]
\[ t_1t_5t_6 = t_6t_3t_2t_4t_6 \]
\[ \implies t_1t_5t_6 = t_6t_3t_2[(12)(35)(46)t_4], \text{ since } \]
\[ [(12)(35)(46)t_4]^{(14)(26)} = [t_1t_2]^{(14)(26)} \]
\[ \implies (12)(35)(46)t_4 = t_4t_6, \text{ by Equation } 5.1 \]
\[ t_1t_5t_6 = (12)(35)(46)[t_6t_3t_2]^{(12)(35)(46)}t_4 \implies t_1t_5t_6 = (12)(35)(46)t_4t_5t_1t_4 \]
\[ t_1t_5t_6 = (12)(35)(46)t_4t_5t_1t_4 \implies t_1t_5t_6 = (12)(35)(46)[t_2t_3t_6]t_4, \text{ since } \]
\[ [t_6t_3t_2]^{(12)(35)(46)} = t_1t_5t_4]^{(12)(35)(46)} \]
\[ \implies t_4t_5t_1 = t_2t_3t_6, \text{ by Equation } 5.4 \]
\[ t_1t_5t_6 = (12)(35)(46)t_2t_3t_6t_4 \]
\[ t_1t_5t_6 = (12)(35)(46)[(15)(23)(46)t_2t_3t_2]t_6t_4, \text{ since } \]
\[ [(12)(36)(45)t_3t_6]^{(163245)} = [t_3t_6t_3]^{(163245)} \]
\[ \implies (15)(23)(46)t_2t_3 = t_2t_3t_2 \]
\[ \implies (15)(23)(46)t_2t_3t_2 = t_2t_3t_2t_2 \]
\[ t_{15}t_6 = (13)(25)t_2t_3t_6t_4 \]
\[ t_{15}t_6 = (13)(25)t_2t_3t_2[(12)(35)(46)t_6], \text{ since } [(12)(35)(46)t_1]^{163245} = [t_1t_2]^{163245} \]
\[ t_{15}t_6 = (13)(25)(12)(35)(46)[t_2t_3t_2]^{(12)(35)(46)}t_6 \]
\[ t_{15}t_6 = (15)(23)(46)t_1t_5t_6 \]
\[ t_{15}t_6 = (15)(23)(46)t_1t_4t_3, \text{ since } [t_6t_3t_2]^{(143)(265)} = [t_1t_5t_4]^{(143)(265)} \]
\[ t_{15}t_6 = t_4t_2t_3, \text{ by Equation 5.4} \]
\[ t_{15}t_6 = (15)(23)(46)t_1t_5t_2 \]
\[ t_{15}t_6 = (15)(23)(46)t_1t_4[(15)(23)(46)t_2t_3t_2], \text{ since } [(12)(36)(45)t_3t_6]^{163245} = [t_3t_6t_3]^{163245} \]
\[ t_{15}t_6 = (15)(23)(46)t_2t_3t_2 = t_2t_3t_2 \]
\[ (15)(23)(46)t_2t_3t_2 = t_2t_3t_2, \text{ by Equation 5.2} \]
\[ t_{15}t_6 = (15)(23)(46)[t_1t_5][15)(23)(46)t_2t_3t_2 \]
\[ t_{15}t_6 = t_5t_6t_2t_3t_2 \]
\[ t_{15}t_6 = (143)(265)t_6t_5t_3t_2, \text{ since } [(143)(265)t_3t_1]^{154236} = [t_1t_5t_4]^{154236} \]
\[ t_{15}t_6 = (143)(265)t_6t_5 = t_6t_5t_2, \text{ by Equation 5.3} \]
\[ t_{15}t_6 = (143)(265)t_6t_5t_3t_2 \]
\[ t_{15}t_6 = (143)(265)t_6[(12)(35)(46)t_5t_2], \text{ since } [(12)(35)(46)t_1]^{(15)(23)(46)} = [t_1t_5]^{(15)(23)(46)} \]
\[ t_{15}t_6 = (12)(35)(46)t_5 = t_5t_3, \text{ by Equation 5.1} \]
\[ t_{15}t_6 = (143)(265)(12)(35)(46)[t_6]^{(12)(35)(46)}t_5t_2 \]
\[ t_{15}t_6 = (163245)t_4t_5t_2 \]
\[ Nt_{15}t_6 = Nt_4t_5t_2 \]

Since \([Nt_{15}t_6]^{(14)(26)} = Nt_4t_5t_2 = Nt_{15}t_6\), then \((14)(26) \in N^{(156)}\).

So, \(N^{(156)} \geq (14)(26) \geq \{e, (14)(26)\}\).

The number of single cosets in \(Nt_{15}t_6N\) is at most \(\frac{|N|}{|N^{(156)}|} = \frac{12}{2} = 6\).
The orbits of $N^{(132)}$ on $\{1, 2, 3, 4, 5, 6\}$ are

$$\{1, 2, 5, 3\}, \{4, 6\}.$$ we take $t_2$ and $t_4$ from each orbit respectively.

We want to determine to which double coset $Nt_1t_3t_2t_1$ and $Nt_1t_3t_2t_4$ belong.

$Nt_1t_3t_2t_2 = Nt_1t_3 \in [13]$ Thus 4 symmetric generators will go to [13].

$Nt_1t_3t_2t_4 = Nt_1t_3t_2t_4$
$$\implies Nt_1t_3t_2t_4 = Nt_1[tt_6t_1t_5], \text{ since}$$
$$[tt_6t_1t_5]^{(163245)} = [tt_1t_5t_4]^{(163245)}$$
$$\implies t_3t_2t_4 = tt_6t_1t_5, \text{ by Equation 5.4}$$

$Nt_1t_3t_2t_4 = Nt_1t_6t_1t_5 \implies Nt_1t_3t_2t_4 = N[(16)(24)(35)t_1t_6]t_5$, since

$$\implies (16)(24)(35)t_1t_6 = t_1t_6t_1, \text{ by Equation 5.2}$$

$Nt_1t_3t_2t_4 = Nt_1t_6t_5 \in [156]$.

Thus 2 symmetric generators will go to [156].

The orbits of $N^{(136)}$ on $\{1, 2, 3, 4, 5, 6\}$ are

$$\{1, 5\}, \{2, 3\}, \{4, 6\}.$$ we take $t_1$, $t_3$, and $t_6$ from each orbit respectively.

We want to determine to which double coset $Nt_1t_3t_6t_1$, $Nt_1t_3t_6t_3$, and $Nt_1t_3t_6t_6$ belong.

$Nt_1t_3t_6t_1 = Nt_1t_3t_6t_1$
$$\implies Nt_1t_3t_6t_1 = Nt_1[tt_2t_4t_5], \text{ since}$$
$$[tt_2t_4t_5]^{(12)(36)(45)} = [tt_1t_5t_4]^{(12)(36)(45)}$$
$$\implies t_3t_6t_1 = t_2t_4t_5, \text{ by Equation 5.4}$$
\[ N_{t_1t_3t_6t_1} = N_{t_1t_2t_4t_5} \]
\[ \implies N_{t_1t_3t_6t_1} = N[(12)(35)(46)t_1]t_4t_5, \text{ by Equation 5.1} \]
\[ \implies N_{t_1t_3t_6t_1} = N_{t_1t_4t_5} \in [136]. \]
Thus 2 symmetric generators will go to [136].

\[ N_{t_1t_3t_6t_3} = N_{t_1t_3t_6t_3} \]
\[ \implies N_{t_1t_3t_6t_3} = N_{t_1}(12)(36)(45)t_3t_6, \text{ by Equation 5.3} \]
\[ \implies N_{t_1t_3t_6t_3} = N(12)(36)(45)[t_1]^{(12)(36)(45)}t_3t_6 \]
\[ \implies N_{t_1t_3t_6t_3} = N_{t_2t_3t_6} \in [154]. \]
Thus 2 symmetric generators will go to [154].

\[ N_{t_1t_3t_6t_6} = N_{t_1t_3} \in [13]. \]
Thus 2 symmetric generators will go to [13].

The orbits of \( N^{(152)} \) on \{1, 2, 3, 4, 5, 6\} are
\[ \{1, 2, 5, 3\}, \{4, 6\}. \]
we take \( t_2 \) and \( t_6 \) from each orbit respectively.
We want to determine to which double coset \( N_{t_1t_5t_2t_2} \) and \( N_{t_1t_5t_2t_6} \) belong.

\[ N_{t_1t_5t_2t_2} = N_{t_1t_5} \in [15]. \]
Thus 4 symmetric generators will go to [15].

\[ N_{t_1t_5t_2t_6} = N_{t_1t_5t_2t_6} \]
\[ \implies N_{t_1t_5t_2t_6} = N_{t_1}(134)(256)t_2t_5t_6, \text{ since} \]
\[ [(143)(265)t_3t_1]^{(15)(23)(46)} = [t_1t_3t_4]^{(15)(23)(46)} \]
\[ \implies (134)(256)t_2t_5 = t_5t_2t_6 \]
\[ \implies (134)(256)t_2t_5t_6 = t_5t_2t_6t_6 \]
\[ \implies (134)(256)t_2t_5t_6 = t_5t_2, \text{ by Equation 5.2} \]
\[ N_{t_1t_5t_2t_6} = N((134)(256))[t_1]^{(134)(256)}t_2t_5t_6 \]
\[ \implies N_{t_1t_5t_2t_6} = N_{t_3t_2t_5t} \in [152]. \]
Thus 2 symmetric generators will go to [152].
The orbits of $N^{(154)}$ on $\{1, 2, 3, 4, 5, 6\}$ are

$$\{1, 6\}, \{2, 4\}, \{3, 5\}.$$ 

We take $t_1$, $t_4$, and $t_3$ from each orbit respectively.

We want to determine to which double coset $N t_1 t_5 t_4 t_1$, $N t_1 t_5 t_4 t_3$, and $N t_1 t_5 t_4 t_3$ belong.

$t_1 t_5 t_4 = t_6 t_3 t_2$, by Equation 5.4

$\Rightarrow t_1 t_5 t_4 t_1 = t_6 t_3 t_2 t_1$

$\Rightarrow t_1 t_5 t_4 t_1 = t_6 t_3 t_2 t_1$

$\Rightarrow t_1 t_5 t_4 t_1 = t_6 t_3 [(12)(35)(46) t_2]$, since

$[(12)(35)(46) t_1]^{(12)(35)(46)} = t_1 t_2$, by Equation 5.1

$t_1 t_5 t_4 t_1 = (12)(35)(46) [t_6 t_3]^{(12)(35)(46)} t_2$

$\Rightarrow t_1 t_5 t_4 t_1 = (12)(35)(46) t_4 t_5 t_2$

$\Rightarrow N t_1 t_5 t_4 t_1 = N t_4 t_5 t_2 \in [156]$.

Thus 2 symmetric generators will go to [156].

$N t_1 t_5 t_4 t_4 = N t_1 t_5 t_4 \in [154]$.

Thus 2 symmetric generators will go to [154].

$t_1 t_5 t_4 = t_6 t_3 t_2$, by Equation 5.4

$\Rightarrow t_1 t_5 t_4 t_3 = t_6 t_3 t_2 t_3$

$\Rightarrow t_1 t_5 t_4 t_3 = t_6 [(15)(23)(46) t_3 t_2]$, since

$[(12)(36)(45) t_3 t_6]^{(14)(26)} = [t_3 t_6 t_3]^{(14)(26)}$

$\Rightarrow (15)(23)(46) t_3 t_2 = t_3 t_2 t_3$, by relation 5.2

$t_1 t_5 t_4 t_3 = (15)(23)(46) t_6^{(15)(23)(46)} t_3 t_2$

$\Rightarrow t_1 t_5 t_4 t_3 = (15)(23)(46) t_4 t_3 t_2$

$\Rightarrow N t_1 t_5 t_4 t_3 = N t_4 t_3 t_2 \in [136]$.

Thus, 2 symmetric generators will go to [136].

The orbits of $N^{(156)}$ on $\{1, 2, 3, 4, 5, 6\}$ are

$$\{3\}, \{5\}, \{1, 4\}, \{2, 6\}.$$
we take $t_3$, $t_5$, $t_1$, and $t_6$ from each orbit respectively.

We want to determine to which double coset $Nt_1t_5t_6t_3$, $Nt_1t_5t_6t_5$, $Nt_1t_5t_6t_1$, and $Nt_1t_5t_6t_6$ belong.

$$t_1t_5t_6t_3 = t_1t_5t_6t_3$$

$\implies t_1t_5t_6t_3 = [(15)(23)(46)t_1t_5t_1]t_6t_3$, since

$$[(12)(36)(45)t_3t_6][^{143}(265)]^{[143](265)} = [t_3t_6t_3][^{143}(265)]$$

$\implies (15)(23)(46)t_5t_5 = t_5t_1$

$\implies (15)(23)(46)t_1t_5t_1 = t_1t_5t_1 \implies (15)(23)(46)t_1t_5t_1 = t_1t_5$, by Equation \ref{eq:5.2}

$$t_1t_5t_6t_3 = (15)(23)(46)t_1t_5t_1t_6t_3$$

$\implies t_1t_5t_6t_3 = (15)(23)(46)t_1t_5[t_5t_4t_2]$, since

$$[t_6t_3t_2]^{(15236)} = [t_1t_3t_4]^{(15236)}$$

$\implies t_1t_6t_3 = t_3t_4t_2$, by Equation \ref{eq:5.4}

$$t_1t_5t_6t_3 = (15)(23)(46)t_1t_5t_5t_4t_2$$

$\implies t_1t_5t_6t_3 = (15)(23)(46)t_1t_4t_2$

$\implies Nt_1t_5t_6t_3 = Nt_1t_4t_2 \in [132]$

Thus 1 symmetric generator will go to $[132]$.

$$Nt_1t_5t_6t_5 = Nt_1t_5t_6t_5$$

$\implies Nt_1t_5t_6t_5 = [Nt_4t_5t_2]t_5$, since $Nt_1t_5t_6 = Nt_4t_5t_2$

$\implies Nt_1t_5t_6t_5 = Nt_4t_5t_2t_5$

$\implies Nt_1t_5t_6t_5 = Nt_4[(134)(256)t_2t_5t_6]t_5$, since

$$[(143)(265)t_3t_1]^{(15)(23)(46)} = [t_1t_3t_4]^{(15)(23)(46)}$$

$\implies (134)(256)t_2t_5 = t_5t_2t_6$

$\implies (134)(256)t_2t_5t_6 = t_5t_2t_6t_6$

$\implies (134)(256)t_2t_5t_6 = t_5t_2$, by Equation \ref{eq:5.3}

$$Nt_1t_5t_6t_5 = N(134)(256)t_4^{(134)(256)}t_2t_5t_6t_5$$

$\implies Nt_1t_5t_6t_5 = Nt_1t_2t_5t_6t_5$

$\implies Nt_1t_5t_6t_5 = N[(12)(35)(46)t_1]t_5t_6t_5$, by Equation \ref{eq:5.1}

$\implies Nt_1t_5t_6t_5 = Nt_1t_5t_6t_5$

$\implies Nt_1t_5t_6t_5 = Nt_1t_5[(134)(256)t_5t_6t_6]$, since

$$[(143)(265)t_3t_1]^{(16)(24)(35)} = [t_1t_3t_4]^{(16)(24)(35)}$$
\[ \Rightarrow (134)(256)t_5t_6 = t_6t_5t_2 \]
\[ \Rightarrow (134)(256)t_5t_6t_2 = t_6t_5t_2t_2 \]
\[ \Rightarrow (134)(256)t_5t_6t_2 = t_6t_5, \text{ by Equation 5.3} \]
\[ N_{t_1t_5t_6t_5} = N(134)(256)[t_1t_5]^{(134)(256)}t_5t_6t_2 \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_3t_6t_5t_6t_2} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N[(12)(36)(45)t_3t_6t_3]t_5t_6t_2, \text{ by Equation 5.2} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_3t_6t_3t_5t_6t_2} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_3t_6([12](35)(46)t_3]t_6t_2, \text{ since} \]
\[ [(12)(35)(46)t_1]^{(134)(256)} = [t_1t_2]^{(134)(256)} \]
\[ \Rightarrow (12)(35)(46)t_3 = t_3t_5, \text{ by Equation 5.1} \]
\[ N_{t_1t_5t_6t_5} = N[(12)(35)(46)[t_3t_6]^{(12)(35)(46)}t_3t_6t_2 \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_4t_3t_6t_3t_2} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_4([12](35)(46)t_5]t_6t_3t_2, \text{ since} \]
\[ [(12)(35)(46)t_1]^{(154236)} = [t_1t_2]^{(154236)} \]
\[ \Rightarrow (12)(35)(46)t_3 = t_5t_5t_3, \text{ by Equation 5.1} \]
\[ N_{t_1t_5t_6t_5} = N[(12)(35)(46)t_4]^{(12)(35)(46)}t_4t_5t_6t_3t_2 \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_6t_5t_6t_3t_2} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_6t_5[1t_1t_4], \text{ by Equation 5.4} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_6t_5[1t_1t_4] \text{t}_4} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_6([15](23)(46)t_5t_1]t_4, \text{ since} \]
\[ \Rightarrow (15)(23)(46)t_5t_3 = t_5t_1t_5, \text{ by Equation 5.2} \]
\[ N_{t_1t_5t_6t_5} = N[(15)(23)(46)t_6]^{(15)(23)(46)}t_5t_1t_4 \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_4t_5t_1t_4} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_2t_3t_6t_4, \text{ since} \]
\[ [t_6t_3t_2]^{(12)(35)(46)} = [t_1t_5t_4]^{(12)(35)(46)} \]
\[ \Rightarrow t_4t_5t_3 = t_2t_3t_6, \text{ by Equation 5.4} \]
\[ N_{t_1t_5t_6t_5} = N_{t_2t_3t_6t_4} \]
\[ \Rightarrow N_{t_1t_5t_6t_5} = N_{t_2t_3[1(12)(35)(46)t_6, \text{ since}} \]
\[(12)(35)(46)t_1\]^{(163245)} = [t_1t_2]^{(163245)}
\[\implies (12)(35)(46)t_6 = t_6t_4, \text{ by Equation 5.1}\]
\[Nt_1t_5t_6t_5 = N(12)(35)(46)[t_2t_3]^{(12)(35)(46)}t_6\]
\[\implies Nt_1t_5t_6t_5 = Nt_1t_5t_6 \in [156]\]
Thus 1 symmetric generator will go to [156].

\[Nt_1t_5t_6 = Nt_4t_5t_2\]
\[\implies Nt_1t_5t_6t_1 = Nt_4t_5t_2t_1\]
\[\implies Nt_1t_5t_6t_1 = Nt_4t_5[(12)(35)(46)t_2], \text{ since}\]
\[[(12)(35)(46)t_1]^{(12)(35)(46)} = [t_1t_2]^{(12)(35)(46)}\]
\[\implies (12)(35)(46)t_2 = t_2t_1, \text{ by Equation 5.1}\]
\[Nt_1t_5t_6t_1 = N(12)(35)(46)[t_4t_5]^{(12)(35)(46)}t_2\]
\[\implies Nt_1t_5t_6t_1 = Nt_6t_3t_2 \in [154]\]
Thus 2 symmetric generators will go to [154].

\[Nt_1t_5t_6t_6 = Nt_1t_5 \in [15]\]
Thus 2 symmetric generators will go to [15].

Below is our completed Cayley Diagram.
Next we will show that $|G| \geq 660$. We compute the action of $G$ on the 55 cosets in $N$ in $G$ that we have found.
Table 5.1: Single Coset Action of $L_2(11)$ Over $D_{12}$

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Thus,
\[ f(x) = (2, 3)(4, 6)(5, 7)(8, 10)(9, 11)(12, 16)(13, 17)(14, 18)(15, 19)(20, 22)(21, 23)(24, 28) \]
\[ f(z) = (2, 3)(4, 7)(5, 6)(8, 11)(9, 10)(12, 19)(13, 18)(14, 17)(15, 16)(20, 23)(21, 22) \]
\[ f(t) = (1, 2)(4, 13)(5, 14)(6, 29)(7, 30)(10, 32)(11, 34)(12, 15)(16, 39)(17, 25)(18, 26)(19, 37) \]

Now \( < f_x, f_y, f_z, f_t > \leq S_{55} \). \( f : G \rightarrow S_{55} \) is a homomorphism, since \( G \) acts on \( X = \{N, N_{t_1}, N_{t_2}, \ldots N_{t_6}t_1t_3\} \) with \( |X| = 55 \). \( < f_x, f_y, f_z, f_t > \) is a homomorphic image of the progenitor if

1. \( f_t \) has exactly 6 conjugates under conjugation by \( < f_x, f_y, f_z > \) and
2. \( < f_x, f_y, f_z > \) acts on \( \{f_{t_1}, f_{t_2}, f_{t_3}, f_{t_4}, f_{t_5}, f_{t_6}\} \) by conjugation as \( S_{55} \).

In addition, if the additional relations \((xt)^3 = (zt^y)^5 = (yt)^5 = 1 \) hold in \( S_{55} \), then
\( < f_x, f_y, f_z, f_t > \) is a homomorphic image of \( G \).

Note that \( |< f_x, f_y, f_z, f_t >| = 660 \).

\[ G/Ker_f \cong Im_f \]
\[ \implies G/Ker_f \cong < f_x, f_y, f_z, f_t > \]
\[ \implies |G/Ker_f| = |< f_x, f_y, f_z, f_t >| = 660 \]
\[ \implies |G| = |< f_x, f_y, f_z, f_t >| \times Ker_f \]
\[ \implies |G| \geq 660 \]

But, from our Cayley Diagram we saw that \( |G| \leq 660 \).

Hence, \( |G| = 660 \).
5.3 \( PGL_2(11) \) as a Homomorphic Image of \( 11^* : m \cdot D_{10} \)

5.3.1 The Construction of \( PGL_2(11) \) Over \( D_{10} \)

Let \( G \cong 11^* : m(D_{10}) \) be a symmetric presentation of \( G \) given by:
\[
<x, y, t | x^{10}, y^2, (x^{-1}y)^2, t^{11}, t^{-1}, t^6, (x^5t)^2, (yt)^3 > \cong PGL(2, 11),
\]
where \( N \cong D_{10} = < x, y | x^{10}, y^2, (x^{-1}y)^2 >, \)
\( x = (1, 3, 7, 15, 9, 19, 17, 13, 5, 11)(2, 12, 6, 14, 18, 20, 10, 16, 8, 4), \) and
\( y = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20). \)

Definition of a double coset: \( NwN = \{ Nwn | n \in N \} \).

Note: \( wn = nn^{-1}wn = nw^n. \)

So, \( Nwn = \{ Nw^n | n \in N \} \).

First we will expand our additional relation.
\[ (yt)^3 = e \]
\[ (yt_1)^3 = e \]
\[ y^3(t_1)^3(yt_1) = e \]
\[ yt_1t_2t_1 = e \]
\[ yt_1t_2t_1t_10 = t_1^{10} \]
\[ yt_1t_2 = t_1^{10} \]
\[ yt_1t_2 = t_{19} \]

Labeling:

\[
\begin{align*}
t_1 &= t_1 & t_6 &= t_2^3 & t_{11} &= t_1^6 & t_{16} &= t_2^8 \\
t_2 &= t_2 & t_7 &= t_1^4 & t_{12} &= t_2^6 & t_{17} &= t_1^9 \\
t_3 &= t_1^2 & t_8 &= t_2^4 & t_{13} &= t_1^7 & t_{18} &= t_2^9 \\
t_4 &= t_2^2 & t_9 &= t_1^5 & t_{14} &= t_2^7 & t_{19} &= t_1^{10} \\
t_5 &= t_1^3 & t_{10} &= t_2^5 & t_{15} &= t_1^8 & t_{20} &= t_2^{10} 
\end{align*}
\]

Our first double coset, \(NeN = \{Ne^n | n \in N\} = \{N\}\), which we will denote by [\(\ast\)]. The orbit of \(N\) on \(\{1,2,3,4,5,6,7,9,10,11,12,13,14,15,16,17,18,19,20\}\) is \(\{1,2,3,4,5,6,8,7,9,\]
\(10,11,12,13,14,15,16,17,18,19,20\}\). We will take a representative from this orbit, say \(t_1\), and determine to which double coset \(Nt_1\) belongs.

**Word of Length 1**

\(Nt_1N\) is a new double coset which we will denote by [1]. \(Nt_1N = \{Nt_1^n | n \in N\}\).

Since the orbit \(\{1,3,2,7,4,12,15,8,6,11,9,16,14,5,19,10,18,13,17,20\}\) contains 20 elements then 20 symmetric generators will go to the new double coset [1].

Now \(N^{(1)} \geq N^1\).

\(N^1 = \{e\}\).  

\(N^{(1)} = \text{Coset Stabiliser in } N \text{ of } Nt_1 = \{n \in N | Nt_1^n = t_1\}\).

We do not have a relation that will increase the Coset Stabiliser \(N^{(1)}\).
Now, since \((Nt_1)^e = Nt_1 \Rightarrow e \in N^{(1)}\), then,
\[ N^1 = N^{(1)} = \text{Coset Stabiliser in } N \text{ of } Nt_1 = \{n \in N | Nt_1^n = t_1\} = \{e\}. \]
Furthermore, the number of single cosets in \(Nt_1N\) is \(\frac{|N|}{|N^{(1)}|} = \frac{20}{1} = 20\).

Therefore, \(Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}\}\)

The orbits of \(N^{(1)}\) on \(\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}\)
are \(\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \) and \(\{20\}\).

We want to see to which double coset \(Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_5, Nt_1t_6, Nt_1t_7, Nt_1t_8, Nt_1t_9, Nt_1t_{10}, Nt_1t_{11}, Nt_1t_{12}, Nt_1t_{13}, Nt_1t_{14}, Nt_1t_{15}, Nt_1t_{16}, Nt_1t_{17}, Nt_1t_{18}, Nt_1t_{19}, \) and \(Nt_1t_{20}\) belong.

\(Nt_1t_1 = Nt_1^2 = Nt_3 \in [1]\).
One symmetric generator will go to [1].

\(Nt_1t_2 = Nt_1^{10} = Nt_{19} \in [1]\), by Equation 5.5.
One symmetric generator will go to [1].

\(Nt_1t_3 = Nt_1t_1^2 = Nt_1^3 = Nt_5 \in [1]\).
One symmetric generator will go to [1].

\(Nt_1t_4N\) is a new double coset which we will denote by [1 4].
One symmetric generator will go to [1 4].

\(Nt_1t_5 = Nt_1t_1^3 = Nt_1^4 = Nt_7 \in [1]\).
One symmetric generator will go to [1].

\(Nt_1t_6N\) is a new double coset which we will denote by [1 6].
One symmetric generator will go to [1 6].
$N_{t_1}t_7 = N_{t_1}t_1^4 = N_{t_1}^5 = N_{t_1}t_9 \in [1]$.
One symmetric generator will go to [1].

$N_{t_1}t_8N$ is a new double coset which we will denote by [1 8].
One symmetric generator will go to [1 8].

$N_{t_1}t_9 = N_{t_1}t_1^5 = N_{t_1}^6 = N_{t_1}t_{11} \in [1]$.
One symmetric generator will go to [1].

$N_{t_1}t_{10}N$ is a new double coset which we will denote by [1 10].
One symmetric generator will go to [1 10].

$N_{t_1}t_{11} = N_{t_1}t_1^6 = N_{t_1}^7 = N_{t_1}t_{13} \in [1]$.
One symmetric generator will go to [1].

$N_{t_1}t_{12}N$ is a new double coset which we will denote by [1 12].
One symmetric generator will go to [1 12].

$N_{t_1}t_{13} = N_{t_1}t_1^7 = N_{t_1}^8 = N_{t_1}t_{15} \in [1]$.
One symmetric generator will go to [1].

$N_{t_1}t_{14} = N_{t_1}t_{14}$
$\implies N_{t_1}t_{14} = N_{t_1}t_2^7$
$\implies N_{t_1}t_{14} = N_{t_1}[x^4y^4t_1^4t_3^3], \text{ by Equation 5.5.}$
$\implies N_{t_1}t_{14} = Nx^4y[t_1]^4y^4t_2^4t_1^3$
$\implies N_{t_1}t_{14} = N_{t_1}^{5t_2^4t_1^3}$
$\implies N_{t_1}t_{14} = N_{t_1}^{gt_1^3}$
$\implies N_{t_1}t_{14} = N_{t_1}^{t_1^3}$
Also,
$N_{t_1}t_{10} \in [110]$
$\implies N[t_{1}t_{10}]^{yx^4} \in [110]$
\[ \Rightarrow N_{t_{18}t_5} \in [110]. \]
Thus, \( N_{t_1t_{14}} \in [110]. \)
One symmetric generator will go to \([1 10]\).

\[ N_{t_{15}} = N_{t_1t_1} = N_{t_1^0} = N_{t_1} \in [1]. \]
One symmetric generator will go to \([1]\).

\begin{align*}
N_{t_1t_{16}} &= N_{t_1t_{16}} \\
\Rightarrow N_{t_1t_{16}} &= N_{t_1t_2^8} \\
\Rightarrow N_{t_1t_{16}} &= N[y_{t_1^{10}t_2^{10}}t_2^8], \text{ by Equation 5.5.} \\
\Rightarrow N_{t_1t_{16}} &= N_{t_1^{10}t_2^2} \\
\Rightarrow N_{t_1t_{16}} &= N_{t_{19}t_{14}}
\end{align*}
Also,
\[ N_{t_1t_8} \in [18] \]
\[ \Rightarrow N[t_{1t_8}]x^5 \in [18] \]
\[ \Rightarrow N_{t_{19}t_{14}} \in [18]. \]
Thus, \( N_{t_1t_{16}} \in [18]. \)
One symmetric generator will go to \([1 8]\).

\[ N_{t_1t_{17}} = N_{t_1t_1^0} = N_{t_1^{10}} = N_{t_1} \in [1]. \]
One symmetric generator will go to \([1]\).

\begin{align*}
N_{t_1t_{18}} &= N_{t_1t_{18}} \\
\Rightarrow N_{t_1t_{18}} &= N_{t_1t_2^9} \\
\Rightarrow N_{t_1t_{18}} &= N[y_{t_1^{10}t_2^{10}}t_2^9], \text{ by Equation 5.5.} \\
\Rightarrow N_{t_1t_{18}} &= N_{t_1^{10}t_2^2} \\
\Rightarrow N_{t_1t_{18}} &= N_{t_{19}t_{16}}
\end{align*}
Also,
\[ N_{t_1t_6} \in [16] \]
\[ \Rightarrow N[t_{1t_6}]x^5 \in [16] \]
\[ \Rightarrow N_{t_{19}t_{16}} \in [16]. \]
Thus, \( N_{t_1t_{18}} \in [16]. \)
One symmetric generator will go to $[1\ 6]$.

\[ Nt_{19}t_{19} = Nt_1t_1^{10} = N \in [\ast]. \]

One symmetric generator will go to $[\ast]$.

\[ Nt_{20}t_{20} = Nt_{19}t_{20} \]
\[ \implies Nt_{20}t_{20} = Nt_1t_1^{10} \]
\[ \implies Nt_{20}t_{20} = N[yt_1^{10}t_2^{10}]t_2^{10}, \text{ by Equation 5.5.} \]
\[ \implies Nt_{19}t_{18} \]

Also,

\[ Nt_{4} \in [14] \]
\[ \implies N[t_1t_4]x^5 \in [14] \]
\[ \implies Nt_{19}t_{18} \in [16]. \]

Thus, $Nt_{20} \in [14]$.

One symmetric generator will go to $[1\ 4]$.

**Word of Length 2**

\[ N^{(14)} = \text{Coset Stabiliser in } N \text{ of } Nt_{4} = \{ n \in N | (Nt_4)^n = t_1t_4 \}. \] We will look for a relation that will increase the Coset Stabiliser $N^{(14)}$.

\[ Nt_{2} = Nt_{19}, \text{ by Equation 5.5} \]
\[ \implies Nt_{12}t_{2} = Nt_1^{10} \]
\[ \implies Nt_{12}t_{2} = Nt_1^{10}t_2 \]
\[ \implies Nt_{2} = Nt_1^{10}t_2 \]
\[ \implies Nt_{4} = Nt_{19}t_2 \]

Since, \((Nt_4)^e = Nt_{14} \Rightarrow e \in N^{(14)}, \) and
\((Nt_4)^x = Nt_{19}t_2 \Rightarrow x^5 \in N^{(14)}, \) then,

\[ N^{(14)} = \text{Coset Stabiliser in } N \text{ of } Nt_{4} = \{ n \in N | (Nt_4)^n = t_1t_4 \} = \{ e, x^5 \}. \]

Furthermore, the number of single cosets in $Nt_{4}N$ is \[
\frac{|N|}{|N^{(14)}|} = \frac{20}{2} = 10.
\]
We find the equal names by conjugating $t_1t_4 \sim t_{19}t_2$ by elements of $N$.

\[
\begin{align*}
    t_1t_4 & \sim t_{18}t_{19} \\
    t_4t_1 & \sim t_{19}t_{18} \\
    t_2t_3 & \sim t_{17}t_{20} \\
    t_3t_2 & \sim t_{20}t_{17} \\
    t_5t_6 & \sim t_{15}t_{15} \\
    t_6t_5 & \sim t_{15}t_{6} \\
    t_7t_12 & \sim t_{10}t_{13} \\
    t_{12}t_7 & \sim t_{13}t_{10}
\end{align*}
\]

$N^{(16)} = \text{Coset Stabiliser in } N$ of $Nt_1t_6 = \{ n \in N | (Nt_1t_6)^n = t_1t_6 \}$. We will look for a relation that will increase the Coset Stabiliser $N^{(16)}$.

$Nt_1t_6 = Nt_1t_6$

$\implies Nt_1t_6 = Nt_1t_2$

$\implies Nt_1t_6 = Nt_1[yx^4t_2^3t_1^7]$, by Equation 5.5

$\implies Nt_1t_6 = N[yx^4t_1^3y^3t_2^3t_1^7]$

$\implies Nt_1t_6 = Nt_2^0t_2^3t_1^7$

$\implies Nt_1t_6 = Nt_2^0t_1^7$

$\implies Nt_1t_6 = N[yx^2t_2^0y^0]t_1^7$, by Equation 5.5

$\implies Nt_1t_6 = Nt_2^0t_1^7$

$\implies Nt_1t_6 = Nt_10t_9$

Since, $(Nt_1t_6)^e = Nt_1t_6 \Rightarrow e \in N^{(16)}$, and

$(Nt_1t_6)x^4y = Nt_{10}t_9 = Nt_1t_6 \Rightarrow x^4y \in N^{(16)}$, then,

$N^{(16)} = \text{Coset Stabiliser in } N$ of $Nt_1t_6 = \{ n \in N | (Nt_1t_6)^n = t_1t_6 \} = \{ e, x^4y \}$.

Furthermore, the number of single cosets in $Nt_1t_6N$ is $\frac{|N|}{|N^{(16)}|} = \frac{20}{2} = 10$.

We find the equal names by conjugating $t_1t_6 \sim t_{19}t_9$ by elements of $N$.

\[
\begin{align*}
    t_1t_6 & \sim t_{10}t_{9} \\
    t_6t_1 & \sim t_{17}t_{8} \\
    t_2t_5 & \sim t_{9}t_{10} \\
    t_5t_2 & \sim t_{18}t_{7} \\
    t_3t_{14} & \sim t_{16}t_{19} \\
    t_{14}t_3 & \sim t_{13}t_{4} \\
    t_7t_{18} & \sim t_{8}t_{17} \\
    t_{18}t_7 & \sim t_{5}t_{2}
\end{align*}
\]

$N^{(18)} = \text{Coset Stabiliser in } N$ of $Nt_1t_8 = \{ n \in N | (Nt_1t_8)^n = t_1t_8 \}$.

We will look for a relation that will increase the Coset Stabiliser $N^{(18)}$.

$Nt_1t_2 = Nt_{19}$, by Equation 5.5
\[ N_t = N_{t_1}^{10} \]
\[ N_t t_2 = N_{t_1}^{10} t_2 \]
\[ N_t t_2^3 = N_{t_1}^{10} t_2^3 \]
\[ N_t t_2^4 = N_{t_1}^{10} t_2^4 \]
\[ N_t t_2^5 = N_{t_1}^{10} t_2^5 \]
\[ N_t t_2^6 = N_{t_1}^{10} t_2^6 \]
\[ N_t t_2^7 = N_{t_1}^{10} t_2^7 \]
\[ N_t t_2^8 = N_{t_1}^{10} t_2^8 \]
\[ N_t t_2^9 = N_{t_1}^{10} t_2^9 \]
\[ N_t t_2^{10} = N_{t_1}^{10} t_2^{10} \]
\[ N_t t_2^{11} = N_{t_1}^{10} t_2^{11} \]
\[ N_t t_2^{12} = N_{t_1}^{10} t_2^{12} \]
\[ N_t t_2^{13} = N_{t_1}^{10} t_2^{13} \]

Since, \((N_t t_8)^e = N_t t_8 \Rightarrow e \in N^{(18)}\), and
\((N_t t_8)^{x^5} = N_{t_2} t_8 = N_t t_8 \Rightarrow x^5 y \in N^{(18)}\), then,
\[ N^{(18)} = \text{Coset Stabiliser in } N \text{ of } N_t t_8 = \{n \in N | (N_t t_8)^n = t_1 t_8 \} = \{e, x^5 y\}. \]
Furthermore, the number of single cosets in \(N_t t_8 N\) is \(\frac{|N|}{|N^{(18)}|} = \frac{20}{2} = 10\).

We find the equal names by conjugating \(t_1 t_8 \sim t_20 t_{13}\) by elements of \(N\).

\[ t_1 t_8 \sim t_20 t_{13} \quad t_3 t_4 \sim t_{10} t_5 \quad t_{14} t_{19} \sim t_5 t_{10} \]
\[ t_8 t_1 \sim t_{15} t_{12} \quad t_4 t_3 \sim t_9 t_6 \quad t_6 t_9 \sim t_{13} t_{20} \]
\[ t_2 t_7 \sim t_{19} t_{14} \quad t_{17} t_{18} \sim t_{12} t_{15} \]
\[ t_7 t_2 \sim t_{16} t_{11} \quad t_{18} t_{17} \sim t_{11} t_{16} \]
\[ N^{(110)} = \text{Coset Stabiliser in } N \text{ of } N_t t_1 0 = \{n \in N | (N_t t_1 0)^n = t_1 t_1 0 \}. \]

We will look for a relation that will increase the Coset Stabiliser \(N^{(110)}\).
\[ N_t t_2 = N_{t_1}^{10}, \text{ by Equation 5.5} \]
\[ N_t t_2 = N_{t_1}^{10} \]
\[ N_t t_2^4 = N_{t_1}^{10} t_2^4 \]
\[ N_t t_2^5 = N_{t_1}^{10} t_2^5 \]
\[ N_t t_2^6 = N_{t_1}^{10} t_2^6 \]
\[ N_t t_2^7 = N_{t_1}^{10} t_2^7 \]
\[ N_t t_2^8 = N_{t_1}^{10} t_2^8 \]
\[ N_t t_2^9 = N_{t_1}^{10} t_2^9 \]
\[ N_t t_2^{10} = N_{t_1}^{10} t_2^{10} \]
\[ N_{t_1 t_{10}} = N_{t_4 t_{15}} \]

Since, \((N_{t_1 t_{10}})^e = N_{t_1 t_{10}} \Rightarrow e \in N^{(110)},\) and
\((N_{t_1 t_{10}})^{xy} = N_{t_4 t_{15}} = N_{t_1 t_{10}} \Rightarrow xy \in N^{(110)},\) then,
\[ N^{(110)} = \text{Coset Stabiliser in } N \text{ of } N_{t_1 t_{10}} = \{ n \in N \mid (N_{t_1 t_{10}})^n = t_{110} \} = \{ e, xy \}. \]
Furthermore, the number of single cosets in \(N_{t_1 t_{10}}N\) is \(\frac{|N|}{|N^{(110)}|} = \frac{20}{2} = 10.\)

We find the equal names by conjugating \(t_{110} \sim t_{415}\) by elements of \(N.\)

\[
\begin{align*}
t_{110} & \sim t_{1415} & t_{518} & \sim t_{1613} & t_{718} & \sim t_{1219} \\
t_{1014} & \sim t_{1314} & t_{185} & \sim t_{1912} & t_{87} & \sim t_{1120} \\
t_{29} & \sim t_{3116} & t_{617} & \sim t_{1514} & t_{176} & \sim t_{2011} \\
t_{92} & \sim t_{1413} & t_{176} & \sim t_{2011} \\
\end{align*}
\]

\[ N^{(112)} = \text{Coset Stabiliser in } N \text{ of } N_{t_1 t_{12}} = \{ n \in N \mid (N_{t_1 t_{12}})^n = t_{112} \}. \]

We will look for a relation that will increase the Coset Stabiliser \(N^{(112)}.\)
\[ N_{t_1 t_2} = N_{t_{19}}, \text{ by Equation 5.5 } \]
\[ \Rightarrow N_{t_1 t_2} = N_{t_1}^{10} \]
\[ \Rightarrow N_{t_1 t_2}^5 = N_{t_1}^{10}t_2^5 \]
\[ \Rightarrow N_{t_1}^{t_6} = N_{t_1}^{t_10t_2} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_{19}t_{10}} \]

Also,
\[ N_{t_1 t_{12}} = N_{t_{19}t_{10}} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_1}^{t_10t_2} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_1}^{t_10}[yx^2t_2^6t_1^2], \text{ by Equation 5.5 } \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{yx^2[t_1^{10}]yx^2t_2^6t_1^2} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_2^6t_1^2} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_2^3t_1^2} \]
\[ \Rightarrow N_{t_1 t_{12}} = N_{t_6t_3} \]

Also,
\[ N_{t_1 t_{12}} = N_{t_6t_3} \]
We find the equal names by conjugating $t_1 t_{12} = N t_2^3 t_1^2$.

$N t_1 t_{12} = N [y x^4 t_2 t_1^7] t_1^2$, by Equation 5.5

$\Rightarrow N t_1 t_{12} = N t_2^3 t_1^3$

$\Rightarrow N t_1 t_{12} = N t_{16} t_{17}$

Thus $N t_1 t_{12} = N t_{19} t_{10} = N t_6 t_3 = N t_{16} t_{17}$

Since, $(N t_1 t_{12}) e = N t_1 t_{12} \Rightarrow e \in N^{(112)}$, and

$(N t_1 t_{12}) x^3 = N t_{19} t_{10} = N t_1 t_{12} \Rightarrow x^5 \in N^{(112)}$, and

$(N t_1 t_{12}) y x^2 = N t_6 t_3 = N t_1 t_{12} \Rightarrow y x^2 \in N^{(112)}$, and

$(N t_1 t_{12}) x^3 y = N t_{16} t_{17} = N t_1 t_{12} \Rightarrow x^3 y \in N^{(112)}$, then,

$N^{(112)} = \text{Coset Stabiliser in } N$ of $N t_1 t_{12} = \{n \in N | (N t_1 t_{12}) n = t_1 t_{12}\} = \{e, x^5, y x^2, x^3 y\}$.

Furthermore, the number of single cosets in $N t_1 t_{12} N$ is $\frac{|N|}{|N^{(112)}|} = \frac{20}{4} = 5$.

We find the equal names by conjugating $t_1 t_{12} \sim t_{19} t_{10} \sim t_6 t_3 \sim t_{16} t_{17}$ by elements of $N$.

$t_1 t_{12} \sim t_{19} t_{10} \sim t_6 t_3 \sim t_{16} t_{17}$

$t_3 t_6 \sim t_{17} t_{16} \sim t_5 t_{19}$

$t_5 t_{11} \sim t_{20} t_9 \sim t_5 t_{14}$

$t_4 t_4 \sim t_{12} t_3 \sim t_6 t_{12}$

$t_7 t_6 \sim t_{13} t_8 \sim t_{18} t_{15} \sim t_4 t_5$

The orbits of $N^{(14)}$ on \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\} are \{1,18\}, \{2,9\}, \{3,14\}, \{4,19\}, \{5,10\}, \{6,7\}, \{8,17\}, \{11,20\}, \{12,15\}, and \{13,16\}.

We want to see to which double coset $N t_1 t_{418}, N t_1 t_{412}, N t_1 t_{414}, N t_1 t_{414}, N t_1 t_{410}, N t_1 t_{416}, N t_1 t_{418}, N t_1 t_{420}, N t_1 t_{412}$, and $N t_1 t_{416}$ belong.

$N t_1 t_{418} = N t_1^2 t_2^9$

$\Rightarrow N t_1 t_{418} = N t_1 \in [1]$. 
Two symmetric generators will go to [1].

\[ N_{t_1}t_4t_2 = N_{t_1}t_2^3t_2 \]
\[ \implies N_{t_1}t_4t_2 = N_{t_1}t_2^3 \]
\[ \implies N_{t_1}t_4t_2 = N_{t_1}t_6 \in [16]. \]

Two symmetric generators will go to [1 6].

\[ N_{t_1}t_4t_{14} = N_{t_1}t_2^7t_2 \]
\[ \implies N_{t_1}t_4t_{14} = N_{t_1}t_2^9 \]
\[ \implies N_{t_1}t_4t_{14} = N_{t_1}t_{18} \in [14]. \]

Two symmetric generators will go to [1 4].

\[ N_{t_1}t_4t_4 = N_{t_1}t_2^2t_2^2 \]
\[ \implies N_{t_1}t_4t_4 = N_{t_1}t_4^4 \]
\[ \implies N_{t_1}t_4t_4 = N_{t_1}t_{8} \in [18]. \]

Two symmetric generators will go to [1 8].

\[ N_{t_1}t_4t_{10} = N_{t_1}t_2^5t_2 \]
\[ \implies N_{t_1}t_4t_{10} = N_{t_1}t_2^7 \]
\[ \implies N_{t_1}t_4t_{10} = N_{t_1}t_{10}^{10t_2}t_2^7, \text{ by Equation 5.5} \]
\[ \implies N_{t_1}t_4t_{10} = N_{t_1}t_6^{10t_2}t_2 \]
\[ \implies N_{t_1}t_4t_{10} = N_{t_1}t_{12} \in [110]. \]

Two symmetric generators will go to [1 10].

\[ N_{t_1}t_4t_6 = N_{t_1}t_2^3t_2^3 \]
\[ \implies N_{t_1}t_4t_6 = N_{t_1}t_2^5 \]
\[ \implies N_{t_1}t_4t_6 = N_{t_1}t_{10} \in [110]. \]

Two symmetric generators will go to [1 10].

\[ N_{t_1}t_4t_8 = N_{t_1}t_2^4t_2^4 \]
\[ \implies N_{t_1}t_4t_8 = N_{t_1}t_2^6 \]
\[ \implies N_{t_1}t_4t_8 = N_{t_1}t_{12} \in [112]. \]
Two symmetric generators will go to $[1 \ 12]$.

$$N_{t_1}t_4t_{20} = N_{t_1}t_2^2t_{10}$$
$$\implies N_{t_1}t_4t_{20} = N_{t_1}t_2.$$  
$$\implies N_{t_1}t_4t_{20} = N_{t_19}, \text{ by Equation 5.5}$$  
$$\implies N_{t_1}t_4t_{20} = N_{t_19} \in [1].$$

Two symmetric generators will go to $[1]\ 4$.

Two symmetric generators will go to $[1]\ 8$.

$$N_{t_1}t_4t_{12} = N_{t_1}t_2^2t_6$$
$$\implies N_{t_1}t_4t_{12} = N_{t_1}t_2^8$$
$$\implies N_{t_1}t_4t_{12} = N[y_{t_1}t_{10}t_{210}]t_2^8, \text{ by Equation 5.5}$$  
$$\implies N_{t_1}t_4t_{12} = N_{t_1}t_2^8$$  
$$\implies N_{t_1}t_4t_{12} = N_{t_19}t_{14} \in [18].$$

Two symmetric generators will go to $[1]\ 8$.

$$N_{t_1}t_4t_{16} = N_{t_1}t_2^2t_8$$
$$\implies N_{t_1}t_4t_{16} = N_{t_1}t_2^{10}$$  
$$\implies N_{t_1}t_4t_{16} = N_{t_1}t_2^8$$  
$$\implies N_{t_1}t_4t_{16} = N_{t_19}t_6^3, \text{ by Equation 5.5}$$  
$$\implies N_{t_1}t_4t_{16} = N_{t_19}t_18 \in [14].$$

Two symmetric generators will go to $[1]\ 4$.

The orbits of $N^{(16)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

are $\{1, 10\}$, $\{2, 17\}$, $\{3, 20\}$, $\{4, 13\}$, $\{5, 8\}$, $\{6, 9\}$, $\{7, 18\}$, $\{11, 16\}$, $\{12, 19\}$, and $\{14, 15\}$.

We want to see to which double coset $N_{t_1}t_6t_{10}, N_{t_1}t_6t_{2}, N_{t_1}t_6t_{3}, N_{t_1}t_6t_{4}, N_{t_1}t_6t_{8}, N_{t_1}t_6t_{6}, N_{t_1}t_6t_{18}, N_{t_1}t_6t_{16}, N_{t_1}t_6t_{12}$, and $N_{t_1}t_6t_{14}$ belong.

$$N_{t_1}t_6t_{10} = N_{t_1}t_2^3t_5$$
$$\implies N_{t_1}t_6t_{10} = N[y_{t_1}t_{10}t_{10}]t_2^5, \text{ by Equation 5.5}$$  
$$\implies N_{t_1}t_6t_{10} = N_{t_1}t_2^5$$  
$$\implies N_{t_1}t_6t_{10} = N_{t_19}t_{14} \in [18].$$
Two symmetric generators will go to [1 8].

\[ N_{t_1} t_6 t_2 = N_{t_1} t_3^2 t_2 \]
\[ \implies N_{t_1} t_6 t_2 = N_{t_1} t_4^4 \]
\[ \implies N_{t_1} t_6 t_2 = N_{t_1} t_8 \in [18]. \]

Two symmetric generators will go to [1 8].

\[ N_{t_1} t_6 t_3 = N_{t_1} t_3^2 t_1^2 \]
\[ \implies N_{t_1} t_6 t_3 = N_{t_1} [y x^4 t_1^2]^2 t_1^2, \text{ by Equation 5.5} \]
\[ \implies N_{t_1} t_6 t_3 = N_{y x^4 [t_1]^2} t_1^4 t_1^4 \]
\[ \implies N_{t_1} t_6 t_3 = N_{t_2^2 t_4^2} t_4 \]
\[ \implies N_{t_1} t_6 t_3 = N_{t_9} t_9 \]
\[ \implies N_{t_1} t_6 t_3 = N_{t_{10}} t_{13} \in [14]. \]

Two symmetric generators will go to [1 10].

\[ N_{t_1} t_6 t_4 = N_{t_1} t_3^2 t_2^2 \]
\[ \implies N_{t_1} t_6 t_4 = N_{t_1} t_5^6 \]
\[ \implies N_{t_1} t_6 t_4 = N_{t_1} t_{10} \in [110]. \]

Two symmetric generators will go to [1 10].

\[ N_{t_1} t_6 t_8 = N_{t_1} t_3^2 t_2^4 \]
\[ \implies N_{t_1} t_6 t_8 = N_{y t_1^{10} t_2^{10}} t_2^7, \text{ by Equation 5.5} \]
\[ \implies N_{t_1} t_6 t_8 = N_{t_1^{10} t_2^6} \]
\[ \implies N_{t_1} t_6 t_8 = N_{t_{19}} t_{12} \in [110]. \]

Two symmetric generators will go to [1 10].

\[ N_{t_1} t_6 t_6 = N_{t_1} t_3^2 t_2^3 \]
\[ \implies N_{t_1} t_6 t_6 = N_{t_1} t_5^6 \]
\[ \implies N_{t_1} t_6 t_6 = N_{t_1} t_{12} \in [112]. \]

Two symmetric generators will go to [1 12].
\(N_{t_1t_6t_{18}} = N_{t_1t_2^3t_2^9}\)
\[\implies N_{t_1t_6t_{18}} = N_{t_1t_2}\]
\[\implies N_{t_1t_6t_{18}} = Ny_1^{10},\] by Equation 5.5
\[\implies N_{t_1t_6t_{18}} = N_{t_19} \in [1].\]
Two symmetric generators will go to [1].

\[N_{t_1t_6t_{16}} = N_{t_1t_2^3t_2^8}\]
\[\implies N_{t_1t_6t_{16}} = N_{t_1} \in [1].\]
Two symmetric generators will go to [1].

\[N_{t_1t_6t_{12}} = N_{t_1t_2^3t_2^6}\]
\[\implies N_{t_1t_6t_{12}} = N_{yt_1^{10}t_2^{10}t_2^{10}},\] by Equation 5.5
\[\implies N_{t_1t_6t_{12}} = N_{t_1^{10}t_2^8}\]
\[\implies N_{t_1t_6t_{12}} = N_{t_19t_{16}} \in [16].\]
Two symmetric generators will go to [1 6].

\[N_{t_1t_6t_{14}} = N_{t_1t_2^3t_2^7}\]
\[\implies N_{t_1t_6t_{14}} = N_{yt_1^{10}t_2^{10}t_2^{10}},\] by Equation 5.5
\[\implies N_{t_1t_6t_{14}} = N_{t_1^{10}t_2^9}\]
\[\implies N_{t_1t_6t_{14}} = N_{t_19t_{18}} \in [14].\]
Two symmetric generators will go to [1 4].

The orbits of \(N^{(18)}\) on \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}
are \{1,20\}, \{2,19\}, \{3,18\}, \{4,17\}, \{5,16\}, \{6,15\}, \{7,14\}, \{8,13\}, \{9,12\}, and \{10,11\}.

We want to see to which double coset \(N_{t_1t_8t_{20}}, N_{t_1t_8t_2}, N_{t_1t_8t_{18}}, N_{t_1t_8t_4}, N_{t_1t_8t_{16}}, N_{t_1t_8t_6}, N_{t_1t_8t_{14}}, N_{t_1t_8t_8}, N_{t_1t_8t_{12}},\) and \(N_{t_1t_8t_{10}}\) belong.

\[N_{t_1t_8t_{20}} = N_{t_1t_2^4t_2^{10}}\]
\[\implies N_{t_1t_8t_{20}} = N_{t_1t_2^3}\]
\[ N t_{1}t_{8}t_{20} = N t_{1}t_{6} \in [16]. \]
Two symmetric generators will go to [1 6].

\[ N t_{1}t_{8}t_{2} = N t_{1}t_{2}t_{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{2} = N t_{1}t_{6}^{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{2} = N t_{1}t_{10} \in [110]. \]
Two symmetric generators will go to [1 10].

\[ N t_{1}t_{8}t_{18} = N t_{1}t_{4}t_{9} \]
\[ \Rightarrow N t_{1}t_{8}t_{18} = N t_{1}t_{2}^{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{18} = N t_{1}t_{4} \in [14]. \]
Two symmetric generators will go to [1 4].

\[ N t_{1}t_{8}t_{4} = N t_{1}t_{2}t_{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{4} = N t_{1}t_{6}^{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{4} = N t_{1}t_{12} \in [12]. \]
Two symmetric generators will go to [1 12].

\[ N t_{1}t_{8}t_{16} = N t_{1}t_{2}t_{2}^{8} \]
\[ \Rightarrow N t_{1}t_{8}t_{16} = N t_{1}t_{2} \]
\[ \Rightarrow N t_{1}t_{8}t_{16} = N y t_{1}^{10}, \text{ by Equation 5.5} \]
\[ \Rightarrow N t_{1}t_{8}t_{16} = N t_{19} \in [1]. \]
Two symmetric generators will go to [1].

\[ N t_{1}t_{8}t_{6} = N t_{1}t_{2}t_{2}^{5} \]
\[ \Rightarrow N t_{1}t_{8}t_{6} = N t_{1}t_{2}^{7} \]
\[ \Rightarrow N t_{1}t_{8}t_{6} = N (y t_{1}^{10} t_{2}^{10})t_{2}^{7}, \text{ by Equation 5.5} \]
\[ \Rightarrow N t_{1}t_{8}t_{6} = N t_{1}^{10}t_{2}^{6} \]
\[ \Rightarrow N t_{1}t_{8}t_{6} = N t_{19}t_{12} \in [110]. \]
Two symmetric generators will go to [1 10].

\[ N t_{1}t_{8}t_{14} = N t_{1}t_{2}t_{2}^{7} \]
\[ \implies N_{t_1} t_{s_1 14} = N_{t_1} \in [1]. \]

Two symmetric generators will go to \([1]\).

\[ N_{t_1} t_{s_8} = N_{t_1} t_{2}^4 t_{2}^4 \]
\[ \implies N_{t_1} t_{s_8} = N_{t_1} t_{2}^8 \]
\[ \implies N_{t_1} t_{s_8} = N[y_{t_1} t_{2}^{10}] t_{2}^6, \text{ by Equation 5.5} \]
\[ \implies N_{t_1} t_{s_8} = N_{t_1} t_{10} t_{7} \]
\[ \implies N_{t_1} t_{s_8} = N_{t_19} t_{14} \in [18]. \]

Two symmetric generators will go to \([1 8]\).

\[ N_{t_1} t_{s_{12}} = N_{t_1} t_{2}^4 t_{2}^6 \]
\[ \implies N_{t_1} t_{s_{12}} = N_{t_1} t_{10} \]
\[ \implies N_{t_1} t_{s_{12}} = N[y_{t_1} t_{10}^{10}] t_{2}^4, \text{ by Equation 5.5} \]
\[ \implies N_{t_1} t_{s_{12}} = N_{t_1} t_{10} t_{0} \]
\[ \implies N_{t_1} t_{s_{12}} = N_{t_19} t_{18} \in [14]. \]

Two symmetric generators will go to \([1 4]\).

\[ N_{t_1} t_{s_{10}} = N_{t_1} t_{2}^5 t_{2}^2 \]
\[ \implies N_{t_1} t_{s_{10}} = N_{t_1} t_{10} \]
\[ \implies N_{t_1} t_{s_{10}} = N[y_{t_1} t_{2}^{10}] t_{2}^0, \text{ by Equation 5.5} \]
\[ \implies N_{t_1} t_{s_{10}} = N_{t_1} t_{10} t_{8} \]
\[ \implies N_{t_1} t_{s_{10}} = N_{t_19} t_{16} \in [16]. \]

Two symmetric generators will go to \([1 6]\).

The orbits of \(N^{(110)}\) on \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}\) are \(\{1, 4\}, \{2, 11\}, \{3, 8\}, \{5, 12\}, \{6, 13\}, \{7, 16\}, \{9, 20\}, \{10, 15\}, \{14, 17\}, \text{ and } \{18, 19\}\).

We want to see to which double coset \(N_{t_1} t_{10} t_{4}, N_{t_1} t_{10} t_{2}, N_{t_1} t_{10} t_{8}, N_{t_1} t_{10} t_{12}, N_{t_1} t_{10} t_{6}, N_{t_1} t_{10} t_{16}, N_{t_1} t_{10} t_{20}, N_{t_1} t_{10} t_{10}, N_{t_1} t_{10} t_{14}, \text{ and } N_{t_1} t_{10} t_{18}\) belong.

\[ N_{t_1} t_{10} t_{4} = N_{t_1} t_{2}^5 t_{2}^2 \]
$$\Rightarrow N_{110} = t_7$$
$$\Rightarrow N_{110} = N[y_1^{10}t_2^{10}]t_2^7, \text{ by Equation 5.5}$$
$$\Rightarrow N_{110} = N_{110}^0$$
$$\Rightarrow N_{110} = N_{110} \in [110].$$
Two symmetric generators will go to $[1 \, 10]$.

$$N_{110}t_2 = N_{t_2}$$
$$\Rightarrow N_{110}t_2 = N_{t_2}^0$$
$$\Rightarrow N_{110}t_2 = N_{110} \in [112].$$
Two symmetric generators will go to $[1 \, 12]$.

$$N_{110}t_8 = N_{t_8}$$
$$\Rightarrow N_{110}t_8 = N_{t_8}^0$$
$$\Rightarrow N_{110}t_8 = N[y_1^{10}t_2^{10}]t_2^8, \text{ by Equation 5.5}$$
$$\Rightarrow N_{110}t_8 = N_{110}^8$$
$$\Rightarrow N_{110}t_8 = N_{110} \in [16].$$
Two symmetric generators will go to $[1 \, 6]$.

$$N_{110}t_{12} = N_{t_{12}}$$
$$\Rightarrow N_{110}t_{12} = N_{1} \in [1].$$
Two symmetric generators will go to $[1 \, 12]$.

$$N_{110}t_6 = N_{t_6}$$
$$\Rightarrow N_{110}t_6 = N_{t_6}^8$$
$$\Rightarrow N_{110}t_6 = N[y_1^{10}t_2^{10}]t_2^8, \text{ by Equation 5.5}$$
$$\Rightarrow N_{110}t_6 = N_{110}^8$$
$$\Rightarrow N_{110}t_6 = N_{110} \in [18].$$
Two symmetric generators will go to $[1 \, 8]$.

$$N_{110}t_{16} = N_{t_{16}}$$
$$\Rightarrow N_{110}t_{16} = N_{t_2}^8$$
$$\Rightarrow N_{110}t_{16} = N_{t_2} \in [14].$$
Two symmetric generators will go to [1 4].

\[ N_{t_1}t_{10}t_{20} = N_{t_1}t_{2}^{10}t_{2}^{10} \]
\[ \implies N_{t_1}t_{10}t_{20} = N_{t_1}t_{2} \]
\[ \implies N_{t_1}t_{10}t_{20} = N_{t_1}t_8 \in [18]. \]

Two symmetric generators will go to [18].

\[ N_{t_1}t_{10}t_{10} = N_{t_1}t_{2}^{5}t_{2}^{5} \]
\[ \implies N_{t_1}t_{10}t_{10} = N_{t_1}t_{2}^{10} \]
\[ \implies N_{t_1}t_{10}t_{10} = N_{t_1}t_{10}^{10}, \text{ by Equation 5.5} \]
\[ \implies N_{t_1}t_{10}t_{10} = N_{t_1}t_{2}^{10} \]
\[ \implies N_{t_1}t_{10}t_{10} = N_{t_1}t_{19}t_{18} \in [14]. \]

Two symmetric generators will go to [14].

\[ N_{t_1}t_{10}t_{14} = N_{t_1}t_{2}^{5}t_{2}^{7} \]
\[ \implies N_{t_1}t_{10}t_{14} = N_{t_1}t_{2} \]
\[ \implies N_{t_1}t_{10}t_{14} = N_{t_1}t_{10}^{10}, \text{ by Equation 5.5} \]
\[ \implies N_{t_1}t_{10}t_{14} = N_{t_1}t_{19} \in [1]. \]

Two symmetric generators will go to [1].

\[ N_{t_1}t_{10}t_{18} = N_{t_1}t_{2}^{5}t_{2}^{3} \]
\[ \implies N_{t_1}t_{10}t_{18} = N_{t_1}t_{2}^{3} \]
\[ \implies N_{t_1}t_{10}t_{18} = N_{t_1}t_{6} \in [16]. \]

Two symmetric generators will go to [16].

The orbits of \( N^{(112)} \) on \( \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\} \)
are \( \{1,6,16,19\}, \{2,7,13,20\}, \{3,12,10,17\}, \{4,15,5,18\}, \text{ and } \{8,9,11,14\}. \)

We want to see to which double coset \( N_{t_1}t_{12}t_{16}, N_{t_1}t_{12}t_{20}, N_{t_1}t_{12}t_{10}, N_{t_1}t_{12}t_{18}, \text{ and } N_{t_1}t_{12}t_{14} \) belong.

\[ N_{t_1}t_{12}t_{16} = N_{t_1}t_{2}^{6}t_{2}^{8} \]
\[ \Rightarrow N_{t1}t_{12}t_{16} = N_{t1}t_3^2 \]
\[ \Rightarrow N_{t1}t_{12}t_{16} = N_{t1}t_6 \in [16]. \]
Two symmetric generators will go to \([1 \ 6]\).

\[ N_{t1}t_{12}t_{20} = N_{t1}t_6t_{10}^6t_2^{10} \]
\[ \Rightarrow N_{t1}t_{12}t_{20} = N_{t1}t_2^5 \]
\[ \Rightarrow N_{t1}t_{12}t_{20} = N_{t1}t_{10} \in [110]. \]
Two symmetric generators will go to \([1 \ 10]\).

\[ N_{t1}t_{12}t_{10} = N_{t1}t_6t_5^6t_2^5 \]
\[ \Rightarrow N_{t1}t_{12}t_{10} = N_{t1}t_1 \]
\[ \Rightarrow N_{t1}t_{12}t_{10} = N_{t1} \in [1]. \]
Two symmetric generators will go to \([1]\).

\[ N_{t1}t_{12}t_{18} = N_{t1}t_6t_8^6t_2^8 \]
\[ \Rightarrow N_{t1}t_{12}t_{18} = N_{t1}t_8^4 \]
\[ \Rightarrow N_{t1}t_{12}t_{18} = N_{t1}t_8 \in [18]. \]
Two symmetric generators will go to \([1 \ 8]\).

\[ N_{t1}t_{12}t_{14} = N_{t1}t_6t_4^6t_2^7 \]
\[ \Rightarrow N_{t1}t_{12}t_{14} = N_{t1}t_2^4 \]
\[ \Rightarrow N_{t1}t_{12}t_{14} = N_{t1}t_4 \in [14]. \]
Two symmetric generators will go to \([1 \ 4]\).

Below is our completed Cayley Diagram.
Thus,

\[ |G| \leq \left( \frac{|N|}{|N|} + \frac{|N|}{|N|} + \frac{|N|}{|N|} + \frac{|N|}{|N|} + \frac{|N|}{|N|} + \frac{|N|}{|N|} \right) \times |N| \]

\[ = \left( \frac{20}{20} + \frac{20}{1} + \frac{20}{2} + \frac{20}{2} + \frac{20}{2} + \frac{20}{4} \right) \times 20 \]

\[ = (1 + 20 + 10 + 10 + 10 + 5) \times 20 \]

\[ = 66 \times 20 \]

\[ = 1320 \]

\[ G \leq 1320 \]

5.3.2 Proof of the Isomorphism
Table 5.2: Single Coset Action of $PGL(2,11)$ Over $D_{10}$

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Table 5.2 – Continued from previous page

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Continued on next page
Now $\langle f_x, f_y, f_t \rangle \leq S_{66}$. $f : G \rightarrow S_{66}$ is a homomorphism, since $G$ acts on $X = \{N, N_{t_1}, N_{t_2}, \ldots N_{t_3t_6}\}$ with $|X| = 66$. $\langle f_x, f_y, f_t \rangle$ is a homomorphic image of the progenitor if

(1) $f_t$ has exactly 2 conjugates under conjugation by $\langle f_x, f_y \rangle$ and
(2) $< f_x, f_y >$ acts on $\{f_t_1, f_t_2\}$ by conjugation as $S_{66}$.

In addition, if the additional relations $(yt)^3 = 1$ hold in $S_{66}$, then $< f_x, f_y, f_t >$ is a homomorphic image of $G$.

Note that $| < f_x, f_y, f_t > | = 1320$.

\[ \frac{G}{\text{Ker} f} \cong \text{Im} f \]

\[ \Rightarrow \frac{G}{\text{Ker} f} \cong < f_x, f_y, f_t > \]

\[ \Rightarrow |\frac{G}{\text{Ker} f}| = | < f_x, f_y, f_t > | = 1320 \]

\[ \Rightarrow |G| = | < f_x, f_y, f_t > | \times \text{Ker} f \]

\[ \Rightarrow |G| \geq 1320 \]

But, from our Cayley Diagram we saw that $|G| \leq 1320$.

Hence, $|G| = 1320$.

### 5.3.3 Building a Map

$L_2(11) = \{ x \mapsto \frac{a + bx}{c + dx}, x \in F_{11} \cup \{\infty\}, a, b, c, d \in F_{11} | ad - bc = 1, \text{ equivalently a nonzero square} \} = < \alpha, \beta, \gamma >$, where $\alpha = x \mapsto x + 1, \beta = x \mapsto kx$, and $\gamma = x \mapsto -\frac{1}{x}$.

Let us start with our mapping for alpha.

$\alpha : x \mapsto x + 1$.

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<td>6</td>
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<tr>
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<td>8</td>
<td>8 + 1 = 9</td>
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</table>
\[9 \mapsto 9 + 1 = 10\]

\[10 \mapsto 10 + 1 = 11 \equiv 1 \mod 11\]

\[\infty \mapsto \infty + 1 = \infty\]

<table>
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<th>3</th>
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</table>

\[\alpha = (\infty)(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)\]

Next, to find our mapping for beta, we will first need to find \(k\). Our nonzero squares in \(F_{11}\) are \(\{1,3,4,5,9\}\).

\[1^2 = 1\]

\[2^2 = 4\]

\[3^2 = 9\]

\[4^2 = 16 \equiv 5 \mod 11\]

\[5^2 = 25 \equiv 3 \mod 11\]

\[6^2 = 36 \equiv 3 \mod 11\]

\[7^2 = 49 \equiv 5 \mod 11\]

\[8^2 = 64 \equiv 9 \mod 11\]

\[9^2 = 81 \equiv 4 \mod 11\]

\[10^2 = 100 \equiv 1 \mod 11\]

To find \(k\), we need a nonzero square that generates all of the other nonzero squares.

Note that

\[3^0 = 1\]

\[3^1 = 3\]

\[3^2 = 9\]

\[3^3 = 27 \equiv 5 \mod 11\]

\[3^4 = 81 \equiv 4 \mod 11\]

\[3^5 = 243 \equiv 1 \mod 11\]
Thus, $k = 3$. Therefore, $\beta : x \mapsto 3x$.

$0 \mapsto 3 \cdot 0 = 0$
$1 \mapsto 3 \cdot 1 = 3$
$2 \mapsto 3 \cdot 2 = 6$
$3 \mapsto 3 \cdot 3 = 9$
$4 \mapsto 3 \cdot 4 = 12 \equiv 1 \mod 11$
$5 \mapsto 3 \cdot 5 = 15 \equiv 4 \mod 11$
$6 \mapsto 3 \cdot 6 = 18 \equiv 7 \mod 11$
$7 \mapsto 3 \cdot 7 = 21 \equiv 10 \mod 11$
$8 \mapsto 3 \cdot 8 = 24 \equiv 2 \mod 11$
$9 \mapsto 3 \cdot 9 = 27 \equiv 5 \mod 11$
$10 \mapsto 3 \cdot 10 = 30 \equiv 8 \mod 11$
$\infty \mapsto 3 \cdot \infty = \infty$

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<td>9</td>
</tr>
<tr>
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<tr>
<td>$\infty$</td>
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$\beta = (\infty)(0)(1, 3, 9, 5, 4)(2, 6, 7, 10, 8)$

$\gamma : x \mapsto -\frac{1}{x}$

$0 \mapsto -\frac{1}{0} = \infty$
$1 \mapsto -\frac{1}{1} = -1 \cdot 1^{-1} = -1 \cdot 1 = -1 \equiv 10 \mod 11$
$2 \mapsto -\frac{1}{2} = -1 \cdot 2^{-1} = -1 \cdot 6 = -6 \equiv 5 \mod 11$
$3 \mapsto -\frac{1}{3} = -1 \cdot 3^{-1} = -1 \cdot 4 = -4 \equiv 7 \mod 11$
$4 \mapsto -\frac{1}{4} = -1 \cdot 4^{-1} = -1 \cdot 3 = -3 \equiv 8 \mod 11$
$5 \mapsto -\frac{1}{5} = -1 \cdot 5^{-1} = -1 \cdot 9 = -9 \equiv 2 \mod 11$
$6 \mapsto -\frac{1}{6} = -1 \cdot 6^{-1} = -1 \cdot 2 = -2 \equiv 9 \mod 11$
\[7 \mapsto -\frac{1}{7} = -1 \cdot 7^{-1} = -1 \cdot 8 = -8 \equiv 3 \mod 11\]
\[8 \mapsto -\frac{1}{8} = -1 \cdot 8^{-1} = -1 \cdot 7 = -7 \equiv 4 \mod 11\]
\[9 \mapsto -\frac{1}{9} = -1 \cdot 9^{-1} = -1 \cdot 5 = -5 \equiv 6 \mod 11\]
\[10 \mapsto -\frac{1}{10} = -1 \cdot 10^{-1} = -1 \cdot 10 = -10 \equiv 1 \mod 11\]
\[\infty \mapsto -\frac{1}{\infty} = 0\]

Table 5.5: \(\gamma : x \mapsto -\frac{1}{x}\)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<th>11</th>
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<tbody>
<tr>
<td>0</td>
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<td>∞</td>
<td>10</td>
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<td>8</td>
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<td>9</td>
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<td>4</td>
<td>6</td>
<td>1</td>
<td>0</td>
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<td></td>
</tr>
</tbody>
</table>

\(\gamma = (0, \infty)(1, 10)(2, 5)(3, 7)(4, 8)(6, 9)\)

Now alpha, beta, and gamma generate \(L_2(11)\). However, since we have \(PGL(2, 11)\) we must find an automorphism for the element of order 2 which is not normal in \(PGL(2, 11)\). Note that
\[
\frac{a(x)+b}{c(x)+d} = \frac{1+0}{0+1} = \frac{1}{x},
\]
where \(ad - bc = 1 - 0 = 1\), is a non-zero square. Therefore our mapping for this automorphism will be

\[\text{aut} : x \mapsto \frac{1}{x}\]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>→</td>
<td>\frac{1}{0} = \infty</td>
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<tr>
<td>1</td>
<td>→</td>
<td>\frac{1}{1} = 1</td>
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<tr>
<td>2</td>
<td>→</td>
<td>\frac{1}{2} = 1 \cdot 2^{-1} = 1 \cdot 6 = 6</td>
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<td>\frac{1}{5} = 1 \cdot 5^{-1} = 1 \cdot 9 = 9</td>
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<td>6</td>
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<td>\frac{1}{6} = 1 \cdot 6^{-1} = 1 \cdot 2 = 2</td>
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<td>9</td>
<td>→</td>
<td>\frac{1}{9} = 1 \cdot 9^{-1} = 1 \cdot 5 = 5</td>
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<tr>
<td>10</td>
<td>→</td>
<td>\frac{1}{10} = 1 \cdot 10^{-1} = 1 \cdot 10 = 10</td>
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</tbody>
</table>
\[ \infty \mapsto \frac{1}{\infty} = 0 \]

<p>| Table 5.6: aut : ( x \mapsto \frac{1}{x} ) |
|-----------------|---|---|---|---|---|---|---|---|---|---|---|---|</p>
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
<th>9</th>
<th>10</th>
<th>( \infty )</th>
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</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>9</td>
<td>2</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

\( aut = (0, \infty)(2, 6)(3, 4)(5, 9)(7, 8) \)

\[
> G<x, y, t>:=\text{Group}<x, y, t|x^{10}, y^2, (x^{-1}y)^2, t^{11}, t^6, (x^5t)^2, (yt)^3>; \\
> \#G; \\
1320 \\
> f, G1, k := \text{CosetAction}(G, \text{sub}<G|x, y>); \\
> S := \text{Sym}(12); \\
> \text{alpha} := S!(11, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10); \\
> \text{beta} := S!(1, 3, 9, 5, 4)(2, 6, 7, 10, 8); \\
> \text{gamma} := S!(11, 12)(1, 10)(2, 5)(3, 7)(4, 8)(6, 9); \\
> \#\text{sub}<S|\text{alpha, beta, gamma}>; \\
660 \\
> \text{aut} := S!(11, 12)(2, 6)(3, 4)(5, 9)(7, 8); \\
> \#\text{sub}<S|\text{alpha, beta, gamma, aut}>; \\
1320 \\
> P := \text{sub}<S|\text{alpha, beta, gamma, aut}>; \\
> s := \text{IsIsomorphic}(G1, P); s; \\
true
\]

### 5.4 \( M_{11} \) as a Homomorphic Image of \( 11^4 : m(4 : 5) \)

Let \( G \cong 11^4 : m(4 : 5) \) be a symmetric presentation of \( G \) given by:

\[
< x, y, t | x^4, xy^4x^3y^3, y^3x^3yx, t^{11}, t^y = t^4, (x^2t^y)^3, (x^3t)^8, (yt^x)^5, (xt^y)^3 \rangle \cong M_{11},
\]

where \( N \cong (4 : 5) =< x, y | x^4, xy^4x^3y^3, y^3x^3yx >, \)

\[
x = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)(21, 22, 23, 24) \\
(25, 26, 27, 28)(29, 30, 31, 32)(33, 34, 35, 36)(37, 38, 39, 40), \text{ and } y = (1, 13, 17, 33, 9) \\
(5, 29, 37, 25, 21)(6, 26, 30, 22, 38)(2, 34, 14, 10, 18)(3, 11, 35, 19, 15)(4, 20, 12, 16, 36) \\
(7, 23, 27, 39, 31)(8, 40, 24, 32, 28). \]
\( G \) is of order 7920, and \( N \) is of order 20. The number of single cosets is equal to \( \frac{|G|}{|N|} = \frac{7920}{20} = 396 \). In the next section we will show how this double coset enumeration of \( G \) over \((C_4 : C_5)\) can be done by performing a double coset enumeration of \( G \) over a maximal subgroup, say \( H \), and \( N \).

### 5.4.1 Manual Double Coset Enumeration over a Maximal Subgroup of Order 120

We will find a maximal subgroup, \( M \), or a conjugate of \( M \) which contains \( f(x), f(y) \) to perform double coset enumeration.

```plaintext
> G<x,y,t>:=Group<x,y,t|x^4,x*y^4*x^3*y^3,
> y^3*x^3*y*x,t^11,t^y=t^4,(x^2*t^y)^3,(x^3*t)^8,
> (y*t^x)^5,
> (x*t^y)^3>;
> #G;
7920
> S:=Sym(40);
> xx:=S!(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)
> (17,18,19,20)(21,22,23,24)(25,26,27,28)
> (29,30,31,32)(33,34,35,36)(37,38,39,40);
> yy:=S!(1,13,17,33,9)(5,29,37,25,21)
> (5,29,37,25,21)
> (6,26,30,22,38)(2,34,14,10,18)(3,11,35,19,15)
> (4,20,12,16,36)
> (7,23,27,39,31)(8,40,24,32,28);
> N:=sub<S|xx,yy>;
> #N;
20
> f,G1,k:=CosetAction(G,sub<G|x>);
> CompositionFactors(G1);
    G
     | M11
  1
> M:=MaximalSubgroups(G1);
> #M;
5
> M;
Conjugacy classes of subgroups
```

---

---
We have found the maximal subgroups of $G$. We need to find which maximal subgroup contains $f(x)$ and $f(y)$.

Using the previous loop, we see that there are 2 maximal subgroups that contain $f(x)$ and $f(y)$.

Let us first examine subgroup number 2, which is of order 120. We need to use our Schreier System to find a representation of this subgroup in words so that we may
perform Double Coset Enumeration of $G$ over $H$.

```plaintext
> N:=G1;
> NN:=G;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..7920]];
> for i in [2..7920] do
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x^-1); end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y^-1); end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
> D:=Conjugates(G1,M[2]|`subgroup`);
> D:=SetToSequence(D);
> f(x) in D[44] and f(y) in D[44];
true
> for g in D[44] do if sub<D[44]|f(x),f(y),g> eq D[44] then gg:=g; end if;
end for;
> Order(gg);
2
> if Order(gg) eq 2 then for i in [1..7920] do if ArrayP[i] eq gg
  then Sch[i]; end if; end for;
x * y * t * x^2 * y^-1 * t^-1 * y^-1
```

Thus our maximal subgroup is $H = \langle x, y, xytx^2y^4t^{10}y^4 \rangle$.

We show how this double coset enumeration of $G$ over $(C_4:C_5)$ can be done using a double coset enumeration of $G$ over $H$ and $N$.

Definition of a double coset: $HwN = \{Hwn|n \in N\}$.

Note: $wn = wn^{-1}wn = nw^n$.

So, $Hwn = \{Hw^n|n \in N\}$.
First we will expand our additional relations.

\[ xytx^2y^4t^{10}y^4 \in H \]
\[ xytx^2y^4t_1^{10}y^4 \in H \]
\[ xytx^2y^4t_{37}y^4 \in H \]
\[ xy[x^2x^{-2}]t_1x^2y^4t_{37}y^4 \in H \]
\[ xyx^2[x^{-2}t_1x^2]y^4t_{37}y^4 \in H \]
\[ xyx^2[t_1^2]y^4t_{37}y^4 \in H \]
\[ xyx^2[t_3]y^4t_{37}y^4 \in H \]
\[ xyx^2[y^4y^{-4}]t_3y^4t_{37}y^4 \in H \]
\[ xyx^2y^4[y^{-4}t_3y^4]t_{37}y^4 \in H \]
\[ xyx^2y^4[y^4]t_3^4t_{37}y^4 \in H \]
\[ xyx^2y^4t_{15}t_{37}y^4 \in H \]
\[ xyx^2y^4[y^4y^{-4}]t_{15}t_{37}y^4 \in H \]
\[ xyx^2y^4[y^{-4}t_{15}t_{37}y^4] \in H \]
\[ xyx^2y^3[y^4_{15}t_{37}] \in H \]
\[ xyx^2y^3t_{19}t_{29} \in H \]
\[ Hxyx^2y^3t_{19}t_{29} = H \]
\[ Hxyx^2y^3t_{31}^8 = H \]
\[ Hxyx^2y^3t_{31}^8t_{41}^8 \in Ht_1^3 \]
\[ Hxyx^2y^3t_3^5t_{11}^8t_1^3 = Ht_1^3 \]
\[ Hxyx^2y^3t_3^5 = Ht_1^3 \]
\[ Hxyx^2y^3t_{19} = Ht_9 \]
\[ Ht_{19} = Ht_9 \]
\[(x^2t^y)^3 = e \]
\[(x^2t_1^y)^3 = e \]
\[(x^3t_3^y)^3 = e \]
\[x^2t_{33}x^2t_{33}x^2t_{33} = e \]
\[x^2(x^2x^{-2})t_{33}x^2t_{33}x^2t_{33} = e \]
\[x^2x^2(x^{-2}t_{33}x^2)t_{33}x^2t_{33} = e \]
\[x^2x^2t_{33}^2t_{33}x^2t_{33} = e \]
\[t_{35}x^2t_{33}x^2t_{33} = e \]
\[t_{35}(x^2x^{-2})t_{33}x^2t_{33} = e \]
\[t_{35}x^2(x^{-2}t_{33}x^2)t_{33} = e \]
\[t_{35}x^2t_{33}^2t_{33}x^2t_{33} = e \]
\[t_{35}x^2t_{35}t_{33} = e \]
\[(x^2x^{-2})t_{35}x^2t_{35}t_{33} = e \]
\[x^2(x^{-2}t_{35}x^2)t_{35}t_{33} = e \]
\[x^2t_{35}^2t_{35}t_{33} = e \]
\[x^2t_{33}t_{35}t_{33} = e \] (5.7)
Our first double coset, $HeN = \{He^n|n \in N\} = \{H\}$, which we will denote by $[\ast]$.

The orbits of $N$ on
\{1,2,3,4,5,6,8,7,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40\}
are \{1,2,13,3,34,14,17,4,11,35,15,10,18,33,20,12,36,19,16,9\}
and \{5,6,29,7,26,30,37,8,23,27,31,22,38,25,40,24,28,39,32,21\}.

We will take a representative from each orbit, say $t_1$ and $t_5$, and determine to which double coset $Ht_1$ and $Ht_5$ belong.

**Word of Length 1**

$Ht_1N$ is a new double coset which we will denote by $[1]$.

$Ht_1N = \{Ht_1^n|n \in N\}$.

Since the orbit \{1, 2, 13, 3, 34, 14, 17, 4, 11, 35, 15, 10, 18, 33, 20, 12, 36, 19, 16, 9\} contains 20 elements then 20 symmetric generators will go to the new double coset $[1]$.

Now $N^{(1)} \geq H^1$.

$N^1 = \{e\}$. 
\( N^{(1)} = \text{Coset Stabiliser in } N \text{ of } Ht_1 = \{ n \in N|Ht_1^n = t_1 \}. \)

We will look for a relation that will increase the Coset Stabiliser \( N^{(1)} \).

\( Ht_{19} = Ht_9 \), by Equation 5.6

\[ [Ht_{19}]^{xy^{-1}x} = [Ht_9]^{xy^{-1}x} \]

\[ \Rightarrow Ht_1 = Ht_{15} \]

Now, since \( Ht_1^e = Ht_1 \Rightarrow e \in N^{(1)} \), and

\( Ht_1^{x^2y^4} = Ht_{15} = Ht_1 \Rightarrow x^2y^4 \in N^{(1)} \), then,

\( N^{(1)} = \text{Coset Stabiliser in } N \text{ of } Ht_1 = \{ n \in N|Ht_1^n = t_1 \} = \{ e, x^2y^4 \} \).

Furthermore, the number of single cosets of \( Ht_1N \) is \( \frac{|N|}{|N^{(1)}|} = \frac{20}{2} = 10 \).

Conjugating by elements in \( N \) gives us the following equal names.

\[

t_1 \sim t_{15} \\
t_2 \sim t_{16} \\
t_3 \sim t_{13} \\
t_4 \sim t_{14} \\
t_9 \sim t_{19} \\
t_{10} \sim t_{20} \\
t_{11} \sim t_{17} \\
t_{12} \sim t_{18} \\
t_{33} \sim t_{35} \\
t_{34} \sim t_{36}
\]

Therefore, \( Ht_1N = \{ Ht_1 = Ht_{15}, Ht_2 = Ht_{16}, Ht_3 = Ht_{13}, Ht_4 = Ht_{14}, \}

\( Ht_9 = Ht_{19}, Ht_{10} = Ht_{20}, Ht_{11} = Ht_{17}, Ht_{12} = Ht_{18}, Ht_{33} = Ht_{35}, Ht_{34} = Ht_{36} \} \)

\( Ht_5N \) is a new double coset which we will denote by \( [5] \).

\( Ht_5N = \{ Ht_5^n | n \in N \} \).

Since the orbit \{5,6,29,7,26,30,37,8,23,27,31,22,38,25,40,24,28,39,32,21\} contains 20 elements then 20 symmetric generators will go to the double coset \( [5] \).

Now \( N^{(5)} \geq H^5 \).

\( N^5 = \{ e \} \).

\( N^{(5)} = \text{Coset Stabiliser in } N \text{ of } Ht_5 = \{ n \in N|Ht_5^n = t_5 \} \).

We will look for a relation that will increase the Coset Stabiliser \( N^{(5)} \).
$Ht_{19} = Ht_9$, by Equation 5.6  
$\Rightarrow Ht_5^5 = Ht_1^3$  
$\Rightarrow Ht_3^{5^5} = Ht_1^{3^3}t_1^3$  
$\Rightarrow Ht_3^{5^3} = Ht_1^6$  
$\Rightarrow H[xyxt_3^6] = Ht_1^6$, since by Equation 5.7  
$x^2t_{33}t_{35}t_{33} = e$, by Equation 5.7  
$[x^2t_{33}t_{35}t_{33}]^{x^2y} = [e]^{x^2y}$  
$\Rightarrow x^2y^2t_{19}t_9t_{19} = e$  
$\Rightarrow x^2y^2t_3^{5^5}t_3^3 = e$  
$\Rightarrow x^2y^2t_3^{5^5}t_3^3 \in N(5)$  
$\Rightarrow x^2y^2t_3^{5^3} = t_3^6$  
$Hxyxt_3^6 = Ht_1^6$  
$\Rightarrow Ht_3^6 = Ht_1^6$  
$\Rightarrow Ht_{23} = Ht_{21}$  
$\Rightarrow [Ht_{23}]^{x^2y} = [Ht_{21}]^{x^2y}$  
$\Rightarrow Ht_5 = Ht_{27}$  

Now, since $Ht_5^e = Ht_5 \Rightarrow e \in N(5)$, and  
$Ht_5^{x^2y^2} = Ht_{27} = Ht_5 \Rightarrow x^2y^2 \in N(5)$, then,  
$N(5) = \text{Coset Stabiliser in } N$ of $Ht_5 = \{n \in N | Ht_5^n = t_5 \} = \{e, x^2y^2 \}$.  
Furthermore, the number of single cosets of $Ht_1N$ is $\frac{|N|}{|N(5)|} = \frac{20}{2} = 10$.

After conjugating by all the elements of $N$, we have the following equal names.

$t_5 \sim t_{27}$  
$t_6 \sim t_{28}$  
$t_7 \sim t_{25}$  
$t_8 \sim t_{26}$  
$t_{21} \sim t_{23}$  
$t_{22} \sim t_{24}$  
$t_{23} \sim t_{21}$  
$t_{29} \sim t_{39}$  
$t_{30} \sim t_{40}$  
$t_{31} \sim t_{37}$  
$t_{32} \sim t_{38}$  

Therefore, $Ht_5N = \{Ht_5 = Ht_{27}, Ht_6 = Ht_{28}, Ht_7 = Ht_{25}, Ht_8 = Ht_{26},$  
$Ht_{21} = Ht_{23}, Ht_{22} = Ht_{24}, Ht_{29} = Ht_{39}, Ht_{30} = Ht_{40}, Ht_{31} = Ht_{37}, Ht_{32} = Ht_{38} \}$
The orbits of \(N^{(1)}\) on 
\[ \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34, 35,36,37,38,39,40\} \]
are \(\{1,15\}, \{2,36\}, \{3,9\}, \{4,18\}, \{5,31\}, \{6,28\}, \{7,21\}, \{8,38\}, \{10,20\}, \{11,33\}, \{12,14\}, \{13,19\}, \{16,34\}, \{17,35\}, \{22,40\}, \{23,25\}, \{24,30\}, \{26,32\}, \{27,37\}, \text{and } \{29,39\}.

We want to see to which double coset \(Ht_1t_1, Ht_1t_2, Ht_1t_3, Ht_1t_4, Ht_1t_5, Ht_1t_6, Ht_1t_7, Ht_1t_8, Ht_1t_{10}, Ht_1t_{11}, Ht_1t_{14}, Ht_1t_{13}, Ht_1t_{16}, Ht_1t_{17}, Ht_1t_{22}, Ht_1t_{25}, Ht_1t_{24}, Ht_1t_{32}, Ht_1t_{37}, \text{and } Ht_1t_{29}\) belong.

\[ Ht_1t_1 = Ht_1t_1 \]
\[ \implies Ht_1t_1 = Ht_1^2 \]
\[ \implies Ht_1t_1 = Ht_5 \]
\[ \implies Ht_1t_1 \in [5], \text{since } Ht_5 \text{ is in } [5]. \]
Two symmetric generators will go to [5].

\(Ht_1t_2N\) is a new double coset which we will label \([1 2]\).
Two symmetric generators will go to \([1 2]\).

\[ Ht_1t_3 = Ht_1t_3 \]
\[ \implies Ht_1t_3 = Ht_{15}t_3, \text{since } Ht_1 = Ht_{15} \]
\[ Ht_1t_3 = Ht_{15}t_3 \]
\[ \implies Ht_1t_3 = Ht_3^4t_3 \]
\[ \implies Ht_1t_3 = Ht_3^5 \]
\[ \implies Ht_1t_3 = Ht_{19} \]
\[ \implies Ht_1t_3 \in [1], \text{since } Ht_{19} \text{ is in } [1]. \]
Two symmetric generators will go to \([1]\).

\(Ht_1t_4N\) is a new double coset which we will label \([1 4]\).
Two symmetric generators will go to \([1 4]\).
\[Ht_1t_5 = Ht_1t_5\]
\[\implies Ht_1t_5 = Ht_1t_1^2\]
\[\implies Ht_1t_5 = Ht_1^3\]
\[\implies Ht_1t_5 = Ht_9\]
\[\implies Ht_1t_5 \in [1], \text{ since } Ht_9 \text{ is in } [1].\]

Two symmetric generators will go to [1].

\[Ht_1t_6N \text{ is a new double coset which we will label } [1 6].\]

Two symmetric generators will go to [1 6].

\[Ht_1t_7 = Ht_1t_7\]
\[\implies Ht_1t_7 = Ht_{15}t_7, \text{ since } Ht_1 = Ht_{15}\]
\[Ht_1t_7 = Ht_{15}t_7\]
\[\implies Ht_1t_7 = Ht_1^4t_3^2\]
\[\implies Ht_1t_7 = Ht_3^6\]
\[\implies Ht_1t_7 = Ht_{23}\]
\[\implies Ht_1t_7 \in [5], \text{ since } Ht_{23} \text{ is in } [5].\]

Two symmetric generators will go to [5].

\[Ht_1t_8 = Ht_1t_8\]
\[\implies Ht_1t_8 = Ht_{15}t_8, \text{ since } Ht_1 = Ht_{15}\]
\[Ht_1t_8 = Ht_{15}t_8\]
\[\implies Ht_1t_8 = Ht_1^4t_3^2\]
\[\implies Ht_1t_8 = Ht_1^4[x^{-1}yt_2t_1^4], \text{ since by Equation 5.8}\]
\[x^3t_{11}t_{10}t_9 = e\]
\[[x^3t_{11}t_{10}t_9]y^{-1}x^{-1} = e^{y^{-1}x^{-1}}\]
\[\implies x^{-1}yt_2t_1t_3t_6 = e\]
\[\implies x^{-1}yt_2t_1^4t_4^9 = e\]
\[\implies x^{-1}yt_2t_1^4t_4^9t_4^2 = t_4^2\]
\[\implies x^{-1}yt_2t_1^4 = t_4^2\]
\[Ht_1t_8 = Ht_3x^{-1}yt_2t_1^4\]
\[\implies Ht_1t_8 = Hx^{-1}y[t_3^4]x^{-1}yt_2t_1^4\]
\[ Ht_1t_8 = Ht_2^3t_2t_1^4 \]
\[ Ht_1t_8 = Ht_2^4t_1^4 \]
\[ Ht_1t_8 = H[xyt_1^0t_1^2]t_1^4, \] since by Equation 5.8
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{-1}y^{-1} = e^{-1}y^{-1} \]
\[ y^{-1}x^{-1}t_{14}t_{33}t_{20} = e \]
\[ y^{-1}x^{-1}t_4^0t_4^5 = e \]
\[ y^{-1}x^{-1}t_4^0t_4^1t_4^0 = t_4^0 \]
\[ y^{-1}x^{-1}t_4^0t_4^2 = t_4^0t_1^2 \]
\[ xy^{-1}x^{-1}t_4^2 = xy^0t_1^2 \]
\[ t_2^1 = xy^0t_1^2 \]
\[ Ht_1t_8 = Hxy^6t_1^0t_1^4 \]
\[ Ht_1t_8 = Ht_4^6t_1^4 \]
\[ Ht_1t_8 = Ht_4^6[x^{-1}y^{-1}t_3^4t_2^0], \] since by Equation 5.8
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]y^{-2} = ey^{-2} \]
\[ x^{-1}y^{-1}t_{15}t_{34}t_{17} = e \]
\[ x^{-1}y^{-1}t_3^4t_2^5 = e \]
\[ x^{-1}y^{-1}t_3^4t_2^1t_1^0 = t_1^0 \]
\[ x^{-1}y^{-1}t_3^4t_2^2 = t_1^6 \]
\[ Ht_1t_8 = Ht_4^6x^{-1}y^{-1}t_3^4t_2^0 \]
\[ Ht_1t_8 = Hx^{-1}y^{-1}[t_4^0]x^{-1}y^{-1}t_3^4t_2^0 \]
\[ Ht_1t_8 = Ht_4^6t_1^0t_2^0 \]
\[ Ht_1t_8 = Ht_3^6t_2^0 \]
\[ Ht_1t_8 = Ht_23t_34 \]

Note that
\[ Ht_1t_6 = Ht_1t_6 \]
\[ Ht_1t_6 = Ht_1t_6 \]
\[ Ht_1t_6 = Ht_1t_6, \] since \( Ht_1 = Ht_1 \)
\[ Ht_1t_6 = Ht_15t_6 \]
\[ Ht_1t_6 = Ht_1t_6 \]
\[ Ht_1t_6 = Ht_1t_6 \]
\[ H_{t1}t_6 = H[yx t_1^6 t_2^2]t_2^2, \text{ since by Equation 5.8} \]
\[ x^3 t_{11}t_{10}t_9 = e \]
\[ [x^3 t_{11}t_{10}t_9]y^{-2} = e y^{-2} \]
\[ \implies x^{-1} y^{-1} t_{15} t_{34} t_{17} = e \]
\[ \implies x^{-1} y^{-1} t_1^4 t_2^5 t_1^5 = e \]
\[ \implies x^{-1} y^{-1} t_3^4 t_2^5 t_1^6 = t_4^6 \]
\[ \implies x^{-1} y^{-1} t_3^4 t_2^5 t_1^7 = t_4^6 t_1^2 \]
\[ \implies y x y^{-1} x^{-1} t_4^5 = y x t_4^6 t_1^2 \]
\[ t_3^4 = y x t_4^6 t_1^2 \]
\[ H_{t1}t_6 = H y x t_1^6 t_2^2 t_2^2 \]
\[ \implies H_{t1}t_6 = H t_1^6 t_2^4 \]
\[ \implies H_{t1}t_6 = H t_3^6 t_4^4, \text{ since } H t_{21} = H t_{23} \]
\[ \implies H t_1^6 = H t_3^6 \]
\[ H_{t1}t_6 = H t_3^6 t_4^4 \]
\[ \implies H_{t1}t_6 = H t_3^6 [x y t_4^6 t_1^2], \text{ since by Equation 5.8} \]
\[ x^3 t_{11}t_{10}t_9 = e \]
\[ [x^3 t_{11}t_{10}t_9]^{-1} y^{-1} = e^{-1} y^{-1} \]
\[ \implies y^{-1} x^{-1} t_{14} t_{33} t_{20} = e \]
\[ \implies y^{-1} x^{-1} t_2^4 t_1^5 t_4 = e \]
\[ \implies y^{-1} x^{-1} t_2^4 t_1^5 t_4 t_4 = t_4^6 \]
\[ \implies y^{-1} x^{-1} t_2^4 t_1^5 t_4 t_1 = t_4^6 t_1^2 \]
\[ \implies x y y^{-1} x^{-1} t_4^5 = x y t_4^6 t_1^2 \]
\[ t_2^4 = x y t_4^6 t_1^2 \]
\[ H_{t1}t_6 = H t_3^6 x y t_4^6 t_1^2 \]
\[ \implies H_{t1}t_6 = H x y t_3^6 x y t_4^6 t_1^2 \]
\[ \implies H_{t1}t_6 = H t_4^6 t_1^2 \]
\[ \implies H_{t1}t_6 = H t_4^6 t_1^2, \text{ since} \]
\[ H t_{12} t_{18} \implies t_3^4 = H t_5^5 \]
\[ H t_{14} t_6 = H t_5^6 t_1^2 \]
\[ \implies H_{t1}t_6 = H [x y^{-1} t_4^7 t_1^6] t_1^2, \text{ since by Equation 5.8} \]
\[ x^3 t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]x^{-1}y = e^{x^{-1}y} \]
\[ \Rightarrow yx^{-1}t_{18}t_{16} = e \]
\[ \Rightarrow yx^{-1}t_5t_4 = e \]
\[ \Rightarrow yx^{-1}t_2t_4t_3 = t_4 \]
\[ \Rightarrow yx^{-1}t_2t_1t_1 = t_4^{-1}t_1^{-1} \]
\[ \Rightarrow xy^{-1}ye^{-1}t_5^2 = xy^{-1}t_4^{-1}t_1^{-1} \]
\[ \Rightarrow t_5^2 = xy^{-1}t_4^{-1}t_1^{-1} \]
\[ Ht_1t_6 = Hxy^{-1}t_4^{-1}t_1^{-1} \]
\[ \Rightarrow Ht_1t_6 = Ht_4t_1 \]
\[ \Rightarrow Ht_1t_6 = Ht_{28}t_1 \]
\[ \Rightarrow Ht_1t_6 = Ht_6t_1, \text{ since } Ht_6 = Ht_{28} \]

\[ Ht_1t_6 = Ht_6t_1 \] (5.9)

Thus,
\[ Ht_1t_8 = Ht_{23}t_{34} \]
\[ \Rightarrow Ht_1t_8 = Ht_{34}t_{23}, \text{ since by Equation 5.9} \]
\[ Ht_1t_6 = Ht_6t_1 \]
\[ \Rightarrow [Ht_1t_6]^{xy} = [Ht_6t_1]^{xy} \]
\[ \Rightarrow Ht_{34}t_{23} = Ht_{23}t_{34} \]
\[ Ht_1t_8 = Ht_{34}t_{23} \]
\[ \Rightarrow Ht_1t_8 \in [16], \text{ since } Ht_{34}t_{23} \text{ is in } [1 6]. \]

Two symmetric generators will go to [1 6].

\[ Ht_{1}t_{10} \text{ is a new double coset which we will label } [1 10]. \]

Two symmetric generators will go to [1 10].

\[ Ht_{1}t_{11} = Ht_{11}t_{11} \]
\[ \Rightarrow Ht_{1}t_{11} = Ht_{15}t_{11}, \text{ since } Ht_1 = Ht_{15} \]
\[ Ht_{1}t_{11} = Ht_{15}t_{11} \]
\[ \Rightarrow Ht_{1}t_{11} = Ht_{3}^{4}t_{3}^{3} \]
\[ \Rightarrow Ht_{1}t_{11} = Ht_{3}^{7} \]
\[ H_{t_1 t_11} = H_{t_11 t_11} \]
\[ H_{t_1 t_11} = H_{t_27} \]
\[ H_{t_1 t_11} \in [5], \text{ since } H_{t_{27}} \text{ is in } [5]. \]

Two symmetric generators will go to \([5]\).
\[ \Rightarrow y^{-1}x^{-1}yt_2 = y^{-1}x^2t_1^7 \]
\[ \Rightarrow t_2 = y^{-1}x^2t_1^7 \]
\[ Ht_{14} = Hy^{-1}x^2t_1^7t_2^2 \]
\[ \Rightarrow Ht_{14} = Ht_2^2 \]
\[ \Rightarrow Ht_{14} = Ht_2^2, \text{ since } \]
\[ Ht_8 = Ht_{26} \]
\[ \Rightarrow Ht_2^2 - Ht_2 \]
\[ Ht_{14} = Ht_2^2 \]
\[ Ht_{14} = Ht_2^2, \text{ since by Equation 5.8} \]
\[ x^3t_1t_10t_9 = e \]
\[ [x^3t_1t_10t_9]x^2y^{-1} = e^2y^{-1} \]
\[ \Rightarrow y^{-1}x^{-1}t_3t_20t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_{14}^5t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_{14}^5t_3t_3^{10} = t_3^{10} \]
\[ \Rightarrow y^{-1}x^{-1}t_{14}^5t_4^{10} = t_3^{10}t_4^{10} \]
\[ \Rightarrow xyt^{-1}x^{-1}t_1^9 = xyt_3^{10}t_4^{10} \]
\[ \Rightarrow t_1^9 = xyt_3^{10}t_4^{10} \]
\[ \Rightarrow Ht_{14} = Hxy[t_2^7]xyt_3^{10}t_4^{10} \]
\[ \Rightarrow Ht_{14} = Hxy[t_2^7]xyt_3^{10}t_4^{10} \]
\[ \Rightarrow Ht_{14} = Ht_3^{10}t_4^{10} \]
\[ \Rightarrow Ht_{14} = Ht_{35}t_{24} \]
\[ \Rightarrow Ht_{14} \in [16], \text{ since } Ht_{35}t_{24} \text{ is in } [1 6]. \]

Two symmetric generators will go to [16].

\[ Ht_{13} = Ht_{13} \]
\[ \Rightarrow Ht_{13} = Ht_1^4 \]
\[ \Rightarrow Ht_{13} = Ht_1^5 \]
\[ \Rightarrow Ht_{13} = Ht_{17} \]
\[ \Rightarrow Ht_{13} \in [1], \text{ since } Ht_{17} \text{ is in } [1]. \]

Two symmetric generators will go to [1].
\[Ht_{16} = Ht_{16}\]
\[\implies Ht_{16} = Ht_{1}t_{4}^{4}\]
\[\implies Ht_{16} = H[y^{-1}t_{3}^{-1}t_{4}^{4}], \text{ since by Equation 5.8}\]
\[x^{3}t_{11}t_{10}t_{9} = e\]
\[\implies y^{-1}t_{1}t_{16}t_{35} = e\]
\[\implies y^{-1}t_{1}t_{4}^{0}t_{3}^{3} = e\]
\[\implies y^{-1}t_{1}t_{4}^{0}t_{3}^{2}t_{3}^{3} = t_{3}^{2}\]
\[\implies y^{-1}t_{1}t_{4}^{7}t_{3}^{7} = t_{3}^{2}t_{4}^{7}\]
\[\implies xy^{-1}y^{-1}t_{1} = xy^{-1}t_{3}^{2}t_{4}^{7}\]
\[\implies t_{1} = xy^{-1}t_{3}^{2}t_{4}^{7}\]
\[Ht_{16} = Hxy^{-1}t_{3}^{2}t_{4}^{7}t_{4}^{4}\]
\[\implies Ht_{16} = Ht_{3}^{2}\]
\[\implies Ht_{16} = Ht_{7}\]
\[\implies Ht_{16} \in [5], \text{ since } Ht_{7} \text{ is in } [5].\]

Two symmetric generators will go to [5].

\[Ht_{17} = Ht_{17}\]
\[\implies Ht_{17} = Ht_{1}t_{5}^{5}\]
\[\implies Ht_{17} = Ht_{1}^{1}\]
\[\implies Ht_{17} = Ht_{21}\]
\[\implies Ht_{17} \in [5], \text{ since } Ht_{21} \text{ is in } [5].\]

Two symmetric generators will go to [5].

\[Ht_{22} = Ht_{12}t_{22}\]
\[\implies Ht_{22} = Ht_{15}t_{22}, \text{ since } Ht_{1} = Ht_{15}\]
\[Ht_{22} = Ht_{15}t_{22}\]
\[\implies Ht_{22} = Ht_{15}t_{22}\]
\[x^{3}t_{11}t_{10}t_{9} = e\]
\[\implies y^{-2} = e^{-2}\]
\[\implies x^{-1}y^{-1}t_{15}t_{34}t_{17} = e\]
\[\implies x^{-1}y^{-1}t_{3}^{3}t_{2}^{5}t_{1} = e\]
\[ x^{-1}y^{-1}t_3^{25}t_1^{11} = t_1^6 \]
\[ x^{-1}y^{-1}t_3^{25}t_1^{11} = t_1^6t_1^2 \]
\[ yxy^{-1}x^{-1}t_4^3 = yxt_1^{6t_1^2} \]
\[ t_3^{t_3} = yxt_1^{6t_1^2} \]

\[ Ht_1t_2 = Hyxt_1^{6t_1^2}t_1^6 \]
\[ \Rightarrow Ht_1t_2 = Ht_1^{6t_1^2} \]
\[ \Rightarrow Ht_1t_2 = Ht_1^{6[x^{-1}t_4^3t_3^3]}, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^x = e^x \]
\[ \Rightarrow x^{-1}t_{12}t_{11}t_{10} = e \]
\[ \Rightarrow x^{-1}t_4^{3}t_3^3 = e \]
\[ \Rightarrow x^{-1}t_4^{3}t_3^3t_2^2 = t_8^2 \]
\[ \Rightarrow x^{-1}t_4^{3}t_3^3 = t_8^2 \]

\[ Ht_1t_2 = Ht_1^{6x^{-1}t_4^3t_3^3} \]
\[ \Rightarrow Ht_1t_2 = Ht_1^{6x^{-1}t_4^3t_3^3} \]
\[ \Rightarrow Ht_1t_2 = Ht_1^{6x^{-1}t_4^3t_3^3} \]
\[ \Rightarrow Ht_1t_2 = Ht_34t_11 \]
\[ \Rightarrow Ht_1t_2 = Ht_34t_11, \text{ since } Ht_34 = Ht_36 \]
\[ Ht_1t_2 = Ht_34t_11 \]
\[ \Rightarrow Ht_1t_2 \in [12], \text{ since } Ht_34t_11 \text{ is in [1 2].} \]

Two symmetric generators will go to [1 2].

\[ Ht_1t_25 = Ht_1t_25 \]
\[ \Rightarrow Ht_1t_25 = Ht_1t_1^7 \]
\[ \Rightarrow Ht_1t_25 = Ht_1^6 \]
\[ \Rightarrow Ht_1t_25 = Ht_29 \]
\[ \Rightarrow Ht_1t_25 \in [5], \text{ since } Ht_29 \text{ is in [5].} \]

Two symmetric generators will go to [5].

\[ Ht_1t_24 = Ht_1t_24 \]
\[ \Rightarrow Ht_1t_24 = Ht_1^{6t_4^3} \]
\( Ht_{1t24} = H[x^{-1}t_3^2t_4^5t_4^4], \) since by Equation 5.8
\( x^3t_{11t10t9} = e \)
\([x^3t_{11t10t9}]x^2y = e^x^2y \)
\implies yx^{-1}t_{11t10t35} = e \)
\implies yx^{-1}t_1^4t_4^9 = e \)
\implies yx^{-1}t_1^4t_3^9t_3^2 = t_3^2 \)
\implies yx^{-1}t_1t_4^7 = t_4^7 \)
\implies xy^{-1}yx^{-1}t_1 = xy^{-1}t_2^3t_4^7 \)
\implies t_1 = xy^{-1}t_3^2t_4^7 \)
\( Ht_{1t30} = Hxy^{-1}t_3^2t_4^5t_4^6 \)
\implies Ht_{1t24} = Hx^{-1}t_3^2t_4^2 \)
\implies Ht_{1t24} = Ht_3^2t_4^2 \)
\implies Ht_{1t24} = Ht_3^2[x^{-1}yt_2t_4^4], \) since by Equation 5.8
\( x^3t_{11t10t9} = e \)
\([x^3t_{11t10t9}]y^{-1}x^{-1} = e^y^{-1}x^{-1} \)
\implies x^{-1}yt_2t_{13t36} = e \)
\implies x^{-1}yt_2t_4^9 = e \)
\implies x^{-1}yt_2t_4^9t_4^2 = t_4^2 \)
\implies x^{-1}yt_2t_4^4 = t_4^4 \)
\( Ht_{1t24} = Ht_3^2x^{-1}yt_2t_4^4 \)
\implies Ht_{1t24} = Hx^{-1}y[t_3^2]^{-1}yt_2t_4^4 \)
\implies Ht_{1t24} = Ht_2^4t_2t_4^4 \)
\implies Ht_{1t24} = Ht_2^4t_4^2 \)
\implies Ht_{1t24} = Ht_4^0t_4^1, \) since
\( Ht_{30} = Ht_{40} \)
\implies Ht_3^0 = Ht_4^4 \)
\( Ht_{1t24} = Ht_4t_4^{10}t_4^4 \)
\( Ht_{1t24} = H[x^{-1}y^{-1}t_2^9t_4^5]t_4^4, \) since by Equation 5.8
\( x^3t_{11t10t9} = e \)
\([x^3t_{11t10t9}]yx^{-1} = e^yx^{-1} \)
\implies x^{-1}y^{-1}t_3^4t_{17t7} = e \)
\implies x^{-1}y^{-1}t_2^9t_4^5t_4^4 = e
\[ x^{-1}y^{-1}t_2^{t_4^5}t_4^{t_4^4} = t_4^{t_4^4} \]
\[ x^{-1}y^{-1}t_2^{t_4^5} = t_4^{t_4^4} \]
\[ Ht_{124} = Hx^{-1}y^{-1}t_2^{t_4^5}t_4^1 \]
\[ \Rightarrow Ht_{124} = Ht_2^{t_4^5} \]
\[ \Rightarrow Ht_{124} = Ht_{34}t_{33} \]
\[ \Rightarrow Ht_{124} = Ht_{36}t_{33}, \text{ since } Ht_{34} = Ht_{36} \]
\[ Ht_{124} = Ht_{36}t_{33} \]
\[ \Rightarrow Ht_{124} \in [110], \text{ since } Ht_{36}t_{33} \text{ is in } [1 10]. \]

Two symmetric generators will go to [1 10].

\[ Ht_{132} = Ht_{132} \]
\[ \Rightarrow Ht_{132} = Ht_1^{t_4^8} \]
\[ \Rightarrow Ht_{132} = H[xy^{-1}t_3^{t_4^2}]t_4^8, \text{ since by Equation 5.8} \]
\[ x^3t_1t_{10}t_9 = e \]
\[ [x^3t_1t_{10}t_9]^{x^2}y = e^{x^2}y \]
\[ \Rightarrow yx^{-1}t_{16}t_{35} = e \]
\[ \Rightarrow yx^{-1}t_1^{t_4^8} = e \]
\[ \Rightarrow yx^{-1}t_1^{t_4^8}t_3^2 = t_3^2 \]
\[ \Rightarrow yx^{-1}t_1^{t_4^8}t_4^7 = t_4^7 \]
\[ \Rightarrow xy^{-1}y^{-1}t_1 = xy^{-1}t_3^{t_4^2} \]
\[ \Rightarrow t_1 = xy^{-1}t_3^{t_4^2} \]
\[ Ht_{132} = Hxy^{-1}t_3^{t_4^2}t_4^8 \]
\[ \Rightarrow Ht_{132} = Ht_3^{t_4^2} \]
\[ \Rightarrow Ht_{132} = Ht_7t_{16} \]
\[ \Rightarrow Ht_{132} = Ht_{25}t_{16}, \text{ since } Ht_7 = Ht_{25} \]
\[ Ht_{132} = Ht_{25}t_{16} \]
\[ Ht_{132} = Ht_{16}t_{25}, \text{ since by Equation 5.9} \]
\[ Ht_{16} = Ht_6t_1 \]
\[ [Ht_{16}]^{y^{-1}} = [Ht_6t_1]^{y^{-1}} \]
\[ \Rightarrow Ht_{16}t_{25} = Ht_{25}t_{16} \]
\[ Ht_{132} = Ht_{16}t_{25} \]
\[ \Rightarrow Ht_{132} \in [16], \text{ since } Ht_{16}t_{25} \text{ is in } [1 6]. \]
Two symmetric generators will go to [1 6].

\[ H_{t_1} t_{37} = H_{t_1} t_{37} \]
\[ \implies H_{t_1} t_{37} = H_{t_1} t_{10} \]
\[ \implies H_{t_1} t_{37} = H_{e} \]
\[ \implies H_{t_1} t_{37} \in [*] \text{, since } H_{e} \text{ is in [*]} \]

Two symmetric generators will go to [*].

\[ H_{t_1} t_{29} = H_{t_1} t_{29} \]
\[ \implies H_{t_1} t_{29} = H_{t_1} t_{8} \]
\[ \implies H_{t_1} t_{29} = H_{t_1} t_{9} \]
\[ \implies H_{t_1} t_{29} = H_{t_{33}} \]
\[ \implies H_{t_1} t_{29} \in [1] \text{, since } H_{t_{33}} \text{ is in [1]} \]

Two symmetric generators will go to [1].

The orbits of \( N^{(5)} \) on
\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40\}
are \{1, 35\}, \{2, 12\}, \{3, 17\}, \{4, 14\}, \{5, 27\}, \{6, 24\}, \{7, 37\}, \{8, 30\}, \{9, 19\}, \{10, 36\},
\{11, 13\}, \{15, 33\}, \{16, 18\}, \{20, 34\}, \{21, 39\}, \{22, 28\}, \{23, 29\}, \{25, 31\}, \{26, 40\},
and \{32, 38\}.

We want to see to which double coset
\( H_{t_5} t_{1}, H_{t_5} t_{2}, H_{t_5} t_{17}, H_{t_5} t_{14}, H_{t_5} t_{5}, H_{t_5} t_{6}, H_{t_5} t_{37}, H_{t_5} t_{30}, H_{t_5} t_{9}, H_{t_5} t_{10}, H_{t_5} t_{13}, H_{t_5} t_{33}, H_{t_5} t_{18}, H_{t_5} t_{20}, H_{t_5} t_{21}, H_{t_5} t_{28}, H_{t_5} t_{29}, H_{t_5} t_{25}, H_{t_5} t_{26}, H_{t_5} t_{32} \) belong.

\[ H_{t_{5}} t_{1} = H_{t_{1}} t_{1} \]
\[ \implies H_{t_5} t_{1} = H_{t_{1}} t_{3} \]
\[ \implies H_{t_5} t_{1} = H_{t_{9}} \]
\[ \implies H_{t_5} t_{1} \in [1] \text{, since } H_{t_{9}} \text{ is in [1]} \]

Two symmetric generators will go to [1].
\[ H_{t_5t_2} = H_{t_1^2t_2} \]
\[ \implies H_{t_5t_2} = H[y^{-1}x^{-1}t_3t_2^4]t_2, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]y^{-1} = ey^{-1} \]
\[ \implies x^{-1}y^{-1}t_3t_{14}t_{33} = e \]
\[ \implies x^{-1}y^{-1}t_3t_2t_1^9 = e \]
\[ \implies x^{-1}y^{-1}t_3t_2t_1^9t_1^2 = t_1^2 \]
\[ \implies x^{-1}y^{-1}t_3t_2^2 = t_1^1 \]
\[ H_{t_5t_2} = Hg^{-1}x^{-1}t_3t_2^4t_2 \]
\[ \implies H_{t_5t_2} = H_{t_3t_2}^5 \]
\[ \implies H_{t_5t_2} = H_{t_3t_2}^5 \]
\[ \implies H_{t_5t_2} = H_{t_3t_2}^5, \text{ since } H_{t_3} = H_{t_13}. \]
\[ H_{t_5t_2} = H_{t_3t_2}^5 \]
\[ \implies H_{t_5t_2} \in [110], \text{ since } H_{t_13t_2}^5 \text{ is in } [1 10]. \]
Two symmetric generators will go to [1 10].

\[ H_{t_5t_17} = H_{t_1^2t_1^5} \]
\[ \implies H_{t_5t_17} = H_{t_1^5} \]
\[ \implies H_{t_5t_17} = H_{t_25} \]
\[ \implies H_{t_5t_17} \in [1], \text{ since } H_{t_25} \text{ is in } [1]. \]
Two symmetric generators will go to [1].

\[ H_{t_5t_{14}} = H_{t_5t_{14}} \]
\[ \implies H_{t_5t_{14}} = H_{t_27t_{14}} \text{, since } H_{t_5} = H_{t_27} \]
\[ H_{t_5t_{14}} = H_{t_27t_{14}} \]
\[ \implies H_{t_5t_{14}} = H_{t_{14}t_{27}}, \text{ since by Equation 5.9} \]
\[ H_{t_{16}} = H_{t_6t_1} \]
\[ \implies [H_{t_1t_6}]^{yx} = [H_{t_6t_1}]^{yx} \]
\[ \implies H_{t_{14}t_{27}} = H_{t_{27}t_{14}} \]
\[ H_{t_5t_{14}} = H_{t_{14}t_{27}} \]
\[ \implies H_{t_5t_{14}} \in [16], \text{ since } H_{t_{14}t_{27}} \text{ is in } [1 6]. \]
Two symmetric generators will go to [1 6].
\[ H_{t5t5} = H_{t1}^2 t_1^2 \]
\[ \implies H_{t5t5} = H_{t1}^4 \]
\[ \implies H_{t5t5} = H_{t13} \]
\[ \implies H_{t5t5} \in [1], \text{ since } H_{t13} \text{ is in } [1]. \]

Two symmetric generators will go to [1].

\[ H_{t5t6} = H_{t1}^2 t_2^2 \]
\[ \implies H_{t5t6} = H_{t1}^2 [x^{-1} y^{-1} t_4 t_3^4], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{-1} x = e^{-1} x \]
\[ \implies x^{-1} y^{-1} t_4 t_{15} t_{34} = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^4 t_2^9 = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^4 t_2^9 t_2^2 = t_2^2 \]
\[ \implies x^{-1} y^{-1} t_4 t_3^4 = t_2^2 \]
\[ H_{t5t6} = H_{t1}^2 x^{-1} y^{-1} t_4 t_3^4 \]
\[ \implies H_{t5t6} = H x^{-1} y^{-1} [t_1^2]^{-1} x^{-1} y^{-1} t_4 t_3^4 \]
\[ \implies H_{t5t6} = H_{t4}^4 t_4 t_3^4 \]
\[ \implies H_{t5t6} = H_{t4}^8 t_4^4 \]
\[ \implies H_{t5t6} = H_{t1}^{10} t_4^2, \text{ since } \]
\[ H_{t32} = H_{t38} \]
\[ \implies H_{t4}^8 = H_{t2}^{10} \]
\[ H_{t5t6} = H_{t2}^{10} t_4^2 \]
\[ \implies H_{t5t6} = H [x^{-1} y t_4^5 t_3^4], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{yx} = e^{yx} \]
\[ \implies x^{-1} y t_{36} t_{19} t_2 = e \]
\[ \implies x^{-1} y t_4^5 t_3^4 t_2 = e \]
\[ \implies x^{-1} y t_4^5 t_3^4 t_2 t_2^{10} = t_2^{10} \]
\[ \implies x^{-1} y t_4^5 t_3^4 = t_2^{10} \]
\[ H_{t5t6} = H x^{-1} y t_4^5 t_3^4 \]
\[ \implies H_{t5t6} = H t_4^9 t_3^4 \]
\[ H_{t5}t_6 = H_{t36}t_{35} \]
\[ H_{t5}t_6 = H_{t34}t_{35}, \text{ since } H_{t34} = H_{t36} \]

\[ H_{t5}t_6 = H_{t34}t_{35} \]
\[ \implies H_{t5}t_6 \in [10], \text{ since } H_{t34}t_{35} \text{ is in [1 10].} \]

Two symmetric generators will go to [1 10].

\[ H_{t5}t_{37} = H_{t1}t_{10} \]
\[ \implies H_{t5}t_{37} = H_{t1} \]
\[ \implies H_{t5}t_{37} \in [1], \text{ since } H_{t1} \text{ is in [1].} \]

Two symmetric generators will go to [1].

\[ H_{t5}t_{30} = H_{t1}t_8^2t_2 \]
\[ H_{t5}t_{30} = H[y^{-1}x^{-1}t_3t_4^2]t_8^2, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]y^{-1} = ey^{-1} \]
\[ \implies x^{-1}y^{-1}t_3t_4t_3 = e \]
\[ \implies x^{-1}y^{-1}t_3t_4^2t_1 = t_2 \]
\[ \implies x^{-1}y^{-1}t_3t_4^2t_1 = t_2^2 \]
\[ H_{t5}t_{30} = Hy^{-1}x^{-1}t_3t_4^2t_2^2 \]
\[ \implies H_{t5}t_{30} = H_{t3}t_2 \]
\[ \implies H_{t5}t_{30} \in [14], \text{ since } H_{t3}t_2 \text{ is in [1 4].} \]

Two symmetric generators will go to [1 4].

\[ H_{t5}t_9 = H_{t1}t_1^3 \]
\[ \implies H_{t5}t_9 = H_{t1}^5 \]
\[ \implies H_{t5}t_9 = H_{t17} \]
\[ \implies H_{t5}t_9 \in [1], \text{ since } H_{t17} \text{ is in [1].} \]

Two symmetric generators will go to [1].

\[ H_{t5}t_{10} = H_{t1}t_3^2 \]
\[ H_{t5}t_{10} = H[y^{-1}x^{-1}t_3t_4^2]t_3^2, \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{y^{-1}} = e^{y^{-1}} \]
\[ \implies x^{-1}y^{-1}t_{3}t_{14}t_{33} = e \]
\[ \implies x^{-1}y^{-1}t_{3}t_{2}t_{1}^{9} = e \]
\[ \implies x^{-1}y^{-1}t_{3}t_{2}t_{1}^{9} = t_{2}^{2} \]
\[ \implies x^{-1}y^{-1}t_{3}t_{2} = t_{2}^{2} \]
\[ Ht_{5}t_{10} = H y^{-1}x^{-1}t_{3}t_{2}t_{1}^{3} \]
\[ \implies Ht_{5}t_{10} = Ht_{3}t_{2}^{2} \]
\[ \implies Ht_{5}t_{10} = Ht_{3}t_{26} \]
\[ \implies Ht_{5}t_{10} = Ht_{13}t_{26}, \text{ since } Ht_{3} = Ht_{13} \]
\[ Ht_{5}t_{10} = Ht_{13}t_{26} \]
\[ \implies Ht_{5}t_{10} \in [16], \text{ since } Ht_{13}t_{26} \text{ is in } [1 6]. \]

Two symmetric generators will go to [1 6].

\[ Ht_{5}t_{13} = Ht_{1}t_{1}^{4} \]
\[ \implies Ht_{5}t_{13} = Ht_{1}^{6} \]
\[ \implies Ht_{5}t_{13} = Ht_{21} \]
\[ \implies Ht_{5}t_{13} \in [5], \text{ since } Ht_{21} \text{ is in } [5]. \]

Two symmetric generators will go to [5].

\[ Ht_{5}t_{33} = Ht_{1}t_{1}^{9} \]
\[ \implies Ht_{5}t_{33} = H \]
\[ \implies Ht_{5}t_{33} \in [*], \text{ since } He \text{ is in } [*]. \]

Two symmetric generators will go to [*].

\[ Ht_{5}t_{18} = Ht_{1}t_{2}^{5} \]
\[ Ht_{5}t_{18} = H[y^{-1}x^{-1}t_{3}t_{2}t_{1}^{9}]t_{2}^{5}, \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{y^{-1}} = e^{y^{-1}} \]
\[ \implies x^{-1}y^{-1}t_{3}t_{14}t_{33} = e \]
\[ \implies x^{-1}y^{-1}t_{3}t_{2}t_{1}^{9} = e \]
\[ \implies x^{-1}y^{-1}t_{3}t_{2}t_{1}^{9} = t_{2}^{2} \]
\[ x^{-1}y^{-1}t_3t_4^4 = t_2^4 \]

\[ Ht_{518} = H_1y^{-1}x^{-1}t_3t_4^4t_2^3 \]

\[ Ht_{518} = Ht_{318} \]

\[ Ht_{518} = Ht_{334} \]

\[ Ht_{518} = Ht_{1334}, \text{ since } Ht_3 = Ht_{13} \]

\[ Ht_{518} = Ht_{1334} \]

\[ Ht_{518} \in \{12\}, \text{ since } Ht_{1334} \text{ is in } \{12\}. \]

Two symmetric generators will go to \{12\}.

\[ Ht_{520} = \frac{H_1^2t_3}{t_4} \]

\[ Ht_{520} = H_1xyt_2t_3^{10}, \text{ since by Equation 5.8} \]

\[ x^3t_{11}t_{10} = e \]

\[ [x^3t_{11}t_{10}]^{xy^{-1}} = e^{xy^{-1}} \]

\[ y^{-1}x^{-1}t_{20}t_3t_{14} = e \]

\[ y^{-1}x^{-1}t_4^5t_3^4 = e \]

\[ y^{-1}x^{-1}t_4^5t_3^4t_2^{10} = t_2^{10} \]

\[ y^{-1}x^{-1}t_4^5t_3^4t_3^{10} = t_2^{10}t_3^{10} \]

\[ xy^{-1}x^{-1}t_4^5 = xyt_2t_3^{10} \]

\[ t_5^5 = xyt_2t_3^{10} \]

\[ Ht_{520} = H_1^2xyt_2t_3^{10} \]

\[ Ht_{520} = Hxy[t_2^2]xyt_2t_3^{10} \]

\[ Ht_{520} = H_1^2t_2t_3^{10} \]

\[ Ht_{520} = H_1t_2t_3^{10} \]

\[ Ht_{520} = Ht_{1039} \]

\[ Ht_{520} \in \{16\}, \text{ since } Ht_{1039} \text{ is in } \{16\}. \]

Two symmetric generators will go to \{16\}.

\[ Ht_{521} = H_1^2t_4^6 \]

\[ Ht_{521} = Ht_{11}^8 \]

\[ Ht_{521} = Ht_{29} \]

\[ Ht_{521} \in \{5\}, \text{ since } Ht_{29} \text{ is in } \{5\}. \]

Two symmetric generators will go to \{5\}.
\[H_{t_5 t_{28}} = H_{t_{27} t_{28}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{27} t_{28}}, \text{ since } H_{t_5} = H_{t_{27}}\]

\[H_{t_5 t_{28}} = H_{t_{27} t_{28}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{3} t_{4}^{7}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{3}^{7} y x^{-1} t_{2}^{5} t_{1}}, \text{ since by Equation 5.8}\]

\[x^{3} t_{11} t_{10} t_{9} = e\]

\[(x^{3} t_{11} t_{10} t_{9})^{-1} = e^{-1}\]

\[\implies y x^{-1} t_{18} t_{1} t_{16} = e\]

\[\implies y x^{-1} t_{2}^{5} t_{1} t_{4}^{4} = e\]

\[\implies y x^{-1} t_{2}^{5} t_{1} t_{4}^{4} t_{4}^{7} = t_{4}^{7}\]

\[\implies y x^{-1} t_{2}^{5} t_{4} = t_{4}^{7}\]

\[H_{t_5 t_{28}} = H_{t_{3}^{7} y x^{-1} t_{2}^{5} t_{1}}\]

\[\implies H_{t_5 t_{28}} = H_{y x^{-1} t_{3}^{7} y x^{-1} t_{2}^{5} t_{1}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{2}^{10} t_{2}^{5} t_{1}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{2}^{5} t_{1}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{14} t_{1}}\]

\[\implies H_{t_5 t_{28}} = H_{t_{4} t_{1}}, \text{ since } H_{t_4} = H_{t_{14}}\]

\[H_{t_5 t_{28}} = H_{t_{4} t_{1}}\]

\[\implies H_{t_5 t_{28}} \in [12], \text{ since } H_{t_{4} t_{1}} \text{ is in } [1 2]\]

Two symmetric generators will go to \([1 2]\).

\[H_{t_5 t_{29}} = H_{t_{1}^{2} t_{1}^{8}}\]

\[\implies H_{t_5 t_{29}} = H_{t_{1}^{10}}\]

\[\implies H_{t_5 t_{29}} = H_{t_{37}}\]

\[\implies H_{t_5 t_{29}} \in [5], \text{ since } H_{t_{37}} \text{ is in } [5].\]

Two symmetric generators will go to \([5]\).

\[H_{t_5 t_{25}} = H_{t_{1}^{2} t_{1}^{5}}\]

\[\implies H_{t_5 t_{25}} = H_{t_{1}^{5}}\]

\[\implies H_{t_5 t_{25}} = H_{t_{33}}\]

\[\implies H_{t_5 t_{25}} \in [1], \text{ since } H_{t_{33}} \text{ is in } [1].\]
Two symmetric generators will go to $[1]$.

\[ Ht_{5t26} = Ht_5^2t_2^7 \]
\[ \implies Ht_{5t26} = H[y^{-1}x^{-1}t_3^2t_4^3]t_2^7, \text{ since by Equation 5.8} \]
\[ x^3t_{11t10t9} = e \]
\[ [x^3t_{11t10t9}]^{-1} = ey^{-1} \]
\[ \implies x^{-1}y^{-1}t_3t_{14t33} = e \]
\[ \implies x^{-1}y^{-1}t_3t_{2t1^0} = e \]
\[ \implies x^{-1}y^{-1}t_3t_{2t1^2} = t_1^2 \]
\[ \implies x^{-1}y^{-1}t_3t_{2} = t_1^2 \]
\[ Ht_{5t26} = H[y^{-1}x^{-1}t_3^2t_4^3] \]
\[ \implies Ht_{5t26} = Ht_3 \]
\[ \implies Ht_{5t26} \in [1], \text{ since } Ht_3 \text{ is in } [1]. \]

Two symmetric generators will go to $[1]$.

\[ Ht_{5t32} = Ht_{5t32}^2 \]
\[ \implies Ht_{5t32} = Ht_{27t32}, \text{ since } Ht_5 = Ht_{27} \]
\[ Ht_{5t32} = Ht_{27t32} \]
\[ \implies Ht_{5t32} = Ht_{3}^7t_4^8 \]
\[ \implies Ht_{5t32} = Ht_{3}^7[x^{-1}t_2^3t_4^3], \text{ since by Equation 5.8} \]
\[ x^3t_{11t10t9} = e \]
\[ [x^3t_{11t10t9}]^{-1} = ex^{-1} \]
\[ \implies x^{-1}t_{10t9t12} = e \]
\[ \implies x^{-1}t_2^3t_4^3 = e \]
\[ \implies x^{-1}t_2^3t_4^3 = t_4^3 \]
\[ \implies x^{-1}t_2^3t_1 = t_4^3 \]
\[ Ht_{5t28} = Ht_3^7x^{-1}t_2^3t_4^3 \]
\[ \implies Ht_{5t32} = Hyx^{-1}[t_3^7]x^{-1}t_2^3t_4^3 \]
\[ \implies Ht_{5t32} = Ht_2^7t_4^3t_1^3 \]
\[ \implies Ht_{5t32} = Ht_2^0t_4^3t_1^3 \]
\[ \implies Ht_{5t32} = Ht_{38t9} \]
\[ \implies Ht_{5t32} = Ht_{9t38}, \text{ since by Equation 5.9} \]
$Ht_1 t_6 = Ht_6 t_1$

$\Rightarrow [Ht_1 t_6]^{y^{-1}} = [Ht_6 t_1]^{y^{-1}}$

$\Rightarrow Ht_5 t_{38} = Ht_{38} t_5$

$Ht_5 t_{32} = Ht_9 t_{38}$

$\Rightarrow Ht_5 t_{32} \in [16]$, since $Ht_9 t_{38}$ is in $[16]$.

Two symmetric generators will go to $[16]$.

**Word of Length 2**

$N^{(12)} = \text{Coset Stabiliser in } N \text{ of } Ht_1 t_2 = \{ n \in N | (Ht_1 t_2)^n = t_1 t_2 \}$.

We will look for a relation that will increase the Coset Stabiliser $N^{(12)}$.

$Ht_1 t_6 = Ht_6 t_1$, by Equation 5.9

$\Rightarrow [Ht_1 t_6]^{x^2} = [Ht_6 t_1]^{x^2}$

$\Rightarrow Ht_3 t_8 = Ht_8 t_3$

$\Rightarrow Ht_3 t_4^2 = Ht_4^2 t_3$

$\Rightarrow Ht_3 t_4^2 t_{10}^4 = Ht_4^2 t_3 t_{10}^4$

$\Rightarrow Ht_3 t_4 = Ht_4^2 t_3 t_{10}^4$

$\Rightarrow Ht_3 t_4 = Ht_4^2 t_3[xyxt_4 t_3^4]$, since by Equation 5.7

$[x^2 t_{33} t_{35} t_{33}]^{-1} y = e y^{-1}$

$\Rightarrow xyxt_4 t_1 t_4 = e$

$\Rightarrow xyxt_4 t_3^4 t_4 = e$

$\Rightarrow xyxt_4 t_3^4 t_{10}^4 t_4 = t_{10}^4$

$\Rightarrow xyxt_4 t_3^4 t_4 = t_{10}^4$

$Ht_3 t_4 = Ht_4^2 t_3[xyxt_4 t_3^4]$

$\Rightarrow Ht_3 t_4 = Hxyx[t_3^2 t_4]^x y x t_4 t_2^4$

$\Rightarrow Ht_3 t_4 = Ht_5^2 t_4^2 t_3 t_2^4$

$\Rightarrow Ht_3 t_4 = Ht_5^2 [xyxt_4 t_3^4] t_4 t_2^4$, since by Equation 5.8

$x^3 t_{11} t_{10} t_9 = e$

$[x^3 t_{11} t_{10} t_9]^{-1} x = e y^{1-x}$

$\Rightarrow x^{-1} y^{-1} t_{17} t_4 t_{15} = e$

$\Rightarrow x^{-1} y^{-1} t_{14} t_3^{14} t_3 = e$

$\Rightarrow x^{-1} y^{-1} t_{14} t_3^{14} t_3^{7} = t_3^7$
\[ x^{-1}y^{-1}t_1^5t_4t_1^{10} = t_3^{7}t_4^{10} \]

\[ yx^{-1}y^{-1}t_1^5 = yxt_3^{7}t_4^{10} \]

\[ t_1^5 = yxt_3^{7}t_4^{10} \]

\[ Ht_3t_4 = Ht_3^{5}yxt_3^{4}t_4t_2^{4} \]

\[ Ht_3t_4 = H(yx[t_2^{8}]yx[t_3^{7}t_2^{4}]t_2^{4}) \]

\[ Ht_3t_4 = Ht_3^{6}t_4^{4}t_2^{4} \]

\[ Ht_3t_4 = Ht_3^{8}t_4^{4} \]

\[ Ht_3t_4 = Ht_1^{4}t_2^{2}, \text{ since} \]

\[ Ht_7 = Ht_{25} \]

\[ Ht_3^{2} = Ht_1^{7} \]

\[ Ht_3t_4 = Ht_3^{4}t_2^{4} \]

\[ Ht_3t_4 = H[x^{-1}yt_3^{5}t_2^{4}]t_2^{4}, \text{ since by Equation 5.8} \]

\[ x^{3}t_{11}t_{10}t_{9} = e \]

\[ [x^{3}t_{11}t_{10}t_{9}]y^{2} = ey^{2} \]

\[ x^{-1}yt_{10}t_{2}t_{13} = e \]

\[ x^{-1}yt_3^{5}t_2^{4}t_1^{4} = e \]

\[ x^{-1}yt_3^{5}t_2^{4}t_1^{4}t_4^{7} = t_1^{7} \]

\[ x^{-1}yt_3^{5}t_2^{4} = t_1^{7} \]

\[ Ht_3t_4 = Hx^{-1}yt_3^{4}t_2^{4}t_1^{4} \]

\[ Ht_3t_4 = Ht_3^{5}t_2^{4} \]

\[ Ht_3t_4 = Ht_1^{3}t_2^{4}, \text{ since} \]

\[ Ht_9 = Ht_{19} \]

\[ Ht_3^{2} = Ht_3^{5} \]

\[ Ht_3t_4 = Ht_1^{3}t_2^{4} \Rightarrow Ht_3t_4 = Ht_{9}t_{18} \]

\[ [Ht_3t_4]x^{2} = [Ht_9t_{18}]x^{2} \]

\[ Ht_1t_2 = Ht_{11}t_{20}. \]

Since, \( Ht_1t_2^{e} = Ht_1t_2 \Rightarrow e \in N^{(12)}, \) and

\( Ht_1t_2^{x^{2}y} = Ht_{11}t_{20} = Ht_1t_2 \Rightarrow x^{2}y \in N^{(12)}, \) then,

\( N^{(12)} = \text{Coset Stabiliser in } N \text{ of } Ht_1t_2 = \{ n \in N | (Ht_1t_2)^{n} = t_1t_2 \} = \{ e, x^{2}y \}. \)

Furthermore, the number of single cosets of \( Ht_1t_2N \) is \( \frac{|N|}{|N^{(12)}|} = \frac{20}{2} = 10. \)
We find the equal names by conjugating $t_1t_2 \sim t_{11}t_{20}$ by elements of $N$.

$t_1t_2 \sim t_{11}t_{20}$  
$t_2t_3 \sim t_{12}t_{17}$  
$t_4t_1 \sim t_{10}t_{19}$  
$t_9t_{18} \sim t_3t_4$

Therefore, $Ht_1t_2N = \{Ht_1t_2 = Ht_{11}t_{20}, Ht_2t_3 = Ht_{12}t_{17}, Ht_4t_1 = Ht_{10}t_{19},$

$Ht_9t_{18} = Ht_3t_4, Ht_13t_{34} = Ht_{35}t_{12}, Ht_14t_{35} = Ht_{36}t_9, Ht_17t_{14} = Ht_{19}t_{16},$

$Ht_{18}t_{15} = Ht_{20}t_{13}, Ht_{34}t_{11} = Ht_{16}t_{33}, Ht_{35}t_{10} = Ht_{15}t_{36}\}$

$N^{(14)} = \text{Coset Stabiliser in } N \text{ of } Ht_1t_4 = \{n \in N | (Ht_1t_4)^n = t_4t_2\}$.

We will look for a relation that will increase the Coset Stabiliser $N^{(14)}$.

$Ht_1t_4 = Ht_2t_4$

$\Rightarrow Ht_1t_4 = H[xy^{-1}t_2^2t_4^7]t_4$, since by Equation 5.8

$x^3t_{11}t_{10}t_9 = e$

$[x^3t_{11}t_{10}t_9]x^2y = e^2y$

$\Rightarrow xy^{-1}t_1t_{16}t_{35} = e$

$\Rightarrow xy^{-1}t_1t_4^3t_3^9 = e$

$\Rightarrow xy^{-1}t_1t_4^3t_3^7 = t_4^7$

$\Rightarrow xy^{-1}t_1t_4^3t_3^7 = t_4^7$

$\Rightarrow xy^{-1}t_1t_4^3t_3^7 = t_4^7$

$\Rightarrow t_1 = xy^{-1}t_3^2t_4^7$

$Ht_1t_4 = Hxy^{-1}t_2^2t_4^7t_4$

$\Rightarrow Ht_1t_4 = Ht_2^8t_4^8$

$\Rightarrow Ht_1t_4 = Ht_3^2[x^{-1}t_2^3t_4^3]$, since by Equation 5.8

$x^3t_{11}t_{10}t_9 = e$

$[x^3t_{11}t_{10}t_9]^{-1} = e^{-1}$

$\Rightarrow x^{-1}t_{10}t_9t_{12} = e$

$\Rightarrow x^{-1}t_2^2t_4^3 = e$
\[ x^{-1}t^3_{14}^{-1}t^3_{14} = t^8_{14} \]
\[ x^{-1}t^3_{14} = t^8_{14} \]
\[ Ht_{14} = Ht^2_{14}x^{-1}t^3_{14} \]
\[ \Rightarrow Ht_{14} = Hx^{-1}[t^2_{14}]^{-1}t^3_{14} \]
\[ \Rightarrow Ht_{14} = Ht^2_{14}t^3_{14} \]
\[ \Rightarrow Ht_{14} = Ht^5_{14} \]
\[ \Rightarrow Ht_{14} = Ht_{18}t_9 \]

Also, \( Ht_{14} = Ht_{18}t_9 \)
\[ \Rightarrow [Ht_{14}]^{yx^{-1}} = [Ht_{18}t_9]^{yx^{-1}} \]
\[ \Rightarrow Ht_{16}t_9 = Ht_{14} \]

\[ Ht_{14} = Ht_{16}t_9 \]
\[ \Rightarrow [Ht_{14}]^{yx} = [Ht_{16}t_9]^{yx} \]
\[ \Rightarrow Ht_{35}t_{14} = Ht_{18}t_9 \]

\[ Ht_{14} = Ht_{16}t_9 \]
\[ \Rightarrow [Ht_{14}]^{yx^{-1}} = [Ht_{16}t_9]^{yx^{-1}} \]
\[ \Rightarrow Ht_{18}t_9 = Ht_{14} \]

Since, \( Ht_{14}^{e} = Ht_{14} \Rightarrow e \in N^{(14)}, \)
\( Ht_{14}^{yx^{-1}} = Ht_{16}t_9 = Ht_{14} \Rightarrow yx^{-1} \in N^{(14)}, \)
\( Ht_{14}^{yx^{-1}} = Ht_{18}t_9 = Ht_{14} \Rightarrow xy^{-1} \in N^{(14)}, \) and
\( Ht_{14}^{yx} = Ht_{35}t_{14} = Ht_{18}t_9 = Ht_{14} \Rightarrow xy^{-1} \in N^{(14)} \)
then,
\( N^{(14)} = \text{Coset Stabiliser in } N \text{ of } Ht_{14} = \{ n \in N | (Ht_{14})^n = t_{14} \} = \{ e, yx^{-1}, xy^{-1}, xy \}. \)

Furthermore, the number of single cosets of \( Ht_{14}N \) is \( \frac{|N|}{|N^{(14)}|} = \frac{20}{4} = 5. \)

We find the equal names by conjugating \( t_{14} \sim t_{35}t_{14} \sim t_{18}t_9 \sim t_{16}t_9 \) by elements of \( N. \)
We will look for a relation that will increase the Coset Stabiliser $N_{Ht}$.

Therefore, $Ht_{14}t_{N} = \{Ht_{14} = Ht_{35}t_{14} = Ht_{18}t_{9} = Ht_{16}t_{19}, \ Ht_{2}t_{1} = Ht_{36}t_{15} = Ht_{19}t_{10} = t_{13}t_{20}, \ Ht_{4}t_{3} = Ht_{34}t_{13} = Ht_{17}t_{12} = Ht_{15}t_{18}, \ Ht_{9}t_{36} = Ht_{11}t_{34} = Ht_{10}t_{33} = Ht_{12}t_{35}, \ Ht_{3}t_{2} = Ht_{33}t_{16} = Ht_{20}t_{11} = t_{14}t_{17}\}$

$N^{(16)} = \text{Coset Stabiliser in } N \text{ of } Ht_{1}t_{6} = \{n \in N|(Ht_{1}t_{6})^{n} = t_{1}t_{6}\}$.

We do not have a relation that will increase the Coset Stabiliser $N^{(16)}$.

$N^{(16)} = \text{Coset Stabiliser in } N \text{ of } Ht_{1}t_{6} = \{n \in N|(Ht_{1}t_{6})^{n} = t_{1}t_{6}\} = \{e\}$.

Furthermore, the number of single cosets of $Ht_{1}t_{6}N$ is $\frac{|N|}{|N^{(16)}|} = \frac{20}{1} = 20$.

$Ht_{1}t_{6}$ conjugated by elements of $N$ gives us the following cosets in $Ht_{1}t_{6}N$.

$Ht_{1}t_{6}N = \{Ht_{1}t_{6}, Ht_{2}t_{7}, Ht_{13}t_{26}, Ht_{4}t_{5}, Ht_{9}t_{38}, Ht_{3}t_{8}, Ht_{34}t_{23}, Ht_{18}t_{31}, Ht_{14}t_{27}, Ht_{17}, t_{30}, \ Ht_{16}t_{25}, Ht_{20}t_{29}, Ht_{36}t_{21}, Ht_{10}t_{39}, Ht_{12}t_{37}, Ht_{33}t_{22}, Ht_{11}t_{40}, Ht_{15}t_{28}, Ht_{35}t_{24}, Ht_{19}t_{32}\}$

$N^{(16)} = \text{Coset Stabiliser in } N \text{ of } Ht_{1}t_{6} = \{n \in N|(Ht_{1}t_{6})^{n} = t_{1}t_{6}\} = \{e\}$.

Furthermore, the number of single cosets of $Ht_{1}t_{6}N$ is $\frac{|N|}{|N^{(16)}|} = \frac{20}{1} = 20$.

$N^{(110)} = \text{Coset Stabiliser in } H \text{ of } Ht_{1}t_{10} = \{n \in N|(Ht_{1}t_{10})^{n} = t_{1}t_{10}\}$.

We will look for a relation that will increase the Coset Stabiliser $N^{(110)}$.

$Ht_{15} = Ht_{1}$
\[\Rightarrow [Ht_{15}]^{x^2y} = [Ht_{1}]^{x^2y}\]
\[\Rightarrow Ht_{17} = Ht_{11}\]
\[\Rightarrow Ht_{1}^{5} = Ht_{3}^{3}\]
\[ Ht_1^5 = Ht_3^2 t_3^{11} \]
\[ Ht_1^5 = Ht_3^2 (t_3 t_3^5) \]
\[ Ht_1^5 = Ht_3^2 t_3^{11} \]
\[ Ht_1^5 = Ht_3^2 t_3^{10} \]
\[ Ht_1^5 = Ht_3^2 t_3^{10} \]
\[ Ht_1^5 = Ht_3^2 t_3^{10} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^y^2 = e y^2 \]
\[ x^{-1} y t_{10} t_2 t_{13} = e \]
\[ x^{-1} y t_3^5 t_2 t_4 = e \]
\[ x^{-1} y t_3^5 t_2 t_4 t_7 = t_1^7 \]
\[ x^{-1} y t_3^5 t_2 t_4 t_10 = t_1^7 t_2^{10} \]
\[ y^{-1} x t^{-1} y t_3^5 = y^{-1} x t_1^7 t_2^{10} \]
\[ t_3^5 = y^{-1} x t_1^7 t_2^{10} \]
\[ Ht_1^5 = Ht_3^2 y^{-1} x t_1^7 t_2^{10} \]
\[ Ht_1^5 = Ht_3^2 y^{-1} x [t_3^5] y^{-1} x t_1^7 t_2^{10} \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_2^{10} t_2 \]
\[ Ht_1^5 t_2 = Ht_4^3 t_1 \]
\[ Ht_1^5 t_2 = Ht_4^3 t_1 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ Ht_1^5 t_2 = Ht_3^2 t_4^7 t_1^7 \]
\[ x_2^2 t_{33} t_{35} t_{33} = e \]
\[ [x_2^2 t_{33} t_{35} t_{33}]^x^2 = e x_2^2 \]
\[ x^2 t_{33} t_{33} t_{35} = e \]
\[ x^2 t_{33}^9 t_{33}^9 = e \]
\[ x^2 t_{33}^9 t_{33}^9 t_{33} = t_3^2 \]
\[ x^2 t_{33}^9 t_{33} = t_3^2 \]

\[ H t_1^5 t_2 = H t_2 t_3^{10} x^2 t_3^9 \]
\[ H t_1^5 t_2 = H x^2 [t_2 t_3^{10}] x^2 t_3^9 \]
\[ H t_1^5 t_2 = H t_4 t_1^{10} t_3^9 \]
\[ H t_1^5 t_2 = H t_4 t_1^{10} t_3^9, \text{ since} \]

\[ H t_4 = H t_4 \]
\[ H t_1^5 t_2 = H t_4 t_1^{10} t_3^9 \]
\[ H t_1^5 t_2 = H [x y t_4^{6} t_4^{2} t_1^{10} t_3^9, \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}] x^{-1} y^{-1} = e x^{-1} y^{-1} \]
\[ y^{-1} x^{-1} t_{14} t_{33} t_{20} = e \]
\[ y^{-1} x^{-1} t_{2} t_{4} t_{5} = e \]
\[ y^{-1} x^{-1} t_{2} t_{4} t_{5} = t_4^6 \]
\[ y^{-1} x^{-1} t_{2} t_{4} t_{5} = t_4^6 t_4^6 \]
\[ x y y^{-1} x^{-1} t_4^6 = x y t_4^6 t_4^2 \]
\[ t_4^6 = x y t_4^6 t_4^2 \]

\[ H t_1^5 t_2 = H x y t_4^{6} t_4^{2} t_1^{10} t_3^9 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9, \text{ since by Equation 5.8} \]

\[ [x^3 t_{11} t_{10} t_{9}] x y x = e x y x \]
\[ x^{-1} y t_{13} t_{36} t_{19} = e \]
\[ x^{-1} y t_{13} t_{36} t_{19} = e \]
\[ x^{-1} y t_{13} t_{36} t_{19} = t_3^6 \]
\[ x^{-1} y t_{13} t_{36} t_{19} = t_3^6 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 x^{-1} y t_4^1 t_4^6 \]
\[ H t_1^5 t_2 = H x^{-1} y [t_4^6 t_1^2 t_3^9] x^{-1} y t_4^1 t_4^6 \]
\[ H t_1^5 t_2 = H t_4^6 t_1^2 t_3^9 \]
\[ \Rightarrow Ht_1^5t_2 = H[x^{-1}y^{-1}t_1^5t_4]t_1^5t_2^4t_4^0, \] since by Equation 5.8
\[ x^3t_1t_1t_0t_9 = e \]
\[ [x^3t_1t_1t_0t_9]x^{y^{-1}x} = e^{xy^{-1}x} \]
\[ \Rightarrow x^{-1}y^{-1}t_1t_4t_1t_5 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_1^5t_4t_3 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_1^5t_4t_3t_7 = \frac{t_7}{3} \]
\[ \Rightarrow x^{-1}y^{-1}t_1^5t_4 = t_3^7 \]
\[ Ht_1^5t_2 = Hx^{-1}y^{-1}t_1^5t_4^2t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = Ht_1^5t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = Ht_1^5[x^2y^{-1}t_1^5t_2^3]t_2^4t_4^0, \] since by Equation 5.7
\[ x^2t_3t_3t_3t_3 = e \]
\[ [x^2t_3t_3t_3]y^{-1}x^{-1} = e^{y^{-1}x^{-1}} \]
\[ \Rightarrow x^2y^{-1}t_2t_0t_20 = e \]
\[ \Rightarrow x^2y^{-1}t_2^3t_4^5 = e \]
\[ \Rightarrow x^2y^{-1}t_2^3t_4^5t_4^6 = t_4^6 \]
\[ \Rightarrow x^2y^{-1}t_2^3t_4^6 = t_4^6 \]
\[ Ht_1^5t_2 = Hx^2y^{-1}t_1^5t_2^3t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = Hx^2y^{-1}t_1^5t_2^3y^{-1}t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = Ht_2^4t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = Ht_1^5t_2^4t_4^0, \] since
\[ Ht_33 = Ht_{35} \]
\[ \Rightarrow Ht_3^0 = Ht_3^0 \]
\[ Ht_1^5t_2 = Ht_1^5t_2^4t_4^0 \]
\[ \Rightarrow Ht_1^5t_2 = H[xyt_1t_1t_1t_0^0t_4^6]t_2^8t_4^0, \] since by Equation 5.8
\[ x^3t_1t_1t_0t_9 = e \]
\[ [x^3t_1t_1t_0t_9]x^{y^{-1}x} = e^{2y^{-1}x} \]
\[ \Rightarrow y^{-1}x^{-1}t_3t_2t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_1^9t_4^5t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_1^9t_4^5t_3^10 = \frac{t_4^10}{3} \]
\[ \Rightarrow y^{-1}x^{-1}t_1^9t_4^5t_4^6 = \frac{t_4^10}{3} \]
\[ \Rightarrow xyt^{-1}x^{-1}t_1^9 = xyt_3^{10}t_4^6 \]
\[ \Rightarrow t_1^9 = xyt_3^{10}t_4^6 \]
\[ H t_1^5 t_2 = H x y t_3^{10} t_4^{10} t_2 t_1 t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_2 t_1 t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_2 t_1 t_4^9, \text{ since } H t_{29} = H t_{39} \]
\[ \Rightarrow H t_1^5 = H t_3^{10} \]
\[ H t_1^5 t_2 = H t_3^{10} t_2 t_1 t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H [x^3 t_3^{10} t_4^9], \text{ since by Equation 5.8} \]
\[ x^3 t_1 t_{10} t_9 = e \]
\[ \Rightarrow x^3 t_3^3 t_2^3 t_1^3 = e \]
\[ \Rightarrow x^3 t_3^3 t_2^3 t_1^3 = t_1^8 \]
\[ \Rightarrow x^3 t_3^3 t_2^3 t_1^3 = t_1^{12} \]
\[ \Rightarrow x^3 t_3^3 = t_1^{12} \]
\[ H t_1^5 t_2 = H x^3 t_3^3 t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_7 = H t_{25} \]
\[ \Rightarrow H t_3^3 = H t_1^7 \]
\[ H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ [x^2 t_3 t_{35} t_{33}] x y^{-1} x = e x y^{-1} x \]
\[ \Rightarrow x^2 y t_3 t_1 t_3 = e \]
\[ \Rightarrow x^2 y t_3 t_1 t_3 = e \]
\[ \Rightarrow x^2 y t_3 t_1 t_3 = t_1^{10} \]
\[ \Rightarrow x^2 y t_3 t_1 t_3 = t_1^{10} t_1 \]
\[ \Rightarrow x^2 y t_3 t_1 t_3 = x^2 y t_3 t_1 t_1 \]
\[ \Rightarrow t_3 = x^2 y t_3 t_1 t_1 \]
\[ H t_1^5 t_2 = H t_1^{10} x^2 y t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H x^2 y [t_1^{10} x^2 y t_3^{10} t_4^9] \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_3^{10} t_4^9 \]
\[ \Rightarrow H t_1^5 t_2 = H t_{35} t_{36} \]
Furthermore, the number of single cosets of $H_t$ gives us the following equal names.

$t_{11} t_{10} \sim t_3 t_{12}$
$t_{21} t_{11} \sim t_4 t_9$
$t_{13} t_{18} \sim t_{11} t_{16}$
$t_{9} t_{14} \sim t_{15} t_{20}$

$t_{34} t_{35} \sim t_{20} t_1$
$t_{10} t_{15} \sim t_{16} t_{17}$
$t_{12} t_{13} \sim t_{14} t_{19}$
$t_{17} t_{2} \sim t_{35} t_{36}$

Since, $H_t t_{10}^{e} = H_t t_{10} \Rightarrow e \in N^{(110)}$,

$H_t t_{10}^{2} = H_t t_{12} = H_t t_{10} \Rightarrow x^2 \in N^{(110)}$, then,

$N^{(110)} = \text{Coset Stabiliser in } N$ of $H_t t_{10} = \{ n \in N | (H_t t_{10} n = t_{11} t_{10}) = \{ e, x^2 \}$. 

Furthermore, the number of single cosets of $H_t t_{10} N$ is $\frac{|N|}{|N(110)|} = 20 \Rightarrow 10$.

$H_t t_{10} N = \{ H_t t_{10} = H_t t_{12}, H_t t_{11} = H_t t_9, H_t t_{13} = H_t t_{16}, H_t t_{14} = H_t t_{20}, H_t t_{34} = H_t t_1, H_t t_{10} = H_t t_{16}, H_t t_{12} = H_t t_{19} t_2 = H_t t_{35} t_{36}, H_t t_{18} = H_t t_{36} t_{33}, H_t t_{19} t_4 = H_t t_{33} t_{34} \}$

The orbits of $N^{(12)}$ are $\{ 1, 11 \}, \{ 2, 20 \}, \{ 3, 13 \}, \{ 4, 34 \}, \{ 5, 23 \}, \{ 6, 40 \}, \{ 7, 29 \}, \{ 8, 26 \}, \{ 9, 35 \}, \{ 10, 16 \}, \{ 12, 18 \}, \{ 14, 36 \}, \{ 15, 17 \}, \{ 19, 33 \}, \{ 21, 27 \}, \{ 22, 32 \}, \{ 24, 38 \}, \{ 25, 39 \}, \{ 28, 30 \}$, and $\{ 31, 37 \}$.

We will check to see where $t_{11} t_{2} t_{1}, t_{11} t_{2} t_{2}, t_{11} t_{2} t_{13}, t_{11} t_{2} t_{4}, t_{11} t_{2} t_{5}, t_{11} t_{2} t_{6}, t_{11} t_{2} t_{9}, t_{11} t_{2} t_{16}, t_{11} t_{2} t_{18}, t_{11} t_{2} t_{14}, t_{11} t_{2} t_{17}, t_{11} t_{2} t_{33}, t_{11} t_{2} t_{21}, t_{11} t_{2} t_{22}, t_{11} t_{2} t_{24}, t_{11} t_{2} t_{25}, t_{11} t_{2} t_{28}$, and $t_{11} t_{2} t_{37}$ belong.

$H_t t_{12} t_{1} = H_t t_{12} t_{1}$
$\Rightarrow H_t t_{12} t_{1} = H_t t_{15} t_{2} t_{1}$, since $H_t = H_{t_{15}}$
$\Rightarrow H_t t_{12} t_{1} = H_t t_{3} t_{2} t_{1}$
$\Rightarrow H_t t_{12} t_{1} = H_t t_{3} [y^{-1} x t_{4} t_{7}^{2} t_{1}] t_{1}$, since by Equation 5.8

$x^{3} t_{11} t_{10} t_{9} = e$
$[x^{3} t_{11} t_{10} t_{9}] y x^{-1} = e y x^{-1}$
\[ x^{-1}yt_2 t_{13} t_{36} = e \]
\[ x^{-1}yt_2 t_1^9 t_4 = e \]
\[ x^{-1}yt_2 t_1^9 t_4^2 t_1 = t_4^2 \]
\[ x^{-1}yt_2 t_1^9 = t_1^2 t_1^7 \]
\[ y^{-1}x^{-1}yt_2 = y^{-1}xt_1^2 t_1^7 \]
\[ t_2 = y^{-1}xt_1^2 t_1^7 \]
\[ H t_1 t_2 t_1 = H t_3 y^{-1}xt_1^2 t_1^7 t_1 \]
\[ H t_1 t_2 t_1 = H y^{-1}x[t_3^3]y^{-1}xt_1^2 t_1^8 \]
\[ H t_1 t_2 t_1 = H t_1^5 t_1^4 t_1^8 \]
\[ H t_1 t_2 t_1 = H t_1^7 t_1^8 \]
\[ H t_1 t_2 t_1 = H t_1^7 [x^3 t_3^3 t_2^3] \times \text{by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ x^3 t_9^3 t_1^3 = e \]
\[ x^3 t_9^3 t_1^3 t_1^4 t_1^4 = t_1^8 \]
\[ x^3 t_9^3 t_2^3 = t_1^8 \]
\[ H t_1 t_2 t_1 = H t_1^7 x^3 t_3^3 t_2^3 \]
\[ H t_1 t_2 t_1 = H x^3 [t_4^7] x^3 t_3^3 t_2 \]
\[ H t_1 t_2 t_1 = H t_3^7 t_3^3 t_2 \]
\[ H t_1 t_2 t_1 = H t_3^10 t_3^2 \]
\[ H t_1 t_2 t_1 = H t_3^10 t_3^8 \]
\[ H t_1 t_2 t_1 = H t_3^10 t_3^8 \]
\[ H t_1 t_2 t_1 = H t_3^10 t_3^2, \text{ since } H t_{10} t_{39} = H t_{39} t_{10} \]
\[ H t_1 t_2 t_1 \in [16], \text{ since } H t_{10} t_{39} \text{ is in } [16]. \]

2 symmetric generators will go to [1 6].

\[ H t_1 t_2 t_2 = H t_1 t_2^2 \]
\[ \Rightarrow H t_1 t_2 t_2 = H t_1 t_6 \]
\[ \Rightarrow H t_1 t_2 t_2 \in [16], \text{ since } H t_1 t_6 \text{ is in } [1 6]. \]

2 symmetric generators will go to [1 6].

\[ H t_1 t_2 t_{13} = H t_1 t_2 t_{13} \]
\[ \Rightarrow H t_1 t_2 t_{13} = H t_1 t_2 t_{13}^4 \]
\[ \Rightarrow H t_1 t_2 t_{13} = H t_1 [y^{-1}xt_1^2 t_1^7] t_1^4, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{y^{-1} x^{-1}} = e^{y^{-1} x^{-1}} \]
\[ \implies x^{-1} y t_2 t_{13} t_{36} = e \]
\[ \implies x^{-1} y t_2 t_4^0 t_0^4 = e \]
\[ \implies x^{-1} y t_2 t_4^0 t_4^0 t_4^2 = t_4^2 \]
\[ \implies x^{-1} y t_2 t_4^0 t_4^7 = t_4^7 \]
\[ \implies y^{-1} x x^{-1} y t_2 = y^{-1} x t_4^2 t_4^7 \]
\[ \implies t_2 = y^{-1} x t_4^2 t_4^7 \]
\[ H t_1 t_2 t_{13} = H t_1 y^{-1} x t_4^2 t_4^7 t_4^4 \]
\[ \implies H t_1 t_2 t_{13} = H y^{-1} x [t_4]^{y^{-1} x} t_4^7 \]
\[ \implies H t_1 t_2 t_{13} = H t_4^3 t_4^2 \]
\[ \implies H t_1 t_2 t_{13} = H t_4^3 t_4^2, \text{ since} \]
\[ H t_20 = H t_{10} \]
\[ \implies H t_4^3 = H t_4^2 \]
\[ H t_1 t_2 t_{13} = H t_4^3 t_4^2 \]
\[ \implies H t_1 t_2 t_{13} = H t_4^7 \]
\[ \implies H t_1 t_2 t_{13} = H t_{28} \]
\[ \implies H t_1 t_2 t_{13} \in [5], \text{ since } H t_{28} \text{ is in } [5] \]
2 symmetric generators will go to [5].

\[ H t_1 t_2 t_4 = H t_1 t_2 t_4 \]
\[ \implies H t_1 t_2 t_4 = H t_{11} t_{20} t_4, \text{ since} \]
\[ H t_1 t_2 = H t_{11} t_{20} \]
\[ H t_1 t_2 t_4 = H t_{11} t_{20} t_4 \]
\[ \implies H t_1 t_2 t_4 = H t_4^3 t_4^5 t_4 \]
\[ \implies H t_1 t_2 t_4 = H t_4^3 t_4^6 \]
\[ \implies H t_1 t_2 t_4 = H t_4^3 t_4^6, \text{ since} \]
\[ H t_{11} = H t_{17} \]
\[ \implies H t_4^3 = H t_4^5 \]
\[ H t_1 t_2 t_4 = H t_4^3 t_4^6 \]
\[ \implies H t_1 t_2 t_4 = H [y x t_4^7 t_4^{10}] t_4^6, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy^{-1}} = e^{xy^{-1}} \]
\[ \Rightarrow x^{-1} y^{-1} t_{17} t_4 t_{15} = e \]
\[ \Rightarrow x^{-1} y^{-1} t_3^5 t_4 t_3^4 = e \]
\[ \Rightarrow x^{-1} y^{-1} t_3^5 t_4 t_3^7 t_3^3 = t_3^7 \]
\[ \Rightarrow x^{-1} y^{-1} t_3^5 t_4 t_3^{10} = t_3^{10} \]
\[ \Rightarrow y x y^{-1} t_3^5 = y t_3^{10} \]
\[ \Rightarrow t_3^5 = y t_3^{10} \]
\[ H t_1 t_2 t_4 = H y x t_3^4 t_4^{10} \]
\[ \Rightarrow H t_1 t_2 t_4 = H t_3^5 t_4^5 \]
\[ \Rightarrow H t_1 t_2 t_4 = H t_1^2 t_4^5, \text{ since} \]
\[ H t_5 = H t_27 \]
\[ \Rightarrow H t_1^2 = H t_3^7 \]
\[ H t_1 t_2 t_4 = H t_1^2 t_4^5 \]
\[ H t_1 t_2 t_4 = H t_1^2 [x y t_3^2 t_3^{10}], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy^{-1}} = e^{xy^{-1}} \]
\[ \Rightarrow y^{-1} x^{-1} t_{20} t_3 t_{14} = e \]
\[ \Rightarrow y^{-1} x^{-1} t_3^5 t_4 t_2^4 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_3^5 t_4 t_2^7 t_2^2 = t_2^7 \]
\[ \Rightarrow y^{-1} x^{-1} t_3^5 t_4 t_3^{10} = t_3^{10} \]
\[ \Rightarrow x y y^{-1} t_3^5 = x t_3^{10} \]
\[ \Rightarrow t_3^5 = x t_3^{10} \]
\[ H t_1 t_2 t_4 = H t_1^2 x y t_3^7 t_3^{10} \]
\[ \Rightarrow H t_1 t_2 t_4 = H x y [t_3^2]^{xy} t_2^7 t_3^{10} \]
\[ \Rightarrow H t_1 t_2 t_4 = H t_3^7 t_2^10 t_3^3 \]
\[ \Rightarrow H t_1 t_2 t_4 = H t_3^{10} t_3^3 \]
\[ H t_1 t_2 t_4 = H t_10 t_39 \]
\[ \Rightarrow H t_1 t_2 t_4 \in [16], \text{ since } H t_{10} t_{39} \text{ is in } [1 6]. \]

2 symmetric generators will go to [1 6].
\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_1^2 t_2 t_3}, \text{ since} \]

\[ H_{t_1} = H_{t_{15}} \]

\[ \Rightarrow H_{t_1} = H_{t_4}^2 \]

\[ H_{t_1 t_2 t_5} = H_{t_1^2 t_2 t_3}^2 \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_1^4 [y^{-1} x t_4^2 t_7]} t_1^2, \text{ since Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9] y^{-1} x e = e y^{-1} x e \]

\[ \Rightarrow x^{-1} y t_2 t_{13} t_{36} = e \]

\[ \Rightarrow x^{-1} y t_2 t_4^2 t_9^2 = e \]

\[ \Rightarrow x^{-1} y t_2 t_4^2 t_9^2 t_4^2 t_3^2 = t_4 \]

\[ \Rightarrow x^{-1} y t_2 t_4^2 t_7 = t_4^2 t_1^2 \]

\[ \Rightarrow y^{-1} x e^{-1} y t_2 = y^{-1} x t_4^2 t_7^2 \]

\[ \Rightarrow t_2 = y^{-1} x t_4^2 t_7^2 \]

\[ H_{t_1 t_2 t_5} = H_{t_2^2 [y^{-1} x t_4^2 t_7]} t_1^2 \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{y^{-1} x [t_4^2 t_3^2]^2} t_1^2 \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_4^2 t_4^2 t_9^2} \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_4^2 t_9^2} \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{y [y x^{-1} t_5^2 t_4]} t_1^9, \text{ since Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9] x^{-1} y e = e x^{-1} y \]

\[ \Rightarrow y x^{-1} t_{18} t_{13} t_{16} = e \]

\[ \Rightarrow y x^{-1} t_{16} t_4^2 t_9^2 = e \]

\[ \Rightarrow y x^{-1} t_{16} t_4^2 t_9^2 t_4^2 t_3^2 = t_4 \]

\[ \Rightarrow y x^{-1} t_{16} t_4^2 t_7 = t_4^7 \]

\[ H_{t_1 t_2 t_5} = H_{y x^{-1} t_5^2 t_4} t_1^9 \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_2^2 t_4^2 t_9^2} \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_{18} t_{37}} \]

\[ \Rightarrow H_{t_1 t_2 t_5} = H_{t_{12} t_{37}}, \text{ since } H_{t_{12}} = H_{t_{18}} \]

\[ H_{t_1 t_2 t_5} \in [16], \text{ since } H_{t_{12} t_{37}} \text{ is in [1 6]} \]

2 symmetric generators will go to [1 6].

\[ H_{t_1 t_2 t_6} = H_{t_1 t_2 t_6} \]
\[ H_{t_1t_2t_6} = H_{t_1t_2t_2} \]
\[ H_{t_1t_2t_6} = H_{t_1t_2}^2 \]
\[ H_{t_1t_2t_6} = H_{t_1t_2}^3 \]
\[ H_{t_1t_2t_6} = H_{t_1t_10} \]
\[ H_{t_1t_2t_6} \in [110] \]

2 symmetric generators will go to [1 10].

\[ H_{t_1t_2t_{29}} = H_{t_1t_2t_{29}} \]
\[ H_{t_1t_2t_{29}} = H_{t_1t_2}^8 \]
\[ H_{t_1t_2t_{29}} = H_{t_1t_2}^8, \text{ since} \]
\[ H_{t_1} = H_{t_{15}} \]
\[ H_{t_1} = t_3^4 \]
\[ H_{t_1t_2t_{29}} = H_{t_1t_2}^4t_1^4 \]
\[ H_{t_1t_2t_{29}} = H_{t_1t_2}^4[y^{-1}xt_1^2t_1^7]t_1^8, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{-1}x^{-1} = e^{-1}x^{-1} \]
\[ \Rightarrow x^{-1}yt_2t_{13}t_{36} = e \]
\[ \Rightarrow x^{-1}yt_2t_1^4t_9^9 = e \]
\[ \Rightarrow x^{-1}yt_2t_1^4t_9^9 = t_4^2 \]
\[ \Rightarrow x^{-1}yt_2t_1^4t_9^9 = t_4^2 \]
\[ \Rightarrow y^{-1}x^{-1}yt_2 = y^{-1}xt_1^2t_1^7 \]
\[ \Rightarrow t_2 = y^{-1}xt_1^2t_1^7 \]
\[ H_{t_1t_2t_{29}} = H_{t_1t_2}^4y^{-1}xt_1^2t_1^7t_1^8 \]
\[ \Rightarrow H_{t_1t_2t_{29}} = H_{t_1t_2}^{-1}x[t_3^4]y^{-1}zt_1^2t_1^4 \]
\[ \Rightarrow H_{t_1t_2t_{29}} = H_{t_1t_2}^4t_1^4 \]
\[ \Rightarrow H_{t_1t_2t_{29}} = H_{t_1t_2}^4t_1^4 \]
\[ \Rightarrow H_{t_1t_2t_{29}} = H_{t_1t_2}^4t_1^4, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{-1}y = e^{-1}y \]
\[ \Rightarrow yx^{-1}t_{18}t_{11}t_{16} = e \]
\[ \Rightarrow yx^{-1}t_{21}^5t_4^4 = e \]
\[ \Rightarrow yx^{-1}t_2^5t_1^4t_4^7 = t_4^7 \]
\[ \Rightarrow yx^{-1}t_2^5t_1^4 = t_4^7 \]
\[ H_{t_1 t_2 t_{29}} = H_y x^{-1} t_5^2 t_1 t_1^4 \]
\[ \implies H_{t_1 t_2 t_{29}} = H_{t_2 t_1^5} \]
\[ \implies H_{t_1 t_2 t_{29}} = H_{t_1 t_8 t_{17}} \]
\[ \implies H_{t_1 t_2 t_{29}} = H_{t_1 t_2 t_{17}}, \text{ since} \]
\[ H_{t_{12}} = H_{t_{18}} \]
\[ H_{t_1 t_2 t_{29}} \in [12], \text{ since } H_{t_1 t_2 t_{17}} \text{ is in } [1 2]. \]

2 symmetric generators will go to [1 2].

\[ H_{t_1 t_2 t_8} = H_{t_1 t_2 t_8} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_1 t_2 t_8} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_1 t_2 t_8} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_3 t_7 t_4} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_3 t_7 t_4} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_3 t_7 t_4} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_3 [y x^{-1} t_1 t_4^2] t_4}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = c \]
\[ [x^3 t_{11} t_{10} t_9]^x = e x^2 y \]
\[ \implies y x^{-1} t_1 t_6 t_{35} = e \]
\[ \implies y x^{-1} t_1 t_4^6 t_3 = e \]
\[ \implies y x^{-1} t_1 t_4^6 t_3 = t_3^2 \]
\[ \implies y x^{-1} t_1 t_4^6 t_3 = t_3^2 \]
\[ H_{t_1 t_2 t_8} = H_{t_3 y x^{-1} t_1 t_4^6 t_3} \]
\[ \implies H_{t_1 t_2 t_8} = H_{y y x^{-1} [t_3] y x^{-1}} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_3 t_1} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_{10} t_1} \]
\[ \implies H_{t_1 t_2 t_8} = H_{t_{20} t_1}, \text{ since} \]
\[ H_{t_{10}} = H_{t_{20}} \]
\[ H_{t_1 t_2 t_8} \in [110], \text{ since } H_{t_{20} t_1} \text{ is in } [1 10]. \]

2 symmetric generators will go to [1 10].

\[ H_{t_1 t_2 t_9} = H_{t_1 t_2 t_9} \]
\[ \implies H_{t_1 t_2 t_9} = H_{t_{15} t_2 t_9}, \text{ since} \]
\[ Ht_1 = Ht_{15} \]
\[ Ht_{12t9} = Ht_{15t2t9} \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 [y^{-1}xt_2^9t_1^1]t_1^3, \] since by Equation 5.8
\[ x^3t_{11t10t9} = e \]
\[ [x^3t_{11t10t9}]^{-1}x^{-1} = e^{-1}x^{-1} \]
\[ \implies x^{-1}yt_2t_{13t36} = e \]
\[ \implies x^{-1}yt_2t_1^4t_1^9 = e \]
\[ \implies x^{-1}yt_2t_1^4t_1^9t_1^2 = t_1^2 \]
\[ \implies x^{-1}yt_2t_1^4t_1^9t_1^7 = t_1^2t_1^7 \]
\[ \implies y^{-1}x^{-1}yt_2 = y^{-1}xt_1^2t_1^7 \]
\[ t_2 = y^{-1}xt_1^2t_1^7 \]
\[ Ht_{12t9} = Ht_{12t2t9}^4 y^{-1}xt_2^9t_1^1 t_1^3 \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 x^{-1}x[t_3^1]y^{-1}xt_1^2t_1^10 \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 t_1^10 \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 t_1^10 \]
\[ \implies Ht_{12t9} = Ht_{12t2t9}^4 y^{-1}xt_1^2t_1^10, \] since by Equation 5.8
\[ x^3t_{11t10t9} = e \]
\[ [x^3t_{11t10t9}]^{-1}y = e^{-1}y \]
\[ \implies y^{-1}t_{14}t_{13}t_{16} = e \]
\[ \implies y^{-1}t_{14}t_{13}t_{16} = e \]
\[ \implies y^{-1}t_{14}t_{13}t_{16} = t_1^7 \]
\[ \implies y^{-1}t_{14}t_{13}t_{16} = t_1^7 \]
\[ t_2 = y^{-1}xt_1^2t_1^7 \]
\[ Ht_{12t9} = Ht_{12t2t9}^4 y^{-1}xt_1^2t_1^10 \]
\[ \implies Ht_{1t2t9} = Ht_{12t2t9}^4 \]
\[ \implies Ht_{1t2t9} \in [1], \] since \( Ht_2 \) is in [1].
2 symmetric generators will go to [1].

\[ Ht_{12t16} = Ht_{12t2t16} \]
\[ \implies Ht_{1t2t16} = Ht_{11t20t16}, \] since
\[ Ht_{1t2} = Ht_{11t20} \]
\[ Ht_{12t16} = Ht_{11t20t16} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^6t_4^2t_4^4} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^6t_4^9} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^6t_4^9}, \text{ since} \]
\[ H_{t_{11}} = H_{t_{17}} \]
\[ \Rightarrow H_{t_3^3} = H_{t_1^5} \]
\[ H_{t_1t_2t_{16}} = H_{t_3^5t_4^9} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{yxt_3^7t_4^{10}t_4^9}, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{xy^{-1}x} = e^{xy^{-1}x} \]
\[ \Rightarrow x^{-1}y^{-1}t_{17}t_4t_{15} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{17}t_4t_{15} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_4t_3^4 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_4t_3^4 = t_3^7 \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_4t_3^{10} = t_3^7t_4^{10} \]
\[ \Rightarrow yx^{-1}y^{-1}t_3^5 = yxt_3^7t_4^{10} \]
\[ \Rightarrow t_3^5 = yxt_3^7t_4^{10} \]
\[ H_{t_1t_2t_{16}} = H_{yxt_3^7t_4^{10}t_4^9} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^7t_4^{10}} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^7t_4^{10}}[x^{-1}t_3^2t_3^2], \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{x^{-1}} = e^{x^{-1}} \]
\[ \Rightarrow x^{-1}t_{10}t_9t_{12} = e \]
\[ \Rightarrow x^{-1}t_{3}^3t_9t_3^3 = e \]
\[ \Rightarrow x^{-1}t_{3}^3t_9t_3^3 = t_4^8 \]
\[ \Rightarrow x^{-1}t_{3}^3t_9t_3^3 = t_4^8 \]
\[ H_{t_1t_2t_{16}} = H_{t_3^5x^{-1}t_3^2t_3^2} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{x^{-1}[t_3^7]^{x^{-1}t_3^2t_3^2}} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_3^7t_4^{3}t_1^3} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_2^3t_1^3} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_2^10t_1^3} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{t_38t_9} \]
\[ \Rightarrow H_{t_1t_2t_{16}} = H_{59t_9}, \text{ by Equation 5.9} \]
\[ H_{t_1t_6} = H_{t_6t_1} \]
\[\Rightarrow [H_{t_1}t_6]^{y^{-1}} = [H_{t_6}t_1]^{y^{-1}}\]
\[\Rightarrow H_{t_9}t_{38} = H_{t_{38}}t_9\]
\[H_{t_1}t_2t_{16} = H_{t_9}t_{38}\]
\[\Rightarrow H_{t_1}t_2t_{16} \in [16], \text{ since } H_{t_9}t_{38} \text{ is in } [1\ 6]\]
2 symmetric generators will go to [1\ 6].

\[H_{t_1}t_2t_{18} = H_{t_1}t_2t_{18}\]
\[\Rightarrow H_{t_1}t_2t_{18} = H_{t_1}t_2t_5\]
\[\Rightarrow H_{t_1}t_2t_{18} = H_{t_1}t_6\]
\[\Rightarrow H_{t_1}t_2t_{18} = H_{t_3}t_6, \text{ since}\]
\[H_{t_1} = H_{t_1}\]
\[\Rightarrow H_{t_1} = H_{t_3}^{4}\]
\[H_{t_1}t_2t_{18} = H_{t_3}^{4}t_6\]
\[H_{t_1}t_2t_{18} = H[yx t_1^{6}t_2^{6}], \text{ since by Equation 5.8}\]
\[x^3 t_11 t_{10} t_9 = e\]
\[[x^3 t_11 t_{10} t_9]^{y^{-2}} = e^{y^{-2}}\]
\[\Rightarrow x^{-1}y^{-1}t_{15}t_{34}t_{17} = e\]
\[\Rightarrow x^{-1}y^{-1}t_{3}^{4}t_2 t_1^{5} = e\]
\[\Rightarrow x^{-1}y^{-1}t_{3}^{4}t_2 t_1^{5}t_{6} = t_1^{6}\]
\[\Rightarrow x^{-1}y^{-1}t_{3}^{4}t_2 t_1^{5}t_{2} = t_1^{6}t_2^{2}\]
\[\Rightarrow yxt^{-1}y^{-1}t_3^{4} = yxt_1^{6}t_2^{2}\]
\[t_3^{4} = yxt_1^{6}t_2^{2}\]
\[H_{t_1}t_2t_{18} = H[yx t_1^{6}t_2^{6}]\]
\[\Rightarrow H_{t_1}t_2t_{18} = H_{t_1}t_6^{8}\]
\[\Rightarrow H_{t_1}t_2t_{18} = H_{t_1}^{6}[x^{-1}t_1^{3}t_3^{2}], \text{ since by Equation 5.8}\]
\[x^3 t_11 t_{10} t_9 = e\]
\[[x^3 t_11 t_{10} t_9]^{x} = e^{x}\]
\[\Rightarrow x^{-1}t_{12}t_11 t_{10} = e\]
\[\Rightarrow x^{-1}t_1^{3}t_3^{2}t_2 = e\]
\[\Rightarrow x^{-1}t_1^{3}t_3^{2}t_2^{8} = t_2^{8}\]
\[\Rightarrow x^{-1}t_1^{3}t_3^{8} = t_2^{8}\]
\[H_{t_1}t_2t_{18} = H_{t_1}^{6}x^{-1}t_1^{3}t_3^{2}\]
\[\begin{align*}
\Rightarrow H_{t_1 t_2 t_18} &= H x^{-1}[t_1^6] x^{-1} t_3^3 t_3^3 \\
\Rightarrow H_{t_1 t_2 t_18} &= H t_4^6 t_3 t_3 \\
\Rightarrow H_{t_1 t_2 t_18} &= H t_4^6 t_3 t_3 \\
\Rightarrow H_{t_1 t_2 t_18} &= H t_2^6 t_3 t_3 \\
\Rightarrow H_{t_1 t_2 t_18} &= H t_3 t_11 \\
\Rightarrow H_{t_1 t_2 t_18} &\in [12], \text{ since } H t_{34} t_{11} \text{ is in } [1 2] \\
2 \text{ symmetric generators will go to } [1 2].
\end{align*}\]

\[\begin{align*}
H_{t_1 t_2 t_{14}} &= H_{t_1 t_2 t_{14}} \\
\Rightarrow H_{t_1 t_2 t_{14}} &= H_{t_15 t_2 t_{14}}, \text{ since } \\
H_{t_1} &= H_{t_{15}} \\
H_{t_1 t_2 t_{14}} &= H_{t_15 t_2 t_{14}} \\
\Rightarrow H_{t_1 t_2 t_{14}} &= H t_{34} t_{12} \\
\Rightarrow H_{t_1 t_2 t_{14}} &= H t_{34} t_{12} \\
\Rightarrow H_{t_1 t_2 t_{14}} &\in [14], \text{ since } H t_{34} t_{12} \text{ is in } [1 4] \\
2 \text{ symmetric generators will go to } [1 4].
\end{align*}\]

\[\begin{align*}
H_{t_1 t_2 t_{17}} &= H_{t_1 t_2 t_{17}} \\
\Rightarrow H_{t_1 t_2 t_{17}} &= H_{t_15 t_2 t_{17}}, \text{ since } \\
H_{t_1} &= H_{t_{15}} \\
H_{t_1 t_2 t_{17}} &= H_{t_15 t_2 t_{17}} \\
\Rightarrow H_{t_1 t_2 t_{17}} &= H t_{34} t_{12} \\
\Rightarrow H_{t_1 t_2 t_{17}} &= H t_{34} t_{12} \\
\Rightarrow H_{t_1 t_2 t_{17}} &= H t_{34} t_{12} \text{, since by Equation 5.8} \\
x^3 t_{11} t_{10} t_9 &= e \\
[x^3 t_{11} t_{10} t_9] y^{-1} x^{-1} &= e y^{-1} x^{-1} \\
\Rightarrow x^{-1} y t_{213} t_{36} &= e \\
\Rightarrow x^{-1} y t_{213} t_{40} &= e \\
\Rightarrow x^{-1} y t_{213} t_{43}^2 &= t_{3}^2 \\
\Rightarrow x^{-1} y t_{213} t_{11}^7 &= t_{3}^2 t_{11}^7 \\
\Rightarrow y^{-1} x^{-1} y t_{2} &= y^{-1} t_{x}^2 t_{4}^7 \\
\Rightarrow t_2 &= y^{-1} x t_{2} t_{4}^7 \\
H_{t_12 t_17} &= H t_{4} y^{-1} x t_{2} t_{4}^7 t_{1}^5
\end{align*}\]
$\Rightarrow H_{t_1t_2t_{17}} = H y^{-1} x [t_3^4] y^{-1} x t_4^2 t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_4^2 t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_1^0 t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_{28} t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_6 t_1$, since
$H_{t_6} = H_{t_{28}}$
$H_{t_1t_2t_{17}} = H t_{28} t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_6 t_1$
$\Rightarrow H_{t_1t_2t_{17}} = H t_{15} t_6$
$\Rightarrow H_{t_1t_2t_{17}} \in [16]$, since $H_{t_1t_6}$ is in $[16]$.

2 symmetric generators will go to $[16]$.

$H_{t_1t_2t_{33}} = H_{t_1}t_2t_{33}$
$\Rightarrow H_{t_1t_2t_{33}} = H_{t_1}t_5t_2t_{33}$, since
$H_{t_1} = H_{t_{15}}$
$H_{t_1t_2t_{33}} = H_{t_{15}}t_2t_{33}$
$\Rightarrow H_{t_1t_2t_{33}} = H_{t_3}t_2t_{1}^0$
$\Rightarrow H_{t_1t_2t_{33}} = H_{t_3}t_2t_{1}^0$, since Equation 5.8
$x^3t_{11}t_{10}t_9 = e$
$[x^3t_{11}t_{10}t_9]y^{-1}x^{-1} = e^{-1}x^{-1}$
$\Rightarrow x^{-1}yt_2t_{13}t_{36} = e$
$\Rightarrow x^{-1}yt_2t_{13}t_{1}^0 = e$
$\Rightarrow x^{-1}yt_2t_{13}t_{4}^0 = t_4^2$
$\Rightarrow x^{-1}yt_2t_{13}t_{1}^7 = t_4^2 t_1^7$
$\Rightarrow y^{-1}x^{-1}yt_2 = y^{-1}x t_4^2 t_1^7$
$\Rightarrow t_2 = y^{-1}x t_4^2 t_1^7$
$H_{t_1t_2t_{33}} = H_{t_3}y^{-1}x t_4^2 t_1^7$
$\Rightarrow H_{t_1t_2t_{33}} = H y^{-1} x [t_3^4] y^{-1} x t_4^2 t_1^5$
$\Rightarrow H_{t_1t_2t_{33}} = H t_4^2 t_1^5$
$\Rightarrow H_{t_1t_2t_{33}} = H t_1^0 t_1^5$
$\Rightarrow H_{t_1t_2t_{33}} = H [y e^{-1} t_2^5 t_1] t_1^5$, since by Equation 5.8
$x^3t_{11}t_{10}t_9 = e$
\[ [x^3 t_{11} t_{10} t_9]^{x-1} y = e^{x-1} y \]
\[ \implies y x^{-1} t_{18} t_{11} t_{16} = e \]
\[ \implies y x^{-1} t_5^4 t_{14} t_4 = e \]
\[ \implies y x^{-1} t_5^2 t_{14} t_4^7 = t_1^7 \]
\[ \implies y x^{-1} t_5^2 t_1 = t_1^7 \]
\[ H t_{12} t_{33} = H y x^{-1} t_5^2 t_1 t_1^5 \]
\[ \implies H t_{12} t_{33} = H t_5^2 t_1 \]
\[ \implies H t_{12} t_{33} = H t_5^2 [x^{-1} y^{-1} t_4 t_2^9], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{y^{-2}} = e^{y^{-2}} \]
\[ \implies x^{-1} y^{-1} t_{15} t_{34} t_{17} = e \]
\[ \implies x^{-1} y^{-1} t_{15} t_{34} t_{17} = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^2 t_1^5 = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^2 t_1^5 t_1^6 = t_1^6 \]
\[ \implies x^{-1} y^{-1} t_4 t_3^2 = t_1^6 \]
\[ H t_{12} t_{33} = H t_5^2 x^{-1} y^{-1} t_4 t_2^9 \]
\[ \implies H t_{12} t_{33} = H x^{-1} y^{-1} [t_2^9] x^{-1} y^{-1} t_4 t_2^9 \]
\[ \implies H t_{12} t_{33} = H t_4 t_3^2 t_2^9 \]
\[ \implies H t_{12} t_{33} = H t_4 t_3^2 \]
\[ \implies H t_{12} t_{33} = H t_4 t_3^2 \]
\[ \implies H t_{12} t_{33} = H [x^{-1} y t_3^2 t_2] t_2^9, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{y^2} = e^{y^2} \]
\[ \implies x^{-1} y t_{19} t_2 t_{13} = e \]
\[ \implies x^{-1} y t_3 t_2 t_1^4 = e \]
\[ \implies x^{-1} y t_3 t_2 t_1^7 = t_1^7 \]
\[ \implies x^{-1} y t_3 t_2 = t_1^7 \]
\[ H t_{12} t_{33} = H x^{-1} y t_3 t_2 t_2^9 \]
\[ \implies H t_{12} t_{33} = H t_3 t_2^{10} \]
\[ \Rightarrow H_{t_1 t_2 t_{33}} = H_{t_{19} t_{38}} \]
\[ \Rightarrow H_{t_1 t_2 t_{33}} = H_{t_9 t_{38}}, \text{ since} \]
\[ H_{t_9} = H_{t_1} \]
\[ H_{t_1 t_2 t_{33}} = H_{t_9 t_{38}} \]
\[ \Rightarrow H_{t_1 t_2 t_{33}} \in [16], \text{ since } H_{t_9 t_{38}} \text{ is in } [1, 6]. \]

2 symmetric generators will go to [1, 6].

\[ H_{t_1 t_2 t_{21}} = H_{t_1 t_2 t_{21}} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{t_{15} t_2 t_{21}}, \text{ since} \]
\[ H_{t_1} = H_{t_{15}} \]
\[ H_{t_1 t_2 t_{21}} = H_{t_{15} t_2 t_{21}} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{t_{15} t_2 t_{21}}^6 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{t_{15} t_2 t_{21}}^6 \]
\[ \text{by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]y^{-1} x^{-1} = e y^{-1} x^{-1} \]
\[ \Rightarrow x^{-1} y t_2 t_{13} t_{36} = e \]
\[ \Rightarrow x^{-1} y t_2 t_{14} t_{4}^9 = e \]
\[ \Rightarrow x^{-1} y t_2 t_{14} t_{4}^9 t_4^2 = t_4^2 \]
\[ \Rightarrow x^{-1} y t_2 t_{14} t_{4}^7 t_4^1 = t_4^7 \]
\[ \Rightarrow y^{-1} x x^{-1} y t_2 = y^{-1} x t_{14} t_{4}^7 \]
\[ \Rightarrow t_2 = y^{-1} x t_{14} t_{4}^7 \]
\[ H_{t_1 t_2 t_{21}} = H_{t_{15} y^{-1} x t_{14} t_{4}^7 t_4^6} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H y^{-1} x [t_{14}^3 y^{-1} x^{-1} t_{14}^2 t_4^1] \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_{15} t_{14} t_{4}^2 t_4^1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_{15} t_{14} t_{4}^2 t_4^1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_{15} t_{14} t_{4}^2 t_4^1 \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]x^{-1} y = e x^{-1} y \]
\[ \Rightarrow y x^{-1} t_{18} t_{1} t_{16} = e \]
\[ \Rightarrow y x^{-1} t_{21} t_{14}^4 = e \]
\[ \Rightarrow y x^{-1} t_{21} t_{14}^4 t_4^7 = t_4^7 \]
\[ \Rightarrow y x^{-1} t_{21} t_{14}^4 = t_4^7 \]
\[ H_{t_1 t_2 t_{21}} = H y x^{-1} t_5^2 t_1 t_2^1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_5^2 t_1^6 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_1 t_5^2 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} \in [14], \text{ since } H_{t_1 t_9} \text{ is in } [14] \]
2 symmetric generators will go to [1 4].

\[ H_{t_1 t_2 t_{21}} = H_{t_1 t_2 t_{21}} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{t_1 t_2 t_6} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{t_1 t_5^2} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} \text{, since by Equation 5.8 } x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy^{-1}} = e^{xy^{-1}} \]
\[ \Rightarrow y^{-1} x^{-1} t_{20} t_3 t_{14} = e \]
\[ \Rightarrow y^{-1} x^{-1} t_5^5 t_3 t_2^4 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_5^6 t_3 t_2^7 = t_2^7 \]
\[ \Rightarrow y^{-1} x^{-1} t_5^3 t_3 = t_2^2 \]
\[ H_{t_1 t_2 t_{21}} = H_{t_1 y^{-1} x^{-1} t_5^5 t_3} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H y^{-1} x^{-1} [t_4 y^{-1} x^{-1} t_4^5 t_3] \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_4^5 t_3 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_1 t_2 t_3, \text{ since } H_{t_32} = H t_{38} \]
\[ H_{t_1 t_2 t_{21}} = H t_4^8 \]
\[ H_{t_1 t_2 t_{21}} = H t_2 t_3^1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H t_2 t_3^1 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H y^{-1} t_4^5 t_3^9 t_3, \text{ since by Equation 5.8 } x^3 t_{11} t_{10} t_9 = e^{xy} \]
\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{xy} \]
\[ \Rightarrow x^{-1} y t_{30} t_{19} t_2 = e \]
\[ \Rightarrow x^{-1} y t_4^6 t_3 t_2 = e \]
\[ \Rightarrow x^{-1} y t_4^6 t_3 t_2 t_2^{10} t_2 = t_2^{10} \]
\[ \Rightarrow x^{-1} y t_4^5 t_3 = t_2^{10} \]
\[ H_{t_1 t_2 t_{21}} = H x^{-1} y t_4^5 t_3 t_3 \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{4 t_3^6} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{36 t_{23}} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} = H_{34 t_{23}}, \text{ since} \]
\[ H_{t_3} = H_{36} \]
\[ H_{t_1 t_2 t_{21}} = H_{34 t_{23}} \]
\[ \Rightarrow H_{t_1 t_2 t_{21}} \in [16], \text{ since } H_{34 t_{23}} \text{ is in } [1 6]. \]
2 symmetric generators will go to [1 6].

\[ H_{t_1 t_2 t_{24}} = H_{t_1 t_2 t_{24}} \]
\[ \Rightarrow H_{t_1 t_2 t_{24}} = H_{t_{11} t_{20} t_{24}}, \text{ since} \]
\[ H_{t_1 t_2} = H_{t_{11} t_{20}} \]
\[ H_{t_1 t_2 t_{24}} = H_{t_{11} t_{20} t_{24}} \]
\[ \Rightarrow H_{t_1 t_2 t_{24}} = H_{t_3^3 t_4^5 t_6^6} \]
\[ \Rightarrow H_{t_1 t_2 t_{24}} = H_{t_3^3} \]
\[ \Rightarrow H_{t_1 t_2 t_{24}} = H_{t_{11}} \]
\[ \Rightarrow H_{t_1 t_2 t_{24}} \in [1], \text{ since } H_{t_{11}} \text{ is in } [1]. \]
2 symmetric generators will go to [1].

\[ H_{t_1 t_2 t_{25}} = H_{t_1 t_2 t_{25}} \]
\[ \Rightarrow H_{t_1 t_2 t_{25}} = H_{t_{15} t_{24} t_{25}}, \text{ since} \]
\[ H_{t_1} = H_{t_{15}} \]
\[ H_{t_1 t_2 t_{25}} = H_{t_{15} t_{24} t_{25}} \]
\[ \Rightarrow H_{t_1 t_2 t_{25}} = H_{t_3^3 t_4^7 t_5^7} \]
\[ \Rightarrow H_{t_1 t_2 t_{25}} = H_{t_3^3 [y^{-1} x t_{4 t_1^2}] t_1^7}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^{-1} x^{-1} = e y^{-1} x^{-1} \]
\[ \Rightarrow x^{-1} y t_{2} t_{13} t_{36} = e \]
\[ \Rightarrow x^{-1} y t_2 t_4 t_9^9 = e \]
\[ \Rightarrow x^{-1} y t_2 t_4 t_9^9 t_4 t_1^2 = t_4^2 \]
\[ \Rightarrow x^{-1} y t_2 t_4 t_9^9 t_4^2 t_1^2 = t_4^2 t_1^7 \]
\[ \Rightarrow y^{-1} x x^{-1} y t_2 = y^{-1} x t_4^2 t_1^7 \]
\[ \Rightarrow t_2 = y^{-1} x t_4^2 t_1^7 \]
\[
H_{t1t2t25} = Ht_{3}t_{4}^{-1}x_{1}t_{4}^{2}t_{1}^{7}
\]
\[
\Rightarrow H_{t1t2t25} = Hy^{-1}x_{1}t_{4}^{3}y^{-1}x_{1}t_{4}^{2}t_{1}^{3}
\]
\[
\Rightarrow H_{t1t2t25} = Ht_{4}^{2}t_{1}^{3}
\]
\[
\Rightarrow H_{t1t2t25} = Ht_{4}^{2}t_{1}^{3}
\]
\[
\Rightarrow H_{t1t2t25} = H[yx^{-1}t_{2}^{3}t_{1}]t_{1}^{3}, \text{ since by Equation 5.8}
\]
\[
x^{3}t_{11}t_{10}t_{9} = e
\]
\[
[x^{3}t_{11}t_{10}t_{9}]^{x^{-1}y} = e^{x^{-1}y}
\]
\[
\Rightarrow yx^{-1}t_{18}t_{1}t_{16} = e
\]
\[
\Rightarrow yx^{-1}t_{2}^{5}t_{1}t_{4} = e
\]
\[
\Rightarrow yx^{-1}t_{2}^{5}t_{1}t_{4}^{2}t_{4}^{7} = t_{4}^{7}
\]
\[
\Rightarrow yx^{-1}t_{2}^{5}t_{1} = t_{4}^{7}
\]
\[
H_{t1t2t25} = H[yx^{-1}t_{2}^{5}t_{1}t_{4}]
\]
\[
\Rightarrow H_{t1t2t25} = Ht_{4}^{2}t_{1}^{4}
\]
\[
\Rightarrow H_{t1t2t25} = Ht_{18}t_{13}
\]
\[
\Rightarrow H_{t1t2t25} = H_{t12}t_{13}, \text{ since}
\]
\[
H_{t12} = H_{t18}
\]
\[
H_{t1t2t25} = H_{t12}t_{13}
\]
\[
\Rightarrow H_{t1t2t25} \in [110], \text{ since } H_{t12}t_{13} \text{ is in } [1 \, 10].
\]

2 symmetric generators will go to [1 \, 10].

\[
H_{t1t2t28} = H_{t1t2t28}
\]
\[
\Rightarrow H_{t1t2t28} = H_{t11}t_{20}t_{28}, \text{ since}
\]
\[
H_{t1t2} = H_{t11}t_{20}
\]
\[
H_{t1t2t28} = H_{t11}t_{20}t_{28}
\]
\[
\Rightarrow H_{t1t2t28} = Ht_{3}^{4}t_{4}^{5}t_{4}^{7}
\]
\[
\Rightarrow H_{t1t2t28} = Ht_{3}^{4}t_{4}^{5}t_{4}^{7}
\]
\[
\Rightarrow H_{t1t2t28} = Ht_{3}^{4}t_{4}^{5}, \text{ since}
\]
\[
H_{t11} = H_{t17}
\]
\[
\Rightarrow H_{t1}^{3} = H_{t1}^{5}
\]
\[
H_{t1t2t28} = H_{t1}^{7}t_{4}
\]
\[
\Rightarrow H_{t1t2t28} = H[yxt_{3}^{7}t_{4}^{10}]t_{4}, \text{ since by Equation 5.8}
\]
\[
x^{3}t_{11}t_{10}t_{9} = e
\]
$[x^3 t_1^9 t_3^7 t_2^9]^{x^{-1} y^{-1} x} = e^{x^{-1} y^{-1} x}$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$

$\Rightarrow x^{-1} y^{-1} t_1 t_2 t_3 t_4 = e$
\[ \Rightarrow H t_1 t_2 t_{37} = H [y x^{-1} t_2^5 t_1] t_1^6, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^{-1} y = e x^{-1} y \]
\[ \Rightarrow y x^{-1} t_{18} t_1 t_{16} = e \]
\[ \Rightarrow y x^{-1} t_2^5 t_1 t_4^4 = e \]
\[ \Rightarrow y x^{-1} t_2^5 t_1 t_4^4 t_1^7 = t_1^7 \]
\[ \Rightarrow y x^{-1} t_2^5 t_1 = t_1^7 \]
\[ H t_1 t_2 t_{37} = H y x^{-1} t_2^5 t_1 t_6 \]
\[ H t_1 t_2 t_{37} = H t_2^5 t_1^4 \]
\[ H t_1 t_2 t_{37} = H t_3 t_1^7, \text{ since} \]
\[ H t_{12} = H t_{18} \]
\[ \Rightarrow H t_2^3 = H t_2^5 \]
\[ H t_1 t_2 t_{37} = H t_3 t_1^7 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H t_4^3 [x^{-1} y t_3^5 t_2], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^2 = e y^2 \]
\[ \Rightarrow x^{-1} y t_{19} t_2 t_{13} = e \]
\[ \Rightarrow x^{-1} y t_3^5 t_2 t_4^4 = e \]
\[ \Rightarrow x^{-1} y t_3^5 t_2 t_4^4 t_1^7 = t_1^7 \]
\[ \Rightarrow x^{-1} y t_3^5 t_2 = t_1^7 \]
\[ H t_1 t_2 t_{37} = H t_4^3 x^{-1} y t_3^5 t_2 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H x^{-1} y [t_3^4] x^{-1} y t_3^5 t_2 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H t_3 t_2^5 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H t_3 t_2^5 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H t_1 t_{17} t_2 \]
\[ \Rightarrow H t_1 t_2 t_{37} = H t_{17}, \text{ since} \]
\[ H t_{11} = H t_{17} \]
\[ H t_1 t_2 t_{37} \in [1 10], \text{ since } H t_{17} t_2 \text{ is in } [1 10]. \]
\[ 2 \text{ symmetric generators will go to } [1 10]. \]

The orbits of \( N^{(14)} \) are \{1, 35, 18, 16\}, \{2, 12, 15, 33\}, \{3, 17, 36, 10\}, \{4, 14, 9, 19\},
\{5, 27, 38, 32\}, \{6, 24, 31, 25\}, \{7, 37, 28, 22\}, \{8, 30, 21, 39\}, \{11, 13, 20, 34\}, and \{23, 29, 40, 26\}. 

We must check to see where the \( t_{14t16}, t_{14t12}, t_{14t36}, t_{14t44}, t_{14t42}, t_{14t24}, t_{14t28}, t_{14t8}, t_{14t20}, \) and \( t_{14t40} \) belong.

\[
Ht_{14t16} = Ht_{14t16} \\
\implies Ht_{14t16} = Ht_{14t4^1} \\
\implies Ht_{14t16} = Ht_{14t5^1} \\
\implies Ht_{14t16} = Ht_{15t20} \\
\implies Ht_{14t16} = Ht_{15t20}, \text{ since } Ht_1 = Ht_{154} \\
Ht_{14t16} = Ht_{15t20} \\
\implies Ht_{14t16} \in [110], \text{ since } Ht_{15t20} \text{ is in } [110].
\]

4 symmetric generators will go to [110].

\[
Ht_{14t12} = Ht_{14t12} \\
\implies Ht_{14t12} = Ht_{14t3^1} \\
\implies Ht_{14t12} = Ht_{14t4^1} \\
\implies Ht_{14t12} = Hx^{-1}t_{3^2}t_{4}^7, \text{ since by Equation 5.8 } x^3t_{11}t_{10}t_9 = e \\
x^3t_{11}t_{10}t_9 |^{x^2y} = e^{x^2y} \\
\implies yx^{-1}t_{14t16}t_{35} = e \\
\implies yx^{-1}t_{14t4^0} = e \\
\implies yx^{-1}t_{14t4^0} \cdot \frac{t_3^2}{t_4^7} = \frac{t_3^2}{t_4^7} \\
\implies yx^{-1}t_{14t4^7} = \frac{t_3^2}{t_4^7} \\
\implies xy^{-1}yx^{-1}t_1 = xy^{-1}t_{3^2}t_4^7 \\
\implies t_1 = xy^{-1}t_{3^2}t_4^7 \\
Ht_{14t12} = Hxy^{-1}t_{3^2}t_4^7t_{4}^1 \\
\implies Ht_{14t12} = Ht_3^1 \\
\implies Ht_{14t12} = Ht_7 \\
\implies Ht_{14t12} \in [5], \text{ since } Ht_7 \text{ is in } [5].
\]

4 symmetric generators will go to [5].
\[ H_{t_{1}t_{4}t_{36}} = H_{t_{1}t_{4}t_{36}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{1}t_{4}t_{4}^{0}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{1}t_{4}^{10}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{1}[x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}]} \], since by Equation 5.8
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{y^{-1}} = e^{y^{-1}} \]
\[ \Rightarrow x^{-1}y^{-1}t_{34}t_{17}t_{4} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}t_{4} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}t_{4}^{10} = t_{4}^{10} \]
\[ \Rightarrow x^{-1}y^{-1}t_{21}^{0} = t_{4}^{10} \]
\[ H_{t_{1}t_{4}t_{36}} = H_{t_{1}x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{x^{-1}y^{-1}[t_{1}]^{x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}^{2}t_{2}^{5}t_{1}^{10}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}^{2}t_{1}^{5}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}^{2}t_{1}^{5}, since} \]
\[ H_{t_{8}} = H_{t_{26}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}t_{1}^{5}} \]
\[ H_{t_{1}t_{4}t_{36}} = H_{t_{2}^{2}t_{1}^{5}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H[t_{1}^{x^{-1}y^{-1}t_{21}^{0}t_{4}^{5}}], since by Equation 5.8 \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{y^{-1}x^{-1}} = e^{y^{-1}x^{-1}} \]
\[ \Rightarrow x^{-1}yt_{21}^{13}t_{36} = e \]
\[ \Rightarrow x^{-1}yt_{21}^{0}t_{4}^{9} = e \]
\[ \Rightarrow x^{-1}yt_{21}^{0}t_{4}^{9}t_{4}^{2} = t_{4}^{2} \]
\[ \Rightarrow x^{-1}yt_{21}^{0}t_{4} = t_{4}^{2} \]
\[ H_{t_{1}t_{4}t_{36}} = H_{x^{-1}yt_{21}^{0}t_{4}^{5}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}t_{1}^{0}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{2}t_{33}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} = H_{t_{16}t_{33}, since} \]
\[ H_{t_{2}} = H_{t_{16}} \]
\[ H_{t_{1}t_{4}t_{36}} = H_{t_{16}t_{33}} \]
\[ \Rightarrow H_{t_{1}t_{4}t_{36}} \in [12], since H_{t_{16}t_{33}} \text{ is in [1 6].} \]
4 symmetric generators will go to [1 6].

\[ H_{t_1 t_3 t_4} = H_{t_1 t_4 t_3} \]
\[ \implies H_{t_1 t_3 t_4} = H_{t_1 t_4 t_3} \]
\[ \implies H_{t_1 t_3 t_4} = H_{t_1} t_4^2 \]
\[ \implies H_{t_1 t_3 t_4} = H[xy^{-1}t_3^2 t_4^7] t_4^2 \], since by Equation 5.8

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{x^2 y} = e^{x^2 y} \]
\[ \implies yx^{-1} t_{11} t_{10} t_{35} = e \]
\[ \implies yx^{-1} t_1 t_4^9 = e \]
\[ \implies yx^{-1} t_1 t_4^9 t_3^2 = t_3^2 \]
\[ \implies yx^{-1} t_1 t_4^7 = t_4^7 \]
\[ \implies xy^{-1} yx^{-1} t_1 = xy^{-1} t_3^2 t_4^7 \]
\[ \implies t_1 = xy^{-1} t_3^2 t_4^7 \]

\[ H_{t_1 t_3 t_4} = Hxy^{-1} t_3^2 t_4^7 \]
\[ \implies H_{t_1 t_3 t_4} = H_{t_1} t_3^0 t_4^9 \]
\[ \implies H_{t_1 t_3 t_4} = H_{t_1} t_4^0, \text{ since} \]

\[ H_{t_7} = H_{t_{25}} \]
\[ \implies H_{t_7} t_3^2 = H_{t_7} t_1^4 \]
\[ H_{t_1 t_4 t_4} = H_{t_7} t_4^0 \]
\[ \implies H_{t_1 t_4 t_4} = H_{t_7} [y^{-1} x t_2^{10} t_3^6], \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{yx} \]
\[ \implies x^{-1} y t_{30} t_{19} t_2 = e \]
\[ \implies x^{-1} y t_4^6 t_3 t_2 = e \]
\[ \implies x^{-1} y t_4^6 t_3^2 t_2^2 = t_2^{10} \]
\[ \implies x^{-1} y t_4^6 t_3^6 = t_2^{10} t_3^6 \]
\[ \implies y^{-1} x x^{-1} y t_4^9 = y^{-1} x t_2^{10} t_3^6 \]
\[ \implies t_4^9 = y^{-1} x t_2^{10} t_3^6 \]

\[ H_{t_1 t_4 t_4} = H_{t_7} y^{-1} x t_2^{10} t_3^6 \]
\[ \implies H_{t_1 t_4 t_4} = H y^{-1} x [t_4^7] y^{-1} x t_2^{10} t_3^6 \]
\[ \implies H_{t_1 t_4 t_4} = H_{t_2} t_2^{10} t_2^{10} t_3^6 \]
\[ H_{t_1 t_4 t_4} = H_{t_2^6 t_3^6} \]
\[ H_{t_1 t_4 t_4} = H_{t_{34} t_{23}} \]
\[ \Rightarrow H_{t_1 t_4 t_4} \in [16], \text{ since } H_{t_{34} t_{23}} \text{ is in } [1 6]. \]
4 symmetric generators will go to [1 6].

\[ H_{t_1 t_4 t_{32}} = H_{t_1 t_4 t_{32}} \]
\[ \Rightarrow H_{t_1 t_4 t_{32}} = H_{t_1 t_4 t_4^8} \]
\[ \Rightarrow H_{t_1 t_4 t_{32}} = H_{t_1 t_4^9} \]
\[ \Rightarrow H_{t_1 t_4 t_{32}} = H_{t_1 t_{36}} \]
\[ \Rightarrow H_{t_1 t_4 t_{32}} = H_{t_{15} t_{36}}, \text{ since } \]
\[ H_{t_1} = H_{t_{15}} \]
\[ H_{t_1 t_4 t_{32}} = H_{t_{15} t_{36}} \]
\[ \Rightarrow H_{t_1 t_4 t_{32}} \in [12], \text{ since } H_{t_{15} t_{36}} \text{ is in } [1 2]. \]
4 symmetric generators will go to [1 2].

\[ H_{t_1 t_4 t_{24}} = H_{t_1 t_4 t_{24}} \]
\[ \Rightarrow H_{t_1 t_4 t_{24}} = H_{t_1 t_4 t_4^6} \]
\[ \Rightarrow H_{t_1 t_4 t_{24}} = H_{t_1 t_4^7} \]
\[ \Rightarrow H_{t_1 t_4 t_{24}} = H_{t_{15} t_{28}}, \text{ since } \]
\[ H_{t_1} = H_{t_{15}} \]
\[ H_{t_1 t_4 t_{24}} = H_{t_{15} t_{28}} \]
\[ \Rightarrow H_{t_1 t_4 t_{24}} \in [16], \text{ since } H_{t_{15} t_{28}} \text{ is in } [1 6]. \]
4 symmetric generators will go to [1 6].

\[ H_{t_1 t_4 t_{28}} = H_{t_1 t_4 t_{28}} \]
\[ \Rightarrow H_{t_1 t_4 t_{28}} = H_{t_1 t_4 t_4^7} \]
\[ \Rightarrow H_{t_1 t_4 t_{28}} = H_{t_1 t_4^8} \]
\[ \Rightarrow H_{t_1 t_4 t_{28}} = H[x y^{-1} t_{4}^7 t_4^8], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^2 y = e x^2 y \]
\[ \Rightarrow y x^{-1} t_{11} t_{10} t_{35} = e \]
\[ \Rightarrow y x^{-1} t_{4}^7 t_4^9 = e \]
\[ yx^{-1}t_1 t_4^8 t_3^2 = t_3^2 \]
\[ yx^{-1}t_1 t_4^7 t_3^4 = t_3^4 \]
\[ yx^{-1}t_1 t_4^9 t_3^2 = t_3^4 \]
\[ xy^{-1}t_1 t_4^7 t_3 = xy^{-1}t_3^4 \]
\[ t_1 = xy^{-1}t_3^4 \]

\[ H_{t_1 t_4 t_2 8} = H x y^{-1} t_3^4 t_4^8 \Rightarrow H_{t_1 t_4 t_2 8} = H t_3^4 \]
\[ H_{t_1 t_4 t_2 8} = H t_1 t_4 t_16 \]
\[ H_{t_1 t_4 t_2 8} = H t_25 t_16, \text{ since } H t_7 = H t_25 \]
\[ H_{t_1 t_4 t_2 8} = H t_10 t_25, \text{ since by Equation 5.9} \]
\[ H_{t_1 t_6} = H t_6 t_1 \]
\[ H_{t_10 t_2} = H t_7 H t_1 \]

\[ H_{t_1 t_4 t_2 8} \in [16], \text{ since } H_{t_10 t_2} \text{ is in } [16]. \]
4 symmetric generators will go to [16].
\[ f^{-1}y^{-1}t_{14}t_{17}t_{4} = e \]
\[ f^{-1}y^{-1}t_{14}^{2}t_{14}^{5} t_{4} = e \]
\[ f^{-1}y^{-1}t_{14}^{2}t_{14}^{5} t_{4} t_{14}^{10} = t_{14}^{10} \]
\[ f^{-1}y^{-1}t_{14}^{2}t_{14}^{5} = t_{14}^{10} \]
\[ Ht_{1}t_{4}t_{8} = Ht_{1}^{2}y^{-1}t_{2}^{9}t_{1}^{5} \]
\[ Ht_{1}t_{4}t_{8} = Hx^{-1}y^{-1}[t_{3}^{2}]x^{-1}y^{-1}t_{2}^{9}t_{1}^{5} \]
\[ Ht_{1}t_{4}t_{8} = Ht_{2}^{10}t_{2}^{9}t_{1}^{5} \]
\[ Ht_{1}t_{4}t_{8} = Ht_{3}^{8}t_{1}^{5} \]
\[ Ht_{1}t_{4}t_{8} = Ht_{30}t_{17} \]
\[ Ht_{1}t_{4}t_{8} = Ht_{17}t_{30}, \text{ since by Equation 5.9} \]
\[ Ht_{1}t_{6} = Ht_{6}t_{1} \]
\[ [Ht_{1}t_{6}]y^{2} = [Ht_{6}t_{1}]y^{2} \]
\[ Ht_{17}t_{30} = Ht_{30}t_{17} \]

\[ Ht_{1}t_{4}t_{8} \in [16], \text{ since } Ht_{17}t_{30} \text{ is in } [16] \]
4 symmetric generators will go to [1 6].

\[ Ht_{1}t_{4}t_{20} = Ht_{1}t_{4}t_{20} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{15}t_{4}t_{20}, \text{ since } Ht_{1} = Ht_{15} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{1}t_{4}t_{20} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{1}^{3}t_{4}t_{4}^{5} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{1}^{3}t_{4}^{6} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{3}^{8}[y^{-1}x^{-1}t_{2}^{9}t_{1}^{5}], \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]x^{-1}y^{-1} = e^{-1}y^{-1} \]
\[ yx^{-1}t_{14}t_{33}t_{20} = e \]
\[ yx^{-1}t_{2}^{4}t_{1}^{9}t_{1}^{5} = e \]
\[ yx^{-1}t_{2}^{4}t_{1}^{9}t_{4}^{6} = t_{4}^{6} \]
\[ yx^{-1}t_{2}^{4}t_{1}^{9} = t_{4}^{6} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{3}^{8}y^{-1}x^{-1}t_{2}^{9}t_{1}^{9} \]
\[ Ht_{1}t_{4}t_{20} = Hg^{-1}x^{-1}[t_{3}^{2}]y^{-1}x^{-1}t_{2}^{9}t_{1}^{9} \]
\[ Ht_{1}t_{4}t_{20} = Ht_{2}^{5}t_{2}^{4}t_{1}^{9} \]
\[ \implies H t_1 t_4 t_{20} = H t_2^9 t_1^9 \]
\[ \implies H t_1 t_4 t_{20} = H t_{34} t_{33} \]
\[ \implies H t_1 t_4 t_{20} = H t_{34} t_{33}, \text{ since } H t_{34} = H t_{36} \]
\[ H t_1 t_4 t_{20} = H t_{34} t_{33} \implies H t_1 t_4 t_{20} \in [110], \text{ since } H t_{34} t_{33} \text{ is in } [1 \, 10]. \]

4 symmetric generators will go to [1 10].

\[ H t_1 t_4 t_{40} = H t_1 t_4 t_{40} \]
\[ \implies H t_1 t_4 t_{40} = H t_1 t_4 t_{10} \]
\[ \implies H t_1 t_4 t_{40} = H t_1 \]
\[ \implies H t_1 t_4 t_{40} \in [1], \text{ since } H t_1 \text{ is in } [1]. \]

4 symmetric generators will go to [1].

\( N^{(16)} \) has 40 single orbits. We will check to see where

\[
t_{1} t_{6} t_{1}, t_{1} t_{6} t_{2}, t_{1} t_{6} t_{3}, t_{1} t_{6} t_{4}, t_{1} t_{6} t_{5}, t_{1} t_{6} t_{6}, t_{1} t_{6} t_{7}, t_{1} t_{6} t_{8}, t_{1} t_{6} t_{9}, t_{1} t_{6} t_{10},
\]
\[
t_{1} t_{6} t_{11}, t_{1} t_{6} t_{12}, t_{1} t_{6} t_{13}, t_{1} t_{6} t_{14}, t_{1} t_{6} t_{15}, t_{1} t_{6} t_{16}, t_{1} t_{6} t_{17}, t_{1} t_{6} t_{18}, t_{1} t_{6} t_{19}, t_{1} t_{6} t_{20},
\]
\[
t_{1} t_{6} t_{21}, t_{1} t_{6} t_{22}, t_{1} t_{6} t_{23}, t_{1} t_{6} t_{24}, t_{1} t_{6} t_{25}, t_{1} t_{6} t_{26}, t_{1} t_{6} t_{27}, t_{1} t_{6} t_{28}, t_{1} t_{6} t_{29}, t_{1} t_{6} t_{30},
\]
\[
t_{1} t_{6} t_{31}, t_{1} t_{6} t_{32}, t_{1} t_{6} t_{33}, t_{1} t_{6} t_{34}, t_{1} t_{6} t_{35}, t_{1} t_{6} t_{36}, t_{1} t_{6} t_{37}, t_{1} t_{6} t_{38}, t_{1} t_{6} t_{39}, \text{ and } t_{1} t_{6} t_{40} \text{ belong.}
\]

\[ H t_{1} t_{6} t_{1} = H t_{1} t_{6} t_{1} \]
\[ \implies H t_{1} t_{6} t_{1} = H t_{6} t_{1}, \text{ since } H t_{1} t_{6} = H t_{6} t_{1} \]

\[ H t_{1} t_{6} t_{1} = H t_{6} t_{1} \]
\[ \implies H t_{1} t_{6} t_{1} = H t_{28} t_{1}^2, \text{ since } H t_{6} = H t_{28} \]

\[ H t_{1} t_{6} t_{1} = H t_{28} t_{1}^2 \]
\[ \implies H t_{1} t_{6} t_{1} = H t_{1} t_{6} t_{1} \implies H t_{1} t_{6} t_{1} = H t_{1}^7 y^{-1} x^{-1} t_{3} t_{2}^{4}, \text{ since by Equation } 5.8 \]
\[ x^{3} t_{1} t_{10} t_{9} = e \]
\[ [x^{3} t_{1} t_{10} t_{9}] y^{-1} = e y^{-1} \]
\[ \implies x^{-1} y^{-1} t_{3} t_{14} t_{33} = e \]
\[ \implies x^{-1} y^{-1} t_{3} t_{14} t_{1}^9 = e \]
\[ \implies x^{-1} y^{-1} t_{3} t_{14} t_{1}^9 t_{1}^2 = t_{2}^2 \]
\[ \implies x^{-1} y^{-1} t_{3} t_{14} = t_{2}^2 \]
\[ H t_{1} t_{6} t_{1} = H t_{1} y^{-1} x^{-1} t_{3} t_{2}^{4} \]
\[ \implies H t_{1} t_{6} t_{1} = H y^{-1} x^{-1} [t_{4}^7] y^{-1} x^{-1} t_{3} t_{2}^{4} \]
\[ \implies H t_{1} t_{6} t_{1} = H s_{3} t_{3} t_{4}^{4} \]
\[ \Rightarrow H_{t1} t_{6} t_{1} = H_{3}^{9} t_{4}^{4} \]
\[ \Rightarrow H_{t1} t_{6} t_{1} = H_{35} t_{14} \]
\[ \Rightarrow H_{t1} t_{6} t_{1} \in [14], \text{ since } H_{35} t_{14} \text{ is in } [1 4]. \]
1 symmetric generator will go to [1 4].

\[ H_{t1} t_{6} t_{2} = H_{t1} t_{6} t_{2} \]
\[ \Rightarrow H_{t1} t_{6} t_{2} = H_{t1} t_{2}^{2} t_{2} \]
\[ \Rightarrow H_{t1} t_{6} t_{2} = H_{t1} t_{2}^{2} \]
\[ \Rightarrow H_{t1} t_{6} t_{2} = H_{t1} t_{10} \]
\[ \Rightarrow H_{t1} t_{6} t_{2} \in [110], \text{ since } H_{t1} t_{10} \text{ is in } [1 10]. \]
1 symmetric generator will go to [1 4].

\[ H_{t1} t_{6} t_{3} = H_{t1} t_{6} t_{3} \]
\[ \Rightarrow H_{t1} t_{6} t_{3} = H_{t1} t_{2}^{2} t_{3} \]
\[ H_{t1} t_{6} t_{3} = H_{t1} [x^{-1} y^{-1} t_{4} t_{3}^{2}] t_{3}, \text{ since by Equation 5.8} \]
\[ x^{3} t_{11} t_{10} t_{9} = e \]
\[ [x^{3} t_{11} t_{10} t_{9}] y^{-1} x = e y^{-1} x \]
\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{15} t_{34} = e \]
\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{3}^{2} t_{2} = e \]
\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{3}^{2} t_{2}^{2} = t_{2}^{2} \]
\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{3}^{2} = t_{2}^{2} \]
\[ H_{t1} t_{6} t_{3} = H_{t1} x^{-1} y^{-1} t_{4} t_{3}^{2} t_{3} \]
\[ \Rightarrow H_{t1} t_{6} t_{3} = H x^{-1} y^{-1} [t_{1}] x^{-1} y^{-1} t_{4} t_{3}^{5} \]
\[ \Rightarrow H_{t1} t_{6} t_{3} = H t_{4}^{9} t_{3}^{5} \]
\[ \Rightarrow H_{t1} t_{6} t_{3} = H t_{4}^{10} t_{3}^{5} \]
\[ \Rightarrow H_{t1} t_{6} t_{3} = H t_{4}^{10} [y^{-1} x t_{1}^{7} t_{4}], \text{ since by Equation 5.8} \]
\[ x^{3} t_{11} t_{10} t_{9} = e \]
\[ [x^{3} t_{11} t_{10} t_{9}] y^{2} = e y^{2} \]
\[ \Rightarrow x^{-1} y t_{19} t_{2} t_{13} = e \]
\[ \Rightarrow x^{-1} y t_{3}^{5} t_{2} t_{1}^{4} = e \]
\[ \Rightarrow x^{-1} y t_{3}^{5} t_{2} t_{1}^{4} t_{1}^{7} = t_{1}^{7} \]
\[ \Rightarrow x^{-1} y t_{3}^{5} t_{2} t_{1}^{10} = t_{1}^{7} t_{1}^{10} \]
\[ \Rightarrow y^{-1}x^{-1}y t_5^3 = y^{-1}x t^7_1 t_2^{10} \]
\[ \Rightarrow t_5^3 = y^{-1}x t^7_1 t_2^{10} \]
\[ H t_1 t_6 t_3 = H t_4^{10} y^{-1}x t^7_1 t_4 \]
\[ \Rightarrow H t_1 t_6 t_3 = H y^{-1}x [t_4^{10}] y^{-1}x t^7_1 t_4 \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_4 t_1^7 t_4 \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_4^9 t_4 \]
\[ \Rightarrow H t_1 t_6 t_3 = H [x y t_3^{10} t_4^{10}] t_4, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^2 y^{-1} = e x^2 y^{-1} \]
\[ \Rightarrow y^{-1}x^{-1} t_{33} t_{20} t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1} t_{14}^4 t_3 t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1} t_{14}^4 t_3 t_3^{10} = t_3^{10} \]
\[ \Rightarrow y^{-1}x^{-1} t_{14}^4 t_3 t_4^{10} = t_4^{10} \]
\[ \Rightarrow x y t^{-1} x^{-1} t_1^0 = x y t_3^{10} t_4^{10} \]
\[ t_1^0 = x y t_3^{10} t_4^{10} \]
\[ H t_1 t_6 t_3 = H x y t_3^{10} t_4^{10} \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_3^{10} t_4^{10} \]
\[ \Rightarrow H t_1 t_6 t_3 = H [x y t_3^{10} t_4^{10}] t_4, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^{-1} y = e x^{-1} y \]
\[ \Rightarrow y x^{-1} t_{18} t_{16} t_1 = e \]
\[ \Rightarrow y x^{-1} t_{2} t_1^2 t_4 = e \]
\[ \Rightarrow y x^{-1} t_{2} t_1^2 t_1^2 t_4^7 = t_4^7 \]
\[ \Rightarrow y x^{-1} t_{2} t_1^2 t_1 = t_4^7 \]
\[ H t_1 t_6 t_3 = H t_3^{10} y x^{-1} t_2^0 t_1 \]
\[ \Rightarrow H t_1 t_6 t_3 = H y x^{-1} t_2^0 t_1 \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_2^0 t_1 \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_6^1 \]
\[ \Rightarrow H t_1 t_6 t_3 = H t_6, \text{ since } H t_1 t_6 = H t_6 t_1 \]
\[ H t_1 t_6 t_3 = H t_1 t_6 \]
\[ \Rightarrow H t_1 t_6 t_3 \in [16], \text{ since } H t_1 t_6 \text{ is in } [16]. \]
1 symmetric generator will go to [1 6].

\[
H t_1 t_6 t_4 = H t_1 t_6 t_4
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_6 t_1 t_4, \text{ since } H t_1 t_6 = H t_6 t_1
\]

\[
H t_1 t_6 t_4 = H t_6 t_1 t_4
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_6^2 t_1 t_4
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_6^2 t_1 t_4
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_6^2 [x y^{-1} t_3^2 t_4^2] t_4, \text{ since by Equation 5.8}
\]

\[
x^3 t_{11} t_{10} t_9 = e
\]

\[
[x^3 t_{11} t_{10} t_9]^2 = x^2 y = e x^2 y
\]

\[
\Rightarrow y x^{-1} t_{11} t_{10} t_{35} = e
\]

\[
\Rightarrow y x^{-1} t_{11} t_4^2 t_3 = e
\]

\[
\Rightarrow y x^{-1} t_{11} t_4^2 t_3 = t_3^2
\]

\[
\Rightarrow y x^{-1} t_{11} t_4^7 = t_3^2 t_4^7
\]

\[
\Rightarrow x y^{-1} y x^{-1} t_1 = x y^{-1} t_3^2 t_4^7
\]

\[
\Rightarrow t_1 = x y^{-1} t_3^2 t_4^7
\]

\[
H t_1 t_6 t_4 = H t_6^2 x y^{-1} t_3^2 t_4^7 t_4
\]

\[
\Rightarrow H t_1 t_6 t_4 = H x y^{-1} [t_2^2] x y^{-1} t_3^2 t_4^8
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_6^2 t_3^2 t_4^8
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_3^{10} t_4^8
\]

\[
\Rightarrow H t_1 t_6 t_4 = H [y^{-1} x^{-1} t_4^9 t_4^5] t_4^8, \text{ since by Equation 5.8}
\]

\[
x^3 t_{11} t_{10} t_9 = e
\]

\[
[x^3 t_{11} t_{10} t_9]^2 = x^2 y^{-1}
\]

\[
\Rightarrow y^{-1} x^{-1} t_4^9 t_3 t_20 t_3 = e
\]

\[
\Rightarrow y^{-1} x^{-1} t_4^9 t_3 = e
\]

\[
\Rightarrow y^{-1} x^{-1} t_4^9 t_3 = t_3^{10}
\]

\[
\Rightarrow y^{-1} x^{-1} t_4^9 t_3 = t_3^{10}
\]

\[
H t_1 t_6 t_4 = H y^{-1} x^{-1} t_4^9 t_4^5 t_4^8
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_4^9 t_3^2 t_4^8
\]

\[
\Rightarrow H t_1 t_6 t_4 = H t_3^2 t_4^8, \text{ since } H t_3^3 = H t_3
\]

\[
\Rightarrow H t_1^9 = H t_3^9
\]
\[\begin{align*}
H_{t1}t_{6}t_{4} &= H_{t3}^{9}t_{4}^{2} \\
H_{t1}t_{6}t_{4} &= H_{t3}^{9}[x^{-1}y_{t2}t_{4}^{4}], \text{ since by Equation 5.8} \\
x^{3}t_{11}t_{10}t_{9} &= e \\
[x^{3}t_{11}t_{10}t_{9}]y^{-1}x^{-1} &= e^{y^{-1}x^{-1}} \\
\implies x^{-1}y_{t2}t_{13}t_{36} &= e \\
\implies x^{-1}y_{t2}t_{1}t_{4}t_{9} &= e \\
\implies x^{-1}y_{t2}t_{1}t_{4}t_{9}t_{1}^{2} &= t_{4}^{2} \\
\implies x^{-1}y_{t2}t_{1}^{2} &= t_{4}^{2} \\
H_{t1}t_{6}t_{4} &= H_{t3}^{9}x^{-1}y_{t2}t_{4}^{4} \\
\implies H_{t1}t_{6}t_{4} &= Hx^{-1}y[t_{3}^{0}]x^{-1}y_{t2}t_{4}^{4} \\
H_{t1}t_{6}t_{4} &= Ht_{1}t_{2}t_{4}^{4} \\
H_{t1}t_{6}t_{4} &= Ht_{5}^{3}t_{4}^{4} \\
H_{t1}t_{6}t_{4} &= Ht_{18}t_{13} \\
H_{t1}t_{6}t_{4} &= Ht_{12}t_{13}, \text{ since } H_{t12} = H_{t18} \\
H_{t1}t_{6}t_{4} &= Ht_{12}t_{13} \\
\implies H_{t1}t_{6}t_{4} &\in [110], \text{ since } H_{t12}t_{13} \text{ is in } [110]. \\
\end{align*}\]

1 symmetric generator will go to [110].
\[ H_{t_1} t_6 t_5 = H_y x^{-1} t_2^5 t_1^3 \]
\[ \Rightarrow H_{t_1} t_6 t_5 = H t_2^5 t_1^4 \]
\[ \Rightarrow H_{t_1} t_6 t_5 = H t_{18} t_{13} \]
\[ \Rightarrow H_{t_1} t_6 t_5 = H t_{12} t_{13}, \text{ since } H_{t_12} = H_{t_{18}} \]
\[ H_{t_1} t_6 t_5 = H t_{12} t_{13} \]
\[ \Rightarrow H_{t_1} t_6 t_5 \in [110], \text{ since } H_{t_{12} t_{13}} \text{ is in [1 10].} \]

1 symmetric generator will go to [1 10].

\[ H_{t_1} t_6 t_6 = H_{t_1} t_6 t_6 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H_{t_15} t_6 t_6, \text{ since } H_{t_1} = H_{t_{15}} \]
\[ H_{t_1} t_6 t_6 = H_{t_15} t_6 t_6 \Rightarrow H_{t_1} t_6 t_6 = H_{t_1} t_2^3 t_2^2 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H t_1 t_2^4 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H t_1 [x y t_4 t_1^2], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^{-1} y^{-1} = e^{-1} x^{-1} y^{-1} \]
\[ \Rightarrow y^{-1} x^{-1} t_{14} t_{33} t_{20} = e \]
\[ \Rightarrow y^{-1} x^{-1} t_2^4 t_1^9 t_4^5 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_2^4 t_1^9 t_4^5 t_6 = t_4^6 \]
\[ \Rightarrow y^{-1} x^{-1} t_2^4 t_1^9 t_4^5 t_6 = t_4^6 t_1^2 \]
\[ \Rightarrow xy^{-1} x^{-1} t_2^4 = xy^{-1} t_1^2 t_4^6 \]
\[ t_4^2 = xy^{-1} t_1^2 t_4^6 \]
\[ H_{t_1} t_6 t_6 = H t_1 [x y t_4 t_1^2] \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H x y [t_1] x y t_4 t_1^2 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H t_1^9 t_4 t_1^2 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H t_1^9 t_4 t_1^2 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H t_2 t_1^2, \text{ since } H_{t_2} = H t_{16} \]
\[ \Rightarrow H_{t_2} = H t_4 \]
\[ H_{t_1} t_6 t_6 = H t_4 t_1^2 \]
\[ \Rightarrow H_{t_1} t_6 t_6 = H [y^{-1} x t_4 t_1^2] t_1^2, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^{-1} y^{-1} = e^{-1} x^{-1} y^{-1} \]
\[\Rightarrow x^{-1}yt_2t_1t_3t_{36} = e\]
\[\Rightarrow x^{-1}yt_2t_4t_9 = e\]
\[\Rightarrow x^{-1}yt_2t_4t_9t_2 = t_4^{2}\]
\[\Rightarrow x^{-1}yt_2t_4t_9t_2 = t_4^{2}\]
\[\Rightarrow y^{-1}tx^{-1}yt_2 = y^{-1}xt_4t_7\]
\[\Rightarrow t_2 = y^{-1}xt_4t_7\]
\[Ht_1t_6t_6 = H\left[y^{-1}xt_4t_7\right]t_4^{2}\]
\[\Rightarrow Ht_1t_6t_6 = Ht_2t_4\]
\[\Rightarrow Ht_1t_6t_6 = Ht_2t_4, \text{ since}\]
\[Ht_8 = Ht_{26}\]
\[\Rightarrow Ht_4^2 = Ht_7^2\]

\[Ht_1t_6t_6 = Ht_{27}[xy^t_3t_4^{10}]\]

Since by Equation 5.8
\[x^3t_1t_1t_0t_9 = e\]
\[x^3t_1t_1t_0t_9]x^2y^{-1} = e^2y^{-1}\]
\[\Rightarrow y^{-1}x^{-1}t_3t_20t_3 = e\]
\[\Rightarrow y^{-1}x^{-1}t_4t_5t_3 = e\]
\[\Rightarrow y^{-1}x^{-1}t_4t_5t_3t_3^{10} = t_3^{10}\]
\[\Rightarrow y^{-1}x^{-1}t_4t_5t_6^{10} = t_3^{10}t_4^{6}\]
\[\Rightarrow xy^{-1}x^{-1}t_4^2 = xy^t_3t_4^{10}\]
\[\Rightarrow t_4^2 = xy^t_3t_4^{10}\]

\[Ht_1t_6t_6 = Ht_{27}[xy^t_3t_4^{10}]\]
\[\Rightarrow Ht_1t_6t_6 = Ht_2t_3x^{10}t_4^{6}\]
\[\Rightarrow Ht_1t_6t_6 = Ht_3x^{10}t_4^{6}\]
\[\Rightarrow Ht_1t_6t_6 = Ht_3t_4^{6}\]
\[\Rightarrow Ht_1t_6t_6 = Ht_{35}t_4^{2}\]
\[\Rightarrow Ht_1t_6t_6 \in [1 6], \text{ since } Ht_{35}t_4^{2} \text{ is in } [1 6]\]
1 symmetric generator will go to [1 6].

\[Ht_1t_6t_7 = Ht_1t_6t_7\]
\[\Rightarrow Ht_1t_6t_7 = Ht_1t_4t_3^{4}\]
\[\Rightarrow Ht_1t_6t_7 = Ht_1[x^{-1}y^{-1}t_4t_3^{4}]t_4^{2}, \text{ since by Equation 5.8}\]
\(x^3 t_{11} t_{10} t_9 = e \)

\([x^3 t_{11} t_{10} t_9]^{y^{-1}x} = e^{y^{-1}x} \)

\[ \implies x^{-1} y^{-1} t_4 t_{15} t_{34} = e \]

\[ \implies x^{-1} y^{-1} t_4 t_{3} t_{2} = e \]

\[ \implies x^{-1} y^{-1} t_4 t_{3} t_{2} t_2 = \frac{t_2^2}{2} \]

\[ \implies x^{-1} y^{-1} t_4 t_{3} = t_2^2 \]

\(H t_{16} t_7 = H t_1 x^{-1} y^{-1} t_4 t_{3} t_2^2 \)

\[ \implies H t_{16} t_7 = H x^{-1} y^{-1} t_1 x^{-1} y^{-1} t_4 t_{3} t_2^2 \]

\[ \implies H t_{16} t_7 = H t_4^9 t_4 t_{3} t_2^2 \]

\[ \implies H t_{16} t_7 = H t_{10} t_4 t_{3} t_2^2 \]

\[ \implies H t_{16} t_7 = H t_5^2 t_4 t_{3} t_2^2 , \text{ since} \]

\(H t_{30} = H t_{40} \)

\[ \implies H t_2^8 = H t_4^{10} \]

\(H t_{16} t_7 = H t_2^8 t_3 t_2^2 \)

\[ \implies H t_{16} t_7 = H t_2^8 t_3 t_2^2 \]

\[ \implies H t_{16} t_7 = H t_2^8 t_3 \]

\[ \implies H t_{16} t_7 = H t_2^8 t_3 \]

\[ \implies H t_{16} t_7 = H [x^{-1} t_4 t_{3} t_2^2] t_3^6 , \text{ since by Equation 5.8} \]

\(x^3 t_{11} t_{10} t_9 = e \)

\([x^3 t_{11} t_{10} t_9]^x = e^x \)

\[ \implies x^{-1} t_{12} t_{11} t_{10} = e \]

\[ \implies x^{-1} t_4 t_3 t_2^3 = e \]

\[ \implies x^{-1} t_4 t_3 t_2^3 t_2^8 = \frac{t_2}{2} \]

\[ \implies x^{-1} t_4 t_3 = \frac{t_2}{2} \]

\(H t_{16} t_7 = H x^{-1} t_4 t_{3} t_2^3 \)

\[ \implies H t_{16} t_7 = H t_4^9 t_3 \]

\[ \implies H t_{16} t_7 = H t_1 t_{35} \]

\[ \implies H t_{16} t_7 \in [14], \text{ since } H t_{12} t_{35} \text{ is in [1 4].} \]

1 symmetric generator will go to [1 4].
\[ H_{t_1} t_{t_6} t_8 = H_{t_7}^1 [xy^{-1} t_3^2 t_7^2] t_4^2, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^2 y = e x^2 y \]
\[ \Rightarrow y x^{-1} t_{11} t_{16} t_{35} = e \]
\[ \Rightarrow y x^{-1} t_{14} t_3^9 = e \]
\[ \Rightarrow y x^{-1} t_{14} t_3^9 t_3^2 = t_3^2 \]
\[ \Rightarrow y x^{-1} t_{14} t_4^7 = t_4^7 \]
\[ \Rightarrow xy^{-1} y x^{-1} t_1 = xy^{-1} t_3^2 t_4^7 \]
\[ \Rightarrow t_1 = xy^{-1} t_3^2 t_4^7 \]
\[ H_{t_1} t_{t_6} t_8 = H_{t_7}^2 xy^{-1} t_3^2 t_4^7 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H x y^{-1} [t_3^2 x y^{-1} t_3^2] t_4^9 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_3^2 t_4^9 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_3^1 t_4^9 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H [y^{-1} x^{-1} t_1^5] t_4^9, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^2 y^{-1} = e x^2 y^{-1} \]
\[ \Rightarrow y^{-1} x^{-1} t_{33} t_{20} t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_1^5 t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_1^5 t_3 t_2 t_9 = t_3^{10} \]
\[ \Rightarrow y^{-1} x^{-1} t_1^5 t_4 = t_3^{10} \]
\[ H_{t_1} t_{t_6} t_8 = H y^{-1} x^{-1} t_1^5 t_4 t_9 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_1^2 t_4 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_{33} t_1^2 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_{33} t_1^2 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 = H t_{35} t_{12}, \text{ since } H t_{33} = H t_{35} \]
\[ H_{t_1} t_{t_6} t_8 = H t_{35} t_{12} \]
\[ \Rightarrow H_{t_1} t_{t_6} t_8 \text{ is in } [1 2], \text{ since } H t_{35} t_{12} \text{ is in } [1 2] \]

1 symmetric generator will go to [1 2].

\[ H_{t_1} t_{t_6} t_9 = H_{t_1} t_{t_6} t_9 \]
\[ \Rightarrow H_{t_1} t_{t_6} t_9 = H t_6 t_1 t_9, \text{ since } H t_1 t_6 = H t_6 t_1 \]
\[ H_{t_1} t_{t_6} t_9 = H_{t_6} t_1 t_9 \]
\[ H_{t_1 t_6 t_9} = H_{t_28 t_1 t_9}, \text{ since } H_{t_6} = H_{t_28} \]

\[ H_{t_1 t_6 t_9} = H_{t_28 t_1 t_9} \implies H_{t_1 t_6 t_9} = H_{t_1^{7} t_1^{3}} \]

\[ \implies H_{t_1 t_6 t_9} = H_{t_1^{7} t_1^{4}} \]

\[ \implies H_{t_1 t_6 t_9} = H[y^{-1} t_1^{5} t_1] t_1^{4}, \text{ since by Equation 5.8} \]

\[ x^3 t_{11 t_10 t_9} = e \]

\[ [x^3 t_{11 t_10 t_9}]^{-1} y = e^{x^{-1} y} \]

\[ \implies y x^{-1} t_{18 t_1 t_16} = e \]

\[ \implies y x^{-1} t_2^{5} t_1 t_4^{4} = e \]

\[ \implies y x^{-1} t_2^{5} t_1 t_2^{4} t_1^{4} t_4^{7} = t_1^{7} \]

\[ \implies y x^{-1} t_2^{5} t_1 = t_1^{7} \]

\[ H_{t_1 t_6 t_9} = H[y x^{-1} t_2^{5} t_1] t_1^{4} \]

\[ \implies H_{t_1 t_6 t_9} = H_{t_1^{5} t_1^{5}} \]

\[ \implies H_{t_1 t_6 t_9} = H_{t_{18 t_1 t_17}} \]

\[ \implies H_{t_1 t_6 t_9} = H_{t_{12 t_1 t_17}}, \text{ since } H_{t_{12}} = H_{t_{18}} \]

\[ H_{t_1 t_6 t_9} = H_{t_{12 t_1 t_17}} \]

\[ \implies H_{t_1 t_6 t_9} \in [1 2], \text{ since } H_{t_{12 t_1 t_17}} \text{ is in } [1 2]. \]

1 symmetric generator will go to [1 2].

\[ H_{t_1 t_6 t_{10}} = H_{t_1 t_6 t_{10}} \]

\[ \implies H_{t_1 t_6 t_{10}} = H_{t_1^{2} t_2^{3}} \]

\[ \implies H_{t_1 t_6 t_{10}} = H_{t_1^{2} t_1} \]

\[ \implies H_{t_1 t_6 t_{10}} = H_{t_{15 t_1 t_18}} \]

\[ \implies H_{t_1 t_6 t_{10}} = H_{t_{15 t_1 t_18}}, \text{ since } H_{t_1} = H_{t_{15}} \]

\[ H_{t_1 t_6 t_{10}} = H_{t_{15 t_1 t_18}} \]

\[ \implies H_{t_1 t_6 t_{10}} \in [1 4], \text{ since } H_{t_{15 t_1 t_18}} \text{ is in } [1 4] \]

1 symmetric generator will go to [1 4].

\[ H_{t_1 t_6 t_{11}} = H_{t_1 t_6 t_{11}} \]

\[ \implies H_{t_1 t_6 t_{11}} = H_{t_1^{2} t_2^{3} t_3} \]

\[ \implies H_{t_1 t_6 t_{11}} = H_{t_1 [x^{-1} y^{-1} t_4 t_3] t_1^{3}}, \text{ since by Equation 5.8} \]

\[ x^3 t_{11 t_10 t_9} = e \]

\[ [x^3 t_{11 t_10 t_9}] y^{-1} x = e^{y^{-1} x} \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_{15}t_{34} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_{15}t_9 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_{15}^2t_2 = t_2^2 \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_3^2 = t_2^2 \]
\[ Ht_{16}t_{11} = Ht_1x^{-1}y^{-1}t_4t_{15}^2t_3 \]
\[ \Rightarrow Ht_{16}t_{11} = Hx^{-1}y^{-1}[t_4]x^{-1}y^{-1}t_4t_3^2 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^0t_4t_3^2 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^1t_3^2 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^2t_3, \text{ since} \]
\[ Ht_{30} = Ht_{40} \]
\[ \Rightarrow Ht_2^0 = Ht_4^0 \]
\[ Ht_{16}t_{11} = Ht_2^0t_3^2 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_2^1t_3^2 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_2^2t_3^2, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{xy^{-1}}x = e^{xy^{-1}}x \]
\[ \Rightarrow [x^{-1}y^{-1}t_{17}t_4t_{15} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_5^2t_4t_3^4 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_5^2t_4t_3^2t_2 = t_3^7 \]
\[ \Rightarrow x^{-1}y^{-1}t_5^2t_4 = t_3^7 \]
\[ Ht_{16}t_{11} = Ht_2^2x^{-1}y^{-1}t_5^2t_4 \]
\[ \Rightarrow Ht_{16}t_{11} = Hx^{-1}y^{-1}[t_5^2]x^{-1}y^{-1}t_5^2t_4 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^0t_5^2t_4 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^1t_5^2t_4 \]
\[ \Rightarrow Ht_{16}t_{11} = Ht_4^2t_5^2t_4, \text{ since} \]
\[ Ht_7 = Ht_{25} \]
\[ \Rightarrow Ht_3^2 = Ht_1^2 \]
\[ Ht_{16}t_{11} = Ht_3^2t_4 \]
\[ \Rightarrow Ht_{16}t_{11} = H[yx^{-1}t_4^2]t_4, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{xy} = e^{xy} \]
\[ \Rightarrow yx^{-1}t_1t_{16}t_{35} = e \]
\[ y x^{-1} t_1 t_4^6 t_3^9 = e \]
\[ y x^{-1} t_1 t_4^6 t_3 t_2^2 = t_3^2 \]
\[ y x^{-1} t_1 t_4^6 = t_3^2 \]

\[ H t_1 t_6 t_{11} = H y x^{-1} t_1 t_4^6 \]
\[ \Rightarrow H t_1 t_6 t_{11} = H t_1 t_4^5 \]
\[ \Rightarrow H t_1 t_6 t_{11} = H t_1 t_20 \]
\[ \Rightarrow H t_1 t_6 t_{11} = H t_1 t_{20}, \text{ since } H t_1 = H t_{15} \]

\[ H t_1 t_6 t_{11} = H t_{15} t_{20} \Rightarrow H t_1 t_6 t_{11} \in [110], \text{ since } H t_{15} t_{20} \text{ is in } [1 10]. \]

1 symmetric generator will go to [1 10].

\[ H t_1 t_6 t_{12} = H t_{16} t_{12} \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_6 t_{12}, \text{ since } H t_1 t_6 = H t_6 t_1 \]
\[ H t_1 t_6 t_{12} = H t_6 t_{12} \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_{16} t_{12} \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_{16} t_{12}^3 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_{16}^3 e \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]

\[ H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^2 = e x^2 y \]
\[ \Rightarrow y x^{-1} t_1 t_{16} t_{35} = e \]
\[ \Rightarrow y x^{-1} t_1 t_4^6 t_3^9 = e \]
\[ \Rightarrow y x^{-1} t_1 t_4^6 t_3^2 = t_3^2 \]
\[ \Rightarrow y x^{-1} t_1 t_4^6 t_3^7 = t_3^7 \]
\[ \Rightarrow x y^{-1} t_1 x^3 t_1 = x y^{-1} t_3^7 \]
\[ \Rightarrow t_1 = x y^{-1} t_3^7 \]

\[ H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]
\[ \Rightarrow H t_1 t_6 t_{12} = H t_3^4 \]

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^2 = e x^2 y \]
\[ \Rightarrow y^{-1} x^{-1} t_3 t_{20} t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_4^5 t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_1^5t_3^3t_{10}^3 = t_3^{10} \]
\[ \Rightarrow y^{-1}x^{-1}t_1^5 = t_3^{10} \]
\[ Ht_{16}t_{12} = Hy^{-1}x^{-1}t_1^5t_{14}^{10} \]
\[ \Rightarrow Ht_{16}t_{12} = Ht_1^6t_4^4 \]
\[ \Rightarrow Ht_{16}t_{12} = Ht_{33}t_{16} \]
\[ \Rightarrow Ht_{16}t_{12} \in [14], \text{ since } Ht_{33}t_{16} \text{ is in } [14]. \]

1 symmetric generator will go to \([14]\).

\[ Ht_{1}t_{6}t_{13} = Ht_{16}t_{13} \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{6}t_{1}t_{13}, \text{ since } Ht_{1}t_{6} = Ht_{6}t_{1} \]
\[ Ht_{1}t_{6}t_{13} = Ht_{6}t_{1}t_{13} \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{2}^2t_{1}t_{1}^4 \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{2}^2t_{1}^5 \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{2}^2[yxt_{3}^4t_{10}^{10}], \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_{9} = e \]
\[ [x^3t_{11}t_{10}t_{9}]^{xy^{-1}x} = e^{xy^{-1}x} \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}7t_{4}t_{15} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_{4}t_{3}^3 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_{4}t_{3}^3 = t_{3}^7 \]
\[ \Rightarrow x^{-1}y^{-1}t_{1}^5t_{4}t_{3}^3 = t_{3}^{10} \]
\[ \Rightarrow yx^{-1}y^{-1}t_{1}^5 = yxt_{4}t_{1}^{10} \]
\[ \Rightarrow t_{1}^5 = yxt_{4}t_{1}^{10} \]
\[ Ht_{1}t_{6}t_{13} = Ht_{2}^2[yxt_{3}^4t_{10}^{10}] \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{2}^2[yxt_{3}^4t_{10}^{10}] \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{3}^{3}t_{4}^{10} \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{3}^{3}t_{4}^{10} \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} = Ht_{11}t_{40} \]
\[ \Rightarrow Ht_{1}t_{6}t_{13} \in [16], \text{ since } Ht_{11}t_{40} \text{ is in } [16]. \]

1 symmetric generator will go to \([16]\).

\[ Ht_{1}t_{6}t_{14} = Ht_{1}t_{6}t_{14} \]
\[ \Rightarrow Ht_{1}t_{6}t_{14} = Ht_{1}t_{2}t_{1}^4 \]
\[ H_{t_1} t_{6t 14} = H_{t_1} t_2^6 \]
\[ H_{t_1} t_{6t 14} = H_{t_1} [y x^{-1} t_{4t 3}^9], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{xy} \]
\[ \implies y x^{-1} t_{10} t_{35} t_{18} = e \]
\[ \implies y x^{-1} t_{4t 3}^9 t_2^5 = e \]
\[ \implies y x^{-1} t_{4t 3}^9 t_2^5 = t_2^6 \]
\[ \implies y x^{-1} t_{4t 3}^9 = t_2^6 \]

\[ H_{t_1} t_{6t 14} = H_{t_1} y x^{-1} t_{4t 3}^9 \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_1} y x^{-1} [t_1]^{y x^{-1} t_{4t 3}^9} \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_1} t_{4t 3}^9 \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_1} t_{4t 3}^9 \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_1} t_{2t 10} t_3^9 \]
\[ H_{t_32} = H_{t_38} \]
\[ \implies H_{t_4}^3 = H_{t_2}^{10} \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_2}^{10} t_3^9 \]
\[ \implies H_{t_1} t_{6t 14} = H [x^{-1} y t_{4t 3}^9 t_3^9], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{yx} = e^{yx} \]
\[ \implies x^{-1} y t_{36} t_{19} t_2 = e \]
\[ \implies x^{-1} y t_{4t 3}^9 t_2^5 t_3^9 = e \]
\[ \implies x^{-1} y t_{4t 3}^9 t_2^5 t_3^9 = t_2^6 \]
\[ \implies x^{-1} y t_{4t 3}^9 = t_2^6 \]

\[ H_{t_1} t_{6t 14} = H x^{-1} y t_{4t 3}^9 t_3^9 \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_1} t_{4t 3}^9 \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_38} t_{11} \]
\[ \implies H_{t_1} t_{6t 14} = H_{t_34t 11}, \text{ since } H_{t_34} = H_{t_36} \]
\[ H_{t_1} t_{6t 14} = H_{t_34t 11} \]
\[ \implies H_{t_1} t_{6t 14} \in [1 2], \text{ since } H_{t_34t 11} \text{ is in [1 2].} \]

1 symmetric generator will go to [1 2].

\[ H_{t_1} t_{6t 15} = H_{t_1} t_{6t 15} \]
\[ \Rightarrow Ht_{16t15} = Ht_{16t15}^{t_2^4t_3^4} \]
\[ \Rightarrow Ht_{16t15} = Ht_1[x^{-1}y^{-1}t_4t_3^4]t_3^4, \text{ since by Equation } 5.8 \]
\[ x^3t_1t_{10}t_9 = e \]
\[ [x^3t_1t_{10}t_9]^{y^{-1}x} = e^{y^{-1}x} \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_{15}t_{34} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_3^4t_2^9 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_3^4t_2^9 = t_2^2 \]
\[ \Rightarrow x^{-1}y^{-1}t_4t_3^4 = t_2^2 \]
\[ Ht_{16t15} = Ht_1x^{-1}y^{-1}t_4t_3^4t_3^4 \]
\[ \Rightarrow Ht_{16t15} = Hx^{-1}[t_1]^{x^{-1}y^{-1}t_4t_3^8} \]
\[ \Rightarrow Ht_{16t15} = Ht_4^9t_3^4t_3^8 \]
\[ \Rightarrow Ht_{16t15} = Ht_4^{t_3^8t_3^4} \]
\[ \Rightarrow Ht_{16t15} = Ht_4^{t_3^8t_3^4}, \text{ since Equation } 5.8 \]

\[ x^3t_1t_{10}t_9 = e \]
\[ [x^3t_1t_{10}t_9]^x = e^x \]
\[ \Rightarrow x^{-1}t_{12}t_{11}t_{10} = e \]
\[ \Rightarrow x^{-1}t_3^4t_3^8t_2^2 = e \]
\[ \Rightarrow x^{-1}t_3^4t_3^8t_2^2 = t_2^2 \]
\[ \Rightarrow x^{-1}t_3^4t_3^8 = t_2^2 \]
\[ \Rightarrow x^{-1}t_3^4 = t_2^2 \]
\[ Ht_{16t15} = Hx^{-1}t_3^4 \]
\[ \Rightarrow Ht_{16t15} = Ht_{12} \]
\[ \Rightarrow Ht_{16t15} \in [1], \text{ since } Ht_{12} \text{ is in } [1]. \]

1 symmetric generator will go to [1].

\[ Ht_{16t16} = Ht_{16t16} \]
\[ \Rightarrow Ht_{16t16} = Ht_6t_{16}, \text{ since } Ht_6 = Ht_1 \]
\[ Ht_{16t16} = Ht_6t_{11t16} \]
\[ Ht_1 t_{16} = Ht_2^2 t_1 t_4^4 \]
\[ Ht_1 t_{16} = Ht_2^2 [xy^{-1} t_3^2 t_4^4] t_4^4, \] since by Equation 5.8
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^2 = e^2 y \]
\[ \Rightarrow yx^{-1} t_{16} t_{35} = e \]
\[ \Rightarrow yx^{-1} t_4 t_3^9 = e \]
\[ \Rightarrow yx^{-1} t_1 t_4 t_3^9 t_4^2 = t_3^2 \]
\[ \Rightarrow yx^{-1} t_1 t_4 t_3^7 t_4 = t_3^7 \]
\[ \Rightarrow xy^{-1} x^{-1} t_1 = xy^{-1} t_3 t_4^7 \]
\[ \Rightarrow t_1 = xy^{-1} t_3 t_4^7 \]
\[ Ht_1 t_{16} = Ht_2^2 xy^{-1} t_3 t_4^4 \]
\[ \Rightarrow Ht_1 t_{16} = Hxy^{-1} t_2^2 [xy^{-1} t_3^2] \]
\[ \Rightarrow Ht_1 t_{16} = Ht_3^2 t_3 \]
\[ \Rightarrow Ht_1 t_{16} = Ht_3^0 \]
\[ \Rightarrow Ht_1 t_{16} = Ht_3^9 \]
\[ \Rightarrow Ht_1 t_{16} \in [5], \text{ since } Ht_3^9 \text{ is in } [5]. \]

1 symmetric generator will go to [5].
\[H_{t1t6t17} = H_t^7 x^{-1} y^{-1} t_{3}^9 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H x^{-1} y^{-1} [t_{4}^7] x^{-1} y^{-1} t_{3}^9 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H t_{33}^6 t_{3}^0 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H t_{3}^0 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H t_{11}^8 t_{2}^0, \text{ since} \]

\[H_{t29} = H_{t39}\]

\[\Rightarrow H_{t1}^8 = H t_{3}^{10}\]

\[H_{t1t6t17} = H t_{33}^8 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H [x^3 t_{33}^3 t_{2}^3 t_{2}^0], \text{ since Equation 5.8}\]

\[x^3 t_{11} t_{10} t_{9} = e\]

\[\Rightarrow x^3 t_{33}^3 t_{3}^3 t_{2}^1 = e\]

\[\Rightarrow x^3 t_{3}^3 t_{2}^3 t_{2}^1 t_{1}^8 = t_{1}^8\]

\[\Rightarrow x^3 t_{33}^3 t_{2}^3 = t_{1}^8\]

\[H_{t1t6t17} = H x^3 t_{33}^3 t_{2}^3 t_{2}^0\]

\[\Rightarrow H_{t1t6t17} = H t_{3}^3 t_{2}\]

\[\Rightarrow H_{t1t6t17} = H t_{11} t_{2}\]

\[\Rightarrow H_{t1t6t17} = H t_{17} t_{2}, \text{ since } H_{t11} = H_{t17}\]

\[H_{t1t6t17} = H t_{17} t_{2}\]

\[\Rightarrow H_{t1t6t17} \in [110], \text{ since } H_{t17} t_{2} \text{ is in } [10]\]

1 symmetric generator will go to [10].

\[H_{t1t6t18} = H_{t1t6t18}\]

\[\Rightarrow H_{t1t6t18} = H t_{15} t_{6} t_{18}, \text{ since } H_{t1} = H_{t15}\]

\[H_{t1t6t18} = H_{t15t6t18}\]

\[\Rightarrow H_{t1t6t18} = H t_{33} t_{3}^2 t_{1}^5\]

\[\Rightarrow H_{t1t6t18} = H t_{3}^5 t_{2}^5\]

\[\Rightarrow H_{t1t6t18} = H [y x t_{1}^5 t_{2}^2] t_{2}^7, \text{ since Equation 5.8}\]

\[x^3 t_{11} t_{10} t_{9} = e\]

\[[x^3 t_{11} t_{10} t_{9}] y^{-2} = e y^{-2}\]

\[\Rightarrow x^{-1} y^{-1} t_{15} t_{34} t_{17} = e\]

\[\Rightarrow x^{-1} y^{-1} t_{33}^9 t_{2}^5 = e\]

\[\Rightarrow x^{-1} y^{-1} t_{33}^9 t_{2}^5 t_{1}^6 = t_{1}^6\]
\[x^{-1}y^{-1}t_3^4t_1^2 = t_1^6t_1^2\]
\[yx^{-1}x^{-1}t_3^4 = yxt_1^6t_1^2\]
\[t_3^4 = yxt_1^6t_1^2\]
\[H_t_1t_6t_18 = Hyxt_1^6t_2^7t_2^7\]
\[H_t_1t_6t_18 = Ht_1^6t_2^9\]
\[H_t_1t_6t_18 = Ht_21t_34\]
\[H_t_1t_6t_18 = Ht_23t_34, \text{ since } Ht_21 = Ht_23\]
\[H_t_1t_6t_18 = Ht_34t_23, \text{ since } Ht_23t_34 = Ht_34t_23\]
\[H_t_1t_6t_18 = Ht_34t_23\]
\[H_t_1t_6t_18 \in [16], \text{ since } Ht_34t_23 \text{ is in } [1, 16]\]

1 symmetric generator will go to [1, 16].

\[H_t_1t_6t_19 = Ht_1t_6t_19\]
\[H_t_1t_6t_19 = Ht_1^6t_3^5\]
\[H_t_1t_6t_19 = Ht_1[x^{-1}y^{-1}t_4t_3^4]t_3^5, \text{ since by Equation 5.8}\]
\[x^3t_{11}t_{10}t_9 = e\]
\[[x^3t_{11}t_{10}t_9]y^{-1}x = e^{y^{-1}x}\]
\[x^{-1}y^{-1}t_4t_15t_{34} = e\]
\[x^{-1}y^{-1}t_4t_3^4t_2^9 = e\]
\[x^{-1}y^{-1}t_4t_3^4t_2^9t_2^5 = t_2^5\]
\[x^{-1}y^{-1}t_4t_3^4t_2^5 = t_2^5\]
\[H_t_1t_6t_19 = Ht_1x^{-1}y^{-1}t_4t_3^4t_3^5\]
\[H_t_1t_6t_19 = Hx^{-1}y^{-1}[t_1]x^{-1}y^{-1}t_4t_3^9\]
\[H_t_1t_6t_19 = Ht_1^9t_4t_3^9\]
\[H_t_1t_6t_19 = Ht_4^1t_3^9\]
\[H_t_1t_6t_19 = Ht_3^9t_4^0, \text{ since}\]
\[H_{30} = H_{30}\]
\[H_t_2^5 = H_{10}^0\]
\[H_t_1t_6t_19 = H_t_1^5t_3^9\]
\[H_t_1t_6t_19 = H[x^{-1}t_4^3t_3^3]t_3^9, \text{ since by Equation 5.8}\]
\[x^3t_{11}t_{10}t_9 = e\]
\[ x^3 t_{11} t_{10} t_9 \] = e^x
\[ \implies x^{-1} t_{12} t_{11} t_{10} = e \]
\[ \implies x^{-1} t_4 t_3 t_2 = e \]
\[ \implies x^{-1} t_4 t_3 t_2 t_3 = t_2^8 \]
\[ \implies x^{-1} t_4 t_3 = t_2^8 \]
\[ H t_1 t_6 t_{19} = H x^{-1} t_4 t_3 t_0^9 \]
\[ \implies H t_1 t_6 t_{19} = H t_4^3 t_3 \]
\[ \implies H t_1 t_6 t_{19} = H t_{12} t_3 \]
\[ \implies H t_1 t_6 t_{19} = H t_{18} t_3, \text{ since } H t_{12} = H t_{18} \]
\[ H t_1 t_6 t_{19} = H t_{18} t_3 \]
\[ \implies H t_1 t_6 t_{19} \in [110], \text{ since } H t_{18} t_3 \text{ is in } [1\ 10]. \]
1 symmetric generator will go to [1\ 10].

\[ H t_1 t_6 t_{20} = H t_1 t_6 t_{20} \]
\[ \implies H t_1 t_6 t_{20} = H t_6 t_1 t_{20}, \text{ since } H t_1 t_6 = H t_6 t_1 \]
\[ H t_1 t_6 t_{20} = H t_6 t_1 t_{20} \]
\[ \implies H t_1 t_6 t_{20} = H t_2^3 t_1 t_4^5 \]
\[ \implies H t_1 t_6 t_{20} = H t_2^3 [x y^{-1} t_3 t_4^7] t_4^5, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] t^2 y = e^x y \]
\[ \implies y x^{-1} t_1 t_{16} t_{35} = e \]
\[ \implies y x^{-1} t_1 t_4 t_3^9 = e \]
\[ \implies y x^{-1} t_1 t_4 t_3^9 = t_2^3 \]
\[ \implies y x^{-1} t_4 t_3^7 = t_4^7 \]
\[ \implies x y^{-1} y x^{-1} t_1 = x y^{-1} t_3 t_4^7 \]
\[ \implies t_1 = x y^{-1} t_3 t_4^7 \]
\[ H t_1 t_6 t_{20} = H t_2^3 x y^{-1} t_3 t_4 \]
\[ \implies H t_1 t_6 t_{20} = H x y^{-1} t_2^3 x y^{-1} t_3 t_4 \]
\[ \implies H t_1 t_6 t_{20} = H t_3^3 t_4 \]
\[ \implies H t_1 t_6 t_{20} = H t_3^3 t_4 \]
\[ \implies H t_1 t_6 t_{20} = H [y^{-1} x^{-1} t_1 t_4^5] t_4, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
$\left[x^3 t_{11} t_{10} t_9\right] x^2 y^{-1} = e x^2 y^{-1}$

$\Rightarrow y^{-1} x^{-1} t_{33} t_{20} t_3 = e$

$\Rightarrow y^{-1} x^{-1} t_{10}^5 t_4 t_3 = e$

$\Rightarrow y^{-1} x^{-1} t_{10}^5 t_4 t_3 t_3 t_3 = t_3^{10}$

$\Rightarrow y^{-1} x^{-1} t_{10}^5 t_4 = t_3^{10}$

$H t_{16} t_{20} = H y^{-1} x^{-1} t_{10}^5 t_4$

$\Rightarrow H t_{16} t_{20} = H t_{16}^6$

$\Rightarrow H t_{16} t_{20} = H t_{33} t_{24}$

$\Rightarrow H t_{16} t_{20} = H t_{35} t_{24}$

$\Rightarrow H t_{16} t_{20} \in [16]$, since $H t_{35} t_{24}$ is in [1 6].

1 symmetric generator will go to [1 6].

$H t_{16} t_{21} = H t_{16} t_{21}$

$\Rightarrow H t_{16} t_{21} = H t_{6} t_{16} t_{21}$, since $H t_{16} t_{6} = H t_{6} o t_{21}$

$\Rightarrow H t_{16} t_{21} = H t_{6} t_{16} t_{21}$

$\Rightarrow H t_{16} t_{21} = H t_{28} t_{16} t_{21}$, since $H t_{6} = H t_{28}$

$H t_{16} t_{21} = H t_{16}^6 t_{16} t_{1}$

$\Rightarrow H t_{16} t_{21} = H t_{16}^7 t_{16}^7$

$\Rightarrow H t_{16} t_{21} = H [y x^{-1} t_{10}^5 t_1] t_1^7$, since by Equation 5.8

$x^3 t_{11} t_{10} t_9 = e$

$[x^3 t_{11} t_{10} t_9] x^{-1} y = e x^{-1} y$

$\Rightarrow y x^{-1} t_{18} t_1 t_{16} = e$

$\Rightarrow y x^{-1} t_{2} t_{16}^5 t_4 = e$

$\Rightarrow y x^{-1} t_{2} t_{16}^5 t_4 t_4 t_4 t_4 = t_4^7$

$\Rightarrow y x^{-1} t_{2} t_1 = t_4^7$

$H t_{16} t_{21} = H y x^{-1} t_{10}^5 t_1 t_7$

$\Rightarrow H t_{16} t_{21} = H t_{16}^5 t_1$

$\Rightarrow H t_{16} t_{21} = H t_{16}^3 t_1$, since

$H t_{16} = H t_{18}$

$\Rightarrow H t_{16} = H t_{18}$

$H t_{16} t_{21} = H t_{16}^3 t_1$
\( H_{t1t6t21} = Ht_{t1}^{3}[x^{-1} t_{3}^{3} t_{2}^{3}] \), since by Equation 5.8

\( x^{3}t_{11}t_{10}t_{9} = e \)

\[ \Rightarrow x^{3} t_{3}^{3} t_{2}^{3} t_{1}^{1} = e \]

\[ \Rightarrow x^{3} t_{3}^{3} t_{2}^{3} t_{1}^{8} = t_{1}^{8} \]

\[ \Rightarrow x^{3} t_{3}^{3} t_{2}^{3} = t_{1}^{8} \]

\( H_{t1t6t21} = Ht_{4}^{3} x^{3} t_{3}^{3} t_{2}^{3} \)

\[ \Rightarrow H_{t1t6t21} = H x^{3} t_{4}^{3} t_{3}^{3} t_{2}^{3} \]

\[ \Rightarrow H_{t1t6t21} = H t_{3}^{3} t_{3}^{3} t_{2}^{3} \]

\[ \Rightarrow H_{t1t6t21} = H t_{6}^{6} t_{3}^{3} \]

\[ \Rightarrow H_{t1t6t21} = H t_{1}^{6} t_{2}^{3} \]

\[ \Rightarrow H_{t21} = H t_{23} \]

\[ \Rightarrow H t_{1}^{6} = H t_{3}^{6} \]

\[ \Rightarrow H_{t1t6t21} = H t_{1}^{6} t_{2}^{3} \]

\( H_{t1t6t21} = H[x^{-1} y^{-1} t_{3}^{3} t_{2}^{3} t_{2}^{3}] \), since by Equation 5.8

\( x^{3}t_{11}t_{10}t_{9} = e \)

\[ [x^{3}t_{11}t_{10}t_{9}] y^{-2} = e y^{-2} \]

\[ \Rightarrow x^{-1} y^{-1} t_{15} t_{34} t_{17} = e \]

\[ \Rightarrow x^{-1} y^{-1} t_{3}^{3} t_{2}^{3} t_{1}^{1} = e \]

\[ \Rightarrow x^{-1} y^{-1} t_{3}^{3} t_{2}^{3} t_{1}^{8} = t_{1}^{8} \]

\[ \Rightarrow x^{-1} y^{-1} t_{3}^{3} t_{2}^{3} = t_{1}^{8} \]

\( H_{t1t6t21} = H x^{-1} y^{-1} t_{3}^{3} t_{2}^{3} t_{2}^{3} \)

\[ \Rightarrow H_{t1t6t21} = H t_{3}^{3} t_{2}^{3} \]

\[ \Rightarrow H_{t1t6t21} = H t_{15} t_{2}^{3} \]

\[ \Rightarrow H_{t1t6t21} = H t_{12}^{3} t_{2} \]

\[ \Rightarrow H_{t1t6t21} = H t_{1} t_{2} \]

\[ \Rightarrow H_{t1t6t21} \in [12], \text{ since } H t_{1} t_{2} \text{ is in [1 2].} \]

1 symmetric generator will go to [1 2].
\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{t_1 [x^{-1} t^{3}_{4,3}]} , \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9]^x = e^x \]

\[ \Rightarrow x^{-1} t_{12} t_{11} t_{10} = e \]

\[ \Rightarrow x^{-1} t^3_{4,3} t^3_{2} = e \]

\[ \Rightarrow x^{-1} t^3_{4,3} t^3_{2} t^8_2 = t^8_2 \]

\[ \Rightarrow x^{-1} t^3_{4,3} = t^8_2 \]

\[ H_{t_1 t_6 t_{22}} = H_{t_1 x^{-1} t^{3}_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{x^{-1} [t_1]^{-1} t^3_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{t_4 t^3_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{t_1 t^3_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{t_1 t_{11}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} = H_{t_2 t_{11}} , \text{ since } H_{t_2} = H_{t_{16}} \]

\[ H_{t_1 t_6 t_{22}} = H_{t_2 t_{11}} \]

\[ \Rightarrow H_{t_1 t_6 t_{22}} \in [1 10] , \text{ since } H_{t_2 t_{11}} \text{ is in } [1 10] . \]

1 symmetric generator will go to \([1 10]\).

\[ H_{t_1 t_6 t_{23}} = H_{t_1 t_6 t_{23}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t_1 t^2_{2} t^6_3} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t_1 [x^{-1} y^{-1} t^{4}_{4,3} t^6_3]} , \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9]^{-1} x = e^{-1} x \]

\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{15} t_{34} = e \]

\[ \Rightarrow x^{-1} y^{-1} t_{4} t_{3} t^9_2 = e \]

\[ \Rightarrow x^{-1} y^{-1} t_{4} t^9_3 t^9_2 t^2_2 = t^8_2 \]

\[ \Rightarrow x^{-1} y^{-1} t_{4} t^9_3 = t^8_2 \]

\[ H_{t_1 t_6 t_{23}} = H_{t_1 x^{-1} y^{-1} t^{4}_{4,3} t^{10}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{x^{-1} y^{-1} [t_1]^{-1} y^{-1} t^{4}_{4,3} t^{10}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t^9_{4,3} t^{10}_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t^{10}_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t^{10}_{4,3}} \]

\[ \Rightarrow H_{t_1 t_6 t_{23}} = H_{t^{8}_{2,3}} , \text{ since} \]
\[ H_{t30} = H_{t40} \]
\[ \Rightarrow H_{t2}^5 = H_{t4}^{10} \]
\[ H_{t16} t_{23} = H_{t2}^{10} t_{3}^{10} \]
\[ H_{t16} t_{23} = H_{t2}^{5} [y^{-1} x^{-1} t_{4}^{9} t_{4}^{5}], \text{ since Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] x^2 y^{-1} = e x^2 y^{-1} \]
\[ \Rightarrow y^{-1} x^{-1} t_{33} t_{20} t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_{14}^{9} t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_{14}^{9} t_3 = e \]
\[ \Rightarrow y^{-1} x^{-1} t_{14}^{9} t_3 = e \]
\[ H_{t16} t_{23} = H_{t2}^{5} y^{-1} x^{-1} t_{14}^{9} \]
\[ \Rightarrow H_{t16} t_{23} = H_{y}^{-1} x^{-1} [t_2^{9}] y^{-1} x^{-1} t_{14}^{9} \]
\[ H_{t16} t_{23} = H_{t11} t_{4}^{10} \]
\[ H_{t16} t_{23} = H_{t11} t_{20} \]
\[ \Rightarrow H_{t16} t_{23} \in [12], \text{ since } H_{t11} t_{20} \text{ is in [1 2]}. \]

1 symmetric generator will go to [1 2].
\[ \Rightarrow t_1 = xy^{-1}t_2^3 t_4^7 \]

\[ Ht_{1} t_{6} t_{24} = Ht_2^2 xy^{-1}t_2^3 t_4^7 t_4^6 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Hxy^{-1}[t_2^2] x y^{-1} t_3^2 t_4^7 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^2 t_4^7 t_4^1 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^2 t_4^2 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^{10} t_4^2 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^{10}[x^{-1}yt_2^4] \], since by Equation 5.8

\[ x^3 t_{11} t_{10} t_{9} = e \]

\[ [x^3 t_{11} t_{10} t_{9}] y^{-1} x^{-1} = e y^{-1} x^{-1} \]

\[ \Rightarrow x^{-1}yt_2 t_{13} t_{36} = e \]

\[ \Rightarrow x^{-1}yt_2 t_{13} t_{49} = e \]

\[ \Rightarrow x^{-1}yt_2 t_{13} t_{49}^2 = t_4^2 \]

\[ \Rightarrow x^{-1}yt_2 t_{13} t_{49} = t_4^2 \]

\[ Ht_{1} t_{6} t_{24} = Ht_3^{10} x^{-1}yt_2 t_{13}^4 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^{10} x^{-1}yt_2 t_{13}^4 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Hx^{-1}yt_2 t_{13}^4[x^{-1}yt_2 t_{13}^4] \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_2^2 t_{13}^4 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^{10} t_4^1 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_3^{10} t_4^1 \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_{13} t_{13} \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} = Ht_{20 t_{13}}, \text{ since } Ht_{13} = Ht_{20} \]

\[ Ht_{1} t_{6} t_{24} = Ht_{20 t_{13}} \]

\[ \Rightarrow Ht_{1} t_{6} t_{24} \in [12], \text{ since } Ht_{20 t_{13}} \text{ is in } [12]. \]

1 symmetric generator will go to [12].
\[ \implies x^3 t_3^2 t_1^2 = e \]
\[ \implies x^3 t_3^2 t_2^3 t_1^8 = t_1^8 \]
\[ \implies x^3 t_3^2 t_2^3 = t_1^8 \]

\[ H t_1 t_6 t_{25} = H t_1^4 x^3 t_3^2 t_2^3 \]
\[ \implies H t_1 t_6 t_{25} = H x^3 [t_1^7] x^3 t_3^2 t_2^3 \]
\[ \implies H t_1 t_6 t_{25} = H t_3^2 t_2^3 \]
\[ \implies H t_1 t_6 t_{25} = H t_3^3 t_1^2 \]
\[ \implies H t_1 t_6 t_{25} = H t_{39} t_{10} \]
\[ \implies H t_1 t_6 t_{25} = H t_{10} t_{39}, \text{ since } H t_{10} t_{39} = H t_{39} t_{10} \]
\[ H t_1 t_6 t_{25} = H t_{10} t_{39} \]
\[ \implies H t_1 t_6 t_{25} \in [16], \text{ since } H t_{10} t_{39} \text{ is in [1 6].} \]

1 symmetric generator will go to [1 6].

\[ H t_1 t_6 t_{26} = H t_1 t_6 t_{26} \]
\[ \implies H t_1 t_6 t_{26} = H t_{15} t_6 t_{26}, \text{ since } H t_1 = H t_{15} \]
\[ H t_1 t_6 t_{26} = H t_{15} t_6 t_{26} \]
\[ \implies H t_1 t_6 t_{26} = H t_3^4 t_2^7 t_1^2 \]
\[ \implies H t_1 t_6 t_{26} = H t_3^4 t_2^9 \]
\[ \implies H t_1 t_6 t_{26} = H [y x t_1^6 t_2^9] t_2^9, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^{-2} = e^y^{-2} \]
\[ \implies x^{-1} y^{-1} t_{15} t_{34} t_{17} = e \]
\[ \implies x^{-1} y^{-1} t_3^4 t_2^6 t_1^5 = e \]
\[ \implies x^{-1} y^{-1} t_3^4 t_2^6 t_1^5 = e \]
\[ \implies x^{-1} y^{-1} t_3^4 t_2^6 t_1^5 = e \]
\[ \implies y x y^{-1} x^{-1} t_3^4 = y x t_1^6 t_2^9 \]
\[ \implies t_3^4 = y x t_1^6 t_2^9 \]
\[ H t_1 t_6 t_{26} = H y x t_1^6 t_2^9 \]
\[ \implies H t_1 t_6 t_{26} = H t_1^6 \]
\[ \implies H t_1 t_6 t_{26} = H t_{21} \]
\[ \implies H t_1 t_6 t_{26} \in [5], \text{ since } H t_{21} \text{ is in [5].} \]

1 symmetric generator will go to [5].
\( H_{t_1t_6t_{27}} = H_{t_1t_6t_{27}} \)

\[ \Rightarrow H_{t_1t_6t_{27}} = H_{t_1t_6t_{27}} \]

\[ \Rightarrow H_{t_1t_6t_{27}} = H_{t_1t_6t_{27}} \]

\[ x^3t_{11}t_{10}t_9 = e \]

\[ [x^3t_{11}t_{10}t_9]^{-1}x = e^{-1}x \]

\[ \Rightarrow x^{-1}y^{-1}t_{4}t_{15}t_{34} = e \]

\[ \Rightarrow x^{-1}y^{-1}t_{4}t_{15}t_{34} = e \]

\[ \Rightarrow x^{-1}y^{-1}t_{4}t_{15}t_{34} = e \]

\[ \Rightarrow x^{-1}y^{-1}t_{4}t_{15}t_{34} = e \]

\[ H_{t_1t_6t_{27}} = H_{t_1x^{-1}y^{-1}t_{4}t_{15}t_{34}} \]

\[ \Rightarrow H_{t_1t_6t_{27}} = H_{x^{-1}y^{-1}t_{4}t_{15}t_{34}} \]

\[ H_{t_1t_6t_{27}} = H_{t_1t_6t_{27}} \]

\[ \Rightarrow H_{t_1t_6t_{27}} = H_{t_1t_6t_{27}} \]

\[ H_{t_1t_6t_{28}} = H_{t_6t_6t_{28}} \]

\[ \Rightarrow H_{t_1t_6t_{28}} = H_{t_6t_6t_{28}} \]

\[ H_{t_1t_6t_{28}} = H_{t_6t_6t_{28}} \]

\[ x^3t_{11}t_{10}t_9 = e \]

\[ [x^3t_{11}t_{10}t_9]^2 = e^2 \]

\[ \Rightarrow yx^{-1}t_{16}t_{35} = e \]

\[ \Rightarrow yx^{-1}t_{16}t_{35} = e \]

\[ \Rightarrow yx^{-1}t_{16}t_{35} = e \]

\[ \Rightarrow yx^{-1}t_{16}t_{35} = e \]

\[ \Rightarrow t_1 = xy^{-1}t_4^{-1}t_9^{-1}t_3^{-1}t_4 \]

\[ H_{t_1t_6t_{28}} = H_{t_2xy^{-1}t_4^{-1}t_9^{-1}t_3^{-1}t_4} \]

\[ \Rightarrow H_{t_1t_6t_{28}} = H_{xy^{-1}[t_2^{-1}xy^{-1}t_3^{-1}t_4]} \]
\[ H t_{16} t_{28} = H t_{3} t_{4} \]
\[ H t_{16} t_{28} = H t_{3} t_{4} \]
\[ H t_{16} t_{28} = H t_{1} t_{4} \], since

\[ H t_{29} t_{39} \]
\[ H t_{1} = H t_{3} \]
\[ H t_{16} t_{28} = H t_{1} t_{4} \]

\[ H t_{16} t_{28} = H t_{1} t_{4} \]

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^x = e^x \]
\[ \implies x^{-1} t_{12} t_{11} t_{10} = e \]
\[ \implies x^{-1} t_{4} t_{3} t_{2} = e \]
\[ \implies x^{-1} t_{4} t_{3} t_{2} = e \]
\[ \implies x^{-1} t_{4} t_{3} t_{2} = e \]
\[ \implies x^{-1} t_{4} t_{3} t_{2} = e \]
\[ \implies x^{-1} t_{4} t_{3} t_{2} = e \]
\[ \implies t_{4} = x t_{2} t_{3} \]

\[ H t_{16} t_{28} = H t_{1} t_{4} x t_{2} t_{3} \]
\[ \implies H t_{16} t_{28} = H x t_{1} t_{2} t_{3} \]
\[ \implies H t_{16} t_{28} = H t_{1} t_{3} t_{2} t_{3} \]
\[ \implies H t_{16} t_{28} = H t_{1} t_{3} t_{2} t_{3} \]
\[ \implies H t_{16} t_{28} = H t_{18} t_{31} \]
\[ \implies H t_{16} t_{28} \in [16], \] since \( H t_{18} t_{31} \) is in \([16]\).

1 symmetric generator will go to \([16]\).

\[ H t_{16} t_{29} = H t_{16} t_{29} \]
\[ \implies H t_{16} t_{29} = H t_{6} t_{1} t_{29}, \] since \( H t_{16} = H t_{6} t_{1} \)
\[ H t_{16} t_{29} = H t_{6} t_{1} t_{29} \]
\[ H t_{16} t_{29} = H t_{28} t_{1} t_{29}, \] since \( H t_{6} = H t_{28} \)
\[ H t_{16} t_{29} = H t_{28} t_{1} t_{29} \]
\[ \implies H t_{16} t_{29} = H t_{1} t_{1} t_{1} t_{1} \]
\[ \implies H t_{16} t_{29} = H t_{1} t_{1} t_{1} t_{1} \]
\[ \implies H t_{16} t_{29} = H [y x^{-1} t_{2} t_{1}] t_{1} t_{1}, \] since by Equation 5.8
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{-1} y = e^{-1} y \]
\[ \Rightarrow yx^{-1}t_{18}t_{16} = e \]
\[ \Rightarrow yx^{-1}t_{5}t_{4}^4 = e \]
\[ \Rightarrow yx^{-1}t_{5}t_{4}^4t_{17}^7 = t_{4}^7 \]
\[ \Rightarrow yx^{-1}t_{5}t_{1} = t_{4}^7 \]
\[ Ht_1t_6t_{29} = Hyx^{-1}t_{5}t_{1}^9 \]
\[ \Rightarrow Ht_1t_6t_{29} = Ht_{5}t_{1}^{10} \]
\[ \Rightarrow Ht_1t_6t_{29} = Ht_{18}t_{37} \]
\[ \Rightarrow Ht_1t_6t_{29} = Ht_{12}t_{37}, \text{ since } Ht_{12} = Ht_{18} \]
\[ Ht_1t_6t_{29} = Ht_{12}t_{37} \]
\[ \Rightarrow Ht_1t_6t_{29} \in [16], \text{ since } Ht_{12}t_{37} \text{ is in } [1, 6]. \]
\[ \Rightarrow Ht_1t_6t_{29} = Ht_{2}t_{1}^{9} \]

1 symmetric generator will go to [1, 6].

\[ Ht_1t_6t_{30} = Ht_{1}t_6t_{30} \]
\[ \Rightarrow Ht_1t_6t_{30} = Ht_{15}t_6t_{30}, \text{ since } Ht_{1} = Ht_{15} \]
\[ Ht_1t_6t_{30} = Ht_{15}t_6t_{30} \]
\[ \Rightarrow Ht_1t_6t_{30} = Ht_{3}t_{2}^{8} \]
\[ \Rightarrow Ht_1t_6t_{30} = Ht_{3}t_{2}^{10} \]
\[ \Rightarrow Ht_1t_6t_{30} = H[yxt_{1}^{6}t_{2}]t_{1}^{10}, \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]y^{-2} = e^{y^{-2}} \]
\[ \Rightarrow x^{-1}y^{-1}t_{15}t_{34}t_{17} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{3}^{4}t_{2}^{5} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{3}^{4}t_{2}^{5}t_{1} = t_{2}^{6} \]
\[ \Rightarrow x^{-1}y^{-1}t_{3}^{4}t_{2}^{5}t_{1} = t_{1}^{6} \]
\[ \Rightarrow yxy^{-1}x^{-1}t_{3}^{4} = yxt_{1}^{6}t_{2}^{6} \]
\[ \Rightarrow t_{3}^{4} = yxt_{1}^{6}t_{2}^{6} \]
\[ Ht_1t_6t_{30} = Hyxt_{1}^{6}t_{2}^{6}t_{1}^{10} \]
\[ \Rightarrow Ht_1t_6t_{30} = Ht_{1}^{6}t_{2} \]
\[ \Rightarrow Ht_1t_6t_{30} = Ht_{1}^{6}[y^{-1}xt_{1}^{6}t_{2}], \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]y^{-1}x^{-1} = e^{y^{-1}x^{-1}} \]
\[ \implies x^{-1}yt_2t_{13}t_{36} = e \]
\[ \implies x^{-1}yt_2t_{14}^{t_9} = e \]
\[ \implies x^{-1}yt_2t_{14}^{t_9}t_{14}^{t_2} = t_{14}^{2} \]
\[ \implies x^{-1}yt_2t_{14}^{t_9} = t_{14}^{2}t_{14}^{t_1} \]
\[ \implies y^{-1}xt^{-1} = t_{14}^{2}t_{14}^{t_1} \]
\[ \implies t_2 = y^{-1}xt_{14}^{2}t_{14}^{t_1} \]
\[ Ht_{14}t_{30} = Ht_{14}^{0}y^{-1}xt_{14}^{2}t_{14}^{t_1} \]
\[ \implies Ht_{14}t_{6}t_{30} = Ht_{14}^{0}y^{-1}xt_{14}^{2}t_{14}^{t_1} \]
\[ \implies Ht_{14}t_{6}t_{30} = Ht_{14}^{0}t_{14}^{t_7} \]
\[ \implies Ht_{14}t_{6}t_{30} = Ht_{14}^{t_7}t_{14}^{t_1} \]
\[ \implies Ht_{14}t_{6}t_{30} = Ht_{14}^{t_1}t_{16}t_{25} \]
\[ \implies Ht_{14}t_{6}t_{30} \in [16], \text{ since } Ht_{14}t_{6}t_{25} \text{ is in } [16]. \]

1 symmetric generator will go to [1 6].

\[
Ht_{14}t_{6}t_{31} = Ht_{14}t_{6}t_{31}
\]
\[ \implies Ht_{14}t_{6}t_{31} = Ht_{14}^{0}t_{14}^{t_8} \]
\[ \implies Ht_{14}t_{6}t_{31} = Ht_{14}[x^{-1}y^{-1}t_{4}t_{3}^{t_9}]t_{3}^{t_8}, \text{ since by Equation 5.8} \]
\[ x^{3}t_{14}t_{10}t_{9} = e \]
\[ [x^{3}t_{14}t_{10}t_{9}]^{-1} = e^{x^{-1}} \]
\[ \implies x^{-1}y^{-1}t_{4}t_{15}t_{34} = e \]
\[ \implies x^{-1}y^{-1}t_{4}t_{3}^{t_9}t_{14}^{t_2} = e \]
\[ \implies x^{-1}y^{-1}t_{4}t_{3}^{t_9}t_{14}^{t_2} = t_{14}^{2} \]
\[ \implies x^{-1}y^{-1}t_{4}t_{3}^{t_9}t_{14}^{t_2} = t_{14}^{2} \]
\[ Ht_{14}t_{6}t_{31} = Ht_{14}x^{-1}y^{-1}t_{4}t_{3}^{t_9} \]
\[ \implies Ht_{14}t_{6}t_{31} = Hx^{-1}y^{-1}[t_{4}]x^{-1}y^{-1}t_{4}t_{3} \]
\[ \implies Ht_{14}t_{6}t_{31} = Ht_{14}^{0}t_{4}t_{3} \]
\[ \implies Ht_{14}t_{6}t_{31} = Ht_{14}^{0}t_{3} \]
\[ \implies Ht_{14}t_{6}t_{31} = Ht_{14}^{8}t_{3}, \text{ since} \]
\[ Ht_{30} = Ht_{40} \]
\[ \implies Ht_{14}^{8}t_{3} = Ht_{14}^{0} \]
\[ Ht_{14}t_{6}t_{31} = Ht_{14}^{8}t_{3} \]
\[ \implies Ht_{14}t_{6}t_{31} = H[x^{-1}t_{4}t_{3}^{t_9}]t_{3}, \text{ since by Equation 5.8} \]
\[x^3t_{11}t_{10}t_9 = e\]
\[[x^3t_{11}t_{10}t_9]^x = e^x\]
\[\implies x^{-1}t_{12}t_{11}t_{10} = e\]
\[\implies x^{-1}t_{4}^3t_{3}^3t_{2}^2 = e\]
\[\implies x^{-1}t_{4}^3t_{3}^3t_{2}^2 = t_8^2\]
\[\implies x^{-1}t_{4}^3t_{3}^3 = t_8^2\]
\[Ht_{1}t_{6}t_{31} = Hx^{-1}t_{4}^3t_{3}t_{3}\]
\[\implies Ht_{1}t_{6}t_{31} = Ht_{4}^3t_{3}\]
\[\implies Ht_{1}t_{6}t_{31} = Ht_{12}t_{15}\]
\[\implies Ht_{1}t_{6}t_{31} = Ht_{18}t_{15}, \text{ since } Ht_{12} = Ht_{18}\]
\[Ht_{1}t_{6}t_{31} = Ht_{18}t_{15}\]
\[\implies Ht_{1}t_{6}t_{31} \in [12], \text{ since } Ht_{18}t_{15} \text{ is in } [12].\]
1 symmetric generator will go to [1 2].

\[Ht_{1}t_{6}t_{32} = Ht_{1}t_{6}t_{32}\]
\[\implies Ht_{1}t_{6}t_{32} = Ht_{6}t_{1}t_{32}, \text{ since } Ht_{1}t_{6} = Ht_{6}t_{1}\]
\[Ht_{1}t_{6}t_{32} = Ht_{6}t_{1}t_{32}\]
\[\implies Ht_{1}t_{6}t_{32} = Ht_{2}^3t_{1}t_{4}^8\]
\[\implies Ht_{1}t_{6}t_{32} = Ht_{2}^3[xy^{-1}t_{4}^3t_{4}^7]t_{4}^8, \text{ since by Equation 5.8}\]
\[x^3t_{11}t_{10}t_9 = e\]
\[[x^3t_{11}t_{10}t_9]^y = e^x\]
\[\implies xy^{-1}t_{1}t_{16}t_{35} = e\]
\[\implies xy^{-1}t_{1}t_{4}^9t_{3}^3 = e\]
\[\implies xy^{-1}t_{1}t_{4}^9t_{3}^3 = t_2^2\]
\[\implies yx^{-1}t_{1}t_{4}^7t_{3}^3 = t_2^2t_{4}^7\]
\[\implies xy^{-1}t_{1}t_{16}t_{35} = xy^{-1}t_{4}^3t_{4}^7\]
\[\implies t_1 = xy^{-1}t_{4}^3t_{4}^7\]
\[Ht_{1}t_{6}t_{32} = Ht_{2}^3xy^{-1}t_{4}^3t_{4}^7t_{4}^8\]
\[\implies Ht_{1}t_{6}t_{32} = Hxy^{-1}[t_{2}^2]xy^{-1}t_{3}^3t_{4}^4\]
\[\implies Ht_{1}t_{6}t_{32} = Ht_{3}^3t_{4}^4\]
\[\implies Ht_{1}t_{6}t_{32} = Ht_{4}^3t_{4}^4\]
\[\implies Ht_{1}t_{6}t_{32} = H[y^{-1}x^{-1}t_{1}t_{4}^3t_{4}^4], \text{ since by Equation 5.8}\]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^x^{-1} = e^x^{-1} \]
\[ \implies y^{-1}x^{-1}t_{33}t_2t_3 = e \]
\[ \implies y^{-1}x^{-1}t^5_{14}t_3 = e \]
\[ \implies y^{-1}x^{-1}t^5_{14}t^3_{33} = t^{10}_{33} \]
\[ \implies y^{-1}x^{-1}t^5_{14} = t^{10}_{33} \]
\[ Ht_{14}t_{32} = H y^{-1} x^{-1} t^5_{14} t^4 \]
\[ \implies H t_{14} t_{6} t_{32} = H t^9_{14} t^4 \]
\[ \implies H t_{14} t_{6} t_{32} = H t_{33} t_{36} \]
\[ \implies H t_{14} t_{6} t_{32} = H t_{35} t_{36} \]
\[ \implies H t_{14} t_{6} t_{32} \in [110], \text{ since } H t_{35} t_{36} \text{ is in [1 10].} \]

1 symmetric generator will go to [1 10].

\[ H t_{14} t_{6} t_{33} = H t_{14} t_{6} t_{33} \]
\[ \implies H t_{14} t_{6} t_{30} = H t_{6} t_{1} t_{33}, \text{ since } H t_{14} t_{6} = H t_{6} t_{1} \]
\[ H t_{14} t_{6} t_{33} = H t_{6} t_{1} t_{33} \]
\[ \implies H t_{14} t_{6} t_{33} = H t_{28} t_{1} t_{33}, \text{ since } H t_{6} = H t_{28} \]
\[ H t_{14} t_{6} t_{33} = H t_{28} t_{1} t_{33} \]
\[ \implies H t_{14} t_{6} t_{33} = H t^7_{14} t^0_{11} \]
\[ \implies H t_{14} t_{6} t_{33} = H [y x^{-1} t^5_{2} t^4_{1}], \text{ since Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^x^{-1} = e^x^{-1} \]
\[ \implies y x^{-1} t_{18} t_{16} = e \]
\[ \implies y x^{-1} t^5_{2} t^4_{1} = e \]
\[ \implies y x^{-1} t^5_{2} t^4_{1} t^7_{33} = t^7_{33} \]
\[ \implies y x^{-1} t^5_{2} t^4_{1} = t^7_{33} \]
\[ H t_{14} t_{6} t_{33} = H y x^{-1} t^5_{2} t^4_{1}, \text{ since Equation 5.8} \]
\[ \implies H t_{14} t_{6} t_{33} = H t^5_{4} \]
\[ \implies H t_{14} t_{6} t_{33} = H t_{18} \]
\[ \implies H t_{14} t_{6} t_{33} \in [1], \text{ since } H t_{18} \text{ is in [1].} \]

1 symmetric generator will go to [1].
\[ H_t t_6 t_{34} = H_t t_6 t_{34} \]
\[ \implies H_t t_6 t_{34} = H_t t_2 t_{30}^0 \]
\[ \implies H_t t_6 t_{34} = H_t t_{1} \]
\[ \implies H_t t_6 t_{34} \in [1], \text{ since } H_t \text{ is in [1].} \]
1 symmetric generator will go to [1].

\[ H_t t_6 t_{35} = H_t t_6 t_{35} \]
\[ \implies H_t t_6 t_{34} = H_t t_2 t_{30}^0 \]
\[ \implies H_t t_6 t_{34} = H_t t_1 t_4 t_3^2 t_{3}^0 \text{, since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}] y^{-1} x = e y^{-1} x \]
\[ \implies x^{-1} y^{-1} t_4 t_1 t_5 t_{34} = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^2 t_{2}^0 = e \]
\[ \implies x^{-1} y^{-1} t_4 t_3^2 t_{2}^0 t_{2}^0 = t_{2}^0 \]
\[ \implies x^{-1} y^{-1} t_4 t_{3}^4 = t_{2}^0 \]
\[ H_t t_6 t_{34} = H_t x^{-1} y^{-1} t_4 t_3^2 t_{3}^0 \]
\[ \implies H_t t_6 t_{34} = H x^{-1} y^{-1} [t_1] x^{-1} y^{-1} t_4 t_{3}^2 \]
\[ \implies H_t t_6 t_{34} = H t_{3}^4 t_{4} t_{3}^2 \]
\[ \implies H_t t_6 t_{34} = H t_{4}^{10} t_{3}^2 \]
\[ \implies H_t t_6 t_{34} = H t_{8}^{8} t_{3}^2, \text{ since} \]
\[ H t_{30} = H t_{40} \]
\[ \implies H t_{8}^{2} = H t_{4}^{10} \]
\[ H t_{1} t_{6} t_{34} = H t_{8}^{8} t_{3}^2 \]
\[ H t_{1} t_{6} t_{34} = H t_{8}^{8} [y x^{-1} t_1 t_4^4], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}] x^2 y = e x^2 y \]
\[ \implies y x^{-1} t_1 t_6 t_{35} = e \]
\[ \implies y x^{-1} t_1 t_4 t_{3}^2 = e \]
\[ \implies y x^{-1} t_1 t_4 t_{3}^2 t_{2}^0 = t_{2}^0 \]
\[ \implies y x^{-1} t_4 t_{3}^4 = t_{3}^4 \]
\[ \implies H_t t_6 t_{34} = H t_{8}^{8} y x^{-1} t_1 t_4^4 \]
\[ H_{t_1t_6t_{34}} = H_{y^{-1}[t_2^2]y^{-1}t_1t_4^2} \]

\[ H_{t_1t_6t_{34}} = H_{t_1^6t_1t_4^4} \]

\[ H_{t_1t_6t_{34}} = H_{t_1^7t_1t_4^4} \]

\[ H_{t_1t_6t_{34}} = H_{t_2^5t_{40}} \]

\[ H_{t_1t_6t_{34}} = H_{t_{40}t_{25}}, \text{ since } H_{t_{25}t_{40}} = H_{t_{40}} = H_{t_{25}} \]

\[ H_{t_1t_6t_{34}} \in [16], \text{ since } H_{t_{40}t_{25}} \text{ is in } [16]. \]

1 symmetric generator will go to [16].

\[ H_{t_1t_6t_{36}} = H_{t_1t_6t_{36}} \]

\[ H_{t_1t_6t_{36}} = H_{t_6t_1t_{36}}, \text{ since } H_{t_1t_6} = H_{t_6t_1} \]

\[ H_{t_1t_6t_{36}} = H_{t_2^2t_1t_4^0} \]

\[ H_{t_1t_6t_{36}} = H_{t_2^2t_1t_4^0} \]

\[ H_{t_1t_6t_{36}} = H_{t_{2}|xy^{-1}[t_2^2t_3^4]t_4^0}, \text{ since by Equation 5.8} \]

\[ x^3t_{11}t_{10}t_9 = e \]

\[ [x^3t_{11}t_{10}t_9]^x^2y = e^x^2y \]

\[ \implies yx^{-1}t_{1}t_{16}t_{35} = e \]

\[ \implies yx^{-1}t_{1}t_{4}^0t_3^0 = e \]

\[ \implies yx^{-1}t_{1}t_{4}^0t_3^2t_3^2 = t_3^2 \]

\[ \implies yx^{-1}t_{1}t_{4}^0t_3^2t_3^4 = t_3^2t_4^2 \]

\[ \implies yx^{-1}y^{-1}t_1 = xy^{-1}t_3^2t_4^2 \]

\[ t_1 = xy^{-1}t_3^2t_4^2 \]

\[ H_{t_1t_6t_{36}} = H_{t_{2}^2xy^{-1}[t_2^2t_3^4]t_4^0} \]

\[ \implies H_{t_1t_6t_{36}} = H_{x^{-1}[t_2^2]y^{-1}t_2^5t_4^5} \]

\[ H_{t_1t_6t_{36}} = H_{t_2^5t_3^4t_4} \]

\[ H_{t_1t_6t_{36}} = H_{t_3^4t_4} \]

\[ H_{t_1t_6t_{36}} = H_{t_1^8t_4^5}, \text{ since } H_{t_{29}} = H_{t_{39}} \]

\[ H_{t_1^8} = H_{t_3^{10}} \]

\[ H_{t_1t_6t_{36}} = H_{t_3^5} \]

\[ H_{t_1t_6t_{36}} = H_{t_1^8t_4^5} \]

\[ H_{t_1t_6t_{36}} = H_{t_1^8[xyt_2^7t_3^{10}]}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy^{-1}} = e^{xy^{-1}} \]
\[ \implies y^{-1} x^{-1} t_{20} t_3 t_{14} = e \]
\[ \implies y^{-1} x^{-1} t^5_4 t^2_2 = e \]
\[ \implies y^{-1} x^{-1} t^5_4 t^4_{t_2} = t^7_2 \]
\[ \implies y^{-1} x^{-1} t^5_4 t^3_{t_3} = t^7_2 t^{10}_{t_3} \]
\[ \implies x y^{-1} x^{-1} t^5_4 = x y t^7_2 t^{10}_{t_3} \]
\[ \implies t^5_4 = x y t^7_2 t^{10}_{t_3} \]
\[ H t_1 t_6 t_{36} = H t^8_1 y t^7_2 t^{10}_{t_3} \]
\[ \implies H t_1 t_6 t_{36} = H x y[t^8_1 y t^7_2 t^{10}_{t_3}] \]
\[ \implies H t_1 t_6 t_{36} = H t^8_1 t^7_2 t^{10}_{t_3} \]
\[ \implies H t_1 t_6 t_{36} = H t^2_2 t^{10}_{t_3} \]
\[ \implies H t_1 t_6 t_{36} = H t^2_2 y^{-1} x^{-1} t^9_1 t_4^5 \]
\[ \implies H t_1 t_6 t_{36} = H y^{-1} x^{-1} [t^9_2] y^{-1} x^{-1} t^9_1 t_4^5 \]
\[ \implies H t_1 t_6 t_{36} = H t^{10}_1 t^5_4 \]
\[ \implies H t_1 t_6 t_{36} = H t^5_1 t_4 \]
\[ \implies H t_1 t_6 t_{36} = H t_{20} t_{29} \]
\[ \implies H t_1 t_6 t_{36} \in [16], \text{ since } H t_{20} t_{29} \text{ is in [1 6].} \]

1 symmetric generator will go to [1 6].

\[ H t_1 t_6 t_{37} = H t_1 t_6 t_{37} \]
\[ \implies H t_1 t_6 t_{37} = H t_6 t_1 t_{37}, \text{ since } H t_1 t_6 = H t_6 t_1 \]
\[ H t_1 t_6 t_{37} = H t^2_2 t^1_1 t^{10}_{t_1} \]
\[ Ht_{16t37} = Ht_2^2 \]
\[ Ht_{16t37} = Ht_6 \]
\[ Ht_{16t37} \in [5], \text{ since } Ht_6 \text{ is in } [5]. \]
1 symmetric generator will go to [5].

\[ Ht_{16t38} = Ht_{16t38} \]
\[ \implies Ht_{16t38} = Ht_1t_2^2t_2^{10} \]
\[ \implies Ht_{16t38} = Ht_1t_2 \]
\[ \implies Ht_{16t38} \in [12], \text{ since } Ht_1t_2 \text{ is in } [1 2]. \]
1 symmetric generator will go to [1 2].

\[ Ht_{16t39} = Ht_{16t39} \]
\[ \implies Ht_{16t39} = Ht_1t_2^2t_3^{10} \]
\[ \implies Ht_{16t39} = Ht_1[x^{-1}y^{-1}t_4t_3^4]t_3^{10}, \text{ since by Equation 5.8} \]
\[ x^3t_{11t10t9} = e \]
\[ [x^3t_{11t10t9}]y^{-1}x = ey^{-1}x \]
\[ \implies x^{-1}y^{-1}t_4t_3t_3t_3 = e \]
\[ \implies x^{-1}y^{-1}t_4t_3t_3t_3 = e \]
\[ \implies x^{-1}y^{-1}t_4t_3t_3t_3 = t_2 \]
\[ \implies x^{-1}y^{-1}t_4t_3t_3 = t_2^2 \]
\[ Ht_{16t39} = Ht_1x^{-1}y^{-1}t_4t_3^4t_3^{10} \]
\[ \implies Ht_{16t39} = Ht_1x^{-1}[t_1]x^{-1}y^{-1}t_4t_3^3 \]
\[ \implies Ht_{16t39} = Ht_1t_3^3 \]
\[ \implies Ht_{16t39} = Ht_4t_3^3 \]
\[ \implies Ht_{16t39} = Ht_4t_3^3 \]
\[ \implies Ht_{16t39} = Ht_{40t11} \]
\[ \implies Ht_{16t39} = Ht_{11t40}, \text{ since } Ht_{11t40} = Ht_{40t11} \]
\[ Ht_{16t39} = Ht_{11t40} \]
\[ \implies t_{16t39} \in [16], \text{ since } Ht_{11t40} \text{ is in } [1 6]. \]
1 symmetric generator will go to [1 6].

\[ Ht_{16t40} = Ht_{16t40} \]
\[ \implies Ht_{16t40} = Ht_6t_1t_40, \text{ since } Ht_1t_6 = Ht_6t_1 \]
\[ \Rightarrow Ht_1t_6t_{40} = Ht_2^2t_1t_4^{10} \]
\[ \Rightarrow Ht_1t_6t_{40} = Ht_2^2[xy^{-1}t_3^2t_4]t_4^{10}, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]x^2y = e^2y \]
\[ \Rightarrow yx^{-1}t_1t_{16}t_{35} = e \]
\[ \Rightarrow yx^{-1}t_1t_4t_3^9 = e \]
\[ \Rightarrow yx^{-1}t_1t_4^3t_3^2 = t_3^2 \]
\[ \Rightarrow yx^{-1}t_1t_4t_3^7 = t_4^7 \]
\[ \Rightarrow xy^{-1}yx^{-1}t_1 = xy^{-1}t_4^2t_4^7 \]
\[ \Rightarrow t_1 = xy^{-1}t_3^2t_4^7 \]
\[ Ht_1t_6t_{40} = Ht_2^2xy^{-1}t_3^2t_4^7t_4^{10} \]
\[ \Rightarrow Ht_1t_6t_{40} = Hxy^{-1}[t_3^2]xy^{-1}t_3^2t_4^6 \]
\[ \Rightarrow Ht_1t_6t_{40} = Ht_3^8t_3^2t_4^6 \]
\[ \Rightarrow Ht_1t_6t_{40} = Ht_3^{10}t_4^6 \]
\[ \Rightarrow Ht_1t_6t_{40} = H[y^{-1}x^{-1}t_4^3t_4^5t_4^6], \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]x^2y^{-1} = e^2y^{-1} \]
\[ \Rightarrow y^{-1}x^{-1}t_{33}t_2t_3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_3^2t_3^3 = e \]
\[ \Rightarrow y^{-1}x^{-1}t_3^3t_3^10 = t_3^{10} \]
\[ \Rightarrow y^{-1}x^{-1}t_3^3t_3^10 = t_3^{10} \]
\[ Ht_1t_6t_{40} = Ht_1^6 \]
\[ \Rightarrow Ht_1t_6t_{40} = Ht_3 \]
\[ \Rightarrow Ht_1t_6t_{40} \in [1], \text{ since } Ht_3 \text{ is in } [1]. \]

1 symmetric generator will go to [1].

The orbits of \( N^{(110)} \) are \{1, 3\}, \{2, 4, 5, 7\}, \{6, 8\}, \{9, 11\}, \{10, 12\}, \{13, 15\}, \{14, 16\}, \{17, 19\}, \{18, 20\}, \{21, 23\}, \{22, 24\}, \{25, 27\}, \{26, 28\}, \{29, 31\}, \{30, 32\}, \{33, 35\}, \{34, 36\}, \{37, 39\}, and \{38, 40\}. We will check to see where \( Ht_1t_{10}t_1, Ht_1t_{10}t_2, Ht_1t_{10}t_3, Ht_1t_{10}t_4, Ht_1t_{10}t_5, Ht_1t_{10}t_6, Ht_1t_{10}t_9, Ht_1t_{10}t_{10}, Ht_1t_{10}t_{13}, Ht_1t_{10}t_{14}, Ht_1t_{10}t_{17}, Ht_1t_{10}t_{18}, Ht_1t_{10}t_{21}, Ht_1t_{10}t_{22}, Ht_1t_{10}t_{25}, Ht_1t_{10}t_{26}, Ht_1t_{10}t_{29}, Ht_1t_{10}t_{30}, Ht_1t_{10}t_{33}, \)
\( H_t1t_{10}t_{34}, H_t1t_{10}t_{37}, H_t1t_{10}t_{38} \) belong.

\[
H_t1t_{10}t_{1} = H_t1t_{10}t_{1} \\
\implies H_t1t_{10}t_{1} = H_t1t_{15}t_{10}t_{1}, \text{ since } H_t1 = H_t15 \\
\implies H_t1t_{10}t_{1} = H_t4t_{3}t_{2}t_{1} \\
\implies H_t1t_{10}t_{1} = H_t4t_{3}[xt_{4}t_{1}^{8}]t_{1}, \text{ since Equation 5.8} \\
x^{3}t_{11}t_{10}t_{9} = e \\
[x^{3}t_{11}t_{10}t_{9}]^{-1} = e^{-1} \\
\implies x^{-1}t_{11}t_{9}t_{12} = e \\
\implies x^{-1}t_{2}^{3}t_{4}^{2} = e \\
\implies x^{-1}t_{2}^{3}t_{4}^{2}t_{1}^{8} = t_{4}^{8} \\
\implies x^{-1}t_{2}^{3}t_{4}^{2}t_{1}^{8} = t_{4}^{8}t_{1}^{2} \\
\implies x^{-1}t_{2}^{3} = x_{4}^{8}t_{1}^{2} \\
\implies t_{2}^{3} = x_{4}^{8}t_{1}^{2} \\
H_t1t_{10}t_{1} = H_t4x_{4}^{8}t_{1}^{2}t_{1} \\
\implies H_t1t_{10}t_{1} = Hx_{4}[t_{3}^{4}]x_{4}^{8}t_{1}^{8} \\
\implies H_t1t_{10}t_{1} = Ht_{4}^{4}t_{4}^{8}t_{1}^{9} \\
\implies H_t1t_{10}t_{1} = Ht_{4}t_{4}^{8}t_{1}^{9} \\
\implies H_t1t_{10}t_{1} = Ht_{4}t_{4}^{8}t_{1}^{9}, \text{ since} \\
H_t4 = H_t14 \\
\implies H_t4 = H_t4^{4} \\
H_t1t_{10}t_{1} = Ht_{4}^{4}t_{4}^{8}t_{1}^{9} \\
H_t1t_{10}t_{1} = H[xyt_{4}^{6}t_{1}^{9}]t_{1}^{9}, \text{ since Equation 5.8} \\
x^{3}t_{11}t_{10}t_{9} = e \\
[x^{3}t_{11}t_{10}t_{9}]^{-1} = e^{-1} \\
\implies y^{-1}x^{-1}t_{14}t_{3}t_{2}t_{20} = e \\
\implies y^{-1}x^{-1}t_{2}^{4}t_{4}^{3}t_{4}^{9} = e \\
\implies y^{-1}x^{-1}t_{2}^{4}t_{4}^{3}t_{4}^{9}t_{6}^{6} = t_{4}^{6} \\
\implies y^{-1}x^{-1}t_{2}^{4}t_{4}^{3}t_{4}^{9}t_{6}^{6} = t_{4}^{6}t_{4}^{2}t_{1}^{2} \\
\implies xy^{-1}x^{-1}t_{2}^{2} = xy_{4}^{6}t_{4}^{2}t_{1}^{2} \\
\implies t_{2}^{2} = xy_{4}^{6}t_{4}^{2}t_{1}^{2} \\
H_t1t_{10}t_{1} = Hxy_{4}^{6}t_{4}^{2}t_{1}^{9}
\[ H_{t_1t_0t_1} = H_{t_4}^6 \]
\[ H_{t_1t_0t_1} = H_{t_24} \]
\[ H_{t_1t_0t_1} \in [5], \text{ since } H_{t_{24}} \text{ is in [5].} \]

2 symmetric generators will go to [5].

\[ H_{t_1t_0t_2} = H_{t_1t_0t_2} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_{15}t_0t_2}, \text{ since } H_1 = H_{t_1} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_{13}^4t_2^3t_1} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_{13}^4t_2^4} \]
\[ \Rightarrow H_{t_1t_0t_2} = H[tyxt_1^6t_2^5t_1^4], \text{ since Equation 5.8} \]
\[ x^3t_1t_0t_9 = e \]
\[ [x^3t_1t_0t_9]^y = e^y \]
\[ \Rightarrow x^{-1}y^{-1}t_1t_3t_4t_17 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{13}^4t_2^5t_1 = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{13}^4t_2^5t_1^6 = t_1^6 \]
\[ \Rightarrow x^{-1}y^{-1}t_{13}^4t_2^5t_1^2 = t_1^6t_1^2 \]
\[ \Rightarrow yxy^{-1}x^{-1}t_3^4 = yxt_1^6t_1^2 \]
\[ \Rightarrow t_3^4 = yxt_1^6t_1^2 \]
\[ H_{t_1t_0t_2} = H[tyxt_1^6t_2^4t_1^4] \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_1^6t_2^4} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_1^6t_3^4}, \text{ since Equation 5.8} \]
\[ x^3t_1t_0t_9 = e \]
\[ [x^3t_1t_0t_9]^y = e^y \]
\[ \Rightarrow yx^{-1}t_1t_3t_5t_18 = e \]
\[ \Rightarrow yx^{-1}t_{13}^4t_2^5t_2 = e \]
\[ \Rightarrow yx^{-1}t_{13}^4t_2^5t_2^6 = t_2^6 \]
\[ \Rightarrow yx^{-1}t_{13}^4t_3^4 = t_2^6 \]
\[ H_{t_1t_0t_2} = H_{t_1^6yxt_1^4t_2^5t_3} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_1^6yxt_3^4t_1^4t_2^5t_3} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_1^6t_3^4t_1^4t_2^5t_3} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_1^6t_3^4t_1^4t_2^5t_3} \]
\[ \Rightarrow H_{t_1t_0t_2} = H_{t_24t_3t_5} \]
\[ Ht_{11}t_{10}t_{5} = Ht_{11}t_{10}t_{5} \]
\[ Ht_{11}t_{10}t_{5} = Ht_{15}t_{10}t_{5}, \text{ since } Ht_{1} = Ht_{15} \]
\[ Ht_{11}t_{10}t_{5} = Ht_{15}t_{10}t_{5} \]
\[ Ht_{11}t_{10}t_{5} = Ht_{15}t_{10}t_{5}^{2} \]
\[ Ht_{11}t_{10}t_{5} = Ht_{15}t_{10}t_{5}^{2}, \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{-1} = e^{-1} \]
\[ y^{-1}x^{-1}t_{14}t_{33}t_{20} = e \]
\[ y^{-1}x^{-1}t_{14}^{2}t_{33}t_{20} = e \]
\[ y^{-1}x^{-1}t_{14}^{3}t_{33}t_{20} = t_{4}^{6} \]
\[ y^{-1}x^{-1}t_{14}^{4}t_{33}t_{20} = t_{4}^{6}t_{2}^{6} \]
\[ xy^{-1}x^{-1}t_{2}^{2} = xyt_{4}^{6}t_{2}^{6} \]
\[ \Rightarrow t_2^4 = x y t_4^2 t_1^2 \]

\[ H t_1 t_{10} t_5 = H x y t_4^2 t_1^{10} \Rightarrow H t_1 t_{10} t_5 = H t_4^6 t_1 \]

\[ \Rightarrow H t_1 t_{10} t_5 = H t_4^6 t_1, \text{ since} \]

\[ H t_{22} = H t_{24} \]

\[ \Rightarrow H t_2^6 = H t_4^6 \]

\[ H t_1 t_{10} t_5 = H t_2^6 t_1 \]

\[ H t_1 t_{10} t_5 = H t_2^6 x t_4^{10}, \text{ since Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9]^2 x = e x^2 y \]

\[ \Rightarrow y x^{-1} t_{11} t_{16} t_{35} = e \]

\[ \Rightarrow y x^{-1} t_{11} t_4 t_3^9 = e \]

\[ \Rightarrow y x^{-1} t_{11} t_4 t_3^2 t_3^2 = t_3^2 \]

\[ \Rightarrow y x^{-1} t_{11} t_4 t_3^7 = t_3^2 t_4^7 \]

\[ \Rightarrow x y^{-1} y x^{-1} t_1 = x y^{-1} t_3^2 t_4^7 \]

\[ \Rightarrow t_1 = x y^{-1} t_3^2 t_4^7 \]

\[ H t_1 t_{10} t_5 = H t_2^6 x y^{-1} t_4^{10} \]

\[ \Rightarrow H t_1 t_{10} t_5 = H x y^{-1} [t_2^6 x y^{-1} t_4^{10}] \]

\[ \Rightarrow H t_1 t_{10} t_5 = H t_3^4 t_3^7 t_4 \]

\[ \Rightarrow H t_1 t_{10} t_5 = H t_3^4 t_3^7 \]

\[ \Rightarrow H t_1 t_{10} t_5 = H t_1 t_{15} t_{28} \]

\[ \Rightarrow H t_1 t_{10} t_5 \in [16], \text{ since } H t_{15} t_{28} \text{ is in [1 6].} \]

2 symmetric generators will go to [1 6].
\[ x^{-1}y^{-1}t_4t_2t_1^5 = e \]
\[ x^{-1}y^{-1}t_4t_2t_1t_6 = t_1^6 \]
\[ x^{-1}y^{-1}t_4^2t_1^2 = t_1^6t_1^2 \]
\[ yxy^{-1}x^{-1}t_3^4 = yxt_1^6t_1^2 \]
\[ t_3^4 = yxt_1^6t_1^2 \]
\[ Ht_1t_10t_6 = Hxyt_1^6t_2t_5^2 \]
\[ Ht_1t_10t_6 = Ht_1^6t_2^2 \]
\[ Ht_1t_10t_6 = Ht_1^6[y^{-1}x^{-1}t_4^5t_3], \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^{xy^{-1}} = e^{xy^{-1}} \]
\[ [y^{-1}x^{-1}t_2t_3t_4t_5 = e] \]
\[ [y^{-1}x^{-1}t_3^2t_4] = e \]
\[ [y^{-1}x^{-1}t_3^2t_4t_5 = t_2^4] \]
\[ [y^{-1}x^{-1}t_3^2t_3 = t_2^7] \]
\[ Ht_1t_{10}t_6 = Ht_1^6y^{-1}x^{-1}t_4^5t_3 \]
\[ Ht_1t_{10}t_6 = Hgy^{-1}x^{-1}[t_1^6]y^{-1}x^{-1}t_4^5t_3 \]
\[ Ht_1t_{10}t_6 = Ht_4^6t_4^5t_3 \]
\[ Ht_1t_{10}t_6 = Ht_4^6t_3 \]
\[ Ht_1t_{10}t_6 \in [4], \text{ since } Ht_4t_3 \text{ is in [4].} \]

2 symmetric generators will go to [4].
\[ t^3_{21} = xt^8_4 \]

\[ Ht_1 t_{10} t_9 = Ht^4_3 xt^8_4 \]

\[ \Rightarrow Ht_1 t_{10} t_9 = Hx[t^4_3 x t^8_4] \]

\[ \Rightarrow Ht_1 t_{10} t_9 = Ht^4_4 t^6_4 \]

\[ \Rightarrow Ht_1 t_{10} t_9 = Ht_4 \]

\[ \Rightarrow Ht_1 t_{10} t_9 \in [1], \text{ since } Ht_4 \text{ is in } [1]. \]

2 symmetric generators will go to [1].

\[ Ht_1 t_{10} t_{10} = Ht_1 t_{10} t_{10} \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = Ht^3_2 t^3_2 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = Ht^3_1 t^3_1 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = Ht_1 [yx^{-1} t^4_4 t^0_3], \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{xy} \]

\[ \Rightarrow yx^{-1} t_{10} t_{35} t_{18} = e \]

\[ \Rightarrow yx^{-1} t^4_4 t^5_2 t^3_2 = e \]

\[ \Rightarrow yx^{-1} t^4_4 t^5_2 t^6_2 = t^6_2 \]

\[ \Rightarrow yx^{-1} t^4_4 t^6_2 = t^6_2 \]

\[ Ht_1 t_{10} t_{10} = Ht_1 yx^{-1} t^4_4 t^0_3 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = H yx^{-1} [t_1]^{yx^{-1}} t^4_4 t^0_3 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = H t^4_4 t^4_3 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = H t^4_3 t^3_3 \]

\[ \Rightarrow Ht_1 t_{10} t_{10} = H t^4_2 t^0_3, \text{ since} \]

\[ Ht_{32} = Ht_{38} \]

\[ \Rightarrow Ht^8_4 = Ht^1_2 \]

\[ Ht_1 t_{10} t_{10} = H t^4_2 t^0_3 \]

\[ Ht_1 t_{10} t_{10} = H [x^{-1} yt^0_4 t^5_3 t^0_3], \text{ since by Equation 5.8} \]

\[ x^3 t_{11} t_{10} t_9 = e \]

\[ [x^3 t_{11} t_{10} t_9]^{yx} = e^{yx} \]

\[ \Rightarrow x^{-1} yt_{36} t_{19} t_{2} = e \]

\[ \Rightarrow x^{-1} yt^0_4 t^5_3 t_{2} = e \]

\[ \Rightarrow x^{-1} yt^0_4 t^5_3 t_{2} t_{10} = t^0_2 \]
\[ \Rightarrow x^{-1}y^{t_4^5_t_3^5} = t_2^{10} \]

\[ Ht_{10}t_{10}t_{10} = Hx^{-1}y^{t_4^5_t_3^5} \]

\[ \Rightarrow Ht_{10}t_{10}t_{10} = Ht_3 \]

\[ \Rightarrow Ht_{10}t_{10}t_{10} = Ht_{36}t_{11} \]

\[ \Rightarrow Ht_{10}t_{10}t_{10} = Ht_{34}t_{11} \text{, since } Ht_{34} = Ht_{36} \]

\[ Ht_{10}t_{10}t_{10} = Ht_{34}t_{11} \]

\[ \Rightarrow Ht_{10}t_{10}t_{10} \in [12], \text{ since } Ht_{34}t_{11} \text{ is in } [1 2]. \]

2 symmetric generators will go to [1 2].

\[ Ht_{10}t_{10}t_{13} = Ht_{10}t_{10}t_{13} \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_{15}t_{10}t_{13}, \text{ since } Ht_{1} = Ht_{15} \]

\[ Ht_{10}t_{10}t_{13} = Ht_{15}t_{10}t_{13} \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_{13} \]

\[ Ht_{10}t_{10}t_{13} = Ht_{13} \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_{13}t_{10}t_{13} \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_{13}[t_4^8t_1^4]t_1^4, \text{ since by Equation 5.8} \]

\[ x^3t_{11}t_{10}t_9 = e \]

\[ [x^3t_{11}t_{10}t_9]^{x^{-1}} = e^{x^{-1}} \]

\[ \Rightarrow x^{-1}t_{10}t_9t_{12} = e \]

\[ \Rightarrow x^{-1}t_2^3t_3^3t_4^3 = e \]

\[ x^{-1}t_2^3t_3^3t_4^3 = t_4^8 \]

\[ \Rightarrow x^{-1}t_2^3t_3^3t_4^3 = t_4^8 \]

\[ \Rightarrow x^{-1}t_2^3t_3^3t_4^3 = x^{-1}t_2^3t_3^3t_4^3 \]

\[ \Rightarrow t_2^3 = xt_4^8t_1^4 \]

\[ Ht_{10}t_{10}t_{13} = Ht_{13}t_4^8t_1^4 \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Hx[t_4^8t_1^4]t_4^8t_1 \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_1t_4^8t_1 \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} = Ht_4t_1 \]

\[ \Rightarrow Ht_{10}t_{10}t_{13} \in [12], \text{ since } Ht_4t_1 \text{ is in } [1 2]. \]

2 symmetric generators will go to [1 2].

\[ Ht_{10}t_{10}t_{14} = Ht_{10}t_{10}t_{14} \]

\[ \Rightarrow Ht_{10}t_{10}t_{14} = Ht_{15}t_{10}t_{14}, \text{ since } Ht_{1} = Ht_{15} \]

\[ Ht_{10}t_{10}t_{14} = Ht_{15}t_{10}t_{14} \]
\[
H_{t_1 t_{10} t_{14}} = H t_1^4 t_2^3 t_2^4
\]
\[
H_{t_1 t_{10} t_{14}} = H [y x t_4^6 t_2^2 t_2^7], \text{ since by Equation 5.8}
\]
\[
x^3 t_{11} t_{10} t_9 = e
\]
\[
[x^3 t_{11} t_{10} t_9]^y = e^y
\]
\[
\Rightarrow x^{-1} y^{-1} t_{15} t_{34} t_{17} = e
\]
\[
\Rightarrow x^{-1} y^{-1} t_3 t_2 t_4 t_1^5 = e
\]
\[
\Rightarrow x^{-1} y^{-1} t_3^4 t_2^5 t_1^6 = t_1^6
\]
\[
\Rightarrow x^{-1} y^{-1} t_3^4 t_2^5 t_1^2 = t_1^6 t_1^2
\]
\[
y x y^{-1} x^{-1} t_3^4 = y x t_4^6 t_1^2
\]
\[
t_3^4 = y x t_4^6 t_1^2
\]
\[
H_{t_1 t_{10} t_{14}} = H y x t_4^6 t_2^2 t_2^7
\]
\[
\Rightarrow H_{t_1 t_{10} t_{14}} = H t_1^4 t_2^9
\]
\[
\Rightarrow H_{t_1 t_{10} t_{14}} = H t_21 t_{34}
\]
\[
\Rightarrow H_{t_1 t_{10} t_{14}} = H t_{23} t_{34}, \text{ since } H_{t_{21}} = H_{t_{23}}
\]
\[
H_{t_1 t_{10} t_{14}} = H t_{23} t_{34}
\]
\[
\Rightarrow H_{t_1 t_{10} t_{14}} = H t_{34} t_{23}, \text{ since by Equation 5.9}
\]
\[
H_{t_{10} t_9} = H t_6 t_1
\]
\[
\Rightarrow [H_{t_{10} t_9}]^{xy} = [H t_6 t_1]^{xy}
\]
\[
\Rightarrow H_{t_3 t_{23}} = H t_{23} t_{34}
\]
\[
H_{t_1 t_{10} t_{14}} = H t_{34} t_{23}
\]
\[
\Rightarrow H_{t_1 t_{10} t_{14}} \in [16], \text{ since } H_{t_{34} t_{23}} \text{ is in } [1 6].
\]

2 symmetric generators will go to [1 6].
\[ x^{-1}t^3_2t^3_4t^8_2 = \frac{t^8_3}{2} \]
\[ x^{-1}t^3_2t^3_4t^8_2 = \frac{t^8_3}{4} \]
\[ x^{-1}t^3_2 = \frac{t^8_3}{4} \]
\[ t^3_2 = xt^8_4 \]

\[ Ht_1t_10t_17 = \frac{Ht_3t^8_4t^8_1}{4} \]
\[ \Rightarrow Ht_1t_10t_17 = Hx[t^4_3]t^8_4t^2_1 \]
\[ \Rightarrow Ht_1t_10t_17 = Ht^1_4t^3_5 \]
\[ \Rightarrow Ht_1t_10t_17 = Ht^1_4t^3_5 \]
\[ \Rightarrow Ht_1t_10t_17 = Ht^4t^5_2 \]
\[ \Rightarrow Ht_1t_10t_17 = Ht^4t^5_2 \]
\[ \Rightarrow Ht_1t_10t_17 \in [16], \text{ since } Ht^5_4t^5_5 \text{ is in } [1, 6]. \]

2 symmetric generators will go to [1 6].

\[ Ht_1t_10t_18 = Ht_1t_10t_18 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht_1t^3_2t^5_2 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht_1t^3_2 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht_1t^3_2 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3, \text{ since by Equation 5.8} \]
\[ x^3t^11_1t^9_1 = e \]
\[ [x^3t^11_1t^9_1]^x = e^x \]
\[ \Rightarrow x^{-1}t^9_1t^11_1 = e \]
\[ \Rightarrow x^{-1}t^3_4t^3_3 = e \]
\[ \Rightarrow x^{-1}t^3_4t^3_3t^3_2 = \frac{t^8_3}{4} \]
\[ \Rightarrow x^{-1}t^3_4t^3_3 = \frac{t^8_3}{2} \]

\[ Ht_1t_10t_18 = Ht_1t^3_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Hx^{-1}[t^3_4]t^3_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^4t^3_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^4t^3_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht^1_4t^3_3 \]
\[ \Rightarrow Ht_1t_10t_18 = Ht_2t^1_11, \text{ since } Ht^2_2t^1_11 = Ht_16 \]
\[ Ht_1t_10t_18 = Ht_2t^1_11 \]
\[ Ht_1t_10t_18 \in [110], \text{ since } Ht_2t^1_11 \text{ is in } [1, 10]. \]

2 symmetric generators will go to [1 10].
$$H_{t_1 t_{10} t_{21}} = H_{t_1 t_{10} t_{21}}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_{15} t_{10} t_{21}},\text{ since } H_{t_1} = H_{t_{15}}$$

$$H_{t_1 t_{10} t_{21}} = H_{t_{15} t_{10} t_{21}}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_3 t_{1} t_{2} t_{1}}^{6}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_3 t_{1} t_{2} t_{1}}^{6}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_3 t_{1} t_{2} t_{1}}^{6},\text{ since by Equation 5.8}$$

$$x^{3} t_{11} t_{10} t_{9} = e$$

$$[x^{3} t_{11} t_{10} t_{9}]^{x^{-1}} = e^{x^{-1}}$$

$$\implies x^{-1} t_{10} t_{9} t_{12} = e$$

$$\implies x^{-1} t_{2} t_{1}^{3} t_{1}^{3} = e$$

$$\implies x^{-1} t_{2} t_{1}^{3} t_{1}^{3} t_{1}^{8} = t_{1}^{8}$$

$$\implies x^{-1} t_{2} t_{1}^{3} t_{1}^{3} = t_{1}^{8}$$

$$\implies x x^{-1} t_{2}^{3} = x t_{4} t_{4}^{8} t_{1}$$

$$\implies t_{2}^{3} = x t_{4} t_{4}^{8} t_{1}$$

$$H_{t_1 t_{10} t_{21}} = H_{t_3 t_{1} t_{2} t_{1}}^{6} x t_{4} t_{4}^{8} t_{1}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_{3} t_{1} t_{2} t_{1}}^{6} x t_{4} t_{4}^{8} t_{1}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_{3} t_{1} t_{2} t_{1}}^{6} x t_{4} t_{4}^{8} t_{1}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_{15}} t_{10} t_{21} = H_{t_{4}} t_{9}$$

$$\implies H_{t_1 t_{10} t_{21}} = H_{t_4} t_{9}$$

$$\implies H_{t_1 t_{10} t_{21}} = [110],\text{ since } H_{t_4} t_{9} \text{ is in } [10].$$

2 symmetric generators will go to [10].

$$H_{t_1 t_{10} t_{22}} = H_{t_1 t_{10} t_{22}}$$

$$\implies H_{t_1 t_{10} t_{22}} = H_{t_{15} t_{10} t_{22}},\text{ since } H_{t_1} = H_{t_{15}}$$

$$H_{t_1 t_{10} t_{22}} = H_{t_{15} t_{10} t_{22}}$$

$$\implies H_{t_1 t_{10} t_{22}} = H_{t_3 t_{1} t_{2} t_{1}}^{6}$$

$$\implies H_{t_1 t_{10} t_{22}} = H_{t_3 t_{1} t_{2} t_{1}}^{6}$$

$$\implies H_{t_1 t_{10} t_{22}} = H_{t_3 t_{1} t_{2}}^{6} t_{2},\text{ since by Equation 5.8}$$

$$x^{3} t_{11} t_{10} t_{9} = e$$

$$[x^{3} t_{11} t_{10} t_{9}]^{y^{-2}} = e^{y^{-2}}$$

$$\implies x^{-1} y^{-1} t_{15} t_{34} t_{17} = e$$

$$\implies x^{-1} y^{-1} t_{4} t_{4}^{9} t_{1}^{5} = e$$
\[ y^{-1}t_3^2t_1^4t_4^5t_6^6 = t_1^6 \]
\[ x^{-1}y^{-1}t_3^4t_2^5t_1^4t_6^7 = t_1^6t_6^2 \]
\[ yxy^{-1}x^{-1}t_4^6 = yxt_4^6t_1^4 \]
\[ t_3^3 = yxt_4^6t_1^4 \]
\[ Ht_1t_10t_22 = Hxyt_1^6t_2^5t_6^2 \]
\[ Ht_1t_10t_22 = Hxyt_1^6 \]
\[ Ht_1t_10t_22 = Hxyt_21 \]
\[ 2 \text{ symmetric generators will go to } [5], \text{ since } Hxyt_1^6t_2^5t_6^2 \text{ is in } [5]. \]

\[ Ht_1t_10t_25 = Ht_1t_10t_25 \]
\[ Ht_1t_10t_25 = Ht_15t_10t_25, \text{ since } Ht_1 = Ht_15 \]
\[ Ht_1t_10t_25 = Ht_15t_10t_25 \]
\[ Ht_1t_10t_25 = Ht_15t_15t_25 \]
\[ Ht_1t_10t_25 = Ht_15t_25t_7 \]
\[ Ht_1t_10t_25 = Ht_15t_15t_25 \]
\[ x^3t_11t_10t_9 = e \]
\[ [x^3t_11t_10t_9]^{x^{-1}} = e^{x^{-1}} \]
\[ Ht_1t_10t_25 = Ht_15t_11t_10t_9 \]
\[ x^{-1}t_15t_10t_9 = e \]
\[ x^{-1}t_2^3t_4^3 = e \]
\[ x^{-1}t_2^3t_3^3t_4^3 = e^8 \]
\[ x^{-1}t_2^3t_3^3t_4^3 = t_4^8 \]
\[ x^{-1}t_2^3t_3^3t_4^3 = t_4^8t_3^1 \]
\[ x^{-1}t_2^3t_3^3t_4^3 = x^8t_4^1 \]
\[ t_3^3 = x^8t_4^1 \]
\[ Ht_1t_10t_25 = Ht_15x^8t_4^1t_1^4 \]
\[ Ht_1t_10t_25 = Hx[t_3^1x^8t_4^1t_1^4] \]
\[ Ht_1t_10t_25 = Ht_15t_3^1x^8t_4^1t_1^4 \]
\[ Ht_1t_10t_25 = Ht_15t_3^1x^8t_4^1t_1^4 \]
\[ Ht_1t_10t_25 = Ht_15t_3^1x^8t_4^1t_1^4 \]
\[ 2 \text{ symmetric generators will go to } [5], \text{ since } Hx[t_3^1x^8t_4^1t_1^4] \text{ is in } [5]. \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{xy} \]
\[ \implies x^{-1} y t_{13} t_{36} t_{19} = e \]
\[ \implies x^{-1} y t_1^6 t_4^5 t_3^3 = e \]
\[ \implies x^{-1} y t_1^6 t_4^5 t_3^3 t_6^6 = t_3^6 \]
\[ \implies x^{-1} y t_1^6 t_4^5 t_3^3 = t_3^2 \]
\[ \implies y x^{-1} y t_1^6 t_4^5 = y x^{-1} t_3^6 t_4^2 \]
\[ \implies t_1^6 = y x^{-1} t_3^6 t_4^2 \]
\[ H t_{11} t_{10} t_{25} = H t_2^3 y x^{-1} t_3^6 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H y^{-1} x [t_2^4]^y x^{-1} t_3^6 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_2^3 t_3^6 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_2^3 t_4^2 \]
\[ H t_{11} t_{10} t_{25} = H t_{11} t_4^2, \text{ since} \]
\[ H t_1 = H t_{15} \]
\[ \implies H t_1 = H t_3^3 \]
\[ H t_{11} t_{10} t_{25} = H t_1^2 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_1^2 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_1 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H [x y^{-1} t_3^6 t_4^2] t_4^2, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9]^{xy} = e^{xy} \]
\[ \implies y x^{-1} t_{11} t_{16} t_{35} = e \]
\[ \implies y x^{-1} t_{11} t_4^6 t_3^3 = e \]
\[ \implies y x^{-1} t_{11} t_4^6 t_3^3 t_6^6 = t_3^6 \]
\[ \implies y x^{-1} t_{11} t_4^6 t_3^3 = t_3^2 \]
\[ \implies y x^{-1} t_{11} t_4^6 t_3^3 t_6^7 = t_3^2 t_4^7 \]
\[ \implies y x^{-1} t_{11} t_4^6 t_3^3 = t_3^2 t_4^7 \]
\[ \implies t_1 = y x^{-1} t_3^6 t_4^2 \]
\[ H t_{11} t_{10} t_{25} = H x y^{-1} t_3^6 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_2^3 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_2^3 t_4^2 \]
\[ \implies H t_{11} t_{10} t_{25} = H t_2^3 t_4^2, \text{ since} \]
\[ H t_7 = H t_{25} \]
\[ \implies H t_7^2 = H t_4^2 \]
\[ H_{t_1t_{10}t_{25}} = Ht_{10}^7t_{15}^9 \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} = Ht_{10}^7[y^{-1}xt_{2}^6t_{3}], \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^y = e^y \]
\[ \Rightarrow x^{-1}yt_{30}t_{19}t_2 = e \]
\[ \Rightarrow x^{-1}yt_{4}^5t_{3}^2t_2 = e \]
\[ \Rightarrow x^{-1}yt_{4}^5t_{3}^2t_{2}^{10} = t_{2}^{10} \]
\[ \Rightarrow x^{-1}yt_{4}^5t_{3}^2 = t_{2}^{10}t_{3}^{6} \]
\[ \Rightarrow y^{-1}xt_4^{-1}yt_{4}^9 = y^{-1}xt_{2}^6t_{3}^{6} \]
\[ \Rightarrow t_{4}^{9} = y^{-1}xt_{2}^6t_{3}^{6} \]
\[ H_{t_1t_{10}t_{25}} = Ht_{10}^7y^{-1}xt_{2}^6t_{3}^{6} \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} = Hy^{-1}x[t_{1}^7]y^{-1}xt_{2}^6t_{3}^{6} \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} = Ht_{2}^{10}t_{2}^{10}t_{3}^{6} \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} = Ht_{2}^{6}t_{3}^{6} \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} = Ht_{34}t_{23} \]
\[ \Rightarrow H_{t_1t_{10}t_{25}} \in [16], \text{ since } Ht_{34}t_{23} \text{ is in } [1 6]. \]

2 symmetric generators will go to [1 6].

\[ H_{t_1t_{10}t_{26}} = Ht_{110}t_{26} \]
\[ \Rightarrow H_{t_1t_{10}t_{26}} = Ht_{15}t_{10}t_{26}, \text{ since } Ht_1 = Ht_{15} \]
\[ H_{t_1t_{10}t_{26}} = Ht_{15}t_{10}t_{26} \]
\[ \Rightarrow H_{t_1t_{10}t_{26}} = Ht_{15}t_{25}t_{2}^7 \]
\[ \Rightarrow H_{t_1t_{10}t_{26}} = Ht_{34}t_{2}^{10} \]
\[ \Rightarrow H_{t_1t_{10}t_{26}} = H[yxt_{1}^6t_{2}^3]t_{2}^{10}, \text{ since by Equation 5.8} \]
\[ x^3t_{11}t_{10}t_9 = e \]
\[ [x^3t_{11}t_{10}t_9]^y = e^y \]
\[ \Rightarrow x^{-1}y^{-1}t_{15}t_{34}t_{17} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{4}^5t_{1} = e \]
\[ \Rightarrow x^{-1}y^{-1}t_{4}^5t_{2}^{10}t_{1} = t_{1}^{6} \]
\[ \Rightarrow x^{-1}y^{-1}t_{4}^5t_{3}^{2} = t_{1}^{6}t_{2}^{10} \]
\[ \Rightarrow yxy^{-1}x^{-1}t_{3}^{4} = yxt_{1}^6t_{2}^{10} \]
\[ \Rightarrow t_{3}^{4} = yxt_{1}^6t_{2}^{10} \]
\[ H_{t_1t_10t_{26}} = H_y x t_1^6 t_2^2 t_3^{10} \]
\[ \implies H_{t_1t_10t_{26}} = H_1^6 t_2 \]
\[ \implies H_{t_1t_10t_{26}} = H_1^6 t_2, \text{ since} \]
\[ H_{t_{21}} = H_{t_{23}} \]
\[ \implies H_{t_{10}}^6 = H_{t_3}^6 \]
\[ H_{t_1t_{10t_{26}}} = H_{t_3}^6 t_2 \]
\[ H_{t_1t_{10t_{26}}} = H_{t_3}^6[y^{-1} x t_4^3 t_1^7], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [x^3 t_{11} t_{10} t_9] y^{-1} x^{-1} = e y^{-1} x^{-1} \]
\[ \implies x^{-1} y t_2 t_{13} t_{36} = e \]
\[ \implies x^{-1} y t_2 t_{14} t_4^9 = e \]
\[ \implies x^{-1} y t_2 t_{14} t_4^2 = t_2 \]
\[ \implies x^{-1} y t_2 t_{14} t_4^7 = t_2^2 \]
\[ \implies y^{-1} x^{-1} y t_2 = y^{-1} x t_4^7 t_1^7 \]
\[ t_2 = y^{-1} x t_4^7 t_1^7 \]
\[ H_{t_{11}t_{10}t_{26}} = H_{t_3}^6 y^{-1} x t_4^7 t_1^7 \]
\[ \implies H_{t_{11}t_{10}t_{26}} = H_{t_3}^6 y^{-1} x t_4^7 t_1^7 \]
\[ \implies H_{t_{11}t_{10}t_{26}} = H_{t_4}^7 t_4^2 t_1^7 \]
\[ \implies H_{t_{11}t_{10}t_{26}} = H_{t_4}^7 t_1^7 \]
\[ \implies H_{t_{11}t_{10}t_{26}} = H_{t_{16}t_{25}} \]
\[ \implies H_{t_{11}t_{10}t_{26}} \in [16], \text{ since } H_{t_{16}t_{25}} \text{ is in } [16]. \]
\[ 2 \text{ symmetric generators will go to } [16]. \]
\[ x^{-1}t_{3}^{3}t_{2}^{3}t_{1}^{8} = t_{4}^{8} \]
\[ x^{-1}t_{3}^{3}t_{2}^{5}t_{1}^{8} = t_{4}^{5}t_{1}^{8} \]
\[ xx^{-1}t_{2}^{3} = x^2t_{2}^{5} \]
\[ t_{2}^{3} = xt_{2}^{5}t_{1}^{8} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{1}^{4}t_{0}t_{29} \]
\[ Ht_{1}t_{0}t_{29} = Hx[t_{4}^{1}x^{8}t_{2}^{5}] \]
\[ Ht_{1}t_{0}t_{29} = Ht_{1}^{4}t_{4}^{5}t_{1}^{8} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{1}^{4}t_{1}^{5}t_{1}^{8} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{1}^{4}t_{1}^{5}, \text{ since} \]
\[ Ht_{4} = Ht_{1}^{4} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{1}^{4}t_{1}^{5}t_{1}^{10} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{2}[yxt_{4}^{7}t_{4}^{10}], \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{0}t_{9} = e \]
\[ [x^{3}t_{11}t_{0}t_{9}]x^{-1}y^{-1} = e^{x^{-1}y^{-1}} \]
\[ x^{-1}y^{-1}t_{1}t_{4}t_{15} = e \]
\[ x^{-1}y^{-1}t_{1}^{-1}t_{4}t_{1}^{4} = e \]
\[ x^{-1}y^{-1}t_{1}^{-1}t_{4}t_{1}^{3} = t_{3}^{7} \]
\[ x^{-1}y^{-1}t_{1}^{-1}t_{4}t_{1}^{10} = t_{1}^{7}t_{1}^{10} \]
\[ yxx^{-1}y^{-1}t_{1}^{5} = yxt_{4}^{10}t_{1}^{10} \]
\[ t_{1}^{5} = yxt_{4}^{10}t_{1}^{10} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{2}yxt_{4}^{7}t_{4}^{10} \]
\[ Ht_{1}t_{0}t_{29} = Hyx[t_{2}^{4}yx^{7}t_{4}^{10}] \]
\[ Ht_{1}t_{0}t_{29} = Ht_{3}^{3}t_{4}^{7}t_{4}^{10} \]
\[ Ht_{1}t_{0}t_{29} = Ht_{4}^{10}t_{4}^{10} \]
\[ Ht_{1}t_{0}t_{29} = H[y^{-1}x^{-1}t_{1}^{9}t_{1}^{5}t_{4}^{10}], \text{ since by Equation 5.8} \]
\[ x^{3}t_{11}t_{0}t_{9} = e \]
\[ [x^{3}t_{11}t_{0}t_{9}]x^{-1}y^{-1} = e^{x^{-1}y^{-1}} \]
\[ y^{-1}x^{-1}t_{3}t_{2}t_{2}t_{3} = e \]
\[ y^{-1}x^{-1}t_{1}^{-1}t_{1}^{4}t_{3} = e \]
\[ y^{-1}x^{-1}t_{1}^{-1}t_{1}^{9}t_{3} = t_{3}^{10} \]
\[ y^{-1}x^{-1}t_{1}^{-1}t_{1}^{9}t_{3} = t_{3}^{10} \]
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\[ H_{t1t_{10}t_{29}} = H y^{-1} x^{-1} t_i^{0} t_j^{10} \]
\[ \Rightarrow H_{t1t_{10}t_{29}} = H t_i^{0} t_j^{4} \]
\[ \Rightarrow H_{t1t_{10}t_{29}} = H t_{33} t_{16} \]
\[ \Rightarrow H_{t1t_{10}t_{29}} \in [14], \text{ since } H t_{33} t_{16} \text{ is in } [14]. \]

2 symmetric generators will go to [14].

\[ H_{t1t_{10}t_{30}} = H_{t1t_{10}t_{30}} \]
\[ \Rightarrow H_{t1t_{10}t_{30}} = H t_i^{3} t_j^{8} \]
\[ \Rightarrow H_{t1t_{10}t_{30}} = H t_{1} \]
\[ \Rightarrow H_{t1t_{10}t_{30}} \in [1], \text{ since } H t_{1} \text{ is in } [1]. \]

2 symmetric generators will go to [1].

\[ H_{t1t_{10}t_{33}} = H_{t1t_{10}t_{33}} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_{15} t_{10} t_{33}, \text{ since } H t_{1} = H t_{15} \]
\[ H_{t1t_{10}t_{33}} = H_{t1t_{10}t_{33}} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_i^{4} t_j^{9} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_{33} [x t_i^{8} t_j^{8}] t_j^{9}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}]^{-1} = e^{-1} \]
\[ \Rightarrow x^{-1} t_{10} t_{9} t_{12} = e \]
\[ \Rightarrow x^{-1} t_i^{3} t_j^{2} e \]
\[ \Rightarrow x^{-1} t_i^{3} t_j^{2} t_j^{8} = t_j^{8} \]
\[ \Rightarrow x^{-1} t_i^{3} t_j^{2} t_j^{8} = t_j^{8} t_j^{8} \]
\[ \Rightarrow x^{-1} t_i^{2} = x t_j^{8} t_j^{8} \]
\[ \Rightarrow t_i^{2} = x t_j^{8} t_j^{8} \]
\[ H_{t1t_{10}t_{33}} = H t_i^{4} x t_j^{8} t_j^{8} t_j^{9} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H x [t_i^{4} t_j^{8} t_j^{6}] \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_i^{4} t_j^{8} t_j^{6} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_i^{4} t_j^{6} \]
\[ \Rightarrow H_{t1t_{10}t_{33}} = H t_i^{4} [x^{-1} y^{-1} t_i^{9} t_j^{9}], \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}]^{-2} = e^{-2} \]
\[ x^{-1}y^{-1}t_{15}t_{34}t_{17} = e \]
\[ x^{-1}y^{-1}t_{3}t_{2}t_{1} = e \]
\[ x^{-1}y^{-1}t_{3}t_{2}t_{1} = t_{1}^{6} \]
\[ x^{-1}y^{-1}t_{3}t_{2} = t_{1}^{6} \]

\[ Ht_{1}t_{10}t_{33} = Ht_{4}x^{-1}y^{-1}t_{3}t_{2}^{6} \]
\[ Ht_{1}t_{10}t_{33} = Hx^{-1}y^{-1}[t_{4}]x^{-1}y^{-1}t_{3}t_{2}^{6} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{3}t_{2}^{6} \]
\[ Ht_{31} = Ht_{37} \]
\[ Ht_{3}^{5} = Ht_{1}^{10} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{1}^{10}t_{2}^{6} \]
\[ Ht_{1}t_{10}t_{33} = H[t_{2}^{6}t_{3}^{9}]t_{2}^{9}, \text{ since Equation 5.8} \]
\[ x^{3}t_{11}t_{10}t_{9} = e \]
\[ [x^{3}t_{11}t_{10}t_{9}]^{y} = e^{y} \]
\[ yx^{-1}t_{35}t_{18}t_{1} = e \]
\[ yx^{-1}t_{3}t_{2}t_{1} = e \]
\[ yx^{-1}t_{3}t_{2}t_{1} = t_{1}^{10} \]
\[ yx^{-1}t_{3}t_{2} = t_{1}^{10} \]
\[ Ht_{1}t_{10}t_{33} = Hx^{-1}t_{3}t_{2}^{9}t_{2}^{6} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{3}t_{2}^{8} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{3}t_{2} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{33}t_{2}^{10}, \text{ since } Ht_{33} = Ht_{35} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{3}t_{10} \]
\[ Ht_{1}t_{10}t_{33} = Ht_{3}t_{10}, \text{ since } Ht_{33}t_{10} \text{ is in [1 2].} \]

2 symmetric generators will go to [1 2].

\[ Ht_{1}t_{10}t_{34} = Ht_{1}t_{10}t_{34} \]
\[ Ht_{1}t_{10}t_{34} = Ht_{1}^{3}t_{2}^{9} \]
\[ Ht_{1}t_{10}t_{34} = Ht_{1} \]
\[ Ht_{1}t_{10}t_{34} = Ht_{1}, \text{ since } Ht_{1} \text{ is in [1].} \]

2 symmetric generators will go to [1].
\[ Ht_1 t_{10} t_{37} = Ht_1 t_{10} t_{37} \]
\[ \implies Ht_1 t_{10} t_{37} = Ht_1 t_{15} t_{10} t_{37}, \text{ since } Ht_1 = Ht_{15} \]
\[ Ht_1 t_{10} t_{37} = Ht_{15} t_{10} t_{37} \]
\[ \implies Ht_1 t_{10} t_{37} = Ht_1^4 t_{10} t_{37}^{10} \]
\[ \implies Ht_1 t_{10} t_{37} = Ht_1^4 [x t_4^8 t_1^8] t_{10}^{10}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}]^{-1} = e^{x^{-1}} \]
\[ \implies x^{-1} t_{10} t_{9} t_{12} = e \]
\[ \implies x^{-1} t_2^3 t_4^8 t_1^4 = e \]
\[ \implies x^{-1} t_2^3 t_4^8 t_1^8 = t_4^8 \]
\[ x^{-1} t_2^3 t_4^8 t_1^4 = t_4^8 t_1^8 \]
\[ x x^{-1} t_2^3 = x t_4^8 t_1^8 \]
\[ t_2^3 = x t_4^8 t_1^8 \]
\[ Ht_1 t_{10} t_{37} = Ht_1^2 x t_4^8 t_1^8 t_{10}^{10} \]
\[ \implies Ht_1 t_{10} t_{37} = H x [t_4^8 t_1^8] t_{10}^{7} \]
\[ \implies Ht_1 t_{10} t_{37} = H t_4^8 t_1^8 t_{10}^{7} \]
\[ \implies Ht_1 t_{10} t_{37} = H t_1^4 t_{10}^{7} \]
\[ \implies Ht_1 t_{10} t_{37} = H t_4 t_{10}^{7}, \text{ since} \]
\[ Ht_4 = H t_{14} \]
\[ \implies Ht_4 = H t_2^4 \]
\[ Ht_1 t_{10} t_{37} = H t_2^4 t_{10}^{7} \]
\[ \implies Ht_1 t_{10} t_{37} = H [x y t_4^6 t_1^2] t_{10}^{7}, \text{ since by Equation 5.8} \]
\[ x^3 t_{11} t_{10} t_{9} = e \]
\[ [x^3 t_{11} t_{10} t_{9}]^{-1} = e^{x^{-1} y^{-1}} \]
\[ \implies y^{-1} x^{-1} t_1 t_{33} t_{20} = e \]
\[ \implies y^{-1} x^{-1} t_2^6 t_4^9 t_1^4 = e \]
\[ \implies y^{-1} x^{-1} t_2^6 t_4^9 t_1^8 t_4^6 = t_4^6 \]
\[ y^{-1} x^{-1} t_2^6 t_4^9 t_1^2 = t_4^6 t_1^2 \]
\[ x y y^{-1} x^{-1} t_2^4 = x y t_4^9 t_1^2 \]
\[ t_2^4 = x y t_4^9 t_1^2 \]
\[ Ht_1 t_{10} t_{37} = H x y t_4^9 t_1^2 t_{10}^{7} \]
\[ \implies Ht_1 t_{10} t_{37} = H t_4^6 t_{10}^{7} \]
\[ \Rightarrow H_{t}t_{10}t_{37} = H_{t}t_{24}t_{33} \]
\[ \Rightarrow H_{t}t_{10}t_{37} = H_{t}t_{22}t_{33}, \text{ since } H_{t_{22}} = H_{t_{24}} \]
\[ \Rightarrow H_{t}t_{10}t_{37} = H_{t}t_{22}t_{33} \]
\[ \Rightarrow H_{t}t_{10}t_{37} = H_{t}t_{33}t_{22}, \text{ since by Equation 5.9} \]
\[ H_{t}t_{6} = H_{t}t_{6}t_{1} \]
\[ \Rightarrow [H_{t_{6}}t_{6}]^{-2} = [H_{t_{6}}t_{1}]^{-2} \]
\[ \Rightarrow H_{t_{33}}t_{22} = H_{t_{22}}t_{33} \]
\[ H_{t}t_{10}t_{37} = H_{t}t_{33}t_{22} \]
\[ \Rightarrow H_{t}t_{10}t_{37} \in [16], \text{ since } H_{t}t_{33}t_{22} \text{ is in [1 6].} \]
2 symmetric generators will go to [1 6].

\[ H_{t}t_{10}t_{38} = H_{t}t_{10}t_{38} \]
\[ \Rightarrow H_{t}t_{10}t_{38} = H_{t}t_{1}t_{2}^{10} \]
\[ \Rightarrow H_{t}t_{10}t_{38} = H_{t}t_{1}t_{2}^{3} \]
\[ \Rightarrow H_{t}t_{10}t_{38} = H_{t}t_{1}t_{6} \]
\[ \Rightarrow H_{t}t_{10}t_{38} \in [16], \text{ since } H_{t}t_{1}t_{6} \text{ is in [1 6].} \]
2 symmetric generators will go to [1 6].

This concludes our double coset enumeration. Below is our completed Cayley Diagram.
Manual Double Coset Enumeration over a Maximal Subgroup of Order 720

Now we will perform Double Coset Enumeration over our other maximal subgroup. Recall that we had 2 maximal subgroups that contained both $f(x)$ and $f(y)$. We will examine subgroup 5.

We see that the order of this subgroup is 720, which is larger than our other subgroup. Next we find a representation of this larger subgroup in words.

```plaintext
> #M[5]'subgroup;
720
> D:=Conjugates(G1,M[5]'subgroup);
> D:=SetToSequence(D);
> f(x) in D[5] and f(y) in D[5];
true
> for g in D[5] do if sub<D[5]|f(x),f(y),g> eq D[5] then gg:=g;
for|if> end if;
for> end for;
> Order(gg);
```
> if Order(gg) = 4 then for i in [1..7920] do if ArrayP[i] = gg then Sch[i]; end if; end for; end if;

> y⁻¹ * t⁻¹ * x * t * y⁻¹ * t

> Order(f(y⁻¹ * t⁻¹ * x * t * y⁻¹ * t));

First we will expand our additional relations.
\[
y^{-1}t^{-1}xty^{-1}t \in H
\]
\[
y^4t^{-1}xty^4t_1 \in H
\]
\[
y^4t_1^0xty^4t_1 \in H
\]
\[
y^4t_{37}x[y^4y^{-4}]t_1y^4t_1 \in H
\]
\[
y^4t_{37}xy^4[y^{-4}t_1y^4]t_1 \in H
\]
\[
y^4t_{37}xy^4[t_1^4]t_1 \in H
\]
\[
y^4t_{37}xy^4t_9t_1 \in H
\]
\[
y^4[(xy^4)(xy^4)^{-1}]t_{37}xy^4t_9t_1 \in H
\]
\[
y^4xy^4[(xy^4)^{-1}t_{37}xy^4]t_9t_1 \in H
\]
\[
y^4xy^4[t_9^4]t_9t_1 \in H
\]
\[
y^4xy^4t_{22}t_9t_1 \in H
\]
\[
xy^2t_{22}t_9t_1 \in H
\]
\[
xy^2t_{22}t_1^3t_1 \in H
\]
\[
xy^2t_{22}t_1^4 \in H
\]
\[
Hxy^2t_{22}t_1^4 = H
\]
\[
Hxy^2t_{22}t_1^4t_1^7 = Ht_1^7
\]
\[
Hxy^2t_{22} = Ht_1^7
\]
\[
Hxy^2t_{22} = Ht_{25}
\]
\[
Ht_{22} = Ht_{25}
\]
\[(x^2t^3)^3 = e\]
\[(x^2t_1^3)^3 = e\]
\[(x^2t_{33})^3 = e\]

\[x^2t_{33}x^2t_{33}x^2t_{33} = e\]
\[x^2(x^2x^{-2})t_{33}x^2t_{33}x^2t_{33} = e\]
\[x^2x^2(x^{-2}t_{33}x^2)t_{33}x^2t_{33} = e\]
\[t_{33}^2t_{33}x^2t_{33} = e\]
\[t_{35}t_{33}x^2t_{33} = e\]
\[t_{35}(x^2x^{-2})t_{33}x^2t_{33} = e\]
\[t_{35}x^2(x^{-2}t_{33}x^2)t_{33} = e\]
\[t_{35}x^2t_{33}^2t_{33} = e\]
\[t_{35}x^2t_{35}t_{33} = e\]
\[(x^2x^{-2})t_{35}x^2t_{35}t_{33} = e\]
\[x^2(x^{-2}t_{35}x^2)t_{35}t_{33} = e\]
\[x^2t_{35}^2t_{35}t_{33} = e\]
\[x^2t_{33}t_{35}t_{33} = e\]
Our first double coset, $HeN = \{He^n | n \in N\} = \{H\}$, which we will denote by $[*]$.

The orbits of $N$ on \{1,2,3,4,5,6,8,7,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25, 26,27,28,29,30,31,32,33,34,35,36,37,38,39,40\} are \{1,2,3,3,4,14,17,4,11,35,15,10,18, 33,20,12,36,19,16,9\} and \{5,6,29,7,26,30,37,8,23,27,31,22,38,25,40,24,28,39,32,21\}.

We will take a representative from each orbit, say $t_1$ and $t_5$, and determine to which double coset $Ht_1$ and $Ht_5$ belong.

**Word of Length 1**

$Ht_1N$ is a new double coset which we will denote by [1].

$Ht_1N = \{Ht_1^n | n \in N\}$.

Since the orbit \{1, 2, 13, 3, 34, 14, 17, 4, 11, 35, 15, 10, 18, 33, 20, 12, 36, 19, 16, 9\} contains 20 elements then 20 symmetric generators will go to the new double coset [1].

Now $N^{(1)} \geq H^1$. 
We will look for a relation that will increase the Coset Stabiliser \( N^{(1)} \).

\[ N^1 = \{ e \} . \]
\[ N^{(1)} = \text{Coset Stabiliser in } N \text{ of } H t_1 = \{ n \in N | H t_1^n = t_1 \} . \]

\( H t_{25} = H t_{22} \), by Equation 5.10

\[ \Rightarrow H t_{17}^7 = H t_2^6 \]
\[ \Rightarrow H t_{17}^7 t_{15}^5 = H t_{21}^6 t_{15}^5 \]
\[ \Rightarrow H t_1 = H t_{21}^6 t_{15}^5 \]
\[ \Rightarrow H t_1 = H t_{21}^6 [ y x t_{43}^7 t_{15}^{10} ] \]

\[ x^3 t_{11} t_{10} t_9 = e \]
\[ [ x^3 t_{11} t_{10} t_9 ] x y^{-1} x = e x y^{-1} x \]
\[ \Rightarrow x^{-1} y^{-1} t_7 t_4 t_{15} = e \]
\[ \Rightarrow x^{-1} y^{-1} t_1^5 t_4 t_3^4 = e \]
\[ \Rightarrow x^{-1} y^{-1} t_1^5 t_4 t_3^4 t_{15}^7 = t_3^7 \]
\[ \Rightarrow x^{-1} y^{-1} t_1^5 t_4 t_3^4 t_{15}^{10} = t_3^7 t_4^{15} \]
\[ \Rightarrow y x x^{-1} y^{-1} t_1^5 = y x t_{34}^7 t_{15}^{10} \]
\[ \Rightarrow t_1^5 = y x t_{34}^7 t_{15}^{10} \]
\[ H t_1 = H t_{22}^6 y x t_{34}^7 t_{15}^{10} \]
\[ \Rightarrow H t_1 = H y x [ t_2^5 ] y x t_{34}^7 t_{15}^{10} \]
\[ \Rightarrow H t_1 = H t_{36}^7 t_{34}^{15} t_{15}^{10} \]
\[ \Rightarrow H t_1 = H t_{36}^7 t_{34}^{10} \]
\[ \Rightarrow H t_1 = H t_{36}^{10} \]

\( H t_{22} = H t_{25} \)

\[ \Rightarrow [ H t_{22} ] x^2 y^{-1} = [ H t_{25} ] x^2 y^{-1} \]
\[ \Rightarrow H t_{40} = H t_{23} \]
\[ \Rightarrow H t_{40}^3 = H t_{23}^3 \]
\[ H t_1 = H t_{40}^3 t_{14}^{10} \]
\[ \Rightarrow H t_1 = H t_{40}^9 t_{14}^{10} \]
\[ \Rightarrow H t_1 = H t_{36} \]
\[ \Rightarrow H t_1 = H t_{36} \]

Also, \( H t_1 = H t_{36} \)

\[ \Rightarrow [ H t_1 ] x^{-1} y^{-1} = [ H t_{36} ] x^{-1} y^{-1} \]
\[ \Rightarrow H t_{36} = H t_{11} . \]
and

\[ Ht_1 = Ht_{36} \]
\[ \implies [Ht_1]^{x^2y} = [Ht_{36}]^{x^2y} \]
\[ \implies Ht_{11} = Ht_{14}, \]

and

\[ Ht_1 = Ht_{36} \]
\[ \implies [Ht_1]^{yx} = [Ht_{36}]^{yx} \]
\[ \implies Ht_{14} = Ht_1. \]

Thus, \( Ht_1 = Ht_{36} = Ht_{11} = Ht_{14} \)

Now, since \( Ht_1^5 = Ht_1 \Rightarrow e \in N^{(1)}, \) and

\[ Ht_1^{x^{-1}y^{-1}} = Ht_{36} = Ht_1 \Rightarrow x^{-1}y^{-1} \in N^{(1)}, \]
\[ Ht_1^{x^2y} = Ht_{11} = Ht_1 \Rightarrow x^2y \in N^{(1)}, \]
\[ Ht_1^{yx} = Ht_{14} = Ht_1 \Rightarrow yx \in N^{(1)}, \]

then, \( N^{(1)} = \text{Coset Stabiliser in } N \) of \( Ht_1 = \{ n \in N | Ht_1^n = t_1 \} = \{ e, x^{-1}y^{-1}, x^2y, yx \}. \)

Furthermore, the number of single cosets of \( Ht_1N \) is \( \frac{|N|}{|N^{(1)}|} = \frac{20}{4} = 5. \)

Conjugating by elements in \( N \) gives us the following equal names.

\[ t_1 \sim t_{36} \sim t_{11} \sim t_{14} \]
\[ t_2 \sim t_{33} \sim t_{12} \sim t_{15} \]
\[ t_3 \sim t_{34} \sim t_9 \sim t_{16} \]

Therefore, \( Ht_1N = \{ Ht_1 = Ht_{36} = Ht_{11} = Ht_{14}, Ht_2 = Ht_{33} = Ht_{12} = Ht_{15}, \]
\[ Ht_3 = Ht_{34} = Ht_9 = Ht_{16}, Ht_4 = Ht_{35} = Ht_{10} = Ht_{13}, \]
\[ Ht_{17} = Ht_{20} = Ht_{19} = Ht_{18} \}

Now \( N^{(5)} \geq N^5. \)
\[ N^5 = \{ e \}. \]
\[ N^{(5)} = \text{Coset Stabiliser in } N \) of \( Ht_5 = \{ n \in N | Ht_5^n = t_5 \}. \)
We will look for a relation that will increase the Coset Stabiliser $N^{(5)}$.

$Ht_{22} = Ht_{25}$, by Equation 5.10

$\implies [Ht_{22}]^{y^{-1}x} = [Ht_{25}]^{y^{-1}x}$

$\implies Ht_5 = Ht_8$.

Also,

$Ht_{22} = Ht_{25}$

$\implies [Ht_{22}]^{y^{-1}x} = [Ht_{25}]^{y^{-1}x}$

$\implies Ht_8 = Ht_7$.

and

$Ht_{22} = Ht_{25}$

$\implies [Ht_{22}]^{y^{-1}} = [Ht_{25}]^{y^{-1}}$

$\implies Ht_7 = Ht_6$.

and

$Ht_{22} = Ht_{25}$

$\implies [Ht_{22}]^{y^2} = [Ht_{25}]^{y^2}$

$\implies Ht_6 = Ht_5$.

Thus $Ht_5 = Ht_6 = Ht_7 = Ht_8$.

Since $Ht_{5}^e = Ht_5 \implies e \in N^{(5)}$,

$Ht_{5}^x = Ht_6 = Ht_5 \implies x \in N^{(5)}$,

$Ht_{5}^{x^2} = Ht_7 = Ht_5 \implies x^2 \in N^{(5)}$,

$Ht_{5}^{x^{-1}} = Ht_8 = Ht_5 \implies x^{-1} \in N^{(5)}$, then,

$N^{(5)} =$ Coset Stabiliser in $N$ of $Ht_5 = \{ n \in N | (Ht_5)^n = t_5 \} = \{ e, x, x^2, x^{-1} \}$.

Furthermore, the number of single cosets of $Ht_5 N$ is $\frac{|N|}{|N^{(5)}|} = \frac{20}{4} = 5$.

Conjugating by elements in $N$ gives us the following equal names.

$t_5 \sim t_6 \sim t_7 \sim t_8 \quad t_{32} \sim t_{25} \sim t_{22} \sim t_{39}$

$t_{29} \sim t_{26} \sim t_{23} \sim t_{40} \quad t_{31} \sim t_{28} \sim t_{21} \sim t_{38}$

$t_{30} \sim t_{27} \sim t_{24} \sim t_{37}$
Thus $Ht_5 N = \{ Ht_5 = Ht_6 = Ht_7 = Ht_8, Ht_{29} = Ht_{26} = Ht_{23} = Ht_{40},$

$Ht_{30} = Ht_{27} = Ht_{24} = Ht_{37}, Ht_{32} = Ht_{25} = Ht_{39},$

$Ht_{31} = Ht_{28} = Ht_{21} = Ht_{38} \}$. 

The orbits of $N^{(1)}$ are $\{ 1, 14, 11, 36 \}, \{ 2, 35, 20, 9 \}, \{ 3, 12, 13, 18 \}, \{ 4, 17, 34, 15 \},$

$\{ 5, 30, 23, 28 \}, \{ 6, 27, 40, 21 \}, \{ 7, 24, 29, 38 \}, \{ 8, 37, 26, 31 \}, \{ 10, 19, 16, 33 \}, $ and 

$\{ 22, 39, 32, 25 \}$. 

We will check to see where $t_1 t_1, t_1 t_9, t_1 t_{13}, t_1 t_{17}, t_1 t_5, t_1 t_{21}, t_1 t_{29}, t_1 t_{37}, t_1 t_{33},$ and $t_1 t_{25}$ belong.

$$Ht_{1} \cdot t_{1} = Ht_{1}^{2}$$

$\implies Ht_{1} \cdot t_{1} = Ht_{5}$

$\implies Ht_{1} \cdot t_{1} \in [5],$ since $Ht_{5}$ is in $[5]$. 

4 Symmetric generators will go to $[5]$. 

$$Ht_{1} \cdot t_{9} = Ht_{1} \cdot t_{1}^{3}$$

$\implies Ht_{1} \cdot t_{9} = Ht_{1}^{4}$

$\implies Ht_{1} \cdot t_{9} = Ht_{13}$

$\implies Ht_{1} \cdot t_{9} \in [1],$ since $Ht_{13}$ is in $[1]$. 

4 Symmetric generators will go to $[1]$. 

$$Ht_{1} \cdot t_{13} = Ht_{1} \cdot t_{1}^{4}$$

$\implies Ht_{1} \cdot t_{13} = Ht_{1}^{5}$

$\implies Ht_{1} \cdot t_{13} = Ht_{17}$

$\implies Ht_{1} \cdot t_{13} \in [1],$ since $Ht_{17}$ is in $[1]$. 

4 Symmetric generators will go to $[1]$. 

$$Ht_{1} \cdot t_{17} = Ht_{1} \cdot t_{1}^{5}$$

$\implies Ht_{1} \cdot t_{17} = Ht_{1}^{6}$

$\implies Ht_{1} \cdot t_{17} = Ht_{21}$

$\implies Ht_{1} \cdot t_{17} \in [5],$ since $Ht_{21}$ is in $[5]$. 

4 Symmetric generators will go to [5].

\[ Ht_1 t_5 = Ht_1 t_1^2 \]
\[ \Rightarrow Ht_1 t_5 = Ht_1^3 \]
\[ \Rightarrow Ht_1 t_5 = Ht_9 \]
\[ \Rightarrow Ht_1 t_5 \in [1], \text{ since } Ht_9 \text{ is in [1].} \]

4 Symmetric generators will go to [1].

\[ Ht_1 t_{21} = Ht_1 t_1^6 \]
\[ \Rightarrow Ht_1 t_{21} = Ht_1^7 \]
\[ \Rightarrow Ht_1 t_{21} = Ht_{25} \]
\[ \Rightarrow Ht_1 t_{21} \in [5], \text{ since } Ht_{25} \text{ is in [5].} \]

4 Symmetric generators will go to [5].

\[ Ht_1 t_{29} = Ht_1 t_1^8 \]
\[ \Rightarrow Ht_1 t_{29} = Ht_1^9 \]
\[ \Rightarrow Ht_1 t_{29} = Ht_{33} \]
\[ \Rightarrow Ht_1 t_{29} \in [1], \text{ since } Ht_{33} \text{ is in [1].} \]

4 Symmetric generators will go to [1].

\[ Ht_1 t_{37} = Ht_1 t_1^{10} \]
\[ \Rightarrow Ht_1 t_{37} = H \]
\[ \Rightarrow Ht_1 t_{37} \in [1], \text{ since } H \text{ is in [*].} \]

4 Symmetric generators will go to [*].

\[ Ht_1 t_{33} = Ht_1 t_1^9 \]
\[ \Rightarrow Ht_1 t_{33} = Ht_1^{10} \]
\[ \Rightarrow Ht_1 t_{33} = Ht_{37} \]
\[ \Rightarrow Ht_1 t_{33} \in [5], \text{ since } Ht_{37} \text{ is in [5].} \]

4 Symmetric generators will go to [5].

\[ Ht_1 t_{25} = Ht_1 t_1^7 \]
\[ H_{t_1}t_{25} = H_{t_1}^8 \]
\[ H_{t_1}t_{25} = H_{t_29} \]
\[ H_{t_1}t_{25} \in [5], \text{ since } H_{t_29} \text{ is in } [5]. \]

4 Symmetric generators will go to [5].

The orbits of \( N^{(5)} \) are \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, \{13, 14, 15, 16\}, \{17, 18, 19, 20\}, \{21, 22, 23, 24\}, \{25, 26, 27, 28\}, \{29, 30, 31, 32\}, \{33, 34, 35, 36\}, \text{ and } \{37, 38, 39, 40\}.

We will check to see where \( t_5t_1, t_5t_5, t_5t_9, t_5t_{13}, t_5t_{17}, t_5t_{21}, t_5t_{25}, t_5t_{29}, t_5t_{33}, \text{ and } t_5t_{37} \) belong.

\[ H_{t_5}t_{1} = H_{t_1}^2t_{1} \]
\[ \implies H_{t_5}t_{1} = H_{t_1}^3 \]
\[ \implies H_{t_5}t_{1} = H_{t_9} \]
\[ \implies H_{t_5}t_{1} \in [1], \text{ since } H_{t_9} \text{ is in } [1]. \]

4 symmetric generators will go to [1].

\[ H_{t_5}t_{5} = H_{t_1}^2t_{1}^2 \]
\[ \implies H_{t_5}t_{5} = H_{t_1}^4 \]
\[ \implies H_{t_5}t_{5} = H_{t_{13}} \]
\[ \implies H_{t_5}t_{5} \in [1], \text{ since } H_{t_{13}} \text{ is in } [1]. \]

4 symmetric generators will go to [1].

\[ H_{t_5}t_{9} = H_{t_1}^2t_{1}^3 \]
\[ \implies H_{t_5}t_{9} = H_{t_1}^5 \]
\[ \implies H_{t_5}t_{9} = H_{t_{17}} \]
\[ \implies H_{t_5}t_{9} \in [1], \text{ since } H_{t_{17}} \text{ is in } [1]. \]

4 symmetric generators will go to [1].

\[ H_{t_5}t_{13} = H_{t_1}^2t_{1}^4 \]
\[ \implies H_{t_5}t_{13} = H_{t_1}^6 \]
\[ H_{t_{13}} = H_{t_{21}} \]
\[ H_{t_{13}} \in [5], \text{ since } H_{t_{21}} \text{ is in [5].} \]
4 symmetric generators will go to [5].

\[ H_{t_{17}} = H_{t_{15}^2 t_{1}^5} \]
\[ H_{t_{17}} \in [5], \text{ since } H_{t_{25}} \text{ is in [5].} \]
4 symmetric generators will go to [5].

\[ H_{t_{21}} = H_{t_{15}^2 t_{1}^6} \]
\[ H_{t_{21}} \in [5], \text{ since } H_{t_{29}} \text{ is in [5].} \]
4 symmetric generators will go to [5].

\[ H_{t_{25}} = H_{t_{15}^2 t_{1}^7} \]
\[ H_{t_{25}} \in [1], \text{ since } H_{t_{33}} \text{ is in [1].} \]
4 symmetric generators will go to [1].

\[ H_{t_{29}} = H_{t_{15}^2 t_{1}^8} \]
\[ H_{t_{29}} \in [5], \text{ since } H_{t_{37}} \text{ is in [5].} \]
4 symmetric generators will go to [5].

\[ H_{t_{33}} = H_{t_{15}^2 t_{1}^9} \]
\[ H_{t_{33}} \in [\ast], \text{ since } H_e \text{ is in [\ast].} \]
4 symmetric generators will go to [\ast].
\[
Ht_{5t37} = Ht_{11t10}^2 \\
\implies Ht_{5t37} = Ht_1 \\
\implies Ht_{5t37} \in [1], \text{ since } Ht_1 \text{ is in } [1].
\]
4 symmetric generators will go to [1].

This conclude our Double Coset Enumeration. Below is our Cayley Diagram.

---

5.5 \((S(4, 3) : 2)\) as a Homomorphict Image of \(2^{*10} : S_5\)

5.5.1 Factor by Center of \(G\)

Let \(G \cong 2^{*20} : (2^4 : S_5)\) be a symmetric presentation of \(G\) given by
\[
G = \langle x, y, t | x^6, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xyx^{-1}, y^{-1}x^{-1}yx^{-1}y^{-1}xy^3x, \\
(x^{-1}y^2x^{-1}y^{-1})^2, t^2, (t, yx^2y^{-2}x^{-1}yx^{-1}), (t, x^{-1}y^{-1}x^{-1}y^{-3}x^{-1}y^{-1}), \\
(t, (yxy^{-1})^3), (t, y^{-1}x^3y^{-2}) \rangle, \text{ where}
\]
x = (1, 16, 9, 2, 15, 10)(3, 18, 11)(4, 17, 12)(5, 6)(7, 14, 19)(8, 13, 20),
y = (1, 14, 15, 2, 13, 16)(3, 19, 6, 8, 18, 10, 4, 20, 5, 7, 17, 9),
\(N = \langle x, y \rangle\), and the order of \(N\) is 1920.
Let us factor the progenitor $2^{*20} : (2^4 : S_5)$ by $[(ytx^3)^6, (ytx^2y^2)^4, (ytx^2y)^8]$.

The Composition Factors of $G$ are given below.

$$
\begin{array}{c|c}
G & \text{Cyclic(2)} \\
* & (2, 3) = S(4, 3) \\
* & \text{Cyclic(2)} \\
1 & \\
\end{array}
$$

However, now our control group has changed.

$$> \#_{\text{sub}<G|x,y>};$$
$$120$$

The order of $N$ is 120 instead of 1920.

We look at our original control group given by

$$<x, y|x^6, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xy^{-1}, y^{-1}x^{-1}yx^{-1}y^{-1}x^{-1}y^3x, (x^{-1}y^2x^{-1}y^{-1})^2>.$$  

Note that $|x| = 6$, $|xy^{-1}| = 4$, $|yxy^{-1}x^{-2}y^{-1}xy^{-1}| = 1$, $|y^{-1}x^{-1}yx^{-1}y^{-1}x^{-1}y^3x| = 1$, and $|x^{-1}y^2x^{-1}y^{-1}| = 2$.

We also note that $|f(x)| = 3$, $|f(xy^{-1})| = 4$, $|f(yxy^{-1}x^{-2}y^{-1}xy^{-1})| = 1$, $|f(y^{-1}x^{-1}yx^{-1}y^{-1}x^{-1}y^3x)| = 1$, and $|f(x^{-1}y^2x^{-1}y^{-1})| = 2$.

So then we change our control group to the following symmetric representation.

$$<x, y|x^3, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xy^{-1}, y^{-1}x^{-1}yx^{-1}y^{-1}x^{-1}y^3x, (x^{-1}y^2x^{-1}y^{-1})^2>.$$
Our new control group of order 120 is a permutation representation of $NN$ over the Stabiliser($N, 1$), which we will call $H$.

Now we check in MAGMA.

Now we check in MAGMA.
true
> s:=IsIsomorphic(N,Sym(5));s;
true

Therefore, \( G = \langle x, y, t | x^3, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xyx^{-1}, y^{-1}x^{-1}yx^{-1}y^{-1}xy^3x, \\
(x^{-1}y^2x^{-1}y^{-1})^2, t^2, (t, yx^2y^{-2}x^{-1}yx^{-1}), (t, x^{-1}y^{-1}x^{-1}y^{-1}xy^{-1}), (t, (yx^{-1})^3), \\
t, y^{-1}x^3y^{-2}, (xyt^{-1})^6, (xyt^2x^2y^2)^4, (xyt^2y)^8 \rangle \)

is isomorphic to

\[
\frac{2^{10} \cdot S_5}{[(xyt^{-1})^6, (xyt^2x^2y^2)^4, (xyt^2y)^8]}
\]

The composition factors of \( G \) are given below.

\[
G \\
| Cyclic(2) \\
* \\
| C(2, 3) = S(4, 3) \\
* \\
| Cyclic(2) \\
1
\]

Now we want to factor \( G \) by \( C_2 \) to obtain the following composition series for \( G \),
\( G = G_1 \supseteq 1 \), where \( G = (G_1/G_2)(G_2/1) = C_2S(4, 3) \).

We find the Normal Lattice of \( G \).

> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------
---
We see that $NL[2]$ is of order 2. We check to see if $NL[2]$ is equal to the center of $G$.

> NL[2] eq Center(G1);
true

We use our Schreier System to write the generators of $NL[2]$ in terms of $x, y$, and $t$.

> IN:=sub<G1|f(x),f(y)>;
> N:=IN;
> #N;
120
> #G;
103680
> N:=G1;
> NN:=G;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..103680]];
> for i in [2..103680] do
> P:=[Id(N): l in [1..#Sch[i]]];
> for j in [1..#Sch[i]] do
> if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
> if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
> if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x^-1); end if;
> if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y^-1); end if;
> if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
> end for;
> PP:=Id(N);
> for k in [1..#P] do
> PP:=PP*P[k]; end for;
> ArrayP[i]:=PP;
> end for;
> for i in [1..103680] do if ArrayP[i] eq NL[2].1 then Sch[i];
> end if; end for;
Id(G)
> for i in [1..103680] do if ArrayP[i] eq NL[2].2 then Sch[i]; end if; end for;
\[
x * t * x * y^-1 * x * t * y * x * t * x * y^-1 * x * t * y^-1 * t * x * t * y
\]
We now have the following presentation for \( G \).
\[
G = <x, y, t | x^3, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xxy^{-1}, y^{-1}x^{-1}yx^{-1}, y^{-1}x^{-1}yx^{-1}, x^{-1}y^2x^{-1}y^{-1} >^2,
\]
\[
t^2, (t, yx^2y^{-2}x^{-1}yx^{-1}), (t, x^{-1}y^3x^{-1}y^{-3}x^{-1}), (t, (yx^{-1})^3), (t, y^{-1}x^3y^{-2}), (xyt^2)^6,
\]
\[
(xyt^2x^2y^2)^4, (xyt^2y)^8, xtxy^{-1}xtxy^{-1}xtxy^{-1}txy >.
\]

The Composition Factors of \( G \) are,
\[
\begin{array}{c|c}
G & \text{Cyclic(2)} \\
* & C(2, 3) = S(4, 3) \\
1 &
\end{array}
\]

**5.5.2 The Construction of \((S(4, 3) : 2)\) Over \( S_5 \)**

We are now ready to perform Double Coset Enumeration on the progenitor
\[ 2^{*10} : S_5, \]
factored by \( (xyt^2x^2y^2)^6, (xyt^2y)^8, xtxy^{-1}xtxy^{-1}xtxy^{-1}txy >. \)

Let \( G \cong 2^{*10} : S_5 \) be a symmetric presentation of \( G \) given by
\[
< x, y, t | x^3, (xy^{-1})^4, yxy^{-1}x^{-2}y^{-1}xxy^{-1}, y^{-1}x^{-1}yx^{-1}, y^{-1}x^{-1}yx^{-1}, x^{-1}y^2x^{-1}y^{-1} >^2,
\]
\[
t^2, (t, yx^2y^{-2}x^{-1}yx^{-1}), (t, x^{-1}y^3x^{-1}y^{-3}x^{-1}), (t, (yx^{-1})^3), (t, y^{-1}x^3y^{-2}), (xyt^2)^6,
\]
\[
(xyt^2x^2y^2)^4, (xyt^2y)^8, xtxy^{-1}xtxy^{-1}xtxy^{-1}txy >, \]
where
\[
N \cong S_5 = < x, y >, x = (1,2)(3,4)(5,6)(7,8)(9,10), \text{ and } y = (1,3,5,7,9,10,8,6,4,2).
\]

We will enter this presentation for \( G \) and label our permutations \( x \) and \( y \) as well as our control group, \( N = < x, y >, \) then verify that we have \( N \cong S_5. \)

\[
> G<x, y, t>:=\text{Group}<x, y, t | x^3, (x*y^{-1})^4, y*x*y^{-1}x^{-2}y^{-1}x^{-2}y^{-1}x*y*
\]
\[
x^{-1}, y^-1*x^-1*y^-1*x^-1*y^{-1}x*y^{-1}x^-1*y^-1>x^-1*y^-1*x^-1*y^-1*x*y^-1
\]
\[
> t^2, (t, y*x^-2*y^-2*x^-1*y^-1), (t, x^-1*y^-1*x^-1*y^-3*x*y^-1),
\]
> (t, (y*x*y^-1)^3), (t, y^-1*x^3*y^-2), (x*y*t^((x^3)))^6, (x*y*t^((x^2*y))^-4),
> (x*y*t^((x^2*y))^-8, x*t*x*y^-1*x*t*y*x*t^*
> x*y^-1*x*t*y^-1*t*x*t*y;)
> S:=Sym(10);
> xx:=S!(1,2,4)(3,5,6)(7,8,10);
> yy:=S!(1,3,2)(4,7,5,9,6,8);
> N:=sub<S|xx,yy>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);

\[
\begin{array}{c}
G \\
| Cyclic(2) \\
| C(2, 3) = S(4, 3) \\
1 \\
\end{array}
\]

> s:=IsIsomorphic(N,Sym(5));s;
true

We then use MAGMA to calculate the number of double cosets of \( G \) over \( N \) as well as to name our \( t_i \)'s. Note that when naming our \( t_i \)'s, \( t_1 = t \), then \( t_2 = t^x \), since \( x = (1,2,4)(3,5,6)(7,8,10) \) takes \( t_1 \) to \( t_2 \). Similarly, \( t_3 = t^y \), since \( y = (1,3,2)(4,7,5,9,6,8) \) takes \( t_1 \) to \( t_3 \), also \( t_4 = t^{x^2} \), since \( x^2 = (1,4,2)(3,6,5)(7,10,8) \) takes \( t_1 \) to \( t_4 \). This process is repeated until all of our \( t_i \)'s have been named.

> #DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
20
> IN:=sub<G1|f(x),f(y)>;
> ts := [Id(G1): i in [1 .. 10] ];
> ts[1]:=f(t); ts[2]:=f(t*x); ts[3]:=f(t*y); ts[4]:=f(t*(x^2));
> ts[5]:=f(t^*(y*x)); ts[6]:=f(t^*(y*x^2)); ts[7]:=f(t^*(x^2*y));
> ts[8]:=f(t^*(x^2*y*x)); ts[9]:=f(t^*(y*x*y));
> ts[10]:=f(t^*(x^2*y*x^2));

So we will have 20 double cosets. The number of single cosets is equal to

\[
\frac{|G|}{|N|} = \frac{51840}{120} = 432.
\]

We will use the following loop to keep count of the single cosets.

It is important in this loop that we input the number of \( t_i \)'s that we have, 10, as well as the number of single cosets that we have, 432. The coset counter will give us a running total of how many single cosets we have thus far, starting with our second double coset [1]. It does not keep count of the 1 single coset in [*].
> prodim:=function(pt, Q, I)
function> v:=pt;
function> for i in I do
function|for> v := v^(Q[i]);
function|for> end for;
function> return v;
function> end function;
> #G/#N;
432
> cst := [null : i in [1 .. Index(G,sub<G|x,y>)]] where null is
> [Integers() | ];
> for i := 1 to 10 do
for> cst[prodim(1, ts, [i])] := [i];
for> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
10

Words of Length 1

Our first double coset is $NcN$, denoted by $[*]$. 

$[*] = \frac{|N|}{|N|} = \frac{120}{120} = 1$ single coset.

> Orbits(N);
[ GSet({0 1, 2, 3, 4, 5, 7, 6, 9, 8, 10 0}) ]

The orbit of $N$ on \{1,2,3,4,5,6,7,8,9,10\} is \{1,2,3,4,5,6,7,8,9,10\}. We pick a representative from the orbit, say 1, and determine the double coset that contains $Nt_1$. 


$Nt_1N$ is a new double coset which we will denote by $[1]$. Since the orbit $\{1,2,3,4,5,6,7,8,9,10\}$ contains ten elements, then ten symmetric generators will go to the new double coset $[1]$. Recall that our coset counter, $m$, was at 10.

We will now examine our double coset $[1]$. Our representative of this double coset is $Nt_1$. The following code labels the point stabiliser of 1 in $N$ as $N1$. Then we label the set $SSS$, which is made up of $t_1$ conjugated by all of the elements of $N$.

```plaintext
> N1:=Stabiliser(N,[1]);
> SSS:={[1]};
> SSS:=SSS^N;
> #SSS;
10
> Seqq:=Setseq(SSS);
```

The next loop tells us if we have any equal names, that is, if any of our cosets in $[1]$ are equal to each other.

```plaintext
> for i in [1..#SSS] do
  for n in IN do
    if ts[1] eq n*ts[Rep(Seqq[i])[1]] then print Rep(Seqq[i]); end if;
  end for;
end for;
```

From the above loop we see that there are not any equal names in $[1]$. Thus the coset stabiliser of 1 in $N$ is equal to the point stabiliser of 1 in $N$. We compute the transversals of $N^{(1)}$ in $N$ and label this as $T1$. Then we use our coset counter to see how many single cosets we have thus far.

```plaintext
> T1:=Transversal(N,N1s);
> for i in [1..#T1] do
  ss:=[1]^T1[i];
  cst[prodim(1, ts, ss)]:=ss;
end for;
> m:=0; for i in [1..432] do if cst[i] ne [] then m:=m+1; end if; end for; m;
10
```
Now we can use MAGMA to find the elements in the set $N^{(1)}$. To find the distinct single cosets in [1], we first find the transversals, then conjugate $Nt_1$ by each of the elements in the set of transversals.

```magma
> #N1s;
12
> Set(N1s);
{
    Id(N1),
    (2, 6, 9, 3, 4, 7) (5, 10, 8),
    (2, 4) (3, 6) (8, 10),
    (2, 9) (3, 7) (5, 10),
    (2, 7, 4, 3, 9, 6) (5, 8, 10),
    (2, 9, 4) (3, 7, 6) (5, 8, 10),
    (4, 9) (5, 8) (6, 7),
    (2, 6) (3, 4) (7, 9) (8, 10),
    (2, 7) (3, 9) (4, 6) (5, 10),
    (2, 4, 9) (3, 6, 7) (5, 10, 8),
    (2, 3) (4, 6) (7, 9),
    (2, 3) (4, 7) (5, 8) (6, 9)
}
> for i in [1..#T1] do ([1]^N1s)^T1[i]; end for;
{[@
    [ 1 ]
}@
{[@
    [ 2 ]
}@
{[@
    [ 3 ]
}@
{[@
    [ 4 ]
}@
{[@
    [ 5 ]
}@
{[@
    [ 6 ]
}@
{[@
    [ 7 ]
}@
\[ N^1 = N^{(1)} = \{e, (2, 6, 9, 3, 4, 7)(5, 10, 8), (2, 4)(3, 6)(8, 10), (2, 9)(3, 7)(5, 10), (2, 9, 4)(3, 7, 6) (5, 8, 10), (2, 7, 4, 3, 9, 6)(5, 8, 10), (2, 6)(3, 4)(7, 9)(8, 10), (4, 9)(5, 8)(6, 7), (2, 7)(3, 9)(4, 6) (5, 10), (2, 4, 9)(3, 6, 7)(5, 10, 8), (2, 3)(4, 6)(7, 9), (2, 3)(4, 7)(5, 8)(6, 9)\}. \]

The number of single cosets in \( N_t^1 N \) is \( \frac{|N|}{|N^{(1)}|} = \frac{120}{10} = 10 \).

\( Nt_1 N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}\} \).

Lastly, we need to compute the orbits of \( N^{(1)} \).

\[
\text{Orbits(N1s);}
\]

\[
\text{[}
\text{\{1\}}
\text{\{5, 10, 8\}}
\text{\{2, 7, 4, 6, 9\}}
\text{\}}
\]

The orbits of the coset stabilier \( N^{(1)} \) on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) are

\( \{1\}, \{5, 10, 8\}, \) and \( \{2, 7, 4, 3, 6, 9\} \).

We take \( t_1, t_5, \) and \( t_2 \) from each orbit respectively.

We want to determine to which double coset \( Nt_1 t_1, Nt_1 t_5, \) and \( Nt_1 t_2 \) belong.

\( Nt_1 t_1 = N \in [\ast] \) (Since our \( t \)'s are of order 2.)

Since the orbit \( \{1\} \) contains one element, then one symmetric generator goes back to the double coset \([\ast]\).
Now, so far we have found the double cosets $[\ast]$ and $[1]$. We will use the following loop to see if $N_{t_1}t_5$, and $N_{t_1}t_2$ belong in these double cosets. If not, then then they will go on to a new double coset.

```magma
> for m,n in IN do if ts[1]*ts[5] eq m*(ts[1]*ts[1])^n
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5] eq m*(ts[1])^n
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2] eq m*(ts[1]*ts[1])^n
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n
for|if> then "true"; break; end if; end for;
```

Since MAGMA did not return any output with the above loop, then we have two new double cosets.

$N_{t_1}t_5N$ is a new double coset which we will denote $[15]$.
Since the orbit $\{5,10,8\}$ contains three elements, then three symmetric generators will go to the new double coset $[15]$.

$N_{t_1}t_2N$ is a new double coset which we will denote $[12]$.
Since the orbit $\{2,7,4,3,6,9\}$ contains six elements, then six symmetric generators will go to the new double coset $[12]$.

**Words of Length 2**

Now we move on to our first double coset of length 2. In the same manner as before we look for equal names.

```magma
> N15:=Stabiliser(N,[1,5]);
> SSS:=[1,5];
> SSS:=SSS^N;
> #SSS;
30
```
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for n in IN do
for if ts[1]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
end if; end for; end for;
[ 1, 5 ]

Since we do not have a relation that will increase our coset stabiliser, then $N_{15} = N^{(15)}$. We input this and check our coset counter.

> T15:=Transversal(N,N15s);
> for i in [1..#T15] do
ss:=[1,5]^T15[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
> m:=0; for i in [1..432] do if cst[i] ne []
then m:=m+1; end if; end for; m;
40

Our coset counter has increased from 10 to 40. We should have 30 distinct single cosets in [15]. We find the distinct single cosets as well as the orbits of $N^{(15)}$.

> [1,5]^N15s;
GSet{@
 [ 1, 5 ]
}@
> for i in [1..#T15] do ([1,5]^N15s)^T15[i]; end for;
{@
 [ 1, 5 ]
}@
{@
 [ 1, 10 ]
}@
{@
 [ 1, 8 ]
}@
{@
 [ 2, 6 ]
}@
Orbits(N15s);
[7, 2]
[7, 4]
[7, 5]
[8, 1]
[8, 6]
[8, 4]
[9, 6]
[9, 5]
[9, 3]
[10, 1]
[10, 2]
[10, 3]
GSet(@ 1 @),
GSet(@ 5 @),
GSet(@ 7, 9 @),
GSet(@ 8, 10 @),
GSet(@ 2, 6, 4, 3 @)
\[ N^{(15)} = \{ e, (24)(36)(810), (26)(34)(79)(810), (23)(46)(79) \}. \] The number of the single cosets in the double coset \( Nt_1t_5N \) is at most \( \frac{|N|}{|N^{(15)}|} = \frac{120}{4} = 30. \)

\[ Nt_1t_5N = \{ Nt_1t_5, Nt_1t_10, Nt_1t_8, Nt_2t_6, Nt_2t_7, Nt_2t_10, Nt_3t_9, Nt_3t_10, Nt_3t_4, Nt_4t_3, Nt_4t_8, Nt_4t_7, Nt_5t_9, Nt_5t_7, Nt_5t_1, Nt_6t_9, Nt_6t_8, Nt_6t_2, Nt_7t_2, Nt_7t_4, Nt_7t_5, Nt_8t_1, Nt_8t_6, Nt_8t_4, Nt_9t_6, Nt_9t_5, Nt_9t_3, Nt_{10}t_1, Nt_{10}t_2, Nt_{10}t_3 \}. \]

Similarly we examine our other double coset \([12]\).

\[
> \text{N12:=Stabiliser(N,[1,2]);} \\
> \text{SSS:=[[1,2]];} \\
> \text{SSS:=SSS^N;} \\
> \text{#SSS;} \\
60 \\
> \text{Seqq:=Setseq(SSS);} \\
> \text{for i in [1..#SSS] do} \\
> \text{for n in IN do} \\
> \text{if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]} \\
> \text{then print Rep(Seqq[i]);} \\
> \text{end if; end for; end for;} \\
> \text{[ 1, 2 ]} \\
> \text{[ 1, 3 ]} \\
\]

\text{MAGMA} tells us that \( Nt_1t_2 = Nt_1t_3 \). So we now have a relation that increases \( N^{(12)} \).

We enter these equal names with the following code and check our coset counter.

\[
> \text{N12s:=N12;} \\
> \text{for n in N do if 1^n eq 1 and 2^n eq 3 then} \\
> \text{N12s:=sub<N|N12s,n>; end if; end for;} \\
> \text{N12s; \#N12s;} \\
\text{Permutation group N12s acting on a set of cardinality 10} \\
\text{(4, 9) (5, 8) (6, 7)} \\
\text{(2, 3) (4, 7) (5, 8) (6, 9)} \\
\text{(2, 3) (4, 6) (7, 9)} \\
4 \\
> \text{#N/#N12s;} \\
30 \\
> \text{T12:=Transversal(N,N12s);} \\
> \text{for i in [1..T12] do}
\]
Our coset counter is now at 70, so we must have 30 distinct single cosets in [12]. We will find the distinct single cosets as well as the orbits of $N^{(12)}$. 

```plaintext
> #N12s;
4
> Set(N12s);
{ (2, 3) (4, 7) (5, 8) (6, 9),
   (4, 9) (5, 8) (6, 7),
   (2, 3) (4, 6) (7, 9),
   Id(N12s) }
> for i in [1..#T12] do ([1,2]^N12s)^T12[i]; end for;
{ @ [ 1, 2 ],
   [ 1, 3 ] @ }
{ @ [ 1, 6 ],
   [ 1, 4 ] @ }
{ @ [ 1, 7 ],
   [ 1, 9 ] @ }
{ @ [ 2, 4 ],
   [ 2, 5 ] @ }
{ @ [ 2, 3 ],
   [ 2, 1 ] @ }
{ @ [ 2, 9 ],
   [ 2, 9 ] @ }
```
[6, 7],
[6, 10]
}
{[6, 5],
[6, 3]
}
{[7, 3],
[7, 8]
}
{[7, 9],
[7, 1]
}
{[7, 10],
[7, 6]
}
{[8, 2],
[8, 9]
}
{[8, 7],
[8, 3]
}
{[8, 5],
[8, 10]
}
{[9, 1],
[9, 7]
}
{[9, 4],
[9, 10]
}
{[9, 8],
[9, 2]
}
{
> Orbits(N12s);

\[
\begin{align*}
\text{GSet}(\emptyset, 1), & \\
\text{GSet}(\emptyset, 10), & \\
\text{GSet}(\emptyset, 2, 3), & \\
\text{GSet}(\emptyset, 5, 8), & \\
\text{GSet}(\emptyset, 4, 9, 7, 6) & \\
\end{align*}
\]

\[
N_{(12)} = \{ e, (23)(47)(58)(69), (49)(58)(67), (23)(46)(79) \}. \]

The number of the single cosets in the double coset \( N_1t_2N \) is at most

\[
\frac{|N|}{|N_{(12)}|} = \frac{120}{4} = 30.
\]

\[
N_{1t_2N} = \{ N_1t_2 = N_1t_3, N_1t_6 = N_1t_4, N_1t_7 = N_1t_9, N_2t_4 = N_2t_5, \}
\]

\[
N_{2t_3} = N_2t_1, N_2t_9 = N_2t_8, N_3t_1 = N_3t_2, N_3t_8 = N_3t_7, N_3t_6 = N_3t_5, \]

\[
N_{4t_1} = N_4t_6, N_4t_5 = N_4t_2, N_4t_4 = N_4t_9, N_5t_2 = N_5t_4, N_5t_10 = N_5t_8, \]

\[
N_{5t_3} = N_5t_6, N_5t_4 = N_5t_1, N_5t_7 = N_5t_610, N_6t_5 = N_6t_3, N_7t_3 = N_7t_8, \]

\[
N_{7t_9} = N_7t_1, N_7t_{10} = N_7t_6, N_8t_2 = N_8t_9, N_8t_7 = N_8t_3, N_9t_5 = N_9t_{10}, \]

\[
N_{9t_1} = N_9t_7, N_9t_4 = N_9t_{10}, N_9t_8 = N_9t_2, N_{10t_6} = N_{10t_7}, N_{10t_4} = N_{10t_9}, \]

\[
N_{10t_8} = N_{10t_5} \}
\]

So far we have \( G = N \cup N_1N \cup N_1t_5N \cup N_1t_2N \).

Which gives us 1 + 10 + 30 + 30 = 71 distinct cosets.

Now we check the orbits of \( N_{(15)} \) and \( N_{(12)} \) to see where our single cosets go.

The orbits of \( N_{(15)} \) on \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} are \{1\}, \{5\}, \{7, 9\}, \{8, 10\}, and \{2, 6, 4, 3\}. We take \( t_1, t_5, t_7, t_8, \) and \( t_2 \) from the orbits of \( N_{(15)} \). We want to de-
termine to which double coset $N_{t_1}t_{5t_1}, N_{t_1}t_{5t_5}, N_{t_1}t_{5t_7}, N_{t_1}t_{5t_8}$ and $N_{t_1}t_{5t_2}$ belong.

The double cosets that we have thus far are $[*]$, [1], [15], and [12]. We will use the following loop to see which cosets will go to these double cosets, those that do not, will form new double cosets. We will make sure to add these new double cosets to our loop as we find them.

> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;

$N_{t_1}t_{5t_1}N$ is a new double coset which we will denote [151].

One symmetric generator goes to the new double coset [151].

> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;
> for |if > then "true"; break; end if; end for;

$N_{t_1}t_{5t_5} \in [1]$

One symmetric generator goes back to [1].

> for |if > then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[7] eq m*(ts[1])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[7] eq m*(ts[1]*ts[5])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[7] eq m*(ts[1]*ts[2])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[7] eq m*(ts[1]*ts[5]*ts[1])ˆn
> for|if> then "true"; break; end if; end for;

\(N_{t1t5t7}N\) is a new double coset which we will denote \([157]\).
Two symmetric generators will go to the new double coset \([157]\).

> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1]*ts[1])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1]*ts[5])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1]*ts[2])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1]*ts[5]*ts[1])ˆn
> for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[8] eq m*(ts[1]*ts[5]*ts[7])ˆn
> for|if> then "true"; break; end if; end for;

\(N_{t1t5t8}N\) is a new double coset which we will denote \([158]\).
Two symmetric generators will go to the new double coset \([158]\).
for if> then "true"; break; end if; end for;
true
> for m,n in IN do if ts[1]*ts[5]*ts[2] eq m*(ts[1]*ts[5]*ts[8])ˆn
for if> then "true"; break; end if; end for;
>
$Nt_1t_2t_3 \in [157].$

Four symmetric generators go to [157].

The orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{1\}$, $\{10\}$, $\{2, 3\}$, $\{5, 8\}$, and $\{4, 9, 7, 6\}$. We take $t_1$, $t_{10}$, $t_2$, $t_5$, and $t_4$ from the orbits of $N^{(12)}$. We want to determine to which double coset $Nt_1t_2t_1$, $Nt_1t_2t_{10}$, $Nt_1t_2t_2$, $Nt_1t_2t_5$, and $Nt_1t_2t_4$ belong.

> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[1])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[2])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5]*ts[1])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5]*ts[7])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5]*ts[8])ˆn
for if> then "true"; break; end if; end for;
>
$Nt_1t_2t_1N$ is a new double coset which we will denote [121].

One symmetric generator will go to the new double coset [121].
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5]*ts[1])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5]*ts[7])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5]*ts[8])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[2]*ts[1])ˆn
for if then "true"; break; end if; end for;
>
\[ N_{t_1 t_2 t_{10}} N \] is a new double coset which we will denote \([1210]\).
One symmetric generator will go to the new double coset \([1210]\).

> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[1])ˆn
for if then "true"; break; end if; end for;
true
> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[2])ˆn
for if then "true"; break; end if; end for;
true
> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5]*ts[1])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5]*ts[7])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5]*ts[8])ˆn
for if then "true"; break; end if; end for;
for if then "true"; break; end if; end for;
>
\[ N_{t_1 t_2 t_2} \in [1]. \]
Two symmetric generators go back to \([1]\).

> for m, n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[1])ˆn
for if then "true"; break; end if; end for;
> for m, n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1])ˆn

for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[2])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[1])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[7])ˆn
for if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[8])ˆn
for if> then "true"; break; end if; end for;
for if> then "true"; break; end if; end for;
>
\[N_{t_1t_2t_5}N\] is a new double coset which we will denote \([125]\).

Two symmetric generators will go to the new double coset \([125]\).

\[N_{t_1t_2t_4} \in [157].\]
Words of Length 3

We continue in the same manner above. Our MAGMA code can be found in appendix.


\[ Nt_1t_5t_1N = \{ Nt_1t_5t_1 = \begin{array}{l}
Nt_2t_6t_2 = Nt_4t_3t_4 = Nt_5t_1t_5 = Nt_6t_2t_6, Nt_3t_9t_3 = \\
Nt_1t_8t_1 = Nt_7t_2t_7 = Nt_2t_7t_2 = Nt_8t_1t_8, Nt_5t_9t_5 = Nt_6t_10t_2 = Nt_8t_4t_8 = \\
Nt_3t_7t_4 = Nt_9t_5t_9 = Nt_10t_2t_10, Nt_6t_9t_6 = Nt_4t_7t_4 = Nt_10t_1t_10 = Nt_1t_10t_1 = \\
Nt_9t_6t_9 = Nt_7t_4t_7, Nt_6t_8t_6 = Nt_5t_7t_5 = Nt_10t_3t_10 = Nt_3t_10t_3 = Nt_8t_6t_8 = Nt_7t_5t_7.
\end{array} \]

\[ N^{(157)} = \{ e, (24)(36)(810) \}. \] The number of the single cosets in the double coset \( Nt_1t_5t_7N \) is at most \( \frac{|N|}{|N^{(157)}|} = \frac{120}{2} = 60. \)

\[ Nt_1t_5t_7N = \{ Nt_1t_5t_7, Nt_1t_5t_9, Nt_1t_8t_6, Nt_1t_8t_4, Nt_1t_10t_3, Nt_1t_10t_2, Nt_2t_6t_8, Nt_2t_6t_9, Nt_2t_10t_3, \\
Nt_2t_10t_1, Nt_2t_7t_5, Nt_2t_7t_4, Nt_3t_9t_5, Nt_3t_9t_6, Nt_3t_4t_8, Nt_3t_4t_7, Nt_3t_10t_2, Nt_3t_10t_1, Nt_4t_3t_10, \\
Nt_4t_3t_9, Nt_4t_7t_5, Nt_4t_7t_2, Nt_4t_8t_6, Nt_4t_8t_1, Nt_5t_9t_6, Nt_5t_9t_3, Nt_5t_1t_10, Nt_5t_1t_8, Nt_5t_7t_4, \\
Nt_5t_7t_2, Nt_6t_9t_3, Nt_6t_9t_5, Nt_6t_2t_7, Nt_6t_2t_10, Nt_6t_8t_1, Nt_6t_8t_4, Nt_7t_2t_10, Nt_7t_2t_6, Nt_7t_5t_9, \\
Nt_7t_5t_1, Nt_7t_4t_8, Nt_7t_4t_3, Nt_8t_1t_10, Nt_8t_1t_5, Nt_8t_4t_7, Nt_8t_4t_3, Nt_8t_9t_9, Nt_8t_6t_2, Nt_9t_6t_8, \\
Nt_9t_6t_2, Nt_9t_3t_10, Nt_9t_3t_4, Nt_9t_5t_7, Nt_9t_5t_1, Nt_10t_1t_8, Nt_10t_1t_5, Nt_10t_3t_9, Nt_10t_3t_4, \\
Nt_10t_2t_7, Nt_10t_2t_6 \} \]

\[ N^{(158)} = \{ e, (24)(36)(810), (26)(34)(79)(810), (23)(46)(79) \}. \] The number of the single cosets in the double coset \( Nt_1t_5t_8N \) is at most \( \frac{|N|}{|N^{(158)}|} = \frac{120}{4} = 30. \)
The number of the single cosets in the double coset $Nt_1t_58N = \{Nt_1t_5t_8 = Nt_1t_5t_10, Nt_1t_10t_5 = Nt_1t_10t_8, Nt_1t_8t_10 = Nt_1t_8t_5, Nt_2t_6t_10 = Nt_2t_6t_7, Nt_2t_7t_6 = Nt_2t_7t_10, Nt_2t_10t_7 = Nt_2t_10t_6, Nt_3t_9t_4 = Nt_3t_9t_10, Nt_3t_10t_9 = Nt_3t_9t_10t_4, Nt_3t_4t_10 = Nt_3t_4t_9, Nt_4t_3t_7 = Nt_4t_3t_8, Nt_4t_8t_3 = Nt_4t_8t_7, Nt_4t_7t_3 = Nt_4t_7t_8, Nt_5t_9t_1 = Nt_5t_9t_7, Nt_5t_7t_1 = Nt_5t_7t_9, Nt_5t_1t_7 = Nt_5t_1t_9, Nt_6t_9t_2 = Nt_6t_9t_8, Nt_6t_8t_2 = Nt_6t_8t_9, Nt_6t_2t_9 = Nt_6t_2t_8, Nt_7t_2t_5 = Nt_7t_2t_4, Nt_7t_4t_5 = Nt_7t_4t_2, Nt_7t_5t_2 = Nt_7t_5t_4, Nt_8t_1t_4 = Nt_8t_1t_6, Nt_8t_6t_1 = Nt_8t_6t_4, Nt_8t_4t_6 = Nt_8t_4t_1, Nt_9t_6t_3 = Nt_9t_6t_5, Nt_9t_5t_3 = Nt_9t_5t_6, Nt_9t_3t_5 = Nt_9t_3t_6, Nt_10t_1t_3 = Nt_10t_1t_2, Nt_10t_2t_1 = Nt_10t_2t_3, Nt_10t_3t_1 = Nt_10t_3t_2\}.

$N^{(121)} = \{e, (49)(58)(67), (123)(456)(798), (123)(486957), (23)(46)(79), (13)(45)(89), (12)(56)(78), (13)(48)(59)(67), (23)(47)(58)(69), (132)(465)(789), (132)(475968), (12)(49)(57)(68)\}$. The number of the single cosets in the double coset $Nt_1t_2t_1N$ is at most \[\frac{|N|}{|N^{(121)}|} = \frac{120}{12} = 10.\]

$N^{(1210)} = \{e, (23)(47)(58)(69), (49)(58)(67), (23)(46)(79)\}$. The number of the single cosets in the double coset $Nt_1t_2t_10N$ is at most \[\frac{|N|}{|N^{(1210)}|} = \frac{120}{4} = 30.\]
So far we have 
\[ N t_9 t_2 t_6, N t_9 t_4 t_3 = N t_9 t_{10} t_3, N t_{10} t_6 t_2 = N t_{10} t_7 t_2, N t_{10} t_8 t_1 = N t_{10} t_5 t_1, N t_{10} t_4 t_3 = N t_{10} t_9 t_3 \}.

\[ N^{(125)} = \{ e, (13)(45)(89), (12)(56)(78), (23)(46)(79), (123)(456)(798), (132)(465)(789) \}. \]

The number of the single cosets in the double coset \( N t_1 t_2 t_5 N \) is at most 
\[ \frac{|N|}{|N^{(125)}|} = \frac{120}{6} = 20. \]

\[ N t_1 t_2 t_5 N = \{ N t_1 t_2 t_5 = N t_2 t_3 t_6 = N t_3 t_1 t_4 = N t_1 t_3 t_5 = N t_3 t_2 t_4 = N t_2 t_1 t_6, N t_3 t_1 t_9 = N t_1 t_2 t_8 = N t_2 t_3 t_7 = N t_3 t_2 t_9 = N t_2 t_1 t_7 = N t_1 t_3 t_8, N t_5 t_2 t_1 = N t_2 t_4 t_6 = N t_4 t_5 t_3 = N t_5 t_4 t_1 = N t_4 t_2 t_3 = N t_2 t_5 t_6, N t_4 t_2 t_8 = N t_2 t_5 t_{10} = N t_5 t_4 t_9 = N t_4 t_5 t_8 = N t_5 t_2 t_9 = N t_2 t_4 t_{10}, N t_4 t_1 t_3 = N t_1 t_6 t_5 = N t_6 t_4 t_2 = N t_4 t_3 t_3 = N t_1 t_4 t_5, N t_6 t_1 t_9 = N t_1 t_4 t_{10} = N t_4 t_6 t_7 = N t_6 t_4 t_9 = N t_4 t_7 t_7 = N t_1 t_6 t_{10}, N t_6 t_1 t_3 = N t_1 t_7 t_8 = N t_7 t_9 t_2 = N t_9 t_7 t_3 = N t_7 t_1 t_2 = N t_1 t_6 t_9, N t_7 t_1 t_4 = N t_1 t_9 t_{10} = N t_9 t_7 t_6 = N t_7 t_9 t_4 = N t_9 t_1 t_6 = N t_1 t_7 t_{10}, N t_7 t_3 t_2 = N t_3 t_8 t_9 = N t_8 t_7 t_1 = N t_8 t_7 t_2 = N t_8 t_3 t_1 = N t_3 t_7 t_9, N t_8 t_7 t_6 = N t_7 t_3 t_5 = N t_3 t_8 t_{10} = N t_8 t_3 t_6 = N t_3 t_7 t_{10} = N t_8 t_8 t_5, N t_8 t_2 t_1 = N t_2 t_9 t_7 = N t_9 t_8 t_3 = N t_8 t_9 t_1 = N t_9 t_2 t_3 = N t_2 t_8 t_7, N t_9 t_2 t_5 = N t_2 t_8 t_{10} = N t_8 t_9 t_4 = N t_9 t_8 t_5 = N t_8 t_2 t_4 = N t_2 t_9 t_{10}, N t_6 t_3 t_2 = N t_3 t_5 t_4 = N t_5 t_6 t_1 = N t_6 t_5 t_2 = N t_5 t_3 t_1 = N t_3 t_6 t_4, N t_5 t_3 t_7 = N t_3 t_6 t_{10} = N t_6 t_3 t_8 = N t_5 t_6 t_7 = N t_6 t_3 t_8 = N t_3 t_5 t_{10}, N t_9 t_4 t_6 = N t_4 t_10 t_7 = N t_{10} t_9 t_1 = N t_{10} t_10 t_6 = N t_{10} t_4 t_1 = N t_4 t_9 t_7, N t_{10} t_4 t_2 = N t_4 t_9 t_8 = N t_{10} t_10 t_5 = N t_{10} t_2 t_2 = N t_9 t_4 t_5 = N t_4 t_10 t_8, N t_5 t_9 t_9 = N t_8 t_{10} t_4 = N t_{10} t_5 t_2 = N t_5 t_10 t_9 = N t_{10} t_8 t_2 = N t_8 t_5 t_4, N t_{10} t_8 t_3 = N t_8 t_5 t_6 = N t_{10} t_5 t_7 = N t_5 t_8 t_7 = N t_8 t_{10} t_6, N t_{10} t_1 t_1 = N t_7 t_6 t_4 = N t_6 t_{10} t_9 = N t_{10} t_6 t_1 = N t_6 t_7 t_9 = N t_{7} t_10 t_4, N t_6 t_7 t_8 = N t_{7} t_10 t_5 = N t_{10} t_6 t_3 = N t_6 t_{10} t_8 = N t_{10} t_7 t_3 = N t_7 t_6 t_5 \}.

So far we have 
\[ G = N \cup N t_1 N \cup N t_5 N \cup N t_1 t_2 N \cup N t_1 t_5 t_1 N \cup N t_1 t_5 t_7 N \cup N t_1 t_5 t_8 N \cup N t_1 t_2 t_1 N \cup N t_1 t_2 t_10 N \cup N t_1 t_2 t_5 N. \]

Which gives us \( 1 + 10 + 30 + 30 + 5 + 60 + 30 + 10 + 30 + 20 = 226 \) distinct cosets.

The orbits of \( N^{(151)} \) on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) are \( \{7, 9, 8, 10\} \) and \( \{1, 2, 4, 3, 5, 6\} \).

We take \( t_7 \) and \( t_1 \) from the orbits of \( N^{(151)} \).

We want to determine to which double coset \( N t_1 t_5 t_1 t_7 \), and \( N t_1 t_5 t_1 t_1 \) belong.
\(N_{t_1 t_5 t_1 t_7} N\) is a new double coset which we will denote \([1517]\).

Four symmetric generators will go to the new double coset \([1517]\).
for m*(ts[1]*ts[5])^n then "true";
for break; end if; end for;
true
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[2])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[5]*ts[1])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[5]*ts[7])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[5]*ts[8])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[2]*ts[1])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[2]*ts[10])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[2]*ts[5])^n then "true";
for break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq
for m*(ts[1]*ts[5]*ts[1]*ts[7])^n then "true";
for break; end if; end for;
>
\[ N_{t_1t_5t_1} \in [15]. \]
Six symmetric generators go back to [15].

The orbits of \( N^{(157)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{1,5,7,9,2,4,3,6\} \), and \( \{8,10\} \). We take \( t_1, t_5, t_7, t_9, t_2, t_3 \) and \( t_8 \) from the orbits of \( N^{(157)} \). We want to determine to which double coset \( N_{t_1t_5t_7t_1}, N_{t_1t_5t_7t_5}, N_{t_1t_5t_7t_7}, N_{t_1t_5t_7t_9}, N_{t_1t_5t_7t_2}, N_{t_1t_5t_7t_3}, \) and \( N_{t_1t_5t_7t_8} \) belong.

\( N_{t_1t_5t_7t_1}N \) is a new double coset which we will denote \([1571]\). One symmetric generator will go to the new double coset [1571].
\(Nt_1t_5t_7t_5N\) is a new double coset which we will denote \([1575]\).
One symmetric generator will go to the new double coset \([1575]\).

\(Nt_1t_5t_7t_7 \in [15]\).
One symmetric generator goes back to \([15]\).

\(Nt_1t_5t_7t_9 \in [1517]\).
One symmetric generator goes to \([1517]\).

\(Nt_1t_5t_7t_2 \in [12]\).
Two symmetric generators go to \([12]\).

\(Nt_1t_5t_7t_3N\) is a new double coset which we will denote \([1573]\).
Two symmetric generators will go to the new double coset \([1573]\).

\(Nt_1t_5t_7t_8 \in [15]\).
Two symmetric generators go to \([15]\).

The orbits of \(N^{(158)}\) on \(\{1,2,3,4,5,6,7,8,9,10\}\) are \(\{1\}, \{5\}, \{7,9\}, \{8,10\}\), and \(\{2,3,6,4\}\).
We take \(t_1, t_5, t_7, t_8\), and \(t_2\) from the orbits of \(N^{(158)}\). We want to determine to which double coset \(Nt_1t_5t_8t_1, Nt_1t_5t_8t_5, Nt_1t_5t_8t_7, Nt_1t_5t_8t_8,\) and \(Nt_1t_5t_8t_2\) belong.

\(Nt_1t_5t_8t_1N\) is a new double coset which we will denote \([1581]\).
One symmetric generator will go to the new double coset \([1581]\).

\(Nt_1t_5t_8t_5N\) is a new double coset which we will denote \([1585]\).
One symmetric generator will go to the new double coset \([1585]\).

\(Nt_1t_5t_8t_7 \in [1573]\).
Two symmetric generators go to \([1573]\).

\(Nt_1t_5t_8t_8 \in [15]\).
Two symmetric generators go back to $[15]$.

$Nt_1t_5t_8t_2 \in [1573]$.

Four symmetric generators go to $[1573]$.

The orbits of $N^{(121)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{10\}, \{1,3,2\}$, and $\{4,9,7,6,8,5\}$. We take $t_{10}, t_1,$ and $t_4$ from the orbits of $N^{(121)}$. We want to determine to which double cosets $Nt_1t_2t_1t_{10}, Nt_1t_2t_1t_1,$ and $Nt_1t_2t_1t_4$ belong.

$Nt_1t_2t_1t_{10}N$ is a new double coset which we will denote $[12110]$.

One symmetric generator will go to the new double coset $[12110]$.

$Nt_1t_2t_1t_1 \in [12]$.

Three symmetric generators go back to $[12]$.

$Nt_1t_2t_1t_4 \in [1575]$.

Six symmetric generators go to $[1575]$.

**Words of Length 4**


The number of the single cosets in the double coset $Nt_1t_5t_1t_7N$ is at most $\frac{|N|}{|N^{(1517)}|} = \frac{120}{6} = 20$.

$Nt_1t_5t_1t_7N = \{Nt_1t_5t_1t_7 = Nt_3t_4t_3t_7 = Nt_6t_2t_6t_7, Nt_8t_4t_8t_7 = Nt_9t_5t_9t_7 = Nt_10t_2t_10t_7, Nt_3t_4t_3t_8 = Nt_2t_6t_2t_8 = Nt_5t_1t_5t_8, Nt_9t_5t_9t_8 = Nt_7t_4t_7t_8 = Nt_10t_1t_10t_8, Nt_3t_9t_3t_5 = Nt_2t_7t_2t_5 = Nt_8t_1t_8t_5, Nt_4t_7t_4t_5 = Nt_6t_9t_6t_5 = Nt_10t_1t_10t_5, Nt_5t_1t_5t_10 = Nt_4t_3t_4t_10 = Nt_6t_2t_6t_10, Nt_9t_3t_9t_10 = Nt_7t_2t_7t_10 = Nt_8t_8t_8t_10, Nt_2t_7t_2t_4 = Nt_1t_8t_1t_4 = Nt_9t_3t_9t_4, Nt_10t_3t_10t_4 = Nt_5t_7t_5t_4, Nt_1t_8t_1t_6 = Nt_3t_9t_3t_6 = Nt_7t_2t_7t_6, Nt_5t_9t_5t_6 = Nt_4t_8t_4t_6 = Nt_10t_2t_10t_6, Nt_2t_6t_2t_9 = Nt_1t_5t_1t_9 = Nt_4t_3t_4t_9, Nt_7t_5t_7t_9 = Nt_8t_6t_8t_9 = Nt_10t_3t_10t_9, Nt_1t_10t_1t_3 = Nt_6t_9t_6t_3 = Nt_7t_4t_7t_3, Nt_2t_10t_2t_3 = Nt_5t_9t_5t_3 = Nt_8t_4t_8t_3, Nt_2t_10t_2t_1 = Nt_9t_5t_9t_1 = Nt_4t_8t_4t_1, Nt_3t_10t_3t_1 = Nt_7t_5t_7t_1 = Nt_6t_8t_6t_1, Nt_1t_10t_1t_2 = \ldots \}$. 
\[ N_{t_6 t_9 t_6 t_9 t_2} = N_{t_4 t_7 t_4 t_2}, N_{t_3 t_10 t_3 t_2} = N_{t_8 t_6 t_8 t_2} = N_{t_5 t_7 t_5 t_2}. \]

\[ N^{(1571)} = \{ e, (24)(36)(810), (19)(24)(310)(68), (19)(38)(610) \}. \] The number of the single cosets in the double coset \( N_{t_1 t_5 t_7 t_1} N \) is at most \( \frac{|N|}{|N^{(1571)}|} = \frac{120}{4} = 30. \)

\[ N_{t_1 t_5 t_7 t_1 N} = N_{t_1 t_5 t_7 t_1} N_{t_9 t_5 t_7 t_9}, N_{t_9 t_5 t_7 t_9} = N_{t_7 t_5 t_1 t_7}, N_{t_1 t_5 t_9 t_1} = N_{t_7 t_5 t_9 t_7}, \]
\[ N_{t_2 t_6 t_8 t_2} = N_{t_9 t_6 t_8 t_9}, N_{t_9 t_6 t_2 t_9} = N_{t_8 t_6 t_2 t_8}, N_{t_2 t_6 t_8 t_2} = N_{t_8 t_6 t_8 t_8}, N_{t_3 t_9 t_5 t_3} = N_{t_6 t_9 t_5 t_6}, N_{t_6 t_9 t_3 t_6} = N_{t_5 t_9 t_3 t_5}, N_{t_3 t_9 t_6 t_3} = N_{t_5 t_9 t_6 t_5}, N_{t_4 t_3 t_10 t_4} = N_{t_9 t_3 t_10 t_9}, \]
\[ N_{t_9 t_3 t_4 t_9} = N_{t_10 t_3 t_4 t_10}, N_{t_4 t_3 t_4 t_4} = N_{t_10 t_3 t_4 t_10}, N_{t_2 t_7 t_4 t_2} = N_{t_5 t_7 t_4 t_5}, N_{t_4 t_7 t_2 t_4} = N_{t_5 t_7 t_2 t_5}, N_{t_2 t_7 t_5 t_2} = N_{t_4 t_7 t_5 t_4}, N_{t_1 t_8 t_4 t_1} = N_{t_6 t_8 t_4 t_6}, N_{t_4 t_8 t_1 t_4} = N_{t_6 t_8 t_1 t_6}, \]
\[ N_{t_4 t_8 t_6 t_4} = N_{t_1 t_8 t_6 t_1}, N_{t_7 t_2 t_10 t_7} = N_{t_6 t_2 t_10 t_6}, N_{t_10 t_2 t_7 t_10} = N_{t_6 t_2 t_7 t_6}, N_{t_7 t_2 t_6 t_7} = N_{t_10 t_2 t_6 t_10}, N_{t_1 t_4 t_10 t_4} = N_{t_5 t_1 t_8 t_5}, N_{t_10 t_1 t_4 t_10} = N_{t_5 t_1 t_8 t_5}, N_{t_10 t_1 t_4 t_10} = N_{t_8 t_1 t_8 t_8}, \]
\[ N_{t_1 t_10 t_2 t_1} = N_{t_3 t_10 t_2 t_3}, N_{t_2 t_10 t_3} = N_{t_3 t_10 t_3 t_2}, N_{t_1 t_2 t_10 t_2} = N_{t_3 t_10 t_1 t_3}, N_{t_2 t_10 t_3 t_2} = N_{t_1 t_10 t_3 t_1}, N_{t_3 t_4 t_8 t_3} = N_{t_7 t_4 t_8 t_7}, N_{t_7 t_4 t_8 t_7}, N_{t_3 t_4 t_8 t_3} = N_{t_8 t_4 t_7 t_8}. \]

\[ N^{(1575)} = \{ e, (24)(36)(810), (294)(376)(5810), (49)(58)(67), (29)(37)(510), (249)(367)(5108) \}. \] The number of the single cosets in the double coset \( N_{t_1 t_5 t_7 t_5 N} \) is at most \( \frac{|N|}{|N^{(1575)}|} = \frac{120}{6} = 20. \)

The distinct single cosets in \( N_{t_1 t_5 t_7 t_5 N} \) are \( \{ N_{t_1 t_5 t_7 t_5} = N_{t_1 t_8 t_6 t_8}, N_{t_1 t_10 t_3 t_10}, N_{t_1 t_5 t_9 t_5} = N_{t_1 t_8 t_4 t_8}, N_{t_1 t_10 t_2 t_10}, N_{t_2 t_9 t_3 t_6} = N_{t_2 t_7 t_5 t_7}, N_{t_2 t_10 t_3 t_10}, N_{t_3 t_9 t_5 t_9} = N_{t_3 t_4 t_8 t_4}, N_{t_3 t_9 t_6 t_9} = N_{t_3 t_4 t_7 t_4} = N_{t_3 t_9 t_10 t_10}, \)
\[ N_{t_4 t_8 t_6 t_8} = N_{t_4 t_3 t_10 t_3} = N_{t_4 t_7 t_5 t_7}, N_{t_4 t_8 t_3 t_9} = N_{t_4 t_7 t_2 t_7}, N_{t_5 t_7 t_4 t_7} = N_{t_5 t_1 t_9 t_1}, N_{t_5 t_9 t_6 t_9}, N_{t_7 t_2 t_7 t_9} = N_{t_5 t_7 t_2 t_7}, N_{t_6 t_8 t_1 t_8} = N_{t_6 t_2 t_7 t_8} = N_{t_6 t_9 t_3 t_9}, N_{t_6 t_9 t_5 t_9} = N_{t_6 t_2 t_10 t_2}, N_{t_6 t_8 t_4 t_8}, N_{t_7 t_4 t_8 t_4} = N_{t_7 t_2 t_10 t_2} = N_{t_7 t_5 t_9 t_5}, \]
\[ N_{t_7 t_5 t_1 t_5} = N_{t_7 t_2 t_6 t_2} = N_{t_7 t_4 t_3 t_4}, N_{t_8 t_6 t_8 t_6} = N_{t_8 t_1 t_10 t_1}, N_{t_8 t_4 t_3 t_4} = N_{t_8 t_1 t_5 t_1} = N_{t_8 t_6 t_2 t_6}, N_{t_9 t_5 t_7 t_5} = N_{t_9 t_3 t_10 t_3}, N_{t_9 t_6 t_8 t_6}, N_{t_9 t_6 t_2 t_6} = N_{t_9 t_3 t_4 t_3} = N_{t_9 t_5 t_1 t_5}, N_{t_10 t_3 t_9 t_3} = N_{t_10 t_2 t_7 t_2} = N_{t_10 t_1 t_8 t_1}, N_{t_10 t_2 t_6 t_2} = N_{t_10 t_1 t_5 t_1} = N_{t_10 t_3 t_4 t_3}. \]

\[ N^{(1573)} = \{ e, (13)(45)(89) \}. \] The number of the single cosets in the double coset \( N_{t_1 t_5 t_7 t_3} N \) is at most \( \frac{|N|}{|N^{(1573)}|} = \frac{120}{2} = 60. \)
$N_{t_1t_5t_7t_3} = \{ N_{t_1t_5t_7t_3} = N_{t_3t_4t_7t_1}, N_{t_5t_1t_10t_4} = N_{t_4t_3t_10t_5}, N_{t_1t_8t_6t_3} = N_{t_3t_9t_6t_1}, N_{t_8t_1t_10t_9} = N_{t_9t_3t_10t_8}, N_{t_5t_9t_4t_4} = N_{t_4t_8t_5t_5}, N_{t_8t_4t_7t_9} = N_{t_9t_5t_7t_8}, N_{t_2t_6t_5t_5} = N_{t_5t_1t_8t_2}, N_{t_6t_2t_7t_1} = N_{t_1t_5t_7t_4}, N_{t_5t_9t_3t_2} = N_{t_2t_10t_3t_5}, N_{t_1t_2t_10t_9} = N_{t_9t_5t_7t_10}, N_{t_6t_9t_3t_1} = N_{t_1t_10t_3t_6}, N_{t_3t_4t_8t_6} = N_{t_2t_6t_3t_6}, N_{t_4t_3t_4t_6} = N_{t_6t_2t_10t_4}, N_{t_9t_6t_8t_7} = N_{t_7t_4t_8t_9}, N_{t_4t_7t_5t_6} = N_{t_5t_9t_5t_4}, N_{t_2t_7t_4t_1} = N_{t_1t_8t_4t_2}, N_{t_7t_2t_10t_8} = N_{t_8t_1t_10t_7}, N_{t_2t_6t_9t_1} = N_{t_1t_5t_9t_2}, N_{t_5t_1t_10t_6} = N_{t_6t_2t_10t_5}, N_{t_7t_5t_9t_8} = N_{t_8t_6t_9t_7}, N_{t_6t_4t_5t_5} = N_{t_5t_7t_4t_6}, N_{t_1t_8t_4t_9} = N_{t_9t_3t_1t_1}, N_{t_3t_9t_5t_8} = N_{t_8t_1t_5t_3}, N_{t_6t_6t_2t_1} = N_{t_1t_10t_2t_9}, N_{t_6t_9t_5t_10} = N_{t_10t_1t_6t_6}, N_{t_8t_6t_2t_3} = N_{t_3t_9t_5t_8}, N_{t_8t_4t_4t_10} = N_{t_10t_3t_4t_6}, N_{t_3t_9t_4t_7} = N_{t_7t_2t_6t_3}, N_{t_2t_7t_4t_9} = N_{t_9t_3t_4t_2}, N_{t_3t_10t_1t_7} = N_{t_7t_5t_1t_3}, N_{t_10t_3t_4t_5} = N_{t_5t_7t_4t_10}, N_{t_2t_10t_1t_9} = N_{t_9t_5t_1t_2}, N_{t_5t_5t_6t_10} = N_{t_10t_2t_5t_5, N_{t_1t_8t_1t_2} = N_{t_2t_10t_1t_4}, N_{t_8t_4t_7t_10} = N_{t_10t_2t_3t_8, N_{t_2t_6t_9t_4} = N_{t_3t_4t_7t_6} = N_{t_6t_2t_7t_3}, N_{t_8t_6t_9t_10} = N_{t_10t_3t_4t_6}, N_{t_6t_9t_3t_3} = N_{t_3t_4t_8t_5} = N_{t_5t_1t_8t_3}, N_{t_10t_3t_9t_7} = N_{t_7t_5t_9t_10}, N_{t_3t_10t_2t_5} = N_{t_5t_7t_2t_3}, N_{t_10t_3t_4t_9} = N_{t_9t_5t_1t_2}, N_{t_5t_6t_3t_7} = N_{t_7t_4t_6t_6}, N_{t_4t_6t_3t_7} = N_{t_8t_6t_1t_9} = N_{t_9t_5t_1t_4, N_{t_5t_9t_3t_8} = N_{t_8t_4t_5t_5}, N_{t_7t_5t_1t_6} = N_{t_6t_8t_1t_7, N_{t_7t_7t_2} = N_{t_8t_6t_2t_5}, N_{t_7t_4t_3t_1} = N_{t_1t_10t_3t_7, N_{t_7t_5t_10} = N_{t_10t_1t_5t_4, N_{t_1t_8t_6t_7} = N_{t_6t_2t_6t_1}, N_{t_2t_7t_5t_8} = N_{t_8t_1t_5t_2, N_{t_10t_2t_6t_4} = N_{t_4t_8t_6t_10, N_{t_2t_10t_3t_8} = N_{t_8t_4t_3t_2}}\).

\[ N_{t_1t_{10}t_8t_1} = N_{t_1t_2t_{10}} = N_{t_9t_6t_5t_9} = N_{t_9t_6t_3t_9} = N_{t_6t_9t_6t_6} = N_{t_1t_{10}t_5t_1} = N_{t_7t_4t_5t_7} = N_{t_7t_4t_2t_7} = N_{t_6t_8t_2t_6} = N_{t_3t_{10}t_3} = N_{t_7t_5t_4t_7} = N_{t_3t_{10}t_4t_3} = N_{t_5t_7t_9t_5} = N_{t_7t_5t_2t_7} = N_{t_8t_6t_1t_8} = N_{t_8t_6t_4t_8} = N_{t_5t_{17}t_1t_1} = N_{t_5t_9t_7t_9} = N_{t_{57}t_7t_7} = N_{t_5t_9t_1t_9} = N_{t_6t_9t_2t_9} = N_{t_6t_2t_8t_2} = N_{t_6t_2t_9t_3} = N_{t_6t_8t_6t_8} = N_{t_7t_4t_2t_4} = N_{t_7t_2t_5t_2} = N_{t_7t_4t_3t_5} = N_{t_7t_5t_2t_5} = N_{t_8t_6t_1t_6} = N_{t_8t_4t_1t_4} = N_{t_9t_5t_3t_5} = N_{t_9t_3t_6t_3} = N_{t_9t_3t_5t_3} = N_{t_9t_6t_5t_6} = N_{t_9t_5t_6t_5} = N_{t_9t_6t_3t_6} = N_{t_{10}t_3t_2t_3} = N_{t_{10}t_2t_1t_2} = N_{t_{10}t_3t_2t_2} = N_{t_{10}t_1t_3t_1} = N_{t_{10}t_3t_2t_1} \]

\[ N^{(1585)} = \{ e, (269347)(5108), (24)(36)(810), (29)(37)(510), (274396)(5810), (294)(376)(5810), (26)(34)(79)(810), (49)(58)(67), (27)(39)(46)(510), (249)(367)(5108), (23)(46)(79), (23)(47)(58)(69) \}. \] The number of the single cosets in the double coset \( N_{t_1t_5t_8t_5} \) is at most \[ \frac{|N|}{|N^{(1581)}|} = \frac{120}{12} = 10. \]

\[ N^{(12110)} = \{ e, (49)(58)(67), (123)(456)(798), (123)(486957), (23)(46)(79), (13)(45)(89), (12)(56)(78), (13)(48)(59)(67), (23)(47)(58)(69), (132)(465)(789), (132)(475968), (12)(49)(57)(68) \}. \] The number of the single cosets in the double coset \( N_{t_1t_2t_1t_10} \) is at most \[ \frac{|N|}{|N^{(1581)}|} = \frac{120}{12} = 10. \]

\[ N_{t_1t_2t_1t_{10}} = \{ N_{t_1t_2t_1t_10} = N_{t_3t_3t_3t_{10}} = N_{t_2t_3t_2t_{10}} = N_{t_2t_1t_2t_{10}} = N_{t_1t_3t_1t_10} = N_{t_3t_2t_3t_{10}} = N_{t_2t_4t_2t_7} = N_{t_5t_2t_5t_7} = N_{t_4t_4t_7} = N_{t_2t_5t_2t_7} = N_{t_5t_4t_5t_7}, N_{t_4t_4t_4t_8} = N_{t_6t_4t_6t_8} = N_{t_1t_6t_1t_8} = N_{t_1t_4t_1t_8} = N_{t_4t_6t_4t_8} = N_{t_6t_1t_6t_8}, N_{t_1t_7t_1t_5} = N_{t_9t_1t_9t_5} = N_{t_7t_7t_7t_5} = N_{t_1t_9t_1t_5} = N_{t_9t_7t_9t_5}, N_{t_3t_8t_3t_4} = N_{t_7t_3t_7t_4} = N_{t_8t_7t_8t_4} = N_{t_5t_3t_8t_4} = N_{t_3t_7t_7t_4} = N_{t_8t_7t_8t_4} = N_{t_9t_3t_8t_4} = N_{t_7t_8t_7t_4} = N_{t_8t_7t_8t_4} = N_{t_9t_8t_9t_8} = N_{t_9t_8t_9t_6} = N_{t_2t_4t_2t_6} = N_{t_2t_8t_2t_4} = N_{t_7t_6t_8t_6} = N_{t_6t_2t_6t_6}, N_{t_3t_3t_3t_9} = N_{t_3t_3t_3t_9} = N_{t_5t_6t_5t_9} = N_{t_5t_6t_5t_9}, N_{t_4t_4t_4t_3} = N_{t_9t_4t_9t_3} = N_{t_10t_9t_10t_3} = N_{t_4t_4t_4t_3} = N_{t_9t_10t_9t_3} = N_{t_5t_10t_5t_3} = N_{t_8t_5t_8t_1} = N_{t_10t_8t_10t_1} = N_{t_10t_5t_10t_1} = \]
\[ Nt_5t_8t_5t_1 = Nt_8t_10t_8t_1, Nt_6t_7t_6t_2 = Nt_10t_6t_10t_2 = Nt_7t_10t_7t_2 = Nt_6t_10t_6t_2 = Nt_{10}t_7t_{10}t_2 \].

So far we have \( G = N \cup Nt_1N \cup Nt_1t_5N \cup Nt_1t_2N \cup Nt_1t_5t_1N \cup Nt_1t_5t_7N \cup Nt_1t_5t_8N \cup Nt_1t_2t_1N \cup Nt_1t_2t_10N \cup Nt_1t_2t_5N \cup Nt_1t_5t_1t_7N \cup Nt_1t_5t_7t_5N \cup Nt_1t_5t_7t_3N \cup Nt_1t_5t_8t_1N \cup Nt_1t_5t_8t_5N \cup Nt_1t_2t_1t_{10}N. \)

Which gives us \( 1+10+30+30+5+60+30+10+30+20+20+30+20+60+5+10+10 = 381 \) distinct cosets.

The orbits of \( N^{(1517)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{7\}, \{1,3,6\}, \{2,4,5\}, \) and \( \{8,10,9\} \).

We take \( t_7, t_1, t_2, \) and \( t_8 \) from the orbits of \( N^{(1517)} \).

\[ Nt_1t_5t_1t_7t_7 \in [151]. \]

One symmetric generator goes back to [151].

\[ Nt_1t_5t_1t_7t_1N \] is a new double coset which we will denote [15171].

Three symmetric generators will go to the new double coset [15171].

\[ Nt_1t_5t_1t_7t_2 \in [125]. \]

Three symmetric generators go to [125].

\[ Nt_1t_5t_1t_7t_8 \in [157]. \]

Three symmetric generators go to [157].

The orbits of \( N^{(1571)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{5\}, \{7\}, \{1,9\}, \{2,4\} \) and \( \{3,6,10,8\} \).

We take \( t_5, t_7, t_1, t_2, \) and \( t_3 \) from the orbits of \( N^{(1571)} \).

\[ Nt_1t_5t_7t_1t_5 \in [1210]. \]

One symmetric generator goes back to [1210].

\[ Nt_1t_5t_7t_1t_7 \in [15171]. \]

One symmetric generator goes to [15171].
$N t_1 t_5 t_7 t_1 t_1 \in [157].$

Two symmetric generators go back to $[157].$

$N t_1 t_5 t_7 t_1 t_2 N$ is a new double coset which we will denote $[15712].$

Two symmetric generators will go to the new double coset $[15712].$

$N t_1 t_5 t_7 t_1 t_3 \in [1210].$

Four symmetric generators go back to $[1210].$

The orbits of $N^{(1575)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}, \{2,4,9\}, \{3,6,7\},$ and $\{5,8,10\}.$

We take $t_1, t_2, t_3,$ and $t_5$ from the orbits of $N^{(1575)}.$

$N t_1 t_5 t_7 t_5 t_1 \in [15712].$

One symmetric generator goes to $[15712].$

$N t_1 t_5 t_7 t_5 t_2 \in [121].$

Three symmetric generators go back to $[121].$

$N t_1 t_5 t_7 t_5 t_3 \in [1210].$

Three symmetric generators go to $[1210].$

$N t_1 t_5 t_7 t_5 t_5 \in [157].$

Three symmetric generators go to $[157].$

The orbits of $N^{(1573)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{2\}, \{6\}, \{7\}, \{10\}, \{1,3\}, \{4,5\},$ and $\{8,9\}.$ We take $t_2, t_6, t_7, t_{10}, t_1, t_4$ and $t_8$ from the orbits of $N^{(1573)}.$

$N t_1 t_5 t_7 t_3 t_2 \in [15712].$

One symmetric generator goes to $[15712].$

$N t_1 t_5 t_7 t_3 t_6 \in [1210].$
One symmetric generator goes to \([1210]\).

\[Nt_1t_3t_7t_3t_7 \in [158].\]

One symmetric generator goes to \([158]\).

\[Nt_1t_5t_7t_3t_{10} \in [125].\]

One symmetric generator goes to \([125]\).

\[Nt_1t_5t_7t_3t_1 \in [157].\]

Two symmetric generators go to \([157]\).

\[Nt_1t_5t_7t_3t_4 \in [15171].\]

Two symmetric generators go to \([15171]\).

\[Nt_1t_5t_7t_3t_8 \in [158].\]

Two symmetric generators go to \([158]\).

The orbits of \(N^{(1581)}\) on \(\{1,2,3,4,5,6,7,8,9,10\}\) are \(\{7,9,8,10\}\) and \(\{1,2,4,3,5,6\}\). We take \(t_7\) and \(t_1\) from the orbits of \(N^{(1581)}\).

\[Nt_1t_5t_8t_1t_7 \in [125].\]

Four symmetric generators go to \([125]\).

\[Nt_1t_5t_8t_1t_1 \in [158].\]

Six symmetric generators go back to \([158]\).

The orbits of \(N^{(1585)}\) on \(\{1,2,3,4,5,6,7,8,9,10\}\) are \(\{1\}\), \(\{5,8,10\}\), and \(\{2,3,7,9,4,6\}\). We take \(t_1, t_5\) and \(t_2\) from the orbits of \(N^{(1585)}\).

\[Nt_1t_5t_8t_5t_1N\] is a new double coset which we will denote \([15851]\).

One symmetric generator will go to the new double coset \([15851]\).
$N t_1 t_5 t_8 t_5 t_5 \in [158]$.  
Three symmetric generators go back to [158].

$N t_1 t_5 t_8 t_5 t_2 \in [15171]$.  
Six symmetric generators go to [15171].

The orbits of $N^{(12110)}$ on \{1,2,3,4,5,6,7,8,9,10\} are \{10\},\{1,3,2\}, and \{4,9,7,6,8,5\}. We take $t_{10}, t_1,$ and $t_4$ from the orbits of $N^{(12110)}$.

$N t_1 t_2 t_4 t_1 t_1 t_{10} \in [121]$.  
One symmetric generator goes back to [121].

$N t_1 t_2 t_4 t_1 t_{10} \in [15171]$.  
Three symmetric generators go to [15171].

$N t_1 t_2 t_4 t_1 t_{10} \in [15712]$.  
Six symmetric generators go to [15712].

**Words of Length 5**

$N^{(15171)} = \{e, (24)(36)(810), (19)(24)(310)(68), (19)(38)(610)\}$. The number of the single cosets in the double coset $N t_1 t_5 t_7 t_1 t_7 t_1 N$ is at most $\frac{|N|}{|N^{(15171)}|} = \frac{120}{4} = 30.$

$N t_1 t_5 t_1 t_7 t_1 = N t_1 t_5 t_7 t_1 t_7 t_1 = N t_9 t_5 t_9 t_7 t_9, N t_9 t_5 t_9 t_1 t_9 = N t_7 t_5 t_7 t_1 t_7, N t_1 t_5 t_1 t_9 t_1 = N t_7 t_5 t_7 t_9 t_7, N t_2 t_6 t_2 t_8 t_2 = N t_9 t_6 t_9 t_6 t_9, N t_9 t_6 t_9 t_2 t_9 = N t_8 t_6 t_8 t_2 t_8, N t_2 t_6 t_2 t_9 t_2 = N t_8 t_6 t_8 t_9 t_8, N t_3 t_9 t_3 t_5 t_3 = N t_6 t_9 t_6 t_5 t_6, N t_6 t_9 t_6 t_3 t_6 = N t_5 t_9 t_5 t_3 t_5, N t_3 t_9 t_3 t_6 t_3 = N t_5 t_9 t_5 t_6 t_5, N t_4 t_3 t_4 t_1 t_4 = N t_9 t_3 t_9 t_1 t_9, N t_9 t_3 t_9 t_4 t_9 = N t_1 t_3 t_1 t_4 t_1, N t_4 t_3 t_4 t_4 t_4 = N t_1 t_3 t_1 t_9 t_1, N t_2 t_7 t_2 t_4 t_2 = N t_5 t_7 t_5 t_4 t_5, N t_4 t_7 t_4 t_2 t_4 = N t_5 t_7 t_5 t_2 t_5, N t_2 t_7 t_2 t_5 t_2 = N t_4 t_7 t_4 t_5 t_4, N t_1 t_8 t_1 t_4 t_1 = N t_6 t_8 t_6 t_4 t_6, N t_4 t_8 t_4 t_1 t_4 = N t_6 t_8 t_6 t_1 t_6, N t_4 t_8 t_4 t_6 t_4 = N t_1 t_8 t_1 t_6 t_1, N t_7 t_2 t_7 t_1 t_7 = N t_6 t_2 t_6 t_1 t_6, N t_1 t_6 t_2 t_6 t_7 t_6 = N t_1 t_6 t_2 t_6 t_7 t_1 = N t_6 t_2 t_6 t_7 t_6 t_7 = N t_1 t_6 t_2 t_6 t_7 t_1 t_6, N t_8 t_1 t_8 t_1 t_8 t_8 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 = N t_5 t_1 t_5 t_1 t_5 t_5, N t_1 t_1 t_1 t_1 t_1 t_1 =
\[ N_{t_4 t_7 t_8 t_7}, N_{t_8 t_4 t_8 t_3 t_8} = N_{t_7 t_4 t_7 t_3 t_7}, N_{t_3 t_4 t_3 t_7 t_3} = N_{t_8 t_4 t_8 t_7 t_8} \].

\[ N^{(15712)} = \{ e, (129)(387)(5106), (29)(37)(510), (19)(38)(610), (192)(378)(5610), (12)(56)(78) \}. \]

The number of the single cosets in the double coset \( N_{t_1 t_5 t_7 t_1 t_2} N \) is at most \[ \frac{|N|}{|N^{(15712)}|} = \frac{120}{6} = 20. \]

\[ N_{t_1 t_5 t_7 t_1 t_2} = \{ N_{t_1 t_5 t_7 t_1 t_2} = N_{t_9 t_6 t_8 t_9 t_1} = N_{t_9 t_5 t_7 t_9 t_2} = N_{t_1 t_10 t_3 t_1 t_9} = N_{t_2 t_9 t_3 t_2 t_9} = N_{t_4 t_9 t_3 t_1 t_4 t_9} = N_{t_4 t_3 t_10 t_4 t_9} = N_{t_5 t_1 t_10 t_5} = N_{t_6 t_9 t_3 t_6 t_9} = N_{t_7 t_2 t_4 t_7 t_2 t_5} = N_{t_8 t_1 t_5 t_8 t_1} = N_{t_9 t_6 t_9 t_1 t_9} = N_{t_3 t_4 t_3 t_1 t_3 t_5} = N_{t_3 t_4 t_3 t_2 t_3 t_5} = N_{t_7 t_2 t_4 t_7 t_2 t_5} = N_{t_3 t_4 t_3 t_2 t_3 t_5} = N_{t_3 t_4 t_3 t_2 t_3 t_5} \]

\[ N^{(15851)} = \{ e \}. \] The number of the single cosets in the double coset \( N_{t_1 t_5 t_8 t_5 t_1} N \) is at most \[ \frac{|N|}{|N^{(15851)}|} = \frac{120}{1} = 1. \]
\[ Nt_{1t5t8t5t1} = \{Nt_{1t5t8t5t1} = Nt_{3t9t4t9t3} = Nt_{4t3t7t3t4} = Nt_{1t8t10t8t1} = Nt_{5t9t1t9t5} = Nt_{2t6t7t6t2} = Nt_{7t2t5t2t7} = Nt_{2t10t7t10t2} = Nt_{3t4t10t4t3} = Nt_{6t9t2t6t6} = Nt_{9t6t3t6t9} = Nt_{4t3t8t3t4} = Nt_{1t8t6t4t8} = Nt_{5t1t9t1t5} = Nt_{4t7t3t7t4} = Nt_{1t5t1t7t5} = Nt_{2t7t10t7t2} = Nt_{8t6t6t8t6} = Nt_{6t8t2t6t6} = Nt_{1t5t10t5t1} = Nt_{7t2t4t2t7} = Nt_{2t10t6t10t2} = Nt_{3t4t9t4t3} = Nt_{10t1t3t1t10} = Nt_{6t2t9t2t6} = Nt_{9t3t6t3t9} = Nt_{7t5t4t5t7} = Nt_{2t7t6t7t2} = Nt_{3t10t9t10t3} = Nt_{6t2t8t2t6} = Nt_{4t8t7t8t4} = Nt_{10t3t2t3t10} = Nt_{4t8t3t8t4} = Nt_{9t5t6t5t9} = Nt_{3t10t4t10t3} = Nt_{8t4t1t4t8} = Nt_{3t9t10t9t3} = Nt_{4t7t3t7t4} = Nt_{1t10t8t10t1} = Nt_{8t1t6t1t8} = Nt_{9t5t3t5t9} = Nt_{5t9t7t9t5} = Nt_{7t5t9t7t5} = Nt_{8t1t4t1t8} = Nt_{7t4t5t4t7} = Nt_{10t2t1t2t10} = Nt_{6t9t8t9t6} = Nt_{5t7t1t7t5} = Nt_{10t1t2t1t10} = Nt_{7t5t2t5t7} = Nt_{10t2t3t2t10} = Nt_{9t6t5t6t9} = Nt_{6t8t9t8t6} = Nt_{6t6t4t6t8} = Nt_{10t3t1t3t10} = Nt_{7t4t2t4t7} = Nt_{2t6t10t6t2}. \]

So now we have the following double cosets, \( G = N \cup Nt_{1N} \cup Nt_{1t5N} \cup Nt_{1t2N} \cup Nt_{1t5t1N} \cup Nt_{1t5t7N} \cup Nt_{1t5t8N} \cup Nt_{1t2t1N} \cup Nt_{1t2t10N} \cup Nt_{1t2t5N} \cup Nt_{1t5t7t5N} \cup Nt_{1t5t7t6N} \cup Nt_{1t5t7t3N} \cup Nt_{1t5t8t1N} \cup Nt_{1t5t8t5N} \cup Nt_{1t2t1t10N} \cup Nt_{1t5t1t7t1N} \cup Nt_{1t5t7t1t2N} \cup Nt_{1t5t8t5t1N}. \)

Which gives us \( 1 + 10 + 30 + 30 + 5 + 60 + 30 + 10 + 30 + 20 + 20 + 30 + 20 + 60 + 5 + 10 + 10 + 30 + 20 + 1 = 432 \) distinct cosets.

We have now found all of our distinct cosets. Thus, there will be no more new double cosets. To show this we can continue in the same manner as above.

The orbits of \( N^{(1517)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{5\}, \{7\}, \{1,9\}, \{2,4\}, \) and \( \{3,6,10,8\} \).
We take \( t_5,t_7,t_1,t_2, \) and \( t_3 \) from the orbits of \( N^{(1517)} \).

\( Nt_{1t5t1t7t1t5} \in [12110]. \)
One symmetric generator will go to \([12110]\).

\( Nt_{1t5t1t7t1t7} \in [1571]. \)
One symmetric generator will go to \([1571]\).

\( Nt_{1t5t1t7t1t1} \in [1517]. \)
Two symmetric generators will go to [1517].

\[ Nt_1t_5t_1t_7t_1t_2 \in [1585]. \]
Two symmetric generators will go to [1585].

\[ Nt_1t_5t_1t_7t_1t_3 \in [1573]. \]
Four symmetric generators will go to [1573].

The orbits of \( N^{(15712)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{4\} \), \( \{1,9,2\} \), \( \{3,7,8\} \), and \( \{5,6,10\} \). We take \( t_4, t_1, t_3, \) and \( t_5 \) from the orbits of \( N^{(15712)} \).

\[ Nt_1t_5t_7t_1t_2t_4 \in [1575]. \]
One symmetric generator will go to [1575].

\[ Nt_1t_5t_7t_1t_2t_1 \in [1571]. \]
Three symmetric generators will go to [1571].

\[ Nt_1t_5t_7t_1t_2t_3 \in [1573]. \]
Three symmetric generators will go to [1573].

\[ Nt_1t_5t_7t_1t_2t_5 \in [12110] \]
Three symmetric generators will go to [12110].

The orbits of \( N^{(15851)} \) on \( \{1,2,3,4,5,6,7,8,9,10\} \) are \( \{1,3,4,5,2,7,6,9,8,10\} \). We take \( t_1 \) from the orbit of \( N^{(15851)} \).

\[ Nt_1t_5t_8t_5t_1t_1 \in [1585]. \]
Ten symmetric generators will go to [1585].

Below is our completed Cayley Diagram.
Figure 5.5: \((S(4, 3) : 2) \text{ Over } 2^{10} : S_5\)
Chapter 6

Transitive Groups

In this chapter we will examine several groups that are transitive on \( n \) letters by using \( \text{NumberOfTransitiveGroups}(n) \) command in MAGMA.

6.1 Transitive Groups on 20 Letters

Using the following code we find that there are 1117 transitive groups on 20 letters.

\[
\text{NumberOfTransitiveGroups}(20);
\]

1117

We will examine some of these groups and write progenitors.

6.1.1 Transitive Group(20,222)

Let \( N \) be transitive group 222 on 20 letters. \( N \) is of order 1920 and is generated by \( xx = (1, 16, 9, 2, 15, 10)(3, 18, 11)(4, 17, 12)(5, 6)(7, 14, 19)(8, 13, 20) \) and \( yy = (1, 14, 15, 2, 13, 16)(3, 19, 6, 8, 18, 10, 4, 20, 5, 7, 17, 9) \).

\[
\text{Generators}(N);
\]

\[
(1, 16, 9, 2, 15, 10)(3, 18, 11)(4, 17, 12)(5, 6)
\]
Next we find a presentation for $N$.

> FPGroup(N);
Finitely presented group on 2 generators
Relations
$.1^6 = \text{Id}($
$.1 * $.2^4 = \text{Id}($
$.2 * $.1^2 * $.2^4 * $.1^2 * $.2 * $.1^2 = \text{Id}($
$.2^4 * $.1^2 * $.2 * $.1^2 * $.2^4 * $.1 * $.2^3 * $.1 = \text{Id}($
$(.1^2 * $.2^2 * $.1^2 * $.2^4)^2 = \text{Id}($

Thus we have the following presentation of $N$ for the progenitor $2^{4\times20} : (2^4 : S_5)$.

(Proof of Isomorphism of $N$ to follow.)

$$N = < x, y | x^6, (xy^5)^4, yxy^5x^4y^5xy^5, y^5x^5y^5xy^3x, (x^5y^2x^5y^5)^2 >.$$ 

Next we will add $t$. Let $t \sim t_1$. Since our $t$'s are of order 2, we add $t^2$ to the presentation. Now we need to look at the stabiliser of 1 that commute with $t$. We label $N_1$ as the stabiliser of 1 in $N$ and find the generators of $N_1$.

> N1:=Stabiliser(N,1);
> Generators(N1);

$$\{ (3, 7, 14) (4, 8, 13) (5, 10, 16) (6, 9, 15) (11, 19, 18) (12, 20, 17), (3, 5, 4, 6) (7, 9, 8, 10) (13, 16, 14, 15) (19, 20), (13, 14) (15, 16) (17, 18) (19, 20), (3, 8, 4, 7) (5, 10, 6, 9) (15, 16) (17, 20, 18, 19) \}$$

We then use our Schreier System to translate these permutations into words.

> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..1920]];
> for i in [2..1920] do
| for P:=[Id(N): l in [1..#Sch[i]]];
| for j in [1..#Sch[i]] do
| for Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
| for Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
| for Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^2-1; end if;
| end do;
| end do;
| end do;
for|for|for|if> if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..1920] do if ArrayP[i] eq N!(3, 7, 14)(4, 8, 13)
for|if> (5, 10, 16)(6, 9, 15)(11, 19, 18)(12, 20, 17) then Sch[i];
for|if> end if; end for;
y * x^2 * y^-2 * x^-1 * y * x^-1
> for i in [1..1920] do if ArrayP[i] eq N!(3, 5, 4, 6)(7, 9, 8, 10)
for|if> (13, 16, 14, 15)(19, 20) then Sch[i]; end if; end for;
x^-1 * y^-1 * x^-1 * y^-3 * x * y^-1
> for i in [1..1920] do if ArrayP[i] eq N!(13, 14)(15, 16)(17, 18)
for|if> (19, 20) then Sch[i]; end if; end for;
(y * x * y^-1)^3
> for i in [1..1920] do if ArrayP[i] eq N!(3, 8, 4, 7)(5, 10, 6, 9)
for|if> (15, 16)(17, 20, 18, 19) then Sch[i]; end if; end for;
y^-1 * x^3 * y^-2

Therefore we have the following presentation,

\[
2^{20} : (2^4 : S_5) = \langle x, y, t | x^6, (xy)^4, yxy^5 x^4 y^5 x y^5 y^5 x y^5 x y^5 x^2, (x^5 y^2 x^5 y^5)^2, \\
t^2, (y^2 x^4 x^5 y x^5), (t, x^5 y^5 x^3 y^5), (t, (y x y^5)^3), (t, y^5 x^3 y^4) \rangle.
\]

Now we add first and second relations to our progenitor in order to find homomorphic images of \(2^{20} : (2^4 : S_5)\).

\[
G = \langle x, y, t | x^6, (xy)^4, yxy^5 x^4 y^5 x y^5, y^5 x y^5 x y^3 x, (x^5 y^2 x^5 y^5)^2, \\
t^2, (y x^2 y^4 x^5 y x^5), (t, x^5 y^5 x^3 y^5), (t, (y x y^5)^3), (t, y^5 x^3 y^4), \\
(xy t)^{r_1}, (xy t (x^2 y)^{r_2}), (xy t (x^2 y)^{r_3}), (xy t (x^2 y)^{r_4}), ((x y)^3 t (x^2 y) t (x^2 y))^{r_5}, \\
((x y)^3 t (x^2 y) t (x^2 y))^{r_6}, ((x y)^3 t (x^2 y) t)^{r_7}, ((x y)^3 t (x^2 y) t)^{r_8}, (x y^5 x t t (x^2 y))^{r_9}, \\
(xy x t t (x^3))^{r_{10}}, (x^3 t (x y x^3 y)^{r_{11}}, (x y x y^5 (x^2 y))^{r_{12}} \rangle.
\]
Table 6.1: $2^{*}20 : (2^4 : S_5)$

<table>
<thead>
<tr>
<th>r1</th>
<th>r2</th>
<th>r3</th>
<th>r4</th>
<th>r5</th>
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<th>r7</th>
<th>r8</th>
<th>r9</th>
<th>r10</th>
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<th>r12</th>
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<td>$M_{12} : 2$</td>
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<td>8</td>
<td></td>
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<td>$(M_{12} \times 2) : 2$</td>
</tr>
</tbody>
</table>

Proof of the Isomorphism for the Shape of $N$

The composition series of $N$ is given below.

$$G = G_1 \supset G_2 \supset G_3 \supset G_4 \supset G_5 \supset 1,$$

The Normal Lattice of $N$ is...
We use the following loop to give us the largest abelian subgroup of $N$.

```plaintext
> NL:=NormalLattice(N);
> for i in [1..4] do if IsAbelian(NL[i]) then i;end if;end for;
1
2
```

$NL[2]$, our largest abelian subgroup of $N$ is of order 16. We will examine possibilities of groups of order 16 to find the isomorphism type of $NL[2]$.

```plaintext
> X:=AbelianGroup(GrpPerm,[2,2,2,2]);
> s:=IsIsomorphic(X,NL[2]);s;
true
```

We have verified that $NL[2] = 2^4$. Since $1920/16=120$, and we do not have a normal subgroup of order 120, we will have a semi-direct product. We will factor by $NL[2]$ and check the isomorphism type of $q$, our factor group.

```plaintext
> q,ff:=quo<N|NL[2]>
> s:=IsIsomorphic(q,Sym(5));s;
true
```

So we will have a semi-direct product $2^4 : S_5$. We need to write a presentation of $2^4$. 

---

![Diagram](attachment:image.png)
A presentation for $S_5$ is $Q = \langle a, b | a^5, b^2, (a^{-1}b)^4, (aba^{-1}b)^2 \rangle$. We find an element of order 5 and an element of order 2, $F$ and $G$, respectively.

```plaintext
for i in NL[4] do if i notin NL[2] and Order(i) eq 5 and sub<N|i,NL[4]> eq N then F:=i; break; end if; end for;
for i in NL[4] do if i notin NL[2] and Order(i) eq 2 and sub<N|i,NL[4]> eq N then G:=i; break; end if; end for;
```

Now we need to find the action of $F$ and $G$ on the generators of $NL[2]$.

```plaintext
> for i in [1..#N1] do if ArrayP[i] eq A^F then print Sch[i]; for|if> end if; end for;
z
> for i in [1..#N1] do if ArrayP[i] eq B^F then print Sch[i]; for|if> end if; end for;
w * x * y * z
> for i in [1..#N1] do if ArrayP[i] eq C^F then print Sch[i]; for|if> end if; end for;
w
> for i in [1..#N1] do if ArrayP[i] eq D^F then print Sch[i]; for|if> end if; end for;
x
> for i in [1..#N1] do if ArrayP[i] eq A^G then print Sch[i]; for|if> end if; end for;
w
> for i in [1..#N1] do if ArrayP[i] eq B^G then print Sch[i]; for|if> end if; end for;
z
> for i in [1..#N1] do if ArrayP[i] eq C^G then print Sch[i]; for|if> end if; end for;
y
> for i in [1..#N1] do if ArrayP[i] eq D^G then print Sch[i]; for|if> end if; end for;
x
```

Finally we will add the presentation of $Q$, along with the action of $a$ and $b$ on the generators of $H = 2^4$, to our presentation of $H$.

$G = w, x, y, z, a, b | w^2, x^2, y^2, z^2, (w, x), (w, y), (w, z), (x, y), (x, z), (y, z), a^5, b^2, (a^{-1}b)^4,$
\((aba^{-2}ba)^2, w^a = z, x^a = wxyz, y^a = w, z^a = x, w^b = w, x^b = z, y^b = y, z^b = x >.\)

We then verify the isomorphism.

\[
> G\langle w, x, y, z, a, b \rangle := \text{Group}\langle w, x, y, z, a, b \mid \text{w}^2, \text{x}^2, \text{y}^2, \text{z}^2, (\text{w}, \text{x}), (\text{w}, \text{y}), (\text{w}, \text{z}), (\text{x}, \text{y}), (\text{x}, \text{z}), (\text{y}, \text{z}) \rangle,
\]

\[
> \text{a}^5, \text{b}^2, (\text{a}^{\text{w}^{-1}}*\text{b}^{-2})*4, (\text{a}^*\text{b}^*\text{a}^{\text{w}^{-1}}*\text{b}^{-2})*2,
\]

\[
> \text{w}^*\text{a}=\text{z}, \text{x}^*\text{a}=\text{w}*\text{y}^*\text{z}, \text{y}^*\text{a}=\text{w}, \text{z}^*\text{a}=\text{x},
\]

\[
> \text{w}^\text{b}=\text{w}, \text{x}^\text{b}=\text{z}, \text{y}^\text{b}=\text{y}, \text{z}^\text{b}=\text{x}>
\]

\[
> f1, \text{G1}, k1 := \text{CosetAction} (G, \text{sub}\langle G | \text{Id} (G) \rangle);
\]

\[
> s, t := \text{IsIsomorphic} (\text{G1}, N);
\]

\[
> s;
\]

\[
> \text{true}
\]

Thus \(N \cong (2^4 : S_5)\).

### 6.1.2 Transitive Group(20,102)

Let \(N\) be transitive group 102 on 20 letters. \(N = (5^2 \cdot 4^2)\) is of order 400 and is generated by \(x = (1, 6, 15, 16, 2, 9, 12, 17, 3, 7, 14, 18, 4, 10, 11, 19, 5, 8, 13, 20),\)
\(y = (1, 8, 12, 17, 4, 10, 11, 18, 2, 7, 15, 19, 5, 9, 14, 20, 3, 6, 13, 16)\), and \(z = (1, 7, 15, 16)\)
\((2, 8, 13, 19)(3, 9, 11, 17)(4, 10, 14, 20)(5, 6, 12, 18)\). In the same manner as before, we find the following presentation for \(G\).

\[
G < x, y, z, t | t^2, (t, yxy^2), (t, (y^{-1}, x^{-1})),
\]

\[
(y^2t)^r1, (x^3t)^r2, (yt)^r3, (y^3t)^r4 >.
\]

<table>
<thead>
<tr>
<th>(r1)</th>
<th>(r2)</th>
<th>(r3)</th>
<th>(r4)</th>
<th>Order of (G)</th>
<th>Shape of (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>2448</td>
<td>(L_2(17))</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>672</td>
<td>(4 \times L_2(7))</td>
</tr>
</tbody>
</table>
Proof of the Isomorphism for the Shape of $N$

The composition series of $N$ is given below.

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq 1,$$


The Normal Lattice of $N$ is

By looking at the normal lattice we see that we will not have a direct extension since we do not have 2 normal subgroups of $N$ whose product will give us $|N| = 400$. We
then find the largest abelian subgroup of $N$.

> for i in [1..22] do if IsAbelian(NL[i]) then i; end if; end for;
1
2
3
6
> NL[6];
Permutation group acting on a set of cardinality 20
Order = 25 = 5^2
  (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(16, 20, 19, 18, 17)
  (6, 9, 7, 10, 8)(11, 14, 12, 15, 13)(16, 19, 17, 20, 18)
> X:=AbelianGroup(GrpPerm,[5,5]);
> s:=IsIsomorphic(X,NL[6]);s;
true

NL[6] ≅ 5^2 is an abelian subgroup of $N$. Therefore we will have a mixed extension $N ≅ 5^2 : Q$. Now we will factor by NL[6] and form a factor group, $q$. We then check the normal lattice of $q$ to find the isomorphism type of $Q$.

q,ff:=quo<N|NL[6]>;

Now, $Q$ is of order 16. We check isomorphism of $Q$.

> s:=IsIsomorphic(q,CyclicGroup(16));s;
false
> s:=IsIsomorphic(q,DirectProduct(CyclicGroup(8),CyclicGroup(2)));s;
false
> s:=IsIsomorphic(q,DirectProduct(CyclicGroup(4),CyclicGroup(4)));s;
true

$Q ≅ 4^2$. So we should have a mixed extension $5^2 : 4^2$. To prove this isomorphism we need a presentation for $Q ≅ 4^2$.

> Q<a,b>:=Group<a,b|a^4,b^4,(a,b)>;
> f1,Q1,k1:=CosetAction(Q,sub<Q|Id(Q)>);
> s,t:=IsIsomorphic(Q1,q); s;
true
Now we need to write the generators of $Q$ into elements of $q$. We will need to look at the transversals of $NL[6]$.

```plaintext
> T:=Transversal(N,NL[6]);
> #T;
16
> A:=t(f1(a));
> B:=t(f1(b));
> for i in [1..#T] do if ff(T[i]) eq A then i; end if; end for;
2
> for i in [1..#T] do if ff(T[i]) eq B then i; end if; end for;
3
```

Now we store $T[2]$ and $T[3]$ as $A$ and $B$, respectively.

```plaintext
> T[2];
(1, 6, 15, 16, 2, 9, 12, 17, 3, 7, 14, 18, 4, 10, 11, 19, 5, 8, 13, 20)
> A:=N!(1, 6, 15, 16, 2, 9, 12, 17, 3, 7, 14, 18, 4, 10, 11, 19, 5,
> 8, 13, 20);
> T[3];
(1, 8, 12, 17, 4, 10, 11, 18, 2, 7, 15, 19, 5, 9, 14, 20, 3, 6, 13, 16)
> B:=N!(1, 8, 12, 17, 4, 10, 11, 18, 2, 7, 15, 19, 5, 9, 14, 20, 3,
> 6, 13, 16);
```

Next we find generators of $NL[6]$ and store them. Recall that $NL[6] \cong 5^2$, so we will need two generators of order 5.

```plaintext
> Order(NL[6].1);
5
> Order(NL[6].2);
5
> D:=NL[6].1;
> E:=NL[6].2;
```

Now we look at our presentation for $Q$ to see if anything has changed once we apply the action of $q$.

```plaintext
> Order(A);
20
> Order(B);
20
```
Recall that our presentation for $Q$ was $< a, b | a^4, b^4, (a, b) >$, thus the order of $a$ and the order of $b$ have been changed by the action of $q$. We will need to write these generators in terms of $D$ and $E$.

```plaintext
> for i, j in [0..5] do if A^4 eq D^i*E^j then i, j; break; end if; end for;
1 4
> for i, j in [0..5] do if B^4 eq D^i*E^j then i, j; break; end if; end for;
3 3
> for i, j in [0..5] do if (A,B) eq D^i*E^j then i, j; break; end if; end for;
1 0
> for n, o in [0..5] do for p, q in [0..20] do if D^A eq D^n*E^o*A^p*B^q then n, o, p, q; break; end if; end for; end for;
0 0 12 8
> for n, o in [0..5] do for p, q in [0..20] do if D^B eq D^n*E^o*A^p*B^q then n, o, p, q; break; end if; end for; end for;
0 0 16 4
> for n, o in [0..5] do for p, q in [0..20] do if E^A eq D^n*E^o*A^p*B^q then n, o, p, q; break; end if; end for; end for;
0 0 8 8
> for n, o in [0..5] do for p, q in [0..20] do if E^B eq D^n*E^o*A^p*B^q then n, o, p, q; break; end if; end for; end for;
0 0 4 4
```

The above loop tell us that $a^4 = d e^4, b^4 = d^3 e, (a, b) = d, d^c = a^{12} b^8, d^b = a^{16} b^4, e^a = a^8 b^5, e^b = a^4 b^4$. We can now add these relations to our presentation of $Q$, along with the 2 generators of $\text{NL}[6]$, say $d$ and $e$, to check our isomorphism of $N$.

```plaintext
> NN<a,b,d,e>:=Group<a,b,d,e|a^4=d*e^4,b^4=d^3*e^3,(a,b)=d,d^c=a^{12}b^8,d^b=a^{16}b^4,e^a=a^8b^5,e^b=a^4b^4>;
```

Thus \( N \cong 5^2 \cdot (4^2) \)

### 6.1.3 Transitive Group(20,121)

Let \( N \) be transitive group 102 on 20 letters. \( N = (5 : 4) \times S_4 \) is of order 480 and is generated by \( x = (1, 4, 2, 3)(5, 12, 18, 15)(6, 11, 17, 16)(7, 9, 20, 14)(8, 10, 19, 13) \) and \( y = (1, 8, 9, 16, 17, 4, 5, 12, 13, 20)(2, 6, 10, 14, 18)(3, 7, 11, 15, 19) \). In the same manner as before, we find the following presentation for \( G \).

\[
G = \langle x, y, t | x^4, yx^{-2}y^2x^2y, yx^{-1}y^{-1}x^{-2}xy^3, x^{-1}y^{-1}x^{-2}y^{-1}x^2yx^2yx^{-1}, \\
x^{-1}y^{-1}x^{-1}y^{-1}x^{-1}y^{-1}xyxy, \\
t^2, (t, yx^{-1}y^2), (t, yxyx^{-1}y), \\
(yx^2y^{-1}x^{-1}y^{-1}x^{-2}t)^{r_1}, (yx^2y^{-1}x^{-1}y^{-1}x^{-2}t^2)^{r_2}, (y^{-1}x^{-1}y^{-1}x^{-1}xt^2)^{r_3}, \\
(y^{-1}x^{-1}y^{-1}x^{-1}xt)^{r_4}, (x^{-1}y^{-1}x^{-1}y^{-1}x^{-1}yt)^{r_5}, (yx^{-1}y^{-1}x^{-1}yt)^{r_6}, \\
(xyx^{-1}y^{-1}x^{-1}yt)^{r_7}, (x^2yt)^{r_8}, (y^3x^{-1}y^{-1}x^2t)^{r_9}, (y^3x^{-1}y^{-1}x^2t^2)^{r_{10}}, \\
(x^2yxy^3t^{y^3}y^3)^{r_{11}}, (x^2yxy^3y^3x)^{r_{12}}, (y^2t)^{r_{13}}, ((xy)^2y^4)^{r_{14}}, ((xy)^2y^4)^{r_{15}}, ((xy)^2t)^{r_{16}} \rangle.
\]

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<th>( r_3 )</th>
<th>( r_4 )</th>
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<th>( r_{11} )</th>
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<th>( r_{13} )</th>
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### Proof of the Isomorphism for the Shape of \( N \)

The composition series of \( N \) is given below.
\[
G = G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq 1,
\]
where
\[
\]

The Normal Lattice of \( N \) is
We see that NL[7] is of order 20 and NL[8] is of order 24. Since 20 \cdot 24 = 480, and both subgroups are normal in $N$, then we have a direct product.

```plaintext
> s:=IsIsomorphic(N,DirectProduct(NL[7],NL[8])); s;
true

Now we will need to find the isomorphism of NL[7] and NL[8].

> NL[7];
Permutation group acting on a set of cardinality 20
Order = 20 = 2^2 \cdot 5
(5, 9, 17, 13)(6, 10, 18, 14)(7, 11, 19, 15)(8, 12, 20, 16)
(5, 17)(6, 18)(7, 19)(8, 20)(9, 13)(10, 14)(11, 15)(12, 16)
(1, 9, 17, 5, 13)(2, 10, 18, 6, 14)(3, 11, 19, 7, 15)(4, 12, 20, 8, 16)
> FPGroup(NL[7]);
Finitely presented group on 3 generators
Relations
$.1^4 = \text{Id}($) 
$.2^2 = \text{Id}($) 
$.1^{-2} \ast $.2 = \text{Id}($) 
($.2 \ast $.3^{-1})^2 = \text{Id}($) 
$.1 \ast $.3^{-1} \ast $.1^{-1} \ast $.3^{-2} = \text{Id}($) 
> G<x,y,z>:=Group<x,y,z|x^4,y^2,x^{-2}y,(y*z^{-1})^2,x*z^{-1}*x^{-1}z^{-2}>;
> f1,G1,k1:=CosetAction(G,sub<G|\text{Id}(G)>);
> s,t:=IsIsomorphic(G1,NL[7]); s;
> nnl:=NormalLattice(G1);
> nnl;
Normal subgroup lattice
-----------------------
---
---
[2] Order 5 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
> Center(G1);
Permutation group acting on a set of cardinality 20
Order = 1

> for i in [1..4] do if IsAbelian(nnl[i]) then i; end if; end for;
1
2
> s:=IsIsomorphic(nnl[2],CyclicGroup(5)); s;
true
> for i in nnl[4] do if i notin nnl[2] and Order(i) eq 4 and
    sub<G1|i,nnl[4]> eq G1 then F:=i;
    break; end if; end for;
> A:=t(f1(x));
> NN<x>:=Group<x|x^5>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N1): i in [1..#N1]];
> for i in [2..#N1] do
    P:=[Id(N1): I in [1..#Sch[i]]];
    for j in [1..#Sch[i]] do
        if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
        if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^(-1); end if;
    end for;
    PP:=Id(N1);
    for k in [1..#P] do
        PP:=PP*P[k]; end for;
    ArrayP[i]:=PP;
end for;
> for i in [1..#N1] do if ArrayP[i] eq A^F then print Sch[i]; end if; end for;
x^2
> H<x,y>:=Group<x,y|x^5,y^4,x*y=x^2>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,nnl[2]); s;
true

> NL[8];
Permutation group acting on a set of cardinality 20
Order = 24 = 2^3 * 3
Hence \( \text{NL}[8] = S_4 \).

Now we add both presentations together and verify the isomorphism of \( N \).

\[
(3, 4) (7, 8) (11, 12) (15, 16) (19, 20) \\
(2, 4, 3) (6, 8, 7) (10, 12, 11) (14, 16, 15) (18, 20, 19) \\
(1, 2) (3, 4) (5, 6) (7, 8) (9, 10) (11, 12) (13, 14) (15, 16) (17, 18) (19, 20) \\
(1, 4) (2, 3) (5, 8) (6, 7) (9, 12) (10, 11) (13, 16) (14, 15) (17, 20) (18, 19)
\]

\[
> s := \text{IsIsomorphic} (\text{NL}[8], \text{Sym}(4)); s;
\]

true

Hence \( \text{NL}[8] = S_4 \).

\[
> \text{FPGroup} (\text{NL}[7]);
\]

Finitely presented group on 3 generators
Relations
\[
$.1^4 = \text{Id}($) \\
$.2^2 = \text{Id}($) \\
$.1^-2 * $.2 = \text{Id}($) \\
(\text{$.2 * $.3^-1} )^2 = \text{Id}($)
\]

\[
> \text{FPGroup} (\text{NL}[8]);
\]

Finitely presented group on 4 generators
Relations
\[
$.1^2 = \text{Id}($) \\
$.2^3 = \text{Id}($) \\
$.3^2 = \text{Id}($) \\
$.4^2 = \text{Id}($) \\
(\text{$.2^-1 * $.1} )^2 = \text{Id}($)
\]

\[
> \text{H<u,v,w,x,y,z,r>:=Group<u,v,w,x,y,z,r|u^4,v^2,u^2*v, (v*w^-1)2,}
\]

\[
> u*w^-1*u^-1*w^-2, x^2, y^3, z^2, r^2, (y^-1*x)^2,}
\]

\[
> y^-1*z*y*r, (x*z)^2, (z*r)^2,}
\]

\[
> y*z*y^-1*z*r, (u,x), (u,y), (u,z), (u,r), (v,x), (v,y), (v,z), (v,r),}
\]

\[
> (w,x), (w,y), (w,z), (w,r)>;}
\]

\[
> \text{f1,H1,k1:=CosetAction(H,sub<H|Id(H)>)};
\]

\[
> s,t:=\text{IsIsomorphic}(H1,N); s;
\]

true

Hence \( N \cong (5:4) \times S_4 \).
### 6.1.4 Transitive Group \((20,10)\)

Let \(N\) be transitive group 10 on 20 letters. \(N = D_{20}\) is of order 40 and is generated by \(x = (1,19)(2,20)(3,18)(4,17)(5,16)(6,15)(7,13)(8,14)(9,12)(10,11)\) and \(y = (1,3,5,8,10,11,14,16,18,19,2,4,6,7,9,12,13,15,17,20)\). In the same manner as before, we find the following presentation,

\[
G = \langle x, y, t | x^2, (y^{-1}x)^2, y^{20}, t^2, (t, xy^{-9}), (y^6t)^{-1}, (yt)^2, (y^2t)^3, (y^7t)^4, (y^9t)^5, \\
(y^9ty^{13})^6, (xt^{14}y^6)^7, (y^5t)^8, (y^9tt^{12})^9, (xt)^{10}, (xt^9)^{11}, (y^{10}tt^y)^{12}, (y^{10}tt^y)^{13}, \\
(y^{10}tt^y)^{14}, (y^9tt^y)^{15}, (y^4tt^y)^{16}, (xy^ty)^{17}, (xt^{9}ty^{19})^{18}, (xt^{14}ty^9)^{19}, (xy^9ty^7)^{20}, \\
(xy^tt^{14})^{21}, (y^9tt^y)^{22}, (xt^{14})^{23}, (y^5tt^y)^{24}, (y^9tt^y)^{25}, (xy^9ty^7)^{26}, (xy^tt^y)^{27}, \\
(xy^9ty^4)^{28}, (y^8tt^y)^{29}, (xt^{13}ty^{17})^{30}, (xy^9ty^7)^{31}, (xyt)^{32}, (y^4tt)^{33}, (y^7tt^y)^{34} \rangle.
\]

|   | r1 | r2 | r3 | r4 | r5 | r6 | r7 | r8 | r9 | r10 | r11 | r12 | r13 | r14 | r15 | r16 | r17 | r18 | r19 | r20 | r21 |
|---|----|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |    |
| 2 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |    |
| 3 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 4 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 5 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 6 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 7 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 8 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 9 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |    |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |    |
Table 6.5: $2^{*20} : D_{20}$ continued

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<td>$PGL(2, 11)$</td>
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</table>

Proof of the Isomorphism for the Shape of $N$

The composition series of $N$ is given below.

\[
\begin{align*}
G & \quad | \quad \text{Cyclic}(2) \\
* & \quad | \quad \text{Cyclic}(5) \\
* & \quad | \quad \text{Cyclic}(2) \\
1 & \quad | \quad \text{Cyclic}(2) \\
\end{align*}
\]

$G = G_1 \vartrianglerighteq G_2 \vartrianglerighteq G_3 \vartrianglerighteq 1$, where

$G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/1) = C_2C_5C_2C_2$. 

The Normal Lattice of $N$ is
s:=IsIsomorphic(N,DihedralGroup(20)); s; true

6.1.5 Transitive Group(20,11)

Let $N$ be transitive group 10 on 20 letters. $N = 2 \cdot D_{10}$ is of order 40 and is generated by $x = (1, 17, 2, 18)(3, 16, 4, 15)(5, 14, 6, 13)(7, 12, 8, 11)(9, 20, 10, 19)$ and $y = (1, 4, 6, 7, 10)(2, 3, 5, 8, 9)(11, 13, 16, 17, 20, 12, 14, 15, 18, 19)$. In the same manner as before, we find the following presentation,

$$G = \langle x, y, t | x^4, (x^{-1}y)^2, (xy^{-1})^2, y^{10}, t^2, (t, y^5), (y^2tx^2y^3xty^3)^r_1, (txty^2)^r_2, (xttx^2y^3x)^r_3,$$
$$ (xytx^2y^3tx^2y^2)^r_4, (yt^6y^3x)^r_5, (xt^3y^4)^r_6, (x^2y^2tx^2y^3tx^2y^2)^r_7, (xyt)^r_8, (y^4ty)^r_9 \rangle.$$

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<td>$2^*PGL(2,13)$</td>
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Proof of the Isomorphism for the Shape of $N$

The composition series of $N$ is given below.

\[
\begin{array}{c|c|c|c|c}
G & \text{Cyclic}(2) & \text{Cyclic}(5) & \text{Cyclic}(2) & 1 \\
| & * & * & * & \\
\end{array}
\]

$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq 1$, where

$G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/1) = C_2C_5C_2C_2$.

The Normal Lattice of $N$ is

```
> Center(N);  
Permutation group acting on a set of cardinality 20  
Order = 2
```
We will factor by the center of $N$ and examine the factor group $q$.

We see that $q$ is isomorphic to $D_{10}$. Thus we will have a central extension of 2 by $D_{10}$. Now we need to write a presentation for $q \cong D_{10}$ and proceed to verify our isomorphism.
> T:=Transversal(G1,NL[2]);
> #T;
20
> T[2];
(1, 2, 6, 4)(3, 9, 13, 8)(5, 11, 14, 7)(10, 16, 21, 17)(12, 15, 22, 19)(18, 25, 29, 24)(20, 27, 30, 23)(26, 32, 36, 33)(28, 31, 37, 35)(34, 39, 40, 38)
> A:=G1!(1, 2, 6, 4)(3, 9, 13, 8)(5, 11, 14, 7)(10, 16, 21, 17)
> (12, 15, 22, 19)(18, 25, 29, 24)(20, 27, 30, 23)(26, 32, 36, 33)(28, 31, 37, 35)(34, 39, 40, 38);
> T[3];
(1, 3, 10, 18, 26, 34, 28, 20, 12, 5)(2, 7, 15, 23, 31, 38, 32, 24, 16, 8)(4, 11, 19, 27, 35, 39, 33, 25, 17, 9)(6, 13, 21, 29, 36, 40, 37, 30, 22, 14)
> B:=G1!(1, 3, 10, 18, 26, 34, 28, 20, 12, 5)(2, 7, 15, 23, 31, 38, 32, 24, 16, 8)(4, 11, 19, 27, 35, 39, 33, 25, 17, 9)(6, 13, 21, 29, 36, 40, 37, 30, 22, 14);
> q;
> ff(A) eq q.1;
true
> ff(B) eq q.2;
true
> NL[2].1;
> C:=G1!(1, 6)(2, 4)(3, 13)(5, 14)(7, 11)(8, 9)(10, 21)(12, 22)
> F;
Finitely presented group F on 2 generators
Relations
  x^2 = Id(F)
  (y^-1 * x)^2 = Id(F)
y^10 = Id(F)
> for i in [1..2] do if A^2 eq C^i then i; end if; end for;
1
> for i in [1..2] do if (B^-1*A)^2 eq C^i then i; end if; end for;
2
> for i in [1..2] do if B^10 eq C^i then i; end if; end for;
2
> H<c,x,y>:=Group<c,x,y|c^2,(y,c),(x,c),x^2=c,(y^-1*x)^2,y^10>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,G1);
> s;
true
Thus $N \cong 2 \cdot D_{10}$.

### 6.1.6 Transitive Group(20,12)

Let $N$ be transitive group 10 on 20 letters. $N = (5 \times D_4)$ is of order 40 and is generated by $x = (1, 13, 8, 20, 4, 15, 10, 12, 5, 18, 2, 14, 7, 19, 3, 16, 9, 11, 6, 17)$ and $y = (1, 19, 9, 18, 7, 15, 5, 13, 4, 11)(2, 20, 10, 17, 8, 16, 6, 14, 3, 12)$. In the same manner as before, we find the following presentation,

$$G = \langle x, y, t | x^4y^2, x^{-1}y^{-1}x^2yx^{-1}, y^{-1}xy^{-1}xy^{-2}, t^2, (t, x^2y^{-1}x), (y^{-1}x^{-1}t)^r_1, (y^{-1}x^{-1}t^y)^r_2, (yxty^{-1})^r_3, (yx^{-1}y^{-1}t)^r_4, (xyxt^1)^r_5, (x^3t)^r_6, (xyxt^17)^r_7 >.$$  

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<th>$r_3$</th>
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<th>$r_6$</th>
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**Proof of the Isomorphism for the Shape of $N$**

The composition series of $N$ is given below.

<table>
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<tr>
<th>$G$</th>
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<tr>
<td>*</td>
<td>Cyclic(2)</td>
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<tr>
<td>*</td>
<td>Cyclic(2)</td>
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</table>
\[ G = G_1 \supseteq G_2 \supseteq G_3 \supseteq 1, \text{ where} \]
\[ G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/1) = C_2C_5C_2C_2. \]

The Normal Lattice of \( N \) is

It is possible that we have a direct extension.

\[ > s:=\text{IsIsomorphic}(N,\text{DirectProduct(CyclicGroup(5),DihedralGroup(4)))};s; \]
true

Thus \( N \cong (5 \times D_4) \).

### 6.1.7 Transitive Group(20,13)

Let \( N \) be transitive group 10 on 20 letters. \( N \) is of order 40 and is generated by \( x = (1,13,18,6)(2,14,17,5)(3,7,16,11)(4,8,15,12)(9,10)(19,20) \) and \( y = (1,3,5,8,10,12,14,16,18,19)(2,4,6,7,9,11,13,15,17,20) \). In the same manner as before, we find the following presentation,
\[ G = \langle x, y, t \mid x^4, x^{-1}y^{-1}x^2y^{-1}x^{-1}, y^{-1}x^{-1}y^{-1}xy^{-2}, t^2, (t, x^2y^2), (x^{-1}t)^{r_1}, \\
(x^{-1}tx^2y^2)^{r_2}, (x^{-1}tx^2y^2)^{r_3}, (xt^2xy^2)^{r_4}, (x^2y^{-1}tx^2y^3)^{r_5}, \\
(xyt^2x^2y^3)^{r_6}, (xyt^2x^2y^3)^{r_7}, (x^{-1}tx^2y^2)^{r_8}, (x^2y^2ty^2)^{r_9}, \\
(x^2y^3x^3)^{r_{10}}, (x^2y^3x^3)^{r_{11}}, (x^2y^3x^3)^{r_{12}}, (y^2tx^2)^{r_{13}}, \\
(x^2y^{-1}t)^{r_{14}}, (y^{-1}x^{-1}ty^2x)^{r_{15}}, (x^2y^{-1}t^2ty^2x)^{r_{16}}, (x^2ty^4x)^{r_{17}}, \\
(xyty^6)^{r_{18}}, (xt^2x^2y^2)^{r_{19}}, (y^{-1}x^{-1}ty^2xtyx)^{r_{20}}, (yt^2x^2ty)^{r_{21}}, \\
(x^2y^4x)^{r_{22}}, (xyt^2x^2x^3)^{r_{23}}, (x^2ty^6)^{r_{24}}, (xyt^3y)^{r_{25}}, (x^2y^2x^2ty^2)^{r_{26}}, \\
(xyty^2x^2ty)^{r_{27}}, (x^{-1}ty^4x)^{r_{28}}, (x^{-1}ty^4x)^{r_{29}} >. \]

Table 6.8: \(2^{*20} : (4^* : 10)\)

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Table 6.9: $2^{*20} : (4^* : 10)$ continued

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</tbody>
</table>

Proof of the Isomorphism for the Shape of $N$

The composition series of $N$ is given below.

$$
\begin{align*}
G & = G_1 \supseteq G_2 \supseteq G_3 \supseteq 1, \text{ where} \\
G & = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/1) = C_2C_5C_2C_2. 
\end{align*}
$$

The Normal Lattice of $N$ is
We find that the center of $N$ is $\text{NL}[2]$. However we see that $\text{NL}[2]$ is not the largest abelian subgroup. Since $\text{NL}[4] = C_{10}$ is the largest abelian subgroup then we will have a mixed extension.

```maple
> Center(N);
Permutation group acting on a set of cardinality 20
Order = 2
   (1, 12) (2, 11) (3, 14) (4, 13) (5, 16) (6, 15) (7, 17) (8, 18) (9, 20) (10, 19)
> NL[2] eq Center(N);
true
> for i in [1..10] do if IsAbelian(NL[i]) then i; end if; end for;
1
2
3
4
> q,ff:=quo<N|NL[4]>;
> nl:=NormalLattice(q);
```
Normal subgroup lattice
----------------------------------
---
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

> s:=IsIsomorphic(q,CyclicGroup(4));
> s;
true
> H<a>:=Group<a|a^4>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,q); s;
true
> T:=Transversal(N,NL[4]);
> #T;
4
> T[2];
(1, 13, 18, 6)(2, 14, 17, 5)(3, 7, 16, 11)(4, 8, 15, 12)(9, 10)(19, 20)
> A:=N!(1, 13, 18, 6)(2, 14, 17, 5)(3, 7, 16, 11)(4, 8, 15, 12)(9, 10)(19, 20);
> q;
Permutation group q acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 2, 3, 4)
    Id(q)
> ff(A) eq q.1;
true
> Order(A);
4
> IsCyclic(NL[4]);
true
> Order(NL[4].1);
10
> NL[4].1;
(1, 3, 5, 8, 10, 12, 14, 16, 18, 19)(2, 4, 6, 7, 9, 11, 13, 15, 17, 20)
> B:=N!(1, 3, 5, 8, 10, 12, 14, 16, 18, 19)(2, 4, 6, 7, 9,
> 11, 13, 15, 17, 20);
> for i in [0..10] do if B^A eq B^i
>    for|if> for|if> then i; break; end if; end for;
> 7
> H<b,a>:=Group<b,a|b^10,a^4,b^a=b^7>;
> f,h,k:=CosetAction(H,sub<H|Id(H)>);
> #h;
> s:=IsIsomorphic(h,N);
> s;
true

Thus $N \cong 4^* : 10$.

6.2 Transitive Groups on 19 Letters

Using the following code we find that there are 8 transitive groups on 19 letters.

> #TransitiveGroups(19);
8

We will examine some of these groups and write progenitors.

6.2.1 Transitive Group(19,2)

Let $N$ be transitive group 10 on 19 letters. $N = (2 : 19)$ is of order 38 and is generated by $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19)$ and $y = (2, 19)(3, 18)(4, 17)(5, 16)(6, 15)(7, 14)(8, 13)(9, 12)(10, 11)$. In the same manner as before, we find the following presentation,

$$G = < x, y, t | y^2, (x^{-1}y)^2, x^{-19}, t^2, (t, xy^2), (x^5t)^{r_1}, (x^6t)^{r_2}, (x^8t)^{r_3}, (x^9t)^{r_4}, (yt^{x^6})^{r_5}, (xt)^{r_6} >.$$  

<table>
<thead>
<tr>
<th>r1</th>
<th>r2</th>
<th>r3</th>
<th>r4</th>
<th>r5</th>
<th>r6</th>
<th>Order</th>
<th>$G$</th>
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<td>0</td>
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<td>$PGL_2(19)$</td>
</tr>
<tr>
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<td>3</td>
<td>3</td>
<td>25308</td>
<td>$L_2(37)$</td>
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</table>

Table 6.10: 2^{20} : (2 : 19)
Proof of the Isomorphism for the Shape of $N$

> $S:=\text{Sym}(19);$  
> $xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19);$  
> $yy:=S!(1, 18)(2, 17)(3, 16)(4, 15)(5, 14)(6, 13)(7, 12)(8, 11)(9, 10);$  
> $N:=\text{sub}\langle N \mid xx, yy \rangle;$  
> $\#N;$  
> 38
> $NL:=\text{NormalLattice}(N);$  
> $NL;$  
Normal subgroup lattice

-----------------------
[3] Order 38 Length 1 Maximal Subgroups: 2
---
[2] Order 19 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

> $\text{IsIsomorphic}(NL[2], \text{CyclicGroup}(19));$
true
> $H<x>:=\text{Group}<x|x^{19}>;$
> $f,H1,k:=\text{CosetAction}(H, \text{sub}\langle H \mid \text{Id}(H) \rangle);$  
> $s:=\text{IsIsomorphic}(NL[2], H1);s;$  
true
> for i in NL[3] do if i notin NL[2] and Order(i) eq 2 and
for|if> $\text{sub}\langle N \mid i, NL[2] \rangle$ eq N then C:=i;
for|if> break; end if; end for;
> $\text{FPGroup}(N);$  
Finitely presented group on 2 generators
Relations
$.1^4 = \text{Id}($
$.1^{-1} * $.2^{-1} * $.1^2 * $.2^{-1} * $.1^{-1} = \text{Id}($
$.2^{-1} * $.1^{-1} * $.2^{-1} * $.1 * $.2^{-2} = \text{Id}($
> $NN<x,y>:=\text{Group}<x,y|y^2,(x^{-1}*y)^2,x^{-19}>;$
> $\text{Sch}:=\text{SchreierSystem}(NN, \text{sub}\langle NN \mid \text{Id}(NN) \rangle);$  
> $\text{ArrayP}:=[\text{Id}(N): i in [1..38]];$
> for i in [2..38] do
for|> $P:=[\text{Id}(N): l in [1..\#Sch[i]]];$
for|for> for j in [1..\#Sch[i]] do
for|for|> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for|> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^−1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..#NN1] do if ArrayP[i] eq A^C then print Sch[i];
for|if> end if; end for;
x^−1
> H<y,x>:=Group<y,x|x^19,y^2,x^y=x^−1>;
> f2,H2,k1:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(H2,N);
true

Thus $N \cong (2:19)$.

6.3 Transitive Groups on 11 Letters

Using the following code we find that there are 8 transitive groups on 11 letters.

> NumberOfTransitiveGroups(11); 8

We will examine some of these groups and write progenitors.

6.3.1 Transitive Group(11,2)

Let $N$ be transitive group 2 on 11 letters. $N = (2:11)$ is of order 22 and is generated by $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$ and $y = (1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$. In the same manner as before, we find the following presentation,

$$G < x, y, t | y^2, (x^{-1}y)^2, x^{-11}, t^2, (t, yx^2), (xt)^{r_1}, (x^2t)^{r_2}, (x^3t)^{r_3}, (x^4t)^{r_4}, (x^5t)^{r_5} >.$$
Table 6.11: $2^{11} : (2 : 11)$

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<th>r5</th>
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<td>1320</td>
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<td>$PGL(2,23)$</td>
</tr>
</tbody>
</table>

Proof of the Isomorphism of $N$

```plaintext
> S:=Sym(11);
> xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11);
> yy:=S!(1, 10)(2, 9)(3, 8)(4, 7)(5, 6);
> N:=sub<S|xx,yy>;
> #N;
22
> NormalLattice(N);

Normal subgroup lattice
-----------------------
[2] Order 11 Length 1 Maximal Subgroups: 1
[1] Order 1 Length 1 Maximal Subgroups:

> CompositionFactors(N);
  G
  | Cyclic(2)
  *
  | Cyclic(11)
  1
> NL:=NormalLattice(N);
> s:=IsIsomorphic(NL[2],CyclicGroup(11));s;
true
> FPGroup(N);
Finitely presented group on 2 generators
Relations
  $.2^2 = Id($)
  ($.1^-1 * $.2)^2 = Id($)
  $.1^-11 = Id($)
> G<x,y>:=Group<x,y|y^2,(x^-1*y)^2,x^11>;
```
> #G;
22
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> #k;
> NL:=NormalLattice(G1);
> H<x>:=Group<x|x^11>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,NL[2]);s;
true
> for i in NL[3] do if i not in NL[2] and Order(i) eq 2 and
for|if> sub<G1|i,NL[3]> eq G1 then E:=i; break; end if; end for;
> A:=t(f1(x));
> N1:=sub<NL[3]|A>;
> NN<a>:=Group<a|a^11>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N1): i in [1..#N1]];
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> for i in [2..#N1] do
for> P:=[Id(N1): I in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
end for;
end for;
for> PP:=Id(N1);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
end for;
for i in [1..#N1] do if ArrayP[i] eq A^E then print Sch[i];
for|if> end if; end for;
> H<x,e>:=Group<x,e|x^11,e^2,x^e=x^-1>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,G1);s;
true

Thus \( N \cong 2 : 11 \).
6.3.2 Transitive Group(11,5)

Let \( N \) be transitive group 2 on 11 letters. \( N = L_2(11) \) is of order 660 and is generated by \( x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11) \) and \( y = (2, 10)(3, 4)(5, 9)(6, 7) \). In the same manner as before, we find the following presentation,

\[
G < x, y, t | y^2, (yx^{-1})^3, x^{-11}, (xyx^{-3}yx^2)^2, t^2, (t, yx), (t, x^2yx^{-3}), (t, y), \\
((yx^3)^2t(x^2))^{r_1}, (yx^5t)^{r_2}, (yx^5tx^3)^{r_3}, (yx^5tx^2)^{r_4}, (xt)^{r_5}, (x^2t)^{r_6} >.
\]

<table>
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<tr>
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<th>r3</th>
<th>r4</th>
<th>r5</th>
<th>r6</th>
<th>Order</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>660</td>
<td>( L_2(11) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>351120</td>
<td>( J_1 \times 2 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>175560</td>
<td>( J_1 )</td>
</tr>
</tbody>
</table>

Proof of the Isomorphism of \( N \)

\[
\begin{align*}
N &:= \text{TransitiveGroup}(11,5); \\
\#N; &\quad 660 \\
\text{Generators}(N); &\quad \{ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11), \\
&\quad (2, 10)(3, 4)(5, 9)(6, 7) \} \\
S &:= \text{Sym}(11); \\
xx &:= S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11); \\
yy &:= S!(2, 10)(3, 4)(5, 9)(6, 7); \\
N &:= \text{sub}\langle S xx, yy \rangle; \\
\#N; &\quad 660 \\
N &:= \text{sub}\langle S xx, yy \rangle; \\
\#N; &\quad 660 \\
\text{CompositionFactors}(N); \\
G &\quad | A(1, 11) = L(2, 11) \\
&\quad 1 \\
s &:= \text{IsIsomorphic}(N, PSL(2,11)); s; \\
true
\end{align*}
\]
6.4 Transitive Groups on 6 Letters

We will examine some of these groups and write progenitors.

6.4.1 Transitive Group(6,3)

Let $N$ be transitive group 3 on 6 letters. $N = 2 \times S_3$ is of order 12 and is generated by $x = (1,4)(2,3)(5,6)$ and $y = (1,2,3,4,5,6)$. In the same manner as before, we find the following presentation,

$$G < x, y, t | x^2, (y^{-1}x)^2, y^6,$$
$$t^2, (t, xy^3),$$
$$(xt^y)^r_1, (xyt^{y^2})^r_2, (xy^t)^r_3, (y^2t)^r_4, (yt)^r_5 >.$$

<table>
<thead>
<tr>
<th>r1</th>
<th>r2</th>
<th>r3</th>
<th>r4</th>
<th>r5</th>
<th>Order</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>0</td>
<td>3</td>
<td>10</td>
<td>483840</td>
<td>$(2 \times 6) : (L_3(4) : 2)$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>8</td>
<td>241920</td>
<td>$6 : (L_3(4) : 2)$</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>5</td>
<td>10</td>
<td>6</td>
<td>1320</td>
<td>$2^4 L_2 (11)$</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>9</td>
<td>51840</td>
<td>$2 : S(4,3)$</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>380160</td>
<td>$2^4 \times M_{12}$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
<td>7</td>
<td>10</td>
<td>322560</td>
<td>$2^3 : L_3(4)$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>4368</td>
<td>$2 \times PGL(2,13)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>2184</td>
<td>$PGL(2,13)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>24360</td>
<td>$PGL(2,29)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>13680</td>
<td>$2 \times PGL(2,19)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>6840</td>
<td>$PGL(2,19)$</td>
</tr>
</tbody>
</table>

Proof of the Isomorphism of $N$

```maple
> N:=TransitiveGroup(6,3);
> #N;
12
> Generators(N);
{
(1, 4) (2, 3) (5, 6),
(1, 2, 3, 4, 5, 6)
}
> S := Sym(6);
> xx := S!(1, 4) (2, 3) (5, 6);
> yy := S!(1, 2, 3, 4, 5, 6);
> N := sub<S|xx, yy>;
> #N;
12
> CompositionFactors(N);
G
  | Cyclic(2)
  *
  | Cyclic(3)
  *
  | Cyclic(2)
  1
> NormalLattice(N);

Normal subgroup lattice
-----------------------

---
---
[3] Order 3 Length 1 Maximal Subgroups: 1
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

> s := IsIsomorphic(N, DirectProduct(Sym(3), CyclicGroup(2))); s;
true

6.4.2 Transitive Group(6,9)

Let \( N \) be transitive group 9 on 6 letters. \( N = S_3 \times S_3 \) is of order 36 and is generated by \( x = (1, 4)(2, 5)(3, 6), y = (2, 4, 6), \) and \( z = (1, 5)(2, 4) \). In the same manner as before, we find the following presentation,
\[ G = \langle x, y, z, t \mid x^2, y^3, z^2, (y^{-1}z)^2, (xz)^2, y^{-1}xy^{-1}xyz, \\
\qquad t^2, (t, y), (t, xyz), \\
\qquad (xyt)^{r_1}, (xyt(xy^3))^{r_2}, (xy^{-1}xyt)^{r_3}, (xyt)^{r_4}, (xzyt)^{r_5}, (t^{xz})^m = xyzxy^{-1} \rangle \]

Table 6.14: \(2^6 : (S_3 \times S_3)\)

<table>
<thead>
<tr>
<th>(r_1)</th>
<th>(r_2)</th>
<th>(r_3)</th>
<th>(r_4)</th>
<th>(r_5)</th>
<th>(m)</th>
<th>Order</th>
<th>(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3753792</td>
<td>(2 \ast L_3(7))</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>451584</td>
<td>(2^4 : (L_2(7) \times L_2(7)) : 2^4)</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>0</td>
<td>760320</td>
<td>(M_{12} : 2) \times 2^2)</td>
</tr>
</tbody>
</table>

**Proof of the Isomorphism of \(N\)**

\[
\begin{align*}
&\text{\(N:=\text{TransitiveGroup}(6,9);\)} \\
&\text{\(#N;\)} \\
&36 \\
&\text{\(NL:=\text{NormalLattice}(N);\)} \\
&\text{\(NL;\)} \\
&\text{Normal subgroup lattice} \\
\end{align*}
\]

\[
\begin{align*}
\text{[10]} & \quad \text{Order 36} \quad \text{Length 1} \quad \text{Maximal Subgroups: 7 8 9} \\
\text{[ 9]} & \quad \text{Order 18} \quad \text{Length 1} \quad \text{Maximal Subgroups: 5 6} \\
\text{[ 8]} & \quad \text{Order 18} \quad \text{Length 1} \quad \text{Maximal Subgroups: 6} \\
\text{[ 7]} & \quad \text{Order 18} \quad \text{Length 1} \quad \text{Maximal Subgroups: 4 6} \\
\text{[ 6]} & \quad \text{Order 9} \quad \text{Length 1} \quad \text{Maximal Subgroups: 2 3} \\
\text{[ 5]} & \quad \text{Order 6} \quad \text{Length 1} \quad \text{Maximal Subgroups: 2} \\
\text{[ 4]} & \quad \text{Order 6} \quad \text{Length 1} \quad \text{Maximal Subgroups: 3} \\
\text{[ 3]} & \quad \text{Order 3} \quad \text{Length 1} \quad \text{Maximal Subgroups: 1} \\
\text{[ 2]} & \quad \text{Order 3} \quad \text{Length 1} \quad \text{Maximal Subgroups: 1} \\
\text{[ 1]} & \quad \text{Order 1} \quad \text{Length 1} \quad \text{Maximal Subgroups:} \\
\end{align*}
\]

\[
\begin{align*}
&\text{\(s:=\text{IsIsomorphic}(G1,\text{DirectProduct}(NL[5],NL[4]));s;\)} \\
&\text{true} \\
&\text{\(NL[5];\)} \\
&\text{Permutation group acting on a set of cardinality 36} \\
&\text{Order = 6 = 2 * 3} \\
&\text{(1, 2)(3, 9)(4, 7)(5, 12)(6, 13)(8, 16)(10, 18)(11, 20)(14, 24) \\
\]
\((32, 34)\) 
\((35)\) 
\((1, 27, 25) (2, 21, 19) (3, 28, 13) (4, 34, 36) (5, 16, 23) (6, 22, 9)\) 
\((7, 30, 32) (8, 12, 17) (10, 24, 35) (11, 33, 26) (14, 18, 31) (15, 29, 20)\)

\> s:=IsIsomorphic(NL[5], Sym(3)); s;
true

s:=IsIsomorphic(G1, DirectProduct(Sym(3), Sym(3))); s;
true
Chapter 7

More Progenitors

7.1 $2^{*36} : (3^2 : D_4)$

$G = < v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^{-2}x, (w^{-1}v)^2, (xy^{-1})^2,$
$vz^{-1}vz, (xz^{-1})^2, (y, z), wy^{-1}w^{-1}yz^{-1},$
$t^2, (t, vy^{-1}w^{-1}), (tvy^{-1}z^{-1})^m,$
$(vt)^r_1, (vt^2)^r_2, (vw_{t^{wz^2v}})^r_3, (yt_{t^{w^3v}})^r_5,$
$(zt^{w^2v})^r_6, (wt^y)^r_7, (vyt_{t^{w^3v}})^r_8 >$

<table>
<thead>
<tr>
<th>r1</th>
<th>r2</th>
<th>r3</th>
<th>m</th>
<th>r5</th>
<th>r6</th>
<th>r7</th>
<th>r8</th>
<th>Order</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>161280</td>
<td>$4^* (2 : L_3(4))$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3916800</td>
<td>$2^2 \times S(4, 4)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>10</td>
<td>9</td>
<td>5</td>
<td>6840</td>
<td>$PGL_2(19)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>4368</td>
<td>$2 \times PGL_2(13)$</td>
</tr>
</tbody>
</table>

7.2 $2^{*110} : L_2(11)$

$G = < x, y, t | x^2, y^3, (y^{-1}xxy)^5, (xy^{-1})^{11}, (xxyxy^{-1}xy^{-1}xy^{-1}x)^2,$
$t^2, (t, xxyxy^{-1}xy^{-1}xy^{-1}), (t, xxyxy^{-1}xyxy^{-1}xy),$
$(tx)^k, (ttxyxy^{-1}xyxyxy)^l, (tt^m) = xxyxy^{-1}xyxyxy^{-1}xy,$
$(xy^{-1}xxyxy^{-1}y^{-1})^r_1, (xxyxy^{-1}tyxyxyxy^{-1}t(yxy^2))^r_2,$
(yxyxyxy^{-1}xxyxy^{-1}xt^y)^r_3, (y^{-1}xxyxy^{-1}xy^{-1}xyyt^{(xy^2)})^r_4, \\
(xy^t)^r_5, ((xy)^2t)^r_6 >

\[
\begin{array}{cccccccc}
    \text{Table 7.2}: 2^{*110} : L_2(11) \\
    \hline
    r1 & r2 & k & l & m & r3 & r4 & r5 & r6 & \text{Order} & G \\
    \hline
    6 & 0 & 4 & 4 & 5 & 0 & 0 & 0 & 0 & 15840 & C_2 \times M_{11} \\
    0 & 0 & 8 & 4 & 3 & 0 & 0 & 0 & 5 & 7920 & M_{11} \\
    \hline
\end{array}
\]

7.3 \quad 2^{*15} : (C_{15} : C_4)

\[
G = < a, b, c, d, t | a^4, b^2, d^3, a^{-2}b, a^{-1}d^{-1}ad^{-1}, (c, d), ac^{-1}a^{-1}c^2, bc^{-1}a^2c^{-1}d, \\
    t^2, (t, ac^{-1}), \\
    (cbt^b)^r_1, (cbt)^r_2, (cbt^{a})^r_3, (ct)^r_4, (c^2dt)^r_5 >
\]

\[
\begin{array}{ccccccccc}
    \text{Table 7.3}: 2^{*15} : (C_{15} : C_4) \\
    \hline
    r1 & r2 & r3 & r4 & r5 & \text{Order} & G \\
    \hline
    2 & 8 & 8 & 6 & 6 & 161280 & C_2 * M_{12} * C_2 * C_2 * C_3 \\
    \hline
\end{array}
\]
Chapter 8

MAGMA Code

8.1 Double Coset Enumeration of \((S(4,3):2)\)

```magma
> G<x,y,t>:=Group<x,y,t|x^3,(x*y^-1)^4,y*x*y^-1*x^-2*y^-1*x*y*x^-1,
> y^-1*x^-1*y*x^-1*y^-1*x*y^3*x,(x^-1*y^2*x^-1*y^-1)^2,t^2,(t,y^*
> x^2*y^-2*x^-1*y*x^-1), (t,x^-1*y^-1*x^-1*y^-3*x*y^-1), (t, (y*x*
> y^-1)^3), (t, y^-1*x^-1*y^-2), (x*y*t^-1(x^-3))^6, (x*y^t^-1(x^2*y*x^2))^4,
> (x*y*t^-1(x^2*y))^8, x * t * x * y^-1 * x * t * y * x * t * x * y^-1
> * x * t * y^-1 * t * x * t * y>;
> #G;
51840
> S:=Sym(10);
> xx:=S!(1,2,4)(3,5,6)(7,8,10);
> yy:=S!(1,3,2)(4,7,5,9,6,8);
> N:=sub<S|xx,yy>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
```

```
G
| Cyclic(2)
*|
| C(2, 3) = S(4, 3)
1
```

```magma
> s:=IsIsomorphic(N,Sym(5)); s
true
> IN:=sub<G1|f(x), f(y)>;
> ts := [Id(G1): i in [1 .. 10] ];
> /* since there are 10 letters */
> ts[1]:=f(t); ts[2]:=f(t*x); ts[3]:=f(t*y); ts[4]:=f(t*(x^2));
> ts[5]:=f(t*(y*x)); ts[6]:=f(t*(y*x^2)); ts[7]:=f(t*(x^2*y));
```
ts[8] := f(t*(x^2*y*x));
> ts[9] := f(t*(y*x*y));
> ts[10] := f(t*(x^2*y*x^2));
prodim := function(pt, Q, I)
function v := pt;
function for i in I do
function for v := v^Q[i];
function for end for;
function return v;
function end function;
> prodim(1, ts, [10]);
432
> cst := [null : i in [1..Index(G, sub<G|x,y>)] where null is
> [Integers() | ];
> for i := 1 to 10 do
> cst[prodim(1, ts, [i])] := [i];
> end for;
> m := 0; for i in [1..432] do if cst[i] ne []
> if then m := m+1; end if; end for; m;
10
> #N1s;
12
> Set(N1s);
{
   Id(N1),
   (2, 6, 9, 3, 4, 7) (5, 10, 8),
   (2, 4) (3, 6) (8, 10),
   (2, 9) (3, 7) (5, 10),
   (2, 7, 4, 3, 9, 6) (5, 8, 10),
   (2, 9, 4) (3, 7, 6) (5, 8, 10),
   (2, 6) (3, 4) (7, 9) (8, 10),
   (4, 9) (5, 8) (6, 7),
   (2, 7) (3, 9) (4, 6) (5, 10),
   (2, 4, 9) (3, 6, 7) (5, 10, 8),
   (2, 3) (4, 6) (7, 9),
   (2, 3) (4, 7) (5, 8) (6, 9)
}
> for i in [1..#T1] do ([1]^N1s)^T1[i]; end for;
> Orbits(N1s);
[
   GSet(@ 1 @),
   GSet(@ 5, 10, 8 @),
   GSet(@ 2, 3, 4, 7, 6, 9 @)
> for m,n in IN do if ts[1]*ts[5] eq m*(ts[1]*ts[1])^n then "true"; for|if> break; end if; end for;
> for m,n in IN do if ts[1]*ts[5] eq m*(ts[1])^n then "true";
> for|if> break; end if; end for;
> for m,n in IN do if ts[1]*ts[2] eq m*(ts[1]*ts[1])^n then "true";
> for|if> break; end if; end for;
> for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then "true";
> for|if> break; end if; end for;
> N15:=Stabiliser(N,[1,5]);
> SSS:={[1,5]};
> SSS:=SSS^N;
> #SSS;
30
> Seqq:=Setseq(SSS);
>
> for i in [1..#SSS] do
  for n in IN do
    for|for> if ts[1]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    for|for> then print Rep(Seqq[i]);
    for|for> end if; end for; end for;
[ 1, 5 ]
>
> N15s:=N15;
> N15s; #N15s;
Permutation group N15 acting on a set of cardinality 10
Order = 4 = 2^2
  (2, 4)(3, 6)(8, 10)
  (2, 6)(3, 4)(7, 9)(8, 10)
4
> #N/#N15s;
30
> T15:=Transversal(N,N15s);
> for i in [1..#T15] do
  ss:=[1,5]^T15[i];
  for> cst[prodim(1, ts, ss)]:=ss;
  for> end for;
  m:=0; for i in [1..432] do if cst[i] ne []
  for> then m:=m+1; end if; end for; m;
40
> #N15s;
Set(N15s);
{
    (2, 4)(3, 6)(8, 10),
    (2, 6)(3, 4)(7, 9)(8, 10),
    (2, 3)(4, 6)(7, 9),
    Id(N15)
}
> for i in [1..#T15] do ([1,5]^N15s)^T15[i]; end for;
> Orbits(N15s);
[ GSet{@ 1 @},
  GSet{@ 5 @},
  GSet{@ 7, 9 @},
  GSet{@ 8, 10 @},
  GSet{@ 2, 4, 6, 3 @}]
> N12:=Stabiliser(N,[1,2]);
> SSS:={[1,2]};
> SSS:=SSS^N;
> #SSS;
60
> Seqq:=Setseq(SSS);
>
> for i in [1..#SSS] do
    for n in IN do
        if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        then print Rep(Seqq[i]); end if;
    end for;
[ 1, 2 ]
[ 1, 3 ]
>
> N12s:=N12;
> for n in N do if 1^n eq 1 and 2^n eq 3 then
    N12s:=sub<N|N12s,n>; end if; end for;
> N12s; #N12s;
Permutation group N12s acting on a set of cardinality 10
(4, 9)(5, 8)(6, 7)
(2, 3)(4, 7)(5, 8)(6, 9)
(2, 3)(4, 6)(7, 9)
4
> #N/#N12s;
\begin{verbatim}
30 > T12:=Transversal(N,N12s);
> for i in [1..#T12] do
  for ss:=[1,2]ˆT12[i] do
    cst[prodim(1, ts, ss)]:=ss;
  end for;
  m:=0; for i in [1..432] do if cst[i] ne []
    then m:=m+1; end if; end for; m;
70 > #N12s;
4 > Set(N12s);
{ (2, 3)(4, 7)(5, 8)(6, 9),
    (4, 9)(5, 8)(6, 7),
    (2, 3)(4, 6)(7, 9),
    Id(N12s) }
> for i in [1..#T12] do ([1,2]ˆN12s)ˆT12[i]; end for;
> Orbits(N12s);
[ GSet{@ 1 @},
  GSet{@ 10 @},
  GSet{@ 2, 3 @},
  GSet{@ 5, 8 @},
  GSet{@ 4, 9, 7, 6 @} ]
> for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(ts[1]*ts[1])ˆn
  then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(ts[1]*)ˆn
  then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(ts[1]*ts[5])ˆn
  then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(ts[1]*ts[2])ˆn
  then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*ts[1])ˆn
  then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*)ˆn
  then "true"; break; end if; end for;
true
> for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*ts[5])ˆn
\end{verbatim}
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*ts[2])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*ts[5]*ts[1])ˆn
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[1]*ts[7])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[1]*ts[8])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[2])ˆn
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[7])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[8])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[5])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[5]*ts[1])ˆn
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[5]*ts[7])ˆn
for|if> then "true"; break; end if; end for;
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[7])ˆn
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[5]*ts[5] eq m*(ts[8])ˆn
for|if> then "true"; break; end if; end for;
true
> for m,n in IN do if ts[1]*ts[5]*ts[2] eq m*(ts[1]*ts[5]*ts[8])
for|if> then "true"; break; end if; end for;

> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[1])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[2]*ts[1] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[5]*ts[1]*ts[1] eq m*(ts[1]*ts[1]*ts[7])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[5]*ts[8])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[1])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[1]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[10] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[1])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[2])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5])
for|if> then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[2])
for|if> then "true"; break; end if; end for;
for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5]*ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5]*ts[7])ˆn
  for if then "true"; break; end if; end for;
  for if then "true"; break; end if; end for;
  for if then "true"; break; end if; end for;
>
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[2])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[7])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[5]*ts[8])ˆn
  for if then "true"; break; end if; end for;
  for if then "true"; break; end if; end for;
  for if then "true"; break; end if; end for;
>
> for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[5])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[5]*ts[1])ˆn
  for if then "true"; break; end if; end for;
> for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[5]*ts[7])ˆn
  for if then "true"; break; end if; end for;
true
> for \( m, n \) in \( \text{IN} \) do if \( ts[1] \times ts[2] \times ts[4] \) eq \( m \times (ts[1] \times ts[5] \times ts[8]) \) then "true"; break; end if; end for;
> for \( m, n \) in \( \text{IN} \) do if \( ts[1] \times ts[2] \times ts[4] \) eq \( m \times (ts[1] \times ts[2] \times ts[1]) \) then "true"; break; end if; end for;
> for \( m, n \) in \( \text{IN} \) do if \( ts[1] \times ts[2] \times ts[4] \) eq \( m \times (ts[1] \times ts[2] \times ts[10]) \) then "true"; break; end if; end for;
> \( N151 := \text{Stabiliser}(N, [1, 5, 1]) \);
> \( SSS := \{[1, 5, 1]\} \);
> \( SSS := SSS \times N \);
> \#SSS;
30
> \( \text{Seqq} := \text{Setseq}(SSS) \);
> for \( i \) in \[1..\#SSS\] do
for|for|for>
for|for|for>
\( n \times ts[\text{Rep(Seqq}[i])] [1] \) \times ts[\text{Rep(Seqq}[i])] [2] \) \times
for|for|for>
\( ts[\text{Rep(Seqq}[i])] [3] \)
for|for|for>
then print \( \text{Rep(Seqq}[i]) \);
for|for|for>
end if; end for; end for;
[ 1, 5, 1 ]
[ 2, 6, 2 ]
[ 4, 3, 4 ]
[ 3, 4, 3 ]
[ 5, 1, 5 ]
[ 6, 2, 6 ]
> \( N151s := N151 \);
> for \( n \) in \( N \) do if \( 1^n \) eq 2 and \( 5^n \) eq 6 and \( 1^n \) eq 2 then
for|if>
\( N151s := \text{sub}<N\times N151s, n> \); end if; end for;
> for \( n \) in \( N \) do if \( 1^n \) eq 4 and \( 5^n \) eq 3 and \( 1^n \) eq 4 then
for|if>
\( N151s := \text{sub}<N\times N151s, n> \); end if; end for;
> for \( n \) in \( N \) do if \( 1^n \) eq 3 and \( 5^n \) eq 4 and \( 1^n \) eq 3 then
for|if>
\( N151s := \text{sub}<N\times N151s, n> \); end if; end for;
> for \( n \) in \( N \) do if \( 1^n \) eq 5 and \( 5^n \) eq 1 and \( 1^n \) eq 5 then
for|if>
\( N151s := \text{sub}<N\times N151s, n> \); end if; end for;
> for \( n \) in \( N \) do if \( 1^n \) eq 6 and \( 5^n \) eq 2 and \( 1^n \) eq 6 then
for|if>
\( N151s := \text{sub}<N\times N151s, n> \); end if; end for;
> \( N151s; \#N151s; \)

Permutation group \( N151s \) acting on a set of cardinality 10
(2, 3) (4, 6) (7, 9)
(2, 4) (3, 6) (8, 10)
(1, 2, 3) (4, 5, 6) (7, 9, 8)
(1, 2, 4) (3, 5, 6) (7, 8, 10)
(1, 2, 5, 6) (3, 4) (7, 9, 8, 10)
(1, 2) (5, 6) (7, 8)
(1, 4, 6) (2, 5, 3) (7, 9, 10)
(1, 4, 2) (3, 6, 5) (7, 10, 8)
(1, 4, 5, 3) (2, 6) (7, 9, 10, 8)
(1, 4) (3, 5) (7, 10)
(1, 3, 2) (4, 6, 5) (7, 8, 9)
(1, 3, 6) (2, 5, 4) (8, 10, 9)
(1, 3, 5, 4) (2, 6) (7, 8, 10, 9)
(1, 3) (4, 5) (8, 9)
(1, 5) (2, 6) (7, 8) (9, 10)
(1, 5) (3, 4) (7, 10) (8, 9)
(1, 5) (2, 4, 6, 3) (7, 8, 9, 10)
(1, 5) (2, 3, 6, 4) (7, 10, 9, 8)
(1, 6, 4) (2, 3, 5) (7, 10, 9)
(1, 6, 3) (2, 4, 5) (8, 9, 10)
(1, 6, 5, 2) (3, 4) (7, 10, 8, 9)
(1, 6) (2, 5) (9, 10)

> #N/#N151s;
5
> T151:=Transversal(N,N151s);
> for i in [1..#T151] do
> ss:=[1,5,1]ˆT151[i];
> for> cst[prodim(1, ts, ss)]:=ss;
> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
> then m:=m+1; end if; end for; m;
> #N151s;
24
> Set(N151s);
{}

(2, 6) (3, 4) (7, 9) (8, 10),
(1, 4) (3, 5) (7, 10),
(1, 6, 3) (2, 4, 5) (8, 9, 10),
(1, 2, 5, 6) (3, 4) (7, 9, 8, 10),
(1, 5) (2, 3, 6, 4) (7, 10, 9, 8),
(1, 3, 6) (2, 5, 4) (8, 10, 9),
(1, 6, 4) (2, 3, 5) (7, 10, 9),
(1, 5) (2, 4, 6, 3) (7, 8, 9, 10),
(1, 3, 5, 4) (2, 6) (7, 8, 10, 9),
\[(1, 6) (2, 5) (9, 10),
(1, 4, 5, 3) (2, 6) (7, 9, 10, 8),
(1, 4, 6) (2, 5, 3) (7, 9, 10),
Id(N151s),
(2, 4) (3, 6) (8, 10),
(1, 3) (4, 5) (8, 9),
(1, 5) (2, 6) (7, 8) (9, 10),
(1, 2) (5, 6) (7, 8),
(1, 2, 3) (4, 5, 6) (7, 9, 8),
(2, 3) (4, 6) (7, 9),
(1, 2, 4) (3, 5, 6) (7, 8, 10),
(1, 4, 2) (3, 6, 5) (7, 10, 8),
(1, 5) (3, 4) (7, 10) (8, 9),
(1, 3, 2) (4, 6, 5) (7, 8, 9),
(1, 6, 5, 2) (3, 4) (7, 10, 8, 9)\}

\(>\) for i in [1..#T151] do ([1,5,1]ˆN151s)ˆT151[i]; end for;
\(>\) Orbits(N151s);

\[
\text{GSet}\{\emptyset 7, 9, 8, 10 \},
\text{GSet}\{\emptyset 1, 2, 4, 3, 5, 6 \}
\]

> N157:=Stabiliser(N,[1,5,7]);
> SSS:={[1,5,7]};
> SSS:=SSSˆN;
> #SSS;
60
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
> for n in IN do
> for|for|if ts[1]*ts[5]*ts[7] eq
> for|for|if n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
> for|for|if then print Rep(Seqq[i]);
> for|for|if end if; end for; end for; [1, 5, 7]
> N157s:=N157;
> N157s; #N157s;
Permutation group N157 acting on a set of cardinality 10
Order = 2
\[(2, 4) (3, 6) (8, 10)\]
2
> #N/#N157s;
T157 := Transversal(N,N157s);
for i in [1..#T157] do
  ss := [1,5,7]^T157[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

m := 0; for i in [1..432] do if cst[i] ne [] then m := m + 1; end if; end for; m;

#N157s;
2
Set(N157s);
{
(2, 4)(3, 6)(8, 10),
Id(N157)
}

for i in [1..#T157] do ([1,5,7]^N157s)^T157[i]; end for;
Orbits(N157s);
[
GSet{@ 1 @},
GSet{@ 5 @},
GSet{@ 7 @},
GSet{@ 9 @},
GSet{@ 2, 4 @},
GSet{@ 3, 6 @},
GSet{@ 8, 10 @}
]

N158 := Stabiliser(N,[1,5,8]);
SSS := ([1,5,8]);
SSS := SSS^N;
#SSS;
60
Seqq := Setseq(SSS);
for i in [1..#SSS] do for n in IN do
for|for> if ts[1]*ts[5]*ts[8] eq
for|for>|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
for|for>|if> then print Rep(Seqq[i]);
for|for>|if> end if; end for; end for;
[ 1, 5, 8 ]
[ 1, 5, 10 ]
N158s := N158;
> for n in N do if 1^n eq 1 and 5^n eq 5 and 8^n eq 10 then
> end if; end for;
> N158s; #N158s;
Permutation group N158s acting on a set of cardinality 10
   (2, 3)(4, 6)(7, 9)
   (2, 6)(3, 4)(7, 9)(8, 10)
   (2, 4)(3, 6)(8, 10)
4
> #N/#N158s;
30
> T158:=Transversal(N,N158s);
> for i in [1..#T158] do
> ss:=[1,5,8]^T158[i];
> cst[prodim(1, ts, ss)]:=ss;
> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
> then m:=m+1; end if; end for; m;
165
> #N158s;
4
> Set(N158s);
{
   (2, 4)(3, 6)(8, 10),
   (2, 6)(3, 4)(7, 9)(8, 10),
   (2, 3)(4, 6)(7, 9),
   Id(N158s)
}
> [1,5,8]^N158s;
GSet{@
   [ 1, 5, 8 ],
   [ 1, 5, 10 ]
}@
> for i in [1..#T158] do ([1,5,8]^N158s)^T158[i]; end for;
> Orbits(N158s);
[ GSet{@ 1 @},
   GSet{@ 5 @},
   GSet{@ 7, 9 @},
   GSet{@ 8, 10 @},
   GSet{@ 2, 3, 6, 4 @}
]
> N121:=Stabiliser(N,[1,2,1]);
> SSS:=[(1,2,1)];
> SSS:=SSS\^N;
> #SSS;
60
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for> for n in IN do
for|for> if ts[1]*ts[2]*ts[1] eq
for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
for|for|if> ts[Rep(Seqq[i])[3]]
for|for|if> then print Rep(Seqq[i]);
for|for|if> end if; end for; end for;
[ 1, 2, 1 ]
[ 3, 1, 3 ]
[ 2, 3, 2 ]
[ 2, 1, 2 ]
[ 3, 2, 3 ]
> N121s:=N121;
> for n in N do if 1^n eq 3 and 2^n eq 1 and 1^n eq 3 then
for|if> N121s:=sub<N|N121s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 2^n eq 3 and 1^n eq 2 then
for|if> N121s:=sub<N|N121s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 2^n eq 1 and 1^n eq 2 then
for|if> N121s:=sub<N|N121s,n>; end if; end for;
> for n in N do if 1^n eq 1 and 2^n eq 3 and 1^n eq 1 then
for|if> N121s:=sub<N|N121s,n>; end if; end for;
> for n in N do if 1^n eq 3 and 2^n eq 2 and 1^n eq 3 then
for|if> N121s:=sub<N|N121s,n>; end if; end for;
> N121s; #N121s;
Permutation group N121s acting on a set of cardinality 10
(4, 9)(5, 8)(6, 7)
(1, 3, 2)(4, 7, 5, 9, 6, 8)
(1, 3, 2)(4, 6, 5)(7, 8, 9)
(1, 2, 3)(4, 8, 6, 9, 5, 7)
(1, 2, 3)(4, 5, 6)(7, 9, 8)
(1, 2)(5, 6)(7, 8)
(1, 2)(4, 9)(5, 7)(6, 8)
(2, 3)(4, 7)(5, 8)(6, 9)
(2, 3)(4, 6)(7, 9)
(1, 3)(4, 8)(5, 9)(6, 7)
(1, 3)(4, 5)(8, 9)
12
> #N/#N121s;
T121:=Transversal(N,N121s);
for i in [1..#T121] do
  ss:=[1,2,1]^T121[i];
  cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..432] do if cst[i] ne [] then m:=m+1; end if; end for; m;
#N121s;
12
Set(N121s);
{
  (4,9)(5,8)(6,7),
  (1,2,3)(4,5,6)(7,9,8),
  (1,2,3)(4,8,6,9,5,7),
  (2,3)(4,6)(7,9),
  (1,3)(4,5)(8,9),
  (1,2)(5,6)(7,8),
  (1,3)(4,8)(5,9)(6,7),
  Id(N121s),
  (2,3)(4,7)(5,8)(6,9),
  (1,3,2)(4,6,5)(7,8,9),
  (1,3,2)(4,7,5,9,6,8),
  (1,2)(4,9)(5,7)(6,8)
}
for i in [1..#T121] do ([1,2,1]^N121s)^T121[i]; end for;
Orbits(N121s);
[
  GSet{10},
  GSet{1,3,2},
  GSet{4,9,7,6,8,5}
]
N1210:=Stabiliser(N,[1,2,10]);
SSS:=[[1,2,10]];
SSS:=SSS\N;
#SSS;
60
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
  for n in IN do
    if ts[1]*ts[2]*ts[10] eq
for if n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]* 
for if ts[Rep(Seqq[i])[3]] 
for if then print Rep(Seqq[i]); 
for if end if; end for; end for; 
[ 1, 2, 10 ] 
[ 1, 3, 10 ] 
N1210s:=N1210; 
for n in N do if 1^n eq 1 and 2^n eq 3 and 10^n eq 10 then 
N1210s:=sub<N|N1210s,n>; end if; end for; 
N1210s; #N1210s;
Permutation group N1210s acting on a set of cardinality 10 
(4, 9)(5, 8)(6, 7) 
(2, 3)(4, 7)(5, 8)(6, 9) 
(2, 3)(4, 6)(7, 9) 
4 
#N/#N1210s; 
30 
T1210:=Transversal(N,N1210s); 
for i in [1..#T1210] do 
ss:=[1,2,10]^T1210[i]; 
cst[prodim(1, ts, ss)]:=ss; 
end for; 
m:=0; for i in [1..432] do if cst[i] ne [] 
then m:=m+1; end if; end for; m; 
205 
#N1210s; 
4 
Set(N1210s); 
{ 
(2, 3)(4, 7)(5, 8)(6, 9), 
(4, 9)(5, 8)(6, 7), 
(2, 3)(4, 6)(7, 9), 
Id(N1210s) 
} 
for i in [1..#T1210] do ([1,2,10]^N1210s)^T1210[i]; end for; 
Orbits(N1210s); 
[ 
GSet{@ 1 @}, 
GSet{@ 10 @}, 
GSet{@ 2, 3 @}, 
GSet{@ 5, 8 @}, 
GSet{@ 4, 9, 7, 6 @} 
]
> N125:=Stabiliser(N,[1,2,5]);
> SSS:={[1,2,5]};
> SSS:=SSS^N;
> #SSS;
120
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for> for n in IN do
for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
for|for|if> ts[Rep(Seqq[i])[3]]
for|for|if> end if; end for; end for;
[ [ 1, 2, 5 ]
[ [ 2, 3, 6 ]
[ [ 3, 1, 4 ]
[ [ 1, 3, 5 ]
[ [ 3, 2, 4 ]
[ [ 2, 1, 6 ]
> N125s:=N125;
> for n in N do if 1^n eq 2 and 2^n eq 3 and 5^n eq 6 then
for|if> N125s:=sub<N|N125s,n>; end if; end for;
> for n in N do if 1^n eq 3 and 2^n eq 1 and 5^n eq 4 then
for|if> N125s:=sub<N|N125s,n>; end if; end for;
> for n in N do if 1^n eq 1 and 2^n eq 3 and 5^n eq 5 then
for|if> N125s:=sub<N|N125s,n>; end if; end for;
> for n in N do if 1^n eq 3 and 2^n eq 2 and 5^n eq 4 then
for|if> N125s:=sub<N|N125s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 2^n eq 1 and 5^n eq 6 then
for|if> N125s:=sub<N|N125s,n>; end if; end for;
> N125s; #N125s;
Permutation group N125s acting on a set of cardinality 10
  (1, 2, 3) (4, 5, 6) (7, 9, 8)
  (1, 3, 2) (4, 6, 5) (7, 8, 9)
  (2, 3) (4, 6) (7, 9)
  (1, 3) (4, 5) (8, 9)
  (1, 2) (5, 6) (7, 8)
6
> #N/#N125s;
20
> T125:=Transversal(N,N125s);
> for i in [1..#T125] do
for> ss:=[1,2,5]^T125[i];
for cst[prodim(1, ts, ss)]:=ss;
for end for;
> m:=0; for i in [1..432] do if cst[i] ne [] then m:=m+1; end if; end for; m;
225
> #N125s;
6
> Set(N125s);
{
    (1, 3) (4, 5) (8, 9),
    (1, 2) (5, 6) (7, 8),
    (2, 3) (4, 6) (7, 9),
    (1, 2, 3) (4, 5, 6) (7, 9, 8),
    Id(N125s),
    (1, 3, 2) (4, 6, 5) (7, 8, 9)
}
> for i in [1..#T125] do ([1,2,5]^N125s)^T125[i]; end for;
> Orbits(N125s);
[ GSet{@ 10 @},
  GSet{@ 1, 2, 3 @},
  GSet{@ 4, 5, 6 @},
  GSet{@ 7, 9, 8 @} ]
/* Checking Orbits */
> Orbits(N151s);
[ GSet{@ 7, 9, 8, 10 @},
  GSet{@ 1, 2, 4, 3, 5, 6 @} ]

> for m,n in IN do for i in [7,1] do if ts[1]*ts[5]*ts[1]*ts[i] eq m*(ts[1]*ts[1])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if ts[1]*ts[5]*ts[1]*ts[i] eq m*(ts[1])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if ts[1]*ts[5]*ts[1]*ts[i] eq m*(ts[1]*ts[5])^n then i;
for|for|if> break; end if; end for; end for;
1
> for m, n in IN do for i in [7, 1] do if
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1] do if
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1] do if
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1] do if
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1] do if
for|for|if> break; end if; end for; end for;
>
> Orbits(N157s);
[
  GSet(1, 0, 0),
  GSet(5, 0, 0),
  GSet(7, 0, 0),
  GSet(9, 0, 0),
  GSet(2, 4, 0),
  GSet(3, 6, 0),
  GSet(8, 10, 0)
]
> for m, n in IN do for i in [1, 5, 7, 9, 2, 3, 8] do if
for|for|if> m*(ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1] eq
for|for|if> m*(ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> m*(ts[1]+ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1] eq
for|for|if> m*(ts[1]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> m*(ts[1]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1] eq
for|for|if> m*(ts[1]+ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> m*(ts[1]*ts[5]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1] eq
for|for|if> m*(ts[1]+ts[5]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> m*(ts[1]*ts[2]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> m*(ts[1]*ts[2]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[i] eq
for|for|if> m*(ts[1]*ts[2]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[1]*ts[7])ˆn then i;
for|for|if> break; end if; end for; end for;
>
for m,n in IN do for i in [1,5,7,9,2,3,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[i] eq
for|for|if> m*(ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
for m,n in IN do for i in [1,5,8,10,2,4,7] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq
for|for|if> m*(ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
for m,n in IN do for i in [1,5,8,10,2,4,7] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq
for|for|if> m*(ts[1])*ts[5]ˆn then i;
for|for|if> break; end if; end for; end for;
>
for m,n in IN do for i in [1,5,8,10,2,4,7] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq
for|for|if> m*(ts[1]*ts[2])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[7])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[10])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
> > for m,n in IN do for i in [1,5,8,10,2,4,7] do if 
for|for|if> ts[1]*ts[5]*ts[8]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[5])ˆn then i;
for m,n in IN do for i in [1,5,8,10,2,4,7] do if
for if> ts[1]*ts[5]*ts[8]*ts[i] eq
for if> m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for if> break; end if; end for; end for;
7
> Orbits(N121s);
[ GSet(@ 10 @),
  GSet(@ 1, 3, 2 @),
  GSet(@ 4, 9, 7, 6, 8, 5 @)
]
> for m,n in IN do for i in [10,1,4] do if
for if> ts[1]*ts[2]*ts[1]*ts[i] eq
for if> m*(ts[1]*ts[1])ˆn then i;
for if> break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
for if> ts[1]*ts[2]*ts[1]*ts[i] eq
for if> m*(ts[1])ˆn then i;
for if> break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
for if> ts[1]*ts[2]*ts[1]*ts[i] eq
for if> m*(ts[1]*ts[5])ˆn then i;
for if> break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
for if> ts[1]*ts[2]*ts[1]*ts[i] eq
for if> m*(ts[1]*ts[5]*ts[1])ˆn then i;
for if> break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
for if> ts[1]*ts[2]*ts[1]*ts[i] eq
for if> m*(ts[1]*ts[5]*ts[7])ˆn then i;
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[8]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[2] \times ts[10]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[2] \times ts[5]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[7] \times ts[1]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[7] \times ts[5]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[3]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[7] \times ts[3]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for; 
for \( m, n \) in IN do for \( i \) in [10, 1, 4] do if 
for \( m \times (ts[1] \times ts[5] \times ts[7] \times ts[3]) \times n \) then \( i \); 
for \( break; \) end if; end for; end for;
for if> break; end if; end for; end for;
for if> m,n in IN do for i in [10, 1, 4] do if
for if> break; end if; end for; end for;
>
> N1517 := Stabiliser(N, [1, 5, 1, 7]);
> SSS := ([1, 5, 1, 7]);
> SSS := SSS^N;
> #SSS;
60
> Seqq := Setseq(SSS);
> for i in [1..#SSS] do
for if> for n in IN do
for if> n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]] *
for if> ts[Rep(Seqq[i])[3]] * ts[Rep(Seqq[i])[4]]
for if> then print Rep(Seqq[i]);
for if> end if; end for; end for;
[ 1, 5, 1, 7 ]
[ 3, 4, 3, 7 ]
[ 6, 2, 6, 7 ]
> N1517s := N1517;
> for n in N do if 1^n eq 3 and 5^n eq 4 and 1^n eq 3 and
for if> 7^n eq 7 then
for if> N1517s := sub<N|N1517s,n>; end if; end for;
> for n in N do if 1^n eq 6 and 5^n eq 2 and 1^n eq 6 and
for if> 7^n eq 7 then
for if> N1517s := sub<N|N1517s,n>; end if; end for;
> N1517s; #N1517s;
Permutation group N1517s acting on a set of cardinality 10
(2, 4)(3, 6)(8, 10)
(1, 3, 6)(2, 5, 4)(8, 10, 9)
(1, 3)(4, 5)(8, 9)
(1, 6, 3)(2, 4, 5)(8, 9, 10)
(1, 6)(2, 5)(9, 10)
6
> #N/#N1517s;
20
> T1517 := Transversal(N, N1517s);
> for i in [1..#T1517] do
for> ss:=[1,5,1,7]^T1517[i];
for> cst[prodim(1, ts, ss)]:=ss;
for> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
245
> #N1517s;
6
> Set(N1517s);
{
(2, 4)(3, 6)(8, 10),
(1, 3, 6)(2, 5, 4)(8, 10, 9),
(1, 3)(4, 5)(8, 9),
(1, 6, 3)(2, 4, 5)(8, 9, 10),
Id(N1517s),
(1, 6)(2, 5)(9, 10)
}
> for i in [1..#T1517] do ([1,5,1,7]^N1517s)^T1517[i]; end for;
> Orbits(N1517s);
[ GSet{@ 7 @},
GSet{@ 1, 3, 6 @},
GSet{@ 2, 4, 5 @},
GSet{@ 8, 10, 9 @} ]
> N1571:=Stabiliser(N,[1,5,7,1]);
> SSS:=[(1,5,7,1)];
> SSS:=SSS`N;
> #SSS;
60
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for|for> for n in IN do
for|for|for> if ts[1]*ts[5]*ts[7]*ts[1] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
for|for|for|if> ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;
[ 1, 5, 7, 1 ]
[ 9, 5, 7, 9 ]
> N1571s:=N1571;
> for n in N do if 1^n eq 9 and 5^n eq 5 and 7^n eq 7
and
for | if { 1 | n \ neq 9 then
  for | if { N1571s := \text{sub}_N|N1571s, n}; end if; end for;
> N1571s; \#N1571s;

Permutation group N1571s acting on a set of cardinality 10
(2, 4)(3, 6)(8, 10)
(1, 9)(2, 4)(3, 10)(6, 8)
(1, 9)(3, 8)(6, 10)

4
> \#N/\#N1571s;
30

> T1571:=\text{Transversal}(N,N1571s);
> for i in \{1..\#T1571\} do
  for ss:=[1,5,7,1]^{T1571[i]};
  for cst[\text{prodim}(1, ts, ss)]:=ss;
  end for;
  m:=0; for i in \{1..432\} do if cst[i] \neq \{} then m:=m+1; end if; end for; m;
275
>
> \#N1571s;
4

> Set(N1571s);
4

> (2, 4)(3, 6)(8, 10),
(1, 9)(2, 4)(3, 10)(6, 8),
(1, 9)(3, 8)(6, 10),
\text{Id}(N1571s)

> for i in \{1..\#T1571\} do \{1,5,7,1\}^{N1571s}^{T1571[i]}; end for;
> Orbits(N1571s);
[
  \text{GSet}\{5\},
  \text{GSet}\{7\},
  \text{GSet}\{1, 9\},
  \text{GSet}\{2, 4\},
  \text{GSet}\{3, 6, 10, 8\}
]

> N1575:=\text{Stabiliser}(N,\{1,5,7,5\});
> SSS:=(\{1,5,7,5\});
> SSS:=SSS^{N};
> \#SSS;
60
> Seqq:=\text{Setseq}(SSS);
for i in [1..#SSS] do
for n in IN do
  if ts[1]*ts[5]*ts[7]*ts[5] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
    ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
  then print Rep(Seqq[i]);
end if; end for; end for;
[ 1, 5, 7, 5 ]
[ 1, 8, 6, 8 ]
[ 1, 10, 3, 10 ]
N1575s:=N1575;
for n in N do if 1^n eq 1 and 5^n eq 8 and 7^n eq 6 and
  5^n eq 8 then
  N1575s:=sub<N|N1575s,n>; end if; end for;
for n in N do if 1^n eq 1 and 5^n eq 10 and 7^n eq 3 and
  5^n eq 10 then
  N1575s:=sub<N|N1575s,n>; end if; end for;
N1575s; #N1575s;
Permutation group N1575s acting on a set of cardinality 10
  (2, 4)(3, 6)(8, 10)
  (4, 9)(5, 8)(6, 7)
  (2, 9, 4)(3, 7, 6)(5, 8, 10)
  (2, 4, 9)(3, 6, 7)(5, 10, 8)
  (2, 9)(3, 7)(5, 10)
6
#N/#N1575s;
20
T1575:=Transversal(N,N1575s);
for i in [1..#T1575] do
  ss:=[1,5,7,5]^T1575[i];
  cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..432] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
295
#N1575s;
6
Set(N1575s);
{ (2, 4)(3, 6)(8, 10),
  (2, 9, 4)(3, 7, 6)(5, 8, 10),
  (4, 9)(5, 8)(6, 7),
  (2, 9)(3, 7)(5, 10),
}
(2, 4, 9)(3, 6, 7)(5, 10, 8),
Id(N1575s)
}

> for i in [1..#T1575] do ([1,5,7,5]^N1575s)^T1575[i]; end for;
> Orbits(N1575s);
[
  GSet{@ 1 @},
  GSet{@ 2, 4, 9 @},
  GSet{@ 3, 6, 7 @},
  GSet{@ 5, 8, 10 @}
]

> N1573:=Stabiliser(N,[1,5,7,3]);
> SSS:={[1,5,7,3]};
> SSS:=SSS^N;
> #SSS;
120
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
  for n in IN do
    for if ts[1]*ts[5]*ts[7]*ts[3] eq n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*
         ts[Rep(Seqq[i])][3]*ts[Rep(Seqq[i])][4] then print Rep(Seqq[i]);
    for end if; end for; end for;
[ 1, 5, 7, 3 ]
[ 3, 4, 7, 1 ]
> N1573s:=N1573;
> for n in N do if 1^n eq 3 and 5^n eq 4 and 7^n eq 7 and
         3^n eq 1 then
    for if N1573s:=sub<N|N1573s,n>; end if; end for;
> N1573s;  #N1573s;
Permutation group N1573s acting on a set of cardinality 10
Order = 2
   (1, 3)(4, 5)(8, 9)
2
> #N/#N1573s;
60
>
> T1573:=Transversal(N,N1573s);
> for i in [1..#T1573] do
  ss:=[1,5,7,3]^T1573[i];
  cst[prodim(1, ts, ss)]:=ss;
  end for;
m:=0; for i in [1..432] do if cst[i] ne [] then m:=m+1; end if; end for; m;

#N1573s;
2
Set(N1573s);
{(1, 3)(4, 5)(8, 9), Id(N1573s)}

for i in [1..#T1573] do ([1,5,7,3]^N1573s)^T1573[i]; end for;

Orbits(N1573s);
[ GSet{@ 2 @},
  GSet{@ 6 @},
  GSet{@ 7 @},
  GSet{@ 10 @},
  GSet{@ 1, 3 @},
  GSet{@ 4, 5 @},
  GSet{@ 8, 9 @}]

N1581:=Stabiliser(N,[1,5,8,1]);
SSS:={[1,5,8,1]};
SSS:=SSS^N;
#SSS;
60
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
  for n in IN do
    if ts[1]*ts[5]*ts[8]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]] then print Rep(Seqq[i]);
  end if;
  end for;
end for;
[ 1, 5, 8, 1 ]
[ 2, 6, 10, 2 ]
[ 4, 3, 7, 4 ]
[ 2, 6, 7, 2 ]
[ 3, 4, 10, 3 ]
[ 4, 3, 8, 4 ]
[ 5, 1, 9, 5 ]
[ 5, 1, 7, 5 ]
[ 1, 5, 10, 1 ]
\[
\begin{align*}
[3, 4, 9, 3] \\
[6, 2, 9, 6] \\
[6, 2, 8, 6] \\
> N1581s := N1581; \\
> \text{for } n \text{ in } N \text{ do if } 1^n \text{ eq } 2 \text{ and } 5^n \text{ eq } 6 \text{ and } 8^n \text{ eq } 10 \\
> \text{for|if> and } 1^n \text{ eq } 2 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 4 \text{ and } 5^n \text{ eq } 3 \text{ and } 8^n \text{ eq } 7 \\
> \text{for|if> and } 1^n \text{ eq } 4 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 2 \text{ and } 5^n \text{ eq } 6 \text{ and } 8^n \text{ eq } 7 \\
> \text{for|if> and } 1^n \text{ eq } 2 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 4 \text{ and } 5^n \text{ eq } 3 \text{ and } 8^n \text{ eq } 10 \\
> \text{for|if> and } 1^n \text{ eq } 3 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 4 \text{ and } 5^n \text{ eq } 4 \text{ and } 8^n \text{ eq } 8 \\
> \text{for|if> and } 1^n \text{ eq } 4 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 5 \text{ and } 5^n \text{ eq } 1 \text{ and } 8^n \text{ eq } 9 \\
> \text{for|if> and } 1^n \text{ eq } 5 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 5 \text{ and } 5^n \text{ eq } 1 \text{ and } 8^n \text{ eq } 7 \\
> \text{for|if> and } 1^n \text{ eq } 5 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 1 \text{ and } 5^n \text{ eq } 5 \text{ and } 8^n \text{ eq } 10 \\
> \text{for|if> and } 1^n \text{ eq } 1 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 3 \text{ and } 5^n \text{ eq } 4 \text{ and } 8^n \text{ eq } 9 \\
> \text{for|if> and } 1^n \text{ eq } 3 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 6 \text{ and } 5^n \text{ eq } 2 \text{ and } 8^n \text{ eq } 9 \\
> \text{for|if> and } 1^n \text{ eq } 6 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> \text{for n in } N \text{ do if } 1^n \text{ eq } 6 \text{ and } 5^n \text{ eq } 2 \text{ and } 8^n \text{ eq } 8 \\
> \text{for|if> and } 1^n \text{ eq } 6 \text{ then} \\
> \text{for|if> } N1581s := \text{sub}<N|N1581s,n>; \text{ end if}; \text{ end for}; \\
> N1581s; #N1581s; \\
\text{Permutation group N1581s acting on a set of cardinality 10} \\
\quad (2, 3)(4, 6)(7, 9) \\
\quad (1, 2, 4)(3, 5, 6)(7, 8, 10) \\
\quad (1, 2, 5, 6)(3, 4)(7, 9, 8, 10) \\
\quad (1, 4, 2)(3, 6, 5)(7, 10, 8)
\[(1, 4, 5, 3)(2, 6)(7, 9, 10, 8)\]
\[(1, 2, 3)(4, 5, 6)(7, 9, 8)\]
\[(1, 2)(5, 6)(7, 8)\]
\[(1, 3, 6)(2, 5, 4)(8, 10, 9)\]
\[(1, 3, 5, 4)(2, 6)(7, 8, 10, 9)\]
\[(1, 4, 6)(2, 5, 3)(7, 9, 10)\]
\[(1, 4)(3, 5)(7, 10)\]
\[(1, 5)(3, 4)(7, 10)(8, 9)\]
\[(1, 5)(2, 4, 6, 3)(7, 8, 9, 10)\]
\[(1, 5)(2, 6)(7, 8)(9, 10)\]
\[(1, 5)(2, 3, 6, 4)(7, 10, 9, 8)\]
\[(2, 6)(3, 4)(7, 9)(8, 10)\]
\[(2, 4)(3, 6)(8, 10)\]
\[(1, 3, 2)(4, 6, 5)(7, 8, 9)\]
\[(1, 3)(4, 5)(7, 8, 9)\]
\[(1, 6, 3)(2, 4, 5)(8, 9, 10)\]
\[(1, 6, 5, 2)(3, 4)(7, 10, 9, 8)\]
\[(1, 6, 4)(2, 3, 5)(7, 10, 9)\]
\[(1, 6)(2, 5)(9, 10)\]

\begin{Verbatim}
(2, 6)(3, 4)(7, 9)(8, 10),
(1, 4)(3, 5)(7, 10),
(1, 6, 3)(2, 4, 5)(8, 9, 10),
(1, 2, 5, 6)(3, 4)(7, 9, 8, 10),
(1, 5)(2, 3, 6, 4)(7, 10, 9, 8),
(1, 6, 4)(2, 3, 5)(7, 10, 9),
(1, 3, 6)(2, 5, 4)(8, 10, 9),
(1, 5)(2, 4, 6, 3)(7, 8, 9, 10),
\end{Verbatim}
\begin{verbatim}
> for i in [1..#T1581] do ([1,5,8,1]^N1581s)^T1581[i]; end for;
> Orbits(N1581s);
> GSet[@ 7, 9, 8, 10 @],
   GSet[@ 1, 2, 4, 3, 5, 6 @]
> N1585:=Stabiliser(N,[1,5,8,5]);
> SSS:={[1,5,8,5]};
> SSS:=SSS^N;
> #SSS;
60
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for|for> for n in IN do
for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
for|for|if> ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
for|for|if> then print Rep(Seqq[i]);
for|for|if> end if; end for; end for;
[ 1, 5, 8, 5 ]
[ 1, 8, 10, 8 ]
[ 1, 5, 8, 5 ]
[ 1, 10, 5, 10 ]
[ 1, 5, 10, 5 ]
[ 1, 10, 8, 10 ]
> N1585s:=N1585;
\end{verbatim}
for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

for n in N do if \(1^n \equiv 1\) and \(5^n \equiv 8\) then
for N \(\subseteq\) \(\{n\in N\}\); end if; end for;

N \(\subseteq\) \(\{n\in N\}\);
Permutation group N1585s acting on a set of cardinality 10

\[
\begin{align*}
(2, 3)(4, 6)(7, 9) \\
(2, 7, 4, 3, 9, 6)(5, 8, 10) \\
(2, 9, 4)(3, 7, 6)(5, 8, 10) \\
(4, 9)(5, 8)(6, 7) \\
(2, 3)(4, 7)(5, 8)(6, 9) \\
(2, 4, 9)(3, 6, 7)(5, 10, 8) \\
(2, 6, 9, 3, 4, 7)(5, 10, 8) \\
(2, 6)(3, 4)(7, 9)(8, 10) \\
(2, 4)(3, 6)(8, 10) \\
(2, 9)(3, 7)(5, 10) \\
(2, 7)(3, 9)(4, 6)(5, 10)
\end{align*}
\]

12

> \#N/\#N1585s;
10

> T1585:=Transversal(N,N1585s);

> for i in [1..#T1585] do
for> ss:=[1,5,8,5]\^T1585[i];
for> cst[prodim(1, ts, ss)]:=ss;
for> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
370
> \#N1585s;
12

> Set(N1585s);
\textbf{Id(N1585s),}
\begin{itemize}
\item (2, 6, 9, 3, 4, 7) (5, 10, 8),
\item (2, 4) (3, 6) (8, 10),
\item (2, 9) (3, 7) (5, 10),
\item (2, 7, 4, 3, 9, 6) (5, 8, 10),
\item (2, 9, 4) (3, 7, 6) (5, 8, 10),
\item (2, 6) (3, 4) (7, 9) (8, 10),
\item (4, 9) (5, 8) (6, 7),
\item (2, 7) (3, 9) (4, 6) (5, 10),
\item (2, 4, 9) (3, 6, 7) (5, 10, 8),
\item (2, 3) (4, 6) (7, 9),
\item (2, 3) (4, 7) (5, 8) (6, 9)
\end{itemize}

\textbf{for i in [1..#T1585] do ([1,5,8,5]^N1585s)^T1585[i]; end for;}
\textbf{Orbits(N1585s);}
\begin{itemize}
\item GSet[@ 1 @],
\item GSet[@ 5, 8, 10 @],
\item GSet[@ 2, 3, 7, 9, 4, 6 @]
\end{itemize}
\textbf{N12110:=Stabiliser(N,[1,2,1,10]);}
\textbf{SSS:={[1,2,1,10]};}
\textbf{SSS:=SSS^N;}
\textbf{#SSS; 60}
\textbf{Seqq:=Setseq(SSS);}
\textbf{for i in [1..#SSS] do for n in IN do for|for if ts[1]*ts[2]*ts[1]*ts[10] eq for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]* for|for|if> ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]] for|for|if> then print Rep(Seqq[i]);
\textbf{[ 1, 2, 1, 10 ] \[ 3, 1, 3, 10 \] \[ 2, 3, 2, 10 \] \[ 2, 1, 2, 10 \] \[ 1, 3, 1, 10 \] \[ 3, 2, 3, 10 \]}
\textbf{N12110s:=N12110;}
\textbf{for n in N do if 1\^{}n eq 3 and 2\^{}n eq 1 and 1\^{}n eq 3 for|if> and 10\^{}n eq 10 then for|if> N12110s:=sub<N|N12110s,n>; end if; end for;
for n in N do if 1^n eq 2 and 2^n eq 3 and 1^n eq 2
for|if> and 10^n eq 10 then
for|if> N12110s:=sub<N|N12110s,n>; end if; end for;
for n in N do if 1^n eq 2 and 2^n eq 1 and 1^n eq 2
for|if> and 10^n eq 10 then
for|if> N12110s:=sub<N|N12110s,n>; end if; end for;
for n in N do if 1^n eq 1 and 2^n eq 3 and 1^n eq 1
for|if> and 10^n eq 10 then
for|if> N12110s:=sub<N|N12110s,n>; end if; end for;
for n in N do if 1^n eq 3 and 2^n eq 2 and 1^n eq 3
for|if> and 10^n eq 10 then
for|if> N12110s:=sub<N|N12110s,n>; end if; end for;
N12110s; #N12110s;
Permutation group N12110s acting on a set of cardinality 10
(4, 9)(5, 8)(6, 7)
(1, 3, 2)(4, 7, 5, 9, 6, 8)
(1, 3, 2)(4, 6, 5)(7, 8, 9)
(1, 2, 3)(4, 8, 6, 9, 5, 7)
(1, 2, 3)(4, 5, 6)(7, 9, 8)
(1, 2)(5, 6)(7, 8)
(1, 2)(4, 9)(5, 7)(6, 8)
(2, 3)(4, 7)(5, 8)(6, 9)
(2, 3)(4, 6)(7, 9)
(1, 3)(4, 8)(5, 9)(6, 7)
(1, 3)(4, 5)(8, 9)
12
> #N/#N12110s;
10
> T12110:=Transversal(N,N12110s);
> for i in [1..#T12110] do
for> ss:=[1,2,1,10]^T12110[i];
for> cst[prodim(1, ts, ss)]:=ss;
for> end for;
> m:=0; for i in [1..432] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
380
> #N12110s;
12
> Set(N12110s);
{
(4, 9)(5, 8)(6, 7),
(1, 2, 3)(4, 5, 6)(7, 9, 8),
(1, 2, 3)(4, 8, 6, 9, 5, 7),
(2, 3)(4, 6)(7, 9),
(1, 3)(4, 5)(8, 9),
(1, 2)(5, 6)(7, 8),
(1, 3)(4, 8)(5, 9)(6, 7),
Id(N12110s),
(2, 3)(4, 7)(5, 8)(6, 9),
(1, 3, 2)(4, 6, 5)(7, 8, 9),
(1, 3, 2)(4, 7, 5, 9, 6, 8),
(1, 2)(4, 9)(5, 7)(6, 8)
}
> for i in [1..#T12110] do ([1,2,1,10]^N12110s)^T12110[i]; end for;
> Orbits(N12110s);
[ GSet{@ 10 @},
  GSet{@ 1, 3, 2 @},
  GSet{@ 4, 9, 7, 6, 8, 5 @}
]
/*Checking Orbits*/
> Orbits(N1517s);
[ GSet{@ 7 @},
  GSet{@ 1, 3, 6 @},
  GSet{@ 2, 4, 5 @},
  GSet{@ 8, 10, 9 @}
]
> for m,n in IN do for i in [7,1,2,8] do if
  for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq
  for|for|if> m*(ts[1]*ts[1])^n then i;
  for|for|if> break; end if; end for; end for;
> for m,n in IN do for i in [7,1,2,8] do if
  for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq
  for|for|if> m*(ts[1])^n then i;
  for|for|if> break; end if; end for; end for;
> for m,n in IN do for i in [7,1,2,8] do if
  for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq
  for|for|if> m*(ts[1]*ts[2])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[2]*ts[10])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[7])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if 
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[i] eq 
for|for|if> m*(ts[1]*ts[5]*ts[1]*ts[7])ˆn then i;
for|for|if> break; end if; end for; end for;
> for m, n in IN do for i in [7, 1, 2, 8] do if
    break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if
    m * (ts[1] * ts[5] * ts[7])^n then i;
    break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if
    break; end if; end for; end for;
>
> for m, n in IN do for i in [7, 1, 2, 8] do if
    m * (ts[1] * ts[2] * ts[10])^n then i;
    break; end if; end for; end for;
>
> Orbits(N1571s);
[
    GSet(@ 5 @),
    GSet(@ 7 @),
    GSet(@ 1, 9 @),
    GSet(@ 2, 4 @),
    GSet(@ 3, 6, 10, 8 @)
]
> for m, n in IN do for i in [5, 7, 1, 2, 3] do if
    m * (ts[1] * ts[1])^n then i;
    break; end if; end for; end for;
>
> for m, n in IN do for i in [5, 7, 1, 2, 3] do if
    m * (ts[1])^n then i;
    break; end if; end for; end for;
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[5])^n then i; for|for|if> break; end if; end for; end for; >
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[2])^n then i; for|for|if> break; end if; end for; end for; >
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[5]*ts[1])^n then i; for|for|if> break; end if; end for; end for; 1
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[5]*ts[7])^n then i; for|for|if> break; end if; end for; end for; >
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[2]*ts[1])^n for|for|if> then i;break; end if; end for; end for; >
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[2]*ts[10])^n for|for|if> then i;break; end if; end for; end for; 3
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq for|for|if> m*(ts[1]*ts[2]*ts[5])^n for|for|if> then i;break; end if; end for; end for; >
> for m,n in IN do for i in [5,7,1,2,3] do if for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq
for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[1])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[5])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[1]*ts[7])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[8]*ts[5])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[1]*ts[8])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[2]*ts[1]*ts[10])ˆn then i;break; end if; end for; end for;

> for m,n in IN do for i in [5,7,1,2,3] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[i] eq m*(ts[1]*ts[5]*ts[1]*ts[7]*ts[1])ˆn then i;break; end if; end for; end for;

> Orbits(N1575s);

[ GSet{@ 1 @},
  GSet{@ 2, 4, 9 @},
  GSet{@ 3, 6, 7 @},
  GSet{@ 10 @} ]
GSet(@ 5, 8, 10 @)

> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[5])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[2])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[8])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq
for|for|if> m*(ts[1]*ts[2]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,2,3,5] do if
for\(i\in [1,2,3,5]\) do if \\
\(m*(ts[1]*ts[5]*ts[7]*ts[5])^n\) then i; \\
end if; end for; end for;

> for \(m,n\) in \(IN\) do for \(i\) in \([1,2,3,5]\) do if \\
\(m*(ts[1]*ts[2]*ts[5])^n\) then i; \\
end if; end for; end for;

> for \(m,n\) in \(IN\) do for \(i\) in \([1,2,3,5]\) do if \\
\(m*(ts[1]*ts[5]*ts[1]*ts[7])^n\) then i; \\
end if; end for; end for;

> for \(m,n\) in \(IN\) do for \(i\) in \([1,2,3,5]\) do if \\
\(m*(ts[1]*ts[8]*ts[5])^n\) then i; \\
end if; end for; end for;

> for \(m,n\) in \(IN\) do for \(i\) in \([1,2,3,5]\) do if \\
\(m*(ts[1]*ts[8]*ts[1])^n\) then i; \\
end if; end for; end for;

> for \(m,n\) in \(IN\) do for \(i\) in \([1,2,3,5]\) do if \\
\(m*(ts[1]*ts[10]*ts[5])^n\) then i; \\
end if; end for; end for;
for m,n in IN do for i in [1,2,3,5] do if ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[5])^n then i; end if; end for; end for;

for m,n in IN do for i in [1,2,3,5] do if ts[1]*ts[5]*ts[7]*ts[5]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[5])^n then i; end if; end for; end for;

for m,n in IN do for i in [1,2,3,5] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[3])^n then i; end if; end for; end for;

for m,n in IN do for i in [1,2,3,5] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[5])^n then i; end if; end for; end for;

for m,n in IN do for i in [1,2,3,5] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[2])^n then i; end if; end for; end for;

Orbits(N1573s);
[
  GSet(@ 2 @),
  GSet(@ 6 @),
  GSet(@ 7 @),
  GSet(@ 10 @),
  GSet(@ 1, 3 @),
  GSet(@ 4, 5 @),
  GSet(@ 8, 9 @)
]

for m,n in IN do for i in [2,6,7,10,1,4,8] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[1])^n then i; end if; end for; end for;

for m,n in IN do for i in [2,6,7,10,1,4,8] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1])^n then i; end if; end for; end for;

for m,n in IN do for i in [2,6,7,10,1,4,8] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[5])^n then i; end if; end for; end for;

for m,n in IN do for i in [2,6,7,10,1,4,8] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[2])^n then i; end if; end for; end for;

for m,n in IN do for i in [2,6,7,10,1,4,8] do if ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq m*(ts[1]*ts[2])^n then i; end if; end for; end for;
for m,n in IN do for i in [2, 6, 7, 10, 1, 4, 8] do if
for if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for if> m*(ts[1]*ts[5]*ts[7])^n then i;
for if> break; end if; end for; end for;

> for m,n in IN do for i in [2, 6, 7, 10, 1, 4, 8] do if
for if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for if> m*(ts[1]*ts[5]*ts[8])^n then i;
for if> break; end if; end for; end for;

> for m,n in IN do for i in [2, 6, 7, 10, 1, 4, 8] do if
for if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for if> m*(ts[1]*ts[2]*ts[1])^n then i;
for if> break; end if; end for; end for;

> for m,n in IN do for i in [2, 6, 7, 10, 1, 4, 8] do if
for if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for if> m*(ts[1]*ts[2]*ts[5])^n then i;
for if> break; end if; end for; end for;
for m,n in IN do for i in [2,6,7,10,1,4,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [2,6,7,10,1,4,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [2,6,7,10,1,4,8] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[3]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> Orbits(N1581s);
[  GSet{@ 7, 9, 8, 10 @},
  GSet{@ 1, 2, 4, 5, 3, 6 @}
]
> for m,n in IN do for i in [7,1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[1])ˆn then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
for if m*(ts[1])ˆn then i;
for if break; end if; end for; end for;
>
for m,n in IN do for i in [7,1] do
for if ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
for if m*(ts[1]*ts[5])ˆn then i;
for if break; end if; end for; end for;
>
for m,n in IN do for i in [7,1] do
for if ts[1]*ts[5]*ts[8]*ts[1]*ts[2]ˆn then i;
for if break; end if; end for; end for;
>
for m,n in IN do for i in [7,1] do
for if ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
for if m*(ts[1]*ts[5]*ts[1])ˆn then i;
for if break; end if; end for; end for;
>
for m,n in IN do for i in [7,1] do
for if break; end if; end for; end for;
>
for m,n in IN do for i in [7,1] do
for if break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[5]*ts[1]*ts[7])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[5]*ts[7]*ts[1])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[5]*ts[7]*ts[5])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[5]*ts[8]*ts[5])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[2]*ts[10])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[7]*ts[1]*ts[1])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [7,1] do if
> ts[1]*ts[5]*ts[8]*ts[1]*ts[i] eq
> m*(ts[1]*ts[5]*ts[7]*ts[2])^n then i;
for | break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,2] do if
> ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq m*(ts[1]*ts[1])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,2] do if
> ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq m*(ts[1]*ts[1])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,2] do if
> ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq m*(ts[1]*ts[1])^n then i;
> break; end if; end for; end for;
>
> for m,n in IN do for i in [1,5,2] do if
> ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq m*(ts[1]*ts[1])^n then i;
> break; end if; end for; end for;

5
for m, n in IN do for i in [1, 5, 2] do if
  m * (ts[1] * ts[2] * ts[1]) ^ n then i;
  break; end if; end for; end for;
end for;
for m,n in IN do for i in [1,5,2] do if
ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq
m*(ts[1]*ts[2]*ts[1]*ts[10])^n then i;
break; end if; end for; end for;
> for m,n in IN do for i in [1,5,2] do if
ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq
m*(ts[1]*ts[5]*ts[1]*ts[7]*ts[1])^n then i;
break; end if; end for; end for;
> for m,n in IN do for i in [1,5,2] do if
ts[1]*ts[5]*ts[8]*ts[5]*ts[i] eq
m*(ts[1]*ts[5]*ts[1]*ts[10])^n then i;
break; end if; end for; end for;
> Orbits(N12110s);
[ GSet{@ 10 @},
  GSet{@ 1, 3, 2 @},
  GSet{@ 4, 9, 7, 6, 8, 5 @} ]
> for m,n in IN do for i in [10,1,4] do if
ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
m*(ts[1]*ts[1])^n then i;
break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
m*(ts[1])^n then i;
break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
m*(ts[5])^n then i;
break; end if; end for; end for;
> for m,n in IN do for i in [10,1,4] do if
ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
m*(ts[2])^n then i;
break; end if; end for; end for;
for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[1])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[7])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[2])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[8])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[10])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[7])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[1])^n then i; for if break; end if; end for; end for;
>
> for m, n in IN do for i in [10, 1, 4] do if ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq m*(ts[1]*ts[5]*ts[7]*ts[5])^n then i;
for | for | if > break; end if; end for; end for;
>
> for m,n in IN do for i in [10,1,4] do if
for | for | if > ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
for | for | if > m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for | for | if > break; end if;
for | for > end for; end for;
>
> for m,n in IN do for i in [10,1,4] do if
for | for | if > ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
for | for | if > m*(ts[1]*ts[5]*ts[8]*ts[5])ˆn then i;
for | for | if > break; end if; end for; end for;
>
> for m,n in IN do for i in [10,1,4] do if
for | for | if > ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
for | for | if > m*(ts[1]*ts[5]*ts[8]*ts[5])ˆn then i;
for | for | if > break; end if; end for; end for;
>
> for m,n in IN do for i in [10,1,4] do if
for | for | if > ts[1]*ts[2]*ts[1]*ts[10]*ts[i] eq
for | for | if > m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for | for | if > break; end if; end for; end for;
1

> N15171:=Stabiliser(N,[1,5,1,7,1]);
> SSS:={[1,5,1,7,1]};
> SSS:=SSSˆN;
> #SSS;
> #SSS;
60
> Seqq:=Setseq(SSS);
> for i in [1..#SSS] do
for n in IN do
for|if> if ts[1]*ts[5]*ts[1]*ts[7]*ts[1] eq
for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
for|if> ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*
for|if> ts[Rep(Seqq[i])[5]]
for|if> then print Rep(Seqq[i]);
for|i> end if; end for; end for;
[ 1, 5, 1, 7, 1 ]
[ 9, 5, 9, 7, 9 ]
> N15171s:=N15171;
> for n in N do if 1^n eq 9 and 5^n eq 5 and 1^n eq
for|if> 9 and 7^n eq 7 and 1^n eq 9 then
for|i> N15171s:=sub<N|N15171s,n>; end if; end for;
> N15171s; #N15171s;
Permutation group N15171s acting on a set of cardinality 10
  (2, 4) (3, 6) (8, 10)
  (1, 9) (2, 4) (3, 10) (6, 8)
  (1, 9) (3, 8) (6, 10)
4
> #N/#N15171s;
30
> T15171:=Transversal(N,N15171s);
> for i in [1..#T15171] do
for ss:=[1,5,1,7,1]^T15171[i];
for cst[prodim(1, ts, ss)]:=ss;
for end for;
> m:=0; for i in [1..432] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
400
>
> [1,5,1,7,1]^N15171s;
GSet{@
  [ 1, 5, 1, 7, 1 ],
  [ 9, 5, 9, 7, 9 ]
}@}
> for i in [1..#T15171] do ([1,5,1,7,1]^N15171s)^T15171[i]; end for;
> Orbits(N15171s);
[
  GSet(@ 5 @),
  GSet(@ 7 @),
  GSet(@ 1, 9 @),
GSet(\{2, 4\}),
GSet(\{3, 6, 10, 8\})

> #N15171s;
4
>
> N15712 := Stabiliser(N, [1, 5, 7, 1, 2]);
> SSS := ([1, 5, 7, 1, 2]);
> SSS := SSS \setminus N;
> #SSS;
120
> Seqq := Setseq(SSS);
> for i in [1..#SSS] do
for
for
for
for
for
for
for
if n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
for
for
if then print Rep(Seqq[i]);
for
for
if end if; end for; end for;
[ 1, 5, 7, 1, 2 ]
[ 9, 6, 8, 9, 1 ]
[ 9, 5, 7, 9, 2 ]
[ 1, 10, 3, 1, 9 ]
[ 2, 10, 3, 2, 9 ]
[ 2, 6, 8, 2, 1 ]
> N15712s := N15712;
> for n in N do if 1^n eq 9 and 5^n eq 6 and 7^n eq 8
for
for
if and 1^n eq 9 and 2^n eq 1 then
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
> for n in N do if 1^n eq 1 and 5^n eq 10 and 7^n eq 7
for
for
if and 1^n eq 1 and 2^n eq 9 then
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 10 and 7^n eq 3
for
for
if and 1^n eq 2 and 2^n eq 9 then
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 6 and 7^n eq 8
for
for
if and 1^n eq 2 and 2^n eq 1 then
for
for
if N15712s := sub<N|N15712s, n>; end if; end for;
> N15712s; #N15712s;
Permutation group $N_{15712s}$ acting on a set of cardinality 10

$$(1, 9, 2)(3, 7, 8)(5, 6, 10)$$

$$(1, 9)(3, 8)(6, 10)$$

$$(2, 9)(3, 7)(5, 10)$$

$$(1, 2, 9)(3, 8, 7)(5, 10, 6)$$

$$(1, 2)(5, 6)(7, 8)$$

6

> $\#N/\#N_{15712s};$

20

> $T_{15712} := \text{Transversal}(N, N_{15712s});$

> for $i$ in [1..#$T_{15712}$] do

> for $ss := [1, 5, 7, 1, 2]^T_{15712}[i]$;

> for $cst[\text{prodim}(1, ts, ss)] := ss;$

> end for;

> end for;

> m := 0; for $i$ in [1..432] do if $cst[i]$ ne []

> if then $m := m+1;$ end if; end for; m;

420

> [$1, 5, 7, 1, 2]^N_{15712s};$

GSet{@

[ 1, 5, 7, 1, 2 ],
[ 9, 6, 8, 9, 1 ],
[ 9, 5, 7, 9, 2 ],
[ 1, 10, 3, 1, 9 ],
[ 2, 10, 3, 2, 9 ],
[ 2, 6, 8, 2, 1 ]
}@}

> for $i$ in [1..#$T_{15712}$] do ($[1, 5, 7, 1, 2]^N_{15712s}$)$^T_{15712}[i] do; end for;

> Orbits($N_{15712s}$);

[ 

GSet{@ 4 @},
GSet{@ 1, 9, 2 @},
GSet{@ 3, 7, 8 @},
GSet{@ 5, 6, 10 @}
]

> $\#N_{15712s};$

6

> $N_{15851} := \text{Stabiliser}(N, [1, 5, 8, 5, 1]);$

> SSS := ([1, 5, 8, 5, 1]);

> SSS := SSS$^N;$

> $\#SSS;$

60
Seqq := Setseq(SSS);
for i in [1..#SSS] do
  for n in IN do
    ts[Rep(Seqq[i])[3]] * ts[Rep(Seqq[i])[4]] *
    ts[Rep(Seqq[i])[5]]
    then print Rep(Seqq[i]);
  end if;
end for;
end for;

[ 1, 5, 8, 5, 1 ]
[ 2, 6, 10, 6, 2 ]
[ 3, 9, 4, 9, 3 ]
[ 4, 3, 7, 3, 4 ]
[ 1, 8, 10, 8, 1 ]
[ 5, 9, 1, 9, 5 ]
[ 2, 6, 7, 6, 2 ]
[ 7, 2, 5, 2, 7 ]
[ 2, 10, 7, 10, 2 ]
[ 3, 4, 10, 4, 3 ]
[ 6, 9, 2, 9, 6 ]
[ 9, 6, 3, 6, 9 ]
[ 4, 3, 8, 3, 4 ]
[ 1, 8, 5, 8, 1 ]
[ 8, 4, 6, 4, 8 ]
[ 5, 1, 9, 1, 5 ]
[ 4, 7, 8, 7, 4 ]
[ 1, 10, 5, 10, 1 ]
[ 5, 1, 7, 1, 5 ]
[ 2, 7, 10, 7, 2 ]
[ 8, 6, 1, 6, 8 ]
[ 9, 3, 5, 3, 9 ]
[ 6, 8, 2, 8, 6 ]
[ 1, 5, 10, 5, 1 ]
[ 7, 2, 4, 2, 7 ]
[ 2, 10, 6, 10, 2 ]
[ 3, 4, 9, 4, 3 ]
[ 10, 1, 3, 1, 10 ]
[ 6, 2, 9, 2, 6 ]
[ 9, 3, 6, 3, 9 ]
[ 7, 5, 4, 5, 7 ]
[ 2, 7, 6, 7, 2 ]
[ 3, 10, 9, 10, 3 ]
[ 6, 2, 8, 2, 6 ]
\[
\begin{align*}
&[ 4, 8, 7, 8, 4 ] \\
&[ 10, 3, 2, 3, 10 ] \\
&[ 4, 8, 3, 8, 4 ] \\
&[ 9, 5, 6, 5, 9 ] \\
&[ 3, 10, 4, 10, 3 ] \\
&[ 8, 4, 1, 4, 8 ] \\
&[ 3, 9, 10, 9, 3 ] \\
&[ 4, 7, 3, 7, 4 ] \\
&[ 1, 10, 8, 10, 1 ] \\
&[ 8, 1, 6, 1, 8 ] \\
&[ 9, 5, 3, 5, 9 ] \\
&[ 5, 7, 9, 7, 5 ] \\
&[ 8, 1, 4, 1, 8 ] \\
&[ 7, 4, 5, 4, 7 ] \\
&[ 10, 2, 1, 2, 10 ] \\
&[ 7, 4, 2, 4, 7 ] \\
&[ 6, 9, 8, 9, 6 ] \\
&[ 5, 7, 1, 7, 5 ] \\
&[ 10, 1, 2, 1, 10 ] \\
&[ 7, 5, 2, 5, 7 ] \\
&[ 10, 2, 3, 2, 10 ] \\
&[ 9, 6, 5, 6, 9 ] \\
&[ 6, 8, 9, 8, 6 ] \\
&[ 8, 6, 4, 6, 8 ] \\
&[ 10, 3, 1, 3, 10 ] \\
\end{align*}
\]

> N15851s := N15851;
> for n in N do if 1^n eq 2 and 5^n eq 6 and 8^n eq 10
for|if> and 5^n eq 6 and 1^n eq 2 then
for|if> N15851s := sub<N|N15851s, n>; end if; end for;
> for n in N do if 1^n eq 3 and 5^n eq 9 and 8^n eq 4
for|if> and 5^n eq 9 and 1^n eq 3 then
for|if> N15851s := sub<N|N15851s, n>; end if; end for;
> for n in N do if 1^n eq 4 and 5^n eq 3 and 8^n eq 7
for|if> and 5^n eq 3 and 1^n eq 4 then
for|if> N15851s := sub<N|N15851s, n>; end if; end for;
> for n in N do if 1^n eq 1 and 5^n eq 8 and 8^n eq 10
for|if> and 5^n eq 8 and 1^n eq 1 then
for|if> N15851s := sub<N|N15851s, n>; end if; end for;
> for n in N do if 1^n eq 5 and 5^n eq 9 and 8^n eq 1
for|if> and 5^n eq 9 and 1^n eq 5 then
for|if> N15851s := sub<N|N15851s, n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 6 and 8^n eq 7

for if and 5^n eq 6 and 1^n eq 2 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 7 and 5^n eq 2 and 8^n eq 5
for if and 5^n eq 2 and 1^n eq 7 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 10 and 8^n eq 7
for if and 5^n eq 10 and 1^n eq 2 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 9 and 5^n eq 6 and 8^n eq 3
for if and 5^n eq 6 and 1^n eq 9 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 4 and 5^n eq 7 and 8^n eq 8
for if and 5^n eq 7 and 1^n eq 4 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 3 and 5^n eq 4 and 8^n eq 10
for if and 5^n eq 4 and 1^n eq 3 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 6 and 5^n eq 9 and 8^n eq 2
for if and 5^n eq 9 and 1^n eq 6 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 8 and 5^n eq 6 and 8^n eq 1
for if and 5^n eq 6 and 1^n eq 8 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 5 and 5^n eq 1 and 8^n eq 7
for if and 5^n eq 1 and 1^n eq 5 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 1 and 5^n eq 8 and 8^n eq 5
for if and 5^n eq 8 and 1^n eq 1 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 8 and 5^n eq 4 and 8^n eq 6
for if and 5^n eq 4 and 1^n eq 8 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 5 and 5^n eq 1 and 8^n eq 7
for if and 5^n eq 1 and 1^n eq 5 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 4 and 5^n eq 7 and 8^n eq 8
for if and 5^n eq 7 and 1^n eq 4 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 1 and 5^n eq 10 and 8^n eq 5
for if and 5^n eq 10 and 1^n eq 1 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 7 and 8^n eq 10
for if and 5^n eq 7 and 1^n eq 2 then
for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 8 and 5^n eq 6 and 8^n eq 1
for if and 5^n eq 6 and 1^n eq 8 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 9 and 5ˆn eq 3 and 8ˆn eq 5
for|if> and 5ˆn eq 3 and 1ˆn eq 9 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 6 and 5ˆn eq 8 and 8ˆn eq 2
for|if> and 5ˆn eq 8 and 1ˆn eq 6 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 1 and 5ˆn eq 5 and 8ˆn eq 10
for|if> and 5ˆn eq 5 and 1ˆn eq 1 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 7 and 5ˆn eq 2 and 8ˆn eq 4
for|if> and 5ˆn eq 2 and 1ˆn eq 7 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 2 and 5ˆn eq 10 and 8ˆn eq 6
for|if> and 5ˆn eq 10 and 1ˆn eq 2 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 3 and 5ˆn eq 4 and 8ˆn eq 9
for|if> and 5ˆn eq 4 and 1ˆn eq 3 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 10 and 5ˆn eq 1 and 8ˆn eq 3
for|if> and 5ˆn eq 1 and 1ˆn eq 10 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 6 and 5ˆn eq 2 and 8ˆn eq 9
for|if> and 5ˆn eq 2 and 1ˆn eq 6 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 9 and 5ˆn eq 3 and 8ˆn eq 6
for|if> and 5ˆn eq 3 and 1ˆn eq 9 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 7 and 5ˆn eq 5 and 8ˆn eq 4
for|if> and 5ˆn eq 5 and 1ˆn eq 7 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 2 and 5ˆn eq 7 and 8ˆn eq 6
for|if> and 5ˆn eq 7 and 1ˆn eq 2 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 3 and 5ˆn eq 10 and 8ˆn eq 9
for|if> and 5ˆn eq 10 and 1ˆn eq 3 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 6 and 5ˆn eq 2 and 8ˆn eq 8
for|if> and 5ˆn eq 2 and 1ˆn eq 6 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1ˆn eq 4 and 5ˆn eq 8 and 8ˆn eq 7
for|if> and 5ˆn eq 8 and 1ˆn eq 4 then
for|if> N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 10 and 5^n eq 3 and 8^n eq 2 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 4 and 5^n eq 8 and 8^n eq 3 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 9 and 5^n eq 5 and 8^n eq 6 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 3 and 5^n eq 10 and 8^n eq 4 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 8 and 5^n eq 1 and 8^n eq 6 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 1 and 5^n eq 9 and 8^n eq 1 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 2 and 5^n eq 3 and 1^n eq 10 then
> for if N15851s:=sub<N|N15851s,n>; end if; end for;
for if and 5^n eq 2 and 1^n eq 10 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 7 and 5^n eq 2 and 8^n eq 2
for if and 5^n eq 4 and 1^n eq 7 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 6 and 5^n eq 9 and 8^n eq 8
for if and 5^n eq 9 and 1^n eq 6 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 9 and 5^n eq 7 and 8^n eq 2
for if and 5^n eq 7 and 1^n eq 9 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 10 and 5^n eq 1 and 8^n eq 2
for if and 5^n eq 1 and 1^n eq 10 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 7 and 5^n eq 4 and 8^n eq 2
for if and 5^n eq 5 and 1^n eq 7 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 10 and 5^n eq 9 and 8^n eq 8
for if and 5^n eq 1 and 1^n eq 10 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 6 and 5^n eq 7 and 8^n eq 9
for if and 5^n eq 7 and 1^n eq 6 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 10 and 5^n eq 3 and 8^n eq 1
for if and 5^n eq 3 and 1^n eq 10 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 6 and 5^n eq 8 and 8^n eq 2
for if and 5^n eq 8 and 1^n eq 6 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 7 and 5^n eq 6 and 8^n eq 4
for if and 5^n eq 6 and 1^n eq 7 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> for n in N do if 1^n eq 10 and 5^n eq 3 and 8^n eq 1
for if and 5^n eq 1 and 1^n eq 10 then
for if N15851s := sub<N|N15851s,n>; end if; end for;
> N15851s; #N15851s;
Permutation group N15851s acting on a set of cardinality 10

(2, 3) (4, 6) (7, 9)
(1, 3, 2) (4, 7, 5, 9, 6, 8)
(1, 3) (4, 8) (5, 9) (6, 7)
(1, 4, 2) (3, 6, 5) (7, 10, 8)
(1, 4, 5, 3) (2, 6) (7, 9, 10, 8)
(2, 7, 4, 3, 9, 6) (5, 8, 10)
(2, 9, 4) (3, 7, 6) (5, 8, 10)
(1, 5, 9, 3, 4, 8) (6, 10, 7)
(1, 5, 9, 6, 8) (2, 4, 10, 7, 3)
(1, 10, 2, 7, 5) (3, 6, 4, 9, 8)
(1, 6, 3) (2, 4, 5) (8, 9, 10)
(1, 6, 5, 2) (3, 4) (7, 10, 8, 9)
(1, 9, 4, 2) (3, 7, 10, 5) (6, 8)
(1, 9, 10, 5, 3) (2, 7, 4, 8, 6)
(1, 7) (2, 10) (3, 6) (4, 8)
(1, 7, 9) (2, 6, 8, 4, 3, 10)
(1, 2, 3) (4, 8, 6, 9, 5, 7)
(1, 2) (4, 9) (5, 7) (6, 8)
(1, 4, 5, 8, 7) (2, 10, 3, 9, 6)
(1, 4, 2, 9) (3, 10) (5, 8, 7, 6)
(1, 10) (2, 8) (3, 5) (4, 7)
(1, 10) (2, 5, 3, 8) (4, 6, 7, 9)
(1, 4, 10, 7) (2, 5, 8, 3) (6, 9)
(1, 4, 9) (3, 5, 8) (6, 10, 7)
(1, 9) (2, 4) (3, 10) (6, 8)
(1, 9, 7) (2, 10, 3, 4, 8, 6)
(1, 3, 5, 10, 9) (2, 6, 8, 4, 7)
(1, 3, 6, 7) (2, 5, 10, 9) (4, 8)
(1, 8) (3, 9) (4, 5) (6, 10)
(1, 8) (2, 9, 7, 3) (4, 10, 6, 5)
(1, 3, 7, 6) (2, 8, 10, 4) (5, 9)
(1, 3, 8, 10, 4) (2, 7, 5, 9, 6)
(1, 4, 9, 2) (3, 6, 10, 8) (5, 7)
(1, 4, 10, 8, 3) (2, 6, 9, 5, 7)
(2, 9) (3, 7) (5, 10)
(2, 7) (3, 9) (4, 6) (5, 10)
(1, 8, 6, 9, 5) (2, 3, 7, 10, 4)
(1, 8, 6, 2, 7, 5) (4, 9, 10)
(1, 9, 7) (2, 8, 3) (4, 10, 6)
(1, 9) (3, 8) (6, 10)
(1, 5, 9, 6, 2, 10) (3, 8, 7)
(1, 5, 9, 3, 10) (2, 8, 7, 6, 4)
(1, 5, 7, 2, 10) (3, 8, 9, 4, 6)
(1, 5, 7, 4, 10) (2, 8, 9)
(1, 8, 4, 3, 9, 5) (6, 7, 10)
(1, 8, 4, 7, 5) (2, 9, 10, 6, 3)
(1, 7, 3, 6) (2, 10) (4, 9, 8, 5)
(1, 7, 8, 5, 4) (2, 6, 9, 3, 10)
(1, 10, 3, 9, 6, 8) (2, 4, 5)
(1, 10, 3, 4, 8) (2, 9, 7, 6, 5)
(1, 6, 4) (2, 7, 5, 9, 3, 10)
(1, 6) (2, 10) (3, 7) (5, 9)
(1, 5, 7, 2, 6, 8)(4, 10, 9)
(1, 5, 7, 4, 8)(2, 3, 6, 10, 9)
(1, 10, 3, 9, 5)(2, 4, 6, 7, 8)
(1, 10, 3, 4, 7, 5)(2, 9, 8)
(1, 7, 9)(2, 3, 8)(4, 6, 10)
(1, 7)(2, 8)(4, 10)
(1, 10)(2, 5)(3, 8)(6, 9)
(1, 10)(2, 8, 3, 5)(4, 9, 7, 6)
(1, 9, 8, 5, 6)(2, 10, 3, 4, 7)
(1, 9, 2, 4)(3, 10)(5, 6, 7, 8)
(1, 6, 5, 8, 9)(2, 7, 4, 3, 10)
(1, 6, 3, 7)(2, 10)(4, 5, 8, 9)
(1, 8, 4, 3, 10)(2, 5, 6, 7, 9)
(1, 8, 4, 7, 2, 10)(3, 5, 6)
(1, 10, 2, 7, 4, 8)(3, 6, 5)
(1, 10, 2, 6, 8)(3, 7, 9, 4, 5)

120
> #N/#N15851s;
1
> T15851:=Transversal(N,N15851s);
> for i in [1..#T15851] do
    ss:=[1,5,8,5,1]^T15851[i];
    cst[prodim(1, ts, ss)]:=ss;
end for;
> m:=0; for i in [1..432] do if cst[i] ne [] then m:=m+1; end if; end for; m;
421
> [1,5,8,5,1]^N15851s;
GSet{@
[ 1, 5, 8, 5, 1 ],
[ 3, 9, 4, 9, 3 ],
[ 4, 3, 7, 3, 4 ],
[ 1, 8, 10, 8, 1 ],
[ 5, 9, 1, 9, 5 ],
[ 2, 6, 7, 6, 2 ],
[ 7, 2, 5, 2, 7 ],
[ 2, 10, 7, 10, 2 ],
[ 3, 4, 10, 4, 3 ],
[ 6, 9, 2, 9, 6 ],
[ 9, 6, 3, 6, 9 ],
[ 4, 3, 8, 3, 4 ],
[ 1, 8, 5, 8, 1 ],
@}

> for i in [1..#T15851] do ([1,5,8,5,1]^N15851s)^T15851[i]; end for;
> Orbits(N15851s);
>
> GSet{@ 1, 3, 4, 5, 2, 7, 6, 9, 8, 10 @}
>
> #N15851s;
120

/*Checking Orbits*/
> Orbits(N15171s);
>
> GSet{@ 5 @},
GSet{@ 7 @},
GSet{@ 1, 9 @},
GSet{@ 2, 4 @},
GSet{@ 3, 6, 10, 8 @}
>
> for m,n in IN do for i in [5,7,1,2,3] do if
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[1])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [5,7,1,2,3] do if
for|for|if> m*(ts[1])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [5,7,1,2,3] do if
for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1] eq
for|for|if> m*(ts[1]*ts[5])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [5,7,1,2,3] do if
for|for|if> m*(ts[1]*ts[2])^n then i;
for|for|if> break; end if; end for; end for;
>
> for m,n in IN do for i in [5,7,1,2,3] do if
for|for|if> ts[1]*ts[5]*ts[1] eq
for|for|if> m*(ts[1]*ts[3])^n then i;
for|for|if> break; end if; end for; end for;
>
>
>
>
>
>
>
>
>
> for m,n in IN do for i in [5,7,1,2,3] do if 
> for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq 
> for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])^n then i; 
> for|for|if> break; end if; end for; end for; 
> for m,n in IN do for i in [5,7,1,2,3] do if 
> for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq 
> for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[5])^n then i; 
> for|for|if> break; end if; end for; end for; 
> for m,n in IN do for i in [5,7,1,2,3] do if 
> for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq 
> for|for|if> m*(ts[1]*ts[2]*ts[1]*ts[10])^n then i; 
> for|for|if> break; end if; end for; end for; 
> for m,n in IN do for i in [5,7,1,2,3] do if 
> for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq 
> for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[5]*ts[1])^n then i; 
> for|for|if> break; end if; end for; end for; 
> for m,n in IN do for i in [5,7,1,2,3] do if 
> for|for|if> ts[1]*ts[5]*ts[1]*ts[7]*ts[1]*ts[i] eq 
> for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[5]*ts[1])^n then i; 
> for|for|if> break; end if; end for; end for; 
> Orbits(N15712s); 
> 
[ 
   GSet(@ 4 @), 
   GSet(@ 1, 9, 2 @), 
   GSet(@ 3, 7, 8 @), 
   GSet(@ 5, 6, 10 @) 
] 
> for m,n in IN do for i in [4,1,3,5] do if
for m,n in IN do for i in [4,1,3,5] do if
ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq m*(ts[1]*ts[5])\n then i;
for|if> break; end if; end for; end for;
> for m,n in IN do for i in [4,1,3,5] do if
ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq m*(ts[1]*ts[5]*ts[7])\n then i;
for|if> break; end if; end for; end for;
> for m,n in IN do for i in [4,1,3,5] do if
ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq m*(ts[1]*ts[5]*ts[8])\n then i;
for|if> break; end if; end for; end for;
> for m,n in IN do for i in [4,1,3,5] do if
ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq m*(ts[1]*ts[2]*ts[10])\n then i;
for|for|if> break; end if;
for|for> end for; end for;
> > for m,n in IN do for i in [4,1,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq
for|for|if> m*(ts[1]*ts[2]*ts[5])ˆn then i;
for|for|if> break; end if;
for|for> end for; end for;
> > for m,n in IN do for i in [4,1,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[1])ˆn then i;
for|for|if> break; end if;
for|for> end for; end for;
> > for m,n in IN do for i in [4,1,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])ˆn then i;
for|for|if> break; end if;
for|for> end for; end for;
> > for m,n in IN do for i in [4,1,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[5])ˆn then i;
for|for|if> break; end if;
for|for> end for; end for;
> > for m,n in IN do for i in [4,1,3,5] do if
for|for|if> ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[1])ˆn then i;
for|for|if> break; end if;
for m,n in IN do for i in [4,1,3,5] do if ts[1]*ts[5]*ts[7]*ts[1]*ts[2]*ts[i] eq m*(ts[1]*ts[2]*ts[1]*ts[10])^n then i; for if break; end if; for> end for; end for;
5
> Orbits(N15851s);
[ GSet{@ 1, 3, 4, 5, 2, 7, 6, 9, 8, 10 @} ]
for if break; end if; end for; end for;
for if break; end if; end for; end for;
> for m,n in IN do for i in [1] do if ts[1]*ts[5]*ts[8]*ts[5]*ts[5]*ts[1]*ts[1] eq m*(ts[1])^n then i;
for if break; end if; end for; end for;
> for m,n in IN do for i in [1] do if ts[1]*ts[5]*ts[8]*ts[5]*ts[5]*ts[1] eq m*(ts[1])^n then i;
for if break; end if; end for; end for;
> for m,n in IN do for i in [1] do if ts[1]*ts[5]*ts[8]*ts[5]*ts[5]*ts[1] eq m*(ts[1])^n then i;
for m, n in IN do for i in [1] do if  
for ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq  
for m*(ts[1]*ts[2])ˆn then i;  
for break; end if; end for; end for;

for m, n in IN do for i in [1] do if  
for ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq  
for m*(ts[1]*ts[5]*ts[1])ˆn then i;  
for break; end if; end for; end for;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[1])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[5])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[7]*ts[3])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
1
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[8]*ts[5])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[2]*ts[1]*ts[10])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m,n in IN do for i in [1] do if
for|for|if> ts[1]*ts[5]*ts[8]*ts[5]*ts[1]*ts[i] eq
for|for|if> m*(ts[1]*ts[5]*ts[1]*ts[7]*ts[1])^n then i;
for|for|if> break; end if;
for|for> end for; end for;
>
> for m, n in IN do for i in [1] do if
for|for|if> break; end if;
for|for> end for; end for;
>
> for m, n in IN do for i in [1] do if
for|for|if> break; end if;
for|for> end for; end for;
>
>
Bibliography


