6-2018

Symmetric Presentations, Representations, and Related Topics

Adam Manriquez
California State University - San Bernardino

Symmetric Presentations, Representations, and Related Topics

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Adam David Manriquez

June 2018

Symmetric Presentations, Representations, and Related Topics

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

by

Adam David Manriquez

June 2018

Approved by:

Dr. Zahid Hasan, Committee Chair

Dr. Lynn Scow, Committee Member

Dr. Rolland Trapp, Committee Member

Dr. Charles Stanton, Chair, Department of Mathematics

Dr. Corey Dunn, Graduate Coordinator

Abstract

The purpose of this thesis is to develop original symmetric presentations of finite non-abelian simple groups, particularly the sporadic simple groups. We have found original symmetric presentations for the Janko group J_1, the Mathieu group M_{12}, the Symplectic groups $S(3,4)$ and $S(4,5)$, a Lie type group $Suz(8)$, and the automorphism group of the Unitary group $U(3,5)$ as homomorphic images of the progenitors $2^{*^{60}} : (2 \times A_5)$, $2^{*^{60}} : A_5$, $2^{*^{56}} : (2^3 : 7)$, and $2^{*^{28}} : (PGL(2,7) : 2)$, respectively. We have also discovered the groups $2^4 : A_5$, $3^4 : S_5$, $PSL(2,31)$, $PSL(2,11)$, $PSL(2,19)$, $PSL(2,41)$, A_8, $3^4 : S_5$, A_5^2, $2 : A_6^2$, $2^\bullet A_5^2$, $PSL(2,49)$, $2^\bullet PSL(3,4)$, $2^8 : A_5$, $PGL(2,19)$, $2^\bullet PSL(2,31)$, $PSL(2,71)$, $2^4 : A_5$, $2^4 : A_6$, $PSL(2,7)$, $3 \times PSL(3,4)$, $PSL(3,4)$, $2^\bullet(M_{12} : 2)$, $3^7 : S_7$, $3^5 : S_5$, S_6, $2^5 : S_6$, $3^5 : S_6$, $2^5 : S_5$, $2^4 : S_6$, and M_{10} as homomorphic images of the permutation progenitors $2^{*^{60}} : (2 \times A_5)$, $2^{*^{60}} : A_5$, $2^{*^{21}} : (7 : 3)$ $2^{*^{60}} : (2 \times A_5)$, $2^{*^{120}} : S_5$, and $2^{*^{144}} : (3^2 : 2^4)$. We have given original proof of the 2^{*^n} Symmetric Presentation Theorem. In addition, we have also provided original proof for the Extension of the Factoring Lemma (involutory and non-involutory progenitors). We have constructed S_5, $PSL(2,7)$, and $U(3,5) : 2$ using the technique of double coset enumeration and by way of linear fractional mappings. Furthermore, we have given proofs of isomorphism types for 7×2^2, $U(3,5) : 2$, $2^\bullet(M_{12} : 2)$, and $(4 \times 2) :^\bullet 2^2$.

Acknowledgements

My dream to become a mathematician was brought about by Keith Johnson, my calculus teacher from Mount San Jacinto Community College. He forever changed the way I would perceive mathematics. Also, I would like to thank everyone that has contributed to my learning over the years. This includes past instructors, but also my peers. If it were not for such talented and influential people such as yourselves, I certainly wouldn't be the mathematician that I am today. To my friend and fishing buddy Charles Baccari, thank you for always being yourself. It didn't matter what was going on in school or life, you always made it enjoyable to be around. Kimberly Miller, I will miss the times we had in class, your brilliancy was always on full display. It was a pleasure to work with someone as smart and as witty as yourself. Charles Seager, I will miss the times we had playing chess. It was always a pleasure winning. To Shirley Juan and Joana Luna, when I first joined the legion of graduate students studying under Dr. Hasan, it was the two of you that extended a helping hand to get me up to speed. For that, I say thank you.

Next, I would like to recognize Dr. Zahid Hasan, my mentor and advisor. Dr. Hasan's approach to mathematics is second to none. Not only is he hard working and brilliant, but he also has the gift of being able to teach mathematics. There have been a countless number of times during lectures or meetings where I have been mesmerized by his natural gifts. His ability to bring out the beauty in math is unworldly. Not to mention, Dr. Hasan's dedication to his students will never be usurped. Dr. Hasan has always been there to help me from day one, no matter what the time of day or night (12am emails). I will forever be indebted to Dr. Hasan for the wealth of knowledge he has bestowed upon me. Dr. Hasan, thank you for everything you have done for me.

To my beautiful boys, Isaiah, Isaac, Israel, and Immanuel, you have brought much delight to me over the years. I am truly blessed to have each and every one of you. I love you boys with all my heart. To my mother-in-law Dawn, thank you for all your help and support you have provided over the years.

Now, to my wonderful wife Nichole, you have been a rock in our marriage. Your unwavering love, kindness, and compassion have been insurmountable. You have supported me in all my academic endeavors. Whether I needed time to study or do research, you were always there to help. I never have to worry with you by my side. You are a superlative wife, mother, and person. I love you!

Table of Contents

List of Tables

List of Figures

Introduction

Every finite simple group is known by the classification of simple groups. However, what is unknown are all the ways by which one can generate these simple groups, particularly the sporadic simple group. Sporadic simple groups are among some of the most intriguing mathematical objects found within the last half century. However, it was not so easy for mathematicians to study these groups for several reasons. Due to the many different ways these sporadic simple groups were discovered, one needs to study a lot of different techniques in order to understand the construction of these groups thoroughly. To much delight, there has been discovery of a standard way at lower levels to construct these simple groups.

Fascinatingly enough, Robert T. Curtis proved that every finite non-abelian simle group is a homomorphic image of an involutory progenitor, $2^{*^n} : N$, where $N \leq S_n$ [Cur07]. Utilizing this fact, we study the involutory progenitor. Through the use of double coset enumeration we can find a permutation representation for any symmetrically generated group. Additionally, we can also construct all of the orbits of N on the right cosets of N in G by double coset enumeration summarized efficiently in a Cayley graph of G over N.

On the other hand, we have also studied the non-involutory progenitor, $m^{*^n} :_m N$, where $N \leq S_k$ for some k. To construct a non-involutory progenitor, we find an irreducible monomial representation of N. The challenge is to determine some/all of the irreducible monomial representations. We attack the subgroups H of N by manually lifting the characters $\dot{\chi}$ of H up to N by way of induction, $\dot{\chi}_H^N \uparrow$. We can determine if the character $\dot{\chi}$ will be an irreducible monomial representation of N if it turns out to be faithful. If so, we have another unique way to capture finitely presented groups.

Once we factor a progenitor by suitable relations and obtain a group G, we need to determine what G is. This leads us to isomorphism types. Knowing that G is finite enables us to determine the group's isomorphism type. Every finite group G has a composition series. That is, every finite group G has composition factors where these composition factors are simple. Moreover, we are able to write G as a product of its composition factors provided we solve the extension problem(s). In addition, these composition factors are unique up to isomorphism by way of the Jordan-Hölder theorem. Having solved G's extension problem, we can immediately determine what G is. Thus, we can determine if we have captured any sporadic simple groups. Therefore, it is important we know how to solve the extension problem(s) of any finite group G.

Chapter 1

Preliminaries

1.1 Definitions

Definition 1.1. *[Rot95] A* **group** *G $(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*
- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.*

Definition 1.2. *[Rot95] Let G be a set. A (binary)* **operation** *on G is a function that assigns each ordered pair of elements of G an element on G.*

Definition 1.3. *[Rot95] For group G, a* **subgroup** *S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$.*

Definition 1.4. *[Rot95] Let H be a subgroup of group G. H is a* **proper** *subgroup of G if $H \neq G$. We denote this as $H < G$.*

Definition 1.5. *[Rot95] A* **symmetric group***, S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions.*

Definition 1.6. *[Rot95] If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$.*

Definition 1.7. *[Rot95] A* **semigroup** *$(G, *)$ is a nonempty set G equipped with an associative operation *.*

Definition 1.8. *[Rot95] If* $x \in X$ *and* $\phi \in S_X$*, then* ϕ **fixes** x *if* $\phi(x) = x$ *and* ϕ **moves** x *if* $\phi(x) \neq x$.

Definition 1.9. *[Rot95] For permutations* $\alpha, \beta \in S_X$*,* α *and* β *are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(a) = a \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y.$$

Definition 1.10. *[Rot95] A permutation which interchanges a pair of elements is a* **transposition**.

Definition 1.11. *[Rot95] In group* G*, if* a,$b \in G$*,* a *and* b **commute** *if* $a * b = b * a$.

Definition 1.12. *[Rot95] A group* G *is* **abelian** *if every pair of elements in* G *commutes with one another.*

Definition 1.13. *[Rot95] Let* X *be a set and* Δ *be a family of words on* X*. A group* G *has* **generators** X *and* **relations** Δ *if* $G \cong F/R$*, where* F *is a free group with basis* X *and* R *is the normal subgroup of* F *generated by* Δ*. We say* $< X|\Delta >$ *is a* **presentation** *of* G.

Definition 1.14. *[Rot95] Let* G *be a group. If* $H \leq G$*, the* **normalizer** *of* H *in* G *is defined by* $N_G(H) = \{a \in G | aHa^{-1} = H\}$

Definition 1.15. *[Cur07] Let* G *be a group and* $T = \{t_1, t_2, ..., t_n\}$ *be a symmetric generating set for* G *with* $|t_i| = m$*. Then if* $N = N_G(\bar{T})$*, then we define the* **progenitor** *to be the semi direct product* $m^{*n} : N$*, where* m^{*n} *is the free product of* n *copies of the cyclic group* C_m.

Definition 1.16. *[Rot95] Let* G *be a group. If* $H \leq G$*, the* **centralizer** *of* H *in* G *is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}.$$

Definition 1.17. *[Rot95] Let* p *be prime. If* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$*, then we say* G *is* **elementary abelian**.

Definition 1.18. *[Rot95] Let* $(G, *)$ *and* $(H, \circ)$ *be groups. The function* $\phi : G \to H$ *is a* **homomorphism** *if* $\phi(a * b) = \phi(a) \circ \phi(b)$*, for all* a,$b \in G$*. An* **isomorphism** *is a bijective homomorphism. We say* G *is isomorphic to* H*,* $G \cong H$*, if there is exists an isomorphism* $f : G \to H$.

Definition 1.19. *[Rot95] Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *.*

Definition 1.20. *[Rot95] Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X.*

Definition 1.21. *[Rot95] Let G be a group. We say G is a* **direct product** *of two subgroups K and Q denoted, $G \cong K \times Q$ if:*

1. *$K \trianglelefteq G$, $Q \trianglelefteq G$;*
2. *$Q \cap K = 1$,*
3. *$G = KQ$;*

Definition 1.22. *[Rot95] G is a* **semi-direct product** *of two subgroups K and Q denoted, $G \cong K : Q$ if:*

1. *$K \trianglelefteq G$;*
2. *K has the complement $Q_1 \cong Q$;*

Definition 1.23. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **central extension** *by H if N is the center of G. We denote this by $G \cong N^{\bullet} H$.*

Definition 1.24. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **mixed extension** *by H if it is a combination of both central extensions and semi-direct products, where N is the normal subgroup of G but not central. We denote this by $G \cong N^{\bullet} : H$.*

Definition 1.25. *[Rot95] Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1} a g | g \in G\}$*

Definition 1.26. *[Rot95] Let G be a group. A* **normal series** *G is a sequence of subgroups*

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1$$

with $G_{i+1} \lhd G_i$. Furthermore, the **factors groups** *of G are given by G_i/G_{i+1} for $i = 0, 1, \ldots, n-1$.*

Definition 1.27. *[Rot95] Let G be a group. A* **composition series** *of G given by:*

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

is a normal series where, for all i, either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_i$.

Definition 1.28. *[Rot95] If group G has a composition series, the factor groups of its series are the* **composition factors** *of G.*

Definition 1.29. *[Rot95] The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$.*

Definition 1.30. *[Rot95] A* **general linear group***, $GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$.*

Definition 1.31. *[Rot95] A* **special linear group***, $SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$.*

Definition 1.32. *[Rot95] A* **projective special linear group***, $PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}.$$

Definition 1.33. *[Rot95] A* **projective general linear group***, $PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}.$$

Definition 1.34. *[Led77]* (**Monomial Character**) *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H.*

Definition 1.35. *[Led77] Suppose that with each element x of the group G there is associated an m by m non-singular matrix*

$$A(x) = (a_{ij}(x)) \quad (i, j = 1, 2, \ldots, m),$$

with coefficients in the field K, in such a way that

$$A(x)A(y) = A(xy) \quad (x, y \in G).$$

*Then $A(x)$ is called a **matrix representation** of G of degree (dimension) m over K.*

Definition 1.36. *[Led77](**Character**) Let $A(x) = (A_{ij}(x))$ be a matrix representation of G of degree m. We consider the character polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the **character** *of $A(x)$.*

Definition 1.37. *[Led77] The sum of squares of the degrees of the s=distinct irreducible characters of G is equal to $|G|$. The* **degree of a character** *χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character.*

Definition 1.38. *[Led77]* (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x))$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. The lifting process preserves irreducibility.*

Definition 1.39. *[Led77]* (**Induced Character**) *Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then*

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G.

Definition 1.40. *[Rot95] Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$.*

Definition 1.41. *[Rot95] Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \lhd G$ to denote K as being normal in G.*

Definition 1.42. *[Rot95] Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St.*

Definition 1.43. *[Rot95] Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G.*

Definition 1.44. *[Rot95] Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$.*

Definition 1.45. *[Rot95] Let N be a group. The* **point stabiliser** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s.}$$

Definition 1.46. *[Rot95] Let N be a group. The* **coset stabiliser** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s.}$$

Definition 1.47. *[Rot95] Let G be a group. The* **center** *of G, $Z(G)$, is the set of all elements in G that commute with all elements of G.*

Definition 1.48. *[Led77] A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1,\pi_2\omega_2,...}$$

where 2^{*n} *denotes a free product of* n *copies of the cyclic group of order 2,* N *is transitive permutation group of degree* n *which permutes the* n *generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators* $\pi_1\omega_1, \pi_2\omega_2, ...$ *have been factored out.*

Theorem 1.49. *[Led77] The number of irreducible character of* G *is equal to the number of conjugacy classes of* G

Theorem 1.50. *[Rot95] Let* $\phi : G \to H$ *be a homomorphism with kernel* K*. Then* K *is a normal subgroup of* G *and* $G/K \cong im\phi$*.*

Theorem 1.51. *[Rot95] Let* N *and* T *be subgroups of* G *with* N *normal. Then* $N \cap T$ *is normal in* T *and* $T/(N \cap T) \cong NT/N$*.*

Theorem 1.52. *[Rot95] Every permutation* $\alpha \in S_n$ *is either a cycle or a product of disjoint cycles.*

Theorem 1.53. *[Rot95] Let* $f : (G, *) \to (G', \circ)$ *be a homomorphism. The following hold true:*

- $f(e) = e'$*, where* e' *is the identity in* G'*,*
- *If* $a \in G$*, then* $f(a^{-1}) = f(a)^{-1}$*,*
- *If* $a \in G$ *and* $n \in \mathbb{Z}$*, then* $f(a^n) = f(a)^n$*.*

Theorem 1.54. *[Rot95] The intersection of any family of subgroups of a group* G *is again a subgroup of* G*.*

Theorem 1.55. *[Rot95] If* $S \leq G$*, then any two right (or left) cosets of* S *in* G *are either identical or disjoint.*

Theorem 1.56. *[Rot95] If* G *is a finite group and* $H \leq G$*, then* $|H|$ *divides* $|G|$ *and* $[G : H] = |G|/|H|$*.*

Theorem 1.57. *[Rot95] If* S *and* T *are subgroups of a finite group* G*, then*

$$|ST||S \cap T| = |S||T|.$$

Theorem 1.58. *[Rot95] If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$.*

Theorem 1.59. *[Rot95] The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$.*

Theorem 1.60. *[Rot95] Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$.*

Theorem 1.61. *[Rot95] If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite.

Theorem 1.62. *[Rot95] If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$.*

Theorem 1.63. *[Rot95] Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n.*

Theorem 1.64. *[Rot95] If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker\rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H.*

Theorem 1.65. *[Rot95] If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set.*

Theorem 1.66. *[Rot95] Every two composition series of a group G are equivalent.*

We will refer to this Theorem as the **Jordan-Hölder Theorem***.*

Theorem 1.67. *[Rot95] Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$.*

1.2 Lemmas

Lemma 1.68. *[Rot95] Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*
- *If $H \lhd G$, then the subsets Hx are blocks of X.*

Chapter 2

The Involutory Progenitor

2.1 Introduction

Robert T. Curtis proved that every finite non-abelian group is a homomorphic image of an involutory progenitor, $2^{*^n} : N$, where $N \leq S_n$. Utilizing this fact, we study the involutory progenitor. The definition of a *progenitor* is that it is a semi-direct product of the following form: $P \cong 2^{*^n} : N = \{\pi\omega | \pi \in N, \omega$ a reduced word in the $t_i\}$, where 2^{*^n} denotes a free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, \ldots, n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutary generators. We refer to N as the *control group* and the involutions as *symmetric generators*.

It should be noted that a progenitor is an infinite group. Therefore, we must find relations to factor a progenitor by in order to capture homomorphic images of finitely presented groups (relations are of the form $\pi\omega$). There are various ways to obtain relations to factor a progenitor by. Although, which ever approach is taken, none can gaurantee a particular group. Therefore, we try all methods of finding relations to factor a progenitor by. We will now illustrate the process by which a progenitor is constructed through the use of MAGMA. In addition, we will show how to factor a progenitor by various relations.

Our desire is to write the progenitor $2^{*^6} : A_6$, where 2^{*^6} denotes the free product of involutions and A_6 is our control group. We start with the representation of A_6.

$A := Alt(6);$

$Generators(A);$

$(1,2,3),(1,2)(3,4,5,6)$

We define our generators and save the group N generated by these elements for later use.

$yy := A!(1,2,3);$

$xx := A!(1,2)(3,4,5,6);$

$N := sub < A|xx, yy >;$

$\#N;$

360

Next, we execute the finitely presented group command on A_6 to garner a presentation for A_6, $FPGroup(Alt(6))$. We translate the given presentation from MAGMA in terms of $x's$ and $y's$ (MAGMA will yield presentations in terms of numerical values such as .1, .2, .3, etc. therefore we translate them into words). Having done so, the presentation for A_6 is as follows:

$$NN < x, y > := Group < x, y|x^4, y^3, (x^{-1}, y^{-1})^2, x^{-1}y^{-1}x^{-2}y^{-1}x^2y^{-1}x^2y^{-1}x^{-1}, (y^{-1}x^{-1})^5 > .$$

We can check the $\#NN$ to make sure we are on the right track (no errors were made). Or, we can utilize Cayley's Theorem [*Rot* 95] in conjunction with Theorem 3.14 [*Rot* 95] which allows us to check if NN is isomorphic to A_6. That is, NN can be imbedded as a subgroup of S_{360}. Moreover, if we choose $Id(NN) = H \leq NN$, then we obtain a faithful representation of NN. Furthermore, we are now able to see if our presentation is correct on A_6. This is how we perform the check in magma.

$f, NN1, k := CosetAction(NN, sub < NN|Id(NN) >);$

$s := IsIsomorphic(NN1, Alt(6));$

$s;$

$true$

Hence, our presentation of A_6 is correct.

We now need to add in the free product of involutions 2^{*6} described by t_i, where $1 \leq i \leq 6$. Note: $|t_i| = 2$. That is, we must make $t \sim t_1$ commute with the one point stabilizer of A_6, denoted A_6^1. That is, $(t, A_6^1) = 1$ means $1^s = 1$ for all $s \in A_6^1$. By doing so, we are saying t has six conjugates or in other words, there are six symmetric generators. N acts on the family of all its subgroups by conjugation. We will show that there is a homorphism from N to the family of all the conjugates of $< (t) >$ in N, thus inducing an action.

Lemma 2.1. *[Rot95] Let $< (t) > \leq m^{*^m} : N$ and X be the family of all the conjugates of $< (t) >$ in N. There is a homomorphism $\phi : N \to S_X$ with $ker\phi \leq N_N(< (t) >)$. (Note: $N_N(< (t) >)$ is the normalizer in N of $< (t) >$).*

Proof: Let $a \in N$, define $\phi_a : X \to X$, by $\phi_a(n^{-1} < (t) > n) = a^{-1}n^{-1} < (t) > na$. Let $b \in N$, and consider the following:

$$\begin{aligned} \phi_a\phi_b(n^{-1} < (t) > n) &= \phi_a(b^{-1}n^{-1} < (t) > nb) \\ &= \phi_{ab}(a^{-1}b^{-1}n^{-1} < (t) > nba) \\ &= \phi_{ab}(n^{-1} < (t) > n). \end{aligned}$$

Thus, ϕ is a homomorphism. Moreover, since $a \in N$, then $a^{-1} \in N$. Hence, ϕ_a has inverse $\phi_{a^{-1}}$ and thus $\phi_a \in S_X$. Furthermore, if $a \in ker\phi$, then

$$\begin{aligned} \phi_a(n^{-1} < (t) > n) &= a^{-1}n^{-1} < (t) > na \\ &= n^{-1} < (t) > n, \quad \textit{for all } n \in N. \end{aligned}$$

Choose $n = e$, then $\phi_a(e^{-1} < (t) > e) = a^{-1}e^{-1} < (t) > ea = a^{-1} < (t) > a$.

Therefore, $a \in N_N(<(t)>)$ *and* $ker\phi \leq N_N(<(t)>)$ *as desired.* $\square$

Continuing, our group A_6 acts on the indices of the $t_i's$ by conjugation. Moreover, we can conclude if A_6 acts on the indices by conjugation, then for $t \in m^{*^n}$, $\mathcal{O}(t)$ is the conjugacy class of t and $A_{6_t} = C_{A_6}(t)$. That is, the stabilizer of t is equal to the centralizer of t in A_6.

Corollary 2.2. *[Rot95] Let* $G = m^{*^n} : N$, *if* N *acts on* m^{*^n} *by conjugation and* $t \in m^{*^n}$, *then* $\mathcal{O}(t)$ *is the conjugacy class of* t *and* $N_t = C_N(t)$.

Proof: Let $t \sim t_1$, *and* N^1 *be the one point stabilizer in* N. *Then*

$$\begin{aligned} N_t &= Nt \\ &= \{n \in N^1 \mid nt = t\} \\ &= \{n \in N^1 \mid t^n = t\}, \qquad \textit{since } N \textit{ acts by conjugation on } m^{*^n} \\ &= \{n \in N^1 \mid n^{-1}tn = t\} \\ &= \{n \in N^1 \mid nt = tn\} \\ &= C_N(t). \end{aligned}$$

Thus, $N_t = C_N(t)$ *as desired.*

Furthermore,

$$\begin{aligned} \mathcal{O}(t) &= \{nt \mid n \in N\} \\ &= \{t^n \mid n \in N\}, \qquad \textit{since } N \textit{ acts by conjugation on } m^{*^n} \\ &= \{n^{-1}tn \mid n \in N\} \\ &= t^N \end{aligned}$$

Thus, $\mathcal{O}(t) = t^N$ as desired. $\square$

Theorem 2.3. *[Rot95] Show that the number of conjugates of t in $N \leq S_n$ is equal to the index of the centralizer of t in N, denoted $|t^N| = [N : C_N(t)]$.*

Proof: (Note: For the purpose of this proof, we shall use left cosets as demonstrated in [Rot 95].) Let $t^N = \{ntn^{-1} | n \in N\}$, and let $C = C_N(t)$ and $N/C = \{nC | n \in N\}$. Define $f : t^N \to N/C$ by $f(ntn^{-1}) = nC$. We need to show that f is well defined and onto, thus we have our desired result. (Well defined) Suppose $ntn^{-1} = hth^{-1}$ $\iff$ $tn = n^{-1}hth^{-1}$ $\iff$ $tn^{-1}h = n^{-1}ht$, hence $n^{-1}h \in C$. This implies, $n^{-1}hC = C$ $\iff$ $nC = hC$. Hence, f is well defined. (Onto) Let $nC \in N/C$, then $ntn^{-1} \in t^N$ such that $f(ntn^{-1}) = nC$, hence f is onto. $\square$

We now conclude the index of the centralizer of t_1 in N is equal to the number of conjugates of t_1 in N which is equal to the index of the one point stabilizer in N. That is, $|t^N| = [N : C_N(t)] = [N : N_t]$. But $\mathcal{O}(t) = t^N$, hence $|\mathcal{O}(t)| = |t^N| = [N : N_t]$. Since A_6 is a transitive G-set, $|\mathcal{O}(t)| = 6$, hence t_1 has six conjugates. Moreover, we can compute the conjugates of t_i explicitly by the right coset representatives of N over N^1, the control group and the one point stabilizer respectively.

We will now describe how to present 2^{*^n} in a symmetric presentation of the progenitor $2^{*^n} : N$.

2.2 2^{*^n} Symmetric Presentation Theorem

Theorem 2.4. *(2^{*^n} Symmetric Presentation Theorem) Let $N \leq S_n$ be transitive and $(t, N^1) = 1$, then $t^a = t^b \iff N^1a = N^1b$ for all $a, b \in N$.*

Proof: Since N is transitive, $|\mathcal{O}(t)| = n$, for all $t \in m^{*^n}$. By previous proof, this implies N has n conjugacy classes. Moreover, this means that $[N : N^1] = n$. Now let $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_n$, we will show that the conjugates of t are obtained by the right coset representatives of N^1. In doing so, we are saying there are n-many $t_{i's}$

and no more. Suppose $t^a = t^b$*, then*

$$\begin{aligned} t^a &= t^b \\ \iff \quad a^{-1}ta &= b^{-1}tb \\ \iff \quad ba^{-1}ta &= tb \\ \iff \quad ba^{-1}tab^{-1} &= t \\ \iff \quad (ab^{-1})^{-1}tab^{-1} &= t \\ \iff \quad t^{ab^{-1}} &= t \end{aligned}$$

Thus, $ab^{-1} \in N^1$*. That is,* $N^1ab^{-1} = N^1 \iff N^1a = N^1b$. $\square$

By viewing the statement of the theorem, if it were the case that two $t's$ were equal under conjugation, then the right cosets must be equal. Looking at the negation of the statement, $t^a \neq t^b \iff N^1a \neq N^1b$, we are saying there are as many $t's$ as there are right cosets.

That is, we can write N as a decomposition of right cosets in terms of the one point stabilizer and the representatives of the n conjugacy classes, denoted by $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_n$. Thus, when we conjugate t by $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_n$, we get n $t_{i's}$.

Currently, our progenitor is given by:
$2^{*^6} : A_6 =< x, y, t | x^4, y^3, (x^{-1}, y^{-1})^2, x^{-1}y^{-1}x^{-2}y^{-1}x^2y^{-1}x^2y^{-1}x^{-1}, (y^{-1}x^{-1})^5, t^2, (t, A_6^1) >$
.

However, we still need to convert the generators of A_6^1 into words. So, we utilize the Schreier System to convert A_6^1's permutations into words. Upon completion, we will have succesfully created a progenitor for $2^{*^6} : A_6$.

$N1 := Stabilizer(N, 1);$

$Generators(N1)$;

$(4, 5, 6)$

$(2, 3, 4)$

$(3, 4, 5)$

Sch:=SchreierSystem(NN,sub< NN | Id(NN)>);
ArrayP:=[Id(N): i in [# N]];
for i in [2.. # N] do P:=[Id(N): l in [1.. # Sch[i]]];
for j in [1.. # Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^{-1}; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^{-1}; end if;
end for;
PP:=Id(N);
for k in [1.. # P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1.. #N] do if ArrayP[i] eq N! (4, 5, 6) then Sch[i];
end if; end for;
$xyx^{-1}yxy$
for i in [1..#N] do if ArrayP[i] eq N! (2, 3, 4) then Sch[i]; end if; end for;
$yx^{-1}yx$
for i in [1..#N] do if ArrayP[i] eq N! (3, 4, 5) then Sch[i];end if;end for;
$xyxyx^{-1}y$

A simple check to see if we converted the generators properly is done by the following,

$N1\ eq\ sub < N|xxyyxx^{-1}yyxxyy,\ yyxx^{-1}yyxx,\ xxyyxxyyxx^{-1}yy >$;
true

Therefore, our progenitor for $2^{*6} : A_6$ is,

$G < x, y, t >:= Group < x, y, t | x^4, y^3, (x^{-1}, y^{-1})^2, x^{-1}y^{-1}x^{-2}y^{-1}x^2y^{-1}x^2y^{-1}x^{-1}, (y^{-1}x^{-1})^5, t^2, (t, xyx^{-1}yxy), (t, yx^{-1}yx), (t, xyxyx^{-1}y) >$.

Again, when we make t commute with the generators of the one point stabilizer in the presentation, we are saying t has 6 conjugates. This follows from the one point stabilizer being a subgroup of N with index 6. Moreover, we can write N as a decomposition of right cosets in terms of the one point stabilizer and the representatives of the six conjugacy classes, denoted $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_6$. Therefore, when we conjugate t by $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_6$, we get six $t_{i's}$.

2.3 Applying the Factoring Lemma

We can perform a check utilizing Lemma 2.13 (Factoring Lemma) [*Gri* 15]. This lemma states that when you factor the progenitor $m^{*n} : N$ by (t_i, t_j), for $1 \leq i \leq j \leq n$, you obtain a homomorphic image of the group $m^n : N$. In other words, to show that our progenitor is correct, we have to show how t_1 commutes with all the other $t_i's$, where $2 \leq i \leq 6$. By taking the order of $m^n : N$, it should be $|m^n| \cdot |N| = |2^6| \cdot |360| = 23040$. We execute this by taking the orbits of the one point stabilizer.

Orbits(N1);
$\{1\}$
$\{2, 3, 4, 5, 6\}$

We add the relation $(t, t^x) = 1$ and take the order of G.

$G < x, y, t >:= Group < x, y, t | x^4, y^3, (x^{-1}, y^{-1})^2, x^{-1}y^{-1}x^{-2}y^{-1}x^2y^{-1}x^2y^{-1}x^{-1}, (y^{-1}x^{-1})^5, t^2, (t, xyx^{-1}yxy), (t, yx^{-1}yx), (t, xyxyx^{-1}y), (t, t^x) >$;

$\#G;$
23040

Therefore, the presentation for the progenitor $2^{*6} : A_6$ is correct.

2.4 Extension of Factoring Lemma

Corollary 2.5. *: Let* $G = \frac{m^{*^n}:N}{t_1 t^x = t^x t_1}$*, then* $Z(G) =< (t_n t_{n-1} \cdots t_1) >$ *and* $|Z(G)| = m$.

Proof: Let $\omega\pi \in G$*, where* ω *is the product of* $t_{i's}$*,* $1 \leq i \leq n$ *in* m^{*^n} *and* π *is a permutation in* N*. It suffices to show that* $(t_n t_{n-1} \cdots t_1)$ *commutes with* π*. Since* $< t_1 > \times < t_2 > \times \cdots \times < t_n >$ *is a direct product, then* $t_i t_j = t_j t_i$*, for all* $1 \leq i, j, \leq n$*. Consider:*

$$\begin{aligned}
\omega\pi &= (t_n t_{n-1} \cdots t_1)\pi \\
&= \pi\pi^{-1}(t_n t_{n-1} \cdots t_1)\pi \\
&= \pi(t_n t_{n-1} \cdots t_1)^{\pi} \\
&= \pi(t_n^{\pi} t_{n-1}^{\pi} \cdots t_1^{\pi}) \\
&= \pi(t_q t_r \cdots t_s) \qquad \textit{where } 1 \leq q, r, s \leq n \\
&= \pi(t_n t_{n-1} \cdots t_1) \\
&= \pi\omega
\end{aligned}$$

Therefore, $(t_n t_{n-1} \cdots t_1)$ *commutes with* π*. We must show that* $Z(G) =< (t_n t_{n-1} \cdots t_1) >$ *is a product of* $t_{i's}$ *all of the same power, has length* n*, and that* $|Z(G)| = m$*.*

Since N *is transitive,* N *has only one orbit. That is, given* $(t_q t_r \cdots t_s)$*,* $1 \leq q, r, s \leq n$ *with length less than* n*, then there is some* t_i *not in* $(t_q t_r \cdots t_s)$*. We can choose a* $\beta \in N$ *that sends any other* t_j*,* $1 \leq j \leq n$ *in* $(t_q t_r \cdots t_s)$ *to* t_i *by conjugation of* β*,* $t_j^{\beta} = t_i$*. Similarly, if there was a* t_i^p*,* $1 \leq p \leq m$ *that so happened to be raised to a different power than some other* t_j*,* $1 \leq j \leq n$*, we could choose some* $\alpha \in N$

that sends $t_i^p \to t_j^p$ by conjugation of α, $(t_i^p)^\alpha = t_j^p$. Hence, $Z(G)$ must be of length n and all $t_{i's}$ must be raised to the same power. We will now prove that $|Z(G)| = m$. If m is composite, then suppose $| < t_1 > \times < t_2 > \times \cdots \times < t_n > | = k$, where $1 < k < m$. This implies $t^k = t_n^{-k} t_{n-1}^{-k} \cdots t_2^{-k}$. Since $< t_1 > \times < t_2 > \times \cdots \times < t_n >$, then $< t_1 > \cap < t_2 > \times \cdots \times < t_n >= \{e\}$. But $t^k = t_n^{-k} t_{n-1}^{-k} \cdots t_2^{-k} \in < t_1 > \cap < t_2 > \times \cdots \times < t_n >= \{e\}$. This implies $|t^k| = e$, a contradiction. If m is prime, then $< t_1 > \times < t_2 > \times \cdots \times < t_n >$ is elementary abelian of order m^n, this implies $|Z(G)| = | < t_1 > \times < t_2 > \times \cdots \times < t_n > | = m$, since all non-trivial elements of an elementary group have order m. Hence, $|Z(G)| = m$. $\square$

2.5 Factoring $2^{*6} : A_6$ By First Order Relations

Given our infinite progenitor $2^{*6} : A_6$, we would now like to factor by suitable first order relations to obtain finite homomorphic images of simple groups. That is, we factor by the smallest normal subgroups to obtain these images. Since the conjugacy classes are normal, we compute the conjugacy classes of our control group. Next, we compute the centralizer of each class representative. Then, find the orbits of the corresponding centralizers. By taking the orbits of the respective conjugacy classes, it will allow us to determine which $t_{i's}$ are in the same orbits. This is a short cut or a way to cut down on listing the respective first order relations as it can be taxing when the groups get much larger in size. That is, the $t_{i's}$ in the same orbits will be repeats of each other since they commute with one another by conjugation of their respective class representative. For example, class [3] has orbit $\{1, 2, 3\}$. This means t_1, t_2, and t_3 commute with each other by conjugation of $(1, 2, 3)(4, 5, 6)$.

```
A:=Alt(6);
C:=Classes(A);
#C;
7
C;
```

The conjugacy classes are as follows:

Conjugacy Classes of group A

$- -$

[1] Order 1 Length 1
Rep Id(A)

[2] Order 2 Length 45
Rep (1, 2)(3, 4)

[3] Order 3 Length 40
Rep (1, 2, 3)(4, 5, 6)

[4] Order 3 Length 40
Rep (1, 2, 3)

[5] Order 4 Length 90
Rep (1, 2, 3, 4)(5, 6)

[6] Order 5 Length 72
Rep (1, 2, 3, 4, 5)

[7] Order 5 Length 72
Rep (1, 3, 4, 5, 2)

for i in [2..$\#C$] do i, Orbits(Centraliser(A,C[i][3])); end for;

2
{ 5, 6 }
{1, 2, 3, 4 }
3

{ 1, 2, 3 }

{ 4, 5, 6 }

4

{ 1, 2, 3 }

{ 4, 5, 6 }

5

{5, 6 }

{ 1, 2, 3, 4 }

6

{6 }

{ 1, 2, 3, 4, 5 }

7

{ 6 }

{ 1, 3, 4, 5, 2 }

```
for j in [2..#C] do
for i in [1..#A] do
if ArrayP[i] eq C[j][3] then Sch[i]; end if; end for;
j;
end for;
```

$y * x^-1 * y * x * y$

2

$x * y^-1 * x^-1 * y^-1 * x$

3

y

4

$x * y^-1 * x^2 * y * x^2$

5

$x * y^-1 * x * y^-1 * x^-1$

6

$x * y * x * y * x^-1$

7

We now right multiply the above words by t and we have the following first order relations:

$(y * x^{-1} * y * x * y * t)$
$(y * x^{-1} * y * x * y * t^{(y^2 * x^2)})$
$(x * y^{-1} * x^{-1} * y^{-1} * x * t)$
$(x * y^{-1} * x^{-}1 * y^{-1} * x * t^{(y^2 * x)})$
$(y * t)$
$(y * t^{(y^2 * x)})$
$(x * y^{-1} * x^2 * y * x^2 * t)$
$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2 * x^2)})$
$(x * y^{-1} * x * y^{-1} * x^{-1} * t)$
$(x * y^{-1} * x * y^{-1} * x^{-1} * t^{(y^2 * x^3)})$
$(x * y * x * y * x^{-1} * t)$
$(x * y * x * y * x^{-1} * t^{(y^2 * x^3)})$

We raise the relations to powers through the use of a loop in MAGMA. MAGMA will return numbers which are the powers we should use for corresponding relations based on the homomorphic images our progenitor captures. Below is our infinite progenitor of $2^{*6} : A_6$:

$G < x, y, t > := Group < x, y, t | x^4, y^3, (x^{-}1, y^{-}1)^2, t^2,$
$x^{-}1 * y^{-}1 * x^{-}2 * y^{-}1 * x^2 * y^{-}1 * x^2 * y^{-}1 * x^{-1}, (y^{-}1 * x^{-1})^5,$
$t^2,$
$(t, x * y * x^{-}1 * y * x * y),$
$(t, y * x^{-}1 * y * x),$
$(t, x * y * x * y * x^{-}1 * y),$
$(t, t^x),$
$(y * x^{-1} * y * x * y * t)^{r1},$
$(y * x^{-1} * y * x * y * t^{(y^2 * x^2)})^{r2},$

$(x * y^{-1} * x^2 * y * x^2 * t)^{r3}$,
$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2*x^2)})^{r4}$,
$(y * t)^{r5}$,
$(y * t^{(y^2*x)})^{r6}$,
$(x * y^{-1} * x^2 * y * x^2 * t)^{r7}$,
$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2*x^2)})^{r8}$,
$(x * y^{-1} * x * y^{-1} * x^{-1} * t)^{r9}$,
$(x * y^{-1} * x * y^{-1} * x^{-1} * t^{(y^2*x^3)})^{r10}$,
$(x * y * x * y * x^{-1} * t)^{r11}$,
$(x * y * x * y * x^{-1} * t^{(y^2*x^3)})^{r12}$
$>$;

2.6 Factoring $2^{*^6} : A_6$ By Second Order Relations

Second order relations are of the form $\pi t_i t_j$, where $\pi \in N$. We will use our first order relations in order to obtain these relations. We need to right multiply our first order relations by a t_j such that t_j is not in the same orbit of t_i. Therefore, we find the orbits of the one point stabilizer.

Stabilizer(Alt(6),1);
Permutation group acting on a set of cardinality 6
Order = 60 = $2^2 * 3 * 5$
(2, 3, 4)
(3, 4, 5)
(4, 5, 6)

Orbits(Stabilizer(Alt(6),1));
{ 1 },
{ 2, 3, 4, 5, 6 }

Our relations are as follows,

$(y * x^{-1} * y * x * y * t)$

$(y * x^{-1} * y * x * y * t^{(y^2 * x^2)})$

$(x * y^{-1} * x^{-1} * y^{-1} * x * t)$

$(x * y^{-1} * x^{-}1 * y^{-1} * x * t^{(y^2 * x)})$

$(y * t)$

$(y * t^{(y^2 * x)})$

$(x * y^{-1} * x^2 * y * x^2 * t)$

$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2 * x^2)})$

$(x * y^{-1} * x * y^{-1} * x^{-1} * t)$

$(x * y^{-1} * x * y^{-1} * x^{-1} * t^{(y^2 * x^3)})$

$(x * y * x * y * x^{-1} * t)$

$(x * y * x * y * x^{-1} * t^{(y^2 * x^3)})$

Our infinite progenitor $2^{*6} : A_6$ factored by second order relations

$G < x, y, t > := Group < x, y, t | x^4, y^3, (x^{-}1, y^{-}1)^2, t^2,$

$x^{-}1 * y^{-}1 * x^{-}2 * y^{-}1 * x^2 * y^{-}1 * x^2 * y^{-}1 * x^{-}1, (y^{-}1 * x^{-}1)^5,$

$t^2,$

$(t, x * y * x^{-}1 * y * x * y),$

$(t, y * x^{-}1 * y * x),$

$(t, x * y * x * y * x^{-}1 * y),$

$(t, t^x),$

$(y * x^{-1} * y * x * y * t * t^y)^{r1},$

$(y * x^{-1} * y * x * y * t^{(y^2 * x^2)} * t)^{r2},$

$(x * y^{-1} * x^2 * y * x^2 * t * t^y)^{r3},$

$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2 * x^2)} * t)^{r4},$

$(y * t * t^y)^{r5},$

$(y * t^{(y^2 * x)} * t)^{r6},$

$(x * y^{-1} * x^2 * y * x^2 * t * t^y)^{r7},$

$(x * y^{-1} * x^2 * y * x^2 * t^{(y^2 * x^2)} * t)^{r8},$

$(x * y^{-1} * x * y^{-1} * x^{-1} * t * t^y)^{r9},$

$(x * y^{-1} * x * y^{-1} * x^{-1} * t^{(y^2 * x^3)} * t)^{r10}$,
$(x * y * x * y * x^{-1} * t * t^y)^{r11}$,
$(x * y * x * y * x^{-1} * t^{(y^2 * x^3)} * t)^{r12}$
$>$;

2.7 Factoring Progenitors Using the Famous Lemma

The motivation behind the Famous Lemma came quite natural. Robert T. Curtis understood that every element of a progenitor, $m^{*^n} : N$ had the form $\pi\omega$, where $\pi \in N$ and ω is a word in the symmetric generators. Then, by factoring by $\pi\omega$ (first order relations), we are saying $\pi\omega = 1$. As a result, we are writing permutations in terms of symmetric generators. Curtis noticed this was switching the outer automorphisms of the free product of $t_{i's}$ into inner automorphisms of its homomorphic image. This lead to the question of whether it was possible to write permutations in N in terms of two symmetriic generators. It turns out you can because of the lemma. Here is the statement of the lemma.

Lemma 2.6. *: (The Famous Lemma) [Cur07]* $N \cap < t_i, t_j > \leq C_N(N_{ij})$, *where* N_{ij} *denotes the two point stbilizer in* N *of* i *and* j.

That is, if you want to write permutations in terms of symmetric generators, then for each two point stabilizer in N, calculate its centralizer in N and write the elements of the centralizer in terms of t_i and t_j. First, we explore the case where the centralizer is not equal to N, then show the case where it unrestricted (equaling N). The Dihedral group of order $2k$ is given by the following presentation, $D_{2k} =< t_i, t_j >= \{t_i^2 = 1, t_j^2 = 1, (t_i t_j)^k = 1\}$. The center of D_{2k} is trivial if k is odd or $(t_i t_j)^{\frac{k}{2}}$, if k is even. After calculating the two point stabilizer in N, find its centralizer in N and write the elements of the centralizer in terms of t_i and t_j in following manner: $(xt_i)^m = 1$, when $2 \nmid m$ and $1^x = 2$ or $(t_i t_j)^m = x$, when $2 \mid m$ and $1^x = 1$ and $2^x = 2$.

2.7.1 Example

Using our progenitor $2^{*6} : A_6$, we calculate the point-stabilizer of 1 and 2.

Stabilizer(Alt(6),[1,2]);

Permutation group acting on a set of cardinality 6

Order = $12 = 2^2 * 3$

(3, 4, 5)

(4, 5, 6)

S:=Stabilizer(Alt(6),[1,2]);

We now calculate the centralizer of the point-stabilizer of two points.

Centralizer(Alt(6),S);

Permutation group acting on a set of cardinality 6 Order = 1

Therefore, the relation will be, $(t_1t_2)^k = 1$, where k is even.

Our progenitor factored by Curtis' Lemma is given below.

$G < x, y, t >:= Group < x, y, t|x^4, y^3, (x^-1, y^-1)^2, t^2, x^-1 * y^-1 * x^-2 * y^-1 * x^2 * y^-1 * x^2 * y^-1 * x^-1, (y^-1 * x^-1)^5, t^2, (t, x * y * x^-1 * y * x * y), (t, y * x^-1 * y * x), (t, x * y * x * y * x^-1 * y), (t, t^x), (t_1t_2)^k = 1 >;$

On the other hand, a natural question that arises is, what happens if the centralizer is N? That is, $N \cap < t_i, t_j >$ is unrestricted. Consider the progenitor $2^{*4} : A_4 =< x, y, t|x^3, y^3, (xy)^2, (t, x) >$, where $t \sim t_1$, $x \sim (2, 3, 4)$, and $y \sim (1, 2, 3)$. Let $N =< x, y >$. Since $S := Stabilizer(Alt(4), [1, 2]) = 1$, this implies $Centralizer(Alt(4), S) = N$. Thus, $N \cap < t_1, t_2 > \leq C_N(N_{12}) = N = A_4$. $N \cap < t_1, t_2 >$ is a subgroup of A_4 that is normalized by the permutation $(1, 2)(3, 4)$. The subgroups of A_4 that are normalized by $(1, 2)(3, 4)$ are e, $< (1, 2)(3, 4) >$, $< (1, 2)(3, 4), (1, 3)(2, 4) >$, and A_4. We desire $G =< t_1, t_2, t_3, t_4 >$ (simple groups are generated by involutions). Now, if $< t_1, t_2 >\geq N \cap < t_1, t_2 >= A_4$, then $< t_1, t_2 >\geq< t_1, t_2, A_4 >=<$

$t_1, t_2, t_3, t_4 >= G$ since $t_1^{A_4} = \{t_1, t_2, t_3, t_4\}$. Thus, $G =< t_1, t_2 >$. Hence, G is dihedral. In order to find 'interesting' groups (simple groups, quotient groups that are simple, etc.), it must be the case that $N \cap < t_1, t_2 >= e$, $N \cap < t_1, t_2 >=< (1,2)(3,4) >$, or $N \cap < t_1, t_2 >=< (1,2)(3,4), (1,3)(2,4) >$. A corollary that follows from the Famous Lemma tells us which relation we should factor our progenitor by to find 'interesting' G.

Corollary 2.7. *[Cur07] (i) If g belongs to N and $i^g = i$ and $j^g = j$, then we should factor the progenitor by the relation $(t_i t_j)^k = g$, for any positive integer k.*
(ii) If g belongs to N and $i^g = j$ and $j^g = i$, then we should factor the progenitor by the relation $(g t_i)^k = 1$, for any odd positive integer k.

Proof: (i) Now $< t_i, t_j >$ is dihedral (since $|t_i| = 2$ and $|t_j| = 2$). Thus, if k is even, Center($< t_i, t_j >$) $=< (t_i t_j)^{(\frac{k}{2})} >$. Then $(t_i t_j)^{(t_i t_j)^{(\frac{k}{2}})} = t_i t_j$. Also, $(t_i, t_j)^\pi = (t_i, t_j)$, where $\pi \in N$ fixed i and j. Thus, if g belongs to N and $i^g = i$ and $j^g = j$, then we should factor the progenitor by the relation $(t_i t_j)^k = g$, for any positive integer k.
(ii) $t_i t_j t_i$, $t_i t_j t_i t_j t_i$, $t_i t_j t_i t_j t_i t_j t_i, \ldots$ interchange t_i and t_j (note: $(t_i t_j)^{t_i t_j t_i} = t_j t_i$, $(t_i t_j)^{t_i t_j t_i t_j t_i} = t_j t_i$, $(t_i t_j)^{t_i t_j t_i t_j t_i t_j t_i} = t_j t_i, \ldots$). Thus, if g belongs to N and $i^g = j$ and $j^g = i$, then we should factor the progenitor by the relation $g = t_i t_j t_i$, $g = t_i t_j t_i t_j t_i$, $g = t_i t_j t_i t_j t_i t_j t_i, \ldots$ Equivalently, we should factor by the relation $(g t_i)^k = 1$, for any odd positive integer k. $\square$

Therefore, based on the above corollary, in the event that $N \cap < t_1, t_2 >= 1$ we should factor by the relation $((1,2)t_1)^m = 1$, where m is a positive odd integer. Or, if it were the case $N \cap < t_1, t_2 >=< (1,2)(3,4) >$ or $N \cap < t_1, t_2 >=< (1,2)(3,4), (1,3)(2,4) >$ we should factor by $(t_1 t_2)^r, (t_1 t_2 t_1)^s, \ldots$ where r and s are positive integers. In summary, first, we find a permutation P in N that interchanges t_i and t_j. Then, find all subgroups of N that are normalized by P. Take all of these subgroups normalized by P, excluding N, then apply the above corollary.

Chapter 3

The Monomial Progenitor

3.1 Introduction

We have been working with involutory progenitors where the symetric generators are of order 2. The complement of the semidirect product permutes the symmetric generators by acting on the indices of the generators of the free product. However, if the symmetric generators had an order higher than 2, say 3, there would be additional automorphisms of the symmetric generators. For example, if we were to fix all the symmetric generators of 3^{*^n} except one, then by sending the one unfixed generator to itself raised to the power of 2, we have created a new automorphism.

Definition 3.1. *[Cur07] Automorphisms which permute the symmetric generators and raise them to such coprime powers are called monomial progenitors.*

A generalized monomial progenitor has the form $m^* :_m N$ where $N \leq S_n$ acts transitively on the n cyclic subgroups and the subscripted m on the colon indicates the action of the control group is monomial. [Cur07]

This is great news. A monomial progenitor is richer (more automorphisms of the symmetric generators) than an involutory progenitor. It can permute the symmetric generators, but also with the added bonus of mapping them to different powers ($t_i \to t_j^m, m < n$). As a result, this gives a monomial progenitor more avenues to search for homomorphic images of finite non-abelian simple groups. This leads us to the thought of how it is we are able to contruct monomial progenitors. It is practical to think that in order to have a monomial progenitor we must first have a monomial representation of our control group.

Definition 3.2. *[Cur07] A monomial representation of a group G is a homomorphism from G into $GL(n, F)$ in which the image of every element of G is an $n \times n$ monomial matrix over F.*

Robert T. Curtis notes that every monomial representation of a group G where G acts transitively on the 1-dimensional subspaces generated by the basis vectors is given by inducing a linear representation of a subgroup H of G. It is the famous mathematician Ferdinand Frobenius that proved this important result. We shall examine this discovery before moving on.

We will now define an induced representation of H up to G (see [Led77] for more details). Frobenious knew that given a finite group G, he could obtain a matrix representation of G. However, he wanted to know if you could take an arbitrary subgroup H of G and some how represent G in terms of H. That is, come up with a matrix representation for a group G by way of an arbitrary subgroup H of G.

Let G be a group where $|G| = g$, and let H be a subgroup of G where $|H| = h$ and $[G : H] = n$. We let $\phi(u),\ u \in H$ be a representation of H with degree q. This implies

$$\phi(h_0)\phi(h_1) = \phi(h_0 h_1),\ h_0, h_1 \in H$$

We now extend our matrix representation to our group G carefully. We note

$$\phi(h_0) = 0,\ \text{if } x \notin H.$$

Let $G = Ht_1 \cup Ht_2 \cup \cdots \cup Ht_n$ be a decomposition of G into right cosets of H where t_i, $1 \leq i \leq n$ are the right transversals of H in G. Then for all $x \in G$ we denote $A(x)$ as an $n \times n$ matrix of degree qn with an array of blocks, each of degree q given by

$$A(x) = \begin{pmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \cdots & \phi(t_1xt_n^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \cdots & \phi(t_2xt_n^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(t_nxt_1^{-1}) & \phi(t_nxt_2^{-1}) & \cdots & \phi(t_nxt_n^{-1}) \end{pmatrix}$$

We want to show that given $x, y \in G$, $A(x)A(y) = A(xy)$. That is, we need to make sure all the blocks are equal. We take an arbitrary block, say, the $(i,j)th$ block and we fix i, j, x, and y. We see that

$$\sum_{r=1}^{n} \phi(t_ixt_r^{-1})\phi(t_ryt_j^{-1}) = \sum_{r=1}^{n} \phi(t_ixt_r^{-1}t_ryt_j^{-1}) = \sum_{r=1}^{n} \phi(t_ixyt_j^{-1}) = \phi(t_ixyt_j^{-1})$$

Naturally, we have two cases that we must analyze, $t_ixyt_j^{-1} \notin H$ and $t_ixyt_j^{-1} \in H$, for all $1 \leq r \leq n$.

i) Suppose $t_ixyt_j^{-1} \notin H$. This implies that $\phi(t_ixyt_j^{-1}) = 0$. However, this implies that $t_ixt_r^{-1} \notin H$ or $t_ryt_j^{-1} \notin H$, for all $1 \leq r \leq n$. Otherwise, we would have a contradiction to our supposition of $t_ixyt_j^{-1} \notin H$. Without loss of generality, suppose $t_ixt_r^{-1} \notin H$, for all $1 \leq r \leq n$. This implies

$$\begin{aligned} & t_ixt_r^{-1} \notin H \\ \iff & t_ix \notin Ht_r, \text{ for all } \ 1 \leq r \leq n. \end{aligned}$$

This is impossible, therefore $t_ixt_r^{-1} \in H$ and $t_ryt_j^{-1} \in H$ for some $1 \leq r \leq n$. Hence, $t_ixyt_j^{-1} \in H$.

ii) Suppose $\nu = t_xyt_j^{-1} \in H$. Then $t_ix \in Ht_s$, for some $1 \leq s \leq n$. Let $u = t_ixt_s^{-1} \in H$, if $r \neq s$, we have $t_ixt_r^{-1} \notin H$. Moreover, the sum on the left, $\sum_{r=1}^{n} \phi(t_ixt_r^{-1})\phi(t_syt_j^{-1})$ reduces to just the case where $r = s$. The term $t_syt_j^{-1} \neq 0$ since

$$t_syt_j^{-1} = t_sx^{-1}t_i^{-1}t_ixyt_j^{-1} = u^{-1}\nu \in H.$$

As a result, this implies

$$\phi(u)\phi(u^{-1}\nu) = \phi(\nu)$$

We have shown for the fixed values i and j, there is one and only one block that is nonzero. Hence, $A(x)$ is a representation of our group G. We say that the representation of G has been induced from the representation $\phi(u)$ of H. This leads us to the idea of monomial representations.

Theorem 3.3. *: (Monomial Character)* $[Isa\ 76]$ *Let χ be a character of G. Then χ is monomial if $\chi = \lambda^G$, where λ is a linear character of some subgroup (not necessarily proper) of G. If χ is monomial then χ is afforded by a monomial representation X of G; that is, each row and columnn of $X(g)$ has exactly one nonzero entry for each $g \in G$. Moreover, the nonzero entries of $X(g)$, for any $g \in G$, are n^{th} roots of unity for some, since G is finite.*

Hence, to achieve a monomial representation, induce a linear character ϕ^H (phi of H), from a subgroup H of our group G up to G. (Note: sometimes we will write just ϕ in place of ϕ^H but with the same intention. If ϕ is used outside of being a linear character of a subgroup H, it will be made explicit.) Continuing, the way in which you define how it is you induce your linear character from your subgroup H up to G is important. This leads us to the following.

Definition 3.4. *[Led77] (Induced Character) Let G be a group and let $H \leq G$ be a subgroup. Let χ be a character of H and let $\chi : G \to \mathbb{C}$ be defined by the following formula*

$$\dot{\chi}(g) = \begin{cases} \chi(g), & g \in H \\ 0 & g \notin H \end{cases}$$

Definition 3.5. *[Led77] (Formula for Induced Character) Let G be a finite group and H be a subgroup such that $[G : H] = \frac{|G|}{|H|} = n$. Let $C_\alpha, \alpha = 1, 2, \cdots, m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, \cdots, m$. Let χ be a character of H and ϕ^G be the* *character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by*

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{C_\alpha \cap H} \phi(\omega), \alpha = 1, 2, \cdots, m.$$

In other words, it is important that $\phi^H = \phi^G$. This is accomplished by making sure ϕ^H yields all the same values as ϕ^G on the m classes of G. Given that ϕ^H is faithful, we can find the monomial matrix representation afforded by the formula introduced previously.

Definition 3.6. *[Led77] (Formula for Monomial Representation) Let ϕ be a linear character of the subgroup H of index n in G and let $G = H \cup Ht_1 \cup Ht_2 \cup \cdots \cup Ht_n$. Let $x \in G$, then the monomial representation of G has the formula.*

$$A(x) = \begin{pmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \cdots & \phi(t_1xt_n^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \cdots & \phi(t_2xt_n^{-1}) \\ \cdots & \cdots & \ddots & \vdots \\ \phi(t_nxt_1^{-1}) & \phi(t_nxt_2^{-1}) & \cdots & \phi(t_nxt_n^{-1}) \end{pmatrix}$$

After computing our monomial matrices on the generators of a group G, we then convert our monomial matrix representation into a permutation representation. The goal is to see where the symmetric generators ($< t_1 > \times < t_2 > \cdots < t_n >$) are being sent. That is, we are figuring out the automorphism of the symmetric generators. To do this, take the place values $a_{i,j}$ within the monomial matrix where i stands for the i^{th} row and j stands for the j^{th} column. Therefore, the automorphisms of the symmetric generators are given by

$$a_{i,j} = 1 \text{ if the automorphism takes } t_i \to t_j$$
$$a_{i,j} = n \text{ if the automorphism takes } t_i \to t_j^n$$

After finding out what the permutation representation is, a presentation can be generated in the regular fashion. Although, there is a slight difference when constructing a monomial progenitor. Typically, one of the symmetric generators is stabilized, then note that the symmetric generator commutes with the generators of the stabilizer.

Depending on the number of conjugates the fixed symmetric generator has lets us know the total number of symmetric generators. However, for a monomial presentation one of the sets of symmetric generators must be fixed, say, $< t_i >$, and be normalized. In doing so, this will explicitly say within the presentation where all the members of the set $< t_i >$ are being sent. This is an important part of the presentation; it is the defining part of the presentation that makes it monomial. Lastly, factoring the monomial progenitor by relations will give way to finite non-abelian simple groups as homomorphic images. By way of example, we will go through this process given an actual group N as our starting point.

3.2 Constructing a Monomial Progenitor Using M_{11}

Construct the monomial progenitor of $3^{*^{11}} :_m M_{11}$. Notice that our control group N is M_{11}. A presentation for M_{11} is given by $< x, y | x^2, y^4, y^{-1}xy^{-2}xy^{-2}xy^2xy^2xy^2xy^{-1}$, $xyxyxy^{-1}xy^{-1}xyxy^{-1}xy^2xyxy^{-1}, xy^{-2}xy^{-1}xyxy^{-2}xy^{-1}xyxy^2xy^{-1}x * y, (xy^{-1})^{11} >$. We must choose a subgroup H of N such that the index of H in N has the same degree as the irreducible character ϕ^N (phi of N). This will ensure that the linear character ϕ^H induced up to N is irreducible. By analyzing the character table of M_{11} in Table 5.1, notice that N has characters $\chi.1, \chi.2, \cdots, \chi.10$. We will select $\chi^5 = \phi^N$ from the character table of N. Note that the degree of ϕ^N is 11. Therefore the subgroup H of N will be such that $[N : H] = \frac{|N|}{|H|} = \frac{7920}{|H|} = 11$. That is, our subgroup H must be of order 720. M_{10} is a proper subgroup of M_{11} and has index 11 since $[M_{11} : M_{10}] = \frac{7920}{720} = 11$. Hence, our monomial matrix representation will be represented by 11×11 matrices. We will select $\chi.2 = \phi^H$ from Table 5.2 and induce up to N. We must make sure we have a faithful monomial representation. In order to do so, we invoke the formula for an induced character.

Table 3.1: Character Table of N

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
$\chi.1$	1	1	1	1	1	1	1	1	1	1
$\chi.2$	10	2	1	2	0	-1	0	0	-1	-1
$\chi.3$	10	-2	1	0	0	1	$\mathbb{Z}_1$	-$\mathbb{Z}_1$	-1	-1
$\chi.4$	10	-2	1	0	0	1	-$\mathbb{Z}_1$	$\mathbb{Z}_1$	-1	-1
$\chi.5$	11	3	2	-1	1	0	-1	-1	0	0
$\chi.6$	16	0	-2	0	1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2\#2$
$\chi.7$	16	0	-2	0	1	0	0	0	$\mathbb{Z}_2\#2$	$\mathbb{Z}_2$
$\chi.8$	44	4	-1	0	-1	1	0	0	0	0
$\chi.9$	45	-3	0	1	0	0	-1	-1	1	1
$\chi.10$	55	-1	1	-1	0	-1	1	1	0	0

Z_1 is the primitive fifth root of unity, Z_2 is the primitive eighth root of unity, and # denotes algebraic conjugation.

Table 3.2: Character Table of N

χ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
$\chi.1$	1	1	1	1	1	1	1	1
$\chi.2$	1	1	1	1	-1	1	-1	-1
$\chi.3$	9	1	0	1	-1	-1	1	1
$\chi.4$	9	1	0	1	1	-1	-1	-1
$\chi.5$	10	2	1	-2	0	0	0	0
$\chi.6$	10	-2	1	0	0	0	$\mathbb{Z}_1$	-$\mathbb{Z}_1$
$\chi.7$	10	-2	1	0	0	0	-$\mathbb{Z}_1$	$\mathbb{Z}_1$
$\chi.8$	16	0	-2	0	0	1	0	0

We now omit the information from Tables 3.1 and 3.2 that we will not need for induction. Tables 3.3 and 3.4 will be the refined tables for N and H, respectively.

Table 3.3: $\chi_{.5}$ of N

ϕ^N	Class	Size	Class Representative
11	C_1	1	Id(N)
3	C_2	165	(3, 6)(4, 50)(5, 38)(7, 10)(8, 9)(11, 53)(12, 19)(13, 21)(14, 39)(15, 17)(16, 32)(20, 45)(22, 27)(23, 37)(24, 43)(25, 31)(26, 34)(28, 44)(29, 46)(33, 48)(35, 51)(36, 40)(41, 47)(54, 55)
2	C_3	440	(2, 4, 50)(3, 19, 46)(5, 39, 9)(6, 29, 12)(7, 52, 10)(8, 14, 38)(11, 51, 21)(13, 35, 53)(15, 55, 48)(16, 40, 45)(17, 33, 54)(18, 23, 37)(20, 36, 32)(22, 30, 27)(24, 43, 42)(25, 41, 34)(26, 47, 31)(28, 44, 49)
-1	C_4	990	(2, 49)(3, 35, 6, 51)(4, 17, 50, 15)(5, 33, 38, 48)(7, 13, 10, 21)(8, 44, 9, 28)(11, 29, 53, 46)(12, 43, 19, 24)(14, 54, 39, 55)(16, 36, 32, 40)(20, 22, 45, 27)(23, 47, 37, 41)(25, 34, 31, 26)(42, 52)
1	C_5	1584	(1, 40, 45, 30, 27)(2, 33, 13, 11, 26)(3, 55, 35, 37, 28)(4, 39, 52, 47, 19)(5, 50, 49, 12, 15)(6, 24, 53, 29, 8)(7, 23, 25, 46, 21)(9, 31, 51, 42, 54)(10, 48, 34, 43, 44)(14, 38, 18, 17, 41)(16, 32, 36, 22, 20)
0	C_6	1320	(2, 44, 4, 49, 50, 28)(3, 6, 19, 29, 46, 12)(5, 17, 39, 33, 9, 54)(7, 10, 52)(8, 48, 14, 15, 38, 55)(11, 13, 51, 35, 21, 53)(16, 47, 40, 31, 45, 26)(18, 22, 23, 30, 37, 27)(20, 25, 36, 41, 32, 34)(24, 42, 43)
-1	C_7	990	(2, 42, 49, 52)(3, 8, 51, 28, 6, 9, 35, 44)(4, 11, 15, 46, 50, 53, 17, 29)(5, 43, 48, 12, 38, 24, 33, 19)(7, 39, 21, 54, 10, 14, 13, 55)(16, 37, 40, 47, 32, 23, 36, 41)(18, 30)(20, 26, 27, 31, 45, 34, 22, 25)
-1	C_8	990	(2, 42, 49, 52)(3, 9, 51, 44, 6, 8, 35, 28)(4, 53, 15, 29, 50, 11, 17, 46)(5, 24, 48, 19, 38, 43, 33, 12)(7, 14, 21, 55, 10, 39, 13, 54)(16, 23, 40, 41, 32, 37, 36, 47)(18, 30)(20, 34, 27, 25, 45, 26, 22, 31)
0	C_9	720	(1, 12, 13, 38, 22, 2, 50, 41, 53, 14, 11)(3, 37, 43, 54, 23, 32, 39, 48, 45, 15, 28)(4, 10, 17, 16, 31, 6, 21, 52, 40, 24, 47)(5, 46, 8, 25, 49, 18, 9, 30, 35, 20, 44)(7, 27, 36, 29, 42, 33, 51, 19, 34, 26, 55)
0	C_{10}	720	(1, 13, 22, 50, 53, 11, 12, 38, 2, 41, 14)(3, 43, 23, 39, 45, 28, 37, 54, 32, 48, 15)(4, 17, 31, 21, 40, 47, 10, 16, 6, 52, 24)(5, 8, 49, 9, 35, 44, 46, 25, 18, 30, 20)(7, 36, 42, 51, 34, 55, 27, 29, 33, 19, 26)

Table 3.4: $\chi_{.2}$ of H

ϕ^H	Class	Size	Class Representative
1	D_1	1	Id(H)
1	D_2	45	(1, 16)(2, 33)(3, 25)(4, 43)(5, 35)(6, 9)(7, 13)(8, 31)(10, 39)(11, 47)(12, 53)(14, 21)(17, 42)(18, 54)(19, 41)(20, 27)(22, 36)(23, 50)(24, 37)(26, 49)(28, 38)(29, 52)(40, 45)(44, 51)
1	D_3	80	(1, 44, 50)(2, 6, 23)(3, 16, 45)(4, 11, 40)(5, 37, 15)(7, 14, 55)(8, 22, 33)(9, 17, 43)(12, 48, 25)(13, 18, 39)(19, 32, 38)(20, 41, 24)(21, 54, 30)(26, 35, 28)(27, 46, 49)(29, 47, 31)(34, 36, 42)(51, 53, 52)
11	D_4	90	(1, 19, 44, 48)(2, 42, 27, 34)(3, 35, 47, 40)(4, 9, 26, 43)(5, 22, 8, 15)(6, 32, 49, 12)(7, 18, 14, 21)(11, 31, 28, 45)(13, 30, 54, 39)(16, 24, 29, 20)(17, 41)(23, 25, 46, 38)(33, 52, 37, 51)(36, 50)
-1	D_5	180	(1, 25, 28, 20)(2, 6, 22, 41)(3, 24, 49, 8)(4, 15)(5, 45, 31, 43)(9, 52, 38, 35)(10, 21, 39, 55)(12, 53, 33, 44)(13, 30, 18, 14)(16, 34, 19, 50)(17, 48, 27, 23)(26, 51, 42, 36)(29, 32, 37, 46)(40, 47)
1	D_6	144	(1, 17, 12, 33, 50)(2, 53, 42, 16, 23)(3, 34, 25, 43, 4)(5, 20, 27, 35, 15)(6, 45, 11, 31, 51)(7, 39, 10, 13, 30)(8, 47, 40, 9, 44)(14, 55, 21, 54, 18)(19, 26, 46, 49, 41)(22, 48, 36, 52, 29)(24, 37, 28, 32, 38)
-1	D_7	90	(1, 34, 19, 2, 44, 42, 48, 27)(3, 28, 35, 45, 47, 11, 40, 31)(4, 5, 9, 22, 26, 8, 43, 15)(6, 52, 32, 37, 49, 51, 12, 33)(7, 30, 18, 54, 14, 39, 21, 13)(10, 55)(16, 25, 24, 46, 29, 38, 20, 23)(17, 36, 41, 50)
-1	D_8	90	(1, 42, 19, 27, 44, 34, 48, 2)(3, 11, 35, 31, 47, 28, 40, 45)(4, 8, 9, 15, 26, 5, 43, 22)(6, 51, 32, 33, 49, 52, 12, 37)(7, 39, 18, 13, 14, 30, 21, 54)(10, 55)(16, 38, 24, 23, 29, 25, 20, 46)(17, 36, 41, 50)

3.3 Inducing Linear Character ϕ^H

We will now use the induced character formula given by $\phi_\alpha^G = \frac{n}{h_\alpha}\sum_{w\in H\cap C_\alpha}\phi(w)$, where $n = \frac{|G|}{|H|} = \frac{9720}{720} = 11$ and h_α is the order of G's conjugacy class with repect to α. If $\omega \in H \cap C_\alpha$, then $\phi(\omega)$ is assigned the value with respect to the conjugacy classes of H's character table. If $\omega \notin H \cap C_\alpha$, then $\phi(\omega) = 0$. We will now verify that we get the same values of G's character table with respect to ϕ^G.

$$\begin{aligned}
\phi_1^G &= \frac{11}{1}\sum_{w\in H\cap C_1}\phi(w) \\
&= 11\sum_{w\in H\cap C_1}\phi(1) \\
&= 11(\phi(1)) \\
&= 11(1) \\
&= 11
\end{aligned}$$

$$\begin{aligned}
\phi_2^G &= \frac{11}{165}\sum_{w\in H\cap C_2}\phi(w) \\
&= \frac{11}{165}\sum_{w\in H\cap C_2}(\phi(1,16)(2,33)(3,25)(4,43)(5,35)(6,9)(7,13)(8,31)(10,39)(11,47) \\
&(12,53)(14,21)(17,42)(18,54)(19,41)(20,27)(22,36)(23,50)(24,37)(26,49)(28,38) \\
&(29,52)(40,45)(44,51)) \\
&= \frac{11}{165}(45(1)) \\
&= \frac{495}{165} \\
&= 3
\end{aligned}$$

$$
\begin{aligned}
\phi_3^G &= \frac{11}{165}\sum_{w\in H\cap C_3}\phi(w)\\
&= \frac{11}{440}\sum_{w\in H\cap C_3}(\phi(1,19,44,48)(2,42,27,34)(3,35,47,40)(4,9,26,43)(5,22,8,15)\\
&(6,32,49,12)(7,18,14,21)(11,31,28,45)(13,30,54,39)(16,24,29,20)(17,41)\\
&(23,25,46,38)(33,52,37,51)(36,50))\\
&= \frac{11}{440}(80(1))\\
&= \frac{880}{440}\\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\phi_4^G &= \frac{11}{165}\sum_{w\in H\cap C_4}\phi(w)\\
&= \frac{11}{990}\sum_{w\in H\cap C_4}(\phi(1,19,44,48)(2,42,27,34)(3,35,47,40)(4,9,26,43)(5,22,8,15)\\
&(6,32,49,12)(7,18,14,21)(11,31,28,45)(13,30,54,39)(16,24,29,20)(17,41)\\
&(23,25,46,38)(33,52,37,51)(36,50)+\phi((1,25,28,20)(2,6,22,41)(3,24,49,8)(4,15)\\
&(5,45,31,43)(9,52,38,35)(10,21,39,55)(12,53,33,44)(13,30,18,14)(16,34,19,50)\\
&(17,48,27,23)(26,51,42,36)(29,32,37,46)(40,47)))\\
&= \frac{11}{990}(90(1)+180(-1))\\
&= \frac{11}{990}(-90)\\
&= \frac{-990}{990}\\
&= -1
\end{aligned}
$$

$$\begin{aligned}
\phi_5^G &= \frac{11}{1584} \sum_{w \in H \cap C_5} \phi(w) \\
&= \frac{11}{1584} \sum_{w \in H \cap C_5} (\phi(1,17,12,33,50)(2,53,42,16,23)(3,34,25,43,4)(5,20,27,35,15) \\
&(6,45,11,31,51)(7,39,10,13,30)(8,47,40,9,44)(14,55,21,54,18)(19,26,46,49,41) \\
&(22,48,36,52,29)(24,37,28,32,38)) \\
&= \frac{11}{1584}(144(1)) \\
&= \frac{1584}{1584} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\phi_6^G &= \frac{11}{165} \sum_{w \in H \cap C_6} \phi(w) \\
&= \frac{11}{1320} \sum_{w \in H \cap C_6} \phi(0) \\
&= \frac{11}{1320}(0) \\
&= 0
\end{aligned}$$

Since $H \cap C_6 = \emptyset$.

$$\begin{aligned}
\phi_7^G &= \frac{11}{990} \sum_{w \in H \cap C_7} \phi(w) \\
&= \frac{11}{990} \sum_{w \in H \cap C_7} (\phi(1,42,19,27,44,34,48,2)(3,11,35,31,47,28,40,45)(4,8,9,15,26,5,43,22) \\
&(6,51,32,33,49,52,12,37)(7,39,18,13,14,30,21,54)(10,55)(16,38,24,23,29,25,20,46) \\
&(17,36,41,50)) \\
&= \frac{11}{990}(90(-1)) \\
&= -1
\end{aligned}$$

$$
\begin{aligned}
\phi_8^G &= \frac{11}{990} \sum_{w \in H \cap C_8} \phi(w) \\
&= \frac{11}{990} \sum_{w \in H \cap C_8} (\phi(1,34,19,2,44,42,48,27)(3,28,35,45,47,11,40,31)(4,5,9,22,26,8,43,15) \\
&(6,52,32,37,49,51,12,33)(7,30,18,54,14,39,21,13)(10,55)(16,25,24,46,29,38,20,23) \\
&(17,36,41,50)) \\
&= \frac{11}{990}(90(-1)) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\phi_9^G &= \frac{11}{165} \sum_{w \in H \cap C_9} \phi(w) \\
&= \frac{11}{720} \sum_{w \in H \cap C_9} \phi(0) \\
&= \frac{11}{720}(0) \\
&= 0
\end{aligned}
$$

Since $H \cap C_9 = \emptyset$.

$$
\begin{aligned}
\phi_{10}^G &= \frac{11}{165} \sum_{w \in H \cap C_{10}} \phi(w) \\
&= \frac{11}{720} \sum_{w \in H \cap C_{10}} \phi(0) \\
&= \frac{11}{720} \\
&= 0
\end{aligned}
$$

Since $H \cap C_10 = \emptyset$.

Hence, the induction from H up to G is $\phi \uparrow_H^G = 11, 3, 2, -1, 1, 0, -1, -1, 0, 0$. We see that we get the same values for $\chi.5$ of M_{11}'s character by inducing $\phi^{M_{10}}$ up to G.

3.4 Verifying the Monomial Matrix Representation

Moving forward, we need to obtain a monomial matrix representation for M_{11}. Let $G = Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4 \cup Ht_5 \cup Ht_6 \cup Ht_7 \cup Ht_8 \cup Ht_9 \cup Ht_{10} \cup Ht_{11}$ be a decomposition of right cosets of H, where the t_i's, $1 \leq i \leq 11$ are the transversals of H in G. We then take the generators of G and compute the formula for monomial representation. M_{11} has two generators; this means we will have two matrices $A(x)$ and $A(y)$. We now define our generators of G and our transversals. Let $x = (2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)(39,46,50,52)(43,49,53,55)(51,54)$ and $y = (1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)$. Now let $t_1 = e$, $t_2 = (2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)(39,46,50,52)(43,49,53,55)(51,54)$, $t_3 = (1,2,4,8,13,23,40,12,3,5,7)(6,11,20,35,45,15,21,37,14,26,19)(9,16,29,38,22,39,41,51,55,34,28)(10,18,25,43,49,50,48,27,36,44,33)(17,32,24,42,52,30,31,46,53,54,47)$, $t_4 = (2,5)(3,7)(4,9)(6,12)(8,14)(10,19)(11,21)(13,24)(15,28)(16,30)(17,33)(18,34)(20,36)(22,40)(23,41)(25,37)(27,45)(31,47)(35,42)(39,50)(43,53)(44,48)(46,52)(49,55)$, $t_5 = (1,2,5)(3,7,4,11,21,9)(6,16,39,53,34,14)(8,18,43,50,30,12)(10,26,19,13,32,24)(15,37,25,28,20,36)(17,42,27,44,52,41)(22,31,51,47,40,29)(23,46,48,45,35,33)(49,54,55)$, $t_6 = (2,7,5,3)(4,12,9,6)(8,19,14,10)(11,28,21,15)(13,33,24,17)(16,40,30,22)(18,37,34,25)(20,45,36,27)(23,47,41,31)(29,38)(35,48,42,44)(39,52,50,46)(43,55,53,49)(51,54)$, $t_7 = (1,2,7,5,4,16,31,17,10,6,3)(8,26,19,18,28,21,20,44,27,15,9)(11,37,43,54,47,46,30,29,38,22,12)(13,42,45,36,35,52,53,49,34,25,14)(23,51,55,50,41,40,39,48,33,32,24)$, $t_8 = (1,3)(4,22)(5,6)(8,26)(9,10)(11,18)(12,15)(13,48)(14,17)(16,41)(19,25)(20,42)(21,27)(23,54)(24,31)(30,38)(32,33)(35,39)(36,44)(37,49)(40,46)(43,51)(47,50)(52,55)$, $t_9 = (1,5,9,14,24,41,22,6,7,2,3)(4,30,29,38,40,50,23,51,49,18,15)(8,26,10,12,21,36,42,27,28,11,25)(13,35,46,16,47,52,43,54,31,33,32)(17,19,34,37,53,55,39,44,45,20,48)$, $t_{10} = (1,14,22,2,5,24,$

$6, 3, 9, 41, 7)(4, 38, 23, 18, 30, 40, 51, 15, 29, 50, 49)(8, 12, 42, 11, 26, 21, 27, 25, 10, 36, 28)$ $(13, 16, 43, 33, 35, 47, 54, 32, 46, 52, 31)(17, 37, 39, 20, 19, 53, 44, 48, 34, 55, 45)$, and $t_{11} =$ $(1, 7, 3)(2, 5, 12, 28, 15, 6)(4, 40, 52, 49, 25, 10)(8, 26, 14, 33, 32, 17)(9, 19, 37, 55, 46, 22)$ $(11, 34, 18, 21, 45, 27)(13, 44, 20, 35, 50, 31)(16, 23, 54, 41, 30, 38)(24, 47, 39, 42, 36, 48)$ $(43, 51, 53)$.

The monomial matrices obtained are given below. We shall prove the nonzero entries for both $A(x)$ and $A(y)$, but we leave out the zero entries.

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For $A(x)$ we have the following

$\phi(t_1xt_2^{-1}) = \phi(e(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,33)$ $(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)(39,$ $46,50,52)(43,49,53,55)(51,54)(2,7,5,3)(4,12,9,6)(8,19,14,10)(11,28,21,15)(13,$ $33,24,17)(16,40,30,22)(18,37,34,25)(20,45,36,27)(23,47,41,31)(29,38)(35,48,$ $42,44)(39,52,50,46)(43,55,53,49)(51,54)) = \phi(e) = 1$

$\phi(t_2xt_4^{-1}) = \phi((2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,33)$ $(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)(39,$ $46,50,52)(43,49,53,55)(51,54)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)$ $(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,$ $44,42,48)(39,46,50,52)(43,49,53,55)(51,54)(2,5)(3,7)(4,9)(6,12)(8,14)(10,19)(11,$ $21)(13,24)(15,28)(16,30)(17,33)(18,34)(20,36)(22,40)(23,41)(25,37)(27,45)(31,47)$ $(35,42)(39,50)(43,53)(44,48)(46,52)(49,55)) = \phi(e) = 1$

$\phi(t_3xt_3^{-1}) = \phi((1,2,4,8,13,23,40,12,3,5,7)(6,11,20,35,45,15,21,37,14,26,$ $19)(9,16,29,38,22,39,41,51,55,34,28)(10,18,25,43,49,50,48,27,36,44,33)$ $(17,32,24,42,52,30,31,46,53,54,47)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,$ $28)(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)$

$(35,44,42,48)(39,46,50,52)(43,49,53,55)(51,54)(1,7,5,3,12,40,23,13,8,4,2)$
$(6,19,26,14,37,21,15,45,35,20,11)(9,28,34,55,51,41,39,22,38,29,16)(10,33,$
$44,36,27,48,50,49,43,25,18)(17,47,54,53,46,31,30,52,42,24,32)) = \phi(e) = 1$

$\phi(t_4xt_6^{-1}) = \phi((2,5)(3,7)(4,9)(6,12)(8,14)(10,19)(11,21)(13,24)(15,28)(16,30)$
$(17,33)(18,34)(20,36)(22,40)(23,41)(25,37)(27,45)(31,47)(35,42)(39,50)(43,53)$
$(44,48)(46,52)(49,55)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,$
$33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)$
$(39,46,50,52)(43,49,53,55)(51,54)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)$
$(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,$
$44,42,48)(39,46,50,52)(43,49,53,55)(51,54)) = \phi(e) = 1$

$\phi(t_5xt_5^{-1}) = \phi((1,2,5)(3,7,4,11,21,9)(6,16,39,53,34,14)(8,18,43,50,30,12)$
$(10,26,19,13,32,24)(15,37,25,28,20,36)(17,42,27,44,52,41)(22,31,51,47,40,$
$29)(23,46,48,45,35,33)(49,54,55)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)$
$(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,$
$44,42,48)(39,46,50,52)(43,49,53,55)(51,54)(1,5,2)(3,9,21,11,4,7)(6,14,34,53,$
$39,16)(8,12,30,50,43,18)(10,24,32,13,19,26)(15,36,20,28,25,37)(17,41,52,44,$
$27,42)(22,29,40,47,51,31)(23,33,35,45,48,46)(49,55,54)) = \phi(e) = -1$

$\phi(t_6xt_1^{-1}) = \phi((2,7,5,3)(4,12,9,6)(8,19,14,10)(11,28,21,15)(13,33,24,17)$
$(16,40,30,22)(18,37,34,25)(20,45,36,27)(23,47,41,31)(29,38)(35,48,42,44)(39,$
$52,50,46)(43,55,53,49)(51,54)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,$
$17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,$
$48)(39,46,50,52)(43,49,53,55)(51,54)e) = \phi(e) = 1$

$\phi(t_7xt_8^{-1}) = \phi((1,2,7,5,4,16,31,17,10,6,3)(8,26,19,18,28,21,20,44,27,15,9)$
$(11,37,43,54,47,46,30,29,38,22,12)(13,42,45,36,35,52,53,49,34,25,14)(23,51,$
$55,50,41,40,39,48,33,32,24)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28$
$)(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,$
$44,42,48)(39,46,50,52)(43,49,53,55)(51,54)(1,3)(4,22)(5,6)(8,26)(9,10)(11,18)$
$(12,15)(13,48)(14,17)(16,41)(19,25)(20,42)(21,27)(23,54)(24,31)(30,38)(32,33)$
$(35,39)(36,44)(37,49)(40,46)(43,51)(47,50)(52,55)) = \phi(e) = 1$

$\phi(t_8xt_9^{-1}) = \phi((1,3)(4,22)(5,6)(8,26)(9,10)(11,18)(12,15)(13,48)(14,17)$
$(16,41)(19,25)(20,42)(21,27)(23,54)(24,31)(30,38)(32,33)(35,39)(36,44)(37,49)$
$(40,46)(43,51)(47,50)(52,55)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,$

$17, 24, 33)(16, 22, 30, 40)(18, 25, 34, 37)(20, 27, 36, 45)(23, 31, 41, 47)(29, 38)(35, 44, 42, 48)(39, 46, 50, 52)(43, 49, 53, 55)(51, 54)(1, 3, 2, 7, 6, 22, 41, 24, 14, 9, 5)(4, 15, 18, 49, 51, 23, 50, 40, 38, 29, 30)(8, 25, 11, 28, 27, 42, 36, 21, 12, 10, 26)(13, 32, 33, 31, 54, 43, 52, 47, 16, 46, 35)(17, 48, 20, 45, 44, 39, 55, 53, 37, 34, 19)) = \phi(e) = 1$

$\phi(t_9xt_{11}^{-1}) = \phi((1, 5, 9, 14, 24, 41, 22, 6, 7, 2, 3)(4, 30, 29, 38, 40, 50, 23, 51, 49, 18, 15)(8, 26, 10, 12, 21, 36, 42, 27, 28, 11, 25)(13, 35, 46, 16, 47, 52, 43, 54, 31, 33, 32)(17, 19, 34, 37, 53, 55, 39, 44, 45, 20, 48)(2, 3, 5, 7)(4, 6, 9, 12)(8, 10, 14, 19)(11, 15, 21, 28)(13, 17, 24, 33)(16, 22, 30, 40)(18, 25, 34, 37)(20, 27, 36, 45)(23, 31, 41, 47)(29, 38)(35, 44, 42, 48)(39, 46, 50, 52)(43, 49, 53, 55)(51, 54)(1, 3, 7)(2, 6, 15, 28, 12, 5)(4, 10, 25, 49, 52, 40)(8, 17, 32, 33, 14, 26)(9, 22, 46, 55, 37, 19)(11, 27, 45, 21, 18, 34)(13, 31, 50, 35, 20, 44)(16, 38, 30, 41, 54, 23)(24, 48, 36, 42, 39, 47)(43, 53, 51)) = \phi(e) = 1$

$\phi(t_{10}xt_{10}^{-1}) = \phi((1, 14, 22, 2, 5, 24, 6, 3, 9, 41, 7)(4, 38, 23, 18, 30, 40, 51, 15, 29, 50, 49)(8, 12, 42, 11, 26, 21, 27, 25, 10, 36, 28)(13, 16, 43, 33, 35, 47, 54, 32, 46, 52, 31)(17, 37, 39, 20, 19, 53, 44, 48, 34, 55, 45)(2, 3, 5, 7)(4, 6, 9, 12)(8, 10, 14, 19)(11, 15, 21, 28)(13, 17, 24, 33)(16, 22, 30, 40)(18, 25, 34, 37)(20, 27, 36, 45)(23, 31, 41, 47)(29, 38)(35, 44, 42, 48)(39, 46, 50, 52)(43, 49, 53, 55)(51, 54)(1, 7, 41, 9, 3, 6, 24, 5, 2, 22, 14)(4, 49, 50, 29, 15, 51, 40, 30, 18, 23, 38)(8, 28, 36, 10, 25, 27, 21, 26, 11, 42, 12)(13, 31, 52, 46, 32, 54, 47, 35, 33, 43, 16)(17, 45, 55, 34, 48, 44, 53, 19, 20, 39, 37)) = \phi(e) = -1$

$\phi(t_{11}xt_7^{-1}) = \phi((1, 7, 3)(2, 5, 12, 28, 15, 6)(4, 40, 52, 49, 25, 10)(8, 26, 14, 33, 32, 17)(9, 19, 37, 55, 46, 22)(11, 34, 18, 21, 45, 27)(13, 44, 20, 35, 50, 31)(16, 23, 54, 41, 30, 38)(24, 47, 39, 42, 36, 48)(43, 51, 53)(2, 3, 5, 7)(4, 6, 9, 12)(8, 10, 14, 19)(11, 15, 21, 28)(13, 17, 24, 33)(16, 22, 30, 40)(18, 25, 34, 37)(20, 27, 36, 45)(23, 31, 41, 47)(29, 38)(35, 44, 42, 48)(39, 46, 50, 52)(43, 49, 53, 55)(51, 54)(1, 3, 6, 10, 17, 31, 16, 4, 5, 7, 2)(8, 9, 15, 27, 44, 20, 21, 28, 18, 19, 26)(11, 12, 22, 38, 29, 30, 46, 47, 54, 43, 37)(13, 14, 25, 34, 49, 53, 52, 35, 36, 45, 42)(23, 24, 32, 33, 48, 39, 40, 41, 50, 55, 51)) = \phi(e) = 1$

For $A(y)$ we have the following

$\phi(t_1xt_1^{-1}) = \phi((e(1, 2)(3, 4)(6, 8)(9, 11)(10, 13)(12, 16)(14, 18)(15, 20)(17, 23)(19, 26)(22, 29)(24, 32)(27, 35)(28, 37)(30, 39)(31, 40)(33, 42)(34, 43)(41, 46)(44, 45)(47, 51)(48, 52)(50, 53)(54, 55)e)) = \phi(e) = 1$

$\phi(t_2xt_3^{-1}) = \phi((2,3,5,7)(4,6,9,12)(8,10,14,19)(11,15,21,28)(13,17,24,33)$
$(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,38)(35,44,42,48)$
$(39,46,50,52)(43,49,53,55)(51,54)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)$
$(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)$
$(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(1,7,5,3,12,40,23,13,8,4,2)(6,19,$
$26,14,37,21,15,45,35,20,11)(9,28,34,55,51,41,39,22,38,29,16)(10,33,44,$
$36,27,48,50,49,43,25,18)(17,47,54,53,46,31,30,52,42,24,32)) = \phi(e) = 1$

$\phi(t_3xt_2^{-1}) = \phi((1,2,4,8,13,23,40,12,3,5,7)(6,11,20,35,45,15,21,37,14,$
$26,19)(9,16,29,38,22,39,41,51,55,34,28)(10,18,25,43,49,50,48,27,36,44,33)$
$(17,32,24,42,52,30,31,46,53,54,47)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)$
$(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)$
$(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(2,7,5,3)(4,12,9,6)(8,19,14,10)$
$(11,28,21,15)(13,33,24,17)(16,40,30,22)(18,37,34,25)(20,45,36,27)(23,47,41,31)$
$(29,38)(35,48,42,44)(39,52,50,46)(43,55,53,49)(51,54)) = \phi(e) = 1$

$\phi(t_4xt_5^{-1}) = \phi((2,5)(3,7)(4,9)(6,12)(8,14)(10,19)(11,21)(13,24)(15,28)$
$(16,30)(17,33)(18,34)(20,36)(22,40)(23,41)(25,37)(27,45)(31,47)(35,42)(39,50)$
$(43,53)(44,48)(46,52)(49,55)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)(15,20)$
$(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)(41,46)$
$(44,45)(47,51)(48,52)(50,53)(54,55)(1,5,2)(3,9,21,11,4,7)(6,14,34,53,39,16)$
$(8,12,30,50,43,18)(10,24,32,13,19,26)(15,36,20,28,25,37)(17,41,52,44,27,42)$
$(22,29,40,47,51,31)(23,33,35,45,48,46)(49,55,54)) = \phi(e) = 1$

$\phi(t_5xt_4^{-1}) = \phi((1,2,5)(3,7,4,11,21,9)(6,16,39,53,34,14)(8,18,43,50,30,12)$
$(10,26,19,13,32,24)(15,37,25,28,20,36)(17,42,27,44,52,41)(22,31,51,47,40,29)$
$(23,46,48,45,35,33)(49,54,55)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)(15,20)$
$(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)(41,46)$
$(44,45)(47,51)(48,52)(50,53)(54,55)(2,5)(3,7)(4,9)(6,12)(8,14)(10,19)(11,21)$
$(13,24)(15,28)(16,30)(17,33)(18,34)(20,36)(22,40)(23,41)(25,37)(27,45)(31,47)$
$(35,42)(39,50)(43,53)(44,48)(46,52)(49,55)) = \phi(e) = 1$

$\phi(t_6xt_7^{-1}) = \phi((2,7,5,3)(4,12,9,6)(8,19,14,10)(11,28,21,15)(13,33,24,17)$ $(16,40,30,22)(18,37,34,25)(20,45,36,27)(23,47,41,31)(29,38)(35,48,42,44)(39,$ $52,50,46)(43,55,53,49)(51,54)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)(15,20)$ $(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)(41,46)$ $(44,45)(47,51)(48,52)(50,53)(54,55)(1,3,6,10,17,31,16,4,5,7,2)(8,9,15,27,$ $44,20,21,28,18,19,26)(11,12,22,38,29,30,46,47,54,43,37)(13,14,25,34,49,$ $53,52,35,36,45,42)(23,24,32,33,48,39,40,41,50,55,51)) = \phi(e) = 1$

$\phi(t_7xt_6^{-1}) = \phi((1,2,7,5,4,16,31,17,10,6,3)(8,26,19,18,28,21,20,44,27,$ $15,9)(11,37,43,54,47,46,30,29,38,22,12)(13,42,45,36,35,52,53,49,34,25,14)$ $(23,51,55,50,41,40,39,48,33,32,24)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,$ $18)(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)$ $(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(2,3,5,7)(4,6,9,12)(8,10,14,19)(11,$ $15,21,28)(13,17,24,33)(16,22,30,40)(18,25,34,37)(20,27,36,45)(23,31,41,47)(29,$ $38)(35,44,42,48)(39,46,50,52)(43,49,53,55)(51,54)) = \phi(e) = 1$

$\phi(t_8xt_8^{-1}) = \phi((1,3)(4,22)(5,6)(8,26)(9,10)(11,18)(12,15)(13,48)(14,17)$ $(16,41)(19,25)(20,42)(21,27)(23,54)(24,31)(30,38)(32,33)(35,39)(36,44)(37,49)$ $(40,46)(43,51)(47,50)(52,55)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)(15,20)$ $(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)(41,46)$ $(44,45)(47,51)(48,52)(50,53)(54,55)(1,3)(4,22)(5,6)(8,26)(9,10)(11,18)(12,15)$ $(13,48)(14,17)(16,41)(19,25)(20,42)(21,27)(23,54)(24,31)(30,38)(32,33)(35,39)$ $(36,44)(37,49)(40,46)(43,51)(47,50)(52,55)) = \phi(e) = 1$

$\phi(t_9xt_1^{-1}0) = \phi((1,5,9,14,24,41,22,6,7,2,3)(4,30,29,38,40,50,23,51,49,$ $18,15)(8,26,10,12,21,36,42,27,28,11,25)(13,35,46,16,47,52,43,54,31,33,32)(17,$ $19,34,37,53,55,39,44,45,20,48)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)$ $(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)$ $(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(1,7,41,9,3,6,24,5,2,22,14)(4,49,$ $50,29,15,51,40,30,18,23,38)(8,28,36,10,25,27,21,26,11,42,12)(13,31,52,46,$ $32,54,47,35,33,43,16)(17,45,55,34,48,44,53,19,20,39,37)) = \phi(e) = 1$

$\phi(t_{10}xt_9^{-1}) = \phi((1,14,22,2,5,24,6,3,9,41,7)(4,38,23,18,30,40,51,15,29,$
$50,49)(8,12,42,11,26,21,27,25,10,36,28)(13,16,43,33,35,47,54,32,46,52,31)$
$(17,37,39,20,19,53,44,48,34,55,45)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,$
$18)(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,$
$43)(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(1,3,2,7,6,22,41,24,14,9,5)(4,$
$15,18,49,51,23,50,40,38,29,30)(8,25,11,28,27,42,36,21,12,10,26)(13,32,33,$
$31,54,43,52,47,16,46,35)(17,48,20,45,44,39,55,53,37,34,19)) = \phi(e) = 1$

$\phi(t_{11}xt_1^{-1}1) = \phi((1,7,3)(2,5,12,28,15,6)(4,40,52,49,25,10)(8,26,14,33,32,$
$17)(9,19,37,55,46,22)(11,34,18,21,45,27)(13,44,20,35,50,31)(16,23,54,41,30,$
$38)(24,47,39,42,36,48)(43,51,53)(1,2)(3,4)(6,8)(9,11)(10,13)(12,16)(14,18)$
$(15,20)(17,23)(19,26)(22,29)(24,32)(27,35)(28,37)(30,39)(31,40)(33,42)(34,43)$
$(41,46)(44,45)(47,51)(48,52)(50,53)(54,55)(1,3,7)(2,6,15,28,12,5)(4,10,25,$
$49,52,40)(8,17,32,33,14,26)(9,22,46,55,37,19)(11,27,45,21,18,34)(13,31,50,$
$35,20,44)(16,38,30,41,54,23)(24,48,36,42,39,47)(43,53,51)) = \phi(e) = 1$

Our matrices were derived from the cyclotomic field 2. However, we need to extend to the field $\mathbb{Z}_3$ in order to obtain all the automorphisms of the symetric generators. In doing so, instead of writing -1 in the fifth column, fifth row, of $A(x)$, we can write 2 in its place. For a_{55}, instead of writing $t_5 \to t_5^{-1}$, we can now write $t_5 \to t_5^2$.

3.5 Converting Matrix Representation into Permutation Representation

We will now convert our monomial matrix representation into a permutation representation. This will allow us to create a presentation for our desired progenitor. There are eleven columns in both matrices $A(x)$ and $A(y)$. This implies that we will have eleven t's (symmetric generators). Moreover, the symmetric generators will be of order 3. We shall label all the symmetric generators in an ascending order. For example, t_i, $1 \leq i \leq 11$ are assigned the numbers 1-11, respectively. Next, t_i^2 , $1 \leq i \leq 11$ are assigned the numbers 12-22, respectively. As a reminder, we find use of $a_{i,j} = 1$ if the automorphism takes $t_i \to t_j$ and $a_{i,j} = n$ if the automorphism takes $t_i \to t_j^n$ to determine where our $t_{i's}$ are being sent.

Table 3.5: Automorphisms of A(x)

1	2	3	4	5	6	7	8	9	10	11
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_4	t_3	t_6	t_5^2	t_1	t_8	t_9	t_{11}	t_{10}^2	t_7
2	4	3	6	16	1	8	9	11	21	7

12	13	14	15	16	17	18	19	20	21	22
t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2	t_8^2	t_9^2	t_{10}^2	t_{11}^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2^2	t_4^2	t_3^2	t_6^2	t_5	t_1^2	t_8^2	t_9^2	t_{11}^2	t_{10}	t_7^2
13	15	14	17	5	12	19	20	22	10	18

We get the permutation $(1, 2, 4, 6)(5, 16)(7, 8, 9, 11)(10, 21)(12, 13, 15, 17)$ $(18, 19, 20, 22)$.

Table 3.6: Automorphisms of A(y)

1	2	3	4	5	6	7	8	9	10	11
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_3	t_2	t_5	t_4	t_7	t_6	t_8	t_{10}	t_9^2	t_{11}
1	3	2	5	4	7	6	8	10	9	11

12	13	14	15	16	17	18	19	20	21	22
t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2	t_8^2	t_9^2	t_{10}^2	t_{11}^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^2	t_3^2	t_2^2	t_5^2	t_4^2	t_7^2	t_6^2	t_8^2	t_{10}^2	t_9	t_{11}^2
12	14	13	16	15	18	17	19	21	20	22

We get the permutation $(2, 3)(4, 5)(6, 7)(9, 10)(13, 14)(15, 16)(17, 18)$ $(20, 21)$.

We must check if our permutations generate a group isomorphic M_{11}.
$S := Sym(22);$

$aa := S!(1,2,4,6)(5,16)(7,8,9,11)(10,21)(12,13,15,17)(18,19,20,22);$

$bb := S!(2,3)(4,5)(6,7)(9,10)(13,14)(15,16)(17,18)(20,21);$

$N := sub < S \mid aa,\ bb >;$

$IsIsomorphic(G, N);$

$true$

We have our presentation for our group N given by $< x, y | x^2, y^4, y^{-1}xy^{-2}xy^{-2}xy^2 xy^2xy^2xy^{-1}, xyxyxy^{-1}xy^{-1}xyxy^{-1}xy^2xyxy^{-1}, xy^{-2}xy^{-1}xyxy^{-2}xy^{-1}xyxy^2xy^{-1}x*y, (xy^{-1})^{11} >$.

3.6 Inserting Symmetric Generators

Fix symmetric generator $t \sim t_1$. This implies we fix the set $< t_1 >$; that is, we stabilize the set $\{1, 12\}$. In a similar fashion to the involutory progenitor, N acts on the family of all its subgroups by conjugation. We will show that there is a homorphism from N to the family of all the conjugates of $< (t) >$ in N, thus inducing an action.

Theorem 3.7. *[Rot95] Let $< (t) > \leq N$ and X be the family of all the conjugates of $< (t) >$ in N. There is a homomorphism $\phi : N \to S_X$ with $ker\phi \leq N_N(< (t) >)$.*

Proof: Let $a \in N$, define $\phi_a : X \to X$, by $\phi_a(n < (t) > n^{-1}) = an < (t) > n^{-1}a^{-1}$. Let $b \in N$, and consider the following:

$$\begin{aligned}\phi_a\phi_b(n < (t) > n^{-1}) &= \phi_a(bn < (t) > n^{-1}b^{-1}) \\ &= abn < (t) > n^{-1}b^{-1}a^{-1} \\ &= \phi_{ab}(n < (t) > n^{-1}).\end{aligned}$$

Thus, ϕ is a homomorphism. Moreover, since $a \in N$, then $a^{-1} \in N$. Hence, ϕ_a has inverse $\phi_{a^{-1}}$ and thus $\phi_a \in S_X$. Furthermore, if $a \in ker\phi$, then

$$\phi_a(n < (t) > n^{-1}) = an < (t) > n^{-1}a^{-1}$$
$$= n < (t) > n^{-1}, \quad \textit{for all } n \in N.$$

Choose $n = e$*, then* $\phi_a(e < (t) > e^{-1}) = ae < (t) > ea^{-1} = a < (t) > a^{-1}$*. Therefore,* $a \in N_N(< (t) >)$ *and* $ker\phi \leq N_N(< (t) >)$ *as desired.* $\square$

Theorem 3.8. *[Rot95] Let* $G = m^{*^n} :_m N$*, if* N *acts by conjugation on the family of all subgroups of* m^{*^n} *and if* $< (t) > \leq G$*, then* $\mathcal{O}(< (t) >) = \{$*all conjugates of* $< (t) >\}$ *and* $G_{<(t)>} = N_G(< (t) >)$*.*
Proof: We will first show that $\mathcal{O}(< (t) >) = \{$*all conjugates of* $< (t) >\}$*.*
Consider:

$$\begin{aligned}\mathcal{O}(< (t) >) &= \{n \in N \mid n < (t) >\} \\ &= \{n \in N \mid \; < (t) >^n\} \\ &= \{n \in N \mid n < (t) > n^{-1}\} \\ &= < (t) >^N\end{aligned}$$

We will now show $G_{<(t)>} = N_G(< (t) >)$*. Consider the following:*

$$\begin{aligned}G_{<(t)>} &= \{n \in N \mid n < (t) > = < (t) >\} \\ &= \{n \in N \mid \; < (t) >^n = < (t) >\}, \quad \textit{since } N \textit{ acts by conjugation on } m^{*^n} \\ &= \{n \in N \mid n < (t) > n^{-1} = < (t) >\} \\ &= N_G(< (t) >).\end{aligned}$$

Thus, the coset stabilizer of $< (t) >$ *is equal to the normalizer of* $< (t) >$ *in* G*.* $\square$

Moreover, the number of conjugates of H in G is $[G : N_G(H)]$ (Corollary 3.21 [*Rot* 95]). Since $N \leq S_n$ is a transitive G-set, this implies N has one orbit. As a result, $< t_1 >$ has 22 conjugates. Continuing, we convert the generators of $N_G(< t_1 >)$ into words. Next, we make t commute with these words and add them into the presentation. However, there is one generator of the stabilizer namely, $z = (1,12)(2,8,10,11)(3,4,9,5)(7,18)(13,19,21,22)(14,15,20,16)$ that sends $1 \rightarrow 12$. In the presentation, raise t to z, but z must be converted into a word. In doing so, we get $t^{(y^2xy^{-1}xy^2)} = t^2$. That is, we have two $t's$ in $< (t) >$. Furthermore, when we make t commute with the generators of the coset stabilizer in the presentation, we are saying t has 22 conjugates. This follows from the coset stabilizer being a subgroup of N with index 11. Moreover, we can write N as a decomposition of right cosets in terms of the coset stabilizer and the representatives of the 11 conjugacy classes, denoted $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_{11}$. Therefore, when we conjugate t by $N = N^1x_1 \cup N^1x_2 \cup \cdots \cup N^1x_{11}$, we get 22 $t_{i's}$.

3.7 Checking the Monomial Progenitor

Our monomial presentation of $3^{11} :_m M_{11}$ is, $< x, y, t | x^2, y^4, t^3, y^{-1}xy^{-2}xy^{-2}xy^2$
$xy^2xy^2xy^{-1}, xyxyxy^{-1}xy^{-1}xyxy^{-1}xy^2xyxy^{-1}, xy^{-2}xy^{-1}xyxy^{-2}xy^{-1}xyxy^2xy^{-1}x * y,$
$(xy^{-1})^{11}, (t, y^{-1}xyxy^{-1}xy^2xy^{-1}xy^2xy^{-1}xyxy), (t, x), (t, xyxyxyxyxyxy^{-1}xy^{-1}xy^{-1}$
$xy^2xy), (t, yxy^2xy^{-1}), (t^{(y^2xy^{-1}xy^2)} = t^2) >$.

We will now perform a check to make sure our monomial progenitor is correct. By applying the Factoring Lemma we should obtain a homomorphic image of the group $3^{11} :_m M_{10}$, where $|3^{11} :_m M_{11}| = 3^{11} \times 7920 = 1403004240$.

$N1 := Stabilizer(N, \{1, 12\});$

$Orbits(N1);$

$GSet\{1, 12\}$

$GSet\{2, 3, 8, 11, 7, 4, 6, 22, 10, 19, 9, 5, 18, 21, 20, 16, 14, 13, 15, 17\}$
$GG < x, y, t >:= Group < x, y, t | x^2, y^4, t^3, y^{-1}xy^{-2}xy^{-2}xy^2xy^2xy^2xy^{-1}, xyxyxy^{-1}x$
$y^{-1}xyxy^{-1}xy^2xyxy^{-1}, xy^{-2}xy^{-1}xyxy^{-2}xy^{-1}xyxy^2xy^{-1}x * y, (xy^{-1})^{11},$
$(t, y^{-1}xyxy^{-1}xy^2xy^{-1}xy^2xy^{-1}xyxy), (t, x), (t, xyxyxyxyxyxy^{-1}xy^{-1}xy^{-1}xy^2xy), (t,$
$yxy^2xy^{-1}), (t^{(y^2xy^{-1}xy^2)} = t^2), (t, t^y) >$
$\#GG;$
1403004240

Thus, we have successfully constructed the monomial progenitor $3^{*^{11}} :_m M_{11}$.

3.8 Extension of Factoring Lemma (Monomial)

Corollary 3.9. *The center $Z(G)$ of $G = \frac{s^{*^n} :_m N}{tt^x = t^x t}$, a monomial progenitor factored by the relation (t, t_i), $1 \leq i \leq n$, is the trivial subgroup $\{e\}$.*
Proof: Since N is transitive on n letters, this implies $Z(G)$ must have length n. Moreover, if $Z(G)$ has its product of $t_{i's}$ raised to different powers, we get a similar contradiction to Extension of Factoring Lemma. Suppose $Z(G) =< (t_n t_{n-1} \cdots t_2 t_1) >$. Since G is a monomial representation, there exists $\xi \in N$ such that $(t_j)^\xi = t_l^r$, where $1 \leq r \leq m$ and $1 \leq j, l \leq n$. Without loss of generality, suppose $(t_n)^\xi = t_{n-1}^r$, where $1 \leq r \leq m$.

Consider,

$$\begin{aligned}
(t_n t_{n-1} \cdots t_2 t_1)\xi &= \xi\xi^{-1}(t_n t_{n-1} \cdots t_2 t_1)\xi \\
&= \xi(t_n t_{n-1} \cdots t_2 t_1)^\xi \\
&= \xi(t_n^\xi t_{n-1}^\xi \cdots t_2^\xi t_1^\xi)\xi \\
&= \xi(t_{n-1}^r t_s \cdots t_t t_p), \qquad \textit{where } 1 \leq s, t, p \leq m \\
&= \xi(t_n t_{n-1}^r \cdots t_2 t_1)
\end{aligned}$$

But $(t_n t_{n-1} \cdots t_2 t_1) \neq (t_n t_{n-1}^r \cdots t_2 t_1)$, thus $Z(G) \neq< (t_n t_{n-1} \cdots t_2 t_1) >$. Hence, $Z(G) = \{e\}$. $\square$

3.9 Lifting All Linear Characters From H/H' up to H

Let G be a group such that $G = 2 \times S_4$. You can obtain a faithful and irreducible monomial representation of G by inducing the linear character $\chi^H.3$ from the proper subgroup $H = 3 : 2^2$ of G up to G.

The task at hand is to find all irreducible linear characters of H. In other words, we need to find all normal subgroups N of H. In doing so, we can associate each character of H/N with its lift to H. Finding characters of H/N is easier than finding characters of H since H/N is smaller in size than H. By finding the characters of H/N, we then utilize the fact that there is a bijective correspndence between the set of characters of H/N and the characters χ of H that satisfy $N \leq Ker\chi$. Moreover, irreducible characters of H/N correspond to irreducible characters of H that have N in their kernel. However, the big question at hand is, is there a way to obtain all linear chracters from a normal subgroup N of H? Quite simply, the answer is yes. It turns out that all the linear characters of H are the lifts to H of the irreducible characters of H/H', where H' denotes the derived subgroup of H. In addition, the number of distinct linear characters of H is equal to $|\frac{H}{H'}|$.

Therefore, given any group H, we can find all linear irreducible characters of H by lifting all irreducible characters of H/H'.

Utilizing the facts from above, we turn our attention back to our group $H = 3 : 2^2$. We want to find all irreducible linear characters of H. This is accomplished by finding the derived subgroup of H, H', and the character table of H/H'. Next, lift all characters of H/H' to H.

The derived subgroup of H is

$$\begin{aligned} H' &=< (2,6,10)(3,7,5)(4,11,9) > \\ &= \{Id(dH), (2,6,10)(3,7,5)(4,11,9), (2,10,6)(3,5,7)(4,9,11)\} \end{aligned}$$

We must now find H/H' and its character table. We obtain H/H' by right multiplication of H to H'. In other words, we are taking the transversal of H' in H and right multiplying them with H'.

That is to say

$$\begin{aligned} H/H' &= \{H', H'(2,5)(3,10)(6,7)(9,11), H'(1,8)(2,5)(3,6)(7,10), H'(1,8)(3,7)(6,10)(9,11)\} \\ &=< H'(2,5)(3,10)(6,7)(9,11), H'(1,8)(2,5)(3,6)(7,10) > \end{aligned}$$

We find a character table of H/H' as follows. Note $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{c, d | c^2 = d^2 = (cd)^2 = 1\}$. Now $H/H' = \mathbb{Z}_2 \times \mathbb{Z}_2$, where $c = H'(2,5)(3,10)(6,7)(9,11)$ and $d = H'(1,8)(2,5)(3,6)(7,10)$.

Let $a = (2,5)(3,10)(6,7)(9,11)$ and $b = (1,8)(2,5)(3,6)(7,10)$, we construct the character table of H/H' in the following manner.

	Rep	$a^0 * b^0$	$a^1 * b^0$	$a^0 * b^1$	$a^1 * b^1$
$(-1)^0 * (-1)^0$	$\chi^{\bullet}_{.1}$	$((-1)^0)^0 * ((-1)^0)^0$	$((-1)^0)^1 * ((-1)^0)^0$	$((-1)^0)^0 * ((-1)^0)^1$	$((-1)^0)^1 * ((-1)^0)^1$
$(-1)^1 * (-1)^0$	$\chi^{\bullet}_{.2}$	$((-1)^1)^0 * ((-1)^0)^0$	$((-1)^1)^1 * ((-1))^0$	$((-1)^1)^0 * ((-1))^1$	$((-1)^1)^1 * ((-1)^0)^1$
$(-1)^0 * (-1)^1$	$\chi^{\bullet}_{.3}$	$((-1)^0)^0 * ((-1)^1)^0$	$((-1)^0)^1 * ((-1)^1)^0$	$((-1)^0)^0 * ((-1)^1)^1$	$((-1)^0)^1 * ((-1)^1)^1$
$(-1)^1 * (-1)^1$	$\chi^{\bullet}_{.4}$	$((-1)^1)^0 * ((-1)^1)^0$	$((-1)^1)^1 * ((-1)^1)^0$	$((-1)^1)^0 * ((-1)^1)^1$	$((-1)^1)^1 * ((-1)^1)^1$

This is equivalent to

Character Table of H/H'

Rep	H'	$H'(2,5)(3,10)(6,7)(9,11)$	$H'(1,8)(2,5)(3,6)(7,10)$	$H'(1,8)(3,7)(6,10)(9,11)$
$\chi^{\bullet}_{.1}$	1	1	1	1
$\chi^{\bullet}_{.2}$	1	-1	1	-1
$\chi^{\bullet}_{.3}$	1	1	-1	-1
$\chi^{\bullet}_{.4}$	1	-1	-1	1

Given the character table of H/H', we see that H will have four irreducible linear characters. We will now lift the characters of H/H' to H. Using the definition of $lift$, $H' \lhd H$, with $\chi^{\bullet}$ being a character of H', then the character of χ of H is given by $\chi(h) = \chi^{\bullet}(h)$, $h \in H$. The representatives for the conjugacy classes of H are:

```
Id(H),
(1, 8)(2, 5)(3, 6)(7, 10),
(2, 5)(3, 10)(6, 7)(9, 11),
(1, 8)(3, 7)(6, 10)(9, 11),
(2, 10, 6)(3, 5, 7)(4, 9, 11),
(1, 8)(2, 7, 6, 5, 10, 3)(4, 9, 11)
```

Now we lift all of the characters of H/H' up to H, evaluated at $h \in H$. We assign a numerical value within the character table of H based on the character $\chi^{\bullet}$ of H/H' evaluated at the conjugacy class representatives. Depending where the conjugacy class representative belongs evaluated by $\chi^{\bullet}$, we assign the respective value.

To help with assigning numerical values to the character table of H, we will list the elements within the sets contained in H/H'.

$< H' >= \{Id(H), (2,6,10)(3,7,5)(4,11,9), (2,10,6)(3,5,7)(4,9,11)\}$

$< H'(2,5)(3,10)(6,7)(9,11) >= \{(2,3)(4,11)(5,6)(7,10), (2,7)(3,6)(4,9)$
$(5,10), (2,5)(3,10)(6,7)(9,11)\}$

$< H'(1,8)(2,5)(3,6)(7,10) >= \{(1,8)(2,3,10,5,6,7)(4,11,9), (1,8)$
$(2,7,6,5,10,3)(4,9,11), (1,8)(2,5)(3,6)(7,10)\}$

$< H'(1,8)(3,7)(6,10)(9,11) >= \{(1,8)(3,7)(6,10)(9,11), (1,8)(2,10)(4,9)$
$(5,7), (1,8)(2,6)(3,5)(4,11)\}$

Note $\chi_i(1) = \chi_i^{\bullet}(H') = 1$, for $1 \le i \le 4$, this implies we can fill the first row and first column with value 1 on the character table of H. We will repeat this process on the remaining characters of H/H'.

$\chi_2((1,8)(2,5)(3,6)(7,10)) = \chi_2^{\bullet}(H'(1,8)(2,5)(3,6)(7,10)) = 1,$
$\chi_2((2,5)(3,10)(6,7)(9,11)) = \chi_2^{\bullet}(H'(2,5)(3,10)(6,7)(9,11)) = -1,$
$\chi_2((1,8)(3,7)(6,10)(9,11)) = \chi_2^{\bullet}(H'(1,8)(3,7)(6,10)(9,11)) = -1,$
$\chi_2((2,10,6)(3,5,7)(4,9,11)) = \chi_2^{\bullet}(H'(2,10,6)(3,5,7)(4,9,11)) = 1$, and
$\chi_2((1,8)(2,7,6,5,10,3)(4,9,11)) = \chi_2^{\bullet}(H'(1,8)(2,7,6,5,10,3)(4,9,11)) = 1,$

$\chi_3((1,8)(2,5)(3,6)(7,10)) = \chi_3^{\bullet}(H'(1,8)(2,5)(3,6)(7,10)) = 1,$
$\chi_3((2,5)(3,10)(6,7)(9,11)) = \chi_3^{\bullet}(H'(2,5)(3,10)(6,7)(9,11)) = 1,$
$\chi_3((1,8)(3,7)(6,10)(9,11)) = \chi_3^{\bullet}(H'(1,8)(3,7)(6,10)(9,11)) = -1,$
$\chi_3(2,10,6)(3,5,7)(4,9,11)() = \chi_3^{\bullet}(H'(2,10,6)(3,5,7)(4,9,11)) = 1$, and
$\chi_3((1,8)(2,7,6,5,10,3)(4,9,11)) = \chi_3^{\bullet}(H'(1,8)(2,7,6,5,10,3)(4,9,11)) = -1,$

$\chi_4((1,8)(2,5)(3,6)(7,10)) = \chi_4^{\bullet}(H'(1,8)(2,5)(3,6)(7,10)) = -1,$
$\chi_4((2,5)(3,10)(6,7)(9,11)) = \chi_4^{\bullet}(H'(2,5)(3,10)(6,7)(9,11)) = -1,$
$\chi_4((1,8)(3,7)(6,10)(9,11)) = \chi_4^{\bullet}(H'(1,8)(3,7)(6,10)(9,11)) = 1,$
$\chi_4((2,10,6)(3,5,7)(4,9,11)) = \chi_4^{\bullet}(H'(2,10,6)(3,5,7)(4,9,11)) = 1$, and
$\chi_4((1,8)(2,7,6,5,10,3)(4,9,11)) = \chi_4^{\bullet}(H'(1,8)(2,7,6,5,10,3)(4,9,11)) = -1,$

Having lifted all of the irreducible linear characters of H/H' up to H, we have successfully found all linear characters of H.

Table 3.7: Character Table of H

Rep	*Id(H)*	$(1,8)(2,5)(3,6)(7,10)$	$(2,5)(3,10)(6,7)(9,11)$	$(1,8)(3,7)(6,10)(9,11)$	$(2,10,6)(3,5,7)(4,9,11)$	$(1,8)(2,7,6,5,10,3)(4,9,11)$
$\chi.1$	1	1	1	1	1	1
$\chi.2$	1	1	-1	-1	1	1
$\chi.3$	1	-1	1	-1	1	-1
$\chi.4$	1	-1	-1	1	1	-1
$\chi.5$						
$\chi.6$						

Chapter 4

Double Coset Enumeration and Related Topics

4.1 Manual Double Coset Enumeration of S_5 over A_4

Our group is given by $S_5 \cong \frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$, we will begin constructing the Cayley graph of S_5 over A_4. Let $t_1 \sim t$, $x \sim (1,3,4)$, and $y \sim (2,3,4)$, where x and y are the generators of A_4 and t represents a symmetric generator. Lastly, let $A_4 \sim N$. We will now expand our relation.
Consider:

$$\begin{aligned}
(y^{-1}xt_1)^4 &= (y^{-1}xt_1)(y^{-1}xt_1)(y^{-1}xt)(y^{-1}xt_1)\\
&= y^{-1}xt_1y^{-1}xt_1(y^{-1}x)^2t_1^{(y^{-1}x)}t_1\\
&= y^{-1}xt_1(y^{-1}x)^3t_1^{(y^{-1}x)^2}t_1^{(y^{-1}x)}t_1\\
&= (y^{-1}x)^4t_1^{(y^{-1}x)^3}t_1^{(y^{-1}x)^2}t_1^{(y^{-1}x)}t_1\\
&= (1,3,2)t_1t_2t_3t_1\\
&= e
\end{aligned}$$

Therefore, our relation is given by $(1,3,2)t_1t_2t_3t_1 = e$. That is to say, $(1,3,2)t_1t_2 = t_1t_3$.

First Double Coset

We consider our first double coset NeN, denoted by $[*]$. This double coset contains one single coset, namely N. Since N is transitive on $\{1,2,3,4\}$, this implies the double coset has a single orbit, $\{t_1.t_2,t_3,t_4\}$. Next, we select a representative from the orbit, say t_1. We wish to see which double coset Nt_1 belongs to. Nt_1 belongs to the double coset Nt_1N. Thus, all four $t_{i's}$ will be moving forward to the new double coset Nt_1N, denoted by [1].

Second Double Coset

We are now at our second double coset, Nt_1N, which 4 $t_{i's}$ moved to. Needing to determine how many unique single cosets [1] contains, we will find the one point stabilizer of 1 in N, denoted $N^{(1)}$. These will be the elements of A_4 that fix 1. The one point stabilizer, $N^1 \geq< (2,3,4) >$. Therefore, the number of unique single cosets in Nt_1N is given by $\frac{|N|}{|N^1|} = \frac{12}{3} = 4$. We can now determine the orbits of Nt_1 by observing which numbers are in the same permutation. This will imply the corresponding $t_{i's}$ will be in the same orbit. For example, given the permutation $(2,3,4)$, we notice $\{2,3,4\}$ are in the same 3-cycle. This implies $\{t_2,t_3,t_4\}$ will also be in the same orbit. Hence the orbits of Nt_1 on $\{1,2,3,4\}$ are $\{t_1\}$ and $\{t_2,t_3,t_4\}$. We will select a member of each orbit to determine which double coset the respective coset belongs to. Notice that $Nt_1t_1 = Ne = N$ and $N \in [*]$. Hence, one symmetric generator goes back. There does not exist a relation having Nt_2, Nt_3, or Nt_4 equaling Nt_i, for some $i \in \{1,2,3,4\}$. As a result, we get a new double coset which we denote [1 2]. Additionally, there will be three symmetric generators moving forward to this new double coset [1 2].

Third Double Coset

At the third double coset, [1 2], we find the number of unique single cosets by finding the coset stabilizer of 1 and 2, denoted $N^{(1\ 2)}$. The coset stabilizer of 1 and 2

is given by $N^{(1\ 2)} =< e >$ since nothing stabilizes 1 and 2. However, we must accout for equal cosets. That is, using our relation, we must see which cosets are equal. Using our relation $(1,3,2)t_1t_2 = t_1t_3$ and left multiplying by N, we have $Nt_1t_2 = Nt_1t_3$, since $(1,3,2) \in N$.
Consider $(1,3,2)t_1t_2 = t_1t_3$ conjugated by $(2,3,4)$.

$$\begin{aligned}
&\Longrightarrow ((1,3,2)t_1t_2)^{(2,3,4)} = (t_1t_3)^{(2,3,4)} \\
&\iff (1,4,3)t_1t_3 = t_1t_4 \\
&\iff Nt_1t_3 = Nt_1t_4
\end{aligned}$$

Hence, $Nt_1t_2 = Nt_1t_3 = Nt_1t_4$. Going back to our coset stabilizer, we see that the permutation $(2,3,4)$ stabilizes 1 and 2. Reason being, $(2,3,4)$ fixes 1 and permutes the last symmetric generator, t_i, $i \in \{2,3,4\}$. We deduced that this was satisfactory since $Nt_1t_2 = Nt_1t_3 = Nt_1t_4$. Hence our coset stabilizer $N^{(1\ 2)} \geq< (2,3,4) >$. As a result, the number of single cosets in Nt_1N is given by $\frac{|N|}{|N^{(1\ 2)}|} = \frac{12}{3} = 4$.
We now wish to find the single cosets in [1 2]. In order to do so, we will conjugate $Nt_1t_2 = Nt_1t_3 = Nt_1t_4$ by all of N. That is, we obtain the transversals of [1 2] by finding all of the right cosets of $N^{(1\ 2)}$ in N. Then, we will conjugate our relation from above by the transversals to find the distinct cosets.

$$\begin{aligned}
N^{(1\ 2)}(e) &= N^{(1\ 2)} \\
N^{(1\ 2)}((1,3,4)) &= \{(1,3,4),(1,3)(2,4),(1,3,2)\} \\
N^{(1\ 2)}((1,4,3)) &= \{(1,4,3),(1,4,2),(1,4)(2,3)\} \\
N^{(1\ 2)}((1,2,4)) &= \{(1,2,4),(1,2,3),(1,2)(3,4)\}
\end{aligned}$$

We have all of the right cosets of $N^{(1\ 2)}$ in N. We will take a representative from the given right cosets and define them as the set of transversals. That is, our transversals are $T = \{e,(1,3,4),(1,4,3),(1,2,4)\}$. We now conjugate our relations above by the transversals of [1 2].

$$\begin{aligned}
N(t_1t_2)^{(e)} = N(t_1t_3)^{(e)} = N(t_1t_4)^{(e)} &\iff Nt_1t_2 = Nt_1t_3 = Nt_1t_4\\
N(t_1t_2)^{(1,3,4)} = N(t_1t_3)^{(1,3,4)} = N(t_1t_4)^{(1,3,4)} &\iff Nt_3t_2 = Nt_3t_4 = Nt_3t_1\\
N(t_1t_2)^{(1,4,3)} = N(t_1t_3)^{(1,4,3)} = N(t_1t_4)^{(1,4,3)} &\iff Nt_4t_2 = Nt_4t_1 = Nt_4t_3\\
N(t_1t_2)^{((1,2,4)} = N(t_1t_3)^{((1,2,4)} = N(t_1t_4)^{((1,2,4)} &\iff Nt_2t_4 = Nt_2t_3 = Nt_2t_1
\end{aligned}$$

We now have the four distinct right cosets in [1 2].

Analyzing the permutations of the coset stabilizer, we see that the orbits of $N^{(1\ 2)}$ on $\{1,2,3,4\}$ are $\{t_1\}$ and $\{t_2,t_3,t_4\}$. We select a member of each orbit to determine which double coset the respective coset belongs to. Notice that $Nt_1t_2t_2 = Nt_1$ and $Nt_1 \in [1]$. Hence, three symmetric generators go back. There does not exist a relation having $Nt_1t_2t_1$ equaling Nt_1t_i for some $i \in \{1,2,3,4\}$. As a result, we get a new double coset which we denote [1 2 1]. Moreover, there will be one symmetric generator moving forward to this new double coset [1 2 1].

Fourth Double Coset

At the fourth double coset, [1 2 1], we find the number of unique single cosets by finding the coset stabilizer of $N^{(1\ 2\ 1)}$. Using our relations obtained from the third double coset, $Nt_1t_2 = Nt_1t_3 = Nt_1t_4$, we right multiply by t_1. We obtain $Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$. We see the permutation $(2,3,4)$ is in the coset stabilizer $N^{(1\ 2\ 1)}$. This would imply we have $\frac{|N|}{|N^{(1\ 2\ 1)}|} = \frac{12}{3} = 4$ distinct right cosets, but the total number of distinct right cosets is given by $\frac{|S_5|}{|A_4|} = \frac{120}{12} = 10$. That is, we have three more right cosets than what we should have (Our total number of distinct right cosets is currently 9). Therefore, we must account for additional equal right cosets. Using $Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$, and conjugating it by the transversals in [1 2 1] in the same fashion as in the third double coset, we get:

$$Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$$
$$Nt_3t_2t_3 = Nt_3t_4t_3 = Nt_3t_1t_3$$
$$Nt_4t_2t_4 = Nt_4t_1t_4 = Nt_4t_3t_4$$
$$Nt_2t_4t_2 = Nt_2t_3t_2 = Nt_2t_1t_2$$

Using our relation, we must show that these four right cosets are in fact equal.
For the sake of notation, let $t_1t_2t_1 \sim t_1t_3t_1 \sim t_1t_4t_1$ imply $Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$.
Using our relation $(1,3,2)t_1t_2t_3t_1 = e$, we will show that all four right cosets are equal.

Consider the following:

$$\begin{aligned}
t_1t_3t_1 &= t_1\underline{t_3t_1} \\
&= t_1(3,1,4)t_3t_4, \text{ since } ((1,3,2)t_1t_2)^{(1,3)(2,4)} = (t_1t_3)^{(1,3)(2,4)} \iff (3,1,4)t_3t_4 = t_3t_1 \\
&= (3,1,4)(3,4,1)t_1(3,1,4)t_3t_4 \\
&= (3,1,4)t_1^{(3,1,4)}t_3t_4 \\
&= (3,1,4)t_4t_3t_4
\end{aligned}$$

Hence, $Nt_1t_3t_1 = Nt_4t_3t_4$ since $(3,1,4) \in N$.

$$\begin{aligned}
t_3t_4t_3 &= t_3\underline{t_4t_3} \\
&= t_3(4,3,1)t_3t_4, \text{ since } ((1,3,2)t_1t_2)^{(1,4,2)} = (t_1t_3)^{(1,4,2)} \iff (4,3,1)t_4t_1 = t_4t_3 \\
&= (4,3,1)(4,1,3)t_3(4,3,1)t_4t_1 \\
&= (4,3,1)t_1^{(4,3,1)}t_4t_1 \\
&= (4,3,1)t_1t_4t_1
\end{aligned}$$

Hence, $Nt_3t_4t_3 = Nt_1t_4t_1$ since $(4,3,1) \in N$.

$$\begin{aligned}
t_4t_1t_4 &= t_4\underline{t_1t_4} \\
&= t_4(1,4,2)t_1t_2, \text{ since } ((1,3,2)t_1t_2)^{(2,4,3)} = (t_1t_3)^{(2,4,3)} \iff t_1t_4 = (1,4,2)t_1t_2 \\
&= (1,4,2)(1,2,4)t_4(1,4,2)t_1t_2 \\
&= (1,4,2)t_1^{(1,4,2)}t_1t_2 \\
&= (1,4,2)t_2t_1t_2
\end{aligned}$$

Hence, $Nt_4t_1t_4 = Nt_2t_1t_2$ since $(1,4,2) \in N$.

We have shown that $Nt_1t_3t_1 = Nt_4t_3t_4$ and $Nt_3t_4t_3 = Nt_1t_4t_1$, therefore $Nt_1t_3t_1 = Nt_4t_3t_4 = Nt_3t_4t_3 = Nt_1t_4t_1$. Additionally, we showed $Nt_4t_1t_4 = Nt_2t_1t_2$, but $Nt_4t_1t_4 = Nt_4t_3t_4$, hence $Nt_4t_1t_4 = Nt_2t_1t_2 = Nt_4t_1t_4 = Nt_4t_3t_4$. That is to say, we have showed $Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1 = Nt_3t_2t_3 = Nt_3t_4t_3 = Nt_3t_1t_3 = Nt_4t_2t_4 = Nt_4t_1t_4 = Nt_4t_3t_4 = Nt_2t_4t_2 = Nt_2t_3t_2 = Nt_2t_1t_2$.
The coset stabilizer $N^{(1\ 2\ 1)} = N$. Therefore, we see that the orbits of $N^{(1\ 2\ 1)}$ on $\{1,2,3,4\}$ are $\{t_1,t_2,t_3,t_4\}$. We select a member of each orbit to determine which double coset the respective coset belongs to. Notice that $Nt_1t_2t_1t_1 = Nt_1t_2$ and $Nt_1t_2 \in [1\ 2]$. Hence, all four symmetric generators go back. Thus, our double coset enumeration of S_5 over A_4 is complete. Furthermore, we have shown that $|G| = (\frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(1\ 2)}|} + \frac{|N|}{|N^{(1\ 2\ 1)}|})|N| \leq (1 + \frac{12}{3} + \frac{12}{3} + \frac{12}{12})12 = (10)12 = 120$.

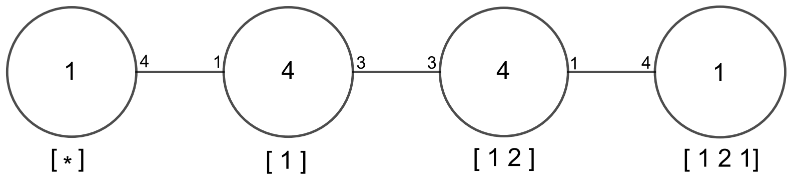

Figure 4.1: Cayley graph of S_5 over A_4

4.2 Proof of $S_5 \cong \frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$

Let $G = \frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$. We will prove that $S_5 \cong G$. We will obtain a permutation representation of G by computing the action of G on the ten right cosets of N in G that we have found. As a result, we will show $|G| \geq 120$. We label our right cosets $N, Nt, Nt_2, Nt_3, Nt_4, Nt_1t_2$, $Nt_3t_2, Nt_4t_2, Nt_2t_4$, and $Nt_1t_2t_1$ by the numbers 1,2,3,...,9, and 10, respectively. Let $t_1 \sim t$, $x \sim (1,3,4)$, and $y \sim (2,3,4)$, where x and y are the generators of A_4 and t represents a symmetric generator. We will now compute the action of the generators of G on the right cosets.

Note: We make use of the relations when computing the action on the right cosets.

$$Nt_1t_2 = Nt_1t_3 = Nt_1t_4$$

$$Nt_3t_2 = Nt_3t_4 = Nt_3t_1$$

$$Nt_4t_2 = Nt_4t_1 = Nt_4t_3$$

$$Nt_2t_4 = Nt_2t_3 = Nt_2t_1$$

$Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1 = Nt_3t_2t_3 = Nt_3t_4t_3 = Nt_3t_1t_3 = Nt_4t_2t_4 = Nt_4t_1t_4 = Nt_4t_3t_4 = Nt_2t_4t_2 = Nt_2t_3t_2 = Nt_2t_1t_2$

Table 4.1: Action of $x \sim (1, 3, 4)$

1	2	3	4	5	6	7	8	9	10
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_1t_2	Nt_3t_2	Nt_4t_2	Nt_2t_4	$Nt_1t_2t_1$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
N	Nt_3	Nt_2	Nt_4	Nt_1	Nt_3t_2	Nt_4t_2	Nt_1t_2	Nt_2t_1	$Nt_3t_2t_3$
1	4	3	5	2	7	8	6	9	10

We obtain the permutation $f_x = (2, 4, 5)(6, 7, 8)$.

Table 4.2: Action of $y \sim (2, 3, 4)$

1	2	3	4	5	6	7	8	9	10
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_1t_2	Nt_3t_2	Nt_4t_2	Nt_2t_4	$Nt_1t_2t_1$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
N	Nt_1	Nt_3	Nt_4	Nt_2	Nt_1t_3	Nt_4t_3	Nt_2t_3	Nt_3t_2	$Nt_1t_3t_1$
1	2	4	5	3	6	8	9	7	10

We obtain the permutation $f_y = (3, 4, 5)(7, 8, 9)$.

Table 4.3: Action of $t \sim t_1$

1	2	3	4	5	6	7	8	9	10
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_1t_2	Nt_3t_2	Nt_4t_2	Nt_2t_4	$Nt_1t_2t_1$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Nt_1	N	Nt_2t_4	Nt_3t_2	Nt_4t_2	$Nt_1t_2t_1$	Nt_3	Nt_4	Nt_2	Nt_1t_2
2	1	9	7	8	10	4	5	3	6

We obtain the permutation $f_t = (1, 2)(3, 9)(4, 7)(5, 8)(6, 10)$.

Now $< f_x, f_y, f_t > \leq S_{10}$. This implies there is a homomorphism f from G to S_{10}, $f : G \overset{Hom}{\rightarrow} S_{10}$. It is easily checked that the order of $< f_x, f_y, f_t >$ is 120. Since the orders of f_x, f_y, and f_{xy} are 3,3, and 2, respectively, $Im(N) =< f_x, f_y >\cong A_4$. We ask the question, is $< f_x, f_y, f_t >$ a homomorphic image of G? f_t must have four conjugates under conjugation by $Im(N) =< f_x, f_y >$. Consider the following:

$f(t_1^x) = f(x^{-1}t_1x) = f(x^{-1})f(t_1)f(x) = f(x)^{-1}f(t_1)f(x) = f(t_1)^{f(x)}$. Thus, the four conjugates are given by, $f(t_1), f(t_1)^{f(x)}, f(t_1)^{f(x^2)}$ and $f(t_1)^{f(y)}$. That is, $Im(N) =< f_x, f_y >$ acts on $\{f_{t_1}, f_{t_2}, f_{t_3}, f_{t_4}\}$ by conjugation as A_4, denoted $f_x : (f_{t_1}, f_{t_3}, f_{t_4})$ and $f_y : (f_{t_2}, f_{t_3}, f_{t_4})$. Furthermore, if our additional relation $(1, 3, 2)t_1t_2 = t_1t_3$ holds in S_{10}, then $< f_x, f_y, f_t >$ is a homomorphic image of G. But, our relation can be expressed as $f((1, 3, 2))f(t_1)f(t_1)^{f((1,2,4))} = f(t_1)f(t_1)^{f((1,3,4))}$, thus, $< f_x, f_y, f_t >$ is a homomorphic image of G. f is a homomorphism from G to S_{10}, then by the First Isomorphism Theorem we have, $G/ker(f) \cong Im(f) =< f_x, f_y, f_t >$. Thus, $|G/ker(f)| \cong | < f_x, f_y, f_t > |$. So, $\frac{|G|}{|ker(f)|} = | < f_x, f_y, f_t > |$, but this implies $|G| = |ker(f)|| < f_x, f_y, f_t > |$. Therefore, $|G| \geq 120$, but from our Cayley diagram, $|G| \leq 120$. Hence, $|G| = 120$ and $ker(f) = 1$. As a result, $G \cong< f_x, f_y, f_t >$.

4.3 Canonical Symmetric Representation Form

The presentation of G is $G =< x, y, t|x^3, y^3, (x^-1 * y^-1)^2, t^2, (t, y), (y^-1 * x * t)^4 >$, where x, y, and t correspond to the permutations f_x, f_y, and f_{t_1}, repectively. $G_1 =< f_x, f_y, f_{t_1} >\leq S_{10}$ is a faithful permutation representation of G. Thus, every element of G is a unique permutation in G_1 on 10 letters. We wish to convert $p \in G_1$ to its canonical symmetric representation form. That is, write p as a permutation of A_4 on four letters followed by a word of length of at most three in the symmetric generators. We want to see where p sends N. So, $1^p = Np = N\omega$ this implies $Np = N\omega$ if and only if $p\omega^{-1} \in N$. $p\omega^{-1} \in N$ is a permutation in N identified by its action on the four cosets whose representatives are of length one (Nt_1, Nt_2, Nt_3, Nt_4). Then, $p\omega^{-1}\omega$ is the permutation p in its canonical symmetric representation form. We will convert the permutation $(2,4)(3,5)(6,7)(8,9)$ into canonical symmetric representation form. Then $(2,4)(3,5)(6,7)(8,9)$ is of the form $(2,4)(3,5)(6,7)(8,9) = n\omega$ where $n \in N$ and ω is a word in the symmetric generators. Observe:

$$1^{(2,4)(3,5)(6,7)(8,9)} = 1 = N = N(2,4)(3,5)(6,7)(8,9)$$

Thus, $(2,4)(3,5)(6,7)(8,9) \in N$. We now calculate the action on the right cosets.

Table 4.4: Action of $(2,4)(3,5)(6,7)(8,9)$

1	2	3	4	5
N	Nt_1	Nt_2	Nt_3	Nt_4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
N	Nt_3	Nt_4	Nt_1	Nt_2
1	4	5	2	3

We obtain the permutation $(1,3)(2,4)$. Hence, our permutation $(2,4)(3,5)(6,7)(8,9) \in G_1$ has the canonical representation form of $(1,3)(2,4)$ in $\frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$.

Next, we take the permutation $(1,4)(2,8,3,6,5,9)(7,10) \in G_1$ and convert it into its canonical form.

$$1^{(1,4)(2,8,3,6,5,9)(7,10)} = 4 = Nt_4 = N(1,4)(2,8,3,6,5,9)(7,10)$$

Thus, $(1,4)(2,8,3,6,5,9)(7,10)t_4 \in N$ and $(1,4)(2,8,3,6,5,9)(7,10)t_4t_4 = (1,4)(2,8,3,6,5,9)(7,10)$ in its canonical symmetric representation form. t_4 is the permutation $f_{t_4} = (1,4)(2,6)(3,9)(5,8)(7,10)$. Then, $(1,4)(2,8,3,6,5,9)(7,10)(1,3)(2,6)(4,7)(5,8)(9,10) = (2,5,3)(6,8,9)$. We now calculate the action of $(2,5,3)(6,8,9)$ on the right cosets.

Table 4.5: Action of $(2,5,3)(6,8,9)$

1	2	3	4	5
N	Nt_1	Nt_2	Nt_3	Nt_4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
N	Nt_4	Nt_1	Nt_3	Nt_2
1	5	2	4	3

We obtain the permutation $(1,4,2)$. Hence, our permutation $(1,4)(2,8,3,6,5,9)(7,10) \in G_1$ has the canonical representation form of $(1,4,2)t_4$ in $\frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$.

4.4 Representation Form

Given $(12)(34)t_1t_2t_1t_3 \in \frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$, we now wish to convert $(12)(34)t_1t_2t_1t_3$ into its permutation form in G_1. By illustrating this example, we can find the class that $(12)(34)t_1t_2t_1t_3$ belongs to in $\frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$. That is, we can find the permutation representation form of $(12)(34)t_1t_2t_1t_3$ in G_1, compute its respective class, then convert all the permutations of the class in G_1 back into its canonical form. Since f is a faithful permutation representation of $\frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$, we can find t_1, t_2, and t_3 by conjugation of f_{t_1} by N under the map f. Thus, it suffices to calculate the action of $(12)(34)$ on the right cosets in $\frac{2^{*^4}:A_4}{(y^{-1}xt)^4}$.

We obtain the permutation $(3,2)(4,5)(6,9)(7,8)$.

Table 4.6: Action of (12)(34)

1	2	3	4	5	6	7	8	9	10
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_1t_2	Nt_3t_2	Nt_4t_2	Nt_2t_4	$Nt_1t_2t_1$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
N	Nt_2	Nt_1	Nt_4	Nt_3	Nt_2t_4	Nt_4t_2	Nt_3t_2	Nt_3t_2	$Nt_2t_1t_2$
1	3	2	5	4	9	8	7	6	10

Lastly, we find t_1, t_2, and t_3 by conjugation of f_{t_1} by N under the map f. Therefore, t_1, t_2, and t_3 are given by $f_{t_1}, f_{t_1^{(1,2,4)}}$, and $f_{t_1^{(1,3,4)}}$, respectively. We now compose the the permutations in G_1, $(3,2)(4,5)(6,9)(7,8)f_{t_1}f_{t_1^{xy^2}}f_{t_1^x} = (3,2)(4,5)(6,9)(7,8)(1,2)(3,9)(4,7)(5,8)(6,10)(1,4)(2,6)(3,9)(5,8)(7,10)(1,2)(3,9)(4,7)(5,8)(6,10)(1,3)(2,6)(4,7)(5,8)(9,10) = (1,9,7,8,6)(2,10,3,4,5)$. Thus, $(12)(34)t_1t_2t_1t_3 = (1,9,7,8,6)(2,10,3,4,5)$.

4.5 Manual Double Coset Enumeration of $PSL(2,7)$ over $(7:3)$

Our group is given by $PSL(2,7) \cong \frac{2^{*^{21}}:(7:3)}{(x^{-1}t^{(xy)})^2,(yt^{(xy^{-1})})^3}$, we will begin constructing the Cayley graph of $PSL(2,7)$ over $(7:3)$. Let $t_1 \sim t$, $x \sim (1,2,4)(3,8,10)(5,13,14)$ $(6,16,9)(7,18,15)(11,21,17)(12,20,19)$, and $y \sim (1,3,9,20,21,15,5)(2,6,17,13,8,19,7)$ $(4,11,10,18,16,14,12)$, where x and y are the generators of $(7:3)$ and t represents a symmetric generator. Lastly, let $(7:3) \sim N$. We will now expand our relations. Consider:

$$\begin{aligned}
(x^{-1}t_1^{(xy)})^2 &= (x^{-1}t_1^{(xy)})(x^{-1}t_1^{(xy)}) \\
&= (x^{-1}t_6)(x^{-1}t_6) \\
&= (x^{-1})^2 t_6^{x^{-1}} t_6 \\
&= xt_9t_6 \\
&= e
\end{aligned}$$

Therefore, our relation is given by $xt_9t_6 = e$. That is to say, $xt_9 = t_6$.

$$\begin{aligned}
(yt_1^{(xy^{-1})})^3 &= (yt_1^{(xy^{-1})})(yt_1^{(xy^{-1})})(yt_1^{(xy^{-1})}) \\
&= (yt_7)(yt_7)(yt_7) \\
&= y^3 t_7^{y^2} t_7^{y} t_7 \\
&= y^3 t_9 t_3 t_7 \\
&= e
\end{aligned}$$

Therefore, our relation is given by $y^3t_9t_3t_7 = e$. That is to say, $y^3t_9t_3 = t_7$.

First Double Coset

We consider our first double coset NeN, denoted by [$*$]. This double coset contains one single coset, namely N. Since N is transitive on $\{1, 2, \cdots, 21\}$, this implies the double coset has a single orbit, $\{t_1 t_2, \cdots, t_{21}\}$. Next, we select a representative from the orbit, say t_1. We wish to see which double coset Nt_1 belongs to. Nt_1 belongs to the double coset Nt_1N. Thus, all 21 $t_{i's}$, $1 \leq i \leq 21$ will be moving forward to the new double coset Nt_1N, denoted by [1].

Second Double Coset

We are now at our second double coset, Nt_1N, which 21 symmetric generators moved to. Needing to determine how many unique single cosets [1] contains, we will find the coset stabilizer of 1 in N, denoted $N^{(1)}$. These will be the elements of $7:3$ that fix 1. The coset stabilizer of 1 is given by, $N^{(1)} \geq < e >$. Therefore, the number of unique single cosets in Nt_1N is, $\frac{|N|}{|N^{(1)}|} = \frac{21}{1} = 21$. But, we know the number of distinct right cosets is given by $\frac{|PSL(2,7)|}{|(7:3)|} = \frac{168}{21} = 8$. This implies we have too many right cosets in the double coset [1]. Thus, we must account for equal cosets. That is, using our relation, we must see which cosets are equal. Using the relation $xt_9 = t_6$ and left multiplying by N we have $Nt_9 = Nt_6$, since $x \in N$. We wish to find all equal right cosets, therefore we conjugate our relation by all of N. Let $t_9 \sim t_6$ denote $Nt_9 = Nt_6$.

Consider the following:

$(t_9 \sim t_6)^{(1,17,18)(2,10,20)(3,7,14)(4,9,13)(5,19,11)(6,12,15)(8,16,21)} \iff t_{13} \sim t_{12}$

$(t_9 \sim t_6)^{(1,2,4)(3,8,10)(5,13,14)(6,16,9)(7,18,15)(11,21,17)(12,20,19)} \iff t_6 \sim t_{16}$

$(t_9 \sim t_6)^{(1,21,3,15,9,5,20)(2,8,6,19,17,7,13)(4,16,11,14,10,12,18)} \iff t_5 \sim t_{19}$

$(t_9 \sim t_6)^{(1,20,5,9,15,3,21)(2,13,7,17,19,6,8)(4,18,12,10,14,11,16)} \iff t_{15} \sim t_8$

$(t_9 \sim t_6)^{(1,10,7)(2,9,12)(3,16,13)(4,17,5)(6,21,18)(8,20,11)(14,19,15)} \iff t_{12} \sim t_{21}$

$(t_9 \sim t_6)^{(1,5,15,21,20,9,3)(2,7,19,8,13,17,6)(4,12,14,16,18,10,11)} \iff t_3 \sim t_2$

$(t_9 \sim t_6)^{(1,6,14)(2,11,15)(3,19,4)(5,8,18)(7,16,20)(9,17,10)(12,21,13)} \iff t_{17} \sim t_{14}$

$(t_9 \sim t_6)^{(1,4,2)(3,10,8)(5,14,13)(6,9,16)(7,15,18)(11,17,21)(12,19,20)} \iff t_{16} \sim t_9$

$(t_9 \sim t_6)^{(1,8,12)(2,16,5)(3,6,11)(4,21,7)(9,19,18)(10,15,13)(14,20,17)} \iff t_{19} \sim t_{11}$

$(t_9 \sim t_6)^{(1,15,20,3,5,21,9)(2,19,13,6,7,8,17)(4,14,18,11,12,16,10)} \iff t_1 \sim t_7$

$(t_9 \sim t_6)^{(1,19,16)(2,14,21)(3,17,12)(4,15,8)(5,6,10)(7,11,9)(13,18,20)} \iff t_7 \sim t_{10}$

$(t_9 \sim t_6)^{(1,16,19)(2,21,14)(3,12,17)(4,8,15)(5,10,6)(7,9,11)(13,20,18)} \iff t_{11} \sim t_5$

$(t_9 \sim t_6)^{(1,14,6)(2,15,11)(3,4,19)(5,18,8)(7,20,16)(9,10,17)(12,13,21)} \iff t_{10} \sim t_1$

$(t_9 \sim t_6)^{(1,7,10)(2,12,9)(3,13,16)(4,5,17)(6,18,21)(8,11,20)(14,15,19)} \iff t_2 \sim t_{18}$

$(t_9 \sim t_6)^{(1,11,13)(2,3,18)(4,6,20)(5,12,7)(8,9,14)(10,19,21)(15,16,17)} \iff t_{14} \sim t_{20}$

$(t_9 \sim t_6)^{(1,3,9,20,21,15,5)(2,6,17,13,8,19,7)(4,11,10,18,16,14,12)} \iff t_{20} \sim t_{17}$

$(t_9 \sim t_6)^{(1,13,11)(2,18,3)(4,20,6)(5,7,12)(8,14,9)(10,21,19)(15,17,16)} \iff t_8 \sim t_4$

$(t_9 \sim t_6)^{(1,9,21,5,3,20,15)(2,17,8,7,6,13,19)(4,10,16,12,11,18,14)} \iff t_{21} \sim t_{13}$

$(t_9 \sim t_6)^{(1,12,8)(2,5,16)(3,11,6)(4,7,21)(9,18,19)(10,13,15)(14,17,20)} \iff t_{18} \sim t_3$

$(t_9 \sim t_6)^{Id(N)} \iff t_9 \sim t_6$

$(t_9 \sim t_6)^{(1,18,17)(2,20,10)(3,14,7)(4,13,9)(5,11,19)(6,15,12)(8,21,16)} \iff t_4 \sim t_{15}$

We can now determine the orbits of the right cosets of Nt_1N by observing which cosets have equal names. This will imply the corresponding $t_{i's}$ will be in the same orbit. By conjugation of our relation shown above, we have the following equal right cosets:

$$t_1 \sim t_{10} \sim t_7$$

$$t_2 \sim t_9 \sim t_{12}$$

$$t_3 \sim t_{16} \sim t_{13}$$

$$t_4 \sim t_{17} \sim t_5$$

$$t_6 \sim t_{21} \sim t_{18}$$

$$t_8 \sim t_{20} \sim t_{11}$$

$$t_{14} \sim t_{19} \sim t_{15}$$

We can now determine that the coset stabilizer is given by $N^{(1)} =< (1, 10, 7)(2, 9, 12)(3, 16, 13)(4, 17, 5)(6, 21, 18)(8, 20, 11)(14, 19, 15) >$. Thus, the number of right cosets in [1] is given by $\frac{|N|}{|N^{(1)}|} = \frac{21}{3} = 7$. Moreover, the orbits of $N^{(1)}$ on $\{1, 2, \cdots, 21\}$ are $\{t_1, t_{10}, t_7\}$, $\{t_2, t_9, t_{12}\}$, $\{t_3, t_{16}, t_{13}\}$, $\{t_4, t_{17}, t_5\}$, $\{t_6, t_{21}, t_{18}\}$, $\{t_8, t_{20}, t_{11}\}$, and $\{t_{14}, t_{19}, t_{15}\}$. We will select a member of each orbit to determine which double coset the respective coset belongs to. Notice that $Nt_1t_1 = Ne = N$ and $N \in [*]$. Hence, 3 symmetric generators go back to the double coset $[*]$. We now use our relations to show the rest of the right cosets loop back to the double coset [1].

$Nt_1t_5 \in [1]$ since

$$\begin{aligned} &(y^3 t_9 t_3 = t_7)^{y^{-2}} \\ &\iff y^{3^{y^{-2}}} t_9^{y^{-2}} t_3^{y^{-2}} = t_7^{y^{-2}} \\ &\iff y^{3^{y^{-2}}} t_1 t_5 = t_8 \in [1]. \end{aligned}$$

Hence, three more symmetric generators loop back to the double coset [1].

$Nt_1t_2 \in [1]$ since

$$\begin{aligned} t_1\underline{t_2} &= t_1 y^{3^y} t_{20} t_9, \text{ since } (y^3 t_9 t_3 = t_7)^y \iff y^{3^y} t_{20} t_9 = t_2 \\ &= t_1^{y^{3^y}} t_{20} t_9 \\ &= t_{20} t_{20} t_9 \\ &= t_9 \in [1]. \end{aligned}$$

Hence, 3 more symmetric generators loop back to the double coset [1].

$Nt_1t_{16} \in [1]$ since

$$\begin{aligned} \underline{t_1} t_{16} &= y^{3^{x^2y^2}} t_{12} t_{16} t_{16}, \text{ since } (y^3 t_9 t_3 = t_7)^{x^2y^2} \iff y^{3^{x^2y^2}} t_{12} t_{16} = t_1 \\ &= y^{3^{x^2y^2}} t_{12} t_{16} t_{16} \\ &= y^{3^{x^2y^2}} t_{12} \in [1]. \end{aligned}$$

Hence, three more symmetric generators loop back to the double coset [1].

$Nt_1t_6 \in [1]$ since

$$\begin{aligned} t_1\underline{t_6} &= t_1 x t_9, \text{ since } x t_9 = t_6 \\ &= x t_1^x t_9 \\ &= x t_2 t_9 \\ &= x\underline{t_2} t_9 \\ &= x y^y t_{20} t_9 t_9, \text{ since } (y^3 t_9 t_3 = t_7)^y \iff y^y t_{20} t_9 = t_2 \\ &= x y^y t_{20} \in [1]. \end{aligned}$$

Hence, three more symmetric generators loop back to the double coset [1].

$Nt_1t_{11} \in [1]$ since

$$\begin{aligned}
t_1\underline{t_{11}} &= t_1x^{xy^4}t_{19}, \text{ since } (xt_9 = t_6)^{xy^4} \iff x^{xy^4}t_{19} = t_{11} \\
&= t_1x^{xy^4}t_{19} \\
&= x^{xy^4}t_1^{x^{xy^4}}t_{19} \\
&= x^{xy^4}t_{17}t_{19} \\
&= x^{xy^4}\underline{(y^{xy})^{-1}y^{xy}}t_{17}t_{19} \\
&= x^{xy^4}(y^{xy})^{-1}\underline{y^{xy}t_{17}t_{19}} \\
&= x^{xy^4}(y^{xy})^{-1}t_{16}, \text{since } (y^3t_9t_3 = t_7)^{xy} \iff y^{xy}t_{17}t_{19} = t_{16} \\
&= x^{xy^4}(y^{xy})^{-1}t_{16} \in [1].
\end{aligned}$$

Hence, three more symmetric generators loop back to the double coset [1].

$Nt_1t_{15} \in [1]$ since

$$\begin{aligned}
t_1\underline{t_{15}} &= t_1x^{x^2y^3}t_4, \text{ since } (xt_9 = t_6)^{x^2y^3} \iff x^{x^2y^3}t_4 = t_{15} \\
&= t_1x^{x^2y^3}t_4 \\
&= x^{x^2y^3}t_1^{x^{x^2y^3}}t_4 \\
&= x^{x^2y^3}t_{19}t_4 \\
&= x^{x^2y^3}\underline{t_{19}}t_4 \\
&= x^{x^2y^3}x^{y^4}t_5t_4, \text{ since } (xt_9 = t_6)^{y^4} \iff x^{y^4}t_5 = t_{19} \\
&= x^{x^2y^3}x^{y^4}t_5\underline{t_4} \\
&= x^{x^2y^3}x^{y^4}t_5x^{xy^3}t_8, \text{ since } (xt_9 = t_6)^{xy^3} \iff x^{xy^3}t_8 = t_4 \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}t_5^{x^{xy^3}}t_8 \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}t_6t_8 \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}\underline{(y^{3^x})^{-1}(y^{3^x})}t_6t_8 \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}(y^{3^x})^{-1}\underline{(y^{3^x})t_6t_8} \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}(y^{3^x})^{-1}t_{18}, \text{ since } (y^3t_9t_3 = t_7)^x \iff y^{3^x}t_6t_8 = t_{18} \\
&= x^{x^2y^3}x^{y^4}x^{xy^3}(y^{3^x})^{-1}t_{18} \in [1].
\end{aligned}$$

Hence, three more symmetric generators loop back to the double coset [1].

Therefore, the remaining symmetric generators loop back to the double coset [1]. As a result, we have shown that $|G| = (\frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|})|N| \leq (1 + \frac{168}{21})21 = (8)21 = 168$.

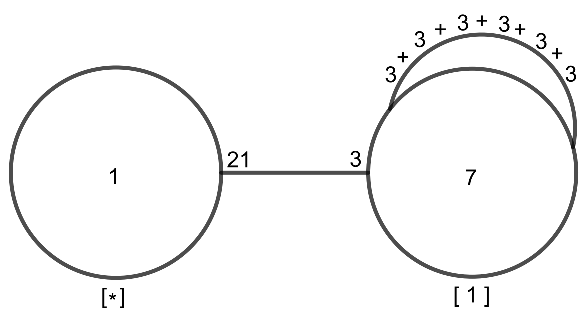

Figure 4.2: Cayley graph of $PSL(2,7)$ over $(7:3)$

4.6 $G \cong PSL(2,7)$ By Way of Linear Fractional Map

Consider $G \cong \frac{2^{*^{21}}:(7:3)}{(x^{-1}t^{(xy)})^2,(yt^{(xy^{-1})})^3}$, where $(7:3)$ is generated by $x \sim (1,2,4)$ $(3,8,10)(5,13,14)(6,16,9)(7,18,15)(11,21,17)(12,20,19)$ and $y \sim (1,3,9,20,21,15,5)$ $(2,6,17,13,8,19,7)(4,11,10,18,16,14,12)$. We will show that $G \cong PSL(2,7)$. Dickson showed that every $PSL(n,q)$ can be represented in the following manner: $PSL(n,q) = \{x \mapsto \frac{ax+b}{cx+d}, x \in F_q \cup \{\infty\}, a,b,c,d \in F_q | ad - bc = 1$ or equivalently a nonzero square$\}$. Thus, $PSL(2,7) = \{x \mapsto \frac{ax+b}{cx+d}, x \in F_7 \cup \{\infty\}, a,b,c,d \in F_7 | ad - bc = 1$ or equivalently a nonzero square $\}$. Moreover, Conway proved that $PSL(n,q) =< \alpha, \beta, \gamma >$, where $\alpha : x \mapsto x+1$, $\beta : x \mapsto kx$, where k is a nonzero square that generates all nonzero squares, and $\gamma : x \mapsto \frac{1}{-x}$. Hence, $PSL(2,7) =< \alpha, \beta, \gamma >$. As a result, we need to find α, β, and γ in order to obtain a representation of $PSL(2,7)$.

Table 4.7: $\alpha : x \mapsto x+1$

0	1	2	3	4	5	6	∞
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
1	2	3	4	5	6	0	∞

Thus, we obtain the permutation $\alpha := (\infty)(0,1,2,3,4,5,6)$.

In order to find β, we need to know all the nonzero squares of F_7. The squares of F_7 are: $1^2, 2^2, 3^2, 4^2, 5^2, 6^2$. After taking modulo 7 into account, we end up with the following nonzero squares: 1,2,4. It is easily checked that 2 generates the nonzero squares. Thus, $\beta : x \mapsto 2x$.

Table 4.8: $\beta : x \mapsto 2x$

0	1	2	3	4	5	6	∞
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	2	4	6	1	3	5	∞

Thus, we obtain the permutation $\beta := (\infty)(0)(1,2,4)(3,6,5)$.

Table 4.9: $\gamma : x \mapsto \frac{1}{-x}$

0	1	2	3	4	5	6	∞
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	6	3	2	5	4	1	0

Thus, we obtain the permutation $\gamma := (0,\infty)(1,6)(2,3)(4,5)$.

Furthermore, we can define the homomorphism $\phi : G \to PSL(2,7)$ on the generators of G by Dickson's method given by: $\phi(x) := \frac{2x-7}{-x+1}$, $\phi(y) := \frac{x}{-x+1}$, and $\phi(t) := \frac{x-1}{5x-1}$. Now $< \phi_x, \phi_y, \phi_t > \leq S_8$. This implies there is a homomorphism ϕ from G to S_8, $\phi : G \overset{Hom}{\longrightarrow} S_8$. It is easily checked that the order of $< \phi_x, \phi_y, \phi_t >$ is 168. Since the orders of ϕ_x, ϕ_y, and $\phi_{xy^{-1}x^{-1}y^2}$ are 3,7, and 1, respectively, this implies $Im(N) =< \phi_x, \phi_y > \cong (7:3)$. We ask the question, is $< \phi_x, \phi_y, \phi_t >$ a homomorphic image of G? Now, if G is a homomorphic image of $PSL(2,7)$, then it must be the case that $\phi(t)$ has 21 conjugates under conjugation by $Im(N) =< \phi_x, \phi_y >$. However, the 21 conjugates are given by, $\phi(t)^{\phi(x^i y^j)}$, where $1 \leq i \leq 3$ and $1 \leq j \leq 7$. Thus, $Im(N) =< \phi_x, \phi_y >$ acts on $\{\phi_{t_1}, \phi_{t_2}, \cdots, \phi_{t_{21}}\}$ by conjugation as $(7:3)$ acts on $\{t_1, t_2, \cdots, t_{21}\}$ by conjugation. Furthermore, it is easily checked that our additional relations $(x^{-1}t^{(xy)})^2$ and $(yt^{(xy^{-1})})^3$ hold in S_8. Thus $< \phi_x, \phi_y, \phi_t >$ is a homomorphic image of G. This implies ϕ is a homomorphism from G to S_8 and by the First Isomorphism Theorem we have, $\frac{G}{ker\phi} \cong Im(\phi) =< \phi_x, \phi_y, \phi_t >$. Thus, $|\frac{G}{ker\phi}| \cong | < \phi_x, \phi_y, \phi_t > |$. So, $\frac{|G|}{|ker\phi|} = | < \phi_x, \phi_y, \phi_t > |$, but this implies $|G| = |ker\phi|| < \phi_x, \phi_y, \phi_t > |$. Therefore, $|G| \geq 168$, but from our Cayley diagram, $|G| \leq 168$. Hence, $|G| = 168$ and $ker\phi = 1$. As a result, $G \cong < \phi_x, \phi_y, \phi_t > = PSL(2,7)$.

4.7 Construction of $U_3(5):2$ over $PGL_2(7):2$

$U_3(5):2$ is a homomorphic image of a semi-direct product between a free product of twenty eight copies of a cyclic group of order 2 and $PGL_2(7):2$ factored by the relations $(z*x^2*y*x^2*y*t^{(y*x^5)})^3$, and $(z*x^2*y*x^2*y*t^{(x^2)})^5$. In other words,

$$U_3(5):2 \cong \frac{2^{*28}:(PGL_2(7):2)}{(z*x^2*y*x^2*y*t^{(y*x^5)})^3,(z*x^2*y*x^2*y*t^{(x^2)})^5}$$

We will begin constructing the Cayley graph of $U_3(5):2$ over $PGL_2(7):2$ by double coset enumeration. Let $t \sim t_1$, $x = (1,8,14,19,23,26,6)(2,9,15,20,24,5,12)$ $(3,10,16,21,4,11,17)(7,13,18,22,25,27,28)$, $y = (1,28)(2,22)(3,18)(4,27)(5,25)$ $(6,13)(8,21)(9,17)(10,26)(11,24)(15,20)(16,19)$, and $z = (1,6)(2,5)(3,4)(8,26)(9,24)$ $(10,21)(11,17)(13,28)(14,23)(15,20)(18,27)(22,25)$, where x, y, and z are the generators of $N = PGL_2(7):2$. We will now expand our relations.

Let $\pi = z*x^2*y*x^2*y$ and note $(y*x^5) = (1,25,20,2,13,23,14)(3,7,27,16,8,10,19)$ $(4,22,5,18,11,15,9)(6,28,26,17,12,24,21)$. Therefore,

$$(z*x^2*y*x^2*y*t_1^{(y*x^5)})^3$$

$$\begin{aligned}
&= (\pi * t_{25})^3 \\
&= \pi^3 * t_{25}^{\pi^2} * t_{25}^{\pi} * t_{25} \\
&= (1,28,3,24,20,15,11,18)(2,12,22,4,26,14,10,27) \\
&\quad (5,17,9,25)(6,21,19,23,16,8,13,7)t_{17}t_9t_{25}
\end{aligned}$$

Since $\pi = (1,24,11,28,20,18,3,15)(2,4,10,12,26,27,22,14)(5,25,9,17)(6,23,13,21,$ $16,7,19,8)$

$\pi^2 = (1,11,20,3)(2,10,26,22)(4,12,27,14)(5,9)(6,13,16,19)(7,8,23,21)(15,24,28,18)$ $(17,25)$,

$\pi^3 = (1,28,3,24,20,15,11,18)(2,12,22,4,26,14,10,27)(5,17,9,25)(6,21,19,23,16,$ $8,13,7)$

Therefore $Nt_{17}t_9t_{25} = N$ for our first relation.

For our second relation, $(z{*}x^2{*}y{*}x^2{*}y{*}t_1^{(x^2)})^5$, we still have $\pi = z{*}x^2{*}y{*}x^2{*}y$ and note $x^2 = (1,14,23,6,8,19,26)(2,15,24,12,9,20,5)(3,16,4,17,10,21,11)(7,18,25,28,13,22,27)$. Then,

$$
\begin{aligned}
&(z * x^2 * y * x^2 * y * t_1^{(x^2)})^5 \\
&\qquad = (\pi * t_{14})^5 \\
&\qquad = \pi^5 * t_{14}^{\pi^4} * t_{14}^{\pi^3} * t_{14}^{\pi^2} * t_{14}^{\pi} * t_{14} \\
&\qquad = (1,18,11,15,20,24,3,28)(2,27,10,14,26,4,22,12) \\
&\qquad\quad (5,25,9,17)(6,7,13,8,16,23,19,21)t_{12}t_{10}t_4t_2t_{14}
\end{aligned}
$$

Since,
$\pi^4 = (1,20)(2,26)(3,11)(4,27)(6,16)(7,23)(8,21)(10,22)(12,14)(13,19)(15,28)(18,24)$,

$\pi^5 = (1,18,11,15,20,24,3,28)(2,27,10,14,26,4,22,12)(5,25,9,17)(6,7,13,8,16,23,19,21)$

Therefore, $Nt_{12}t_{10}t_4t_2t_{14} = N$

To determine the number of unique cosets of $PGL_2(7):2$ in $U_3(5):2$, we must calculate the index of $PGL_2(7):2$ in $U_3(5):2$. The index of $PGL_2(7):2$ in $U_3(5):2$ is given by $\frac{|U_3(5):2|}{|PGL_2(7):2|} = \frac{252{,}000}{336} = 750$. Hence, we have a total of 750 unique single cosets.

First Double Coset

We consider the first double coset NeN, denoted by $[*]$. This double coset contains one single coset, namely, N. Since N is transitive on $\{1,2,3,...,28\}$ this implies the double coset has a single orbit, $\{t_1,t_2,t_3,...,t_{28}\}$. Next, we select a representative from the orbit, say t_1. We wish to see which double coset $Nt_1N = \{Nt_1^n | n \in N\} = \{N, Nt_1, Nt_2, ..., Nt_{28}\}$. That is, all 28 $t_{i's}$ will be moving foward to the new double coset Nt_1N, denoted by [1].

Second Double Coset

We are now at our second double coset Nt_1N which 28 $t_{i's}$ moved to. Needing to determine how many unique single cosets [1] contains, we will find the coset stabilizer of 1, denoted by $N^{(1)}$ in $PGL_2(7):2$. These will be the elements of $PGL_2(7):2$ which fix 1. $N^{(1)} = <(2,3)(4,7)(5,6)(8,9)(10,13)(11,12)(15,22)(16,21)(17,20)(18,19)$ $(23,28)(24,27),(2,8)(3,9)(4,11)(5,10)(6,13)(7,12)(15,16)(17,18)(19,20)(21,22)$ $(24,27)(25,26),(2,6,7)(3,5,4)(8,12,13)(9,11,10)(14,26,25)(15,21,27)(16,24,22)$ $(17,28,18)(19,20,23)>$ therefore, the number of single cosets in Nt_1N is given by, $\frac{|N|}{|N^{(1)}|} = \frac{336}{12} = 28$. We can now determine the orbits of Nt_1 by observing the numbers in the permutations. This will imply the corresponding $t_{i's}$ will be in the same orbit. For example, the permutation (2, 6, 7)(3, 5, 4)(8, 12, 13)(9, 11, 10)(14, 26, 25)(15, 21, 27)(16, 24, 22)(17, 28, 18)(19, 20, 23), we can see that $\{2,6,7\}$ are in a 3-cycle together, this implies $\{t_2,t_6,t_7\}$ will be in the same orbit. Moving on to the next 3-cycle, (3, 5, 4), we can conclude $\{t_3,t_5,t_4\}$ will be in another orbit. If we keep repeating this process throughout the entire permutation and the rest of the generators of $N^{(1)}$, we can see the orbits of $N^{(1)}$ on $\{1,2,...,28\}$ are $\{t_1\}, \{t_{14},t_{26},t_{25}\}, \{t_{15},t_{22},t_{16},t_{21},t_{24},t_{27}\}, \{t_{17},t_{20},t_{18},t_{28},t_{19}, t_{23}\}$, and $\{t_2,t_3,t_8,t_6,t_9,t_5,t_{12},t_{13},t_7,t_{11},t_{10},t_4\}$. We now choose a member of each orbit to determine which double coset the respective coset belongs. $Nt_1t_1 = Ne = N \in [*]$, since the $t_{i's}$ are involutions. Moving forward, $Nt_1t_24 \in [1]$. By using our relation $Nt_{17}t_9 = t_{25}$ if we conjugate it by (1, 23, 22, 17)(2, 5, 27, 18)(3, 26, 13, 15)(4, 20, 28, 8) (6, 11, 25, 14)(7, 16)(9, 24)(10, 19, 21, 12) $\in$ N. We get $Nt_1t_{24} = Nt_{14}$, but $Nt_14 \in [1]$, therefore $Nt_1t_24 \in [1]$ implying six symmetric generators loop back to [1]. There does not exist a relation having Nt_1t_{17}, Nt_1t_{14}, or Nt_1t_2 equaling Nt_i for some $i \in \{1,2,...,28\}$. As a result we see that we get three new double cosets [1 17], [1 14], and [1 2]. Moreover, there will be 6, 3, and 12 symmetric generators moving forward, respectively.

Third Double Coset

At the third double coset, [1 17], we find the number of unique single cosets by finding the coset stabilizer of 1 and 17. $N^{(1,17)} = <(2,12)(3,11)(4,10)(5,9)(6,8)(7,13)$

$(14,26)(15,24)(16,21)(18,28)(19,23)(22,27) >$, therefore, the number of unique single cosets in [1 17] will be at most, $\frac{|N|}{|N^{(1,17)}|} = \frac{336}{2} = 168$. After calculating the right cosets of $N^{(117)}$ in MAGMA (also known as the transversals of the double coset [1 17]), we see that there are in fact 168 single cosets in the double coset [1 17]. We can now determine the orbits of $N^{(117)}$ by analyzing the permutations of the coset stabilizer $N^{(117)}$. The orbits on $\{1,2,...,28\}$ are $\{t_1\},\{t_{17}\},\{t_{20}\},\{t_{25}\},\{t_2,t_{12}\},\{t_3,t_{11}\},\{t_4,t_{10}\},\{t_5,t_9\}$, $\{t_6,t_8\},\{t_7,t_{13}\},\{t_{14},t_{26}\},\{t_{15},t_{24}\},\{t_{16},t_{21}\},\{t_{18},t_{28}\},\{t_{19},t_{23}\}$, and $\{t_{22},t_{27}\}$. We select a member of each orbit to determine which double coset the repective coset belongs. Using MAGMA, we see that, $t_1t_{17}t_1 \in [1,17], t_1t_{17}t_{17} \in [1], t_1t_{17}t_{20} \in [1,17], t_1t_{17}t_5 \in [1,2], t_1t_{17}t_6 \in [1,2], t_1t_{17}t_7 \in [1,17], t_1t_{17}t_{14} \in [1,17], t_1t_{17}t_{15} \in [1,17], t_1t_{17}t_{19} \in [1,2]$, $t_1t_{17}t_{16} \in [1,17,2], t_1t_{17}t_{18} \in [1,17,2]$, and $t_1t_{17}t_{22} \in [1,17]$. By looking at the orbits of the respective representatives, we see 1 symmetric generator goes back to [1], 6 symmetric generators move to [1 2], 1 symmetric generator moves to [1 17 25], 2 symmetric generators move to [1 17 2], 2 symmetric generators move to [1 17 4], 6 symmetric generators move to [1 17 3] and the remaining 10 symmetric generators loop back to the double coset [1 17].

Fourth Double Coset

At the fourth double coset, [1 2], we find the number of unique single cosets by finding the coset stabilizer of 1 and 2. $N^{(1\ 2)} =< e >$, therefore the number of unique single cosets in [1 2] are at most $\frac{|N|}{|N^{(1\ 2)}|} = \frac{336}{1} = 336$. However, we have to account for equal right cosets using our relations. Using MAGMA, we find out that we have two equal names. We have that $t_1t_2 = (y*x^{-2}*y*x^3)t_{16}t_{11}$, therefore we must also include permutations g $\in N^{(12)}$. This implies $\frac{|N|}{|N^{(12)}|} = \frac{336}{2} = 168$, hence the most number of single cosets in [1 2] is 168. After calculating the right cosets of $N^{(12)}$ in MAGMA, we see that there are in fact 168 single cosets in the double coset [1 2]. We now determine the orbits of $N^{(12)}$ on $\{1,2,...,28\}$ are $\{t_5\}$, $\{t_8\}$, $\{t_{19}\}$, $\{t_{28}\}$, $\{t_1,t_{16}\}$, $\{t_2,t_{11}\}$, $\{t_3,t_{23}\}$, $\{t_4,t_{20}\}$, $\{t_6,t_{27}\}$, $\{t_7,t_{26}\}$, $\{t_9,t_{15}\}$, $\{t_{10},t_{14}\}$, $\{t_{12},t_{18}\}$, $\{t_{13},t_{17}\}$, $\{t_{21},t_{25}\}$, and $\{t_{22},t_{24}\}$. We select a member of each orbit to determine which double coset the respective coset belongs. Using MAGMA, we see that, $t_1t_2t_5 \in$ [1 17 3], $t_1t_2t_8 \in$ [1 17 4], $t_1t_2t_{19} \in$ [1 14], $t_1t_2t_{28} \in$ [1 2], $t_1t_2t_1 \in$ [1 17 25], $t_1t_2t_2 \in$ [1], $t_1t_2t_3 \in$ [1 17 3], $t_1t_2t_4 \in$ [1 17 3], $t_1t_2t_6 \in$ [1 17 3],$t_1t_2t_7 \in$ [1 17], $t_1t_2t_9 \in$ [1 2], $t_1t_2t_{10} \in$ [1 17], $t_1t_2t_{12} \in$ [1 17

2], $t_1t_2t_{13} \in$ [1 17 25], $t_1t_2t_{21} \in$ [1 17], and $t_1t_2t_{22} \in$ [1 2]. Analyzing the orbits of the respective representatives, we see that 2 symmetric generators go back to [1], 6 move to [1 17], 1 moves to [1 14], 2 move to [1 17 2], 1 moves to [1 17 4], 7 move to [1 17 3], 4 move to [1 17 25], and 5 loop back to [1 2].

Fifth Double Coset

At the fifth double coset, [1 14], we find the number of unique single cosets by finding the coset stabilizer of 1 and 14. $N^{(1\ 14)} = <$(2,8)(3,9)$(4,11)(5,10)(6,13)(7,12)$ $(15,16)(17,18)(19,20)(21,22)(24,27)(25,26), (2,3)(4,7)(5,6)(8,9)(10,13)(11,12)$ $(15,22)(16,21)(17,20)(18,19)(23,28)(24,27) >$, therefore, the number of unique single cosets in [1 14] will be at most, $\frac{|N|}{|N^{(1\ 14)}|} = \frac{336}{4} = 84$. Additionally, we still must account for equal right cosets using our relation. Using MAGMA, we determine that we have four equal names. We have $t_1t_{14} = (y*x^{-2}*y*x^3)t_{16}t_{11} = (y*x^2*y*x^3)t_{24}t_{27} = (x^{-2}*y*x^2*y*x)t_{27}t_{24}$, therefore we must also include the permutations g $\in N^{(1\ 14)}$ such that $(t_1t_4)^g = ((y*x^{-2}*y*x^3)t_{16}t_{11})^g = ((y*x^2*y*x^3)t_{24}t_{27})^g = ((x^{-2}*y*x^2*y*x)t_{27}t_{24})^g$ implies $t_1t_2 = (y*x^{-2}*y*x^3)t_{16}t_{11} = (y*x^2*y*x^3)t_{24}t_{27} = (x^{-2}*y*x^2*y*x)t_{27}t_{24}$ or any permutation that sends $1 \to 14$ and $14 \to 1$ or $1 \to 24$ and $14 \to 27$ or $1 \to 27$ and $14 \to 24$. Upon further inspection, we find that sixteen permutations satisfy the given criterion.

$Id(N)$,

$(2,8)(3,9)(4,11)(5,10)(6,13)(7,12)(15,16)(17,18)(19,20)(21,22)(24,27)(25,26)$

$(2,3)(4,7)(5,6)(8,9)(10,13)(11,12)(15,22)(16,21)(17,20)(18,19)(23,28)(24,27)$

$(1,14)(2,9)(4,21)(5,20)(6,19)(7,22)(10,17)(11,16)(12,15)(13,18)(23,26)(25,28)$

$(1,14)(2,3,9,8)(4,22,12,16)(5,19,13,17)(6,20,10,18)(7,21,11,15)(23,25,28,26)(24,27)$

$(1,14)(2,8,9,3)(4,16,12,22)(5,17,13,19)(6,18,10,20)(7,15,11,21)(23,26,28,25)(24,27)$

$(1,14)(3,8)(4,15)(5,18)(6,17)(7,16)(10,19)(11,22)(12,21)(13,20)(23,25)(26,28)$

$(1,24,14,27)(2,23,8,26,9,28,3,25)(4,15,16,11,12,21,22,7)(5,10,17,20,13,6,19,18)$

$(1,24)(2,26)(3,28)(4,12)(5,17)(7,21)(8,23)(9,25)(11,15)(13,19)(14,27)(18,20)$

$(1,24,14,27)(2,28,8,25,9,23,3,26)(4,21,16,7,12,15,22,11)(5,6,17,18,13,10,19,20)$

$(1,24)(2,25)(3,23)(5,19)(6,10)(7,15)(8,28)(9,26)(11,21)(13,17)(14,27)(16,22)$

$(1,27)(2,28)(3,25)(4,22)(5,13)(6,18)(8,26)(9,23)(10,20)(12,16)(14,24)(15,21)$

$(1,27,14,24)(2,25,3,28,9,26,8,23)(4,7,22,21,12,11,16,15)(5,18,19,6,13,20,17,10)$

$(1,27)(2,23)(3,26)(4,16)(6,20)(7,11)(8,25)(9,28)(10,18)(12,22)(14,24)(17,19)$

$(1,27,14,24)(2,26,3,23,9,25,8,28)(4,11,22,15,12,7,16,21)(5,20,19,10,13,18,17,6)$

Therefore, we also include these permutations in $N^{(1\ 2)}$. This implies $\frac{|N|}{|N^{(1\ 2)}|} = \frac{336}{16} = 21$, hence the most number of single cosets in [1 14] is 21. After calculating the right cosets of $N^{(1\ 14)}$ in MAGMA, we see that there are in fact 21 single cosets in the double coset [1 14]. We now determine the orbits of $N^{(1\ 14)}$, we see that the orbits of $N^{(1\ 14)}$ on {1,2,...,28} are $\{t_1, t_{14}, t_{24}, t_{27}\}$, $\{t_2, t_3, t_9, t_8, t_{23}, t_{26}, t_{28}, t_{25}\}$, and $\{t_5, t_6, t_{13}, t_{20}, t_{19}, t_{17}, t_{18}, t_{10}\}$. We select a member of each orbit and determine which double coset the respective coset belongs. Using MAGMA, we see that $t_1t_{14}t_{14} \in [1]$, $t_1t_{14}t_2 \in [1\ 17\ 25]$, $t_1t_{14}t_4 \in [1\ 2]$ and $t_1t_{14}t_5 \in [1\ 17\ 3]$. By evaluating the orbits of the respective representatives, we see that 4 symmetric generators move to [1], 8 move to [1 17 25], 8 move to [1 2] and the remaining 8 move to [1 17 3].

Sixth Double Coset

At the sixth double coset, [1 17 25], we find the number of unique single cosets by finding the coset stabilizer of 1, 17, and 25. $N^{(1\ 17\ 25)} = <$ (2,12)(3,11)(4,10)(5,9)(6,8)(7,13)(14,26)(15,24)(16,21)(18,28)(19,23)(22,27)$>$, therefore, the number of unique single cosets in [1 17 25] will be at most, $\frac{|N|}{|N^{(1\ 17\ 25)}|} = \frac{336}{2} = 168$. Keeping in mind, we still must account for equal right cosets using our relations. Using MAGMA, we find out that we have three equal names. We have that $t_1t_{17}t_{25} = (y*x*y*x^{-1})t_7t_{15}t_{12} = (y*x^3)t_{13}t_{24}t_{27}$ therefore we must also include permutations g $\in N^{(1\ 17\ 25)}$ such that $(t_1t_{17}t_{25})^g = ((y*x*y*x^{-1})t_7t_{15}t_{12})^g = ((y*x^3)t_{13}t_{24}t_{27})^g$ implying $t_1t_{17}t_{25} = (y*x*y*x^{-1})t_7t_{15}t_{12} = (y*x^3)t_{13}t_{24}t_{27}$ or any permutation that sends $1 \to 7$, $17 \to 15$, and $25 \to 12$ or $1 \to 13$, $17 \to 24$, and $25 \to 2$. Upon further inspection, we find that six permutations satisfy the given criterion.

$Id(N)$

$(2,12)(3,11)(4,10)(5,9)(6,8)(7,13)(14,26)(15,24)(16,21)(18,28)(19,23)(22,27)$

$(1,7,13)(2,25,12)(3,22,9)(4,28,8)(5,27,11)(6,18,10)(14,19,21)(15,24,17)(16,23,26)$

$(1,7)(3,5)(4,6)(8,18)(9,27)(10,28)(11,22)(12,25)(14,16)(15,17)(19,26)(21,23)$

$(1,13,7)(2,12,25)(3,9,22)(4,8,28)(5,11,27)(6,10,18)(14,21,19)(15,17,24)(16,26,23)$

$(1,13)(2,25)(3,27)(4,18)(5,22)(6,28)(8,10)(9,11)(14,23)(16,19)(17,24)(21,26)$

Therefore, we also include these permutations in $N^{(1,17,25)}$. This implies $\frac{|N|}{|N^{(1,17,25)}} = \frac{336}{6} = 56$, hence the most number of single cosets in [1 17 25] is 56. After calculating the right cosets of $N^{(1,17,25)}$ in MAGMA, we see that there are in fact 56 single cosets in the double coset [1 17 25]. We now determine the orbits of $N^{(1,17,25)}$ by analyzing the permutations of the coset stabilizer $N^{(1,17,25)}$, we see the orbits of $N^{(1,17,25)}$ on {1, 2, ..., 28} are $\{t_{20}\}, \{t_1, t_7, t_{13}\}, \{t_2, t_{12}, t_{25}\}, \{t_{15}, t_{24}, t_{17}\}, \{t_3, t_{11}, t_{22}, t_5, t_9, t_{27}\}$, $\{t_4, t_{10}, t_{28}, t_6, t_8, t_{18}\}$, and $\{t_{14}, t_{26}, t_{19}, t_{16}, t_{21}, t_{23}\}$. We select a member of each orbit to determine which double coset the representative coset belongs. Using MAGMA, we see that, $t_1t_{17}t_{25}t_{20} \in [1\ 17\ 25]$, $t_1t_{17}t_{25}t_1 \in [1\ 17\ 25]$, $t_1t_{17}t_{25}t_2 \in [1\ 17]$, $t_1t_{17}t_{25}t_{15} \in [1\ 14]$, $t_1t_{17}t_{25}t_3 \in [1\ 2]$, $t_1t_{17}t_{25}t_4 \in [1\ 2]$, and $t_1t_{17}t_{25}t_{14} \in [1\ 17\ 3]$. By evaluating the orbits of the respective representatives, we see that we see 4symmetric generators loop back to [1 17 25], 3 symmetric generators move to [1 17], 3 symmetric generators move to [1 14], 6 symmetric generators move to [1 2], and 6 symmetric generators move to [1 17 3].

Seventh Double Coset

At the seventh double coset, [1 17 3], we find the number of unique single cosets by finding the coset stabilizer of 1, 17, and 3. $N^{(1,17,3)} = < Id(N) >$, therefore, the number of unique single cosets in [1 17 3] will be at most, $\frac{|N|}{N^{(1,17,3)}} = \frac{336}{1} = 336$. Not forgetting, we still must account for equal right cosets using our relations. Using MAGMA, we find out that we have two equal names. We have that $t_1t_{17}t_3 = (y * x^3 * y * x^2)t_{16}t_{13}t_{23}$, therefore we must also include permutations $g \in N^{(1,17,3)}$ such that $(t_1t_{17}t_3)^g = ((y * x^3 * y * x^2)t_{16}t_{13}t_{23})^g$ implies $t_1t_{17}t_3 = ((y * x^3 * y * x^2))^g t_{16}t_{13}t_{23}$ or any permutation which sends $1 \to 16$, $17 \to 13$, and $3 \to 23$. Upon further inspection, we find that two permutations satisfy the given criterion.

$Id(N)$

$(1,16)(2,11)(3,23)(4,20)(6,27)(7,26)(9,15)(10,14)(12,18)(13,17)(21,25)(22,24)$

Therefore, we also include these permutations to $N^(1,17,3)$. This implies $\frac{|N|}{N^{(1,17,3)}} = \frac{336}{2} = 168$, hence the most number of single cosets in [1 17 3] is 168. After calculating the right cosets of $N^(1,17,3)$ in MAGMA, we see that there are in fact 168 single cosets in the double coset [1 17 3]. We now determine the orbits of $N^(1,17,3)$ by analyzing the permutations of the coset stabilizer $N^(1,17,3)$, we see the orbits of $N^(1,17,3)$ on {1,2,...,28} are $\{t_5\}$, $\{t_8\}$, $\{t_{19}\}$, $\{t_{28}\}$, $\{t_1, t_{16}\}$, $\{t_2, t_{11}\}$, $\{t_3, t_{23}\}$, $\{t_4, t_{20}\}$, $\{t_6, t_{27}\}$, $\{t_7, t_{26}\}$, $\{t_9, t_{15}\}$, $\{t_{10}, t_{14}\}$, $\{t_{12}, t_{18}\}$, $\{t_{13}, t_{17}\}$, $\{t_{21}, t_{25}\}$, and $\{t_{22}, t_{24}\}$. We select a member from each orbit to determine which double coset the respective coset belongs. Using MAGMA, we see that, $t_1t_{17}t_3t_5 \in$ [1 17 3], $t_1t_{17}t_3t_8 \in$ [1 14], $t_1t_{17}t_3t_{19} \in$ [1 17 4], $t_1t_{17}t_3t_{28} \in$ [1 2], $t_1t_{17}t_3t_1 \in$ [1 17 2], $t_1t_{17}t_3t_2 \in$ [1 17], $t_1t_{17}t_3t_3 \in$ [1 17], $t_1t_{17}t_3t_4 \in$ [1 17 25], $t_1t_{17}t_3t_6 \in$ [1 17], $t_1t_{17}t_3t_7 \in$ [1 17 3], $t_1t_{17}t_3t_9 \in$ [1 17 2], $t_1t_{17}t_3t_{10} \in$ [1 6], $t_1t_{17}t_3t_{12} \in$ [1 17], $t_1t_{17}t_3t_{13} \in$ [1 2], $t_1t_{17}t_3t_{21} \in$ [1 17 3], and $t_1t_{17}t_3t_{22} \in$ [1 17 2]. By evaluating the orbits of the respective representatives we see that 5 symmetric generators loop back to [1 17 3], 6 symmetric generators go back to [1 17], 1 symmetric generators goes back to [1 14], 7 symmetric generators go back to [1 2], 2 symmetric generators go back to [1 17 25], 6 symmetric generators go back to [1 17 2], and 1 symmetric generators goes back to [1 17 4].

Eighth Double Coset

At the eighth double coset, [1 17 2], we find the number of unique single cosets by finding the coset stabilizer of 1, 17, and 2. $N^{(1,17,2)} = < Id(N) >$, therefore, the number of unique single cosets in [1 17 2] will be at most, $\frac{|N|}{|N^{(1,17,2)}} = \frac{336}{1} = 336$. We still must account for equal right cosets using our relations. Using MAGMA, we find out that we have equal names. We have that $t_1t_{17}t_2 = (y * x * y * x^{-2} * y)t_3t_{24}t_{19} = (y * x^2 * y * x^{-3})t_9t_{26}t_{11}$, therefore we must also include permutations g $\in N^{(1,17,2)}$ such that $(t_1t_{17}t_2)^g = ((y * x * y * x^{-2} * y)t_3t_{24}t_{19})^g = ((y * x^2 * y * x^{-3})t_9t_{26}t_{11})^g$ implies $t_1t_{17}t_2 = ((y * x * y * x^{-2} * y))^g t_3t_{24}t_{19} = ((y * x^2 * y * x^{-3}))^g t_9t_{26}t_{11}$ or any permutation that sends $1 \to 3$, $17 \to 24$, and $2 \to 19$ or $1 \to 9$, $17 \to 26$, and $2 \to 11$. Upon further inspection, we find that two permutations satisfy the given criterion.

$Id(N)$

$(1,3,9)(2,19,11)(4,20,8)(5,14,10)(6,21,12)(7,22,13)(15,23,16)(17,24,26)(18,25,27)$

$(1,9,3)(2,11,19)(4,8,20)(5,10,14)(6,12,21)(7,13,22)(15,16,23)(17,26,24)(18,27,25)$

Therefore, we also include these permutations in $N^{(1,17,2)}$. This implies $\frac{|N|}{|N^{(1,17,2)}}$ $= \frac{336}{3} = 112$. After calculating the right cosets of $N^{(1,17,2)}$ in MAGMA, we see that there are in fact 112 single cosets in the double coset [1 17 2]. We now determine the orbits of $N^{(1,17,2)}$ by analyzing the permutations of the coset stabilzer $N^{(1,17,2)}$, and see the orbits of $N^{(1,17,2)}$ on {1,2,...,28} are $\{t_{28}\}, \{t_1, t_3, t_9\}, \{t_2, t_{19}, t_{11}\}, \{t_4, t_{20}, t_8\}, \{t_5, t_{14}, t_{10}\}$, $\{t_6, t_{21}, t_{12}\}, \{t_7, t_{22}, t_{13}\}, \{t_{15}, t_{23}, t_{16}\}, \{t_{17}, t_{24}, t_{26}\}$, and $\{t_{18}, t_{25}, t_{27}\}$. We select a member of each orbit to determine which double coset the respective coset belongs. Using MAGMA, we see that $t_1t_{17}t_2t_{28} \in [1\ 17\ 4]$, $t_1t_{17}t_2t_1 \in [1\ 17\ 3]$, $t_1t_{17}t_2t_2 \in [1\ 17]$, $t_1t_{17}t_2t_4 \in [1\ 17\ 2]$, $t_1t_{17}t_2t_5 \in [1\ 17\ 3]$, $t_1t_{17}t_2t_6 \in [1\ 17\ 2]$, $t_1t_{17}t_2t_7 \in [1\ 17\ 3]$, $t_1t_{17}t_2t_{15} \in [1\ 17]$, $t_1t_{17}t_2t_{17} \in [1\ 2]$, and $t_1t_{17}t_2t_{18} \in [1\ 17\ 2]$. By evaluating the orbits of the respective representatives, we see that 12 symmetric generators loop back to [1 17 2], 9 symmetric generators go back to [1 17 3], 3 symmetric generators go back to [1 17], 3 symmetric generators go back to [1 2], and 1 symmetric generators goes back to [1 17 4].

Ninth Double Coset

At the ninth double coset, [1 17 4], we find the number of unique single cosets by finding the coset stabilizer of 1, 17, and 4. $N^{(1,17,4)} =< Id(N) >$, therefore, the number of unique single cosets in [1 17 4] will be at most, $\frac{|N|}{|N^{(1,17,4)}|} = \frac{336}{1} = 336$. Again, we still must account for equal right cosets using our relations. Using MAGMA, we find out that we have twelve equal names. We have that $t_1t_{17}t_4 = (x^2*y*x^{-3}*y*x)t_7t_{15}t_{28} = (x^3*y*x^{-3}*y)t_{13}t_{24}t_8 = (x^2*y*x^{-3}*y*x)t_1t_{19}t_{12} = (Id(N))t_{13}t_{14}t_{25} = (x^3*y*x^{-3}*y)t_7t_{21}t_2 = (Id(N))t_{11}t_{21}t_{23} = (x^3*y*x^{-3}*y)t_{27}t_{19}t_{16} = (x^2*y*x^{-3}*y*x)t_5t_{14}t_{26} = (x^3*y*x)t_5t_{24}t_3 = (x^3*y*x)t_{27}t_{17}t_{22} = (x^3*y*x)t_{11}t_{15}t_9$, therefore we must also include permutations g $\in N^{(1,17,4)}$ such that $(t_1t_{17}t_4)^g = ((x^2*y*x^{-3}*y*x)t_7t_{15}t_{28})^g = ((x^3*y*x^{-3}*y)t_{13}t_{24}t_8)^g = ((x^2*y*x^{-3}*y*x)t_1t_{19}t_{12})^g = ((Id(N))t_{13}t_{14}t_{25})^g = ((x^3*y*x^{-3}*y)t_7t_{21}t_2)^g = ((Id(N))t_{11}t_{21}t_{23})^g = ((x^3*y*x^{-3}*y)t_{27}t_{19}t_{16})^g = ((x^2*y*x^{-3}*y*x)t_5t_{14}t_{26})^g = ((x^3*y*x)t_5t_{24}t_3)^g = ((x^3*y*x)t_{27}t_{17}t_{22})^g = ((x^3*y*x)t_{11}t_{15}t_9)^g$

implies $t_1t_{17}t_4 = (x^2*y*x^{-3}*y*x)^g t_7t_{15}t_{28} = (x^3*y*x^{-3}*y)^g t_{13}t_{24}t_8 = (x^2*y*x^{-3}*y*x)^g t_1t_{19}t_{12} = (Id(N))^g t_{13}t_{14}t_{25} = (x^3*y*x^{-3}*y)^g t_7t_{21}t_2 = (Id(N))^g t_{11}t_{21}t_{23} = (x^3*y*x^{-3}*y)^g t_{27}t_{19}t_{16} = (x^2*y*x^{-3}*y*x)^g t_5t_{14}t_{26} = (x^3*y*x)^g t_5t_{24}t_3 = (x^3*y*x)^g t_{27}t_{17}t_{22} = (x^3*y*x)^g t_{11}t_{15}t_9$ or any permutation that sends $1 \to 7$, $17 \to 15$, and $4 \to 28$, $1 \to 13$, $17 \to 24$, and $4 \to 8$, $1 \to 13$, $17 \to 24$, and $4 \to 8$, $1 \to 1$, $17 \to 19$, and $4 \to 12$, $1 \to 7$, $17 \to 21$, and $4 \to 2$, $1 \to 11$, $17 \to 21$, and $4 \to 23$, $1 \to 27$, $17 \to 19$, and $4 \to 16$, $1 \to 5$, $17 \to 14$, and $4 \to 26$, $1 \to 5$, $17 \to 24$, and $4 \to 3$, $1 \to 27$, $17 \to 17$, $4 \to 22$, or $1 \to 11$, $17 \to 15$, and $4 \to 9$. Upon further inspectation, we find that twelve permutations satisfy the given criterion.

$Id(N)$

$(1,7,13)(2,25,12)(3,22,9)(4,28,8)(5,27,11)(6,18,10)(14,19,21)(15,24,17)(16,23,26)$

$(1,13,7)(2,12,25)(3,9,22)(4,8,28)(5,11,27)(6,10,18)(14,21,19)(15,17,24)(16,26,23)$

$(2,9)(3,8)(4,12)(5,13)(6,10)(7,11)(15,21)(16,22)(17,19)(18,20)(23,28)(25,26)$

$(1,13,11)(2,22,26)(3,28,16)(4,25,23)(5,7,27)(6,18,20)(8,9,12)(14,21,17)(15,19,24)$

$(1,7,5)(2,3,4)(8,22,23)(9,25,16)(10,18,20)(11,13,27)(12,28,26)(14,19,15)(17,21,24)$

$(1,11,13)(2,26,22)(3,16,28)(4,23,25)(5,27,7)(6,20,18)(8,12,9)(14,17,21)(15,24,19)$

$(1,27)(2,23)(3,26)(4,16)(6,20)(7,11)(8,25)(9,28)(10,18)(12,22)(14,24)(17,19)$

$(1,5,11)(2,16,8)(3,23,12)(4,26,9)(6,20,10)(7,27,13)(14,15,17)(19,24,21)(22,25,28)$

$(1,5,7)(2,4,3)(8,23,22)(9,16,25)(10,20,18)(11,27,13)(12,26,28)(14,15,19)(17,24,21)$

$(1,27)(2,28)(3,25)(4,22)(5,13)(6,18)(8,26)(9,23)(10,20)(12,16)(14,24)(15,21)$

$(1,11,5)(2,8,16)(3,12,23)(4,9,26)(6,10,20)(7,13,27)(14,17,15)(19,21,24)(22,28,25)$

Therefore, we also include these permutations in $N^{(1,17,4)}$. This implies $\frac{|N|}{|N^{(1,17,4)}|} = \frac{336}{12} = 28$. After calculating the right cosets of $N^{(1,17,4)}$ in MAGMA, we see that there are in fact 28 single cosets in the double coset [1 17 4]. We now determine the orbits of $N^{(1,17,4)}$ by analyzing the permutations of the coset stabilizer $N^{(1,17,4)}$, we see the orbits of $N^{(1,17,4)}$ on {1,2,...,28} are $\{t_6, t_{18}, t_{10}, t_{20}\}, \{t_1, t_7, t_{13}, t_{11}, t_{27}, t_5\}$, $\{t_2, t_{25}, t_{12}\}, \{t_{14}, t_{19}, t_{21}t_{17}, t_{24}, t_{15}\}$, and $\{t_9, t_{22}, t_3, t_{26}, t_{23}, t_{16}, t_4, t_{28}, t_8\}$. We select a member of each orbit to determine which double coset the respective coset belongs. Using MAGMA, we see that, $t_1t_{17}t_4t_6 \in$ [1 17 2], $t_1t_{17}t_4t_1 \in$ [1 2], $t_1t_{17}t_4t_{14} \in$ [1 17 3],

$t_1t_{17}t_4t_2 \in [1\ 17]$. By evaluating the orbits of the respective representatives, we see that 4 symmetric generators go back to [1 17 2], 6 symmetric generators go back to [1 2], 6 symmetric generators go back to [1 17 2], and 12 symmetric generators go back to [1 17].

We discover that we have no new double cosets. Therefore, this completes the double coset enumeration of $U_3(5):2$ over $PGL_2(7):2$.

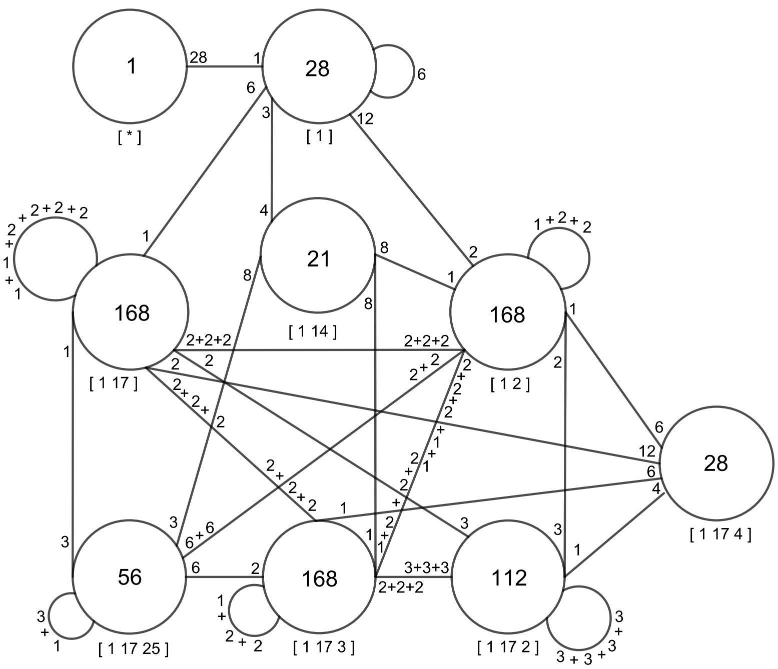

Figure 4.3: Cayley graph of $U_3(5):2$ over $PGL_2(7):2$

Chapter 5

Isomorphism Types

5.1 Introduction

Every finite group G is a subgroup of S_n. Suppose $G \leq S_n$, how is it that we describe G or by what means do we identify G by? Given that G is any finite group of S_n, identifying G can be rather challenging if we do not have any familiarity with G. For example, if you had the permutation representation and presentation of say, M_{24}, it might not be so obvious that they were the same group. However, if you had a way to identify a group regardless if it were in permutation representation or in its finitely presented form, you could quickly determine that both groups were M_{24}. This notion of how we define a finite group led to isomorphism tyes. Simply, the *isomorphism type* of a group G, is a particular name we identify G by.

We will now discuss how we find a group's isomorphism type. To have a better forsight, the analog of isomorphism types is prime factorization of some natural number. Mathematicians searched for ways to uniquely define these finite groups. Every finite group has a composition series, we can start off by calling a group by its respective composition series. For example, S_3 has the following composition series: $S_3 \rhd \ < (1,2,3) > \ \rhd \ < e >$. The very question that comes to mind is, is defining a group by its composition series good enough? The answer is no. Here is an example of why.

Example: The composition series of S_3 and $\mathbb{Z}_6$ are the following: $S_3 \triangleright\ < (1,2,3) > \triangleright\ < e >$ and $\mathbb{Z}_6 \triangleright \{0,2,4\}_6 \triangleright \{e\}_6$. We see that noth $< (1,2,3) >$ and $\{e\}_6$ are isomorphic to a cyclic group of order 3, dnoted C_3. Also, $< e >$ and $\{e\}_6$ are isomorphic to a cyclic group of order 1, denoted C_1. Therefore, the composition series of S_3 and $\mathbb{Z}_6$ become $S_3 \triangleright C_3 \triangleright C_1$ and $\mathbb{Z}_6 \triangleright C_3 \triangleright C_1$. It appears that S_3 and $\mathbb{Z}_6$ have the same composition series. However, we cannot name these groups by their respective composition series since S_3 and $\mathbb{Z}_6$ are not isomorphic. The reason, S_3 is non-abelian yet $\mathbb{Z}_6$ is abelian.

Hence, it is not enough to determine a groups isomorphism type by its composition series. Using a correspondence theorem we will show that composition factors are simple.

Theorem 5.1. *: (Correspondence Theorem) Let $K \triangleleft G$ and let $\nu : G \to G/K$ be the natural map. Then $S \mapsto \nu(S) = S/K$ is a bijection from the family of all those subgroups S of G which contain K to the family of all the subgroups of G/K. [Rot95]*

$$Moreover,\ if\ we\ denote\ S/K\ by\ S^*,\ then:$$

(i) $T \leq S \iff T^* \leq S^*,$ *and then* $[S : T] = [S^* : T^*]$ *and*
(ii) $T \triangleleft S \iff T^* \triangleleft S^*,$ *and then* $S/T \cong S^*/T^*.$

Theorem 5.2. *: Let G be a group, then H is maximal normal in $G \iff G/H$ is simple. [Rot95]*

Proof: ($\Rightarrow$) *Assume H is maximal normal in G. Suppose there exists $K/H \triangleleft G/H$ (w.t.s. $K/H = 1$ or $K/H = G/H$). Since $K/H \triangleleft G/H$, this imlies $K \triangleleft G$. Moreover, $H \triangleleft K$ by definition of a factor set. However, H is maximal normal in G. Hence, $H = K$ or $H = G$. From either of the two cases we see we get $K/H = 1$ or $K/H = G/H$.*
($\Leftarrow$) *Assume G/H is simple (w.t.s. H is maximal normal in G). Let $K \triangleleft G$ such that $H \leq K \leq G$. Then $K/H \triangleleft G/H$, but G/H is simple. This implies $K/H = 1$ or $K/H = G/H$. If $K/H = 1$, then $K = H$ or if $K/H = G/H$, then $K = G$. Hence, H is maximal normal in G.* $\square$

Therefore, given a groups composition series, we can derive its composition factors. More importantly, these composition factors are simple. This is nemeficial sinve we know everuthing there is to know about simple groups. By the classification of finite simple groups theorem, it states that if a group G is simple, then it belongs to one of the four classes of simple groups. These four classes include the cyclic groups of prime order, alternating groups of at least degree 5, groups of Lie type, or a sporadic group. In addition, the Jordan-Hölder theorem states, every two composition series of a group G are equivalent. In other words, given a finite group G with two composition series, these two series will be of the same length and have the same compositiion factors. We will illustrate this idea by way of an example.

Example: We list three composition series of C_{12}, a cyclic group of order 12.

1) $C_{12} \triangleright C_6 \triangleright C_2 \triangleright C_1$
2) $C_{12} \triangleright C_4 \triangleright C_2 \triangleright C_1$
3) $C_{12} \triangleright C_6 \triangleright C_3 \triangleright C_1$

The composition factors are as follow:
1) C_2, C_3, C_2
2) C_3, C_2, C_2
3) C_2, C_2, C_3

Notice that the composition factors are the same.

We will now illustrate how we can write a group G by its composition factors. Let G be a group with corresponding factor groups $K_0/K_1 = Q_1, \ldots, K_{n-2}/K_{n-1} = Q_{n-1}, K_{n-1}/K_n = Q_n$. Notice $K_n = 1$, this implies $K_{n-1} = Q_n$. Moreover, K_{n-2}/K_{n-1} $= Q_{n-2}$, we get K_{n-2} from Q_n and Q_{n-1} provided we solve the extension problem, $K_{n-2} = Q_n \cdot Q_{n-1}$. Continuing, we can recapture K_{n-3} given $K_{n-3}/K_{n-2} = K_{n-3}/Q_n \cdot Q_{n-1} = Q_{n-3}$ provided we solve the extension problem, $K_{n-3} = Q_n \cdot Q_{n-1} \cdot Q_{n-2}$. It is clear, if we keep repeating this process up to $G = K_0$ we can recapture G by the following $K_0/Q_nQ_{n-1}\cdot\cdots\cdot Q_2 = Q_1$ provided we solve the extension problem , $K_0 = (Q_n\cdot Q_{n-1}\cdot\cdots\cdot Q_2)\cdot Q_1$. To great avail, we can now write our finite group G as a product of simple groups provided we solve the extension problems of G.

We now turn our attention back to S_3 and $\mathbb{Z}_6$. We are familiar with the composition series and composition factors of S_3 and $\mathbb{Z}_6$. We write S_3 and $\mathbb{Z}_6$ as a product of their composition factors: $S_3 = C_2{\cdot}C_3$ and $\mathbb{Z}_6 = C_2{\cdot}C_3$. We now solve their extension types. $C_3 \lhd S_3$ and $C_2 \cap C_3 = \emptyset$, thus $S_3 = C_2 : C_3$. Furthermore, $C_3 \lhd \mathbb{Z}_6$ and $C_2 \lhd \mathbb{Z}_6$, thus $\mathbb{Z}_6 = C_2 \times C_3$. Just as suspected, S_3 and $\mathbb{Z}_6$ have different isomorphism types. S_3 is a semi-direct product which implies a non-abelian group. Also, a direct product implies an abelian group. As a result, we now have a means to identify any finite group G.

5.2 Direct Product

5.2.1 $G \cong 7 \times 2^2$

Let $G =< (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28), (1,19,9,28,17,8,25,15,6,23,14,4,22,12)(2,20,10,27,18,7,26,16,5,24,13,3,21,11) >$, a permutation group acting on a set of cardinality 28 of order 28, then G has the isomorphism type $G \cong 7 \times 2^2$. We begin by analyzing the composition factors and the normal lattice of $G2$.

$CompositionFactors(G)$

G
$|\quad Cyclic(7)$
$*$
$|\quad Cyclic(2)$
$*$
$|\quad Cyclic(2)$
1

$NormalLattice(G)$

$- -$

[10] $Order$ 28 $Length$ 1 $MaximalSubgroups$: 6 7 8 9

$- - -$

[9] $Order$ 14 $Length$ 1 $Maximal\ Subgroups$: 4 5

[8] $Order$ 14 $Length$ 1 $Maximal\ Subgroups$: 2 5

[7] $Order$ 14 $Length$ 1 $Maximal\ Subgroups$: 3 5

[6] $Order$ 4 $Length$ 1 $Maximal\ Subgroups$: 2 3 4

$- - -$

[5] $Order$ 7 $Length$ 1 $Maximal\ Subgroups$: 1

[4] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

[3] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

[2] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

$- - -$

[1] $Order$ 1 $Length$ 1 $Maximal\ Subgroups$:

We find that the largest abelian subgroup of G is G. That is, all subgroups of G are normal in G. Therefore, we need to find two disjoint subgroups K and Q such that $G = K \cdot Q$. From the normal lattice of G, we see that $NL[5] \cap NL[6] = \emptyset$ and we suspect $G \cong NL[5] \times NL[6]$. $NL[5]$ is a cyclic group of order 7, its presentation is $< a|a^7 >$. Furthermore, $NL[6]$ is a group of order 4. Its isomorphism type is either $NL[6] \cong 4$ or $NL[6] \cong 2^2$. Looking at the normal lattice of $NL[6]$ we suspect $NL[6] \cong 2^2$.

$NormalLattice(NL[5])$

$- -$

[5] $Order$ 4 $Length$ 1 $Maximal\ Subgroups$: 2 3 4

$- - -$

[4] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

[3] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

[2] $Order$ 2 $Length$ 1 $Maximal\ Subgroups$: 1

$- - -$

[1] $Order$ 1 $Length$ 1 $Maximal\ Subgroups$:

To check, we combine the presentation of [2] and [3], normal subgroups of $NL[5]$, both of order 2. In addition, we make make the generators commute. We run the following check:

$HH < b, c > := Group < b, c | b^2, c^2, (b * c)^2 >;$

$f, H, k := CosetAction(HH, sub < HH | Id(HH) >);$

$s := IsIsomorphic(H, NL[6]);$

$true$

Hence, $NL[6] \cong 2^2$.

Now, we need to combine the presentation of $NL[5]$ and $NL[6]$. We have the presentation of 7×2 as being $< a, b, c | a^7, b^2, c^2, (b * c)^2, (a, b), (a, c) >$. We run the following check to see if G is isomorphic to a representation of our finitely presented group 7×2^2.

$HH < a, b, c > := Group < a, b, c | a^7, b^2, c^2, (b * c)^2, (a, b), (a, c) >;$

$f, H, k := CosetAction(HH, sub < HH | Id(HH) >);$

$s := IsIsomorphic(H, G1);$

Hence, $G \cong 7 \times 2^2$.

5.3 Semi-Direct Product

5.3.1 $G \cong U_3(5) : 2$

Let $G \cong \frac{2^{*^{28}}:(PGL(2,7):2)}{(z*x^2*y*x^2*y*t^{(y*x^5)})^3,(z*x^2*y*x^2*y*t^{(x^2)})^5}$, then G has the isomorphism type $G \cong U_3(5) : 2$. A representation of G is given by $G1$, where $G1$ is a permutation group acting on a set of cardinality 750 and $|G1| = 252000$. $G1$ is currently represented on 750 letters after we performed the coset action command on G, $f1, G1, k1 := CosetAction(G, sub < G|x, y, z >);$. We would like to find the minimum degree that G can be represented on.

We find G can be represented by $G2 =< (1,2,6,13,3,8,12)(4,5,11,9,18,14,7)$ $(10,20,32,24,22,29,42)(15,23,16,25,35,34,27)(17,19,26,21,33,38,31)(28,40,44,45,$ $36,30,43)(37,46,39,41,48,49,47), (1,3)(2,6)(4,9)(7,14)(10,21)(15,24)(16,26)(17,28)$ $(19,30)(20,23)(22,33)(25,36)(27,38)(29,42)(31,44)(34,45)(37,47)(39,46)(41,49)(48,50),$ $(1,4)(2,7)(3,9)(5,12)(6,14)(8,11)(10,22)(13,18)(15,23)(16,27)(19,31)(20,24)(21,33)$ $(25,34)(26,38)(29,42)(30,44)(36,45)(37,46)(39,47)(40,43)(41,49), (1,5)(3,10)(6,15)$ $(7,16)(8,17)(9,19)(11,13)(12,14)(18,29)(20,22)(21,26)(24,34)(25,37)(27,39)(28,41)$ $(30,45)(31,38)(32,35)(33,46)(40,47)(42,50)(48,49) >$, a permutation group acting on a set of cardinality 50 and where $|G2| = 252000$. We run a check to show that $G2 \cong G1$.

$IsIsomorphic(G1, G2);$

$true$

We begin by analyzing the composition factors and the normal lattice of $G2$.

$CompositionFactors(G2)$

$$
\begin{array}{ll}
G & \\
| & Cyclic(2) \\
* & \\
| & 2A(2,5) \quad = U(3,5) \\
1 &
\end{array}
$$

We see that $G2 \cong U_3(5) \cdot 2$. $NormalLattice(G2)$

$- -$

[3] $Order$ 252000 $Length$ 1 $Maximal\ Subgroups : 2$

$- - -$

[2] $Order$ 126000 $Length$ 1 $Maximal\ Subgroups : 1$

− − −

[1] *Order* 1 *Length* 1 *Maximal Subgroups* :

We find that the largest abelian group or the smallest normal subgroup of $G2$ is $NLG2[2] = U_3(5) =< (1, 16, 7, 5)(2, 13, 23, 11)(3, 19, 14, 50)(4, 37, 44, 25)(6, 15, 28, 41)(8, 22, 39, 31)(9, 10, 42, 12)(17, 38, 27, 20)(18, 21, 46, 32)(26, 29, 35, 33)(30, 45)(36, 43)(40, 49)(47, 48), (1, 5, 15, 6)(2, 39, 43, 27)(3, 38, 46, 24)(4, 14, 23, 12)(7, 16, 40, 47)(8, 29, 42, 11)(9, 20, 37, 26)(10, 34, 33, 31)(13, 50, 18, 17)(19, 21, 25, 22)(28, 48)(30, 45)(36, 44)(41, 49), (2, 25)(3, 26)(4, 27)(6, 7)(10, 47)(13, 30)(14, 45)(15, 44)(16, 43)(17, 22)(18, 31)(19, 20)(21, 28)(29, 41)(33, 46)(34, 48)(35, 49)(36, 50)(37, 39)(38, 40), (1, 26, 12, 16, 3)(2, 34, 9, 19, 24)(4, 45, 6, 15, 30)(5, 10, 7, 14, 21)(8, 41, 42, 29, 49)(17, 48, 18, 50, 28)(20, 22, 27, 44, 39)(23, 46, 31, 38, 33)(25, 37, 47, 36, 40), (1, 8, 13, 2, 12, 3, 6)(4, 14, 9, 5, 7, 18, 11)(10, 29, 24, 20, 42, 22, 32)(15, 34, 25, 23, 27, 35, 16)(17, 38, 21, 19, 31, 33, 26)(28, 30, 45, 40, 43, 36, 44)(37, 49, 41, 46, 47, 48, 39), (1, 7, 17, 45, 31)(2, 47, 15, 46, 14)(3, 39, 48, 25, 22)(4, 40, 38, 9, 27)(6, 18, 33, 21, 26)(10, 34, 16, 24, 50)(11, 42, 35, 41, 49)(12, 23, 13, 19, 30)(28, 44, 36, 43, 37), (1, 3)(2, 8, 6, 13)(4, 14, 7, 9)(5, 18)(10, 29, 27, 25)(15, 44, 31, 24)(16, 32, 26, 43)(17, 34, 45, 28)(19, 40, 30, 35)(20, 22)(21, 36, 38, 42)(23, 33)(37, 39, 49, 48)(41, 46, 47, 50), (1, 6, 22, 14, 18)(2, 17, 26, 37, 34)(3, 7, 31, 46, 28)(4, 27, 38, 40, 9)(10, 44, 33, 45, 25)(11, 42, 49, 29, 35)(12, 23, 30, 20, 13)(15, 50, 16, 39, 21)(24, 36, 47, 48, 43) >$, a permutation group of order 126000.

We factor $G2$ by $U_3(5)$, in doing so, $G2/U_3(5) \cong Q = \{U_3(5), U_3(5)(1, 4)(2, 7)(3, 9)(5, 12)(6, 14)(8, 11)(10, 22)(13, 18)(15, 23)(16, 27)(19, 31)(20, 24)(21, 33)(25, 34)(26, 38)(29, 42)(30, 44)(36, 45)(37, 46)(39, 47)(40, 43)(41, 49)\}$, where Q is a permutation group acting on a set of cardinality 2 and $|Q| = 2$.

To fulfill the semi-direct product requirement, we must calculate the action Q on the generators of $U_3(5)$. As a result, we will need the transversals of $U_3(5)$ in $G2$ in order to see which right coset representative corresponds to the generator of the cyclic group of order 2 . However, we can see there is only one nontrivial right coset representative. Hence, $k \sim (1, 4)(2, 7)(3, 9)(5, 12)(6, 14)(8, 11)(10, 22)(13, 18)(15, 23)(16, 27)(19, 31)(20, 24)(21, 33)(25, 34)(26, 38)(29, 42)(30, 44)(36, 45)(37, 46)(39, 47)(40, 43)(41, 49)$. We now calculate the action of a on the generators of $U_3(5)$. Let $a \sim (1, 16, 7, 5)(2, 13, 23, 11)(3, 19, 14, 50)(4, 37, 44, 25)(6, 15, 28, 41)(8, 22, 39, 31)(9, 10, 42, 12)(17, 38, 27, 20)(18, 21, 46, 32)(26, 29, 35, 33)(30, 45)(36, 43)(40, 49)(47, 48)$, $b \sim (1, 5, 15, 6)(2, 39, 43, 27)(3, 38, 46, 24)$

$(4,14,23,12)(7,16,40,47)(8,29,42,11)(9,20,37,26)(10,34,33,31)(13,50,18,17)(19,21,25,22)(28,48)(30,45)(36,44)(41,49)$, $c \sim (2,25)(3,26)(4,27)(6,7)(10,47)(13,30)(14,45)(15,44)(16,43)(17,22)(18,31)(19,20)(21,28)(29,41)(33,46)(34,48)(35,49)(36,50)(37,39)(38,40)$, $d \sim (1,26,12,16,3)(2,34,9,19,24)(4,45,6,15,30)(5,10,7,14,21)(8,41,42,29,49)(17,48,18,50,28)(20,22,27,44,39)(23,46,31,38,33)(25,37,47,36,40)$, $e \sim (1,8,13,2,12,3,6)(4,14,9,5,7,18,11)(10,29,24,20,42,22,32)(15,34,25,23,27,35,16)(17,38,21,19,31,33,26)(28,30,45,40,43,36,44)(37,49,41,46,47,48,39)$, $f \sim (1,7,17,45,31)(2,47,15,46,14)(3,39,48,25,22)(4,40,38,9,27)(6,18,33,21,26)(10,34,16,24,50)(11,42,35,41,49)(12,23,13,19,30)(28,44,36,43,37)$, $i \sim (1,3)(2,8,6,13)(4,14,7,9)(5,18)(10,29,27,25)(15,44,31,24)(16,32,26,43)(17,34,45,28)(19,40,30,35)(20,22)(21,36,38,42)(23,33)(37,39,49,48)(41,46,47,50)$, and $j \sim (1,6,22,14,18)(2,17,26,37,34)(3,7,31,46,28)(4,27,38,40,9)(10,44,33,45,25)(11,42,49,29,35)(12,23,30,20,13)(15,50,16,39,21)(24,36,47,48,43)$, denote the generators of $U_3(5)$.

Consider the following:

$a^k = (1,46,30,34)(2,12,4,27)(3,22,29,5)(6,50,9,31)(7,18,15,8)(10,47,19,11)(13,33,37,32)(14,23,28,49)(16,24,17,26)(21,38,42,35)(36,44)(39,48)(40,45)(41,43)$

$b^k = (1,6,15,5)(2,27,43,39)(3,24,46,38)(4,12,23,14)(7,47,40,16)(8,11,42,29)(9,26,37,20)(10,31,33,34)(13,17,18,50)(19,22,25,21)(28,48)(30,45)(36,44)(41,49)$

$c^k = (1,16)(2,14)(6,36)(7,34)(9,38)(10,17)(13,19)(18,44)(21,37)(22,39)(23,30)(24,31)(25,48)(26,43)(27,40)(28,33)(35,41)(42,49)(45,50)(46,47)$

$d^k = (1,36,14,23,44)(2,6,33,12,22)(3,31,20,7,25)(4,38,5,27,9)(10,16,30,47,24)(11,49,29,42,41)(13,50,28,17,48)(15,37,19,26,21)(34,46,39,45,43)$

$e^k = (1,6,3,12,2,13,8)(4,11,18,7,5,9,14)(10,32,22,42,20,24,29)(15,16,35,27,23,25,34)(17,26,33,31,19,21,38)(28,44,36,43,40,45,30)(37,39,48,47,46,41,49)$

$f^k = (1,43,26,3,16)(2,17,36,19,4)(5,15,18,31,44)(6,7,39,23,37)(8,29,35,49,41)(9,47,48,34,10)(13,21,33,38,14)(20,50,22,25,27)(28,30,45,40,46)$

$i^k = (1,6,2,3)(4,9)(7,11,14,18)(10,24)(12,13)(15,21)(16,34,22,42)(17,25,36,28)(19,20,23,30)(26,29,33,45)(27,32,38,40)(31,43,44,35)(37,39,50,49)(41,48,46,47)$

$j^k = (1,16,26,43,3)(2,19,37,28,9)(4,14,10,6,13)(5,15,44,24,18)(7,17,38,46,25)(8,29,41,42,35)(20,45,39,48,40)(21,36,34,22,30)(23,50,27,47,33)$.

A presentation for $U_3(5)$ is is given by $< a,b,c,d,e,f,i,j | a^4, b^4, c^2, d^5, e^7, f^5, i^4, j^5, c*f^{-1}*c*j, j^{-1}*f^{-1}*c*f^{-1}, (b^{-1}*a^{-1})^3, a*j^{-1}*b^{-1}*a^{-1}*i*e^{-1}, f^{-1}*e^{-1}*b*a^{-1}*f^{-1}*e, f^{-1}*j*c*a*i^2*a^{-1}, i*j*c*b*i^2*c, f*b^{-1}*d^{-1}*c*f^{-1}*d, (b^{-1}*d^{-2})^2, j*d^2*c*b^2*j, c*b^{-2}*d*b^{-1}*f^2, f*b^{-2}*d*b*j^{-1}*b, f^2*d^2*f^{-1}*b, (c*i^{-1}*d)^2, j*i^{-1}*d*a^{-1}*j*i, (e^{-1}*a)^3, e*a^{-2}*e^{-1}*a*b^2*a^{-1}, (e^{-1}*b^{-1})^3, a*e^{-2}*d^{-1}*a^2*f, b*f^{-1}*e^{-1}*b^{-1}*f*e, e^{-1}*b^{-2}*e*b*a^2*b^{-1}, f*b^{-2}*e*d^{-1}*i*a^{-1}, (e*i*e)^2, j^{-1}*i^{-1}*e*a*i^2*j^{-1}, (i^{-1}*d^{-1})^3, (i^{-1}*d)^3, j*b^{-1}*i*d*a^2*e^{-1}, (i*e^{-1})^3, j^{-1}*i^{-1}*j*b*i^2*f, j^{-1}*i^{-2}*j*e*i*e, b^{-1}*i^{-1}*a^{-2}*j^{-1}*d*a^{-1}, e^2*b^{-1}*a^{-1}*d^{-1}*a*f, b*a^{-1}*b*a^-1*e*i*e >$. We now add the presentation $< k | k^2 >$ of $G2/U_3(5) \cong Q$ to the presentation of $U_3(5)$ along with the action just calculated. However, we must convert the permutation obtained from the action of Q on the generators of $U_3(5)$ into words in terms of the generators of $U_3(5)$. We run a Schreier system that converts the aforementioned action into words in terms of generators of $U_3(5)$. We get the following:

$$\begin{aligned}
a^k &= e^{-1} * a^{-1} * e \\
b^k &= b^{-1} \\
c^k &= c^b \\
d^k &= f^{-1} * d * j * d^{-1} \\
e^k &= e^{-1} \\
f^k &= j^{-1} * d^{-1} * j \\
i^k &= e^{-2} * i * e^{-1} \\
j^k &= c * f * b.
\end{aligned}$$

Therefore, we have the following presentation: $< a, b, c, d, e, f, i, j, k | a^4, b^4, c^2, d^5, e^7, f^5, i^4, j^5, c*f^{-1}*c*j, j^{-1}*f^{-1}*c*f^{-1}, (b^{-1}*a^{-1})^3, a*j^{-1}*b^{-1}*a^{-1}*i*e^{-1}, f^{-1}*e^{-1}*b*a^{-1}*f^{-1}*e, f^{-1}*j*c*a*i^2*a^{-1}, i*j*c*b*i^2*c, f*b^{-1}*d^{-1}*c*f^{-1}*d, (b^{-1}*d^{-2})^2, j*d^2*c*b^2*j, c*b^{-2}*d*b^{-1}*f^2, f*b^{-2}*d*b*j^{-1}*b, f^2*d^2*f^{-1}*b, (c*i^{-1}*d)^2, j*i^{-1}*d*a^{-1}*j*i, (e^{-1}*a)^3, e*a^{-2}*e^{-1}*a*b^2*a^{-1}, (e^{-1}*b^{-1})^3, a*e^{-2}*d^{-1}*a^2*f, b*f^{-1}*e^{-1}*b^{-1}*f*e, e^{-1}*b^{-2}*e*b*a^2*b^{-1}, f*b^{-2}*e*d^{-1}*i*a^{-1}, (e*i*e)^2, j^{-1}*i^{-1}*e*a*i^2*j^{-1}, (i^{-1}*d^{-1})^3, (i^{-1}*d)^3, j*b^{-1}*i*d*a^2*e^{-1}, (i*e^{-1})^3, j^{-1}*i^{-1}*j*b*i^2*f, j^{-1}*i^{-2}*j*e*i*e, b^{-1}*i^{-1}*a^{-2}*j^{-1}*d*a^{-1}, e^2*b^{-1}*a^{-1}*d^{-1}*a*f, b*a^{-1}*b*a^-1*e*i*e, k^2, a^k = e^{-1}*a^{-1}*e, b^k = b^{-1}, c^k = c^b, d^k = f^{-1}*d*j*d^{-1}, e^k = e^{-1}, f^k = j^{-1}*d^{-1}*j, i^k = e^{-2}*i*e^{-1}, j^k = c*f*b. >$.
We need to see if $G2$ is isomorphic to a representation of our finitely presented group $U_3(5):2$.

$HH < a, b, c, d, e, f, i, j, k >:= Group < a, b, c, d, e, f, i, j, k|$

$a^4,$

$b^4,$

$c^2,$

$d^5,$

$e^7,$

$f^5,$

$i^4,$

$j^5,$

$c * f^{-1} * c * j,$

$j^{-1} * f^{-1} * c * f^{-1},$

$(b^{-1} * a^{-1})^3,$

$a * j^{-1} * b^{-1} * a^{-1} * i * e^{-1},$

$f^{-1} * e^{-1} * b * a^{-1} * f^{-1} * e,$

$f^{-1} * j * c * a * i^2 * a^{-1},$

$i * j * c * b * i^2 * c,$

$f * b^{-1} * d^{-1} * c * f^{-1} * d,$

$(b^{-1} * d^{-2})^2,$

$j * d^2 * c * b^2 * j,$

$c * b^{-2} * d * b^{-1} * f^2,$

$f * b^{-2} * d * b * j^{-1} * b,$

$f^2 * d^2 * f^{-1} * b,$

$(c * i^{-1} * d)^2,$

$j * i^{-1} * d * a^{-1} * j * i,$

$(e^{-1} * a)^3,$

$e * a^{-2} * e^{-1} * a * b^2 * a^{-1},$

$(e^{-1} * b^{-1})^3,$

$a * e^{-2} * d^{-1} * a^2 * f,$

$b * f^{-1} * e^{-1} * b^{-1} * f * e,$

$e^{-1} * b^{-2} * e * b * a^2 * b^{-1},$

$f * b^{-2} * e * d^{-1} * i * a^{-1},$

$(e * i * e)^2,$

$j^{-1} * i^{-1} * e * a * i^2 * j^{-1},$

$(i^{-1} * d^{-1})^3,$

$(i^{-1} * d)^3,$

$j * b^{-1} * i * d * a^2 * e^{-1},$

$(i * e^{-1})^3,$

$j^{-1} * i^{-1} * j * b * i^2 * f,$

$j^{-1} * i^{-2} * j * e * i * e,$

$b^{-1} * i^{-1} * a^{-2} * j^{-1} * d * a^{-1},$

$e^2 * b^{-1} * a^{-1} * d^{-1} * a * f,$

$b * a^{-1} * b * a^-1 * e * i * e,$

$k^2,$

$a^k = e^{-1} * a^{-1} * e,$

$b^k = b^{-1},$

$c^k = c^b,$

$d^k = f^{-1} * d * j * d^{-1},$

$e^k = e^{-1},$

$f^k = j^{-1} * d^{-1} * j,$

$i^k = e^{-2} * i * e^{-1},$

$j^k = c * f * b >;$

$f, H, k := CosetAction(HH, sub < HH|Id(HH) >);$

$IsIsomorphic(H, G2);$

$true$

Hence, $G2 \cong U_3(5) : 2$. We say, G is a semi-direct product of $U_3(5)$ by a cyclic group of order 2. Or, G is isomorphic to the automorphism group of $U_3(5)$.

5.4 Central Extension

5.4.1 $G \cong 2^{\bullet}(M_{12} : 2)$

Let $G \cong \frac{2^{*^{60}}:S_5}{(yt^{y^3x})^4,(yt^x)^2,(yt^{xy^{-2}xy^{-1}})^8} =< x, y, t | x^2, y^6, (y * x * y^-1 * x)^2, (x * y^-1)^5, t^2, (t, (y * x * y)^2), (y * t^(y^3 * x))^4, (y * t^(x))^2, (y * t^(x * y^-2 * x * y^-1))^8 >$, then G has the isomorphism type $G \cong 2^{\bullet}M_{12} : 2$. A representation of G is given by $G1$, where $G1$ is a permutation group acting on a set of cardinality 3168 and $|G1| = 380160$. $G1$ is currently represented on 3168 letters after we performed the coset action command on G, $f1, G1, k1 := CosetAction(G, sub < G|x, y >);$. We would like to find the minimum degree that G can be represented on.

We find G can be represented by $G2 =< (1,2)(3,6)(4,5)(7,12)(8,15)(9,16)(10,19)$ $(11,21)(13,24)(14,27)(17,31)(18,34)(20,36)(22,37)(23,32)(25,30)(26,40)(28,29)$ $(33,42)(35,41)(38,44)(39,43)(45,47)(46,48)(1,3,7,13,25,2)(4,5,9,17,32,21)$ $(6,10,18,34,41,30)(8,15,20,11,16,29)(12,22,37,24,39,42)(14,28,36,44,46,48)$ $(19,35,43,45,47,33)(23,38,27,31,26,40)(1,4)(2,5)(3,8)(6,11)(7,14)(9,18)$ $(10,20)(12,23)(13,26)(15,19)(16,30)(17,33)(21,35)(22,32)(24,31)(25,36)(27,39)$ $(28,34)(29,41)(37,44)(38,42)(40,43)(45,48)(46,47) >$, a permutation group acting on a set of cardinality 48 and where $|G2| = 380160$. We run a check to show that $G2 \cong G1$.
$IsIsomorphic(G1, G2);$
$true$
We begin by analyzing the composition factors and the normal lattice of $G2$.
$CompositionFactors(G2)$

G
$|\quad Cyclic(2)$
$*$
$|\quad M12$
$*$
$|\quad Cyclic(2)$
1

We see that $G2 \cong 2 \cdot M_{12} \cdot 2$.

$NormalLattice(G2)$

$- -$

[7] $Order$ 380160 $Length$ 1 $Maximal\ Subgroups$: 4 5 6

$---$

[6] $Order$ 190080 $Length$ 1 $Maximal\ Subgroups$: 3

[5] $Order$ 190080 $Length$ 1 $Maximal\ Subgroups$: 2 3

[4] $Order$190080 $Length$ 1 $Maximal\ Subgroups$: 3

$---$

[3] $Order$ 95040 $Length$ 1 $Maximal\ Subgroups$: 1

$---$

[2] $Order$ 2 $Length$1$Maximal\ Subgroups$: 1

$---$

[1] $Order$ 1 $Length$ 1 $Maximal\ Subgroups$:

We find that the largest abelian group or the smallest normal subgroup of $G2$ is $NLG2[2] =< (1,5)(2,4)(3,9)(6,16)(7,17)(8,18)(10,29)(11,30)(12,31)(13,32)(14,33)(15,34)(19,28)(20,41)(21,25)(22,26)(23,24)(27,42)(35,36)(37,40)(38,39)(43,44)(45,46)(47,48) >$, a group of order 2. It turns out that $NLG2[2] = Z(G2)$. This implies we will have a central extension in our isomorphism type. We factor $G2$ by $Z(G2)$, in doing so, $G2/Z(G2) \cong Q_1$, where Q_1 is a permutation group acting on a set of cardinality 144 and $|Q1| = 190080$. Q_1 is currently represented on 144 letters. We would like to find the minimum degree that Q_1 can be represented by. We find Q_1 can be represented by $G3 =< (1,2)(3,6)(4,8)(5,9)(7,12)(10,13)(11,16)(14,21)(15,22)(17,23)(18,20)(19,24), (1,3,7,9,8,2)(4,6,11,17,23,15)(5,10,14,12,19,24)(13,20,18,21,16,22), (1,4)(2,5)(3,8)(6,12)(7,9)(10,15)(11,18)(13,21)(14,20)(16,22)(17,23)(19,24) >$, a permutation group acting on a set of cardinality 24 and $|G2| = 190080$. We run a check to show that $G3 \cong Q_1$.

$IsIsomorphic(Q1, G3)$;

$true$

We continue by analyzing the composition factors and the normal lattice of $G3$.

$CompositionFactors(G3)$

G

$|\quad Cyclic(2)$

$*$

$|\quad M12$

1

We see that $G3 \cong M_{12} \cdot 2$.

$NormalLattice(G3)$

$- -$

[3] $Order$ 190080 $Length$ 1 $Maximal\ Subgroups$: 2

$- - -$

[2] $Order$ 95040 $Length$ 1 $Maximal\ Subgroups$: 1

$- - -$

[1] $Order$ 1 $Length$ 1 $Maximal\ Subgroups$:

We see that $NLG3[2] = M_{12} =< (1,17)(2,9)(3,10)(4,24)(5,16)(6,8)(7,18)(11,22)$ $(12,21)(13,15)(14,19)(20,23), (3,10,12)(4,9,11)(5,13,17)(6,19,15)(7,18,14)(21,23,22) >$ is normal in $G3$. There is not another normal subgroup of order 2. Hence, we will have a semi-direct product with a cyclic group of order 2. When we factor $G3$ by M_{12}, we have $G3/M_{12} = \{M_{12}, M_{12}(1,2)(3,6)(4,8)(5,9)(7,12)(10,13)(11,16)(14,21)(15,22)(17,23)$ $(18,20)(19,24)\}$, a cyclic group of order 2 whose presentation is given by $< a|a^2 >$. To fulfill the semi-direct product, we must calculate the action of the aforementioned cyclic group on the generators of M_{12}. As a result, we will need the transversals of M_{12} in $G3$ in order to see which right coset representative corresponds to the generator of the cyclic group of order 2 . However, we can see there is only one nontrivial right coset representative. Hence, $a \sim (1,2)(3,6)(4,8)(5,9)(7,12)(10,13)(11,16)(14,21)(15,22)(17,23)(18,20)$ $(19,24)$. We now calculate the action of a on the generators of M_{12}. Let $b \sim (1,17)(2,9)(3,10)$ $(4,24)(5,16)(6,8)(7,18)(11,22)(12,21)(13,15)(14,19)(20,23)$ and $c \sim (3,10,12)(4,9,11)$ $(5,13,17)(6,19,15)(7,18,14)(21,23,22)$, the two generators of M_{12}. Consider the following:

$$b^a = (1,5)(2,23)(3,4)(6,13)(7,14)(8,19)(9,11)(10,22)(12,20)(15,16)(17,18)(21,24)$$
$$c^a = (3,24,22)(5,16,8)(6,13,7)(9,10,23)(12,20,21)(14,17,15).$$

We now convert these permutations into words in terms of the generators of M_{12}. A presentation for M_{12} is given by $< b, c|b^2, c^3, (b*c^{-1})^{11}, (c*b*c^{-1}*b)^6, (c*b*c^{-1}*b*c*b*c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b)^2, (c^{-1}*b*c^{-1}*b*c*b*c*b)^5 >$. We run a Schreier system that converts the aforementioned action, b^a and c^a into words in terms of generators of M_{12}. We get the following:

$$
\begin{aligned}
b^a = & c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c^{-1}*b \\
& *c*b*c*b*c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1} \\
c^a = & c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b*c \\
& *b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b
\end{aligned}
$$

We now add the presentation $< a|a^2 >$ to the presentation of M_{12} along with the action just calculated. We get, $M_{12} : 2 =< a, b, c|b^2, c^3, (b*c^{-1})^{11}, (c*b*c^{-1}*b)^6, (c*b*c^{-1}*b*c*b*c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b)^2, (c^{-1}*b*c^{-1}*b*c*b*c*b)^5, a^2, c^a = c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b, b^a = c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b*c*b*c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1} >$. We check to see if $Q_1 \cong M_{12} : 2$.

$HH < a, b, c >:= Group < a, b, c|$

b^2,

c^3,

$(b*c^{-1})^{11}$,

$(c*b*c^{-1}*b)^6$,

$(c*b*c^{-1}*b*c*b*c*b*c*b*c^{-1}*$

$b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b)^2$,

$(c^{-1}*b*c^{-1}*b*c*b*c*b)^5$,

a^2,

$c^a =$

$c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b*c$

$*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c*b$,

$b^a =$

$c*b*c*b*c^{-1}*b*c*b*c^{-1}*b*c^{-1}*b*c^{-1}*b*c^{-1}*b$

$* c * b * c * b * c * b * c * b * c^{-1} * b * c * b * c^{-1} * b * c^{-1} >;$

$f2, H, k2 := CosetAction(HH, sub < HH|Id(HH) >);$

$IsIsomorphic(q1, H);$

$true$

Hence, $Q_1 \cong M_{12} : 2$.

Now that we have shown $Q_1 \cong M_{12} : 2$, we need to show $G2 \cong 2^{\bullet}M_{12} : 2$. The presentation of Q_1 is given $< a, b, c|a^2, b^6, c^2, (b * a * b^{-1} * a)^2, (b^{-1} * a * c * a)^2, (b^{-1} * c)^4, (a * b^{-1})^5, (c * a)^5, b^{-1} * c * b^{-2} * a * b^2 * c * b^2 * a * b^{-1} >$. We need to know if we can write the central element in terms of permutations in Q_1. So, we find the transversals T, of $Z(G2)$ in $G2$ and select the transversals that correspond to the generators of Q_1. When we compute the factor group $G2/Z(G2)$, we use $q1, ff1 := quo < G2|NLG2[2] >$. From this we know: $ff1(T[2])eqq1.1;$

$true$

$ff1(T[3])eqq1.2;$

$true$

$ff1(T[4])eqq1.3;$

$true$

$A := T[2]; B := T[3]; C := T[4];$

$A;$

$(1, 2)(3, 6)(4, 5)(7, 12)(8, 15)(9, 16)(10, 19)(11, 21)(13, 24)(14, 27)(17, 31)(18, 34)(20, 36)(22, 37)(23, 32)(25, 30)(26, 40)(28, 29)(33, 42)(35, 41)(38, 44)(39, 43)(45, 47)(46, 48)$

$B;$

$(1, 3, 7, 13, 25, 2)(4, 5, 9, 17, 32, 21)(6, 10, 18, 34, 41, 30)(8, 15, 20, 11, 16, 29)(12, 22, 37, 24, 39, 42)(14, 28, 36, 44, 46, 48)(19, 35, 43, 45, 47, 33)(23, 38, 27, 31, 26, 40)$

$C;$

$(1, 4)(2, 5)(3, 8)(6, 11)(7, 14)(9, 18)(10, 20)(12, 23)(13, 26)(15, 19)(16, 30)(17, 33)(21, 35)(22, 32)(24, 31)(25, 36)(27, 39)(28, 34)(29, 41)(37, 44)(38, 42)(40, 43)(45, 48)(46, 47)$.

That is, $ff1(T[2])$ eq $q1.1$, $ff1(T[3])$ eq $q1.2$, and $ff1(T[4])$ eq $q1.3$ implies $a \sim A :=$ $T[2] = (1,2)(3,6)(4,5)(7,12)(8,15)(9,16)(10,19)(11,21)(13,24)(14,27)(17,31)(18,34)$ $(20,36)(22,37)(23,32)(25,30)(26,40)(28,29)(33,42)(35,41)(38,44)(39,43)(45,47)(46,48)$, $b \sim B := T[3] = (1,3,7,13,25,2)(4,5,9,17,32,21)(6,10,18,34,41,30)(8,15,20,11,16,29)$ $(12,22,37,24,39,42)(14,28,36,44,46,48)(19,35,43,45,47,33)(23,38,27,31,26,40)$, and $c \sim C := T[4] = (1,4)(2,5)(3,8)(6,11)(7,14)(9,18)(10,20)(12,23)(13,26)(15,19)(16,30)$ $(17,33)(21,35)(22,32)(24,31)(25,36)(27,39)(28,34)(29,41)(37,44)(38,42)(40,43)(45,48)$ $(46,47)$, the transversals that correspond with each generator of Q_1.

We now write the presentation of Q_1 in terms of A, B, and C and check if the order of the relations change. If so, this implies the central element can be written in terms of that relation, where $Z(G2)$ has the presentation of $< d|d^2 >$. We perform the check: $Order(A)$;

2

$Order(B)$;

6

$Order(C)$;

2

$Order(B * A * B^{-1} * A)$;

2

$Order(B * A * B^{-1} * A)$;

2

$Order(B^{-1} * A * C * A)$;

2

$Order(B^{-1} * C)$;

4

$Order(A * B^{-1})$;

5

$Order(C * A)$;

10

$Order(B^{-1} * C * B^{-2} * A * B^2 * C * B^2 * A * B^{-1})$;

1

$(C * A)^5$ eq $NLG2[2].2$;

true

$NLG2[2].2$ is the permutation $(1,5)(2,4)(3,9)(6,16)(7,17)(8,18)(10,29)(11,30)(12,31)$ $(13,32)(14,33)(15,34)(19,28)(20,41)(21,25)(22,26)(23,24)(27,42)(35,36)(37,40)(38,39)$ $(43,44)(45,46)(47,48)$, the generator of $Z(G2)$. Therefore, we can write the central element in terms of the relation $(C*A)^5$. That is, $(c*a)^5 = d$. We now combine the presentation of Q_1 and $Z(G2)$. We now have the presentation of $2^{\bullet}M_{12}:2$ given by $< a,b,c,d|a^2, b^6, c^2, d^2, (b*a*b^{-1}*a)^2, (b^{-1}*a*c*a)^2, (b^{-1}*c)^4, (a*b^{-1})^5, (c*a)^5 = d, b^{-1}*c*b^{-2}*a*b^2*c*b^2*a*b^{-1}, (a,d), (b,d), (c,d) >$. We need to see if $G2$ is isomorphic to a representation of our finitely presented group $2^{\bullet}M_{12}:2$. $GG < a,b,c,d >:=$ $Group < a,b,c,d|$

$a^2,$

$b^6,$

$c^2,$

$d^2,$

$(b*a*b^{-1}*a)^2,$

$(b^{-1}*a*c*a)^2,$

$(b^{-1}*c)^4,$

$(a*b^{-1})^5,$

$(c*a)^5 = d,$

$b^{-1}*c*b^{-2}*a*b^2*c*b^2*a*b^{-1},$

$(a,d),(b,d),(c,d)$

$>;$

$f4, GG, k4 := CosetAction(GG, sub < GG|Id(GG) >);$

$IsIsomorphic(G2, GG);$

true

Hence, $G2 \cong 2^{\bullet}M_{12}:2$. We say, G is centrally extended of a cyclic group of order two by the automorphism group of M_{12}.

5.5 Mixed Extension

5.5.1 $G \cong (4 \times 2) :^{\bullet} 2^2$

Let $G =< (2,6)(3,7),(1,2,3,4,5,6,7,8) >$, where G is a permutation group acting on a set of cardinality 8 and $|G| = 32$, then G has the isomorphism type $(4 \times 2) :^{\bullet} 2^2$. We begin by analyzing the composition factors and the normal lattice of G.

$CompositionFactors(G)$

G
$|\quad Cyclic(2)$
$*$
$|\quad Cyclic(2)$
$*$
$|\quad Cyclic(2)$
$*$
$|\quad Cyclic(2)$
$*$
$|\quad Cyclic(2)$
1

$NormalLattice(G)$

$- -$

[12] $Order$ 32 $Length$ 1 $Maximal\ Subgroups$: 9 10 11

$- - -$

[11] $Order$ 16 $Length$ 1 $Maximal\ Subgroups$: 7

[10] $Order$ 16 $Length$ 1 $MaximalSubgroups$: 7

[9] $Order$ 16 $Length$ 1 $MaximalSubgroups$: 6 7 8

$- - -$

[8] $Order$ 8 $Length$ 1 $Maximal\ Subgroups$: 3

[7] $Order$ 8 $Length$ 1 $Maximal\ Subgroups$: 3 4 5

[6] $Order$ 8 $Length$ 1 $Maximal\ Subgroups$: 3

$- - -$

[5] $Order$ 4 $Length$ 1 $Maximal\ Subgroups$: 2

$[4]$ $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2

$[3]$ $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2

$---$

$[2]$ $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

$---$

$[1]$ $Order$ 1 $Length$ 1 $Maximal$ $Subgroups$:

By the look of the composition factors of G, it isn't so clear as to what the isomorphism type may be. Therefore, we begin by taking the largest abelian subgroup from the normal lattice. We see that $NL[7] =< (1,3,5,7)(2,4,6,8),(2,6)(4,8),(1,5)(3,7) >$ is the largest abelian group from the normal lattice of G. Now we look at the normal lattice of $NL[7]$ to determine its isomorphism type.

$NormalLattice(NL7)$

$-----------------------$

$[8]$ $Order$ 8 $Length$ 1 $Maximal$ $Subgroups$: 5 6 7

$---$

$[7]$ $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2

$[6]$ $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2

$[5]$ $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2 3 4

$---$

$[4]$ $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

$[3]$ $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

$[2]$ $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

$---$

$[1]$ $Order$ 1 $Length$ 1 $Maximal$ $Subgroups$:

Notice $NL7[7] \triangleleft NL[7]$ and $NL7[3] \triangleleft NL[7]$ with $NL7[7] \cap NL7[3] = \emptyset$ thus, $NL[7] \cong NL7[7] \times NL7[3]$. But $NL7[7] \cong 4$ and $NL7[3] \cong 2$, a cyclic group of order 4 and 2, respectively. Therefore, $NL[7] \cong (4 \times 2)$. We now factor G by $NL[7] =< (1,3,5,7)(2,4,6,8),(2,6)(4,8),(1,5)(3,7) >$, then $G/NL[7] \cong Q =< (1,2)(3,4),(1,3)(2,4) >$, where Q is a permutation group acting on a set of cardinality 4 and $|Q| = 4$.

We look at the normal lattice of Q.

$NormalLattice(Q)$

$-------------------------$

[5] $Order$ 4 $Length$ 1 $Maximal$ $Subgroups$: 2 3 4

$---$

[4] $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

[3] $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

[2] $Order$ 2 $Length$ 1 $Maximal$ $Subgroups$: 1

$---$

[1] $Order$ 1 $Length$ 1 $Maximal$ $Subgroups$:

We see from the normal lattice that Q has two disjoint normal subgroups of order 2. This implies $Q \cong 2^2$. We suspect G is a semi-direct product of (4×2) by 2^2. So, we continue and calculate the action of Q on the generators of (4×2). We take the transversals of (4×2) in G. So, we find the transversals T, of (4×2) in G and select the transversals that correspond to the generators of Q. When we compute the factor group $G/NL[7]$, we use $q, ff := quo < G|NL[7] >$. From this we know:

$ff(T[2])eqq.1;$

$true$

$ff(T[3])eqq.2;$

$true$

$A := T[2];$

$A;$

$(2,6)(3,7)$

$B := T[3];$

$B;$

$(1,2,3,4,5,6,7,8)$

We begin by labeling the generators of $NL[7] \cong (4 \times 2)$. Let $c \sim (1,3,5,7)(2,4,6,8)$, $d \sim$

$(2,6)(4,8)$, and $e \sim (1,5)(3,7)$ denote the generators of (4×2). Consider the following:

$$\begin{aligned}
c^a &= (1,7,5,3)(2,8,6,4) \\
&= c^{-1} \\
c^b &= (1,3,5,7)(2,4,6,8) \\
&= c \\
d^a &= (2,6)(4,8) \\
&= d \\
d^b &= (1,5)(3,7) \\
&= e \\
e^a &= (1,5)(3,7) \\
&= e \\
e^b &= (2,6)(4,8) \\
&= d
\end{aligned}$$

Now that we have calculated the action of Q on the generators of (4×2), we combine the presentation of $NL[7]$ and Q along with the action just calculated. A presentation for (4×2) is $< c, d, e | d^2, e^2, c^4, d * c^2 * e, c^{-1} * d * c * d >$ and a presentation for 2^2 is $< a, b | a^2, b^2, (a * b)^2 >$. With everything together, the presentation is $< a, b, c, d, e | a^2, b^2, (a*b)^2, d^2, e^2, c^4, d*c^2*e, c^{-1}*d*c*d, c^a = c^{-1}, c^b = c, d^a = d, d^b = e, e^a = e, e^b = d >$. However, when we try and construct a representation of our finitely presented group, we find that our group is not isomorphic to G. That is, we don't have a semi-direct product between (4×2) and 2^2. Instead, we will have a mixed extension. Now, we must determine the central elemnt of (4×2) that can be written in terms of a permutation in Q. So, we now write the presentation of Q in terms of A and B and check if the order of the relations change. If so, this implies a central element can be written in terms of that relation. We perform the check:

$Order(A)$;

2

$Order(B);$

8

$Order(A * B);$

8

$true$

This implies that our relations b^2 and $(a * b)^2$ are equal to a central element in (4×2). We need to find which central elements our relations are equal to.

for i, k in [0..1] do for j in [0..7] do if B^2 eq $C^i * D^j * E^k$ then i, j, k;
break; end if; end for;end for;

1 0 0

for i, k in [0..1] do for j in [0..7] do if $(A * B)^2$ eq $C^i * D^j * E^k$ then
i, j, k; break; end if; end for;end for;

1 0 1

The above lets us know that $b^2 = c$ and $(a * b)^2 = c * e$. We now add the extra relations to our presentation from above, $< a, b, c, d, e | a^2, b^2 = c, (a * b)^2 = c * e, d^2, e^2, c^4, d * c^2 * e, c^-1 * d * c * d, c^a = c^-1, c^b = c, d^a = d, d^b = e, e^a = e, e^b = d >$. We now check to see if G is isomorphic to a representation of our finitely presented group.

$HH < a, b, c, d, e >:= Group < a, b, c, d, e | a^2, b^2 = c, (a * b)^2 = c * e,$

$d^2,$

$e^2,$

$c^4,$

$d * c^2 * e,$

$c^-1 * d * c * d,$

$c^a = c^{-1},$

$c^b = c,$

$d^a = d,$

$d^b = e,$

$e^a = e,$

$e^b = d >;$

$f1, H, k1 := CosetAction(HH, sub < HH | Id(HH) >);$

$IsIsomorphic(G, H);$

$true$

Hence, $G \cong (4 \times 2) :^{\bullet} 2^2$.

Chapter 6

Progenitor Charts

6.1 Introduction

For given progenitor charts below, we let $t \sim t_1$. Therefore, we make the symmetric generator commute with the one point stabilizer (or coset stabilizer).

6.1.1 $2^{*^{60}} : (2 \times A_5)$

Table 6.1: $2^{*^{60}} : (2 \times A_5)$

a	b	c	d	e	G
3	0	0	6	3	J_1
4	5	2	8	11	M_{12}
0	0	2	5	6	$2^4 : A_5$
0	0	2	6	4	$3^4 : S_5$
0	0	3	0	3	$PSL(2,31)$
0	3	0	0	3	$PSL(2,11)$
0	5	9	0	2	$PSL(2,19)$
0	0	0	7	2	$PSL(2,41)$

```
G<x,y,z,t>:=Group<x,y,z,t|x^2,y^6,z^2,y^3*z,(x*z)^2,(x*y^-1)^5,
t^2,
(t,x*y*x*y^-1*x*y*x*y^-1*x),
(y * x*t)^a,
((y * x)^2*t)^b,
(y*t)^c,
(y * x * y*t)^d,
(y^-1 * x * y * x * y^-1*t)^e>;
Generators of our control group are given by:
x:=(1, 2)(3, 9)(4, 7)(5, 12)(6, 13)(8, 16)(10, 20)(11, 18)(14,
26)(15, 24)(17, 29)(19, 32)(21, 35)(22, 36)(23, 37)(27, 42)
(28, 43)(30, 46)(31, 44)(33, 49)(34, 50)(39, 54)(40, 53)(41,
55)(45, 52)(48, 59)(51, 58)(56, 60),
y:=(1, 3, 10, 4, 11, 5)(2, 6, 14, 7, 15, 8)(9, 17, 30, 18, 31,
19)(12, 21, 34, 20, 33, 22)(13, 23, 38, 24, 39, 25)(16, 27,
41, 26, 40, 28)(29, 43, 56, 44, 55, 45)(32, 47, 58, 46, 57,
48)(35, 51, 60, 49, 59, 52)(36, 53, 54, 50, 42, 37),
z:=(1, 4)(2, 7)(3, 11)(5, 10)(6, 15)(8, 14)(9, 18)(12, 20)(13,
24)(16, 26)(17, 31)(19, 30)(21, 33)(22, 34)(23, 39)(25, 38)
(27, 40)(28, 41)(29, 44)(32, 46)(35, 49)(36, 50)(37, 54)(42,
53)(43, 55)(45, 56)(47, 57)(48, 58)(51, 59)(52, 60)
```

6.1.2 $2^{*^{60}} : A_5$

Table 6.2: $2^{*^{60}} : A_5$

a	b	c	d	e	f	g	h	i	G
3	4	3	0	0	0	0	0	0	$S(3,4)$
0	2	6	5	0	0	0	0	0	$S(4,5)$
3	3	0	4	0	0	0	0	0	A_8
0	0	0	0	4	0	2	6	6	$3^4 : S_5$
0	0	0	0	0	0	2	5	3	A_5^2
0	0	0	0	0	0	4	5	2	$2 : A_6^2$
0	0	0	0	0	0	0	2	3	$2^{\bullet}A_5^2$
0	0	0	0	0	0	2	2	7	$PSL(2,49)$
0	0	0	0	5	0	4	2	3	$2^{\bullet}PSL(3,4)$
0	0	0	0	5	0	3	3	2	$2^8 : A_5$

```
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y^-1)^5,
t^2,(t,x*y^-1*x*y*x),
(y*t^(x^y))^a,
(y*t^(y * x * y * x * y^-1 * x * y^-1))^b,
(y*t^(x * y * x))^c,
(y*t^(y^-1 * x * y * x * y^-1))^d,
(y*t^(y^-1 * x * y^-1 * x * y * x))^e,
(y*t^(x * y * x * y^-1 * x * y * x))^f,
(y*t^(x * y * x * y^-1 * x * y * x))^g,
(y*t^((y * x * y^-1 * x)^2))^h,
(y*t^(x * y * x * y^-1 * x * y * x * y^-1 * x))^i>;

Generators of our control group are given by:

 x:=(1, 2)(3, 7)(4, 8)(5, 9)(10, 15)(11, 16)(12, 17)(13, 18)
(14, 19)(20, 25)(21, 22)(23, 26)(24, 27)(28, 30),
y:=(1, 3, 4)(2, 5, 6)(7, 10, 11)(8, 12, 13)(9, 14, 15)(16,
20, 21)(17, 22, 23)(18, 24, 19)(25, 27, 28)(26, 29, 30)
```

Table 6.3: $2^{*^{60}} : A_5$

a	b	c	d	e	f	g	h	k	G
0	5	5	2	0	0	0	0	0	$PSL(2, 19)$
0	4	6	2	0	0	0	0	0	$PSL(2, 19) : 2$
0	2	8	4	0	0	0	0	0	$2^{\bullet}PSL(2, 31)$
2	0	7	4	0	0	0	0	0	$PSL(2, 71)$
0	0	0	0	3	3	6	0	2	$2^4 : A_5$
0	0	0	0	3	4	5	0	2	$2^4 : A_6$

```
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y^-1)^5,
t^2,(t,x*y^-1*x*y*x),
((x * y)^2*t^(y * x * y * x * y^-1))^a,
((x * y)^2*t^(x * y * x * y^-1 * x * y * x * y^-1 * x))^b,
((x * y)^2*t^(y * x * y * x * y^-1 * x * y^-1))^c,
((x * y)^2*t^(y^-1 * x * y * x * y^-1 * x * y^-1))^d,
(y*t^(x^y))^e,
(y*t^((x * y)^2))^f,
(y*t^(x * y^-1 * x * y))^g,
(y*t^(y * x * y * x * y^-1))^h,
(y*t^(y * x * y * x * y^-1))^i>;
```

Generators of our control group are given by:

```
x:=(1, 2)(3, 7)(4, 8)(5, 9)(10, 15)(11, 16)(12, 17)(13, 18)(14,
19)(20, 25)(21, 22)(23, 26)(24, 27)(28, 30),
y:=(1, 3, 4)(2, 5, 6)(7, 10, 11)(8, 12, 13)(9, 14, 15)(16, 20,
21)(17, 22, 23)(18, 24, 19)(25, 27, 28)(26, 29, 30)
```

6.1.3 $2^{*^{56}} : (2^3 : 7)$

Table 6.4: $2^{*^{56}} : (2^3 : 7)$

a	b	c	d	G
2	7	7	5	$Suz(8)$

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^7,x^2,y^2,z^2,w^(-1)*x*w*y,
(x*y)^2,(x*z)^2,(y*z)^2,
w*x*w^(-1)*x*z, w*z*w^-1*y*z,
t^2,
(w^-1*t^(z * w^-2))^a,
(w^-1*t^(w * y * w^2))^b,
(w^-1*t^(w^-2 * y * w^-1))^c,
(w^-1*t^(y))^d>;
```

```
Generators of our control group are given by:

 w:=(1, 2, 7, 22, 42, 21, 6)(3, 9, 27, 48, 50, 33, 13)(4, 14,
34, 45, 51, 35,16)(5, 17, 36, 43, 52, 37, 18)(8, 24, 46, 56,
41, 31,12)(10, 29, 49, 55, 40, 32, 15)(11, 26, 23, 44, 54,
39, 20)(19,30, 28, 25, 47, 53, 38),
 x:=(1, 3)(2, 8)(4, 11)(5, 12)(6, 16)(7, 23)(9, 17)(10, 26)(13,
20)(14, 28)(15, 30)(18, 32)(19, 31)(21, 38)(22, 43)(24, 29)
(25, 36)(27, 34)(33, 40)(35, 41)(37, 39)(42, 55)(44, 47)(45,
49)(46, 48)(50, 53)(51, 52)(54, 56),
 y:=(1, 4)(2, 9)(3, 11)(5, 15)(6, 19)(7, 24)(8, 17)(10, 28)(12,
30)(13, 32)(14, 26)(16, 31)(18, 20)(21, 40)(22, 44)(23, 29)
(25, 34)(27, 36)(33, 38)(35, 37)(39, 41)(42, 52)(43, 47)(45,
48)(46, 49)(50, 56)(51, 55)(53, 54),
 z:=(1, 5)(2, 10)(3, 12)(4, 15)(6, 20)(7, 25)(8, 26)(9, 28)(11,
30)(13, 16)(14, 17)(18, 19)(21, 41)(22, 45)(23, 36)(24, 34)
(27, 29)(31, 32)(33, 37)(35, 38)(39, 40)(42, 50)(43, 49)(44,
48)(46, 47)(51,54)(52, 56)(53, 55)
```

6.1.4 $2^{*^{21}} : (7 : 3)$

Table 6.5: $2^{*^{21}} : (7 : 3)$

a	b	c	d	e	G
0	2	0	0	3	$PSL(2, 7)$
3	3	0	5	0	$3 \times PSL(3, 4)$
0	3	3	7	5	$PSL(3, 4)$

```
G<x,y,t>:=Group<x,y,t|x^3,x*y^-1*x^-1*y^2,
t^2,
(x^-1*t^(x * y^-2))^a,
(x^-1*t^(x * y))^b,
(y*t)^c,
(y*t^(y * x^-1))^d,
(y*t^(x * y^-1))^e>;

Generators of our control group are given by:

x:=(1, 2, 4)(3, 8, 10)(5, 13, 14)(6, 16, 9)(7, 18, 15)(11, 21,
17)(12, 20, 19),
y:=(1, 3, 9, 20, 21, 15, 5)(2, 6, 17, 13, 8, 19, 7)(4, 11, 10,
18, 16, 14, 12)
```

6.1.5 $2^{*^{60}} : (2 \times A_5)$

Table 6.6: $2^{*^{120}} : (2 \times A_5)$

a	b	c	d	G
8	6	2	8	$2^{\bullet}M_{12}$
5	2	8	11	$PSL(2, 41)$
0	2	5	6	$PSL(2, 19)$
0	2	6	4	$3^7 : S_7$
0	3	0	3	$3^5 : S_5$

```
G<x,y,z,t>:=Group<x,y,z,t|x^2,y^3,z^2,y^-1*z*y*z,(x*z)^2,
(x*y^-1)^5,
t^2,
(t,y * x * z * y^-1 * x * y * x * y^-1 * x * y),
(x*y*z*t^((x * y^-1)^2))^a,
(z*y^-1*x*y^-1*x*t)^b,
(z*y^-1*x*y^-1*x*t^(y))^c,
(z*y^-1*x*y^-1*x*t^(y^-1 * x * y * x * y))^d>;

Generators of our control group are given by:

 x:=(1, 2)(3, 9)(4, 7)(5, 12)(6, 13)(8, 16)(10, 18)(11, 20)(14,
24)(15, 26)(17, 29)(19, 32)(21, 35)(22, 36)(23, 37)(25, 40)
(27, 43)(28, 44)(30, 45)(31, 47)(33, 48)(34, 50)(38, 53)(39,
54)(41, 55)(42, 57)(46, 60)(49, 51)(52, 59)(56, 58),
 y:=(1, 3, 5)(2, 6, 8)(4, 10, 11)(7, 14, 15)(9, 17, 19)(12, 21,
22)(13, 23, 25)(16, 27, 28)(18, 30, 31)(20, 33, 34)(24, 38,
39)(26, 41, 42)(29, 44, 46)(32, 48, 49)(35, 51, 47)(36, 52,
37)(40, 55, 56)(43, 58, 54)(45, 57, 59)(50, 60, 53),
 z:=(1, 4)(2, 7)(3, 10)(5, 11)(6, 14)(8, 15)(9, 18)(12, 20)(13,
24)(16, 26)(17, 30)(19, 31)(21, 33)(22, 34)(23, 38)(25, 39)
(27, 41)(28, 42)(29, 45)(32, 47)(35, 48)(36, 50)(37, 53)(40,
54)(43, 55)(44, 57)(46, 59)(49, 51)(52, 60)(56, 58)
```

6.1.6 $2^{*^{120}} : S_5$

Table 6.7: $2^{*^{60}} : S_5$

a	b	c	d	e	f	g	h	k	l	G
3	2	2	0	0	0	0	0	0	0	S_6
3	2	4	0	0	0	0	0	0	0	$2^5 : S_6$
0	0	3	2	6	0	0	0	0	0	$3^5 : S_6$
0	0	0	0	0	0	3	0	0	3	$2^5 : S_5$
0	0	0	0	0	0	0	3	3	4	$2^4 : S_6$

```
G<x,y,t>:=Group<x,y,t|x^2,y^6,(y*x*y^-1*x)^2,(x*y^-1)^5,
t^2,
(t,(y * x * y)^2),
(x*t^(y * x * y))^a,
(x*t^(y))^b,
(x*t^(y^-1 * x))^c,
((x * y^2)^2*t^(y * x * y))^d,
((x * y^2)^2*t^(y^3 * x))^e>;

Generators of our control group are given by:

 x:=(1, 2)(3, 7)(4, 9)(5, 11)(6, 13)(8, 17)(10, 20)(12, 23)(14,
27)(15, 28)(16, 22)(18, 31)(19, 25)(21, 33)(24, 37)(26, 39)
(29, 42)(30, 43)(32, 46)(34, 48)(35, 49)(36, 47)(38, 50)(40,
54)(41, 56)(44,51)(45, 55)( 52, 59)(53, 58)(57, 60),
 y:=(1, 3, 8, 18, 10, 4)(2, 5, 12, 24, 14, 6)(7, 15, 9, 19, 29,
16)(11, 21, 17, 30, 34, 22)(13, 25, 38, 32, 20, 26)(23, 35,
50, 46, 51, 36)(27, 40, 55, 43, 48, 41)(28, 39, 54, 57, 47,
33)(31, 44, 53, 37, 52, 45)(42, 56, 59, 60, 58, 49)
```

6.1.7 $3^{*^3} : S_3$

Table 6.8: $3^{*^3} : S_3$

a	b	c	G
7	0	0	$2^{\bullet}A_7$
13	6	0	$PGL(2,25)$

```
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^3,(t,y),(t,x^2*y*x^2),
t^(x^2)=t^2,(x*y*x*y^2*x*y*x*y^-1*x*y^-1*x*y^-1*x*y
*x*t^(y^-1 * x * y^2 * x * y * x))^a, (y * x * y^-1 * x *
y^2 * x * y * x * y * x * y^-1 * x * y * x * y^-1 *
 x*t^(y^-1 * x))^b>;
```

6.1.8 $2^{*^{28}} : (PGL(2,7) : 2)$

Table 6.9: $2^{*^{28}} : (PGL(2,7) : 2)$

a	b	G
3	5	$U(3,5) : 2$

```
G<x,y,z,t>:=Group<x,y,z,t|x^7,y^2,z^2,(x^-1*z)^2,
(y*z)^2,(y*x^-1)^3,(x*y*x^2)^4,
t^2,
(t,x * y * z * x^2 * y * x^-1),
(t,x^2 * y * z * x^3),
(t,x^-2 * y * x^2 * y * x^-1),
(z * x^2 * y * x^2 * y*t^(y*x^5))^a,
(z * x^2 * y * x^2 * y*t^(x^2))^b>;

Generators of our control group are given by:

 x:=(1, 2, 5, 11, 12, 7, 4)(3, 8, 9, 16, 20, 17, 10)(6, 13, 18,
22, 19, 14, 15)(21, 25, 24, 26, 28, 27, 23),
 y:=(1, 3)(2, 6)(4, 9)(7, 14)(10, 13)(11, 15)(16, 19)(17, 21)
(18, 23)(20, 24)(22, 26)(27, 28),
 z:=(1, 4)(2, 7)(3, 9)(5, 12)(6, 14)(10, 16)(13, 19)(17, 20)
(18, 22)(21, 24)(23, 26)(27, 28)
```

6.1.9 $2^{*^{144}} : (3^2 : 2^4)$

Table 6.10: $2^{*^{144}} : (3^2 : 2^4)$

a	b	c	G
3	8	3	M_{10}

```
G<u,v,w,x,y,z,t>:=Group<u,v,w,x,y,z,t|u^2,v^4,w^4,x^2,y^3,z^3,
v^-2*x,w^-1*v^2*w^-1,v^-1*w^-1*v*w^-1,
(u*w^-1)^2,(x*y^-1)^2,u*z^-1*u*z,(x*z^-1)^2,
(y,z),v^-1*u*v^-1*u*w^-1,v*y^-1*v^-1*y*z,
u*y^-1*u*y^-1*z,
t^2,
(y * u*t)^a,
(v * u*t)^b,
(v^-1 * u*t)^c>;
```

Generators of our control group are given by:

```
 u:=(1, 2)(3, 21)(4, 18)(5, 14)(6, 36)(7, 16)(8, 48)(9, 13)(10,
57)(11,20)(12, 25)(15, 47)(17, 22)(19, 42)(23, 113)(24, 106)
(26, 124)(27,108)(28, 99)(29, 93)(30, 83)(31, 95)(32, 131)
(33, 76)(34, 132)(35,78)(37, 133)(38, 92)(39, 100)(40, 134)
(41, 73)(43, 105)(44, 87)(45,135)(46, 85)(49, 139)(50, 137)
(51, 140)(52, 138)(53, 80)(54, 72)(55, 97)(56, 74)(58, 141)
(59, 94)(60, 142)(61, 71)(62, 81)(63, 107)(64, 104)(65, 136)
(66, 102)(67, 127)(68, 121)(69, 116)(70, 123)(75,117)(77,
128)(79, 130)(82, 129)(84, 120)(86, 111)(88, 115)(89, 110)
(90, 126)(91, 112)(96, 119)(98, 118)(101, 122)(103, 109)
(114,144)(125, 143),
 x:=(1, 5)(2, 14)(3, 8)(4, 9)(6, 34)(7, 35)(10, 32)(11, 33)(12,
17)(13, 18)(15, 77)(16, 78)(19, 75)(20, 76)(21, 48)(22, 25)
(23, 51)(24, 52)(26, 49)(27, 50)(28, 55)(29, 56)(30, 53)(31,
54)(36, 132)(37, 40)(38, 41)(39, 114)(42, 117)(43, 45)(44,
46)(47, 128)(57, 131)(58, 60)(59, 61)(62, 125)(63, 65)(64,
66)(67, 90)(68, 91)(69, 88)(70, 89)(71, 94)(72, 95)(73, 92)
(74, 93)(79, 82)(80, 83)(81, 143)(84, 86)(85, 87)(96, 98)(97,
99)(100, 144)(101, 103)(102, 104)(105,135)(106, 138)(107,
136)(108, 137)(109, 122)(110, 123)(111, 120)(112,121)
(113, 140)(115, 116)(118, 119)(124, 139)(126, 127)(129,
130)(133, 134)(141, 142),
 y:=(1, 6, 10)(2, 15, 19)(3, 23, 26)(4, 28, 30)(5, 32, 34)(7,
```

39,47)(8, 49,51)(9, 53, 55)(11, 42, 62)(12, 67, 69)(13, 71, 73)
(14, 75, 77)(16, 81, 36)(17, 88, 90)(18, 92, 94)(20, 57, 100)
(21, 105, 107)(22, 109, 111)(24, 58, 43)(25, 120, 122)(27, 63,
37)(29, 44, 38)(31, 59, 64)(33, 125, 117)(35, 128, 114)(40, 65,
50)(41, 46, 56)(45, 60, 52)(48, 136, 135)(54, 66, 61)(68, 96,
84)(70, 101, 79)(72, 85, 80)(74, 97, 102)(76, 144, 131)(78,
132, 143)(82, 103, 89)(83, 87, 95)(86, 98, 91)(93, 104, 99)
(106, 133, 113)(108, 124, 141)(110, 118, 115)(112, 126, 129)
(116, 119, 123)(121, 130, 127)(134, 138, 140)(137,142, 139),
v:=(1, 4, 5, 9)(2, 13, 14, 18)(3, 22, 8, 25)(6, 38, 34, 41)(7,
44, 5, 46)(10, 59, 32, 61)(11, 64, 33, 66)(12, 48, 17, 21)(15,
80, 77,83)(16, 85, 78, 87)(19, 97, 75, 99)(20, 102, 76, 104)
(23, 115, 51, 116)(24, 118, 52, 119)(26, 126, 49, 127)(27,
129, 50, 130)(28, 117, 55, 42)(29, 125, 56, 62)(30, 128, 53,
47)(31, 114, 54, 39)(36, 73, 132, 92)(37, 110, 40, 123)(43,
111, 45, 120)(57, 71, 131, 94)(58, 112, 60, 121)(63, 109, 65,
122)(67, 139, 90, 124)(68, 142, 91, 141)(69, 140, 88, 113)(70,
134, 89, 133)(72, 144, 95, 100)(74, 143, 93, 81)(79, 137, 82,
108)(84, 135, 86, 105)(96, 138, 98, 106)(101, 136, 103, 107),
w:=(1, 3, 5, 8)(2, 12, 14, 17)(4, 25, 9, 22)(6, 37, 34, 40)(7,
43,35, 45)(10, 58, 32, 60)(11, 63, 33, 65)(13, 21, 18, 48)(15,
79, 77, 82)(16, 84, 78, 86)(19, 96, 75, 98)(20, 101, 76, 103)
(23, 114, 51, 39)(24, 117, 52, 42)(26, 125, 49, 62)(27, 128,
50,47)(28, 119, 55, 118)(29, 127, 56, 126)(30, 130, 53, 129)
(31, 116, 54, 115)(36, 70, 132, 89)(38, 123, 41, 110)(44, 120,
46,111)(57, 68, 131, 91)(59, 121, 61, 112)(64, 122, 66, 109)
(67,143, 90, 81)(69, 144, 88, 100)(71, 141, 94, 142)(72, 113,
95,140)(73, 133, 92, 134)(74, 124, 93, 139)(80, 108, 83, 137)
(85, 105, 87, 135)(97, 106, 99, 138)(102, 107, 104, 136),
z:=(1, 7, 11)(2, 16, 20)(3, 24, 27)(4, 29, 31)(5, 33, 35)(6,
39, 42)(8, 50, 52)(9, 54, 56)(10, 47, 62)(12, 68, 70)(13, 72,
74)(14,76, 78)(15, 81, 57)(17, 89, 91)(18, 93, 95)(19, 36,
100)(21, 106, 108)(22, 110, 112)(23, 58, 63)(25, 121, 123)
(26, 43, 37)(28, 44, 59)(30, 38, 64)(32, 125, 128)(34, 117,
114)(40, 45, 49)(41, 53, 66)(46, 55, 61)(48, 137, 138)(51,
65, 60)(67, 96, 101)(69, 84, 79)(71, 85, 97)(73, 80, 102)(75,
144, 132)(77, 131, 143)(82, 86, 88)(83, 92, 104)(87, 94, 99)
(90, 103, 98)(105, 133, 124)(107, 113, 141)(109, 118, 126)
(111, 115, 129)(116,120, 130)(119, 122, 127)(134, 135,
139)(136, 142, 140)

Appendix A

MAGMA Code for Monomial Representation of M_{11} and Monomial Presentation of $3^{*^{11}} :_m M_{11}$

```
/*Note: Permutations and transversals change every time one
signs in or out of MAGMA*/
G:=PrimitiveGroup(55,4);
/*M11*/

S:=Subgroups(G);
CG:=CharacterTable(G);

for i in [1..#CG] do i, CG[i][1]; end for;
/*
1 1
2 10
3 10
4 10
5 11
6 16
7 16
8 44
9 45
10 55
```

```
*/

S:=Subgroups(G);
for i in [1..#S] do i, Index(G,S[i]`subgroup); end for;
/*
36 55
37 22
38 11
39 1
*/

H:=S[38]`subgroup;
 Generators(H);
/*
    (1, 16)(2, 33)(3, 25)(4, 43)(5, 35)(6, 9)(7, 13)(8, 31)
(10, 39)(11, 47)(12, 53)(14, 21)(17, 42)(18, 54)(19, 41)
(20, 27)(22, 36)(23, 50)(24, 37)(26, 49)(28, 38)(29, 52)
(40, 45)(44, 51), (1, 25, 28, 20)(2, 6, 22, 41)(3, 24, 49,
8)(4, 15)(5, 45, 31, 43)(9, 52, 38, 35)(10, 21, 39, 55)(12,
53, 33, 44)(13, 30, 18, 14)(16, 34, 19, 50)(17, 48, 27,
23)(26, 51, 42, 36)(29, 32, 37, 46)(40, 47)
*/
m:=H!(1, 16)(2, 33)(3, 25)(4, 43)(5, 35)(6, 9)(7, 13)(8,
31)(10, 39)(11, 47)(12, 53)(14, 21)(17, 42)(18, 54)(19,
41)(20, 27)(22, 36)(23, 50)(24, 37)(26, 49)(28, 38)(29,
52)(40, 45)(44, 51);

n:=H!(1, 25, 28, 20)(2, 6, 22, 41)(3, 24, 49, 8)(4, 15)(5,
45, 31, 43)(9, 52, 38, 35)(10, 21, 39, 55)(12, 53, 33, 44)
(13, 30, 18, 14)(16, 34, 19, 50)(17, 48, 27, 23)(26, 51,
42, 36)(29, 32, 37, 46)(40, 47);

K:=sub<G|m,n>;

CH:=CharacterTable(H);

CG;
/*
Character Table of Group G

----------------------------------------------------
Class |    1   2   3   4    5    6   7   8    9   10
Size  |    1 165 440 990 1584 1320 990 990  720  720
Order |    1   2   3   4    5    6   8   8   11   11
```

```
----------------------------------------------------
p  =  2    1   1   3   2    5    3   4   4   10    9
p  =  3    1   2   1   4    5    2   7   8    9   10
p  =  5    1   2   3   4    1    6   8   7    9   10
p  = 11    1   2   3   4    5    6   7   8    1    1
----------------------------------------------------
X.1    +    1   1   1   1    1    1   1   1    1    1
X.2    +   10   2   1   2    0   -1   0   0   -1   -1
X.3    0   10  -2   1   0    0    1  Z1 -Z1   -1   -1
X.4    0   10  -2   1   0    0    1 -Z1  Z1   -1   -1
X.5    +   11   3   2  -1    1    0  -1  -1    0    0
X.6    0   16   0  -2   0    1    0   0   0   Z2 Z2#2
X.7    0   16   0  -2   0    1    0   0   0 Z2#2   Z2
X.8    +   44   4  -1   0   -1    1   0   0    0    0
X.9    +   45  -3   0   1    0    0  -1  -1    1    1
X.10   +   55  -1   1  -1    0   -1   1   1    0    0


Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

Z1     = (CyclotomicField(8: Sparse := true)) !
[ RationalField() | 0, 1, 0, 1 ]

Z2     = (CyclotomicField(11: Sparse := true)) !
[ RationalField() | 0, 1, 0, 1, 1, 1, 0, 0, 0, 1 ]
*/
CH;
/*Character Table of Group H

-------------------------------------
Class |    1  2  3  4   5   6   7   8
Size  |    1 45 80 90 180 144  90  90
Order |    1  2  3  4   4   5   8   8
-------------------------------------
p  =  2    1  1  3  2   2   6   4   4
p  =  3    1  2  1  4   5   6   7   8
p  =  5    1  2  3  4   5   1   8   7
-------------------------------------
X.1   +    1  1  1  1   1   1   1   1
X.2   +    1  1  1  1  -1   1  -1  -1
```

```
X.3   +     9  1  0  1  -1  -1   1   1
X.4   +     9  1  0  1   1  -1  -1  -1
X.5   +    10  2  1 -2   0   0   0   0
X.6   0    10 -2  1  0   0   0  Z1 -Z1
X.7   0    10 -2  1  0   0   0 -Z1  Z1
X.8   +    16  0 -2  0   0   1   0   0


Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

Z1     = (CyclotomicField(8: Sparse := true)) !
[ RationalField() | 0, 1, 0, 1 ]*/

for i in [1..#CH] do if Induction(CH[i],G) eq CG[5]
then i; end if; end for;
/*2*/

I:=Induction(CH[2],G);
I eq CG[5];
/*true*/

CH[2];
/*
( 1, 1, 1, 1, -1, 1, -1, -1 )
*/

/*Our base field is Z3 but we need to extend to Z7*/
PrimitiveRoot(7);
/*3*/

/*By calculation our elt of order 3 in Z7 is 3*/

T:=Transversal(G,H);
#T;
/*11
(2, 3, 5, 7)(4, 6, 9, 12)(8, 10, 14, 19)(11, 15, 21, 28)(13,
17, 24, 33)(16, 22, 30, 40)(18, 25, 34, 37)(20, 27, 36, 45)
(23, 31, 41, 47)(29, 38)(35, 44, 42, 48)(39, 46, 50, 52)
(43, 49, 53, 55)(51, 54), (1, 2, 4, 8, 13, 23, 40, 12, 3, 5,
7)(6, 11, 20, 35, 45, 15, 21, 37, 14, 26, 19)(9, 16, 29, 38,
```

```
22, 39, 41, 51, 55, 34, 28)(10, 18, 25, 43, 49, 50, 48, 27,
36, 44, 33)(17, 32, 24, 42, 52, 30, 31, 46, 53, 54, 47),

(2, 5)(3, 7)(4, 9)(6, 12)(8, 14)(10, 19)(11, 21)(13, 24)(15,
28)(16, 30)(17, 33)(18, 34)(20, 36)(22, 40)(23, 41)(25,
37)(27, 45)(31, 47)(35, 42)(39, 50)(43, 53)(44, 48)(46,
52)(49, 55),

(1, 2, 5)(3, 7, 4, 11, 21, 9)(6, 16, 39, 53, 34, 14)(8, 18,
43, 50, 30, 12)(10, 26, 19, 13, 32, 24)(15, 37, 25, 28,
20, 36)(17, 42, 27, 44, 52, 41)(22, 31, 51, 47, 40, 29)
(23, 46, 48, 45, 35, 33)(49, 54, 55),

(2, 7, 5, 3)(4, 12, 9, 6)(8, 19, 14, 10)(11, 28, 21, 15)
(13, 33, 24, 17)(16, 40, 30, 22)(18, 37, 34, 25)(20, 45,
36, 27)(23, 47, 41, 31)(29, 38)(35, 48, 42, 44)(39, 52,
50, 46)(43, 55, 53, 49)(51, 54),

(1, 2, 7, 5, 4, 16, 31, 17, 10, 6, 3)(8, 26, 19, 18, 28, 21,
20, 44, 27, 15, 9)(11, 37, 43, 54, 47, 46, 30, 29, 38, 22,
12)(13, 42, 45, 36, 35, 52, 53, 49, 34, 25, 14)(23, 51,
55, 50, 41, 40, 39, 48, 33, 32, 24),

(1, 3)(4, 22)(5, 6)(8, 26)(9, 10)(11, 18)(12, 15)(13, 48)
(14, 17)(16, 41)(19, 25)(20, 42)(21, 27)(23, 54)(24, 31)
(30, 38)(32, 33)(35, 39)(36, 44)(37, 49)(40, 46)(43, 51)
(47, 50)(52, 55),

(1, 5, 9, 14, 24, 41, 22, 6, 7, 2, 3)(4, 30, 29, 38, 40, 50,
23, 51, 49, 18, 15)(8, 26, 10, 12, 21, 36, 42, 27, 28, 11, 2
5)(13, 35, 46, 16, 47, 52, 43, 54, 31, 33, 32)(17, 19, 34,
37, 53, 55, 39, 44, 45, 20, 48),

(1, 14, 22, 2, 5, 24, 6, 3, 9, 41, 7)(4, 38, 23, 18, 30, 40,
51, 15, 29, 50, 49)(8, 12, 42, 11, 26, 21, 27, 25, 10, 36,
28)(13, 16, 43, 33, 35, 47, 54, 32, 46, 52, 31)(17, 37, 39,
20, 19, 53, 44, 48, 34, 55, 45),

(1, 7, 3)(2, 5, 12, 28, 15, 6)(4, 40, 52, 49, 25, 10)(8, 26,
14, 33, 32, 17)(9, 19, 37, 55, 46, 22)(11, 34, 18, 21, 45,
27)(13, 44, 20, 35, 50, 31)(16, 23, 54, 41, 30, 38)(24, 47,
39, 42, 36, 48)(43, 51, 53)
*/
```

```
Generators(G);

C:=CyclotomicField(2);
GG:=GL(11,C);

A:=[[C.1,0,0,0,0,0,0,0,0,0,0]: i in [1..11]];
for i,j in [1..11] do A[i,j]:=0; end for;
for i,j in [1..11] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[2](T[i]*xx*T[j]^-1); end if; end for;

GG!A;
/*
[ 0  1  0  0  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0  0  0]
[ 0  0  0  0  0  1  0  0  0  0  0]
[ 0  0  0  0 -1  0  0  0  0  0  0]
[ 1  0  0  0  0  0  0  0  0  0  0]
[ 0  0  0  0  0  0  0  1  0  0  0]
[ 0  0  0  0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  0  0  0  1]
[ 0  0  0  0  0  0  0  0  0 -1  0]
[ 0  0  0  0  0  0  1  0  0  0  0]
*/

B:=[[C.1,0,0,0,0,0,0,0,0,0,0]: i in [1..11]];
for i,j in [1..11] do B[i,j]:=0; end for;
for i,j in [1..11] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[2](T[i]*yy*T[j]^-1); end if; end for;

GG!B;
/*
[1 0 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 1]
*/
```

```
HH:=sub<GG|A,B>;
#HH;
7920
/*IsIsomorphic(G,HH);*/
/*
true
*/

S:=Sym(22);
aa:=S!(2,3)(4,5)(6,7)(9,10)(13,14)(15,16)(17,18)(20,21);
bb:=S!(1,2,4,6)(5,16)(7,8,9,11)(10,21)(12,13,15,17)(18,19,20,
22);



 Order(aa);
 Order(bb);
 Order(aa*bb);

N:=sub<S|aa,bb>;
/*IsIsomorphic(G,N);*/

NN<x,y>:=Group<x,y|x^2,y^4,y^-1*x*y^-2*x*y^-2*x*y^2*x*y^2
*x*y^2*x*y^-1,x*y*x*y*x*y^-1*x*y^-1*x*y*x*y^-1*x*y^2*x*y*x*
y^-1,x*y^-2*x*y^-1*x*y*x*y^-2*x*y^-1*x*y*x*y^2*x*y^-1*x*y,
(x*y^-1)^11>;

N1:=Stabilizer(N,{1,12});
Generators(N1);
/*
 (3, 7)(4, 20)(6, 8)(9, 15)(10, 11)(14, 18)(17, 19)(21, 22),
    (2, 3)(4, 5)(6, 7)(9, 10)(13, 14)(15, 16)(17, 18)(20, 21),
    (4, 6)(5, 7)(8, 22)(9, 10)(11, 19)(15, 17)(16, 18)(20, 21),
    (3, 11)(4, 21)(5, 17)(6, 16)(7, 9)(10, 15)(14, 22)(18, 20),
    (1, 12)(2, 8, 10, 11)(3, 4, 9, 5)(7, 18)(13, 19, 21, 22)
(14, 15, 20, 16)
*/

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [#N]];
for i in [2..#N] do P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=aa; end if;
```

```
if Eltseq(Sch[i])[j] eq 2 then P[j]:=bb; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=bb^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq N!(3, 7)(4, 20)(6, 8)
(9, 15)(10, 11)(14, 18)(17, 19)(21, 22)  then Sch[i]; end if;
end for;
/*
y^-1*x*y*x*y^-1*x*y^2*x*y^-1*x*y^2*x*y^-1*x*y*x*y
*/
for i in [1..#N] do if ArrayP[i] eq N!(2, 3)(4, 5)(6, 7)
(9, 10)(13, 14)(15, 16)(17, 18)(20, 21)  then Sch[i]; end if;
end for;
/*
x
*/
for i in [1..#N] do if ArrayP[i] eq N!(4, 6)(5, 7)(8, 22)
(9, 10)(11, 19)(15, 17)(16, 18)(20, 21)  then Sch[i]; end if;
end for;
/*
x*y*x*y*x*y*x*y*x*y*x*y^-1*x*y^-1*x*y^-1*x*y^2*x*y
*/
for i in [1..#N] do if ArrayP[i] eq N!(3, 11)(4, 21)(5, 17)
(6, 16)(7, 9)(10, 15)(14, 22)(18, 20)  then Sch[i]; end if;
end for;
/*
y*x*y^2*x*y^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!(1, 12)(2, 8, 10, 11)
(3, 4, 9, 5)(7, 18)(13, 19, 21, 22)(14, 15, 20, 16)  then
Sch[i]; end if; end for;
/*
y^2*x*y^-1*x*y^2
*/

/*We have the monomial presentation of 3*11:m M11*/

G<x,y,t>:=Group<x,y,t|x^2,y^4,y^-1*x*y^-2*x*y^-2*x
*y^2*x*y^2*x*y^2*x*y^-1, x*y*x*y*x*y^-1*x*y^-1*x*
```

```
y*x*y^-1*x*y^2*x*y*x*y^-1,x*y^-2*x*y^-1*x*y*x*y^-2
*x*y^-1*x*y*x*y^2*x*y^-1*x*y,(x*y^-1)^11,t^3,
(t,y^-1*x*y*x*y^-1*x*y^2*x*y^-1*x*y^2*x*y^-1*x*y*x*y),
(t,x),
(t,x*y*x*y*x*y*x*y*x*y*x*y^-1*x*y^-1*x*y^-1*x*y^2*x*y),
(t,y*x*y^2*x*y^-1),
(t^(y^2*x*y^-1*x*y^2)=t^2)>;
```

Appendix B

MAGMA Code for DCE of A_4 over S_5

```
S:=Sym(4);
xx:=S!(1, 3, 4);
yy:=S!(2, 3, 4);
N:=sub<S|xx,yy>;

G<x,y,t>:=Group<x,y,t|x^3,y^3,(x^-1*y^-1)^2,
t^2,(t,y),(y^-1*x*t)^4>;

f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

A:=[Id(G1): i in [1..3]];
A[1]:=f(t);
A[2]:=f(t * x * t);
A[3]:=f( t * x * t * x^-1 * t);

/*Image of N "IN"*/
IN:=sub<G1|f(x),f(y)>;

#G/#IN;

ts := [ Id(G1): i in [1 .. 4] ];
ts[1]:=f(t);
```

```
ts[2]:=f(t^(x * y^-1 * x));
ts[3]:=f(t^(y^-1 * x));
ts[4]:=f(t^(x * y));

prodim := function(pt, Q, I)
 v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
    for i := 1 to 4 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;

for i in [1..10] do if cst[i] ne [] then m:=m+1;
end if;
end for; m;

Orbits(N);

/*=====================[1]==========================*/

N1:=Stabiliser(N,1);
SSS:={[1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1s:=N1;

T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
```

```
then m:=m+1; end if; end for;  m;

Orbits(N1s);
/*
GSet{@ 1 @},
GSet{@ 2, 3, 4 @}
*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/

/*======================[1 2]=========================*/

N12:=Stabiliser(N1,2);
SSS:={[1,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s:=N12;
/*
[ 1, 2 ]
[ 1, 3 ]
[ 1, 4 ]
*/
for g in N do if 1^g eq 1 and 2^g eq 3
then N12s:=sub<N|N12s,g>; end if; end for;

for g in N do if 1^g eq 1 and 2^g eq 4
then N12s:=sub<N|N12s,g>; end if; end for;

T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
```

```
Orbits(N12s);
/*
GSet{@ 1 @},
GSet{@ 2 @},
GSet{@ 3 @},
GSet{@ 4 @}
*/

for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[1] eq m*(A[i])^n then i; end if; end for;
end for;
/*3*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[2] eq m*(A[i])^n then i; end if; end for;
end for;
/*1*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[3] eq m*(A[i])^n then i; end if; end for;
end for;
/*1*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[4] eq m*(A[i])^n then i; end if; end for;
end for;
/*1*/

/*======================[1 2 1]======================*/

N121:=Stabiliser(N12,1);
SSS:={[1,2,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N121s:=N121;
/*
[ 1, 2, 1 ]
[ 3, 2, 3 ]
[ 1, 3, 1 ]
[ 4, 2, 4 ]
[ 4, 3, 4 ]
```

```
[ 3, 4, 3 ]
[ 1, 4, 1 ]
[ 2, 3, 2 ]
[ 2, 4, 2 ]
[ 4, 1, 4 ]
[ 3, 1, 3 ]
[ 2, 1, 2 ]
*/

for g in N do if 1^g eq 3 and 2^g eq 2 and 1^g eq 3
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 1 and 2^g eq 3 and 1^g eq 1
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 4 and 2^g eq 2 and 1^g eq 4
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 4 and 2^g eq 3 and 1^g eq 4
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 3 and 2^g eq 4 and 1^g eq 3
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 1 and 2^g eq 4 and 1^g eq 3
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 2 and 2^g eq 3 and 1^g eq 2
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 2 and 2^g eq 4 and 1^g eq 2
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 4 and 2^g eq 1 and 1^g eq 4
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 3 and 2^g eq 1 and 1^g eq 3
then N121s:=sub<N|N121s,g>; end if; end for;

for g in N do if 1^g eq 2 and 2^g eq 1 and 1^g eq 2
then N121s:=sub<N|N121s,g>; end if; end for;

T121:=Transversal(N,N121s);
for i in [1..#T121] do
```

```
ss:=[1,2,1]^T121[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N121s);
/*
GSet{@ 1 @},
GSet{@ 2 @},
GSet{@ 3 @},
GSet{@ 4 @}
*/

for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[1]*ts[1] eq m*(A[i])^n then i; end if;
 end for;
end for;
/*2*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[1]*ts[2] eq m*(A[i])^n then i; end if;
end for;
end for;
/*2*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[1]*ts[3] eq m*(A[i])^n then i; end if;
end for;
end for;
/*2*/
for i in [1..3] do for m,n in IN do
if ts[1]*ts[2]*ts[1]*ts[4] eq m*(A[i])^n then i; end if;
 end for;
end for;
/*2*/
```

Appendix C

MAGMA Code for DCE of $U(3,5):2$ over $PGL(2,7):2$

```
/*
 CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  2A(2, 5)                  = U(3, 5)

*/
```

```
G<x,y,z,t>:=Group<x,y,z,t|x^7,y^2,z^2,(x^-1*z)^2,
(y*z)^2,(y*x^-1)^3,(x*y*x^2)^4,
t^2,
(t,x * y * z * x^2 * y * x^-1),
(t,x^2 * y * z * x^3),
(t,x^-2 * y * x^2 * y * x^-1),
(z * x^2 * y * x^2 * y*t^(y*x^5))^3,
(z * x^2 * y * x^2 * y*t^(x^2))^5>;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
```

```
#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
/*9*/

DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
/*
<GrpFP, Id(G), GrpFP>, <GrpFP, t * x * t * y * t * x^-1 * t,
 GrpFP>,<GrpFP, t * x^2 * t * x * t, GrpFP>, <GrpFP, t * y
 * t * x * t, GrpFP>,<GrpFP, t * y * t, GrpFP>, <GrpFP, t,
GrpFP>, <GrpFP, t * x * t, GrpFP>,<GrpFP, t * x^2 * t,
GrpFP>, <GrpFP, t * x * t * x^-1 * t, GrpFP>

N:=PrimitiveGroup(28,1);
Generators(N);
PSL(2,7):2
*/

S:=Sym(28);

xx:=S!(1, 8, 14, 19, 23, 26, 6)(2, 9, 15, 20, 24, 5, 12)
(3, 10, 16, 21, 4, 11,17)(7, 13, 18, 22, 25, 27, 28);
yy:=S!(1, 28)(2, 22)(3, 18)(4, 27)(5, 25)(6, 13)(8, 21)(9, 17)
(10, 26)(11,24)(15, 20)(16, 19);
zz:=S!(1, 6)(2, 5)(3, 4)(8, 26)(9, 24)(10, 21)(11, 17)(13, 28)
(14, 23)(15,20)(18, 27)(22, 25);

N:=sub<S|xx,yy,zz>;

FPGroup(N);
/*
Finitely presented group on 3 generators
Relations
    $.1^7 = Id($)
    $.2^2 = Id($)
    $.3^2 = Id($)
    ($.1^-1 * $.3)^2 = Id($)
    ($.2 * $.3)^2 = Id($)
    ($.2 * $.1^-1)^3 = Id($)
    ($.1 * $.2 * $.1^2)^4 = Id($)
*/


N1:=Stabilizer(N,1);
Generators(N1);
```

```
/*
(2, 13)(3, 10)(4, 9)(5, 11)(6, 12)(7, 8)(14, 25)(15, 22)
(16, 27)(17, 28)(20, 23)(21, 24),
(2, 11)(3, 12)(4, 13)(5, 8)(6, 9)(7, 10)(14, 26)(15, 27)
(17, 20)(18, 23)(19, 28)(22, 24),
(2, 12)(3, 11)(4, 10)(5, 9)(6, 8)(7, 13)(14, 26)(15, 24)
(16, 21)(18, 28)(19, 23)(22, 27)
*/



A:=[Id(G1): i in [1..8]];
A[1]:=f(t * x * t * y * t * x^-1 * t);
A[2]:=f(t * x^2 * t * x * t);
A[3]:=f(t * y * t * x * t);
A[4]:=f(t * y * t);
A[5]:=f(t);
A[6]:=f(t * x * t);
A[7]:=f(t * x^2 * t);
A[8]:=f(t * x * t * x^-1 * t);




/*Image of N "IN"*/
IN:=sub<G1|f(x),f(y),f(z)>;
#IN;
/*336*/
#G;

#G/#IN;
/*750*/



ts := [ Id(G1): i in [1 .. 28] ];

ts[1]:=f(t);
ts[2]:=f(t^(x^-1*y*x^3*y*x^2));
ts[3]:=f(t^(x^-1*y*x*y));
ts[4]:=f(t^(x^3*y*x^2));
ts[5]:=f(t^(x^-1*y*x^2*y*x^-2));
ts[6]:=f(t^(x^-1));
ts[7]:=f(t^(x^-3*y*x^2*y*x^-1));
```

```
ts[8]:=f(t^(x));
ts[9]:=f(t^(x^3*y*x^-3*y));
ts[10]:=f(t^(x^-2*y));
ts[11]:=f(t^(z*x^2*y*x^2));
ts[12]:=f(t^(x^-1*y*x^2*y*x^-1));
ts[13]:=f(t^(x^-1 * y));
ts[14]:=f(t^(x^2));
ts[15]:=f(t^(x*y*x^2*y*x^-2));
ts[16]:=f(t^(x*y*x^-1));
ts[17]:=f(t^(x^3 * y * x^-3));
ts[18]:=f(t^(y^x));
ts[19]:=f(t^(x^3));
ts[20]:=f(t^(x*y*x^2*y*x^-2*y));
ts[21]:=f(t^(x * y));
ts[22]:=f(t^(x^-1*y*x^2));
ts[23]:=f(t^(x^-3));
ts[24]:=f(t^(x^3*y*z*x^3*y));
ts[25]:=f(t^(z*x*y*x^-2));
ts[26]:=f(t^(x^-2));
ts[27]:=f(t^(z * x * y * x^-1));
ts[28]:=f(t^(z * x * y));



prodim := function(pt, Q, I)
 v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y,z>)]]
where null is [Integers() | ];
    for i := 1 to 28 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;

for i in [1..750] do if cst[i] ne [] then m:=m+1; end if;
end for; m;

Orbits(N);
```

```
=============================================
[1]

N1:=Stabiliser(N,1);
SSS:={[1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1s:=N1;


T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N1s);
/*
    GSet{@ 1 @},
    GSet{@ 14, 25, 26 @},
    GSet{@ 15, 16, 22, 21, 27, 24 @},
    GSet{@ 17, 18, 28, 19, 20, 23 @},
    GSet{@ 2, 8, 13, 9, 7, 3, 6, 5, 4, 12, 11, 10 @}
*/


for i in [1..8] do for m,n in IN do
if ts[1]*ts[1] eq m*(A[i])^n then i; end if; end for; end for;
/**/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14] eq m*(A[i])^n then i; end if; end for; end for;
/*7*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[15] eq m*(A[i])^n then i; end if; end for; end for;
/*5*/
for i in [1..8] do for m,n in IN do
```

```
if ts[1]*ts[17] eq m*(A[i])^n then i; end if; end for; end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2] eq m*(A[i])^n then i; end if; end for; end for;
/*6*/


================================================
[1 14]

N114:=Stabiliser(N1,14);
SSS:={[1,14]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[14] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N114s:=N114;
/*
[ 1, 14 ]
[ 14, 1 ]
[ 24, 27 ]
[ 27, 24 ]
*/

for g in N do if 1^g eq 14 and 14^g eq 1
then N114s:=sub<N|N114s,g>; end if; end for;

for g in N do if 1^g eq 24 and 14^g eq 27
then N114s:=sub<N|N114s,g>; end if; end for;

for g in N do if 1^g eq 27 and 14^g eq 24
then N114s:=sub<N|N114s,g>; end if; end for;


T114:=Transversal(N,N114s);
for i in [1..#T114] do
ss:=[1,14]^T114[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
```

```
then m:=m+1; end if; end for;  m;

Orbits(N114s);
/*
GSet{@ 1, 14, 24, 27 @},
    GSet{@ 2, 3, 9, 8, 23, 26, 28, 25 @},
    GSet{@ 4, 7, 12, 21, 22, 16, 15, 11 @},
    GSet{@ 5, 6, 13, 20, 19, 17, 18, 10 @}
*/

for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[14] eq m*(A[i])^n then i; end if; end for;
end for;
/*5*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2] eq m*(A[i])^n then i; end if; end for;
end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[4] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/



================================================
[1 17]

N117:=Stabiliser(N1,17);
SSS:={[1,17]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[17] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N117s:=N117;
```

```
T117:=Transversal(N,N117s);
for i in [1..#T117] do
ss:=[1,17]^T117[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;


Orbits(N117s);

/*
     GSet{@ 1 @},
    GSet{@ 17 @},
    GSet{@ 20 @},
    GSet{@ 25 @},
    GSet{@ 2, 12 @},
    GSet{@ 3, 11 @},
    GSet{@ 4, 10 @},
    GSet{@ 5, 9 @},
    GSet{@ 6, 8 @},
    GSet{@ 7, 13 @},
    GSet{@ 14, 26 @},
    GSet{@ 15, 24 @},
    GSet{@ 16, 21 @},
    GSet{@ 18, 28 @},
    GSet{@ 19, 23 @},
    GSet{@ 22, 27 @}
*/

for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[1] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[17] eq m*(A[i])^n then i; end if; end for;
end for;
/*5*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[20] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
```

```
if ts[1]*ts[17]*ts[25] eq m*(A[i])^n then i; end if; end for;
end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2] eq m*(A[i])^n then i; end if; end for;
end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[3] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[4] eq m*(A[i])^n then i; end if; end for;
end for;
/*1*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[5] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[6] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[7] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[14] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[15] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[16] eq m*(A[i])^n then i; end if; end for;
 end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[18] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
```

```
if ts[1]*ts[17]*ts[19] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[22] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/



=============================================


[1 2]

N12:=Stabiliser(N1,2);
SSS:={[1,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s:=N12;

/*
[ 1, 2 ]
[ 16, 11 ]
*/

for g in N do if 1^g eq 16 and 2^g eq 11
then N12s:=sub<N|N12s,g>; end if; end for;


T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
```

```
Orbits(N12s);

/*
 GSet{@ 5 @},
    GSet{@ 8 @},
    GSet{@ 19 @},
    GSet{@ 28 @},
    GSet{@ 1, 16 @},
    GSet{@ 2, 11 @},
    GSet{@ 3, 23 @},
    GSet{@ 4, 20 @},
    GSet{@ 6, 27 @},
    GSet{@ 7, 26 @},
    GSet{@ 9, 15 @},
    GSet{@ 10, 14 @},
    GSet{@ 12, 18 @},
    GSet{@ 13, 17 @},
    GSet{@ 21, 25 @},
    GSet{@ 22, 24 @}
*/

for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[5] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[8] eq m*(A[i])^n then i; end if; end for;
end for;
/*1*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[19] eq m*(A[i])^n then i; end if; end for;
end for;
/*7*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[28] eq m*(A[i])^n then i; end if; end for;
 end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[1] eq m*(A[i])^n then i; end if; end for;
end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[2] eq m*(A[i])^n then i; end if; end for;
```

```
 end for;
/*5*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[3] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[4] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[6] eq m*(A[i])^n then i; end if; end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[7] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[9] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[10] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[12] eq m*(A[i])^n then i; end if; end for;
end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[13] eq m*(A[i])^n then i; end if; end for;
end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[21] eq m*(A[i])^n then i; end if; end for;
end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[22] eq m*(A[i])^n then i; end if; end for;
end for;
/*6*/
```

```
==================================================


[1 14 2]


N1142:=Stabiliser(N114,2);
SSS:={[1,14,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[14]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1142s:=N1142;

/*
[ 1, 14, 2 ]
[ 14, 1, 2 ]
[ 4, 21, 6 ]
[ 12, 15, 17 ]
[ 21, 4, 6 ]
[ 15, 12, 17 ]
*/

for g in N do if 1^g eq 14 and 14^g eq 1
and 2^g eq 2 then N1142s:=sub<N|N1142s,g>; end if; end for;

for g in N do if 1^g eq 4 and 14^g eq 21
and 2^g eq 6 then N1142s:=sub<N|N1142s,g>; end if; end for;

for g in N do if 1^g eq 12 and 14^g eq 15
and 2^g eq 17 then N1142s:=sub<N|N1142s,g>; end if; end for;

for g in N do if 1^g eq 21 and 14^g eq 4
and 2^g eq 6 then N1142s:=sub<N|N1142s,g>; end if; end for;

for g in N do if 1^g eq 15 and 14^g eq 12
and 2^g eq 17 then N1142s:=sub<N|N1142s,g>; end if; end for;
```

```
T1142:=Transversal(N,N1142s);
for i in [1..#T1142] do
ss:=[1,14,2]^T1142[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N1142s);

/*
GSet{@ 27 @},
    GSet{@ 2, 6, 17 @},
    GSet{@ 3, 8, 24 @},
    GSet{@ 9, 19, 10 @},
    GSet{@ 1, 14, 4, 12, 21, 15 @},
    GSet{@ 5, 18, 7, 28, 26, 16 @},
    GSet{@ 11, 22, 25, 13, 20, 23 @}
*/


for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[27] eq m*(A[i])^n then i; end if;
 end for;
end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[2] eq m*(A[i])^n then i; end if;
end for;
end for;
/*7*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[3] eq m*(A[i])^n then i; end if;
end for;
 end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[9] eq m*(A[i])^n then i; end if;
 end for;
 end for;
/*8*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[1] eq m*(A[i])^n then i; end if;
```

```
 end for;
end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[5] eq m*(A[i])^n then i; end if;
end for;
end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[2]*ts[11] eq m*(A[i])^n then i; end if;
end for;
end for;
/*6*/



============================================================




[1 14 5]


N1145:=Stabiliser(N114,5);
SSS:={[1,14,5]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[14]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1145s:=N1145;

/*
[ 1, 14, 5 ]
[ 27, 24, 5 ]
*/

for g in N do if 1^g eq 27 and 14^g eq 24
and 5^g eq 5 then N1145s:=sub<N|N1145s,g>; end if;
end for;
```

```
T1145:=Transversal(N,N1145s);
for i in [1..#T1145] do
ss:=[1,14,5]^T1145[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N1145s);

/*
GSet{@ 5 @},
    GSet{@ 13 @},
    GSet{@ 15 @},
    GSet{@ 21 @},
    GSet{@ 1, 27 @},
    GSet{@ 2, 23 @},
    GSet{@ 3, 26 @},
    GSet{@ 4, 16 @},
    GSet{@ 6, 20 @},
    GSet{@ 7, 11 @},
    GSet{@ 8, 25 @},
    GSet{@ 9, 28 @},
    GSet{@ 10, 18 @},
    GSet{@ 12, 22 @},
    GSet{@ 14, 24 @},
    GSet{@ 17, 19 @}
*/

for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[5] eq m*(A[i])^n then i; end if;
 end for; end for;
/*7*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[13] eq m*(A[i])^n then i; end if;
end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[15] eq m*(A[i])^n then i; end if;
end for; end for;
/*1*/
for i in [1..8] do for m,n in IN do
```

```
if ts[1]*ts[14]*ts[5]*ts[21] eq m*(A[i])^n then i; end if;
end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[1] eq m*(A[i])^n then i; end if;
end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[2] eq m*(A[i])^n then i; end if;
end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[3] eq m*(A[i])^n then i; end if;
end for; end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[4] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[6] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[7] eq m*(A[i])^n then i; end if;
end for; end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[8] eq m*(A[i])^n then i; end if;
 end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[9] eq m*(A[i])^n then i; end if;
end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[10] eq m*(A[i])^n then i; end if;
 end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[12] eq m*(A[i])^n then i; end if;
end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
```

```
if ts[1]*ts[14]*ts[5]*ts[14] eq m*(A[i])^n then i; end if;
end for; end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[14]*ts[5]*ts[17] eq m*(A[i])^n then i; end if;
end for; end for;
/*8*/



=============================================================



[1 17 2]


N1172:=Stabiliser(N117,2);
SSS:={[1,17,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[17]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1172s:=N1172;

/*
[ 1, 17, 2 ]
[ 3, 24, 19 ]
[ 9, 26, 11 ]
*/


for g in N do if 1^g eq 3 and 17^g eq 24
and 2^g eq 19 then N1172s:=sub<N|N1172s,g>; end if; end for;

for g in N do if 1^g eq 9 and 17^g eq 26
and 2^g eq 11 then N1172s:=sub<N|N1172s,g>; end if; end for;
```

```
T1172:=Transversal(N,N1172s);
for i in [1..#T1172] do
ss:=[1,17,2]^T1172[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N1172s);

/*
 GSet{@ 28 @},
    GSet{@ 1, 3, 9 @},
    GSet{@ 2, 19, 11 @},
    GSet{@ 4, 20, 8 @},
    GSet{@ 5, 14, 10 @},
    GSet{@ 6, 21, 12 @},
    GSet{@ 7, 22, 13 @},
    GSet{@ 15, 23, 16 @},
    GSet{@ 17, 24, 26 @},
    GSet{@ 18, 25, 27 @}
*/


for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[28] eq m*(A[i])^n then i; end if;
end for; end for;
/*1*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[1] eq m*(A[i])^n then i; end if;
end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[2] eq m*(A[i])^n then i; end if;
end for; end for;
/*4*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[4] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[5] eq m*(A[i])^n then i; end if;
 end for; end for;
/*2*/
```

```
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[6] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[7] eq m*(A[i])^n then i; end if;
end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[15] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[17] eq m*(A[i])^n then i; end if;
end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[17]*ts[2]*ts[18] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/



===============================================================


[1 2 8]


N128:=Stabiliser(N12,8);
SSS:={[1,2,8]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N128s:=N128;

/*
[ 1, 2, 8 ]
```

```
[ 16, 11, 8 ]
[ 1, 11, 5 ]
[ 11, 5, 1 ]
[ 16, 2, 5 ]
[ 2, 8, 1 ]
[ 8, 1, 2 ]
[ 8, 16, 11 ]
[ 5, 16, 2 ]
[ 5, 1, 11 ]
[ 2, 5, 16 ]
[ 11, 8, 16 ]
*/


for g in N do if 1^g eq 16 and 2^g eq 11
and 8^g eq 8 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 1 and 2^g eq 11
and 8^g eq 5 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 11 and 2^g eq 5
and 8^g eq 1 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 16 and 2^g eq 2
and 8^g eq 5 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 2 and 2^g eq 8
and 8^g eq 1 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 8 and 2^g eq 1
and 8^g eq 2 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 8 and 2^g eq 16
and 8^g eq 11 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 5 and 2^g eq 16
and 8^g eq 2 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 5 and 2^g eq 1
and 8^g eq 11 then N128s:=sub<N|N128s,g>; end if; end for;

for g in N do if 1^g eq 2 and 2^g eq 5
and 8^g eq 16 then N128s:=sub<N|N128s,g>; end if; end for;
```

```
for g in N do if 1^g eq 11 and 2^g eq 8
and 8^g eq 16 then N128s:=sub<N|N128s,g>; end if; end for;


T128:=Transversal(N,N128s);
for i in [1..#T128] do
ss:=[1,2,8]^T128[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..750] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N128s);

/*
GSet{@ 3, 23, 12, 18 @},
    GSet{@ 1, 16, 11, 2, 8, 5 @},
    GSet{@ 19, 28, 25, 24, 22, 21 @},
    GSet{@ 4, 20, 13, 27, 17, 15, 10, 14, 26, 7, 6, 9 @}
*/

for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[8]*ts[3] eq m*(A[i])^n then i; end if;
end for; end for;
/*3*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[8]*ts[8] eq m*(A[i])^n then i; end if;
 end for; end for;
/*6*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[8]*ts[19] eq m*(A[i])^n then i; end if;
end for; end for;
/*2*/
for i in [1..8] do for m,n in IN do
if ts[1]*ts[2]*ts[8]*ts[4] eq m*(A[i])^n then i; end if;
 end for; end for;
/*4*/
```

Appendix D

MAGMA Code for Isomorphism Type of $G \cong 7 \times 2^2$

```
G1:=TransitiveGroup(28,2);
G1;
/*
Permutation group acting on a set of cardinality 28
Order = 28 = 2^2 * 7
    (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)
(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28),
    (1, 19, 9, 28, 17, 8, 25, 15, 6, 23, 14, 4, 22, 12)(2,
20, 10, 27, 18, 7, 26, 16, 5, 24, 13, 3, 21, 11)
*/
CompositionFactors(G1);
NL:=NormalLattice(G1);
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
end for;
NL;

/*
Normal subgroup lattice
-----------------------

[10]  Order 28  Length 1  Maximal Subgroups: 6 7 8 9
---
[ 9]  Order 14  Length 1  Maximal Subgroups: 4 5
[ 8]  Order 14  Length 1  Maximal Subgroups: 2 5
```

```
[ 7]  Order 14  Length 1  Maximal Subgroups: 3 5
[ 6]  Order 4   Length 1  Maximal Subgroups: 2 3 4
---
[ 5]  Order 7   Length 1  Maximal Subgroups: 1
[ 4]  Order 2   Length 1  Maximal Subgroups: 1
[ 3]  Order 2   Length 1  Maximal Subgroups: 1
[ 2]  Order 2   Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1   Length 1  Maximal Subgroups:
*/

IsAbelian(TransitiveGroup(28,2));
X:=DirectProduct(NL[5],NL[6]);
s:=IsIsomorphic(X,G);

NL6:=NormalLattice(NL[6]);
X:=AbelianGroup(GrpPerm,[2,2]);
s:=IsIsomorphic(X,NL[6]);

HH<b,c>:=Group<b,c|b^2,c^2,(b*c)^2>;
f,H,k:=CosetAction(HH,sub<HH|Id(HH)>);
s:=IsIsomorphic(H,NL[6]);


HH<a,b,c>:=Group<a,b,c|a^7,b^2,c^2,(b*c)^2,(a,b),(a,c)>;
f,H,k:=CosetAction(HH,sub<HH|Id(HH)>);
s:=IsIsomorphic(H,G1);
```

Appendix E

MAGMA Code for Isomorphism Type of $\frac{2^{*^{28}}:(PGL(2,7):2)}{(z*x^2*y*x^2*y*t^{(y*x^5)})^3,(z*x^2*y*x^2*y*t^{(x^2)})^5} \cong U_3(5):2$

```
G<x,y,z,t>:=Group<x,y,z,t|x^7,y^2,z^2,(x^-1*z)^2,
(y*z)^2,(y*x^-1)^3,(x*y*x^2)^4,
t^2,
(t,x * y * z * x^2 * y * x^-1),
(t,x^2 * y * z * x^3),
(t,x^-2 * y * x^2 * y * x^-1),
(z * x^2 * y * x^2 * y*t^(y*x^5))^3,
(z * x^2 * y * x^2 * y*t^(x^2))^5>;


f,G1,k:=CosetAction(G,sub<G|x,y,z>);
G1;
Permutation group G1 acting on a set of cardinality 750
Order = 252000 = 2^5 * 3^2 * 5^3 * 7

SL:=Subgroups(G1);

/*This is the set of subgroups.*/
```

```
T:={x`subgroup : x in SL};

/*These are all the subgroups that give faithful permutation
representation.*/

TrivCore:={H:H in T |#Core(G1,H) eq 1};

/*Now we want minimal degree*/

mdeg := Min({Index(G1,H):H in TrivCore});

/*We want all the subgroups that give way to minimum degree.*/

Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;

f,G2,K := CosetAction(G1,H);

G2;
Permutation group G2 acting on a set of cardinality 50
Order = 252000 = 2^5 * 3^2 * 5^3 * 7
    (1, 2, 6, 13, 3, 8, 12)(4, 5, 11, 9, 18, 14, 7)(10, 20, 32,
24, 22, 29, 42)(15, 23, 16, 25, 35, 34, 27)(17, 19, 26, 21,
33, 38, 31)(28, 40, 44, 45, 36, 30, 43)(37, 46, 39, 41, 48,
49, 47),
    (1, 3)(2, 6)(4, 9)(7, 14)(10, 21)(15, 24)(16, 26)(17, 28)
(19, 30)(20, 23)(22, 33)(25, 36)(27, 38)(29, 42)(31, 44)
(34, 45)(37, 47)(39, 46)(41, 49)(48, 50),
    (1, 4)(2, 7)(3, 9)(5, 12)(6, 14)(8, 11)(10, 22)(13, 18)(15,
23)(16, 27)(19, 31)(20, 24)(21, 33)(25, 34)(26, 38)(29, 42)
(30, 44)(36, 45)(37, 46)(39, 47)(40, 43)(41, 49),
    (1, 5)(3, 10)(6, 15)(7, 16)(8, 17)(9, 19)(11, 13)(12, 14)
(18, 29)(20, 22)(21, 26)(24, 34)(25, 37)(27, 39)(28, 41)
(30, 45)(31, 38)(32, 35)(33, 46)(40, 47)(42, 50)(48, 49)
CompositionFactors(G2);
    G
    |  Cyclic(2)
    *
    |  2A(2, 5)                    = U(3, 5)
    1

NL:=NormalLattice(G2);
```

```
Normal subgroup lattice
-----------------------

[3]  Order 252000  Length 1  Maximal Subgroups: 2
---
[2]  Order 126000  Length 1  Maximal Subgroups: 1
---
[1]  Order 1       Length 1  Maximal Subgroups:
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
 end for;

NL[2];
Permutation group acting on a set of cardinality 50
Order = 126000 = 2^4 * 3^2 * 5^3 * 7

S:=Sym(50);
     A:=S!(1, 16, 7, 5)(2, 13, 23, 11)(3, 19, 14, 50)(4, 37,
44, 25)(6, 15, 28, 41)(8, 22, 39, 31)(9, 10, 42, 12)(17, 38,
27, 20)(18, 21, 46, 32)(26, 29, 35, 33)(30, 45)(36, 43)
(40, 49)(47, 48);
    B:=S!(1, 5, 15, 6)(2, 39, 43, 27)(3, 38, 46, 24)(4, 14,
23, 12)(7, 16, 40, 47)(8, 29, 42, 11)(9, 20, 37, 26)(10, 34,
 33, 31)(13, 50, 18, 17)(19, 21, 25, 22)(28, 48)(30, 45)
(36, 44)(41, 49);
    C:=S!(2, 25)(3, 26)(4, 27)(6, 7)(10, 47)(13, 30)(14, 45)
(15, 44)(16, 43)(17, 22)(18, 31)(19, 20)(21, 28)(29, 41)
(33, 46)(34, 48)(35, 49)(36, 50)(37, 39)(38, 40);
    D:=S!(1, 26, 12, 16, 3)(2, 34, 9, 19, 24)(4, 45, 6, 15,
30)(5, 10, 7, 14, 21)(8, 41, 42, 29, 49)(17, 48, 18, 50,
28)(20, 22, 27, 44, 39)(23, 46, 31, 38, 33)(25, 37, 47,
36, 40);
    E:=S!(1, 8, 13, 2, 12, 3, 6)(4, 14, 9, 5, 7, 18, 11)(10,
29, 24, 20, 42, 22, 32)(15, 34, 25, 23, 27, 35, 16)(17,
38, 21, 19, 31, 33, 26)(28, 30, 45, 40, 43, 36, 44)(37,
49, 41, 46, 47, 48, 39);
    F:=S!(1, 7, 17, 45, 31)(2, 47, 15, 46, 14)(3, 39, 48,
25, 22)(4, 40, 38, 9, 27)(6, 18, 33, 21, 26)(10, 34, 16,
24, 50)(11, 42, 35, 41, 49)(12, 23, 13, 19, 30)(28, 44,
36, 43, 37);
    I:=S!(1, 3)(2, 8, 6, 13)(4, 14, 7, 9)(5, 18)(10, 29, 27,
25)(15, 44, 31, 24)(16, 32, 26, 43)(17, 34, 45, 28)(19,
40, 30, 35)(20, 22)(21, 36, 38, 42)(23, 33)(37, 39, 49,
48)(41, 46, 47, 50);
   J:=S!(1, 6, 22, 14, 18)(2, 17, 26, 37, 34)(3, 7, 31, 46,
```

```
 28)(4, 27, 38, 40, 9)(10, 44, 33, 45, 25)(11, 42, 49, 29,
 35)(12, 23, 30, 20, 13)(15, 50, 16, 39, 21)(24, 36, 47,
48, 43);

N:=sub<S|A,B,C,D,E,F,I,J>;

FPGroup(NL[2]);
NN<a,b,c,d,e,f,i,j>:=Group<a,b,c,d,e,f,i,j|a^4 ,
    b^4 ,
    c^2 ,
    d^5 ,
    e^7 ,
    f^5 ,
    i^4 ,
    j^5 ,
    c * f^-1 * c * j ,
    j^-1 * f^-1 * c * f^-1 ,
    (b^-1 * a^-1)^3 ,
    a * j^-1 * b^-1 * a^-1 * i * e^-1 ,
    f^-1 * e^-1 * b * a^-1 * f^-1 * e ,
    f^-1 * j * c * a * i^2 * a^-1 ,
    i * j * c * b * i^2 * c ,
    f * b^-1 * d^-1 * c * f^-1 * d ,
    (b^-1 * d^-2)^2 ,
    j * d^2 * c * b^2 * j ,
    c * b^-2 * d * b^-1 * f^2 ,
    f * b^-2 * d * b * j^-1 * b ,
    f^2 * d^2 * f^-1 * b ,
    (c * i^-1 * d)^2 ,
    j * i^-1 * d * a^-1 * j * i ,
    (e^-1 * a)^3 ,
    e * a^-2 * e^-1 * a * b^2 * a^-1 ,
    (e^-1 * b^-1)^3 ,
    a * e^-2 * d^-1 * a^2 * f ,
    b * f^-1 * e^-1 * b^-1 * f * e ,
    e^-1 * b^-2 * e * b * a^2 * b^-1 ,
    f * b^-2 * e * d^-1 * i * a^-1 ,
    (e * i * e)^2 ,
    j^-1 * i^-2 * e * a * i^2 * j^-1 ,
    (i^-1 * d^-1)^3 ,
    (i^-1 * d)^3 ,
    j * b^-1 * i * d * a^2 * e^-1 ,
    (i * e^-1)^3 ,
    j^-1 * i^-1 * j * b * i^2 * f ,
```

```
    j^-1 * i^-2 * j * e * i * e ,
    b^-1 * i^-2 * a^-2 * j^-1 * d * a^-1 ,
    e^2 * b^-1 * a^-1 * d^-1 * a * f ,
    b * a^-1 * b * a^-1 * e * i * e
>;
f,H,k:=CosetAction(NN,sub<NN|Id(NN)>);
s:=IsIsomorphic(H,NL[2]);

q,ff:=quo<G2|NL[2]>;
q;
Permutation group q acting on a set of cardinality 2
Order = 2
    Id(q)
    Id(q)
    (1, 2)
    (1, 2)
NLq:=NormalLattice(q);
X:=AbelianGroup(GrpPerm,[2]);
s:=IsIsomorphic(X,q);

T:=Transversal(G2,NL[2]);
ff(T[2]) eq q.3;

K:=T[2];
/*
(1, 4)(2, 7)(3, 9)(5, 12)(6, 14)(8, 11)(10, 22)(13, 18)
(15, 23)(16, 27)(19, 31)(20, 24)(21, 33)(25, 34)
(26, 38)(29, 42)(30, 44)(36, 45)(37, 46)(39, 47)
(40, 43)(41, 49)
*/

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [#N]];
for i in [2..#N] do P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
if Eltseq(Sch[i])[j] eq 7 then P[j]:=I; end if;
if Eltseq(Sch[i])[j] eq 8 then P[j]:=J; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
```

```
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=E^-1; end if;
if Eltseq(Sch[i])[j] eq -6 then P[j]:=F^-1; end if;
if Eltseq(Sch[i])[j] eq -7 then P[j]:=I^-1; end if;
if Eltseq(Sch[i])[j] eq -8 then P[j]:=J^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq N!A^K then Sch[i];
end if; end for;
/*
a^k=e^-1 * a^-1 * e
*/
for i in [1..#N] do if ArrayP[i] eq N!B^K then Sch[i];
end if; end for;
/*
b^k=b^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!C^K then Sch[i];
end if; end for;
/*
c^k=c^b
*/
for i in [1..#N] do if ArrayP[i] eq N!D^K then Sch[i];
end if; end for;
/*
d^k=f^-1 * d * j * d^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!E^K then Sch[i];
end if; end for;
/*
e^k=e^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!F^K then Sch[i];
end if; end for;
/*
f^k=j^-1 * d^-1 * j
*/
for i in [1..#N] do if ArrayP[i] eq N!I^K then Sch[i];
```

```
end if; end for;
/*
i^k=e^-2 * i * e^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!J^K then Sch[i];
end if; end for;
/*
j^k=c * f * b
*/

HH<a,b,c,d,e,f,i,j,k>:=Group<a,b,c,d,e,f,i,j,k|a^4 ,
    b^4 ,
    c^2 ,
    d^5 ,
    e^7 ,
    f^5 ,
    i^4 ,
    j^5 ,
    c * f^-1 * c * j ,
    j^-1 * f^-1 * c * f^-1 ,
    (b^-1 * a^-1)^3 ,
    a * j^-1 * b^-1 * a^-1 * i * e^-1 ,
    f^-1 * e^-1 * b * a^-1 * f^-1 * e ,
    f^-1 * j * c * a * i^2 * a^-1 ,
    i * j * c * b * i^2 * c ,
    f * b^-1 * d^-1 * c * f^-1 * d ,
    (b^-1 * d^-2)^2 ,
    j * d^2 * c * b^2 * j ,
    c * b^-2 * d * b^-1 * f^2 ,
    f * b^-2 * d * b * j^-1 * b ,
    f^2 * d^2 * f^-1 * b ,
    (c * i^-1 * d)^2 ,
    j * i^-1 * d * a^-1 * j * i ,
    (e^-1 * a)^3 ,
    e * a^-2 * e^-1 * a * b^2 * a^-1 ,
    (e^-1 * b^-1)^3 ,
    a * e^-2 * d^-1 * a^2 * f ,
    b * f^-1 * e^-1 * b^-1 * f * e ,
    e^-1 * b^-2 * e * b * a^2 * b^-1 ,
    f * b^-2 * e * d^-1 * i * a^-1 ,
    (e * i * e)^2 ,
    j^-1 * i^-2 * e * a * i^2 * j^-1 ,
    (i^-1 * d^-1)^3 ,
    (i^-1 * d)^3 ,
```

```
    j * b^-1 * i * d * a^2 * e^-1 ,
    (i * e^-1)^3 ,
    j^-1 * i^-1 * j * b * i^2 * f ,
    j^-1 * i^-2 * j * e * i * e ,
    b^-1 * i^-2 * a^-2 * j^-1 * d * a^-1 ,
    e^2 * b^-1 * a^-1 * d^-1 * a * f ,
    b * a^-1 * b * a^-1 * e * i * e,
k^2,
a^k=e^-1 * a^-1 * e,
b^k=b^-1,
c^k=c^b,
d^k=f^-1 * d * j * d^-1,
e^k=e^-1,
f^k=j^-1 * d^-1 * j,
i^k=e^-2 * i * e^-1,
j^k=c * f * b
>;
f,H,k:=CosetAction(HH,sub<HH|Id(HH)>);
IsIsomorphic(H,G2);
```

Appendix F

MAGMA Code for Isomorphism Type of $\frac{2^{*^{60}}:S_5}{(yt^{y^3x})^4,(yt^x)^2,(yt^{xy^{-2}xy^{-1}})^8} \cong 2^{\bullet}M_{12}:2$

```
G<x,y,t>:=Group<x,y,t|x^2,y^6,(y*x*y^-1*x)^2,(x*y^-1)^5,
 t^2,
 (t,(y * x * y)^2),

 (y*t^(y^3 * x))^4,
 (y*t^(x))^2,
 (y*t^(x * y^-2 * x * y^-1))^8
 >;

f1,G1,k1:=CosetAction(G,sub<G|x,y>);
G1;
Permutation group G1 acting on a set of cardinality 3168
Order = 380160 = 2^8 * 3^3 * 5 * 11

SL:=Subgroups(G1);

/*This is the set of subgroups.*/

T:={x`subgroup : x in SL};

/*These are all the subgroups that give faithful permutation
```

```
representation.*/

TrivCore:={H:H in T |#Core(G1,H) eq 1};

/*Now we want minimal degree*/

mdeg := Min({Index(G1,H):H in TrivCore});

/*We want all the subgroups that give way to minimum degree.*/

Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;

f,G2,K := CosetAction(G1,H);
IsIsomorphic(G1,G2);

G2;
Permutation group G2 acting on a set of cardinality 48
Order = 380160 = 2^8 * 3^3 * 5 * 11

   (1, 2)(3, 6)(4, 5)(7, 12)(8, 15)(9, 16)(10, 19)(11, 21)
(13, 24)(14, 27)(17, 31)(18, 34)(20, 36)(22, 37)(23, 32)
(25, 30)(26, 40)(28, 29)(33, 42)(35, 41)(38, 44)(39, 43)
(45, 47)(46, 48)
    (1, 3, 7, 13, 25, 2)(4, 5, 9, 17, 32, 21)(6, 10, 18, 34,
41, 30)(8, 15, 20, 11, 16, 29)(12, 22, 37, 24, 39, 42)(14,
28, 36, 44, 46, 48)(19, 35, 43, 45, 47, 33)(23, 38, 27,
31, 26, 40)
    (1, 4)(2, 5)(3, 8)(6, 11)(7, 14)(9, 18)(10, 20)(12, 23)
(13, 26)(15, 19)(16, 30)(17, 33)(21, 35)(22, 32)(24, 31)
(25, 36)(27, 39)(28, 34)(29, 41)(37, 44)(38, 42)(40, 43)
(45, 48)(46, 47)

CompositionFactors(G2);
 G
    |  Cyclic(2)
    *
    |  M12
    *
    |  Cyclic(2)
    1
```

```
NLG2:=NormalLattice(G2);
 Center(G2);
Center(G2) eq NLG2[2];

Permutation group acting on a set of cardinality 48
Order = 2
  (1, 5)(2, 4)(3, 9)(6, 16)(7, 17)(8, 18)(10, 29)(11, 30)
(12, 31)(13, 32)(14, 33)(15, 34)(19, 28)(20, 41)(21, 25)
(22, 26)(23, 24)(27, 42)(35, 36)(37, 40)(38, 39)(43, 44)
(45, 46)(47, 48)
Normal subgroup lattice
-----------------------

[7]  Order 380160  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 190080  Length 1  Maximal Subgroups: 3
[5]  Order 190080  Length 1  Maximal Subgroups: 2 3
[4]  Order 190080  Length 1  Maximal Subgroups: 3
---
[3]  Order 95040   Length 1  Maximal Subgroups: 1
---
[2]  Order 2       Length 1  Maximal Subgroups: 1
---
[1]  Order 1       Length 1  Maximal Subgroups:

q1,ff1:=quo<G2|NLG2[2]>;
Permutation group q1 acting on a set of cardinality 144
Order = 190080 = 2^7 * 3^3 * 5 * 11


SL:=Subgroups(q1);

/*This is the set of subgroups.*/

T:={x`subgroup : x in SL};

/*These are all the subgroups that give faithful permutation
representation.*/

TrivCore:={H:H in T |#Core(q1,H) eq 1};

/*Now we want minimal degree*/

mdeg := Min({Index(q1,H):H in TrivCore});
```

```
/*We want all the subgroups that give way to minimum
degree.*/

Good := {H: H in TrivCore| Index(q1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;

f,G3,K := CosetAction(q1,H);
s,t:=IsIsomorphic(G3,q1);

G3;
Permutation group G3 acting on a set of cardinality 24
Order = 190080 = 2^7 * 3^3 * 5 * 11
   (1, 2)(3, 6)(4, 8)(5, 9)(7, 12)(10, 13)(11, 16)(14, 21)
(15, 22)(17, 23)(18, 20)(19, 24)
    (1, 3, 7, 9, 8, 2)(4, 6, 11, 17, 23, 15)(5, 10, 14, 12,
19, 24)(13, 20, 18, 21, 16, 22)
    (1, 4)(2, 5)(3, 8)(6, 12)(7, 9)(10, 15)(11, 18)(13, 21)
(14, 20)(16, 22)(17, 23)(19, 24)

CompositionFactors(G3);

    G
    |  Cyclic(2)
    *
    |  M12
    1

NLG3:=NormalLattice(G3);
Normal subgroup lattice
-----------------------

[3]  Order 190080  Length 1  Maximal Subgroups: 2
---
[2]  Order 95040   Length 1  Maximal Subgroups: 1
---
[1]  Order 1       Length 1  Maximal Subgroups:

Generators(NLG3[2]);
    (1, 17)(2, 9)(3, 10)(4, 24)(5, 16)(6, 8)(7, 18)(11, 22)
(12, 21)(13, 15)(14, 19)(20, 23),
    (3, 10, 12)(4, 9, 11)(5, 13, 17)(6, 19, 15)(7, 18, 14)
```

```
(21, 23, 22)

BB:=G3!NLG3[2].1;
CC:=G3!NLG3[2].2;
N:=sub<G3|BB,CC>;
N eq NLG3[2];

G4,ff2:=quo<G3|NLG3[2]>;
TG3:=Transversal(G3,NLG3[2]);
ff2(TG3[2])  eq G4.1;
TG3[2];
(1, 2)(3, 6)(4, 8)(5, 9)(7, 12)(10, 13)(11, 16)(14, 21)
(15, 22)(17, 23)(18, 20)(19, 24)
AA:=TG3[2];

NN<b,c>:=Group<b,c|b^2 ,
c^3 ,
(b * c^-1)^11 ,
(c * b * c^-1 * b)^6 ,
(c * b * c^-1 * b * c * b * c * b * c * b * c^-1 *
b * c * b * c^-1 * b * c^-1 * b * c^-1 * b)^2 ,
(c^-1 * b * c^-1 * b * c * b * c * b)^5>;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [#N]];
for i in [2..#N] do P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=BB; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=CC; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=CC^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq N!CC^AA then Sch[i];
end if; end for;
/*
c^a=
c * b * c * b * c^-1 * b * c^-1 * b * c * b * c * b * c^-1
 * b * c^-1 *b * c^-1 * b * c^-1 * b * c * b * c * b * c^-1
* b * c * b * c^-1
```

```
*/

for i in [1..#N] do if ArrayP[i] eq N!BB^AA then Sch[i];
end if; end for;
/*
b^a=
c * b * c * b * c^-1 * b * c * b * c^-1 * b * c^-1 * b *
c^-1 * b * c^-1 * b * c * b * c * b * c * b * c * b * c^-1
* b * c * b * c^-1 *b * c^-1
*/

HH<a,b,c>:=Group<a,b,c|
b^2 ,
c^3 ,
(b * c^-1)^11 ,
(c * b * c^-1 * b)^6 ,
(c * b * c^-1 * b * c * b * c * b * c * b * c^-1 *
b * c * b * c^-1 * b * c^-1 * b * c^-1 * b)^2 ,
(c^-1 * b * c^-1 * b * c * b * c * b)^5 ,
a^2,
c^a=
c * b * c * b * c^-1 * b * c * b * c^-1 * b * c^-1 * b *
c^-1 * b * c * b* c * b * c^-1 * b * c * b * c^-1 * b *
c^-1 * b * c^-1 * b * c * b,
b^a=
c * b * c * b * c^-1 * b * c * b * c * b * c * b * c^-1
* b * c * b * c * b * c^-1 * b * c^-1 * b * c * b * c^-1
* b * c^-1 * b * c^-1 * b * c *b * c^-1 * b * c^-1>;

f2,H,k2:=CosetAction(HH,sub<HH|Id(HH)>);
IsIsomorphic(q1,H);

T:=Transversal(G2,sub<G2|NLG2[2]>);

ff1(T[2]) eq q1.1;
ff1(T[3]) eq q1.2;
ff1(T[4]) eq q1.3;
A:=T[2]; B:=T[3]; C:=T[4];
A;
(1, 2)(3, 6)(4, 5)(7, 12)(8, 15)(9, 16)(10, 19)(11, 21)
(13, 24)(14, 27)(17, 31)(18, 34)(20, 36)(22, 37)(23,
32)(25, 30)(26, 40)(28, 29)(33, 42)(35, 41)(38, 44)
(39, 43)(45, 47)(46, 48)
B;
```

```
(1, 3, 7, 13, 25, 2)(4, 5, 9, 17, 32, 21)(6, 10, 18, 34,
41, 30)(8, 15, 20, 11, 16, 29)(12, 22, 37, 24, 39, 42)
(14, 28, 36, 44, 46, 48)(19, 35, 43, 45, 47, 33)(23,
38, 27, 31, 26, 40)
C;
(1, 4)(2, 5)(3, 8)(6, 11)(7, 14)(9, 18)(10, 20)(12, 23)
(13, 26)(15, 19)(16, 30)(17, 33)(21, 35)(22, 32)(24,
31)(25, 36)(27, 39)(28, 34)(29, 41)(37, 44)(38, 42)
(40, 43)(45, 48)(46, 47)
Order(A);
Order(B);
Order(C);
Order(B * A * B^-1 * A);
Order(B * A * B^-1 * A);
Order(B^-1 * A * C* A);
Order(B^-1 *C);
Order(A *B^-1);
Order(C *A);
Order(B^-1 * C * B^-2 * A * B^2 * C * B^2 * A * B^-1);
(C*A)^5 eq NLG2[2].2;

GG<a,b,c,d>:=Group<a,b,c,d|
a^2 ,
b^6 ,
c^2 ,
d^2 ,
(b * a * b^-1 * a)^2 ,
(b^-1 * a * c * a)^2 ,
(b^-1 * c)^4 ,
(a * b^-1)^5 ,
(c * a)^5=d ,
b^-1 * c * b^-2 * a * b^2 * c * b^2 * a * b^-1,
(a,d),(b,d),(c,d)
>;

f4,GG,k4:=CosetAction(GG,sub<GG|Id(GG)>);
IsIsomorphic(G2,GG);
```

Appendix G

MAGMA Code for Isomorphism Type of $G \cong (4 \times 2) :^{\bullet} 2^2$

```
G:=TransitiveGroup(8,16);
/*
Permutation group N acting on a set of cardinality 8
Order = 32 = 2^5
*/
CompositionFactors(G);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

NL:=NormalLattice(G);

Normal subgroup lattice
-----------------------

[12]  Order 32  Length 1  Maximal Subgroups: 9 10 11
---
[11]  Order 16  Length 1  Maximal Subgroups: 7
```

```
[10]  Order 16  Length 1  Maximal Subgroups: 7
[ 9]  Order 16  Length 1  Maximal Subgroups: 6 7 8
---
[ 8]  Order 8   Length 1  Maximal Subgroups: 3
[ 7]  Order 8   Length 1  Maximal Subgroups: 3 4 5
[ 6]  Order 8   Length 1  Maximal Subgroups: 3
---
[ 5]  Order 4   Length 1  Maximal Subgroups: 2
[ 4]  Order 4   Length 1  Maximal Subgroups: 2
[ 3]  Order 4   Length 1  Maximal Subgroups: 2
---
[ 2]  Order 2   Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1   Length 1  Maximal Subgroups:

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
 end for;

NL7:=NormalLattice(NL[7]);
X:=AbelianGroup(GrpPerm,[4,2]);
X:=DirectProduct(NL7[3],NL7[7]);
s:=IsIsomorphic(X,NL[7]);

(NL[7]);
Permutation group acting on a set of cardinality 8
Order = 8 = 2^3
    (1, 3, 5, 7)(2, 4, 6, 8)
    (2, 6)(4, 8)
    (1, 5)(3, 7)
C:=G!(1, 3, 5, 7)(2, 4, 6, 8);
D:=G!(2, 6)(4, 8);
E:=G!(1, 5)(3, 7);
N:=sub<G|C,D,E>;

q,ff:=quo<G|NL[7]>;
q;
Permutation group q acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 2)(3, 4)
    (1, 3)(2, 4)
NLq:=NormalLattice(q);
Normal subgroup lattice
-----------------------
```

```
[5]  Order 4  Length 1  Maximal Subgroups: 2 3 4
---
[4]  Order 2  Length 1  Maximal Subgroups: 1
[3]  Order 2  Length 1  Maximal Subgroups: 1
[2]  Order 2  Length 1  Maximal Subgroups: 1
---
[1]  Order 1  Length 1  Maximal Subgroups:

X:=AbelianGroup(GrpPerm,[2,2]);
s:=IsIsomorphic(X,q);

HH<a,b>:=Group<a,b|a^2,b^2,(a*b)^2>;
f1,H,k1:=CosetAction(HH,sub<HH|Id(HH)>);
IsIsomorphic(X,H);

T:=Transversal(N,NL[7]);

ff(T[2]) eq q.1;
ff(T[3]) eq q.2;

A:=T[2];
/*
(2, 6)(3, 7)
*/
B:=T[3];
/*
(1, 2, 3, 4, 5, 6, 7, 8)
*/

FPGroup(NL[7]);
NN<c,d,e>:=Group<c,d,e|
d^2,
e^2,
c^4,
d * c^2 *e,
c^-1 * d* c*d>;

f1,H,k1:=CosetAction(NN,sub<NN|Id(NN)>);
IsIsomorphic(NL[7],H);

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [#N]];
for i in [2..#N] do P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=C^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq N!C^A then Sch[i];
end if; end for;
/*
c^a=c^-1
*/
for i in [1..#N] do if ArrayP[i] eq N!C^B then Sch[i];
end if; end for;
/*
c^b=c
*/
for i in [1..#N] do if ArrayP[i] eq N!D^A then Sch[i];
end if; end for;
/*
d^a=d
*/
for i in [1..#N] do if ArrayP[i] eq N!D^B then Sch[i];
end if; end for;
/*
d^b=e
*/
for i in [1..#N] do if ArrayP[i] eq N!E^A then Sch[i];
end if; end for;
/*
e^a=e
*/
for i in [1..#N] do if ArrayP[i] eq N!E^B then Sch[i];
end if; end for;
/*
e^b=d
*/

Order(A);
2
```

```
Order(B);
8
Order(A*B);
8

for i,k in [0..1] do for j in [0..7] do if
T[3]^2 eq C^i*D^j*E^k then i,j,k;
break; end if; end for;end for;
/*1 0 0*/
for i,k in [0..1] do for j in [0..7] do if
(T[2]*T[3])^2 eq C^i*D^j*E^k then i,j,k;
break; end if; end for;end for;
/*1 0 1*/

HH<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2=c,(a*b)^2=c*e,
d^2,
e^2,
c^4,
d * c^2 *e,
c^-1 * d* c*d,
c^a=c^-1,
c^b=c,
d^a=d,
d^b=e,
e^a=e,
e^b=d >;

f1,H,k1:=CosetAction(HH,sub<HH|Id(HH)>);
IsIsomorphic(G,H);
true
```

Bibliography

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. *The Magma algebra system. I. The user language*, volume 24. 1997. Computational algebra and number theory (London, 1993).

[Bra97] John N. Bray. *Symmetric presentations of sporadic groups and related topics*. University of Birmingham, England, 1997.

[Cur07] Robert T. Curtis. *Symmetric generation of groups*. volume 111 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2007.

[Gri15] Dustin J. Grindstaff. Permutation presentations of non-abelian simple groups, 2015.

[Hum96] John F. Humphreys. *A Course in Group Theory*. Oxford University Press, Oxford, 1996.

[JL93] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, 1993.

[Lam15] Leonard B. Lamp. Permutation presentations and generations, 2015.

[Led77] Walter Lederman. *Introduction to group characters*. Cambridge University Press, 1977.

[Rot95] Joseph J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[WB99] Robert Wilson and John Bray. Atlas of finite group representations. http://brauer.maths.qmul.ac.uk/Atlas/v3/, 1999. [Online; accessed February-2014].

[Wed08] Extensions of groups. http://www.weddslist.com/groups/extensions/ext.html, 2008. [Online; accessed December-2013].

[Why06] Sophie Whyte. *Symmetric Generation: Permutation Images and Irreducible Monomial Representations.* The University of Birmingham, 2006.