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PROGENITORS, SYMMETRIC PRESENTATIONS AND CONSTRUCTIONS

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Progenitors, Symmetric Presentations and Constructions

A Thesis

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In Partial Fulfillment

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in

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by

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Abstract

In this project, we searched for new constructions and symmetric presentations of important groups, nonabelian simple groups, their automorphism groups, or groups that have these as their factor groups. My target non-abelian simple groups included sporadic groups, linear groups, and alternating groups. In addition, we discovered finite groups as homomorphic images of progenitors and proved some of their isomorphism type and original symmetric presentations. In this thesis we found original symmeric presentations of M_{12}, J_1 and the simplectic groups $S(4,4)$ and $S(3,4)$ on various control groups. Using the technique of double coset enumeration we constucted J_2 as a homomorphic image of the permutation progenitor $2^{*10} : (10 \times 2)$. From our monomial progenitor $11^{*4} : (2 : 4)$ we found a homomorphic image of M_{11}. In the following chapters we will discuss how we went about obtaining homomorphic images, some constructions of the Cayley Diagrams, and how we solved some extension problems.

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Now more than ever I believe that dreams come true. Those who do not dream, have no future.

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Introduction

We will begin our discussion of control groups and how a progenitor of infinite order is constructed from a control group N. We used the computer based program $MAGMA$ to help facilitate the construction of such progenitors to obtain homomorphic images of varios interesting groups, and thus to research these groups in more detail. With the help of $MAGMA$ we performed double coset enumeration on groups such as $S_6 : C_2$ over our control group $\frac{2^{*10}:2\times(5:4)}{[y^{-1}*x^{-1}*t]^6,[x^{-1}*y^{-1}*x*t]^4,[x^2*t]^8,[x*t]^5}$. We also proved the isomorphism types of some groups such as $G \cong 3^4 : \cdot(S_5 \times 2)$. We discuss monomial progenitors in more detail, and finally overview all homomorphic images obtained from our progenitors.

We begin by defining the progenitor. A progenitor is a semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi \ w \mid \pi \in N,\ w$ a reduced word in the $t_i\}$, where 2^{*n} denotes the free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, ..., n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators. We refer to the subgroup N as the control subgroup and to the involutory generators of the free product as the symmetric generators. The unique progenitor is then factored by the appropriate relations that produce finite homomorphic images. In the continuing section, I will demonstrate how the process is done.

Definition 0.1. *[Led87] A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1,\pi_2\omega_2,...}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive *permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1\omega_1, \pi_2\omega_2, ...$ have been factored out.*

Before factoring the progenitor $m^{*n} : N$, where m is the order of $t_{i's}$, n is the number of $t_{i's}$, and N is the control group, by necessary relations, we need to write a permutation progenitor. Since the progenitor $m^{*n} : N$ is infinite we write a permutation progenitor where we take N to be transitive on n letters. So we have a general form of a permutation progenitor in the following form:

$$< x, y, t| < x, y >\cong N, t^m, (t, N^i) >, \text{where } N^i \text{ is the stabiliser of } i \text{ in } N$$

Since t commutes with the stabiliser of i in N, (t, N^i), we can obtain the number of conjugates of t. Using the definition we have that $[G : C_g(a)]$ is the number of conjugates of H in G. So to find the index of the centraliser of N and t also denoted as $Centraliser[N, t]$, we are going to calculate $[G : C_g(a)]$. Note that the index of the $Centraliser[N, t]$ is equal to the number of conjugates of t and also equal to the stabiliser of a single point in N. Applying this concept we are going to find the permutation progenitor of the following example, $2^{*10} : (5^2 : 2)$.

EXAMPLE: In this example, we will illustrate how to write a progenitor for the infinite progenitor $2^{*10} : (5^2 : 2)$. Our control group $N = (5^2 : 2)$ is transitive on 10 letters and $(5^2 : 2) =< (2, 4, 6, 8, 10), (1, 6)(2, 7)(3, 8)(4, 9)(5, 10) >$ where the generators of N are $x \sim (2, 4, 6, 8, 10)$ and $y \sim (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$. Then the presentation of $(5^2 : 2)$ is

$$< x, y|x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y >$$

Now we let t be a symmetric generator where $t \sim t_1$ and is of order 2. Since we let t be t_1 we must compute the stabiliser of the single point 1 in N, denoted N^1. So $N^1 =< (2, 4, 6, 8, 10) >$. Notice that this element that fixes 1 is the generator x. Then we write $(t, N^i) = (t, x)$ to denote that N commutes with x our point stabiliser. Thus, our permutation progenitor of $2^{*10} : (5^2 : 2)$ is given as follows:

$$< x, y, t | x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y, t^2, (t, x) >$$

In the continuing chapters we will apply this procedure to find permutation progenitors for the progenitors of the form $m^{*n} : N$. In the next example we will show how to facor the above progenitor by first order relations.

Chapter 1

Definitions, Theorems, and Lemmas

1.1 Preliminaries

1.1.1 Definitions

Definition 1.1. *[Rot95] A* **group** *G $(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*
- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.*

Definition 1.2. *[Rot95] Let G be a set. A (binary)* **operation** *on G is a function that assigns each ordered pair of elements og G an element on G.*

Definition 1.3. *[Rot95] For group G, a* **subgroup** *S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$.*

Definition 1.4. *[Rot95] Let H be a subgroup of group G. H is a* **proper** *subgroup of G if $H \neq G$. We denote this as $H < G$.*

Definition 1.5. *[Rot95] A* **symmetric group***, S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions.*

Definition 1.6. *[Rot95] If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$.*

Definition 1.7. *[Rot95] A* **semigroup** $(G, *)$ *is a nonempty set G equipped with an associative operation *.*

Definition 1.8. *[Rot95] If $x \in X$ and $\phi \in S_X$, then ϕ* **fixes** *x if $\phi(x) = x$ and ϕ* **moves** *x if $\phi(x) \neq x$.*

Definition 1.9. *[Rot95] For permutations $\alpha, \beta \in S_X$, α and β are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\textit{if } \alpha(x) \neq x, \textit{ then } \beta(a) = a \textit{ and if } \alpha(y) = y, \textit{ then } \beta(y) \neq y.$$

Definition 1.10. *[Rot95] A permutation which interchanges a pair of elements is a* **transposition**.

Definition 1.11. *[Rot95] In group G, if a,b $\in G$, a and b* **commute** *if $a * b = b * a$.*

Definition 1.12. *[Rot95] A group G is* **abelian** *if every pair of elements in G commutes with one another.*

Definition 1.13. *[Rot95] Let X be a set and Δ be a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G.*

Definition 1.14. *[Cur07] Let G be a group and $T = t_1, t_2, ..., t_n$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, we define the* **progenitor** *to be the semi-direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_n.*

Definition 1.15. *[Rot95] Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by $N_G(H) = \{a \in G | aHa^{-1} = H\}$*

Definition 1.16. *[Rot95] Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \textit{ for all } h \in H\}.$$

Definition 1.17. *[Rot95] Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we say G is* **elementary abelian**.

Definition 1.18. *[Rot95] Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \to H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all a,b $\in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to H, $G \cong H$, if there exists an isomorphism $f : G \to H$.*

Definition 1.19. *[Rot95] Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *.*

Definition 1.20. *[Rot95] Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X.*

Definition 1.21. *[Rot95] Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$*

Definition 1.22. *[Rot95] The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$.*

Definition 1.23. *[Rot95] A* **general linear group**, *$GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$.*

Definition 1.24. *[Rot95] A* **special linear group**, *$SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$.*

Definition 1.25. *[Rot95] A* **projective special linear group**, *$PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}.$$

Definition 1.26. *[Rot95] A* **projective general linear group**, *$PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}.$$

Definition 1.27. *[Led87]*(**Monomial Character**) *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H.*

Definition 1.28. *[Led87]*(**Character**) *Let $A(x) = (A_{ij}(x))$ be a matrix representation of G of degree m. We consider the character polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of A(x), abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the **character** *of $A(x)$.*

Definition 1.29. *[Led87] The sun of squares of the degrees of the s=distinct irreducible characters of G is equal to $|G|$. The* **degree of a character** *χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character.*

Definition 1.30. *[Led87]* (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x)$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. Then the lifting process preserves irreducibility.*

Definition 1.31. *[Led87]* (**Induced Character**) *Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then*

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G.

Definition 1.32. *[Led87] Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let C_α, $\alpha = 1, 2, ..., m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, 3, ..., m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \alpha = 1, 2, 3, ..., m.$$

Definition 1.33. *[Rot95] Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$.*

Definition 1.34. *[Rot95] Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \lhd G$ to denote K as being normal in G.*

Definition 1.35. *[Rot95] Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St.*

Definition 1.36. *[Rot95] Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G.*

Definition 1.37. *[Rot95] Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK|k \in K\}$ is determined by $g \in G$.*

Definition 1.38. *[Rot95] Let N be a group. The* **point stabiliser** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s.}$$

Definition 1.39. *[Rot95] Let N be a group. The* **coset stabiliser** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s.}$$

Definition 1.40. *[Rot95] Let G be a group. The* **center** *of G, $Z(G)$, is the set of all elements in G that commute with all elements of G.*

1.1.2 Theorems

Theorem 1.41. *[Led87] The number of irreducible character of G is equal to the number of conjugacy classes of G*

Theorem 1.42. *[Rot95] Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$.*

Theorem 1.43. *[Rot95] Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$.*

Theorem 1.44. *[Rot95] Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles.*

Theorem 1.45. *[Rot95] Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*
- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*
- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$.*

Theorem 1.46. *[Rot95] The intersection of any family of subgroups of a group G is again a subgroup of G.*

Theorem 1.47. *[Rot95] If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint.*

Theorem 1.48. *[Rot95] If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$.*

Theorem 1.49. *[Rot95] If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|.$$

Theorem 1.50. *[Rot95] If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$.*

Theorem 1.51. *[Rot95] The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$.*

Theorem 1.52. *[Rot95] Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$.*

Theorem 1.53. *[Rot95] If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite.

Theorem 1.54. *[Rot95] If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$.*

Theorem 1.55. *[Rot95] Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n.*

Theorem 1.56. *[Rot95] If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker \rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H.*

Theorem 1.57. *[Rot95] If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set.*

Theorem 1.58. *[Rot95] Every two composition series of a group G are equivalent.*
We will refer to this Theorem as the **Jordan-Hölder Theorem**.

Theorem 1.59. *[Rot95] Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$.*

1.1.3 Lemmas

Lemma 1.60. *[Rot95] Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*

- *If $H \lhd G$, then the subsets Hx are blocks of X.*

Lemma 1.61. *(Curtis Lemma)*

[Rot95] $N \cap < t_i, t_j > \leq C_N(N^{ij})$ where N_{ij} denotes the stabilizer in N of the two points i and j.

Note:
If the $|t_i| = 2$, $|t_j| = 2$, and $|t_i t_j|$=n, then $< t_i, t_j >= D_{2n}$. The Dihedral group of order $2n$. We also know the center of D_{2n}:

$$Center(D_{2n}) = \begin{cases} 1, & \text{if n is odd} \\ < (t_i, t_j)^{\frac{n}{2}} >, & \text{if n is even.} \end{cases}$$

Lemma 1.62. *[Rot95]*

(i) If g belongs to N and $i^g = i$ and $j^g = j$ then we should factor the progenitor by the relation $(t_i t_j)^k = g$ for any positive integer k. .

(ii) if g belongs to N and $i^g = j$ and $j^g = i$ then we should factor the progenitor by the relation $(g t_i)^k = 1$ for any odd positive integer k.
In other words we have:

$$= \begin{cases} (t_i t_j)^k = g & \textit{where } k \textit{ is even and fixes 1 and 2} \\ (g t_i)^k = 1 & \textit{where } k \textit{ is odd and } g \textit{ sends 1 to 2} \end{cases}$$

Chapter 2

Methods on Finding Progenitors

2.1 Permutation Progenitors

2.2 Factoring $m^* : N$ by the First Order Relations

In order to factor the progenitor, $m^{*n} : N$, by all the first order relations, first, we compute the conjugacy classes of our group N. Then we compute the centralisers of representatives of each non-identity class. Lastly, we determine the orbits for each representative. The detailed work for factoring a progenitor by the first order relations is shown in this section.

EXAMPLE In order to factor $2^{*10} : (5^2 : 2)$ by first order relations we must follow a series of steps. We will demonstrate the procedure here; we begin by using the following codes in MAGMA.

```
S:=Sym(10);
xx:=S!(2, 4, 6, 8, 10);
yy:=S!(1, 6)(2, 7)(3, 8)(4, 9)(5, 10);
N:=sub<S|xx,yy>;
#N;
/*50*/
FPGroup(N);
 Finitely presented group on 2 generators
Relations
    Relations
    .1^5 = Id
    .2^2 = Id
```

```
.1^-1 * .2 * .1^-1 * .2 * .1 * .2 * .1 * .2 = Id
```

We must convert this in terms of x and y to get the presentation of $2^{*10} : (5^2 : 2)$.

$$< x, y | x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y >$$

We now have to find the conjugacy classes of our control group N. We can find the conjugacy classes of N by taking any element k of N and conjugating it by all elements of N. We do this for all elements, but since we sometimes have to compute these conjugacy classes for large groups we use the assistance of $MAGMA$. We do this in MAGMA as follows:

```
C:=Classes(N);
#C;
20
for i in [2..20] do
i, Orbits(Centraliser(N,C[i][3]));
end for;
for j in [2..20] do
C[j][3];
for i in [1..50] do if ArrayP[i] eq C[j][3] then Sch[i];
end if; end for; end for;
```

This information is given in the following table.

Table 1.1: Conjugacy Classes of $N = 5^2 : 2$

Class	Representative of the class	# of elements in the class	Orbits
C_1	Identity	1	{1},{2},{3},{4},{5},{6},...,{10}
C_2	(y)=(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)	5	{1, 6, 10, 5, 9, 4, 8, 3, 7, 2 }
C_3	$(xy)^2$=(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)	1	{1, 6, 8, 10, 3, 2, 5, 4, 7, 9 }
C_4	$(x^2y)^2$=(1, 5, 9, 3, 7)(2, 6, 10, 4, 8)	1	{1, 6, 8, 10, 3, 2, 5, 4, 7, 9 }
C_5	$(yx^{-2})^2$ =(1, 7, 3, 9, 5)(2, 8, 4, 10, 6)	1	{1, 6, 8, 10, 3, 2, 5, 4, 7, 9}
C_6	$(yx^{-1})^2$=(1, 9, 7, 5, 3)(2, 10, 8, 6, 4)	1	{1, 6, 8, 10, 3, 2, 5, 4, 7, 9 }
C_7	(yxy)=(1, 3, 5, 7, 9)	2	{1, 3, 5, 7, 9 },{2, 10, 8, 6, 4 }
C_8	(yx^2y)=(1, 5, 9, 3, 7)	2	{ 1, 5, 9, 3, 7 }, {2, 10, 8, 6, 4}
C_9	$(yx^{-2}y)$=(1, 7, 3, 9, 5)	2	{1, 7, 3, 9, 5}, {2, 10, 8, 6, 4}
C_{10}	$(yx^{-1}y)$=(1, 9, 7, 5, 3)	2	{ 1, 9, 7, 5, 3 }, {2, 10, 8, 6, 4 }

To find all first order relations we take a representative from each class $C_2, ..$ C_{20} (since C_1 is the identity class) and right multiply it by a representative from each orbit until we complete all twenty classes. Let us illustrate an example of obtaining the first order relation for C_2. From table (1.1) we see the representative for this class is y and the representative of the first orbit is {1, 6, 10, 5, 9, 4, 8, 3, 7, 2 }. We will right multiply by 6 in this case, but any representative would have worked. We have $y * t$ where $t \sim t_1$. Now we find a permutation in terms of x and y that takes t_1 to get t_6 and that is y. So we get the first relation to be $(yt)^a$. Following the same process we are able to obtain all possible first order relations which are:

Table 1.1: Conjugacy Classes of $N = 5^2 : 2$

Class	Representative of the class	# of elements in the class	Orbits
C_{11}	$(xyx^{-1}y)$=(1, 9, 7, 5, 3)(2, 4, 6, 8, 10)	2	{1, 9, 7, 5, 3},{2, 4, 6, 8, 10}
C_{12}	$(x^2yx^{-2}y)$=(1, 7, 3, 9, 5)(2, 6, 10, 4, 8)	2	{1, 7, 3, 9, 5},{2, 6, 10, 4, 8 }
C_{13}	$(xyx^2y)^2$=(1, 5, 9, 3, 7)(2, 4, 6, 8, 10)	2	{1, 5, 9, 3, 7},{2, 4, 6, 8, 10}
C_{14}	$(x^2yx^{-1}y)^2$=(1, 9, 7, 5, 3)(2, 6, 10, 4, 8)	2	{1, 9, 7, 5, 3}, {2, 6, 10, 4, 8}
C_{15}	$(yxyx^{-2})^2$ =(1, 3, 5, 7, 9)(2, 8, 4, 10, 6)	2	{1, 3, 5, 7, 9}, {2, 8, 4, 10, 6}
C_{16}	$(yx^{-2}yx^{-1})^2$=(1, 7, 3, 9, 5)(2, 10, 8, 6, 4)	2	{1, 7, 3, 9, 5}, {2, 10, 8, 6, 4}
C_{17}	(yx)=(1, 8, 3, 10, 5, 2, 7, 4, 9, 6)	5	{1, 8, 3, 10, 5, 2, 7, 4, 9, 6}
C_{18}	(xyx^2)=(1, 10, 7, 6, 3, 2, 9, 8, 5, 4)	5	{ 1, 10, 7, 6, 3, 2, 9, 8, 5, 4},
C_{19}	$(x^{-2}yx^{-1})$=(1, 4, 5, 8, 9, 2, 3, 6, 7, 10)	5	{1, 4, 5, 8, 9, 2, 3, 6, 7, 10}
C_{20}	$(x^{-1}y)$=(1, 6, 9, 4, 7, 2, 5, 10, 3, 8)	5	{1, 6, 9, 4, 7, 2, 5, 10, 3, 8}

$((x*y)^2*t)^b, ((x^2*y)^2*t)^c, ((y*x^{-2})^2*t)^d, ((y*x^{-1})^2*t)^e, (y*x*y*t)^f, (y*x^2*y*t)^g, (y*x^{-2}*y*t)^h, (y*x^{-1}*y*t)^i, (x*y*x^{-1}*y*t)^j, (x^2*y*x^{-2}*y*t)^k, (x*y*x^2*y*t)^l, (x^2*y*x^{-1}*y*t)^u, (y*x*y*x^{-2}*t)^n, (y*x^{-2}*y*x^{-1}*t)^o, (y*x*t)^p, (x*y*x^2*t)^q, (x^{-2}*y*x^{-1}*t)^r, (x^{-1}*y*t)^s$.

To obtain finite homomorphic images, we use $MAGMA$ to run $a, ..., s$ for numerical values we choose, for example up to 10. In other words, the highest value for $a, ..., s$ we will see is 10. An example of a finite group given through $MAGMA$ with this progenitor is as follows:

```
a:=0;b:=0;c:=0;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;
```

```
m:=0;n:=0;o:=0;p:=0; q:=0;r:=0;s:=4;

G<x,y,t>:=Group<x,y,t| x^5,y^2,x^-1*y*x^-1*y*x*y*x*y,
(t,x), t^2,
(y*t)^a,
((x * y)^2*t)^b,
((x^2 * y)^2*t)^c,
((y * x^-2)^2*t)^d,
((y * x^-1)^2*t)^e,
(y * x * y*t)^f,
(y * x^2 * y*t)^g,
(y * x^-2 * y*t)^h,
(y * x^-1 * y*t)^i,
(x * y * x^-1 * y*t)^j,
(x^2 * y * x^-2 * y*t)^k,
(x * y * x^2 * y*t)^l,
(x^2 * y * x^-1 * y*t)^m,
(y * x * y * x^-2*t)^n,
(y * x^-2 * y * x^-1*t)^o,
(y * x*t)^p,
(x * y * x^2*t)^q,
(x^-2 * y * x^-1*t)^r,
(x^-1 * y*t)^s
>;
G;
1000
f, G1, k:=CosetAction(G,sub<G|x,y>);
k;
1

CompositionFactors(G1);
 G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    1
```

This group of order 1000 is isomorphic to $5^3 : D_8$. Likewise we find many more finite homomorphic images. A complete list can be found under Chapter 8.

We will now demonstrate how to write progenitors using other methods and factoring these progenitors by first order relations. Particularly, monomial progenitors and the Curtis Lemma progenitors.

2.3 Curtis Lemma Progenitors

In order to find a finite homomorphic image we take a progenitor of the form $m^{*n} : N$ and factor by relations. However, finding simple groups factor by relators can be difficult since we want to produce interesting groups. To help find such images, Robert Curtis discovered a lemma where the elements of the control group of N can be written in terms of symmetric generators. In this section we are going to use the Curtis Lemma to generate symmetric presentations for progenitors to find homomorphic images. We find the stabilizer of two elements say 1 and 2 and we determine the centralizer of the two elements. Note: The relations found with the Curtis Lemma are considered as additional relations.

Example Factoring $2^{*36} : (3^2 : 2^3)$ by Curtis Lemma Relations:
A presentation for G is given by: $< v^2, w^4, x^2, y^3, z^3, w^{-2} * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x * z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1} >$ As before, we let $t \sim t_1$ where t is of order 2, and find the permutation that stabilises the subgroup $< t_1 >$ and write it this permutation in terms of v, w, x, y, z.

```
S:=Sym(36);
vv:=S!(1, 5)(2, 6)(3, 7)(4, 8)(9, 17)(10, 19)(11, 18)(12, 20)
(13, 16)(21, 29)(22, 30)(23, 31)(24, 32)(25, 28)(33, 36);
ww:=S!(1, 6, 9, 19)(2, 5, 10, 17)(3, 8, 11, 20)(4, 7, 12, 18)
(13, 14, 16,15)(21, 34, 29, 27)(22, 36, 31, 28)(23, 33, 30, 25)
(24, 35, 32, 26);
xx:=S!(1, 9)(2, 10)(3, 11)(4, 12)(5, 17)(6, 19)(7, 18)(8, 20)
(13, 16)(14,15)(21, 29)(22, 31)(23, 30)(24, 32)(25, 33)
(26, 35)(27, 34)(28, 36);
yy:=S!(1, 13, 12)(2, 14, 11)(3, 15, 10)(4, 16, 9)(5, 24, 25)
(6, 22, 26)(7, 23, 27)(8, 21, 28)(17, 33, 32)(18, 34, 30)
(19, 35, 31)(20, 36, 29);
zz:=S!(1, 17, 21)(2, 18, 22)(3, 19, 23)(4, 20, 24)(5, 9, 29)
(6, 11, 30)(7, 10, 31)(8, 12, 32)(13, 33, 28)(14, 34, 26)
```

```
(15, 35, 27)(16, 36, 25);
N:=sub<S|vv,ww,xx,yy,zz>;
#N;
NN<v,w,x,y,z>:=Group<v,w,x,y,z|v^2,w^4,x^2,y^3,z^3,
w^-2*x,(w^-1*v)^2,(x*y^-1)^2,v*z^-1*v*z,(x*z^-1)^2,
(y,z),w*y^-1*w^-1*y*z^-1>;
N:=sub<S|vv,ww,xx,yy,zz>;
N;
N1:=Stabiliser(N,1);
(2, 3)(6, 7)(9, 29)(10, 30)(11, 31)(12, 32)(13, 28)(14, 27)
(15, 26)(16, 25)(17, 21)(18, 23)(19, 22)(20, 24)(34, 35)  =
v * x * z^-1
```

Therefore we have, $N^1 = \{(2,3)(6,7)(9,29)(10,30)(11,31)(12,32)(13,28)(14,27)(15,26)(16,25)(17,21)(18,23)(19,22)(20,24)(34,35)\} = \{v * x * z^-1\}$. Since this permutation stabilises one, t_1 will commute with this relation. We now add this to our presentation of G: $< v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^{-2} * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x * z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1}, t^2, (t, v * x * z^{-1}) >$.

Now, according to the Curtis Lemma, we need to find the centraliser of the two point stabiliser. Let the point stabiliser of the two elements 1 and 2 be $N^{t_1 t_2}$, which we will denote as $N^{(12)}$. Keep in mind we can always stabilise any two points, such as one and three for example.

We compute the centraliser of the stabiliser of $N^{(12)}$ as follows:

```
N12:=Stabiliser(N,[1,2]);
Cent:=Centraliser(N,N12);
```

If we find that no permutation centralises $N^{(12)}$, then we look for a permutation that normalises the point stabiliser, which happens to be our case. In this case we find that $(1,2)(3,4)(5,19)(6,17)(7,20)(8,18)(9,10)(11,12)(13,14)(15,16)(21,34)(22,33)(23,36)(24,35)(25,31)(26,32)(27,29)(28,30)$ normalises $N^{(12)}$. Again, we now add this relation to our progenitor, which we do after we convert to terms of v, w, x, y, z. This relation is: $v * w^-1$ and will be labeled as the letter m. This relation is the relation that is required to produce homomorphic images according to Curtis. If this special relation produces even numbers when ran in $MAGMA$, this progenitor is promising. If we obtain odd numbers, we will not produce homomorphic images. Our progenitor now looks like this: $< v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^{-2} * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x *$

$z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1}, t^2, (t, v * x * z^{-1}), (v * w^-1 * t)^m >$. We complete our progenitor by adding necessary relations to our progenitor as we normally do by finding the classes of our control group and following the process as discussed in section 2.2. We have: $< v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^{-2} * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x * z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1}, t^2, (t, v * x * z^{-1}), (v * w^-1 * t)^m, (v * y^{-1} * z^{-1} * t)^a, (v * w * t)^b, (x * t)^c, (y * t)^d, (z * t)^e, (w * t)^f, (y * v * t)^g, (v * w * z * t)^h >$;

With this progenitor we obtain interesting groups such as $4^{\cdot} : S_4$. A complete list of homomorphic images for this progenitor can be found in chapter 8.

2.4 Monomial Progenitor $11^{*2} :_m D_{10}$

We will demonstrate how to construct a monomial presentation of
$11^{*2} :_m D_{10}$. A presentation for D_{10} is given by the following:

$$< x, y | y^2, (x^-1 * y)^2, x^5 >.$$

To construct a monomial presentation we first must induce an irreducible linear character from a subgroup H of G. To obtain an irreducible character we choose a subgroup H of G with an index equal to the degree of an irreducible character of G. Consider the character table of $G = D_{10}$ in Table 1 and note G has characters $\chi_{.1}, \chi_{.2}, ..., \chi_{.4}$. We proceed using $\chi_{.4}$ which has a degree of 2 and look for a subgroup of order 5 so that $\frac{|G|}{|H|} = 2$. Thus we get the following index:

$$[G : H] = [D_{10} : C_5] = 2$$

If a matrix representation exists it will be represented by 2×2 matrices, since the index of our two groups is 2.

Verifying the Induction

We produce a character table for C_5 in Table 2. We will verify the induction $\chi_{.2}$ of C_5 to $\chi_{.3}$ of D_{10} by considering the irreducible characters ϕ (of H) and ϕ^G (of G). $G = D_{10}$ is generated by xx and yy where $xx = (1, 3, 5, 7, 9)(2, 4, 6, 8, 10)$ and $yy = (1, 4)(2, 3)(5, 10)(6, 9)(7, 8)$. Using our definition of induction along with the following equivalencies: $1 = 1$, $Z_1\#1 = 4$, $Z_1\#2 = 5$, $Z_1\#3 = 9$, $Z_1\#4 = 3$, we can reproduce ϕ^G using ϕ (of H).

$\phi^G_\alpha = \frac{n}{h_\alpha}\sum_{w\in H\cap C_\alpha}\phi(w)$, where $n = \frac{|G|}{|H|} = \frac{10}{5} = 2$.

$\phi^G_1 = \frac{2}{1}\sum_{w\in H\cap C_1}\phi(w)$

which implies $\phi^G_1 = \frac{2}{1}(\phi(1)) = 2(1) = 10$.

$\phi^G_2 = \frac{2}{5}\sum_{w\in H\cap C_2}\phi(w)$

$\phi^G_2 = \frac{2}{5}\sum_{w\in H\cap C_2}\phi$

which implies $\phi^G_2 = \frac{2}{5}(\phi(0)) = \frac{2}{5}(0) = 0$.

$\phi^G_3 = \frac{2}{2}\sum_{w\in H\cap C_3}\phi(w)$

$\phi^G_3 = 1\sum_{w\in H\cap C_3}(\phi(1,3,5,7,9)(2,4,6,8,10) + \phi(1,9,7,5,3)(2,10,8,6,4))$

which implies $\phi^G_3 = 1(-3-4-1) = 1(-7) = -7 \equiv 4 \pmod{11}$.

$\phi^G_4 = \frac{2}{2}\sum_{w\in H\cap C_4}\phi(w)$

$\phi^G_4 = \frac{2}{2}\sum_{w\in H\cap C_4}(\phi(1,5,9,3,7)(2,6,10,4,8) + \phi(1,7,3,9,5)(2,8,4,10,6))$

which implies $\phi^G_4 = 1(-5-9-1) = 1(-15) = -15 \equiv 7 \pmod{11}$.

Therefore, $\phi\uparrow^G_H =\ 2, 0, 4, 7$ and we have verified that $\chi_{.2}$ of C_5 induces $\chi_{.3}$ of D_{10}.

Table 1.2: Character Table of G

χ	C_1	C_2	C_3	C_4
$\chi.1$	1	1	1	1
$\chi.2$	1	-1	1	1
$\chi.3$	2	0	Z_1	$Z_1\#2$
$\chi.4$	2	0	$Z_1\#2$	Z_1

# denotes algebraic conjugation.
Z_1 is the primitive fifth root of unity.

Table 1.3: Character Table of H

χ	D_1	D_2	D_3	D_4	D_5
$\chi.1$	1	1	1	1	1
$\chi.2$	1	Z_1	$Z_1\#2$	$Z_1\#3$	$Z_1\#4$
$\chi.3$	1	$Z_1\#2$	$Z_1\#4$	Z_1	$Z_1\#3$
$\chi.4$	1	$Z_1\#3$	Z_1	$Z_1\#4$	$Z_1\#2$
$\chi.5$	1	$Z_1\#4$	$Z_1\#3$	$Z_1\#2$	Z_1

# denotes algebraic conjugation.
Z_1 is the primitive fifth root of unity.

Table 1.4: $\chi_{.3}$ of G

ϕ^G	Class	Size	Class Representative
2	C_1	1	Id(G)
0	C_2	5	(1, 4)(2, 3)(5, 10)(6, 9)(7, 8)
Z_1	C_3	2	(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)
$Z_1\#2$	C_4	2	(1, 5, 9, 3, 7)(2, 6, 10, 4, 8)

Table 1.5: $\chi_{.2}$ of H

ϕ	Class	Size	Class Representative
1	D_1	1	Id(H)
$Z_1\#1$	D_2	1	(1, 7, 3, 9, 5)(2, 8, 4, 10, 6)
$Z_1\#2$	D_3	1	(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)
$Z_1\#3$	D_4	1	(1, 9, 7, 5, 3)(2, 10, 8, 6, 4)
$Z_1\#4$	D_5	1	(1, 5, 9, 3, 7)(2, 6, 10, 4, 8)

Through induction, we now verify the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix},$$

$$A(yy) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Verifying the Monomial Representation

$G =< e, (1,4)(2,3)(5,10)(6,9)(7,8), (1,3,5,7,9)(2,4,6,8,10) >$ and
$H =< e, (1,7,3,9,5)(2,8,4,10,6) >$. Since H is a subgroup of G whose index is equal to the degree of G, we have that: $G = H \cup Ht_1 \cup Ht_2$, where the $t_i's$ are transversals of G acting on H. The transversals of G are labeled as follows: $t_1 = e$, and $t_2 = (1,4)(2,3)(5,10)(6,9)(7,8)$.

We will now use the following formula to verify the matrices:

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) \end{bmatrix}$$

$a_{11} : \phi(t_1xt_1^{-1}) = \phi(x^{t_1}) = \phi(x^e) = \phi(x) = \phi((1,3,5,7,9)(2,4,6,8,10)) = 4$

$a_{12} : \phi(t_1xt_2^{-1}) = \phi(ext_2^{-1}) =$
$\phi((1,3,5,7,9)(2,4,6,8,10) * (1,4)(2,3)(5,10)(6,9)(7,8)) =$
$\phi((1,2)(3,10)(4,9)(5,8)(6,7)) = 0$

$a_{21} : \phi(t_2xt_1^{-1}) = \phi(t_2xe) = \phi((1,4)(2,3)(5,10)(6,9)(7,8) *$
$(1,3,5,7,9)(2,4,6,8,10)) = \phi(1,6)(2,5)(3,4)(7,10)(8,9) = 0$

$a_{22} : \phi(t_2xt_2^{-1}) = \phi(x^{t_2}) = \phi((1,3,5,7,9)(2,4,6,8,10)^{(1,4)(2,3)(5,10)(6,9)(7,8)})$
$= \phi((3,1,9,7,5)(4,2,10,8,6)) = 9$

Likewise for $A(yy)$:

$$A(yy) = \begin{bmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) \end{bmatrix}$$

$a_{11} : \phi(t_1yt_1^{-1}) = \phi(y^{t_1}) = \phi(y^e) = \phi(y) = \phi((1,4)(2,3)(5,10)(6,9)(7,8)) = 0$

$a_{12} : \phi(t_1yt_2^{-1}) =$
$\phi(eyt_2^{-1}) = \phi(e * (1,4)(2,3)(5,10)(6,9)(7,8) * (1,4)(2,3)(5,10)(6,9)(7,8))$
$= \phi((1,4)(2,3)(5,10)(6,9)(7,8)) = 1$

$a_{21} : \phi(t_2yt_1^{-1}) = \phi((1,4)(2,3)(5,10)(6,9)(7,8) * y * e) =$
$\phi((1,4)(2,3)(5,10)(6,9)(7,8) * (1,4)(2,3)(5,10)(6,9)(7,8)) = 1$

$a_{22} : \phi(t_2yt_2^{-1}) = \phi(y^{t_2}) = \phi((1,4)(2,3)(5,10)(6,9)(7,8)^{(1,4)(2,3)(5,10)(6,9)(7,8)})$
$= \phi(y) = 0$

Each ϕ of H corresponded with a conjugacy class of either H or G. If the element belonged in a conjugacy class from H (seen in table 5.4) we wrote the value of ϕ for that class, otherwise, we obtained 0. Therefore the matrix representation of $A(xx)$ and $A(yy)$ respectively are as follows:

$$A(xx) = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$
$$A(yy) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To prove the faithful representation of $D_{10} = < x, y | y^2, (x^-1 * y)^2, x^5 >$, where $|D_{10}| = 10$, we simply check the order of each matrix representation: $|A(x)| = 5$, and $|A(y)| = 2$, then $|A(x)||A(y)| = 10$. which is the order of our index. We con now conclude that $G = < x, y > \cong < A(x), A(y) >$. Now, to finalize the process, all we need is to construct a permutation representation to build a monomial progenitor in hopes of obtaining homomorphic images of interesting groups.

Constructing a Permutation Representation

We worked in Z_{11} on matrices of degree 2×2, which implies we are working with 2 $t_{i's}$ of order 11. Since we have a semi-direct product in our progenitor, the elements of D_{10} will act as an automorphism on $< t_1 > * < t_2 >$. So, $a_{i,j} = a \iff t_i \to t_j^a$, since this is an automorphism. Therefore, for our $A(xx)$ we have:

$$A(xx) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where t_1 corresponds to column 1 and t_2 to column 2 We will label the entries of the matrix as follows: $a_{12} = a, a_{22} = b, a_{21} = c$, and $a_{22} = d$. Then,

$$a_{11} = a \iff t_1 \to t_1^a \qquad\qquad a_{12} = a \iff t_1 \to t_2^b$$

$$a_{21} = a \iff t_1 \to t_1^c \qquad\qquad a_{22} = a \iff t_1 \to t_2^d$$

We can now construct a table with our $t_{i's}$ with nonzero entries to obtain the permutation representation. Keep in mind we are working $\mathbb{Z}_{11}$. We will have a total of 20 $t_{i's}$ for $A(xx)$.

For $\mathbf{a_{11}}$

$$t_1^1 \to t_1^5$$
$$t_1^2 \to (t_1^5)^2 = t_1^{10}$$
$$t_1^3 \to (t_1^5)^3 = t_1^{15} = t_1^4$$
$$t_1^4 \to (t_1^5)^4 = t_1^{20} = t_1^9$$
$$t_1^5 \to (t_1^5)^5 = t_1^{25} = t_1^3$$
$$t_1^6 \to (t_1^5)^6 = t_1^{30} = t_1^8$$
$$t_1^7 \to (t_1^5)^7 = t_1^{35} = t_1^3$$
$$t_1^8 \to (t_1^5)^8 = t_1^{40} = t_1^7$$
$$t_1^9 \to (t_1^5)^9 = t_1^{45} = t_1^1$$
$$t_1^{10} \to (t_1^5)^{10} = t_1^{50} = t_1^6$$

Likewise, for

For $\mathbf{a_{22}}$

$$t_2^1 \to t_2^9$$
$$t_2^2 \to (t_2^9)^2 = t_2^{18} = t_2^7$$
$$t_2^3 \to (t_2^9)^3 = t_2^{27} = t_2^5$$
$$t_2^4 \to (t_2^9)^4 = t_2^{36} = t_2^3$$
$$t_2^5 \to (t_2^9)^5 = t_2^{45} = t_2^1$$
$$t_2^6 \to (t_2^9)^6 = t_2^{54} = t_2^{10}$$
$$t_2^7 \to (t_2^9)^7 = t_2^{63} = t_2^8$$
$$t_2^8 \to (t_2^9)^8 = t_2^{72} = t_2^6$$
$$t_2^9 \to (t_2^9)^9 = t_2^{81} = t_2^4$$
$$t_2^{10} \to (t_2^9)^{10} = t_2^{90} = t_2^2$$

Now we are ready to find our permutations from the following table:

Therefore, our permutation representation is the following:

$A(xx) = <(1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)(4,14,16,12,20)>$.

Table 1.6: Permutation Table of A(xx)

#	t_i	Mapping to t_j^a	Element of Permutation
1	t_1	t_1^5	9
2	t_2	t_2^9	18
3	t_1^2	t_2^9	19
4	t_2^2	t_2^9	14
5	t_1^3	t_2^9	7
6	t_2^3	t_2^9	10
7	t_1^4	t_2^9	17
8	t_2^4	t_2^9	6
9	t_1^5	t_2^9	5
10	t_2^5	t_2^9	2
11	t_1^6	t_2^9	15
12	t_2^6	t_2^9	20
13	t_1^7	t_2^9	3
14	t_2^7	t_2^9	16
15	t_1^8	t_2^9	13
16	t_2^8	t_2^9	12
17	t_1^9	t_2^9	1
18	t_2^9	t_2^9	8
19	t_1^{10}	t_2^9	11
20	t_2^{10}	t_2^9	4

For our $A(yy)$ we would have:

$$t_1 \to t_2^1$$

$$t_2 \to t_1$$

Thus, we would apply the same process and our permutation representation would be: $A(yy) = <(1,2)(3,4)(6,5)(8,7)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)>$.

This demonstrates that our presentation is correct since we have $|A(xx) * A(yy)| = 10 = |G|$.

The Monomial Progenitor:

To build the monomial progenitor, we simply need to compute the stabiliser (N, t_1, t_1^2)
We are looking for what element in N fix our $t_{1's}$. The work is as follows:

```
S:=Sym(20);
xx:=S!(1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)
(4,14,16,12,20);
yy:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)
(15,16)(17,18)(19,20);
N:=sub<S|xx,yy>;

Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,17,19});
Stabiliser(N,{1,3,5,7,9,11,13,15,17,19});
A:=Normaliser!(1, 17, 7, 5, 9)(2, 10, 6, 8, 18)
(3, 13, 15, 11, 19)(4, 20, 12, 16,14);
Normaliser eq sub<N|A>;
for i in [1..#N] do if ArrayP[i] eq A then Sch[i];
 end if; end for;
```

```
/*x^-1*/
FPGroup(Normaliser);
/*  $.1^-5 = Id($)*/

Normaliser eq sub<N|xx,xx^-5>;

This original progenitor for G is the following:

G<x,y>:=Group<x,y|y^2,(x^-1*y)^2,x^5>;

New progenitor for monomial presentation of G:
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-5,t^11,
(t,x^-2),t^x=t^5,(t,t^y)>;
#G;
/*10*/
```

To verify that our progenitor is correct, we use the Grindstaff Lemma as follows:

```
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-5,t^11,(t,x^-2),(t,t^y)>;
#G;
1210
 Index(G,sub<G|x,y>);
121
```

This proves we have the right progenitor since the index of G is 121 which is the index of the group we are working with 11^{*2}. Also, our order of $G = 10$ and the index of $(G)| \times |G| = 1210$. Thus, we have successfully constructed a monomial progenitor of D_{10}. We then factor our progenitor by the appropriate relations in hopes of obtaining homomorphic images.

2.4.1 Factoring $11^{*2} :_m D_{10}$ by First Order Relations

To factor our monomial progenitor by first order relations, we begin by using our new permutation representation obatined from the process mentioned above.

```
S:=Sym(20);
xx:=S! (1, 9, 17, 5, 13)(2, 14, 6, 18, 10)(3, 11, 19, 7, 15)
(4, 16, 8, 20, 12);
yy:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
```

```
(17,18)(19,20);
N:=sub<S|xx,yy>;
```

As in the process of any other progenitor, we run the Schreier System to convert our permutations into words.

```
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
```

In a permutation progenitor we need only to fix or stabilise one element, usually t_1, but in a monomial progenitor, we fix the set of t_i, in other words we can fix any of the following sets: $< t_1 >, < t_2 >, ... < t_{10} >$. Recall that $x \sim (1,9,17,5,13)(2,14,6,18,10)(3,11,19,7,15)(4,16,8,20,12)$, and
$y \sim (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)$. We let $t \sim t_1$, and we will fix $< t_1 >= \{1, 9, 17, 5, 13\}$.

```
Normaliser:=Stabiliser(N,{1, 9, 17, 5, 13});
/*(1, 5, 9, 13, 17)(2, 18, 14, 10, 6)(3, 7, 11, 15, 19)
(4, 20, 16, 12, 8)*/
Stabiliser(N, {1, 9, 17, 5, 13});
A:=Normaliser! (1, 5, 9, 13, 17)(2, 18, 14, 10, 6
)(3, 7, 11, 15, 19)(4, 20, 16, 12, 8);
Normaliser eq sub<N|A>;
true
```

Since we need to add the stabiliser of our group $< t_1 >$ in our presentation, we must convert it into words.

```
for i in [1..#N] do if ArrayP[i] eq A then Sch[i]; end if;
 end for;
```

```
/*x^-2
t^(x^-2);
```

Therefore we will add this to our presentation. Note we must also let $MAGMA$ know how we labeled our $t_{i's}$. Notice that if we conjugate $t^{(x^{-2})} = t^{(1,5,9,13,17)(2,18,14,10,6)(3,7,11,15,19)(4,20,16,12,8)} = 5$. We have:

```
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^5,t^11,t^(x^-2)=
t^3,(t,t^(y))>;
#G;
/*1210*/
Index(G,sub<G|x,y>);
/*121*/
```

This confirms that our progenitor is correct. Now we find the orbits of each conjugacy class of N as with our other progenitors. The following gives us our first order relations:

```
Classes(N);
/*Conjugacy Classes of group N
----------------------------
[1]     Order 1       Length 1
        Rep Id(N)

[2]     Order 2       Length 5
        Rep (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)

[3]     Order 5       Length 2
        Rep (1, 9, 17, 5, 13)(2, 14, 6, 18,
 10)(3, 11, 19, 7, 15)(4, 16, 8, 20,12)

[4]     Order 5       Length 2
        Rep (1, 17, 13, 9, 5)(2, 6, 10,
14, 18)(3, 19, 15, 11, 7)(4, 8, 12, 16, 20)
#C;
/*4*/
 for i in [2..4] do
 i,Orbits(Centralizer(N,C[i][3]));
 end for;
/*2 [
    GSet{@ 1, 2 @},
    GSet{@ 3, 4 @},
```

```
    GSet{@ 5, 6 @},
    GSet{@ 7, 8 @},
    GSet{@ 9, 10 @},
    GSet{@ 11, 12 @},
    GSet{@ 13, 14 @},
    GSet{@ 15, 16 @},
    GSet{@ 17, 18 @},
    GSet{@ 19, 20 @}
]
3
    GSet{@ 1, 9, 17, 5, 13 @},
    GSet{@ 2, 14, 6, 18, 10 @},
    GSet{@ 3, 11, 19, 7, 15 @},
    GSet{@ 4, 16, 8, 20, 12 @}
]
4 [
    GSet{@ 1, 17, 13, 9, 5 @},
    GSet{@ 2, 6, 10, 14, 18 @},
    GSet{@ 3, 19, 15, 11, 7 @},
    GSet{@ 4, 8, 12, 16, 20 @}*/
 for j in [2..4] do
 C[j][3];
 for i in [1..10] do if ArrayP[i] eq C[j][3]
 then Sch[i]; end if;
 end for; end for;
/*(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)
(13, 14)(15, 16)(17, 18)(19, 20)
y
(1, 9, 17, 5, 13)(2, 14, 6, 18, 10)
(3, 11, 19, 7, 15)(4, 16, 8, 20, 12)
x
x * t
(1, 17, 13, 9, 5)(2, 6, 10, 14, 18)
(3, 19, 15, 11, 7)(4, 8, 12, 16, 20)
x^2*/
```

Notice, we need to account for each set of orbits. For this progenitor, we have not inserted all of the first order relations.

```
for a,b,c,d,e,f,g,h,i in [0..10] do
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-5,
t^11,t^(x^-2)=t^3,
(y*t)^a,
(y*t^2)^b,
```

```
(y*t^3)^c,
(y*t^4)^d,
(y*t^5)^e,
(y*t^6)^f,
(y*t^8)^g,
(y*t^9)^h,
(y*t^10)^i>;
if #G gt 10 then a,b,c,d,e,f,g,h,i;
#G;
end if;
end for;
```

Notice we used $t, .., t^{10}$ since from our labeling we have $t \sim 1, t^2 \sim 3, ...t^{10} \sim 19$. You can find all homomorphic images obtained from the progenitors created with the above methods in Chapter 7.

Chapter 3

Isomorphism Types of Some Groups

In Chapter 3 we will solve some extension problems, meaning we will determine the isomorphism type of some groups. To prove that one group is isomorphic to another, we not only have to look at the order of the group, but also consider its structure. For example consider the composition factors of two groups both of order 3916800:

```
 G1
    |  C(2, 4)                          = S(4, 4)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

and

```
    G2
    |  Cyclic(2)
    *
    |  C(2, 4)                          = S(4, 4)
    *
    |  Cyclic(2)
    1
```

We would be tempted to say that the groups are isomorphic to each other because they have the same order, but after investigating their composition factors, we find that group 1 is isomorphic to $4^{\cdot}S_4$ and group 2 is isomorphic to $2^{\cdot}(S_4 : 2)$. We will demonstrate the process required to solve these extension problems, namely 4 types: direct products, semi-direct products, central extensions and mixed extensions.

3.1 Preliminaries

$$G = H_0 \geq H_1 \cdots H_m = 1$$

is a refinement of a normal series

$$G = H_0 \geq H_1 \cdots H_m = 1$$

if $G_0, G_1, ..., G_n$ is a subsequence of $H_0, H_1, ..., H_m$. A **composition series** is a normal series

$$G = G_0 \geq G_1 \cdots G_n = 1$$

in which, for all i either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_1$.
Jordan Hölder Theorem: Every two composition series of a group G are equivalent.
If G has a composition series, then the factor groups of this series are called the **composition factors** of G.
If K $\geq$ G, then a (**right**) **transversal** of K in G is a subset T of G consisting of one element from each right coset of K in G. If K and Q are groups, then an **extension** of K by Q is a group G having a normal subgroup $K_1 \cong$ K with G/K_1 $\cong$ Q. If H and K are groups, then their **direct product**, denoted by H $\times$ K, is the group with all elements ordered pairs (h, k), where $h \in$ H and $k \in$ K and with

operations

$$(h,k)(h',k') = (hh',kk')$$

A group G is a **semi − direct product** of the subgroups K by the subgroups Q, denoted by G = K : Q, if K is normal in G and K has a complement $Q_1 \cong$ Q. There are other another two extensions we need to consider. For instance, a **central extension** of K by Q is an extension G of K by Q with K $\leq$ Z(G). A **mixed extension** combines the properties of both a semi-direct product and central extension, where G = NK and N is a normal subgroup of a group G but is not central. The **dihedral group** D_{2n}, for $2_n \geq 4$, is a group of order $2n$ which is generated by two elements of order 2.

3.2 Direct Products

Let us begin with this simple extension problem. We will solve the extesion problem for a control group N of order 120 given by the following presentation: $< x, y, z | x^3, y^2, z^5, x^{-1} * y * x * y, y * z^{-1} * y * z, (z^{-1} * x)^3, (x^{-1} * z^{-2})^2 >$. Notice here we do not mention any type of relations, since our control groups are simply presentations of finite groups. In our case, N is generated by x, y and z. We begin by finding the compostion factors of N which are:

```
G
|  Alternating(5)
*
|  Cyclic(2)
1
```

Now we must look at the normal lattice of N, this gives us all the normal subgroups of N.

```
   Normal subgroup lattice
-----------------------

[4]  Order 120  Length 1  Maximal Subgroups: 2 3
---
```

```
[3]  Order 60   Length 1  Maximal Subgroups: 1
---
[2]  Order 2    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

This is one of the most significant pieces of information we can obtain. Notice that our normal lattice is telling us that we have a subgroup of order 1, order 2, order 60 and lastly G is the order of our group which is 120. From here, we apply our definitions. We know that if there are two normal subgroups such that their product equals the order of our group in question, then we most likely have a direct product. In this case notice that subgroup [2] and subgroup [3] are of order 2 and 60 respectively. Therefore we verify with $MAGMA$ if $N \cong 2 \times A_5$:

```
   IsIsomorphic(N,DirectProduct(AlternatingGroup(5),
CyclicGroup(2)));
true Homomorphism of GrpPerm: N, Degree 10,
 Order 2^3 * 3 * 5
into GrpPerm: $, Degree 7, Order 2^3 * 3 * 5 induced by
    (2, 4, 10)(5, 7, 9) |--> (2, 4, 5)
    (1, 6)(2, 7)(3, 8)(4, 9)(5, 10) |--> (6, 7)
    (1, 3, 5, 7, 9)(2, 4, 6, 8, 10) |--> (1, 3, 5, 2, 4)
```

We are given the corresponding mappings above that confirm that $N \cong 2 \times A_5$. Next we investigate central extensions.

3.3 Central Extensions with Minimal Degree Permutation Representation

3.3.1 Isomorphism Type of $G \cong \frac{2^{*60}:S_5}{(v*w^{-1}*t)^2,(v*w*z*t)^3}$

.

From our control group $N =< v, w, x, y, z | v^2, w^4, x^2, y^3, z^3, w^{-2}*x, (w^{-1}*v)^2, (x*y^{-1})^2, v*z^{-1}*v*z, (x*z^{-1})^2, (y, z), w*y^{-1}*w^{-1}*y*z^{-1} > \cong S_5$ factored by the relations: $(v*w^{-1}*t)^2, (v*w*z*t)^3$, we found the following finite homomorphic image: 0 0 0 0 0 0 0 0 3 2 , 1440. This group G is of order 1440, and the numbers 3 and 2 are the first order relations mentioned above, to obtain this finite group G. This group G factored by the above relations looks as follows:

$$G < v, w, x, y, z, t >:= Group < v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^-2 * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x * z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1}, (t, v * x * z^{-1}), t^2, (v * w^{-1} * t)^2, (v * w * z * t)^3 >;$$

We use the following command in $MAGMA$, where f is the mapping from the presentation of G given above, k is the kernel of f, and $G1$ is the name we give the permutation group image. $f, G1, k := CosetAction(G, sub < G|v, w, x, y, z >)$. This mapping gives us a permutation representation, but not necessarily one of minimal degree. For example, we have the following generators of $G1$ without using minimal degree representation:

```
Permutation group G1 acting on a set of cardinality 20
Order = 1440 = 2^5 * 3^2 * 5
    (2, 3)(4, 6)(7, 10)(8, 14)(12, 15)(13, 17)
    (2, 3, 4, 6)(7, 11, 10, 9)(8, 15, 12, 14)(13, 18, 17, 16)
    (2, 4)(3, 6)(7, 10)(8, 12)(9, 11)(13, 17)(14, 15)(16, 18)
    (2, 5, 4)(3, 7, 9)(6, 11, 10)(8, 13, 16)(12, 18, 17)(14,
    19, 15)(2, 6, 7)(3, 4, 10)(5, 11, 9)(8, 15, 17)(12, 13,
    14)(16, 19, 18)(1, 2)(3, 8)(4, 5)(6, 12)(7, 13)(9, 10)
    (11, 18)(14, 20)(15, 19)(16, 17)
```

Now, to obtain a permutation representation of $G1$ of the minimal degree, we use the following code in $MAGMA$ which finds a subgroup H in $G1$ whose generators produce all $G1$ on less letters.

```
 SL := Subgroups(G1);
 T := {X`subgroup: X in SL};
 #T;
194
TrivCore := {H:H in T| #Core(G1,H) eq 1};
 mdeg := Min({Index(G1,H):H in TrivCore});
 Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
 #Good;
/*4*/
```

```
 H := Rep(Good);
 #H;
/*120*/
 f,G1,K := CosetAction(G1,H);
G1;
/*Permutation group G1 acting on a set of cardinality 12
Order = 1440 = 2^5 * 3^2 * 5
    (1, 2)(3, 9)(4, 7)(5, 11)(6, 8)(10, 12)
    (1, 3, 4, 8)(2, 6, 7, 9)(5, 12)(10, 11)
    (1, 4)(2, 7)(3, 8)(6, 9)
    (1, 4, 10)(5, 9, 6)
    (1, 4, 10)(2, 7, 12)(3, 8, 11)(5, 9, 6)
    (1, 5)(2, 8)(3, 7)(4, 11)(6, 12)(9, 10)*/
```

Notice now the cardinality of our group is reduced from 20 to 12. We will now prove the isomorphism type of G.

Proof. The composition factors of G are:

```
  G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(2)
    1
```

The composition series for G is:
$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 = 1$.
The composition factors are:

$$\begin{aligned} G &= (G_0/G_1)(G_1/G_2)(G_2/G_3) \\ &= (G_0/G_1)(G_1/G_2)G_3 \\ &= C_2A_5C_2 \end{aligned}$$

We will investigate the normal subgroup lattice of G to get an idea of the isomorphism type.

```
Normal subgroup lattice
-----------------------
```

```
[7]  Order 1440  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 720   Length 1  Maximal Subgroups: 3
[5]  Order 720   Length 1  Maximal Subgroups: 2 3
[4]  Order 720   Length 1  Maximal Subgroups: 3
---
[3]  Order 360   Length 1  Maximal Subgroups: 1
---
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

We usually begin an extension problem by factoring G by the largest abelian subgroup. In this case, G has a center, which also happens to be the largest abelian group of G. Our center is $NL[2]$ which by the normal subgroup lattice we see that it is of order 2. Thus, our center is isomorphic to C_2. Now, we factor G by the center.

```
 q,ff:=quo<G1|NL[2]>;
 q;

/*Permutation group q acting on a set of cardinality 10
Order = 720 = 2^4 * 3^2 * 5
    (2, 3)(4, 6)(7, 8)
    (2, 3, 4, 6)(7, 9, 8, 10)
    (2, 4)(3, 6)(7, 8)(9, 10)
    (2, 5, 4)(3, 7, 10)(6, 9, 8)
    (2, 6, 7)(3, 4, 8)(5, 9, 10)
    (1, 2)(4, 5)(8, 10)*/
```

We are left with a group of order 720 with the generators shown above.

We denote this factor group q. We have an idea of what q might be isomorphic to, S_6 since this group is of order 720 as well. We verify:

```
IsIsomorphic(q,SymmetricGroup(6));
/*true*/
```

We can now begin to construct the presentation of G with the information obtained. We will write the presentation of q with the generators: a,b,c,d,e,f. The central element will be represented as z. Since we know that the center commutes with all elements of the group, we write the presentation as follows:

```
H<a,b,c,d,e,f,z>:=Group<a,b,c,d,e,f,z| a^2, b^4, c^2, d^3, e^3,
f^2, b^-2*c, (b^-1*a)^2, (c*d^-1)^2, a*e^-1*a*e, (c*e^-1)^2,
(d,e), b*d^-1*b^-1*d*e^-1, (b^-1*f*a)^2, (c*f*d^-1)^2,
e^-1*f*e*b*f*b^-1, f*b^-1*f*b*f*a,
d^-1*f*d^-1*a*b^-1*f*d^-1*f, z^2, (z,a),(z,b),(z,c),
(z,d),(z,e),(z,f)>;
```

We now verify that our presentation is isomorphic to G.

```
f,H1,k:=CosetAction(H,sub<H|Id(H)>);

s,t:=IsIsomorphic(G1,H1);
s;
/*true*/
```

Thus, we have $G \cong 2^{*}S_6$. □

Next, we investigate a more complicated isomorphism type, namely semi-direct produts.

3.4 Semi-Direct Products

3.4.1 Isomorphism Type of $G \cong \frac{2^{*10}:(5\times 10)}{(x^{-2}*y*x^{-1}*t)^5,(x^{-1}*y*t)^5}$

.

Our control group $N =< x, y|x^5, y^2, x^{-1}*y*x^{-1}*y*x*y*x*y >\cong Z_5\times D_{10}$. We constructed the following infinite progenitor $G =< x, y, t|x^5, y^2, x^{-1} * y * x^{-1} * y*x*y*x*y, (t, x), t^2 >$, which when factored by the following necessary relations: $(x^{-2}*y*x^{-1}*t)^5, (x^{-1}*y*t)^5$ produces a group G of order 6250. Note: we will use the following loop to give us a permutation representation of G of the minimal degree which is originally of cardinality 125. We use this loop when we have groups of large cardinality, in order to work with the best presentation of that group G.

```
SL := Subgroups(G1);
 T := {X`subgroup: X in SL};
 #T;
/*228*/
 TrivCore := {H:H in T| #Core(G1,H) eq 1};
 mdeg := Min({Index(G1,H):H in TrivCore});
 Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
 #Good;
/*5*/
 H := Rep(Good);
 #H;
/*250*/
 f,G1,K := CosetAction(G1,H);
G1;
 /* Originally Permutation group G1 acting on a set of
 cardinality 125
Order = 6250 = 2 * 5^5 , now
Permutation group G1 acting on a set of cardinality 25
Order = 6250 = 2 * 5^5
    (1, 2, 4, 8, 5)(3, 6, 9, 13, 10)(7, 11, 14, 18, 15)
     (12, 16, 19, 22, 20),
    (2, 5)(3, 7)(4, 8)(6, 11)(9, 14)(10, 15)(12, 17)
    (13, 18)(16, 21)(19, 23)(20,24)(22, 25),
    (1, 3)(2, 6)(4, 9)(5, 10)(7, 12)(8, 13)(11, 16)(14, 19)
    (15, 20)(18, 22)(21,24)(23, 25)*/
```

We will now prove the isomorphism type of G.

Proof. The composition factors of G are:

```
G
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
1
```

The composition series for G is:
$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6$ where $G_6 = 1$.

The composition factors are:

$$\begin{aligned} G &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6) \\ &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6) \\ &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/1) \\ &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)G_5 \\ &= C_2C_5C_5C_5C_5C_5 \end{aligned}$$

The normal lattice of G1 is

```
NL:=NormalLattice(G1);
 NL;

Normal subgroup lattice
-----------------------

[9]  Order 6250  Length 1  Maximal Subgroups: 7 8
---
[8]  Order 3125  Length 1  Maximal Subgroups: 5 6
[7]  Order 1250  Length 1  Maximal Subgroups: 5
---
[6]  Order 625   Length 1  Maximal Subgroups: 4
[5]  Order 625   Length 1  Maximal Subgroups: 4
---
[4]  Order 125   Length 1  Maximal Subgroups: 3
---
[3]  Order 25    Length 1  Maximal Subgroups: 2
---
[2]  Order 5     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

We take the largest abelian group which we find through running the following loop:

```
for i in [1..#NL] do
```

```
if IsAbelian(NL[i]) then i;
end if; end for;
1
2
3
4
6
```

Here we see that the largest abelian group is 6, which refers to NL[6] of order 625 from our normal lattice of $G1$. We first need to find the isomorphism type of NL[6], which has several possibilities, such as $5^3 \times 5, 5^4$, etc. We check and find the following:

```
 NL[6];
/*Permutation group H acting on a set of cardinality 25
Order = 625 = 5^4
Generators are:
 C:(3, 9, 10, 6, 13)(12, 22, 16, 20, 19),
 D:(3, 10, 13, 9, 6)(7, 18, 11, 15, 14)(12, 20, 22, 19, 16),
 E:(1, 8, 2, 5, 4)(7, 11, 14, 18, 15)(12, 22, 16, 20, 19)
     (17, 24, 25, 23, 21),
 F:(1, 4, 5, 2, 8)(3, 9, 10, 6, 13)(7, 11, 14, 18, 15)
    (12, 16, 19, 22, 20)*/
X:=[5,5,5,5];
 IsIsomorphic(NL[6],AbelianGroup(GrpPerm,X));
/*true Mapping from: GrpPerm: H to GrpPerm: $, Degree 20,
Order 5^4
Composition of Mapping from: GrpPerm: H to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 20, Order 5^48*/
```

We find that it is 5^4. We have that, $G_2 = 5^4$. Since the order of G is 6250, and we have that the order of NL[6] is 625, then (G_0/G_1) must be of order 10. This composition factor, we call q. Let us investigate the composition factors and normal subgroups of q to find what q is isomorphic to.

```
CompositionFactors(q);
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    1
```

```
and
Normal subgroup lattice
-----------------------

[3]  Order 10  Length 1  Maximal Subgroups: 2
---
[2]  Order 5   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

 q;
Generators:   A:(2, 3)(4, 5),
                      B:(1, 2)(3, 4)
Order 10
```

Notice that the only normal subgroup of q is of order 5, since we do not have a normal subgroup of order 2, we will not have a direct product of $C_2 \times C_5$. Also, since C_5 is not the center of q, then we will most likely have a semidirect product of $5:2$. We ask $MAGMA$ if q is abelian:

```
 IsAbelian(q);
/*false*/
```

Therefore, q must be the nonabelian group D_{10}. We verify our assumption:

```
IsIsomorphic(DihedralGroup(5),q);
/*true Mapping from: GrpPerm: $, Degree 5,
 Order 2 * 5 to GrpPerm: q
Composition of Mapping from: GrpPerm: $, Degree 5,
Order 2 * 5 to
GrpPC andMapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: q*/
```

We find that q is isomorphic to D_{10}. We now have that this is either a mixed extension or a semidirect product. We determined this by studying our normal lattice. There is no normal subgroup of order 10 in G that intersects with $NL[6]$ of order 625; therefore, we do not have a direct product. Also, our q is not the center of G, so we do not have a central extension. We are not sure if we have a mixed extension, but we will investigate this possibility if a semi- direct product is not the case. To complete our work, we must now find a presentation for the action of D_{10} on the generators of 5^4, which is done as follows:

```
 <c,d,e,f,a,b|c^5,d^5,e^5,f^5, (c,d),(c,e),(c,f),
(d,e),(d,f), (e,f),a^2,b^2,(a*b)^5, c^a?,d^a?,
e^a?,f^a?,c^b?,d^b,e^b,f^b?>
```

Here c,d,e and f represent elements of $NL[6]$, and a and b represent elements of D_{10}. We need to find the action of D_{10} on NL[6], in other words, what is $C^A, ..., F^A$, and $C^B, ..., F^B$? Since Q is not normal in G, we do not know how the elements of Q act on the elements of the normal subgroup K. We need to find which permutations of our normal subgroup K map to the elements of Q to find the action. This is done by using the transversals of G over K and fnding which transversals map to the generators of Q.

```
 T:=Transversal(G1,NL[6]);
 ff(T[2]) eq q.2;
/*true*/

T[2] = (2, 5)(3, 7)(4, 8)(6, 11)(9, 14)(10, 15)(12, 17)
(13, 18)(16, 21)(19, 23)(20, 24)(22, 25)

q.2 = (2, 3)(4, 5)

 ff(T[3]) eq q.3;
/*true*/

T[3] = (1, 3)(2, 6)(4, 9)(5, 10)(7, 12)(8, 13)(11, 16)
           (14, 19)(15, 20)(18, 22)(21,24)(23, 25)

q.3 = (1, 2)(3, 4)
```

Therefore $T[2]$ maps to the first generator of Q and $T[3]$ maps to the second generator of Q. Now, we must verify if the relations of Q match the relations of the transversals. The relations of Q are:

```
    a^2 = Id($)
     b^2 = Id($)
    (b * a)^5 = Id($)
```

The order of the transversals match:

```
Order(T[2]);
/*2*/
Order(T[3]);
/*2*/
 Order(T[2]*T[3]);
/*5*/
```

We now have verified that we have a semi - direct product of K by Q. If the relations did not match, we would have a mixed extension. To complete the presentation of our group, we run the following loop to find $C^A, C^B, D^A, ..., F^A$ and F^B as follows:

```
 for i,j,k,l in [1..5] do if C^A eq C^i*D^j*E^k*F^l then
i,j,k,l; end if; end for;
1 2 5 5
 C^A eq C*D^2;
true
for i,j,k,l in [1..5] do if C^B eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
1 5 5 5
C^B eq C;
true
 for i,j,k,l in [1..5] do if D^A eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 4 5 5
 for i,j,k,l in [1..5] do if D^B eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 4 5 2
 for i,j,k,l in [1..5] do if E^A eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 5 2 4
 for i,j,k,l in [1..5] do if E^B eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 5 4 2
 for i,j,k,l in [1..5] do if F^A eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 5 3 3
 for i,j,k,l in [1..5] do if F^B eq C^i*D^j*E^k*F^l then
 i,j,k,l; end if; end for;
5 5 5 1
```

The numbers represent C, D, E and F. The generators of K are of order 5, therefore any 5 on the loop means the identity element and the action of Q on that element of K does not affect our presentation. For example, the first loop produced $1, 2, 5, 5$ which means $C * D^2$ since E and F are of order 5. Finally, we have our new presentation of our group which we label as H. We check the order and check if it is isomorphic to our original group G.

```
 H<c,d,e,f,a,b>:=Group<c,d,e,f,a,b| a^5,b^5,c^5,d^5,(a,b),(a,c),
(a,d),(b,c),(b,d),(c,d),e^2,f^2,(e*f)^5, a^e=a^4*b*c,
a^f=a^4*c^2, b^e=b^2*c^3,b^f=b^4*c^4,c^e=b^4*c^3,
c^f=c,d^e=a*b^2*c^2*d,d^f=a^4*c*d>;

 #H;
6250
 f,g,k:=CosetAction(H,sub<H|Id(H)>);
 s:=IsIsomorphic(G1,g);
 s;
true
```

Therefore, the isomorphism type of our group G is $5^4 : D_{10}$.

$\square$

Finally, we investigate the last extension problem type, mixed extensions.

3.5 Mixed Extensions

3.5.1 Isomorphism Type of $G \cong \frac{2^{*10}:(2\times(5:4)}{(x*t)^3}$

From our control group $N =< x, y|y^4, y^{-2} * x^{-1} * y^2 * x^{-1}, y^{-1} * x^3 * y * x^{-1} >\cong 2 \times (C_5 : 4)$ we construct the following progenitor: $G < x, y, t >:= Group < x, y, t|y^4, y^{-2} * x^{-1} * y^2 * x^{-1}, y^{-1} * x^3 * y * x^{-1}, t^2, (t, x^{-1} * y^{-1} * x) >$ which when factored by the realtion $(x * t)^3$ we obtain a finite homomorphic image of a group $|G| = 600$. When we use the following command:

```
 f,G1,k:=CosetAction(G,sub<G|x,y>);
```

we obtain the permutation representation of degree 15, whose generators are:

```
 (2, 3, 6, 7, 13, 8, 4, 9, 10, 5)(11, 15)(12, 14)
    (2, 4, 10, 6)(3, 7, 9, 8)(11, 12, 15, 14)
    (1, 2)(3, 5)(4, 11)(6, 12)(7, 9)(10, 14)(13, 15)
```

We will now prove the isomorphism type of G.

Proof. The composition factors of G are:

```
G
  |  Cyclic(2)
  *
  |  Cyclic(3)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(5)
  *
  |  Cyclic(5)
  1
```

The composition series for G is:
$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6$ where $G_6 = 1$.
The composition factors are:

$$
\begin{aligned}
G &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6)\\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6)\\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/1)\\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)G_5\\
&= C_2C_3C_2C_2C_5C_5
\end{aligned}
$$

The normal lattice of G1 is

```
NL:=NormalLattice(G1);
 NL;

Normal subgroup lattice
-----------------------

[12]  Order 600  Length 1  Maximal Subgroups: 9 10 11
---
[11]  Order 300  Length 1  Maximal Subgroups: 7
[10]  Order 300  Length 1  Maximal Subgroups: 6 7 8
[ 9]  Order 300  Length 1  Maximal Subgroups: 5 7
```

```
---
[ 8]  Order 150  Length 1  Maximal Subgroups: 4
[ 7]  Order 150  Length 1  Maximal Subgroups: 3 4
[ 6]  Order 150  Length 1  Maximal Subgroups: 4
[ 5]  Order 100  Length 1  Maximal Subgroups: 3
---
[ 4]  Order 75   Length 1  Maximal Subgroups: 2
[ 3]  Order 50   Length 1  Maximal Subgroups: 2
---
[ 2]  Order 25   Length 1  Maximal Subgroups: 1
[ 1]  Order 1    Length 1  Maximal Subgroups:
```

We take the largest abelian group which we find through running the following loop:

```
for i in [1..#NL] do
if IsAbelian(NL[i]) then i;
end if; end for;
1
2
```

Here we see that the largest abelian group is 2, which refers to NL[2] of order 25 from our normal lattice of $G1$. From the definition of mixed extension, NL[2] would be our K). We first need to find the isomorphism type of NL[2].

```
 NL[2];
Permutation group H acting on a set of cardinality 25
Order = 25 = 5^2*/
X:=[5,5];
 IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
true Mapping from: GrpPerm: H to GrpPerm: $, Degree 10,
Order 5^2
Composition of Mapping from: GrpPerm: H to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 10,
 Order 5^2
```

We find that it is 5^2. We have that $(G_4/G_5)(G_5/G_6) = 5^2$. Since the order of G is 600, and we have that the order of NL[2] is 25, then $(G_0/G_1)(G_1/G_2)$ $(G_2/G_3)(G_3/G_4)$ must be of order 24. This composition factor we call q, which would be our Q from the definition.

```
#G1;
600
 #q;
24
 IsAbelian(q);
false
```

In order to determine what the isomorphism type of q may be we will look at the normal lattice. We see that there is a normal subgroup of order 6 and of order 4 and also a normal subgroup of order 12 and of order 2. Therefore we will check if we have a direct product.

```
Normal subgroup lattice
-----------------------


[11]  Order 24  Length 1  Maximal Subgroups: 8 9 10
---
[10]  Order 12  Length 1  Maximal Subgroups: 6
[ 9]  Order 12  Length 1  Maximal Subgroups: 4 6
[ 8]  Order 12  Length 1  Maximal Subgroups: 5 6 7
---
[ 7]  Order 6   Length 1  Maximal Subgroups: 3
[ 6]  Order 6   Length 1  Maximal Subgroups: 2 3
[ 5]  Order 6   Length 1  Maximal Subgroups: 3
[ 4]  Order 4   Length 1  Maximal Subgroups: 2
---
[ 3]  Order 3   Length 1  Maximal Subgroups: 1
[ 2]  Order 2   Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1   Length 1  Maximal Subgroups:


IsIsomorphic(q, DirectProduct(DihedralGroup(3),CyclicGroup(4)));
true Mapping from: GrpPerm: q to GrpPerm: $, Degree 7,
 Order 2^3 * 3
Composition of Mapping from: GrpPerm: q to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 7,
Order 2^3 * 3
```

We find that q is isomorphic to the direct product of the Dihedral group 6, denoted D_6, and the Cyclic group 4 denoted C_4. Now, we must determine the action of $(D_6 \times C_4)$ on 5^2. We will check if we have a mixed extension or a semi-direct product. For either a mixed extension or a semi - direct product, we need to find the transversals of the normal subgroup K to relate the elements of q to K since q is not normal in G.

Let us check which transversals map to the three elements of q.

```
T:=Transversal(G1,NL[2]);
ff(T[2]) eq q.1
true
ff(T[3]) eq q.2
true
ff(T[4]) eq q.3
true
```

Therefore we will use $T[2], T[3]$ and $T[4]$ to represent $q.1, q.2$ and $q.3$. From our relations we check the orders of the generators of q as follows:

```
FPGroup(q);
/*Finitely presented group on 3 generators
Relations
    $.1^2 = Id($)
    $.2^4 = Id($)
    $.3^2 = Id($)
    $.2^-1 * $.1 * $.2 * $.1 = Id($)
    $.2^-1 * $.3 * $.2 * $.3 = Id($)
    ($.3 * $.1)^3 = Id($)*/
```

Now, we will check if the order of the relations of q match when we map the transversals of K in G.

```
Order(T[2]^2); /*5*/ change
Order(T[3]^4); /*1*/ match
Order(T[4]^2); /*1*/ match
Order(T[3]^-1*T[2]*T[3]*T[2]);
/*5*/ change
 Order(T[3]^-1*T[4]*T[3]*T[4]);
/*5*/ change
Order((T[4]*T[2])^3); /*1*/ match
```

The only relations that changed above were:

```
Order(T[2]^2); /*5*/
Order(T[3]^-1*T[2]*T[3]*T[2]);
/*5*/
 Order(T[3]^-1*T[4]*T[3]*T[4]);
/*5*/
```

Now we know that we will have a mixed extension. To complete a presentation for a mixed extension we need to complete two steps: 1) Find the action of the generators of q on the generators of NL[2].
2) Write the elements of q as products of the elements of NL[2].

We will label the generators of NL[2] below as A and B to begin step 1:

```
Generators(NL[2]);
A:=G1!(1, 11, 12, 14, 15)(2, 6, 13, 4, 10)(3, 9, 7, 5, 8);
B:=G1!(2, 10, 4, 13, 6)(3, 5, 9, 8, 7);
```

We named the transversals as follows for our presentation:

```
T[2] = c
T[3] = d
T[4] = e
```

To complete step 1), we find the action of the generators of q on the generators of NL[2].

```
for i,j in [1..5] do if A^T[2] eq A^i*B^j then i,j; end if;
end for;     /*4 1*/
for i,j in [1..5] do if A^T[3] eq A^i*B^j then i,j; end if;
end for;      /*2 5*/
for i,j in [1..5] do if A^T[4] eq A^i*B^j then i,j; end if;
 end for;     /*2 4*/
for i,j in [1..5] do if B^T[2] eq A^i*B^j then i,j; end if;
 end for;  /*5 1*/
for i,j in [1..5] do if B^T[3] eq A^i*B^j then i,j; end if;
end for;    /*5 2*/
 for i,j in [1..5] do if B^T[4] eq A^i*B^j then i,j; end if;
end for;     /*3 3*/
```

Our presentation of step 1) is the following:

$a^c = a^4 * b,$

$a^d = a^2,$

$b^c = b,$

$b^d = b^2,$

$a^e = a^2 * b^4,$

$b^e = a^3 * b^3$

The following loop determines how to represent the generators of q as products of NL[2] from step 2:

```
for i,j in [1..5] do if T[2]^2 eq A^i*B^j then i,j; end if;
 end for;
/*5 4*/
for i,j in [1..5] do if T[3]^-1*T[2]*T[3]*T[2] eq
A^i*B^j then i,j; end if; end for;
/*5 1*/
 for i,j in [1..5] do if T[3]^-1*T[4]*T[3]*T[4]
eq A^i*B^j then i,j; end if; end for;
/*1 4*/
```

The presentation of the generators of q as products of NL[2] will therefore be:

$c^2 = b^4,$

$d^4 = id,$

$e^2 = id,$

$d^{-1} * c * d * c = b,$

$d^{-1} * e * d * e = a * b^4,$

$(e * c)^3 = id$

Now we can complete our presentation as follows:

$H < c, d, e, a, b >:= Group < c, d, e, a, b | a^5, b^5, (a, b), c^2 = b^4, d^4, e^2, d^{-1} * c * d * c = b, d^{-1} * e * d * e = a * b^4, (e * c)^3, a^c = a^4 * b, a^d = a^2, b^c = b, b^d = b^2, a^e = a^2 * b^4, b^e = a^3 * b^3 >;$

Then we check to make sure that our new presentation matches our $G1$

```
#H;
/*600*/
f2,G2,k2:=CosetAction(H,sub<H|Id(H)>);
 #G2;
600
 IsIsomorphic(G2,G1);
/*true Mapping from: GrpPerm: G2 to GrpPerm: G1
```

```
Composition of Mapping from: GrpPerm: G2 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G1*/
```

Therefore we have a mixed extension of $5^2 :\cdot D_6 \times Z_4$

$\square$

3.5.2 Isomorphism Type of $G \cong \frac{2^{*60}:S_5}{(y^2*t)^2,(y*x*t)^6,(y*t)^6}$

.

From our control group $N =< x, y|x^2, y^6, (y*x*y^-1*x)^2, (x*y^-1)^5 >\cong S_5$ we found the following finite homomorphic image 0 0 2 0 6 6 2, 19440 when N is factored by the following necessary relations:$(y^2 * t)^2, (y * x * t)^6, (y * t)^6$ and the additional relation $(y^2 * x * y^-2)^2$. A presentation of G looks like: $G < x, y >:= Group < x, y, t|x^2, y^6, (y * x * y^{-1} * x)^2, (x * y^{-1})^5, (t, x^y), t^2, (y^2 * x * y^{-2})^2, (y^2 * t)^2, (y * x * t)^6, (y * t)^6 >$ We now express thesymmetric presentation above from words to permutations of degree 162 with the command:

```
f, G1, k:=CosetAction(G,sub<G|x,y>);
```

We will now prove the isomorphism type of G.

Proof. The composition factors of G are:

```
G
 |  Cyclic(2)
 *
 |  Cyclic(2)
 *
 |  Alternating(5)
 *
 |  Cyclic(3)
 *
 |  Cyclic(3)
 *
 |  Cyclic(3)
 *
 |  Cyclic(3)
 1
```

The composition series for G is:
$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7$ where $G_7 = 1$.

The composition factors are:

$$\begin{aligned}
G &= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6)(G_6/G_7) \\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6)(G_6/1) \\
&= (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)(G_5/G_6)G_7 \\
&= C_2C_2A_5C_3C_3C_3C_3
\end{aligned}$$

The normal lattice of G is

```
NL:=NormalLattice(G);
 NL;

Normal subgroup lattice
-----------------------

[8]  Order 19440  Length 1  Maximal Subgroups: 5 6 7
---
[7]  Order 9720   Length 1  Maximal Subgroups: 4
[6]  Order 9720   Length 1  Maximal Subgroups: 4
[5]  Order 9720   Length 1  Maximal Subgroups: 3 4
---
[4]  Order 4860   Length 1  Maximal Subgroups: 2
---
[3]  Order 162    Length 1  Maximal Subgroups: 2
---
[2]  Order 81     Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

It is ideal to begin an extension problem by factoring G by the largest abelian group if possible or factoring by the center. In this case, G does not have a center, thus we take the largest abelian group which we find through running the following loop:

```
for i in [1..#NL] do
if IsAbelian(NL[i]) then i;
end if; end for;
1
2
```

Here we see that the largest abelian group is 2, which refers to NL[2] of order 81 from our normal lattice of G. We first need to find the isomorphism type of NL[2], which has several possibilities, such as $3^3 \times 3, 3^2 \times 3^2, 3^4$, etc. We check and find the following:

```
 NL[2];
Permutation group acting on a set of cardinality 162
Order = 81 = 3^4
X:=[3,3,3,3];
 IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
/*true Mapping from: GrpPerm: H to GrpPerm: $, Degree 12,
Order 3^4
Composition of Mapping from: GrpPerm: H to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 12,
Order 3^4*/
```

We find that $NL[2]$ is isomorphic to the abelian group $3 \times 3 \times 3 \times 3$ or 3^4. Thus, we have that $G_3 = 3^4$. Since the order of G is 19440, and we have that the order of NL[2] is 81, then $(G_0/G_1)(G_1/G_2)(G_2/G_3)$ must be of order 240. We confirm this when we factor G by $NL[2]$ as follows:

```
 q,ff:=quo<G1|NL[2]>;
 q;
/*Permutation group q acting on a set of cardinality 24
Order = 240 = 2^4 * 3 * 5
```

This composition factor we call q. Since we do not have an idea of what q might be isomorphic to, we must check the normal lattice for q.

```
nl:=NormalLattice(q);
nl;
Normal subgroup lattice
-----------------------


[7]  Order 240  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 120  Length 1  Maximal Subgroups: 3
[5]  Order 120  Length 1  Maximal Subgroups: 2 3
[4]  Order 120  Length 1  Maximal Subgroups: 3
---
[3]  Order 60   Length 1  Maximal Subgroups: 1
---
```

```
[2]  Order 2    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

The most convenient method to approach this type of problem is to first check if we have a direct product. Direct products are easily found by simply studying our normal lattice. Recall that a direct product of $G = K \times Q$ requires both K and Q to be normal in G. In our case, we want to see if q is composed of a direct product. Since we a have a normal subgroup of order 2, and three subgroups of order 120, we find we most likely have a direct product. We check as follows:

```
E:=DirectProduct(nl[2],nl[4]);
 IsIsomorphic(E,q);
/*true*/
```

We see that $nl[2] \times nl[4]$ gives us q. We need to find what $nl[2]$ and $nl[4]$ are ismorphic to. Recall that S_5 is a group of order 120, so we check the following:

```
IsIsomorphic(nl[4],SymmetricGroup(5));
/*true*/

IsIsomorphic(nl[2],CyclicGroup(2));

/*true Mapping from: GrpPerm: $, Degree 24, Order 2 to GrpPerm:
 Degree 2, Order 2
Composition of Mapping from: GrpPerm: $, Degree 24, Order 2 to
 GrpPC and Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 2, Order 2*/
```

We find that $q \cong S_5 \times 2$. Now we need to write a presentation for q. Since we know that $nl[4] \cong S_5$ and $nl[2] \cong C_2$, we run the following in $MAGMA$ to give us a presentation for each, so that we may complete the presentation of q.

```
FPGroup(SymmetricGroup(5));
Finitely presented group on 2 generators
Relations
    $.1^5 = Id($)
    $.2^2 = Id($)
    ($.1^-1 * $.2)^4 = Id($)
    ($.1 * $.2 * $.1^-2 * $.2 * $.1)^2 = Id($)
and FPGroup(nl[2]);
```

```
Finitely presented group on 2 generators
Relations
    $.2^2 = Id($)
    $.1 = Id($)
```

A presentation for S_5 is : $S < e, f >:= Group < e, f|e^5, f^2, (e^{-1} * f)^4, (e * f * e^{-2} * f * e)^2 >$, and a presentation for C_2 is: $C < g, h >:= Group < g, h|h^2, g >$. We know that since we have a direct product, the elements of S_5 will commute with the elements of C_2 as follows: $D < e, f, g, h >:= Group < e, f, g, h|e^5, f^2, (e^{-1} * f)^4, (e * f * e^{-2} * f * e)^2, h^2, g, (g, e), (g, f), (h, e), (h, f) >$.

We check to make sure we have the correct presentation for q:
$D < e, f, g, h >:= Group < e, f, g, h|e^5, f^2, (e^{-1}*f)^4, (e*f*e^{-2}*f*e)^2, h^2, g, (g, e), (g, f), (h, e), (h, f) >$

```
ff2,dd,kk2:=CosetAction(D,sub<D|Id(D)>);
s,t:=IsIsomorphic(q,dd);

s;
true
```

Now that we have a presentation for q, let us write a presentation for $NL[2]$ since this will be needed in the future. We run the following command in $MAGMA$:

```
FPGroup(NL[2]);
Finitely presented group on 5 generators
Relations
    $.2^3 = Id($)
    $.3^3 = Id($)
    $.4^3 = Id($)
    $.5^3 = Id($)
    ($.2, $.3) = Id($)
    ($.2, $.4) = Id($)
    ($.3, $.4) = Id($)
    ($.2, $.5) = Id($)
    ($.3, $.5) = Id($)
    ($.4, $.5) = Id($)
    $.1 = Id($)
```

Our presentation is: $NL[2] =< w, x, y, z|w^3, x^3, y^3, z^3, (w, x), (w, y), (x, z), (w, z), (x, z), (y, z) >$. Now, our final task is to find the action of $Q = S_5 \times 2$ on K, which in this case is 3^4. We can easily rule out a central extension since we had no center in G, and a direct product as well since we have no normal subgroup of order 240. We determine that we must have a semi - direct product or a mixed extension. If the transversals of $G/NL[2]$ can be written as products of elements of K, then we will have a mixed extension, or if this is not the case, we will simply have a semi - direct product.

```
T:=Transversal(G1,NL[2]);
 ff(T[2]) eq q.1;
/*true*/
 ff(T[3]) eq q.2;
/*true*/
 ff(T[4]) eq q.3;
/*true*/
```

Notice we need the transversals of G over $NL[2]$. This will help us write the elements of q in terms of $NL[2]$. We check if the mapping from transversals $T[2], T[3]$ and $T[4]$ map to our elements of our group q. They indeed do, therefore we can write the transversals as elements of q, which we will show later on. For now, we find the action of these transversals of G on $NL[2]$.

```
Generators(NL[2]);

A:=G1!(1, 161, 155)(2, 143, 162)(3, 144, 146)
(4, 149, 147)(5, 145, 123)(6,119, 156)(7, 134,
 140)(8, 122, 158)(9, 96, 157)(10, 80, 56)(11,111,
 74)(12, 101, 77)(13, 120, 151)(14, 108, 63)
(15, 154, 114)(16,105, 76)(17, 81, 98)(18, 82,
70)(19, 33, 110)(20, 79, 57)(21, 51,138)(22, 55,
 127)(23, 153, 115)(24, 159, 87)(25, 39, 136)(26,
 160, 88)(27, 141, 139)(28, 102, 75)(29, 53, 131)
(30, 103, 97)(31, 73, 112)(32, 45, 113)(34, 109,
 35)(36, 148, 47)(37, 126, 92)(38, 125, 66)(40,
118, 68)(41, 124, 59)(42, 142, 137)(43, 106,
85)(44, 116,150)(46, 83, 133)(48, 117, 152)(49,
 60, 130)(50, 52, 128)(54, 72,129)(58, 135,
69)(61, 121, 100)(62, 91, 99)(64, 107, 67)(65,
94,89)(71, 104, 86)(78, 90, 93)(84, 95, 132);

B:=G1!(1, 153, 151)(2, 144, 152)(3, 117, 162)(4,
126, 133)(5, 135, 109)(6, 80, 78)(7, 107, 129)(8,
 108, 89)(9, 105, 104)(10, 93, 156)(11,154, 84)
```

```
(12, 75, 106)(13, 161, 115)(14, 94, 158)(15, 132,
 74)(16,71, 157)(17, 57, 124)(18, 159, 61)(19, 38,
 137)(20, 59, 81)(21, 148, 53)(22, 136, 45)(23,
 120, 155)(24, 100, 70)(25, 113, 55)(26, 99, 97)
(27, 73, 130)(28, 85, 101)(29, 138, 36)(30, 160,
62)(31,60, 139)(32, 127, 39)(33, 125, 42)(34,
 123, 58)(35, 145, 69)(37, 83, 147)(40, 150, 50)
(41, 98, 79)(43, 77, 102)(44, 52, 118)(46,149,
92)(47, 131, 51)(48, 143, 146)(49, 141, 112)(54,
 134, 67)(56, 90, 119)(63, 65, 122)(64, 72, 140)
(66, 142, 110)(68, 116, 128)(76,86, 96)(82, 87,
 121)(88, 91, 103)(95, 111, 114);

 C:=G1!(1, 154, 158)(2, 149, 139)(3, 126, 112)(4, 141,
 162)(5, 125, 127)(6, 81, 100)(7, 118, 138)(8, 161, 114)
(9, 101, 99)(10, 57, 82)(11, 94,151)(12, 91, 157)(13,
 111, 89)(14, 153, 84)(15, 122, 155)(16, 75,103)(17,
121, 156)(18, 56, 79)(19, 113, 34)(20, 70, 80)(21,
134, 68)(22, 145, 66)(23, 132, 63)(24, 78, 59)(25,
 58, 137)(26, 104,85)(27, 143, 147)(28, 97, 105)(29,
 129, 52)(30, 76, 102)(31, 144,92)(32, 109, 33)(35,
110, 45)(36, 107, 44)(37, 73, 146)(38, 55,123)(39,
135, 42)(40, 51, 140)(41, 159, 90)(43, 160, 86)(46,
 60,152)(47, 64, 150)(48, 83, 130)(49, 117, 133)
(50, 131, 72)(53, 54,128)(61, 119, 98)(62, 96, 77)
(65, 120, 74)(67, 116, 148)(69, 142,
136)(71, 106, 88)(87, 93, 124)(95, 108, 115);

D:=G1!(1, 159, 157)(2, 145, 150)(3, 135, 128)(4, 125,
148)(5, 116, 162)(6,102, 132)(7, 130, 113)(8, 124, 99)
(9, 161, 87)(10, 85, 111)(11,56, 106)(12, 154, 90)(13,
82, 104)(14, 98, 103)(15, 78, 77)(16,153, 61)(17, 97,
108)(18, 71, 151)(19, 138, 83)(20, 160, 65)(21,133,
33)(22, 64, 139)(23, 100, 76)(24, 96, 155)(25, 129, 73)
(26,89, 57)(27, 55, 107)(28, 95, 156)(29, 37, 137)(30,
 63, 81)(31,136, 72)(32, 134, 49)(34, 118, 48)(35, 40,
 152)(36, 147, 38)(39,54, 112)(41, 91, 158)(42, 53,
126)(43, 74, 80)(44, 143, 123)(45,140, 60)(46, 110,
51)(47, 149, 66)(50, 144, 69)(52, 146, 58)(59,62,
122)(67, 141, 127)(68, 117, 109)(70, 86, 120)(75,
 84, 119)(79,88, 94)(92, 142, 131)(93, 101, 114)
(105, 115, 121);

T[2];
```

```
H:=G1!(2, 3)(6, 10)(8, 14)(9, 16)(11, 15)(13, 23)(18, 24)
(21, 36)(22, 39)(25,45)(26, 30)(27, 49)(29, 51)(32, 55)(33,
 38)(35, 58)(37,46)(40, 52)(42,66)(44, 68)(47, 53)(54, 64)
(56, 78)(57, 81)(59, 79)(60, 73)(61, 87)(62, 88)(63, 89)
(71, 96)(75, 101)(76, 104)(77, 106)(82, 100)(84, 114)(85,
102)(93, 119)(94, 122)(98, 124)(99, 103)(107, 134)(109,
123)(110,137)(111, 132)(112, 139)(117, 143)(126, 149)
(128, 150)(129, 140)(133,147)(135, 145)(146, 152)
(151, 155)(153, 161);

 T[3];

I:=G1!(2, 4)(3, 5, 7)(6, 11, 9, 17, 13, 12)(8, 15)(10,18,20)
(14, 24, 43, 23, 41, 26)(16, 28, 30)(19, 33, 32, 45, 35, 34)
(21, 37, 22, 40, 31, 38)(25, 46,47)(29, 52, 50, 72, 54, 53)
(36, 58, 83, 64, 39, 60)(42, 49, 44, 69, 48,67)(51, 73, 55)
(56, 79, 57, 82, 70, 80)(59, 85, 63)(61, 65, 62)(66, 68, 92)
(71, 95, 93)(74, 99, 98, 89, 77, 100)(75, 102, 76, 105, 97,
 103)(78,106, 132, 124, 104, 108)(81, 111, 101)(84, 87,
 86, 115, 90, 88)(91, 119, 94, 96, 121, 120)(107, 135,
130)(109, 113, 110)(112, 123, 138, 146, 127,140)(116,
142, 117)(118, 144, 125, 134, 126, 145)(128, 131, 129)
(133,150, 136, 152, 148, 137)(139, 147)(141, 143)(151,
 157, 156)(153, 159,160)(154, 161)(155, 158);

T[4];

J:=G1!(1, 2)(3, 6)(4, 8)(5, 9)(7, 13)(10, 19)(11, 21)(12, 22)
(14, 25)(15, 27) (16,29)(17, 31)(18, 32)(20, 35)(23, 42)(24,
44)(26, 48)(28, 50)(30, 54)(33,56)(34, 57)(36, 59)(37, 61)
(38, 62)(39, 63)(40, 65)(41, 47)(43, 46)(45,70)(49, 71)(51,
 74)(52, 75)(53, 76)(55, 77)(58, 84)(60, 86)(64, 90)(66,91)
(67, 93)(68, 94)(69, 95)(72, 97)(73, 98)(78, 107)(79, 109)
(80,110)(81, 112)(82, 113)(83, 85)(87, 116)(88, 117)(89,
 118)(92, 121)(96,123)(99, 125)(100, 126)(101, 127)(102,
 128)(103, 129)(104, 130)(105,131)(106, 133)(108, 136)
(111, 138)(114, 141)(115, 142)(119, 146)(120,140)(122,
 147)(124, 148)(132, 135)(134, 151)(137, 153)(139, 154)
(143,155)(144, 156)(145, 157)(149, 158)(150, 159)(152,
 160)(161, 162);
```

The following code tells $MAGMA$ to give us the action of the transversals on the generators $NL[2]$ which are of order 3. The numbers we obtain represent the action of the transversals of $G/NL[2]$ on $NL[2]$, which are in order of A, B, C and D. For example, in the first set we would obtain that $A^H = B$ since A, C and D are order 3 which is the identity.

```
for i,j,k,l in [1..3] do if A^H eq A^i*B^j*C^k*D^l then i,j,k,l;
 end if; end for;
3 1 3 3
for i,j,k,l in [1..3] do if A^I eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
3 3 1 3
for i,j,k,l in [1..3] do if A^J eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
2 3 3 3
for i,j,k,l in [1..3] do if B^H eq A^i*B^j*C^k*D^l then i,j,k,l;
 end if; end for;
1 3 3 3
for i,j,k,l in [1..3] do if B^I eq A^i*B^j*C^k*D^l then i,j,k,l;
 end if; end for;
3 3 3 1
for i,j,k,l in [1..3] do if B^J eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
1 1 1 1
for i,j,k,l in [1..3] do if C^H eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
3 3 1 3
for i,j,k,l in [1..3] do if C^I eq A^i*B^j*C^k*D^l then i,j,k,l;
 end if; end for;
1 3 3 3
for i,j,k,l in [1..3] do if C^J eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
3 3 2 3
for i,j,k,l in [1..3] do if D^H eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
3 3 3 1
for i,j,k,l in [1..3] do if D^I eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
2 2 2 2
for i,j,k,l in [1..3] do if D^J eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
3 3 3 2
```

Therefore, we have this presentation so far: $Group < w, x, y, z, e, f, g | w^3,$ $x^3, y^3, z^3, (w, x), (w, y), (x, z), (w, z), (x, z), (y, z), e^2, f^6, g^2, (f^{-1} * g)^2, f^{-2} * e * f^2 *$

$g*e*g, (f*e*f^{-1}*e)^2, e*f^{-3}*e*f^{-1}*g*e*f^{-1}*g, (e*f^{-1})^5, w^e = x, w^f = y, w^g = w^2, x^e = w, x^f = z, x^g = w*x*y*z, y^e = y, y^f = w, y^g = y^2, z^e = z, z^f = w^2*x^2*y^2*z^2, z^g = z^2 >$ Here A,B,C,D are represented by w, x, y, z and H,I,J are e, f, g.

We have completed the semi - direct part of our presentation. A characteristic of a mixed extension is that the elements of Q may be written as products of the elements of the normal subgroup K. We will test the elements of Q and their relations. When writing the elements of Q as the transversals of $G/NL[2]$ we may have that the order of the relation changes. If this is the case, we must find what element of the normal subgroup of $NL[2]$ this is, which shows that we indeed have a mixed extension.

Recall that $T[2] \to q.1, T[3] \to q.2$ and $T[4] \to q.3$. We check by running the following in $MAGMA$:

```
FPGroup(q);
Finitely presented group on 3 generators
Relations
    $.1^2 = Id($)
    $.2^6 = Id($)
    $.3^2 = Id($)
    ($.1 * $.2^-1)^5 = Id($)
    ($.2^-1 * $.3)^2 = Id($)
    ($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)
    $.2^-2 * $.1 * $.2^2 * $.3 * $.1 * $.3 = Id($)
    $.1 * $.2^-3 * $.1 * $.2^-1 * $.3 * $.1 * $.2^-1 * $.3
    = Id($)


Order(T[2]^-1*T[3]);
Order is 5,  same as in q, thus does not change.

 Order(T[3]^-1*T[4])^2;
/*36, changes, then run code*/

for i,j,k,l in [1..3] do
if (T[3]^-1*T[4])^2 eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
  1 3 2 3
Thus, (f^-1*g)^2=w*y^2
```

```
Order(T[3]*T[2]*T[3]^-1*T[2]);
The order is 2, the same as in the presentation of q.
Then, nothing changes.

Order(T[3]^-2*T[2]*T[3]^2*T[4]*T[2]*T[4]);
    /* Order is 3. It changes, so run code */

for i,j,k,l in [1..3] do
if (T[3]^-2*T[2]*T[3]^2*T[4]*T[2]*T[4]) eq
 A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
 2 1 1 1
Thus, f^-2*e*f^2*g*e*g=w^2*x*y*z.

Order(T[2]*T[3]^-3*T[2]*T[3]^-1*T[4]*T[2]*T[3]^-1*T[4]);
/* Order is 3. It changes, therefore, we run the code. */

for i,j,k,l in [1..3] do
if T[2]*T[3]^-3*T[2]*T[3]^-1*T[4]*T[2]*T[3]^-1*T[4] eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
 1 2 3 3
Thus, e*f^-3*e*f^-1*g*e*f^-1*g=w*x^2.
```

Now we have the complete presentation of G, which we verify using $MAGMA$:

```
H<w,x,y,z,e,f,g>:=Group<w,x,y,z,e,f,g|
w^3,x^3,y^3,z^3,(w,x),(w,y),(x,z),(w,z),(x,z),(y,z),
e^2,f^6,g^2,(f^-1*g)^2=w*y^2,f^-2*e*f^2*g*e*g=w^2*x*y*z,
(f*e*f^-1*e)^2,e*f^-3*e*f^-1*g*e*f^-1*g=w*x^2,(e*f^-1)^5,
w^e=x,w^f=y,w^g=w^2,x^e=w,x^f=z,x^g=w*x*y*z,
y^e=y,y^f=w,y^g=y^2,z^e=z,z^f=w^2*x^2*y^2*z^2,z^g=z^2>;
#H;
 19440
#G1;
19440
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
true
```

We have successfully solved the extension problem for G, in which we found that $G \cong 3^4 \cdot\!\!: (S_5 \times 2)$.

$\square$

Chapter 4

Double Coset Enumeration

Definition 4.1. *[Cur07] Let H and K be subgroups of the group G and define a relation on G as follows:*

$$x \sim y \iff \exists h \in H \text{ and } k \in K \text{ such that } y = hxk$$

where $\sim$ is an equivalence relation and the equivalence classes are the sets of the following form

$$HxK = \{hxk | h \in H, k \in K\} = \cup_{k \in K} Hxk = \cup_{h \in H} hxK$$

Such a subset of G is called a **double coset**. *Now we consider the double coset of the form NxN, where $x = \pi w$ for some $n \in N$ and w is a reduced word in the $t_{i's}$. Thus $NxN = N\pi wN = NwN = [w]$.*

Definition 4.2. *[Rot95] Let N be a group. The* **point stabiliser** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s.}$$

Definition 4.3. *[Rot95] Let N be a group. The* **coset stabiliser** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s.}$$

Since we usually work with large groups, double coset enumeration is a quicker way to determine the number of single cosets in a group G. Once we find the number of single cosets in G, we can determine that the order of G is at least the product of the number of single cosets and the order of our control group N. Let us begin with a simple example:

4.1 Construction of $G \cong S_6 : C_2$

Consider the infinite group G represented by $< x, y, t | y^4, y^{-2} * x^{-1} * y^2 * x^{-1}, y^{-1} * x^3 * y * x^{-1}, t^2, (t, x^{-1} * y^{-1} * x) >$ obtained from our control group $N = 2 \times (5 : 4)$. N is generated by $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ and $y \sim (1, 7, 9, 3)(2, 4, 8, 6)$. Recall that our $t_{i's}$ are of order 2. When we factor G by the following relations: $(y^{-1} * x^{-1} * t)^6, (x^{-1} * y^{-1} * x * t)^4, (x^2 * t)^8, (x * t)^5$, we obtain a finite homomorphic image of order 1440. We will now demonstrate how we construct double coset enumeration on G.

We begin the process of double coset enumeration by first getting an idea of how many double cosets we can expect to obtain. This is an easy check as $\frac{|G|}{|N|}$ = the number of single cosets. In our case we should have $\frac{|G|}{|N|} = 1440/36 = 36$ single cosets. Before we continue, it is important to mention that we found that $t_1 \sim t_6, t_2 \sim t_7, t_3 \sim t_8, t_4 \sim t_9$, and $t_5 \sim t_{10}$. The relations are verified as follows:

```
ts[6] eq(2, 10)(3, 9)(4, 8)(5, 7)t_1,
ts[7] eq (1, 3)(4, 10)(5, 9)(6, 8)t_2,
ts[8] eq (1, 4, 5, 2)(3, 8)(6, 9, 10, 7)t_3,
ts[9] eq (1, 7)(2, 6)(3, 5)(8, 10)t_4,
ts[10] eq (1, 9)(2, 8)(3, 7)(4, 6)t_5
```

Therefore instead of working on 10 $t's$ or 10 letters, we will be working with 5.

Consider the double coset denoted as $[*]$ is $NeN = N$. The number of singles cosets in $[*]$ can be determined by the number of elements in N that fix the coset NeN. Since every element of N fixes the coset NeN (since N is transitive), we find the number of distinct single cosets in $NeN = \frac{|N|}{|N^{(e)}|} = \frac{40}{40} = 1$. In order to move forward, we choose a representative from the orbit

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In this case we will choose 1. There are ten elements in the orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, therefore, all ten symmetric generators will move forward.

Next we will investigate the double coset Nt_1N denoted as [1]. Conjugating our coset Nt_1 by all elements of N gives us all the single cosets that live in [1]. Those are: $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}\}$. We find the point stabiliser $N^1 =< (2, 10)(3, 9)(4, 8)(5, 7), (2, 4, 10, 8)(3, 7, 9, 5) >$ and $|N^1| = 4$. However, consider the following relation: $t_6 = (2, 10)(3, 9)(4, 8)(5, 7)t_1$. This implies that $Nt_6 = N(2, 10)(3, 9)(4, 8)(5, 7)t_1 = Nt_1$. So any permutation that sends t_1 to t_6 will be in the coset stabilizing group $N^{(1)}$. Thus, our coset stabiliser may increase. This implies $N^{(1)} \geq< N^1, (1, 6)(2, 3, 10, 9)(4, 7, 8, 5) >$. We find $|N^{(1)}| = 8$. The number of distinct singles cosets in [1] are $\frac{|N|}{|N^{(1)}|} = \frac{40}{8} = 5$. The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{1, 6\}$ and $\{2, 10, 4, 9, 3, 5, 7, 8\}$. Now choosing a representative from each orbit, we have two possible new double cosets, but, $Nt_1t_1 = NeN$ which lives in $[*]$ since our order of $ti's$ is 2, and since there are two elements in orbit $\{1, 6\}$, two symmetric generators will return to $[*]$. We have one potential new double coset Nt_1t_2N from the orbit $\{2, 10, 4, 9, 3, 5, 7, 8\}$. Therefore eight symmetric generators will move forward to this new double coset we denote as [12].

Consider the new double coset $Nt_1t_2N \in [1, 2]$. We find $N^{12} =< e >$, since no element in N fixes the two points $(1, 2)$. However, if we conjugate Nt_1t_2 by all elements of N we get all the single cosets that live in $[1, 2]$. There are a total of 40 single cosets in $[1, 2]$, but we find the following: $Nt_1t_2 = Nt_8t_7$. Thus, any element that sends Nt_1t_2 to Nt_8t_7 will be in the coset stabilizing group $N^{(12)}$. We find $(Nt_1t_2)^{(1,8)(2,7)(3,6)(4,5)(9,10)} = Nt_8t_7$. So $N^{(12)} \geq< N^{12}, (1, 8)(2, 7)(3, 6)(4, 5)(9, 10) >$. We find $|N^{(12)}| = 2$. Then number of distinct singles cosets in [12] are $\frac{|N|}{|N^{(12)}|} = \frac{40}{2} = 20$. The orbits of N^{12} on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}$. Therefore, we have five possible new double cosets, but we find the following: $Nt_1t_2t_1 = Nt_1t_2t_3 = Nt_1t_2 \in [12]$. Therefore, four symmetric generators will loop back into [12]. Also $Nt_1t_2t_2 = Nt_1 \in [1]$, this implies that two symmetric generators will return to [1].

Therefore the only remaining possible new double cosets are: $Nt_1t_2t_4$ and $Nt_1t_2t_5$. This implies that four elements will advance to a new double coset, but we find that the two single cosets $Nt_1t_2t_4 = Nt_1t_2t_5$, therefore we will only consider one of these new double cosets. We will work with $Nt_1t_2t_4N \in [124]$.
Consider the new double coset $Nt_1t_2t_4N$. We find $N^{124} =< e >$. However, consider we find that $Nt_1t_2t_4 = Nt_5t_2t_6$. Any element that sends $t_1t_2t_4$ to $t_5t_2t_6$ would be in the coset stabilizing group $N^{(124)}$. We find $(Nt_1t_2t_4)^{(1,5,3,9)(4,6,10,8)} = Nt_5t_2t_6$. So $N^{(124)} \geq< N^{124}, (1,5,3,9)(4,6,10,8) >$. We find $|N^{(124)}| = 4$. The number of distinct singles cosets in [124] are $\frac{|N|}{|N^{(124)}|} = \frac{40}{4} = 10$. The orbits of $N124$ are: $\{1,5,3,9\}, \{2\}, \{4,6,10,8\}, \{7\}$. Then, we have four possible new double cosets, but we find: $Nt_1t_2t_4t_1 = Nt_1t_2t_4t_4 = Nt_1t_2 \in [12]$, and $Nt_1t_2t_4t_2 = Nt_1t_2t_4t_7 = Nt_1t_2t_4 \in [124]$. Then eight symmetric generators will return to [12] and two symmetric generators will loop back into [124]. Since there are no possible new double cosets to investigate, our group is closed under right multiplication of $t_i's$. Since the order of G over N is 36, this implies we must have 36 single cosets all together. The following Cayley Diagram illustrates the correct result.

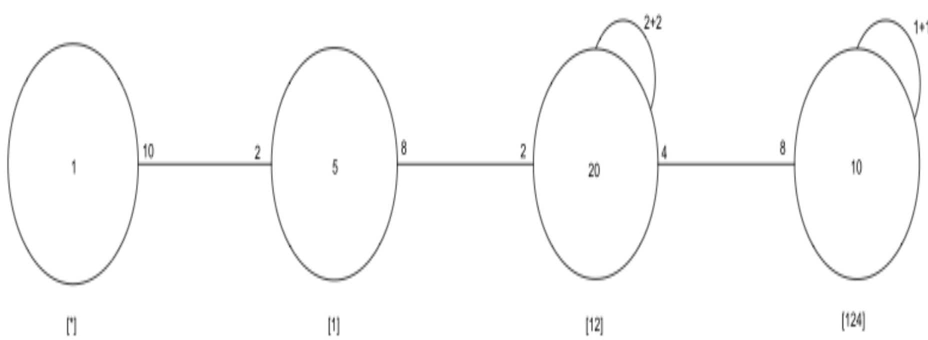

Figure 4.1: Cayley graph of $2^{*10} : (S_6 : C_2)$

4.2 Construction of $5^4 : D_{10}$

We will construct a Caley Diagram of the group $G \cong 5^4 : D_{10}$. Consider the group:

$$G < x, y, t >:= Group < x, y, t | x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y, (t, x), t^2, (x^{-2} * y * x^{-1} * t)^5, (x^{-1} * y * t)^5 > .$$

obtained from our control group $N \cong C_5 \times D_{10}$. $N =< x, y >$ where $x = (2, 4, 6, 8, 10)$, and $y = (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$. We will let $t \sim t_1$.

The double coset denoted as $[*]$ is $NeN = N$, where $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ The number of single cosets in $[*]$ is the number of right cosets which can be determined by $\frac{|N|}{|N|} = \frac{50}{50} = 1$ In order to move forward, we choose a representative from the orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In this case we will choose 1.There are 10 elements in the orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, therefore, ten symmetric generators will move forward.

Consider the new double coset Nt_1N denoted as [1]. Now the point stabilizer, denoted N^1 is equal to $< (2, 4, 6, 8, 10) >$, and $|N^1| = 5$. Any element that fixes 1 or in other words, sends 1 to itself, will be in the coset stabilizing group denoted $N^{(1)}$. We find only four elements that do so, therefore $N^{(1)} \geq < N^1, e, (2, 10, 8, 6, 4), (2, 6, 10, 4, 8), (2, 8, 4, 10, 6) >$. We find $|N^{(1)}| = 5$. The number of distinct singles cosets in [1] are $\frac{|N|}{|N^{(1)}|} = \frac{50}{5} = 10$. The orbits of $N^{(1)}$ are: $\{1\}$,$\{3\}$,$\{5\}$,$\{7\}$,$\{9\}$ and $\{2, 4, 6, 8, 10\}$. Choosing a representative from each orbit, we have six possible new double cosets, but, $Nt_1t_1 = NeN$ which lives in $[*]$, and since there is only one element in orbit $\{1\}$, one symmetric generator will return to $[*]$. Thus, we have five possible new double cosets to investigate. Since none of these double cosets were equal to each other or to Nt_1 they are indeed new double cosets, so we proceed. We now choose one representative from their respective orbits:$\{3\}$,$\{5\}$,$\{7\}$,$\{9\}$ and $\{2, 4, 6, 8, 10\}$. In this case we chose Nt_1t_3, Nt_1t_5, Nt_1t_7, Nt_1t_9 and Nt_1t_2. Therefore, nine symmetric generators will move forward.

Consider the new double coset Nt_1t_2N denoted by [12]. Now, we have that $N^{12} =< e >$. We find that Nt_1t_2 is only equal to itself, and since the only element in the point stabiliser is the identity, the coset stabilizer $N^{(12)} = N^{12} =< e >$. Then $|N^{(12)}| = 1$. The number of distinct singles cosets in [12] are $\frac{|N|}{|N^{(12)}|} = \frac{50}{1} = 50$. The orbits of $N^{(12)}$ are: $\{1\}$,$\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, $\{10\}$. Choosing a representative from each orbit, we have ten possible new double cosets, but after further investigation we find that : $Nt_1t_2t_2 = Nt_1$, therefore one orbit will go back to [1]. We also found that $Nt_1t_2t_1, Nt_1t_2t_3, Nt_1t_2t_5, Nt_1t_2t_7, Nt_1t_2t_9 = Nt_1t_2$, which means that five orbits loop back into [12]. Then four single orbits move forward to the new double cosets $Nt_1t_2t_4N, Nt_1t_2t_6N, Nt_1t_2t_8N$, and $Nt_1t_2t_{10}$.

Consider the new double coset Nt_1t_3N denoted as [13]. Now, $N^{13} =< e >$. Consider the relation $Nt_1t_3 = Nt_3t_5 = Nt_5t_7 = Nt_7t_9 = Nt_9t_1$. These double cosets are equal to each other, and are known as equal names. Any element that sends Nt_1t_3 to any of the following: $Nt_3t_5, Nt_5t_7, Nt_7t_9$ or Nt_9t_1, will be in the coset stabilizing group $N^{(13)}$. We find twenty four elements that do so, therefore $N^{(13)} \geq< N^{13}, (1,3,5,7,9)(2,4,6,8,10)... >$. Then $|N^{(13)}| = 25$. The number of distinct single cosets in [13] are $\frac{|N|}{|N^{(13)}|} = \frac{50}{25} = 2$. The orbits of $N^{(13)}$ are: $\{1,3,9,5,7\}$, and $\{2,4,10,6,8\}$. Then, we have two possible new double cosets, but after further investigation we find the following: $Nt_1t_3t_1 = Nt_1$. Therefore, five orbits go back to [1] and $Nt_1t_3t_2 = Nt_1t_2t_{10}$, then the double coset [132] collapses. Thus, the double coset [13] does not expand any further.

Consider the new double coset Nt_1t_5N denoted by [15]. Now, we have that $N^{15} =< e >$. Consider the relation $Nt_1t_5 = Nt_5t_9 = Nt_9t_3 = Nt_3t_7 = Nt_7t_1$. Any element that sends Nt_1t_5 to any of the following: $Nt_5t_9, Nt_9t_3, Nt_3t_7$ or Nt_7t_1, will be in the coset stabilizing group $N^{(15)}$. We find twenty four elements that do so, therefore $N^{(15)} \geq< N^{15}, (1,5,9,3,7)(2,4,6,8,10)... >$. Then $|N^{(15)}| = 25$. The number of singles cosets in [15] are $\frac{|N|}{|N^{(15)}|} = \frac{50}{25} = 2$. The orbits of $N^{(15)}$ are: $\{1,5,9,3,7\}$ and $\{2,4,10,6,8\}$. Choosing a representative from each orbit we have two possible new double cosets, but we find that:$Nt_1t_5t_1 = Nt_1$, therefore 5 orbits will go back to [1]. We also found that $Nt_1t_5t_2 = Nt_1t_2t_8$, which means that this double coset will collapse. Since there are no more orbits left to check, the double coset [15] ends here.

Consider the new double coset Nt_1t_7N denoted by [17]. We have that $N^{17} =< e >$. However, we have the following relation: $Nt_1t_7 = Nt_3t_9 = Nt_5t_1 = Nt_7t_3 = Nt_9t_5$. Any element that sends Nt_1t_7 to any of the following Nt_3t_9, Nt_5t_1, Nt_7t_3, or Nt_9t_5 will be in the coset stabilizing group $N^{(17)}$. We find twenty four elements that do so, therefore $N^{(17)} \geq < N^{17}, (1,3,5,7,9)(2,4,6,8,10)... >$. Then $|N^{(17)}| = 25$.The number of distinct singles cosets in [17] are $\frac{|N|}{|N^{(17)}|} = \frac{50}{25} = 2$. The orbits of $N^{(17)}$ are: $\{1,3,5,7,9\}$ and $\{2,4,10,6,8\}$. Choosing a representative from each orbit, we have two possible new double cosets, but we find that: $Nt_1t_7t_1 = Nt_1$, therefore, 5 orbits will go back to [1]. Also, $Nt_1t_7t_2 = Nt_1t_2t_6$, which means that this double coset will collapse. Since there are no more orbits left to check, the double coset [17] ends here.

Consider the new double coset Nt_1t_9 denoted by [19]. We have that $N^{19} =< e >$. Consider the relation $Nt_1t_9 = Nt_3t_1 = Nt_5t_3 = Nt_7t_5 = Nt_9t_7$. Any element that sends Nt_1t_9 to any of the following: $Nt_3t_1, Nt_5t_3, Nt_7t_5$ or Nt_9t_7. , will be in the coset stabilizing group $N^{(19)}$. We find twenty four elements that do so, therefore $N^{(19)} \geq < N^{19}, (1,5,9,3,7)(2,4,6,8,10)... >$. Then $|N^{(19)}| = 25$. The number of singles cosets in [19] are $\frac{|N|}{|N^{(19)}|} = \frac{50}{25} = 2$. The orbits of $N^{(19)}$ are: $\{1,5,3,9,7\}$ and $\{2,4,10,6,8\}$. Choosing a representative from each orbit, we have two possible new double cosets, but after further investigation we find that: $Nt_1t_9t_1 = Nt_1$, therefore 5 orbits will go back to [1]. We also find that $Nt_1t_9t_2 = Nt_1t_2t_4$, which means that this double coset will collapse. Since there are no more orbits left to check, the double coset [19] ends here.

We now go back to the new double cosets that extended from [12]. Consider $Nt_1t_2t_4N$ denoted as [124]. $N^{124} =< e >$. Consider the relation $Nt_1t_2t_4 = Nt_1t_4t_6 = Nt_1t_6t_8 = Nt_1t_8t_{10} = Nt_1t_{10}t_2$. Any element that sends $Nt_1t_2t_4$ to any of the following: $Nt_1t_4t_6, Nt_1t_6t_8, Nt_1t_8t_{10}$ or $Nt_1t_{10}t_2$ will be in the coset stabilizing group $N^{(124)}$. We find a total of four elements that do so. For example $Nt_1t_2t_4^{(2,10,8,6,4)} = Nt_1t_{10}t_2$, therefore $N^{(124)} \geq < N^{124}, (2,4,6,8,10), (2,6,10,4,8)$, $(2,8,4,10,6), (2,10,8,6,4) >$. The number of single cosets in [124] are $\frac{|N|}{|N^{(124)}|} = \frac{50}{5} = 10$. The orbits of [124] are:$\{1\}, \{3\}, \{5\}, \{9\}, \{2,4,6,8,10\}$. After investigating, we find that $Nt_1t_2t_4t_2 = Nt_1t_2$, so five orbits go back to [12]. Also

$Nt_1t_2t_4t_5 = Nt_1t_2t_{10}t_7$, and $Nt_1t_2t_4t_3 = Nt_1t_9$ therefore two of these double cosets collapse. We have three new double cosets extending from [124] which are: $Nt_1t_2t_4t_1, Nt_1t_2t_4t_7$ and $Nt_1t_2t_4t_9$, each extending with a single orbit from [124].

Consider $Nt_1t_2t_6N$ denoted as [126] which extended from [12]. Now, $N^{126} =< e >$. Consider the relation $Nt_1t_2t_6 = Nt_1t_4t_8 = Nt_1t_6t_{10} = Nt_1t_8t_2$ $= Nt_1t_{10}t_4$. Any element that sends $Nt_1t_2t_6$ to any of the following: $Nt_1t_4t_8$, $Nt_1t_6t_{10}, Nt_1t_8t_2$ or $Nt_1t_{10}t_4$ will be in the coset stabilizing group $N^{(126)}$. We find four elements that do so, therefore $N^{(126)} \geq< N^{126}, (2, 4, 6, 8, 10)... >$. The number of single cosets in [126] are $\frac{|N|}{|N^{(126)}|} = \frac{50}{5} = 10$. The orbits of [126] are:$\{1\}, \{3\}, \{5\}, \{9\}, \{2, 4, 6, 8, 10\}$. After investigating we find that $Nt_1t_2t_6t_2 = Nt_1t_2$, so five orbits go back to [12]. Also $Nt_1t_2t_6t_3 = Nt_1t_2t_4t_9, Nt_1t_2t_6t_5 = Nt_1t_7, Nt_1t_2t_6t_7 = Nt_1t_2t_{10}t_5$ and $Nt_1t_2t_6t_9 = Nt_1t_2t_8t_3$ therefore four double cosets collapse. We have only one new double coset extending from [126], $Nt_1t_2t_4t_1$ which extends with a single orbit of $\{1\}$.

Consider $Nt_1t_2t_8N$ denoted as [128] which extended from [12]. Now, $N^{128} =< e >$. Consider the relation $Nt_1t_2t_8 = Nt_1t_4t_{10} = Nt_1t_6t_2 = Nt_1t_8t_4 = Nt_1t_{10}t_6$. Any element that sends $Nt_1t_2t_8$ to any of the following: $Nt_1t_4t_{10}, Nt_1t_6t_2$, $Nt_1t_8t_4$ or $Nt_1t_{10}t_6$ will be in the coset stabilizing group $N^{(128)}$. We find four elements that do so, therefore $N^{(128)} \geq< N^{128}, (2, 4, 6, 8, 10)... >$. The number of single cosets in [128] are $\frac{|N|}{|N^{(128)}|} = \frac{50}{5} = 10$. The orbits of [128] are:$\{1\}, \{3\}, \{5\}, \{7\}$, $\{9\}, \{2, 4, 6, 8, 10\}$. After investigating we find that $Nt_1t_2t_8t_2 = Nt_1t_2$, so five orbits go back to [12]. Also $Nt_1t_2t_8t_5 = Nt_1t_2t_4t_7$, and $Nt_1t_2t_8t_7 = Nt_1t_5$ therefore two of these double cosets collapse. We have three new double cosets extending from [128] which are: $Nt_1t_2t_8t_1, Nt_1t_2t_8t_3$ and $Nt_1t_2t_8t_9$, each extending with a single orbit of $\{1\}, \{3\}$ and $\{9\}$.

Consider $Nt_1t_2t_{10}N$ denoted as [1210] which extended from [12]. Now, $N^{1210} =< e >$. Consider the relation $Nt_1t_2t_{10} = Nt_1t_4t_2 = Nt_1t_6t_4 = Nt_1t_8t_6 = Nt_1t_{10}t_8$. Any element that sends $Nt_1t_2t_{10}$ to any of the following: $Nt_1t_4t_2, Nt_1t_6t_4$, $Nt_1t_8t_6$ or $Nt_1t_{10}t_8$ will be in the coset stabilizing group $N^{(1210)}$. We find four elements that do so, therefore $N^{(1210)} \geq< N^{1210}, (2, 4, 6, 8, 10)... >$. The number of single cosets in [1210] are $\frac{|N|}{|N^{(1210)}|} = \frac{50}{5} = 10$. The orbits of [1210] are:$\{1\}, \{3\}, \{5\}, \{9\}, \{2, 4, 6, 8, 10\}$. After investigating, we find that $Nt_1t_2t_{10}t_2 = Nt_1t_2$, so five orbits go back to [12]. Also, $Nt_1t_2t_{10}t_3 = Nt_1t_2t_8t_9$, and $Nt_1t_2t_{10}t_9 = Nt_1t_3$ therefore two of these double cosets collapse. We have three new double cosets extending from [1210] which are: $Nt_1t_2t_{10}t_1, Nt_1t_2t_{10}t_5$ and $Nt_1t_2t_{10}t_7$, each extending with a single orbit of $\{1\}, \{5\}$ and $\{7\}$.

Consider $Nt_1t_2t_4t_1N$ denoted as [1241] which extended from [124]. Now, $N^{1241} =< e >$. We find fourty nine equal names of $Nt_1t_2t_4t_1$. Any element that sends $Nt_1t_2t_4t_1$ to any of the forty nine equal names found will be in the coset stabilizing group $N^{(1241)}$. For example, consider the relation $Nt_1t_2t_4t_1 = Nt_1t_4t_6t_1$. Conjugation by the following element gives the desired result: $Nt_1t_2t_4t_1^{(2,4,6,8,10)} = Nt_1t_4t_6t_1$. Simililarly we can find the the other forty eight elements that give us the remaining relations. Then, $N^{(1241)} \geq < N^{1241}, (2, 4, 6, 8, 10)... >$. Therefore the number of single cosets in [1241] are $\frac{|N|}{|N^{(1241)}|} = \frac{50}{50} = 1$. The orbits of [1241] are: $\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}$. After investigating we find that $Nt_1t_2t_4t_1 = Nt_1t_2t_4$, and $Nt_1t_2t_4t_2 = Nt_1t_2t_4$, so ten orbits go back to [124]. Therefore, the double coset [1241] does not continue.

Consider $Nt_1t_2t_4t_7N$ denoted as [1247] which extended from [124]. Now, $N^{1247} =< e >$. We find twenty four equal names of $Nt_1t_2t_4t_7$. Any element that sends $Nt_1t_2t_4t_7$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(1247)}$. We find twenty four elements that do so. Therefore, $N^{(1241)} \geq < N^{1241}, (1, 9, 7, 5, 3)(2, 4, 6, 8, 10)... >$. The number of single cosets in [1241] are $\frac{|N|}{|N^{(1247)}|} = \frac{50}{25} = 2$. The orbits of [1247] are:$\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8, 10\}$. After investigating, we find that $Nt_1t_2t_4t_7t_1 = Nt_1t_2t_4$, therefore five orbits return to [124]. Also, $Nt_1t_2t_4t_7t_2 = Nt_1t_2t_8$, which implies that the double coset $Nt_1t_2t_4t_7t_2$ collapses.

Consider $Nt_1t_2t_4t_9N$ denoted as [1249] which extended from [124]. Now, $N^{1249} =< e >$. We find twenty four equal names of $Nt_1t_2t_4t_9$. Any element that sends $Nt_1t_2t_4t_9$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(1249)}$. We find twenty four elements that do so. Then, $N^{(1249)} \geq < N^{1249}, (2, 4, 6, 8, 10)... >$. Therefore, the number of single cosets in

[1249] are $\frac{|N|}{|N^{(1249)}|} = \frac{50}{25} = 2$. The orbits of [1249] are:$\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}$. After investigating, we find that $Nt_1t_2t_4t_9t_1 = Nt_1t_2t_4$, therefore five orbits return to [124]. Also, $Nt_1t_2t_4t_9t_2 = Nt_1t_2t_6$, which implies that the double coset $Nt_1t_2t_4t_9t_2$ collapses. Since there are no more possible new double cosets to investigate, [124] does not expand any further. Now we continue to the double coset [126] and investigate further.

Consider $Nt_1t_2t_6t_1N$ denoted as [1261] which extended from [126]. Now, $N^{1261} =< e >$. We find forty nine equal names of $Nt_1t_2t_6t_1$. Any element that sends $Nt_1t_2t_6t_1$ to any of the forty nine equal names found will be in the coset stabilizing group $N^{(1261)}$. We find forty nine elements that do so. Then, $N^{(1261)} \geq< N^{1261}, (1, 7, 3, 9, 5)(2, 10, 8, 6, 4)... >$. Therefore, the number of single cosets in [1261] are $\frac{|N|}{|N^{(1261)}|} = \frac{50}{50} = 1$. The orbits of [1261] are:$\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}$. After investigating, we find that $Nt_1t_2t_6t_1t_1 = Nt_1t_2t_6$, and $Nt_1t_2t_6t_1t_2 = Nt_1t_2t_6$. Therefore, all ten orbits return to [126] and thus the double coset [1261] does not extend any further.

Consider $Nt_1t_2t_8t_1N$ denoted as [1281] which extended from [128]. Now, $N^{1281} =< e >$. We find forty nine equal names of $Nt_1t_2t_8t_1$. Any element that sends $Nt_1t_2t_8t_1$ to any of the forty nine equal names found will be in the coset stabilizing group $N^{(1281)}$. We found forty nine elements that do so. Then, $N^{(1281)} \geq< N^{1281}, (2, 4, 6, 8, 10)... >$. Therefore, the number of single cosets in [1281] are $\frac{|N|}{|N^{(1281)}|} = \frac{50}{50} = 1$. The orbits of [1281] are:$\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}$. After investigating, we find that $Nt_1t_2t_8t_1t_1 = Nt_1t_2t_8$, and $Nt_1t_2t_8t_1t_2 = Nt_1t_2t_8$. Therefore, all ten orbits return to [128] and thus the double coset [1281] does not extend any further.

Consider $Nt_1t_2t_8t_3N$ denoted as [1283] which extended from [128]. Now, $N^{1283} =< e >$. We find twenty four equal names of $Nt_1t_2t_8t_3$. Any element that sends $Nt_1t_2t_8t_3$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(1283)}$. We find twenty four elements that do so, then, $N^{(1283)} \geq< N^{1283}, (1, 3, 5, 7, 9)... >$. Therefore, the number of single cosets in [1283] are $\frac{|N|}{|N^{(1283)}|} = \frac{50}{25} = 2$. The orbits of [1283] are: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$, $\{10\}$. After investigating we find that $Nt_1t_2t_8t_3t_1 = Nt_1t_2t_8t_3t_3 = Nt_1t_2t_8t_3t_5 =$

$Nt_1t_2t_8t_3t_7 = Nt_1t_2t_8t_3t_9 = Nt_1t_2t_8$ and $Nt_1t_2t_8t_3t_2 = Nt_1t_2t_8t_3t_4 = Nt_1t_2t_8t_3t_6 = Nt_1t_2t_8t_3t_8 = Nt_1t_2t_8t_3t_{10} = Nt_1t_2t_6$. Therefore five orbits return to [128] and the remaining five double cosets collapse, respectively.

Consider $Nt_1t_2t_8t_9N$ denoted as [1289] which extended from [128]. Now, $N^{1289} =< e >$. We find twenty four equal names of $Nt_1t_2t_8t_9$. Any element that sends $Nt_1t_2t_8t_9$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(1289)}$. We find twenty four elements that do so, then, $N^{(1289)} \geq< N^{1289}, (1, 3, 5, 7, 9)... >$. Therefore the number of single cosets in [1289] are $\frac{|N|}{|N^{(1289)}|} = \frac{50}{25} = 2$. The orbits of [1289] are:$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}$. After investigating, we find that $Nt_1t_2t_8t_3t_1 = Nt_1t_2t_8t_3t_3 = Nt_1t_2t_8t_3t_5 = Nt_1t_2t_8t_3t_7 = Nt_1t_2t_8t_3t_9 = Nt_1t_2t_8$ and $Nt_1t_2t_8t_3t_2 = Nt_1t_2t_8t_3t_4 = Nt_1t_2t_8t_3t_6 = Nt_1t_2t_8t_3t_8 = Nt_1t_2t_8t_3t_{10} = Nt_1t_2t_{10}$. Therefore five orbits return to [128] and the remaining five double cosets collapse.

Consider $Nt_1t_2t_{10}t_1N$ denoted as [12101] which extended from [1210]. Now, $N^{12101} =< e >$. We find forty nine equal names of $Nt_1t_2t_{10}t_1$. Any element that sends $Nt_1t_2t_{10}t_1$ to any of the forty nine equal names found will be in the coset stabilizing group $N^{(12101)}$. We find forty nine elements that do so, then, $N^{(12101)} \geq< N^{12101}, (1, 3, 5, 7, 9)(2, 4, 6, 8, 10)... >$. Therefore the number of single cosets in [12101] are $\frac{|N|}{|N^{(12101)}|} = \frac{50}{50} = 1$. The orbits of [12101] are:$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}$. After investigating we find that $Nt_1t_2t_{10}t_1t_1 = Nt_1t_2t_{10}t_1t_2 = Nt_1t_2t_{10}t_1t_3 = Nt_1t_2t_{10}t_1t_4 = Nt_1t_2t_{10}t_1t_5 = Nt_1t_2t_{10}t_1t_6 = Nt_1t_2t_{10}t_1t_7 = Nt_1t_2t_{10}t_1t_8 = Nt_1t_2t_{10}t_1t_9 = Nt_1t_2t_{10}t_1t_{10} = Nt_1t_2t_{10}$. Therefore, all ten orbits return to [1210].

Consider $Nt_1t_2t_{10}t_5N$ denoted as [12105] which extended from [1210]. Now, $N^{12105} =< e >$. We find twenty four equal names of $Nt_1t_2t_{10}t_5$. Any element that sends $Nt_1t_2t_{10}t_5$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(12105)}$. We find twenty four elements that do so, then, $N^{(12105)} \geq< N^{12105}, (2, 8, 4, 10, 6)... >$. Therefore the number of single cosets in [12105] are $\frac{|N|}{|N^{(12105)}|} = \frac{50}{25} = 2$. The orbits of [12105] are:$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}$. After investigating we find that $Nt_1t_2t_{10}t_5t_1 = Nt_1t_2t_{10}t_5t_3 = Nt_1t_2t_{10}t_5t_5 = Nt_1t_2t_{10}t_5t_7 = Nt_1t_2t_{10}t_5t_9 = Nt_1t_2t_{10}$. and $Nt_1t_2t_{10}t_5t_2$

$= Nt_1t_2t_{10}t_5t_4 = Nt_1t_2t_{10}t_5t_6 = Nt_1t_2t_{10}t_5t_8 = Nt_1t_2t_{10}t_5t_{10} = Nt_1t_2t_6$. Therefore five orbits return to [1210] and the remaining five double cosets collapse.

Consider $Nt_1t_2t_{10}t_7N$ denoted as [12107] which extended from [1210]. Now, $N^{12107} =< e >$. We find twenty four equal names of $Nt_1t_2t_{10}t_7$. Any element that sends $Nt_1t_2t_{10}t_7$ to any of the twenty four equal names found will be in the coset stabilizing group $N^{(12107)}$. We find twenty four elements that do so, then, $N^{(12107)} \geq< N^{12107}, (1,3,5,7,9)(2,4,6,8,10)... >$. Therefore, the number of single cosets in [12107] are $\frac{|N|}{|N^{(12107)}|} = \frac{50}{25} = 2$. The orbits of [12107] are: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}$. After investigating we find that $Nt_1t_2t_{10}t_7t_1 = Nt_1t_2t_{10}t_7t_3 = Nt_1t_2t_{10}t_7t_5 = Nt_1t_2t_{10}t_7t_7 = Nt_1t_2t_{10}t_7t_9 = Nt_1t_2t_{10}$. and $Nt_1t_2t_{10}t_7t_2 = Nt_1t_2t_{10}t_7t_4 = Nt_1t_2t_{10}t_7t_6 = Nt_1t_2t_{10}t_7t_8 = Nt_1t_2t_{10}t_7t_{10} = Nt_1t_2t_4$. Therefore five orbits return to [1210] and the remaining five double cosets collapse. Since there are no new possible new double cosets to investigate, our group is closed under right multiplication of $t_i's$. The construction of this Caley Diagram is complete. The order of G over N is 125, which is verified through adding the number of single cosets found in the double cosets as illustrated in our Caley Diagram below:

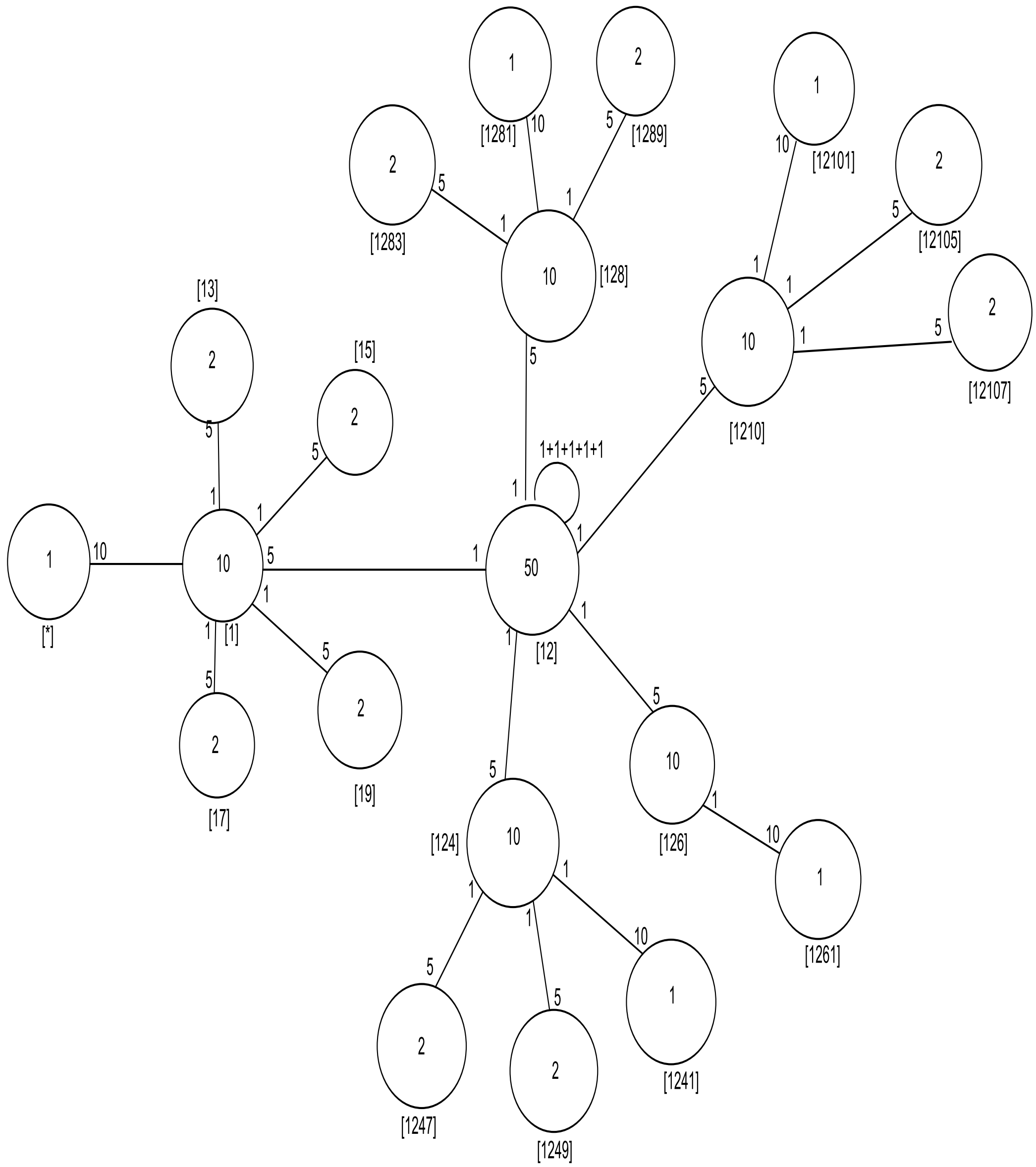

Figure 4.2: Cayley graph of $5^4 : D_{10}$

Chapter 5

Double Coset Enumeration Over Maximal Subgroups

5.1 Construction of $U(3,4)$ over $M = A_5 : C_5$

Definition 5.1. *(Maximal Subgroup): A subgroup $M \neq 1 \leq G$ is a maximal normal subgroup of G if there is no normal subgroup N of G with $M < N < G$.*

A symmetric presentation for the group $G = 2^{*10} : (A_5 : C_5)$ is given by: $< x, y, t | x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y, t^3, (t, x), (y * x^{-2} * y * x^{-1} * t)^2, (x^{-2} * y * x^{-1} * t)^5 >$, where $N =< x, y >, x \sim (2,4,6,8,10), y \sim (1,6)(2,7)(3,8)(4,9)(5,10)$, and $t \sim t_1$. We will prove that the above progenitor factored by the following relations: $(y * x^{-2} * y * x^{-1} * t)^2$ and $(x^{-2} * y * x^{-1} * t)^5$, gives $U(3,4)$. Note that our $t_{i's}$ will be of order 3 in this case, but the process of double coset enumeration is the same. Normally we would perform double coset enumeration of G over N, but this would result in a Caley diagram with 34 double cosets in our case. Therefore to complete the process of double coset enumeration on such a large group, we find a maximal subgroup, such that $N < M < G$. We found a maximal subgroup M generated by our control group $N = 5^2 : 2$ where $M = f(x), f(y), f(t^{-1} * y * x^{-1} * t^{-1} * y * t^{-1} * y * x * y * t^{-1} * y * t * y * t)$ which is isomorphic to $A_5 : C_5$.

Double Coset Enumeration To construct a manual double coset enumeration of G over the maximal subgroup M and N, we denote $[w]$ to be the double coset $MwN = \{Mw^n | n \in N\}$, where w is a word in $t_{i's}$.

MeN

We begin with the double coset MeN, denoted $[*]$, which is equal to $\{Me^n | e \in N\} = \{Me | e \in N\} = \{M\}$. Here the coset representative for $[*]$ is M. Since N is transitive on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}\}$, it contains a single orbit:

$$\mathcal{O} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

The number of distinct single cosets in $[*]$ is determined by dividing the number of N by the coset stabiliser of MeN, which is all N. Thus, $\frac{|N|}{|N|} = \frac{50}{50} = 1$ We right multiply M by a representative of the single orbit, and in this case we chose 1. Thus we have a new double coset Mt_1N, which we will denote as [1].

$\mathbf{Mt_1N}$

Next we will investigate the double coset $[1] = \{Mt_1^n | n \in N\} = \{Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7, Mt_8, Mt_9, Mt_{10}\}$. The coset stabiliser group $N^{(1)}$ is equal to the point stabiliser of N, denoted N^1. The point stabiliser is all such elements that fix 1. In this case, $N^{(1)} = N^1 =< (2, 4, 6, 8, 10) >$. Thus, the number of single cosets of Mt_1N is at most $\frac{|N|}{|N^{(1)}|} = \frac{50}{5} = 10$. Therefore, 10 single cosets live in [1]. Now, the orbits of [1] are:

$$\mathcal{O} = \{\{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{2, 4, 6, 8, 10\}\},$$

which were found by looking at the genertaors of $N^{(1)}$. Now, as in the normal process of double coset enumeration, we must find where these orbits go by right multiplying Mt_1N by a representative from each orbit.We find that $Mt_1t_5 = Id$, $Mt_1t_7N = Mt_1t_3$ and $Mt_1t_9N = Mt_1t_1 = Mt_1N$. Therefore two symmetric generators will loop back into [1], one will return to $[*]$. Two symmetric generators move forward to the new double cosets Mt_1t_3N, and five symmetric generators move forward to Mt_1t_2N.

$\mathbf{Mt_1t_3N}$

Consider the double coset $[13] = Mt_1t_3N = \{M(t_1t_3)^n|n \in N\} = \{Mt_1t_3, Mt_6t_8, Mt_8t_{10}, Mt_{10}t_2, Mt_3t_5, Mt_2t_4, Mt_5t_7, Mt_4t_6, Mt_7t_9, Mt_9t_1\}$. The coset stabiliser of the double coset Mt_1t_3, which we will denote [13] is : $N^{13} \geq N^{(13)} = \{e, (2,4,6,8,10)\}$. Thus, the number of single cosets of Mt_1t_3N is at most $\frac{|N|}{|N^{(13)}|} = \frac{50}{5} = 10$. Now, the orbits of [13] are:

$$\mathcal{O} = \{\{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{2,4,6,8,10\}\},$$

which were found by looking at the generators of $N^{(13)} = (2,4,6,8,10)$. Choosing a representative from each orbit and right multiplying to Mt_1t_3, we have 6 possible new double cosets, but we find the following:

$$\begin{aligned}
Mt_1t_3t_1 &= Mt_1 \in [1]\\
Mt_1t_3t_3 &= Mt_1t_3 \in [13]\\
Mt_1t_3t_5 &= Mt_1t_3 \in [13]\\
Mt_1t_3t_7 &= Mt_1 \in [1]
\end{aligned}$$

Based on the information found, two symmetric generators return to [1] and two loop back into [13]. Now the remaining orbits form two new double cosets which are: $Mt_1t_3t_2$ and $Mt_1t_3t_9$.

$\mathbf{Mt_1t_3t_2N}$

Consider the double coset $[132] = Mt_1t_3t_2N = \{M(t_1t_3t_2)^n|n \in N\} = \{Mt_1t_3t_2, Mt_1t_3t_4, Mt_6t_8t_7, Mt_1t_3t_6, Mt_6t_8t_9, Mt_8t_{10}t_7, Mt_1t_3t_8, Mt_6t_8t_1, Mt_8t_{10}t_9, Mt_{10}t_2t_7, Mt_{10}t_2t_7, Mt_3t_5t_2, Mt_1t_3t10, Mt_6t_8t_3, Mt_8t_{10}t_1, Mt_{10}t_2t_9, Mt_3t_5t_4, Mt_2t_4t_7, Mt_5t_7t_2, Mt_6t_8t_5, Mt_8t_{10}t_3, Mt_{10}t_2t_1, Mt_3t_5t_6, Mt_2t_4t_9, Mt_5t_7t_4, Mt_4t_6t_7, Mt_7t_9t_2, Mt_8t_{10}t_5, Mt_{10}t_2t_3, Mt_3t_5t_8, Mt_2t_4t_1, Mt_5t_7t_6, Mt_4t_6t_9, Mt_7t_9t_4, Mt_9t_1t_2, Mt_{10}t_2t_5, Mt_3t_5t_{10}, Mt_2t_4t_3, Mt_5t_7t_8, Mt_4t_6t_1, Mt_7t_9t_6, Mt_9t_1t_4, Mt_2t_4t_5, Mt_5t_7t_{10}, Mt_4t_6t_3, Mt_7t_9t_8, Mt_9t_1t_6, Mt_4t_6t_5, Mt_7t_9t_{10}, Mt_9t_1t_8, Mt_9t_1t_{10}\}$. We find there exists a relation that sends $Mt_1t_3t_2 = Mt_6t_8t_7$. Thus the coset stabiliser will increase since any relation that sends $t_1 \rightarrow t_6, t_3 \rightarrow t_8$, and $t_2 \rightarrow t_7$ will be in the coset stabiliser. The coset stabiliser of the double coset $Mt_1t_3t_2$ is : $N^{132} \geq N^{(132)} = \{e, (1,6)(2,7)(3,8)(4,9)(5,10)\}$. Thus the number of single cosets of $Mt_1t_3t_2N$ is at most $\frac{|N|}{|N^{(132)}|} = \frac{50}{2} = 25$. Now, the orbits of [132] are:

$$\mathcal{O} = \{\{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}\},$$

which were found by looking at the generators of $N^{(132)} = (1,6)(2,7)(3,8)(4,9)$ $(5,10)$. Choosing a representative from each orbit and right multiplying to $Mt_1t_3t_2$, we have 5 possible new double cosets, but we find the following using **MAGMA**:

$$\begin{aligned}
Mt_1t_3t_2t_1 &= Mt_1t_3 \in [13]\\
Mt_1t_3t_2t_2 &= Mt_1t_3t_2 \in [132]\\
Mt_1t_3t_2t_3 &= Mt_1t_2t_5 \in [125]\\
Mt_1t_3t_2t_4 &= Mt_1t_2t_5 \in [125]\\
Mt_1t_3t_2t_5 &= Mt_1t_3t_2 \in [132]
\end{aligned}$$

Based on the information found, two symmetric generators return to [13] and four loop back into [132]. The remaining double cosets collapse, since they live in another double coset not connected to this branch. Since there are no new possible double cosets to investigate we stop here and investigate other branches.

$\mathbf{Mt_1t_3t_9N}$

Consider the double coset $[139] = Mt_1t_3t_9N = \{M(t_1t_3t_9)^n | n \in N\} = \{Mt_1t_3t_9, Mt_6t_8t_4, Mt_8t_{10}t_6, Mt_{10}t_2t_8, Mt_3t_5t_1, Mt_2t_4t_{10}, Mt_5t_7t_3, Mt_4t_6t_2, Mt_7t_9t_5, Mt_9t_1t_7\}$. We find that $Mt_1t_3t_9 = Mt_3t_5t_1 = Mt_5t_7t_3 = Mt_7t_9t_5 = Mt_9t_1t_7$. Thus, the coset stabiliser will increase since any relation that, for example, sends $t_1 \to t_3, t_3 \to t_5$, and $t_9 \to t_1$ will be in the coset stabiliser. Likewise, this will occur for the other double cosets that $Mt_1t_3t_9$ is equal to. The coset stabiliser of the double coset $Mt_1t_3t_9$ is : $N^{139} \geq N^{(139)} = \{e, (2,4,6,8,10), (1,3,5,7,9)$ $...(1,3,5,7,9)(2,4,6,8,10)\}$. Thus the number of single cosets of $Mt_1t_3t_9N$ is at most $\frac{|N|}{|N^{(139)}|} = \frac{50}{25} = 2$. Now the orbits of [139] are:

$$\mathcal{O} = \{\{1, 3, 5, 7, 9\}, \{2, 4, 10, 6, 8\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_3t_9$, we have 2 possible new double cosets, but we find there exists a relation that shows:

$$Mt_1t_3t_9t_9 = Mt_1t_3 \in [13]$$
$$Mt_1t_3t_9t_2 = Mt_1t_3t_9 \in [139]$$

Based on the information found, five symmetric generators return to [13] and five symmetric generators loop back into [139]. Since there are no more symmetric generators, this branch ends here.

$\mathbf{Mt_1t_2N}$

Consider the double coset $[12] = Mt_1t_2N = \{M(t_1t_2)^n|n \in N\} = \{Mt_1t_2, Mt_1t_4, ...Mt_9t_{10}\}$. Here we omit to list the 50 single cosets that live inside this double coset Mt_1t_2. The coset stabiliser of the double coset Mt_1t_2 is : $N^{12} \geq N^{(12)} = \{e\}$. Thus, the number of single cosets of Mt_1t_2N is at most $\frac{|N|}{|N^{(12)}|} = \frac{50}{1} = 50$. Now the orbits of [12] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\},$$

Choosing a representative from each orbit and right multiplying to Mt_1t_2, we have 10 possible new double cosets, but we find the following:

$$Mt_1t_2t_2 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_6 = Mt_1 \in [1]$$
$$Mt_1t_2t_7 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_{10} = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_3 = Mt_1t_2t_1 \in [123]$$
$$Mt_1t_2t_8 = Mt_1t_2t_4 \in [124]$$
$$Mt_1t_2t_9 = Mt_1t_2t_5 \in [125]$$

Based on the information found, one symmetric generator returns to [1] and three symmetric generators loop back into [12]. The information above results in three new double cosets $Mt_1t_2t_1, Mt_1t_2t_4$ and $Mt_1t_2t_5$ in which two symmetric generators will move forward to each double coset.

$\mathbf{Mt_1t_2t_1N}$

Consider the double coset $[121] = \{M(t_1t_2t_1)^n|n \in N\} = \{Mt_1t_2t_1,$ $Mt_1t_4t_1, ...Mt_9t_{10}t_9\}$. We omit the complete list of 50 single cosets that live in [121]. We find that $Mt_1t_2t_1 = Mt_{10}t_3t_{10}$. Thus, the coset stabiliser will increase since any relation that sends $t_1 \to t_{10}, t_2 \to t_3$, and $t_9 \to t_{10}$ will be in the coset stabiliser. The coset stabiliser of the double coset $Mt_1t_2t_1$ is : $N^{121} \geq N^{(121)} = \{e, (1, 10)(2, 3)(4, 5)(6, 7)(8, 9)\}$. Thus the number of single cosets of $Mt_1t_2t_1N$ is at most $\frac{|N|}{|N^{(121)}|} = \frac{50}{2} = 25$. Now, the orbits of [121] are:

$$\mathcal{O} = \{\{1, 10\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1$, we have 5 possible new double cosets, but we find there exists a relation that verifies the following:

$$Mt_1t_2t_1t_1 = Mt_1t_2t_1t_2 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_1t_4 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_1t_6 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_1t_8 = Mt_1t_2 \in [12]$$

Based on the information found above, two symmetric generators loop back into [121] and four return to [12]; the rest collapse. Since there are no more symmetric generators to investigate, this branch ends here.

$\mathbf{Mt_1t_2t_4N}$

Consider the double coset $[124] = \{M(t_1t_2t_4)^n|n \in N\} = \{Mt_1t_2t_4,$ $Mt_1t_4t_6, ...Mt_9t_{10}t_2\}$. We omit the complete list of 50 single cosets that live in

[124]. We find there exists a relation that sends $Mt_1t_2t_4 = Mt_6t_7t_9$. Thus, the coset stabiliser will increase since any relation that sends $t_1 \to t_6, t_2 \to t_7$, and $t_4 \to t_9$ will be in the coset stabiliser. The coset stabiliser of the double coset $Mt_1t_2t_4$ is : $N^{124} \geq N^{(124)} = \{e, (1,6)(2,7)(3,8)(4,9)(5,10)\}$. Thus the number of single cosets of $Mt_1t_2t_4N$ is at most $\frac{|N|}{|N^{(124)}|} = \frac{50}{2} = 25$. Now the orbits of [124] are:

$$\mathcal{O} = \{\{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_4$, we have five possible new double cosets, but we find the following:

$$Mt_1t_2t_4t_1 = Mt_1t_2t_4t_4 = Mt_1t_2t_4 \in [124]$$
$$Mt_1t_2t_4t_2 = Mt_1t_2t_4t_3 = Mt_1t_2 \in [12]$$

Based on the information found above, four symmetric generators loop back into [124], and four return to [12]. $Mt_1t_2t_4t_5N$ is a new double coset, therefore two symmetric generators move forward.

$\mathbf{Mt_1t_2t_5N}$

Consider the double coset $[125] = \{M(t_1t_2t_5)^n | n \in N\} = \{Mt_1t_2t_5, ...Mt_9t_{10}t_3\}$. We omit the complete list of single the single cosets that live in [125], but there exist a total of 50 of these. The coset stabiliser of the double coset $Mt_1t_2t_1$ is : $N^{125} \geq N^{(125)} = \{e\}$. Thus, the number of single cosets of $Mt_1t_2t_5N$ is at most $\frac{|N|}{|N^{(125)}|} = \frac{50}{1} = 50$. Now, the orbits of [125] are:

$$\mathcal{O} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_5$, we

have 10 possible new double cosets, but we find the following:

$$Mt_1t_2t_5t_1 = Mt_1t_2t_5t_9 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_5t_2 = Mt_1t_2t_5t_8 = Mt_1t_3t_2 \in [132]$$
$$Mt_1t_2t_5t_3 = Mt_1t_2t_5t_7 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_5t_4 = Mt_1t_2t_5t_5 = Mt_1t_2t_5t_6 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_5t_{10} = Mt_1t_2t_4t_5 \in [1245]$$

Based on the information found above, 3 symmetric generators loop back into [125], and two return to [12]. The remaining double cosets collapse. Thus, this branch ends here.

$\mathbf{Mt_1t_2t_4t_5N}$

Consider the double coset $[1245] = Mt_1t_2t_4t_5N = \{M(t_1t_2t_4t_5)^n | n \in N\} = \{Mt_1t_2t_4t_5, ...Mt_9t_{10}t_2t_3\}$. The coset stabiliser of the double coset $Mt_1t_2t_4t_5$ is : $N^{1245} \geq N^{(1245)} = \{e, (1,3,5,7,9)(2,4,6,8,10), (1,5,9,3,7)(2,6,10,4,8), (1,7,3,9,5)$ $(2,8,4,10,6), (1,9,7,5,3)(2,10,8,6,4)\}$. We find that $Mt_1t_2t_4t_5 = Mt_3t_4t_6t_7 = Mt_5t_6t_8t_9 = Mt_7t_8t_{10}t_1 = Mt_9t_{10}t_2t_3$. Thus the coset stabiliser will increase since any relation that, for example, sends $t_1 \to t_3, t_2 \to t_4, t_4 \to t_6$ and $t_5 \to t_7$ will be in the coset stabiliser. Thus, the number of single cosets of $Mt_1t_2t_4t_5N$ is at most $\frac{|N|}{|N^{(1245)}|} = \frac{50}{5} = 10$. Now, the orbits of [1245] are:

$$\mathcal{O} = \{\{1,3,5,7,9\}, \{2,4,10,6,8\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_4t_5$, we have two possible new double cosets, but we find the following:

$$Mt_1t_2t_4t_5t_1 = Mt_1t_2t_4 \in [124]$$
$$Mt_1t_2t_4t_5t_2 = Mt_1t_2t_5 \in [125]$$

Based on the information found, five symmetric generators return to [124] and five loop back into [1245]. Since there are no more orbits to investigate, this branch ends here. Thus we have shown that this group is closed under right multiplication since the index of $\frac{|G|}{|M|}$ tells us we should have 208 single cosets all together. Summing up all the single cosets in each double coset yields the desired result. A Caley graph illustrating the results is provided below:

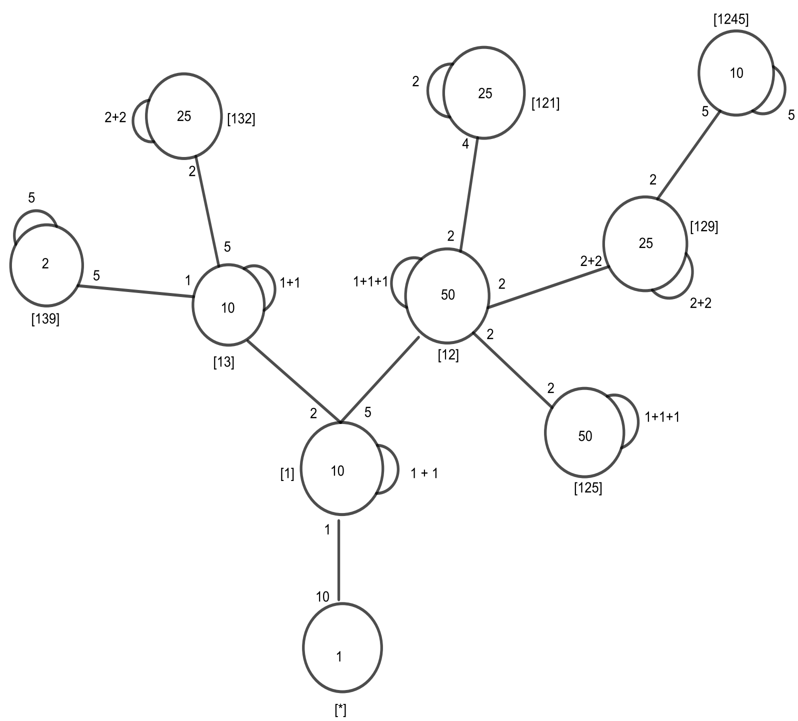

Figure 5.1: Cayley graph of $U(3,4)$ over $A_5 : C_5$

5.2 Construction of J_2 over $M = (10:2):A_5$

Definition 5.2. *(Maximal Subgroup): A subgroup $M \neq 1 \leq G$ is a maximal normal subgroup of G if there is no normal subgroup N of G with $M < N < G$.*

A presentation for the group $G = 2^{*10} : ((10:2):A_5)$ is given by:

$G < x, y, t >:= Group < x, y, t | x^5, y^2, x^{-1} * y * x^{-1} * y * x * y * x * y, (t, x), t^2,$
$(y * x^{-2} * y * x^{-1} * t)^3, (x^{-1} * y * t)^6 >,$

where $N =< x, y >, x \sim (2,4,6,8,10), y \sim (1,6)(2,7)(3,8)(4,9)(5,10)$, and $t \sim t_1$. We will prove that the above progenitor factored by the following relations: $(y * x^{-2} * y * x^{-1} * t)^3$, and $(x^{-1} * y * t)^6$, gives J_2 of order 1209600.
Let $\phi = (y * x^{-2} * y * x^{-1}) = (1,7,3,9,5)(2,10,8,6,4)$, then we have $(\phi t_1)^3 = (1,9,7,5,3)(2,6,10,4,8)$. Likewise, we let $\beta = (x^{-1} * y * t) = (1,6,9,4,7,2,5,10,3,8)$, then we obtain $(\beta t_1)^6 = (1,5,9,3,7)(2,6,10,4,8)$.
Now each relation will be expanded as follows:

$$1 = (\phi * t_1)^3$$
$$1 = \phi^3 t_1^{\phi^2} t_1^{\phi} t_1$$
$$1 = (1,9,7,5,3)(2,6,10,4,8)t_3 t_7 t_1$$
$$t_1 = (1,9,7,5,3)(2,6,10,4,8)t_3 t_7$$

We will denote $t_1 = (1,9,7,5,3)(2,6,10,4,8)t_3t_7$ as relation (1). Likewise,

$$1 = (\beta * t_1)^6$$
$$1 = \beta^6 t_1^{\beta^5} t_1^{\beta^4} t_1^{\beta^3} t_1^{\beta^2} t_1^{\beta} t_1$$
$$1 = (1,6,9,4,7,2,5,10,3,8)t_2t_7t_4t_9t_6t_1$$
$$t_1t_6t_9 = (1,6,9,4,7,2,5,10,3,8)t_2t_7t_4$$

$t_1t_6t_9 = (1,6,9,4,7,2,5,10,3,8)t_2t_7t_4$ will be denoted as relation (2). Normally we would perform double coset enumeration of G over N, but this would result in a Caley diagram with 512 double cosets in our case. Therefore to complete the process of double coset enumeration on such a large group, we find a maximal subgroup, such that $N < M < G$. We found a maximal subgroup M of order 1200 generated by our control group $N = 5^2 : 2$ where $M = f(x), f(y), f(y*t*y*x*t*y*t*y*x^{-1}*y*t*x^{-2}*y*x*t*y*t)$ which is isomorphic to $((10:2):A5)$.

Double Coset Enumeration To construct a manual double coset enumeration of G over the maximal subgroup M and N, we denote $[w]$ to be the double coset $MwN = \{Mw^n | n \in N\}$, where w is a word in $t_{i's}$.

MeN

We begin with the double coset MeN, denoted $[*]$, which is equal to $\{Me^n | e \in N\} = \{Me | e \in N\} = \{M\}$. Here the coset representative for $[*]$ is M. Since N is transitive on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}\}$, it contains a single orbit:

$$\mathcal{O} = \{1,2,3,4,5,6,7,8,9,10\}$$

The number of distinct single cosets in $[*]$ is determined by dividing the number of N by the coset stabiliser of MeN, which is all N. Thus, $\frac{|N|}{|N|} = \frac{50}{50} = 1$ We right multiply M by a representative of the single orbit, and in this case we chose 1. Thus we have a new double coset Mt_1N, which we will denote as [1].

$\mathbf{Mt_1N}$

Next we will investigate the double coset $[1] = \{Mt_1^n | n \in N\} = \{Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7, Mt_8, Mt_9, Mt_{10}\}$. The coset stabiliser group $N^{(1)}$ is equal to the point stabiliser of N, denoted N^1. The point stabiliser is all such elements that fix 1. In this case, $N^{(1)} = N^1 =< (2, 4, 6, 8, 10) >$. Thus the number of single cosets of Mt_1N is at most $\frac{|N|}{|N^{(1)}|} = \frac{50}{5} = 10$. Therefore 10 single cosets live in [1]. Now, the orbits of [1] are:

$$\mathcal{O} = \{\{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{2, 4, 6, 8, 10\}\},$$

which were found by looking at the generators of $N^{(1)}$. Now as in the normal process of double coset enumeration, we must find where these orbits go by right multiplying Mt_1N by a representative from each orbit. Since our $t_{i's}$ are of order 2, we find that $Mt_1t_1N = N \in [*]$, and one symmetric generator goes back to $[*]$.

Now by using the relations expanded above, namely (1), if we right multiply both sides of this relation by t_7 we have: $t_1t_7 = (1, 9, 7, 5, 3)(2, 6, 10, 4, 8)t_3$. This implies that $Mt_1t_7 = Mt_3$ since $(1, 9, 7, 5, 3)(2, 6, 10, 4, 8)$ gets absorbed by N, but Mt_3 is $\in$ [1] as we saw above, therefore the symetric generator $\{7\}$ will return to [1]. We also find that $Mt_1t_5 = Mt_1N$. To prove this, we will demonstrate the example for this case in particular. To prove that there exists a relation that sends Mt_1t_5 to Mt_1, we begin by conjugating our relation

$e = (1, 9, 7, 5, 3)(2, 6, 10, 4, 8)t_3t_7t_1$ by one of the transversals of [1], which were:

$$
\begin{aligned}
&\{e,\\
&(1,6)(2,7)(3,8)(4,9)(5,10),\\
&(1,8,3,10,5,2,7,4,9,6),\\
&(1,4,9,2,7,10,5,8,3,6),\\
&(1,10,5,4,9,8,3,2,7,6),\\
&(1,3,5,7,9),\\
&(1,9,7,5,3),\\
&(1,2,7,8,3,4,9,10,5,6),\\
&(1,5,9,3,7),\\
&(1,7,3,9,5)\}
\end{aligned}
$$

.

We do this to try to find a new relation that sends t_3t_7 to t_1t_5 to show that $Mt_1t_5 = Mt_1$. We find that conjugation by $(1, 9, 7, 5, 3)$ to our relation (1) yields the desired result. We will demonstrate the process here:

$$
\begin{aligned}
&e = (1,9,7,5,3)(2,6,10,4,8)t_3t_7t_1\\
&t_1 = (1,9,7,5,3)(2,6,10,4,8)t_3t_7\\
&(1,9,7,5,3)(2,6,10,4,8)^{((1,9,7,5,3)}t_3^{((1,9,7,5,3)}t_7^{((1,9,7,5,3)} = t_1^{((1,9,7,5,3)}\\
&(9,7,5,3,1)t_1t_5 = t_9
\end{aligned}
$$

We know that $Mt_9 \in [1]$, though, so therefore we have shown that $Mt_1t_5 = Mt_1$. Now, we find the $Mt_1t_3N = Mt_1t_9N, Mt_1t_5 = Mt_1t_7N = Mt_1N$, which implies that two symmetric generators loop back into [1]. Finally we find that both Mt_1t_9N and Mt_1t_2N are new double cosets.

$\mathbf{Mt_1t_9N}$

Consider the double coset $[19] = Mt_1t_9N = \{M(t_1t_9)^n | n \in N\} = \{Mt_1t_9, Mt_6t_4, Mt_8t_6, Mt_{10}t_8, Mt_3t_1, Mt_2t_{10}, Mt_5t_3, Mt_4t_2, Mt_7t_5, Mt_9t_7\}$. The coset stabiliser of the double coset Mt_1t_9, which we will denote [19] is : $N^{19} \geq N^{(19)} = \{e, (2, 4, 6, 8, 10)\}$. Thus the number of single cosets of Mt_1t_9N is at most $\frac{|N|}{|N^{(19)}|} = \frac{50}{5} = 10$. Now the orbits of [19] are:

$$\mathcal{O} = \{\{1\}, \{3\}, \{5\}, \{7\}, \{9\}, \{2, 4, 6, 8, 10\}\},$$

which were found by looking at the generators of $N^{(19)} = (2, 4, 6, 8, 10)$. Choosing a representative from each orbit and right multiplying to Mt_1t_9, we have 6 possible new double cosets, but we find the following using the techinique of conjugation by the transversals of [19] from above:

$$Mt_1t_9t_3 = Mt_1 \in [1]$$
$$Mt_1t_9t_5 = Mt_1t_9 \in [19]$$
$$Mt_1t_9t_7 = Mt_1t_9 \in [19]$$

Also, $Mt_1t_9t_9 = Mt_1 \in [1]$ since our $t_{i's}$ are of order 2. Based on the information found, two symmetric generators return to [1] and two loop back into [19]. Now the remaining orbits form two new double cosets which are: $Mt_1t_9t_2$ and $Mt_1t_9t_1$.

$\mathbf{Mt_1t_9t_1N}$

Consider the double coset $[191] = Mt_1t_9t_1N = \{M(t_1t_9t_1)^n | n \in N\} = \{Mt_1t_9t_1, Mt_6t_4t_6, Mt_8t_6t_8, Mt_{10}t_8t_{10}, Mt_3t_1t_3, Mt_2t_{10}t_2, Mt_5t_3t_5, Mt_4t_2t_4, Mt_7t_5t_7, Mt_9t_7t_9\}$. We find through **MAGMA** that $Mt_1t_9t_1 = Mt_3t_1t_3 = Mt_5t_3t_5 = Mt_7t_5t_7 = Mt_9t_7t_9$. Thus, the coset stabiliser will increase since any relation that, for example, sends $t_1 \rightarrow t_3, t_9 \rightarrow t_1$, and $t_1 \rightarrow t_3$, will be in the coset stabiliser. We find that there exists 25 such relations that show why the single cosets found in **MAGMA** are equal. The coset stabiliser of the double coset $Mt_1t_9t_1$ is : $N^{191} \geq N^{(191)} = \{e, (2, 4, 6, 8, 10), (1, 3, 5, 7, 9), ...$

$(1,9,7,5,3)(2,8,4,10,6)\}$. Thus the number of single cosets of $Mt_1t_9t_1N$ is at most $\frac{|N|}{|N^{(191)}|} = \frac{50}{25} = 2$. Now, the orbits of [191] are:

$$\mathcal{O} = \{\{1,3,5,7,9\},\{2,4,10,6,8\}\},$$

which were found by looking at the generators of $N^{(191)} = (1,3,5,7,9)(2,10,8,6,4)$, ...$(1,7,3,9,5)(2,10,8,6,4)$. Choosing a representative from each orbit and right multiplying to $Mt_1t_9t_1$, we have two possible new double cosets, but we find the following using **MAGMA**:

$$Mt_1t_9t_1t_1 = Mt_1t_9 \in [19]$$
$$Mt_1t_9t_1t_2 = Mt_1t_9t_1 \in [191]$$

Based on the information found, one symmetric generator returns to [19] and one loops back into [191]. Since there are no new possible double cosets to investigate we stop here and ivestigate other branches.

$\mathbf{Mt_1t_9t_2N}$

Consider the double coset $[192] = Mt_1t_9t_2N = \{M(t_1t_9t_2)^n | n \in N\} = \{Mt_1t_9t_2, Mt_1t_9t_4, Mt_6t_4t_7, Mt_1t_9t_6, Mt_6t_4t_9, Mt_8t_6t_7, Mt_1t_9t_8, Mt_6t_4t_1, Mt_8t_6t_9,$ $Mt_{10}t_8t_7, Mt_3t_1t_2, Mt_1t_9t_{10}, Mt_6t_4t_3, Mt_8t_6t_1, Mt_{10}t_8t_9, Mt_3t_1t_4, Mt_2t_{10}t_7, Mt_5t_3t_2,$ $Mt_6t_4t_5, Mt_8t_6t_3, Mt_{10}t_8t_1, Mt_3t_1t_6, Mt_2t_{10}t_9, Mt_5t_3t_4, Mt_4t_2t_7, Mt_7t_5t_2, Mt_8t_6t_5,$ $Mt_{10}t_8t_3t, Mt_3t_1t_8, Mt_2t_{10}t_1, Mt_5t_3t_6, Mt_4t_2t_9, Mt_7t_5t_4, Mt_9t_7t_2, Mt_{10}t_8t_5,$ $Mt_3t_1t_{10}, Mt_2t_{10}t_3, Mt_5t_3t_8, Mt_4t_2t_1, Mt_7t_5t_6, Mt_9t_7t_4, Mt_2t_{10}t_5, Mt_5t_3t_{10}, Mt_4t_2t_3,$ $Mt_7t_5t_8, Mt_9t_7t_6, Mt_4t_2t_5, Mt_7t_5t_{10}, Mt_9t_7t_8, Mt_9t_7t_{10}\}$.

The coset stabiliser of the double coset $Mt_1t_9t_2$ is : $N^{192} \geq N^{(192)} = \{e\}$. Thus the number of single cosets of $Mt_1t_9t_2N$ is at most $\frac{|N|}{|N^{(192)}|} = \frac{50}{1} = 50$. Now the orbits of [192] are:

$$\mathcal{O} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_9t_2$, we

have 10 possible new double cosets, but we find the following using **MAGMA**:

$$Mt_1t_9t_2t_2 = Mt_1t_9 \in [19]$$
$$Mt_1t_9t_2t_3 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_9t_2t_4 = Mt_1t_9t_2t_{10} \in [192]$$
$$Mt_1t_9t_2t_5 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_9t_2t_6 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_9t_2t_7 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_9t_2t_8 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_9t_2t_9 = Mt_1t_9t_2 \in [192]$$

Based on the information found, one symmetric generator returns to [19] and four symmetric generators loop back into [192]. The remaining generators will collapse. The new double cosets found for [192] are $Mt_1t_9t_2t_1$ and $Mt_1t_9t_2t_{10}$.

$\mathbf{Mt_1t_9t_2t_1N}$

Consider the double coset $[192] = Mt_1t_9t_2t_1N = \{M(t_1t_9t_2t_1)^n | n \in N\}$ $= \{Mt_1t_9t_2t_1, Mt_1t_9t_4t_1, Mt_6t_4t_7t_6, Mt_1t_9t_6t_1, Mt_6t_4t_9t_6, Mt_8t_6t_7t_8, Mt_1t_9t_8t_1,$ $Mt_6t_4t_1t_6, Mt_8t_6t_9t_8, Mt_{10}t_8t_7t_{10}, Mt_3t_1t_2t_3, Mt_1t_9t_{10}t_1, Mt_6t_4t_3t_6, Mt_8t_6t_1t_8,$ $Mt_{10}t_8t_9t_{10}, Mt_3t_1t_4t_3, Mt_2t_{10}t_7t_2, Mt_5t_3t_2t_5, Mt_6t_4t_5t_6, Mt_8t_6t_3t_8, Mt_{10}t_8t_1t_{10},$ $Mt_3t_1t_6t_3, Mt_2t_{10}t_9t_2, Mt_5t_3t_4t_5, Mt_4t_2t_7t_4, Mt_7t_5t_2t_7, Mt_8t_6t_5t_8, Mt_{10}t_8t_3t_{10},$ $Mt_3t_1t_8t_3, Mt_2t_{10}t_1t_2, Mt_5t_3t_6t_5, Mt_4t_2t_9t_4, Mt_7t_5t_4t_7, Mt_9t_7t_2t_9, Mt_{10}t_8t_5t_{10},$ $Mt_3t_1t_{10}t_3, Mt_2t_{10}t_3t_2, Mt_5t_3t_8t_5, Mt_4t_2t_1t_4, Mt_7t_5t_6t_7, Mt_9t_7t_4t_9, Mt_2t_{10}t_5t_2,$ $Mt_5t_3t_{10}t_5, Mt_4t_2t_3t_4, Mt_7t_5t_8t_7, Mt_9t_7t_6t_9, Mt_4t_2t_5t_4, Mt_7t_5t_{10}t_7, Mt_9t_7t_8t_9,$ $Mt_9t_7t_{10}t_9\}$.

The coset stabiliser of the double coset $Mt_1t_9t_2t_1$ is : $N^{1921} \geq N^{(1921)} = \{e\}$. Thus the number of single cosets of $Mt_1t_9t_2t_1N$ is at most $\frac{|N|}{|N^{(1921)}|} = \frac{50}{1} = 50$. Now the orbits of [1921] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_9t_2t_1$, we have 10 possible new double cosets, but we find the following using **MAGMA**:

$$\begin{aligned}
&Mt_1t_9t_2t_1t_1 = Mt_1t_9t_2 \in [192]\\
&Mt_1t_9t_2t_1t_3 = Mt_1t_2t_1t_{10}t_7 \in [12110]\\
&Mt_1t_9t_2t_1t_4 = Mt_1t_2t_7 \in [127]\\
&Mt_1t_9t_2t_1t_5 = Mt_1t_2t_1t_{10} \in [12110]\\
&Mt_1t_9t_2t_1t_6 = Mt_1t_2t_5t_8 \in [1258]\\
&Mt_1t_9t_2t_1t_7 = Mt_1t_2t_1t_{10} \in [12110]\\
&Mt_1t_9t_2t_1t_8 = Mt_1t_2t_7t_6 \in [1276]\\
&Mt_1t_9t_2t_1t_9 = Mt_1t_2t_1t_{10}t_7 \in [121107]\\
&Mt_1t_9t_2t_1t_{10} = Mt_1t_2t_5 \in [125]
\end{aligned}$$

Based on the information found, one symmetric generator returns to [192] and the 8 symmetric generators shown above collapse. Thus, only one symmetric generator moves forward to $Mt_1t_9t_2t_1t_2$.

$\mathbf{Mt_1t_9t_2t_1t_2N}$

Consider the double coset $[19212] = Mt_1t_9t_2t_1t_2N = \{M(t_1t_9t_2t_1t_2)^n | n \in N\} = \{Mt_1t_9t_2t_1t_2, Mt_1t_9t_4t_1t_4, Mt_6t_4t_7t_6t_7, Mt_1t_9t_6t_1t_6, Mt_6t_4t_9t_6t_9, Mt_8t_6t_7t_8t_7, Mt_1t_9t_8t_1t_8, Mt_6t_4t_1t_6t_1, Mt_8t_6t_9t_8t_9, Mt_{10}t_8t_7t_{10}t_7, Mt_3t_1t_2t_3t_2, Mt_1t_9t_{10}t_1t_{10}, Mt_6t_4t_3t_6t_3, Mt_8t_6t_1t_8t_1, Mt_{10}t_8t_9t_{10}t_9, Mt_3t_1t_4t_3t_4, Mt_2t_{10}t_7t_2t_7, Mt_5t_3t_2t_5t_2, Mt_6t_4t_5t_6t_5, Mt_8t_6t_3t_8t_3, Mt_{10}t_8t_1t_{10}t_1, Mt_3t_1t_6t_3t_6, Mt_2t_{10}t_9t_2t_9, Mt_5t_3t_4t_5t_4, Mt_4t_2t_7t_4t_7, Mt_7t_5t_2t_7t_2, Mt_8t_6t_5t_8t_5, Mt_{10}t_8t_3t_{10}t_3, Mt_3t_1t_8t_3t_8, Mt_2t_{10}t_1t_2t_1, Mt_5t_3t_6t_5t_6, Mt_4t_2t_9t_4t_9, Mt_7t_5t_4t_7t_4, Mt_9t_7t_2t_9t_2, Mt_{10}t_8t_5t_{10}t_2, Mt_3t_1t_{10}t_3t_{10}, Mt_2t_{10}t_3t_2t_3, Mt_5t_3t_8t_5t_8, Mt_4t_2t_1t_4t_1, Mt_7t_5t_6t_7t_6, Mt_9t_7t_4t_9t_4, Mt_2t_{10}t_5t_2t_5, Mt_5t_3t_{10}t_5t_{10}, Mt_4t_2t_3t_4t_3, Mt_7t_5t_8t_7t_8, Mt_9t_7t_6t_9t_6, Mt_4t_2t_5t_4t_5, Mt_7t_5t_{10}t_7t_{10}, Mt_9t_7t_8t_9t_8, Mt_9t_7t_{10}t_9t_{10}\}$.

The coset stabiliser of the double coset $Mt_1t_9t_2t_1t_2$ is : $N^{19212} \geq N^{(19212)} = \{e\}$. Thus the number of single cosets of $Mt_1t_9t_2t_1t_2N$ is at most $\frac{|N|}{|N^{(19212)}|} = \frac{50}{1} = 50$. Now the orbits of [19212] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_9t_2t_1t_2$, we have 10 possible new double cosets, but we find the following using **MAGMA**:

$$\begin{aligned}
Mt_1t_9t_2t_1t_2t_1 &= Mt_1t_9t_2t_1t_2 \in [19212]\\
Mt_1t_9t_2t_1t_2t_2 &= Mt_1t_9t_2t_1 \in [1921]\\
Mt_1t_9t_2t_1t_2t_3 &= Mt_1t_9t_2t_1t_2 \in [19212]\\
Mt_1t_9t_2t_1t_2t_4 &= Mt_1t_2t_1t_2t_9 \in [12129]\\
Mt_1t_9t_2t_1t_2t_5 &= Mt_1t_9t_2t_1t_2 \in [19212]\\
Mt_1t_9t_2t_1t_2t_6 &= Mt_1t_2t_7t_6 \in [1276]\\
Mt_1t_9t_2t_1t_2t_7 &= Mt_1t_9t_2t_1t_2 \in [19212]\\
Mt_1t_9t_2t_1t_2t_8 &= Mt_1t_2t_5t_8 \in [1258]\\
Mt_1t_9t_2t_1t_2t_9 &= Mt_1t_9t_2t_1t_2 \in [19212]\\
Mt_1t_9t_2t_1t_2t_{10} &= Mt_1t_2t_1t_2t_3 \in [12123]
\end{aligned}$$

Based on the information found, one symmetric generator returns to [1921] and the 5 symmetric generators loop back into [19212]. The remaining symmetric generators shown above collapse. Since there are no new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_9t_2t_1t_{10}N}$

Consider the double coset $[192110] = Mt_1t_9t_2t_1t_{10}N = \{M(t_1t_9t_2t_1t_{10})^n| n \in N\} = \{Mt_1t_9t_2t_{10}, Mt_1t_9t_4t_2, Mt_6t_4t_7t_5, Mt_1t_9t_6t_4, Mt_6t_4t_9t_7, Mt_8t_6t_7t_5, Mt_1t_9t_8t_6, Mt_6t_4t_1t_9, Mt_8t_6t_9t_7, Mt_{10}t_8t_7t_5, Mt_3t_1t_2t_{10}, Mt_1t_9t_{10}t_8, Mt_6t_4t_3t_1, Mt_8t_6t_1t_9, Mt_{10}t_8t_9t_7, Mt_3t_1t_4t_2, Mt_2t_{10}t_7t_5, Mt_5t_3t_2t_{10}, Mt_6t_4t_5t_3, Mt_8t_6t_3t_1, Mt_{10}t_8t_1t_9, Mt_3t_1t_6t_4, Mt_2t_{10}t_9t_7, Mt_5t_3t_4t_2, Mt_4t_2t_7t_5, Mt_7t_5t_2t_{10}, Mt_8t_6t_5t_3, Mt_{10}t_8t_3t_1, Mt_3t_1t_8t_6, Mt_2t_{10}t_1t_9, Mt_5t_3t_6t_4, Mt_4t_2t_9t_7, Mt_7t_5t_4t_2, Mt_9t_7t_2t_{10}, Mt_{10}t_8t_5t_3, Mt_3t_1t_{10}t_8, Mt_2t_{10}t_3t_1, Mt_5t_3t_8t_6, Mt_4t_2t_1t_9, Mt_7t_5t_6t_4, Mt_9t_7t_4t_2,$

$Mt_2t_{10}t_5t_3, Mt_5t_3t_{10}t_8, Mt_4t_2t_3t_1, Mt_7t_5t_8t_6, Mt_9t_7t_6t_4, Mt_4t_2t_5t_3, Mt_7t_5t_{10}t_8,$
$Mt_9t_7t_8t_6, Mt_9t_7t_{10}t_8\}$.

The coset stabiliser of the double coset $Mt_1t_9t_2t_{10}$ is : $N^{19210} \geq N^{(19210)} = \{e\}$. Thus the number of single cosets of $Mt_1t_9t_2t_{10}N$ is at most $\frac{|N|}{|N^{(19210)}|} = \frac{50}{1} = 50$. Now the orbits of [19210] are:

$$\mathcal{O} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_9t_2t_{10}$, we have 10 possible new double cosets, but we find the following using **MAGMA**:

$$Mt_1t_9t_2t_{10}t_1 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_9t_2t_{10}t_2 = Mt_1t_2t_1t_8t_1 \in [12181]$$
$$Mt_1t_9t_2t_{10}t_3 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_9t_2t_{10}t_4 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_9t_2t_{10}t_5 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_9t_2t_{10}t_6 = Mt_1t_9t_2t_{10} \in [19210]$$
$$Mt_1t_9t_2t_{10}t_7 = Mt_1t_9t_2t_{10} \in [19210]$$
$$Mt_1t_9t_2t_{10}t_8 = Mt_1t_9t_2t_{10} \in [19210]$$
$$Mt_1t_9t_2t_{10}t_9 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_9t_2t_{10}t_{10} = Mt_1t_9t_2 \in [192]$$

Based on the information found, two symmetric generators return to [192] and the three symmetric generators loop back into [19210]. The remaining symmetric generators shown above cause those double cosets to collapse. Since there are no new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_2N}$

Consider the double coset $[12] = Mt_1t_2N = \{M(t_1t_2)^n | n \in N\} = \{Mt_1t_2,$ $Mt_1t_4, Mt_6t_7, Mt_1t_6, Mt_6t_9, Mt_8t_7, Mt_1t_8, Mt_6t_1, Mt_8t_9, Mt_{10}t_7, Mt_3t_2, Mt_1t_{10},$ $Mt_6t_3, Mt_8t_1, Mt_{10}t_9, Mt_3t_4, Mt_2t_7, Mt_5t_2, Mt_6t_5, Mt_8t_3, Mt_{10}t_1, Mt_3t_6, Mt_2t_9,$ $Mt_5t_4, Mt_4t_7, Mt_7t_2, Mt_8t_5, Mt_{10}t_3, Mt_3t_8, Mt_2t_1, Mt_5t_6, Mt_4t_9, Mt_7t_4, Mt_9t_2,$ $Mt_{10}t_5, Mt_3t_{10}, Mt_2t_3, Mt_5t_8, Mt_4t_1, Mt_7t_6, Mt_9t_4, Mt_2t_5, Mt_5t_{10}, Mt_4t_3, Mt_7t_8,$ $Mt_9t_6, Mt_4t_5, Mt_7t_{10}, Mt_9t_8, Mt_9t_{10}\}$. The coset stabiliser of the double coset Mt_1t_2 is : $N^{12} \geq N^{(12)} = \{e)\}$. Thus, the number of single cosets of Mt_1t_2N is at most $\frac{|N|}{|N^{(12)}|} = \frac{50}{1} = 50$. Now, the orbits of [12] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to Mt_1t_2, we have 10 possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_2 = Mt_1 \in [1]$$
$$Mt_1t_2t_3 = Mt_1t_2t_9 \in [129]$$
$$Mt_1t_2t_4 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_2t_6 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_8 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_{10} = Mt_1t_2t_1t_8 \in [1218]$$

Based on the information found, one symmetric generator returns to [1] and two loop back into [12]. The symmetric generators mentioned above cause those double cosets to collapse. Now the remaining orbits form five new double cosets which are: $Mt_1t_2t_1, Mt_1t_2t_5, Mt_1t_2t_7, Mt_1t_2t_9$, and $Mt_1t_2t_1t_8$. Since the double coset $Mt_1t_2t_{10} = Mt_1t_2t_1t_8$, two symmetric generators will move forward to $Mt_1t_2t_1t_8$.

$\mathbf{Mt_1t_2t_1N}$

Consider the double coset $[121] = Mt_1t_2t_1N = \{M(t_1t_2t_1)^n | n \in N\} = \{Mt_1t_2t_1,$ $Mt_1t_4t_1, Mt_6t_7t_6, Mt_1t_6t_1, Mt_6t_9t_6, Mt_8t_7t_8, Mt_1t_8t_1, Mt_6t_1t_6, Mt_8t_9t_8, Mt_{10}t_7t_{10},$

$Mt_3t_2t_3, Mt_1t_{10}t_1, Mt_6t_3t_6, Mt_8t_1t_8, Mt_{10}t_9t_{10}, Mt_3t_4t_3, Mt_2t_7t_2, Mt_5t_2t_5, Mt_6t_5t_6,$ $Mt_8t_3t_8, Mt_{10}t_1t_{10}, Mt_3t_6t_3, Mt_2t_9t_2, Mt_5t_4t_5, Mt_4t_7t_4, Mt_7t_2t_7, Mt_8t_5t_8, Mt_{10}t_3t_{10},$ $Mt_3t_8t_3, Mt_2t_1t_2, Mt_5t_6t_5, Mt_4t_9t_4, Mt_7t_4t_7, Mt_9t_2t_9, Mt_{10}t_5t_{10}, Mt_3t_{10}t_3, Mt_2t_3t_2,$ $Mt_5t_8t_5, Mt_4t_1t_4, Mt_7t_6t_7, Mt_9t_4t_9, Mt_2t_5t_2, Mt_5t_{10}t_5, Mt_4t_3t_4, Mt_7t_8t_7, Mt_9t_6t_9,$ $Mt_4t_5t_4, Mt_7t_{10}t_7, Mt_9t_8t_9, Mt_9t_{10}t_9\}$. The coset stabiliser of the double coset $Mt_1t_2t_1$ is : $N^{121} \geq N^{(121)} = \{e)\}$. Thus the number of single cosets of $Mt_1t_2t_1N$ is at most $\frac{|N|}{|N^{(121)}|} = \frac{50}{1} = 50$. Now the orbits of [121] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1$, we have 10 possible new double cosets, but we find the following using **MAGMA** :

$$\begin{aligned}
&Mt_1t_2t_1t_1 = Mt_1t_2 \in [12]\\
&Mt_1t_2t_1t_3 = Mt_1t_2t_1t_9 \in [1219]\\
&Mt_1t_2t_1t_4 = Mt_1t_2t_1t_{10} \in [12110]\\
&Mt_1t_2t_1t_5 = Mt_1t_2t_7 \in [127]\\
&Mt_1t_2t_1t_6 = Mt_1t_2t_1t_8 \in [1218]\\
&Mt_1t_2t_1t_7 = Mt_1t_2t_5 \in [125]
\end{aligned}$$

Based on the information found, two symmetric generators move forward to $Mt_1t_2t_1t_{10}$ and two move forward to $Mt_1t_2t_1t_9$. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbits form four new double cosets which are: $Mt_1t_2t_1t_2, Mt_1t_2t_1t_8, Mt_1t_2t_1t_9, Mt_1t_2t_1t_{10}$.

$\mathbf{Mt_1t_2t_1t_2N}$

Consider the double coset $[1212] = Mt_1t_2t_1t_2N = \{M(t_1t_2t_1t_2)^n | n \in N\} =$ $\{Mt_1t_2t_1t_2, Mt_1t_4t_1t_4, Mt_6t_7t_6t_7, Mt_1t_6t_1t_6, Mt_6t_9t_6t_9, Mt_8t_7t_8t_7, Mt_1t_8t_1t_8,$ $Mt_6t_1t_6t_1, Mt_8t_9t_8t_9, Mt_{10}t_7t_{10}t_7, Mt_3t_2t_3t_2, Mt_1t_{10}t_1t_{10}, Mt_6t_3t_6t_3, Mt_8t_1t_8t_1,$ $Mt_{10}t_9t_{10}t_9, Mt_3t_4t_3t_4, Mt_2t_7t_2t_7, Mt_5t_2t_5t_2, Mt_6t_5t_6t_5, Mt_8t_3t_8t_3, Mt_{10}t_1t_{10}t_1,$ $Mt_3t_6t_3t_6, Mt_2t_9t_2t_9, Mt_5t_4t_5t_4, Mt_4t_7t_4t_7, Mt_7t_2t_7t_2, Mt_8t_5t_8t_5, Mt_{10}t_3t_{10}t_3,$ $Mt_3t_8t_3t_8, Mt_2t_1t_2t_1, Mt_5t_6t_5t_6, Mt_4t_9t_4t_9, Mt_7t_4t_7t_4, Mt_9t_2t_9t_2, Mt_{10}t_5t_{10}t_5,$

$Mt_3t_{10}t_3t_{10}, Mt_2t_3t_2t_3, Mt_5t_8t_5t_8, Mt_4t_1t_4t_1, Mt_7t_6t_7t_6, Mt_9t_4t_9t_4, Mt_2t_5t_2t_5,$ $Mt_5t_{10}t_5t_{10}, Mt_4t_3t_4t_3, Mt_7t_8t_7t_8, Mt_9t_6t_9t_6, Mt_4t_5t_4t_5, Mt_7t_{10}t_7t_{10}, Mt_9t_8t_9t_8,$ $Mt_9t_{10}t_9t_{10}\}$. The coset stabiliser of the double coset $Mt_1t_2t_1t_2$ is : $N^{1212} \geq N^{(1212)} = \{e)\}$. Thus the number of single cosets of $Mt_1t_2t_1t_2N$ is at most $\frac{|N|}{|N^{(1212)}|} = \frac{50}{1} = 50$. Now the orbits of [1212] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_2$, we have 10 possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_1t_2t_2 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_1t_2t_4 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_1t_2t_5 = Mt_1t_2t_5t_2 \in [1252]$$
$$Mt_1t_2t_1t_2t_6 = Mt_1t_2t_1t_8 \in [1218]$$
$$Mt_1t_2t_1t_2t_7 = Mt_1t_2t_7t_2 \in [1272]$$
$$Mt_1t_2t_1t_2t_8 = Mt_1t_2t_1t_8 \in [1218]$$
$$Mt_1t_2t_1t_2t_{10} = Mt_1t_2t_1t_2 \in [1212]$$

Based on the information found, two symmetric generators loop back to [1212] and one returns to [121]. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbits form three new double cosets which are: $Mt_1t_2t_1t_2t_1, Mt_1t_2t_1t_2t_9$ and $Mt_1t_2t_1t_2t_3$

$\mathbf{Mt_1t_2t_1t_2t_3N}$

Consider the double coset $[12123] = Mt_1t_2t_1t_2t_3N = \{M(t_1t_2t_1t_2t_3)^n | n \in N\}$ $= \{Mt_1t_2t_1t_2t_3, Mt_1t_4t_1t_4t_3, Mt_6t_7t_6t_7t_8, Mt_1t_6t_1t_6t_3, Mt_6t_9t_6t_9t_8, Mt_8t_7t_8t_7t_{10},$ $Mt_1t_8t_1t_8t_3, Mt_6t_1t_6t_1t_8, Mt_8t_9t_8t_9t_{10}, Mt_{10}t_7t_{10}t_7t_2, Mt_3t_2t_3t_2t_5, Mt_1t_{10}t_1t_{10}t_3,$ $Mt_6t_3t_6t_3t_8, Mt_8t_1t_8t_1t_{10}, Mt_{10}t_9t_{10}t_9t_2, Mt_3t_4t_3t_4t_5, Mt_2t_7t_2t_7t_4, Mt_5t_2t_5t_2t_7,$ $Mt_6t_5t_6t_5t_8, Mt_8t_3t_8t_3t_{10}, Mt_{10}t_1t_{10}t_1t_2, Mt_3t_6t_3t_6t_5, Mt_2t_9t_2t_9t_4, Mt_5t_4t_5t_4t_7,$ $Mt_4t_7t_4t_7t_6, Mt_7t_2t_7t_2t_9, Mt_8t_5t_8t_5t_{10}, Mt_{10}t_3t_{10}t_3t_2, Mt_3t_8t_3t_8t_5, Mt_2t_1t_2t_1t_4,$ $Mt_5t_6t_5t_6t_7, Mt_4t_9t_4t_9t_6, Mt_7t_4t_7t_4t_9, Mt_9t_2t_9t_2t_1, Mt_{10}t_5t_{10}t_5t_2, Mt_3t_{10}t_3t_{10}t_5,$

$Mt_2t_3t_2t_3t_4, Mt_5t_8t_5t_8t_7, Mt_4t_1t_4t_1t_6, Mt_7t_6t_7t_6t_9, Mt_9t_4t_9t_4t_1, Mt_2t_5t_2t_5t_4,$ $Mt_5t_{10}t_5t_{10}t_7, Mt_4t_3t_4t_3t_6, Mt_7t_8t_7t_8t_9, Mt_9t_6t_9t_6t_1, Mt_4t_5t_4t_5t_6, Mt_7t_{10}t_7t_{10}t_9,$ $Mt_9t_8t_9t_8t_1, Mt_9t_{10}t_9t_{10}t_1\}$. We find through **MAGMA** that $Mt_1t_2t_1t_2t_3 = Mt_1t_2t_1t_2t_4$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_1t_2t_3$ is : $N^{12123} \geq N^{(12123)} = \{e, (1,2)(3,4)(5,6)(7,8)(9,10)\}$, since this element in N sends $t_1 \to t_2, t_2 \to t_1, t_1 \to t_2, t_2 \to t_1$, and $t_3 \to t_4$. Thus the number of single cosets of $Mt_1t_2t_1t_2t_3N$ is at most $\frac{|N|}{|N^{(12123)}|} = \frac{50}{2} = 25$. Now the orbits of [12123] are:

$$\mathcal{O} = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_2t_3$, we have five possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_1t_2t_3t_1 = Mt_1t_9t_2t_1t_2 \in [19212]$$
$$Mt_1t_2t_1t_2t_3t_3 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_1t_2t_3t_5 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_2t_1t_2t_3t_7 = Mt_1t_2t_1t_2t_9 \in [12129]$$
$$Mt_1t_2t_1t_2t_3t_9 = Mt_1t_2t_7t_2 \in [1272]$$

Based on the information found, two symmetric generators return to [1212]. The symmetric generators mentioned above cause those double cosets to collapse. Since there are no more possible new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_2t_1t_2t_9N}$

Consider the double coset $[12129] = Mt_1t_2t_1t_2t_9N = \{M(t_1t_2t_1t_2t_9)^n | n \in N\}$ $= \{Mt_1t_2t_1t_2t_9, Mt_1t_4t_1t_4t_9, Mt_6t_7t_6t_7t_4, Mt_1t_6t_1t_6t_9, Mt_6t_9t_6t_9t_4, Mt_8t_7t_8t_7t_6,$ $Mt_1t_8t_1t_8t_9, Mt_6t_1t_6t_1t_4, Mt_8t_9t_8t_9t_6, Mt_{10}t_7t_{10}t_7t_8, Mt_3t_2t_3t_2t_1, Mt_1t_{10}t_1t_{10}t_9,$ $Mt_6t_3t_6t_3t_4, Mt_8t_1t_8t_1t_6, Mt_{10}t_9t_{10}t_9t_8, Mt_3t_4t_3t_4t_1, Mt_2t_7t_2t_7t_{10}, Mt_5t_2t_5t_2t_3,$ $Mt_6t_5t_6t_5t_4, Mt_8t_3t_8t_3t_6, Mt_{10}t_1t_{10}t_1t_8, Mt_3t_6t_3t_6t_1, Mt_2t_9t_2t_9t_{10}, Mt_5t_4t_5t_4t_3,$ $Mt_4t_7t_4t_7t_2, Mt_7t_2t_7t_2t_5, Mt_8t_5t_8t_5t_6, Mt_{10}t_3t_{10}t_3t_8, Mt_3t_8t_3t_8t_1, Mt_2t_1t_2t_1t_{10},$ $Mt_5t_6t_5t_6t_3, Mt_4t_9t_4t_9t_2, Mt_7t_4t_7t_4t_5, Mt_9t_2t_9t_2t_7, Mt_{10}t_5t_{10}t_5t_8, Mt_3t_{10}t_3t_{10}t_1,$

$Mt_2t_3t_2t_3t_{10}, Mt_5t_8t_5t_8t_3, Mt_4t_1t_4t_1t_2, Mt_7t_6t_7t_6t_5, Mt_9t_4t_9t_4t_7, Mt_2t_5t_2t_5t_{10},$
$Mt_5t_{10}t_5t_{10}t_3, Mt_4t_3t_4t_3t_2, Mt_7t_8t_7t_8t_5, Mt_9t_6t_9t_6t_7, Mt_4t_5t_4t_5t_2, Mt_7t_{10}t_7t_{10}t_5,$
$Mt_9t_8t_9t_8t_7, Mt_9t_{10}t_9t_{10}t_7\}$. We find that $Mt_1t_2t_1t_2t_9 = Mt_1t_2t_1t_2t_{10}$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_1t_2t_9$ is : $N^{12129} \geq N^{(12129)} = \{e, (1,2)(3,4)(5,6)(7,8)(9,10)\}$, since this element in N sends $t_1 \to t_2, t_2 \to t_1, t_1 \to t_2, t_2 \to t_1$, and $t_9 \to t_{10}$. Thus the number of single cosets of $Mt_1t_2t_1t_2t_9N$ is at most $\frac{|N|}{|N^{(12129)}|} = \frac{50}{2} = 25$. Now the orbits of [12129] are:

$$\mathcal{O} = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_2t_9$, we have 5 possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_1t_2t_9t_1 = Mt_1t_9t_2t_1t_2 \in [19212]$$
$$Mt_1t_2t_1t_2t_9t_3 = Mt_1t_9t_2t_1t_2 \in [19212]$$
$$Mt_1t_2t_1t_2t_9t_5 = Mt_1t_2t_5t_2 \in [1252]$$
$$Mt_1t_2t_1t_2t_9t_7 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_2t_1t_2t_9t_9 = Mt_1t_2t_1t_2 \in [1212]$$

Based on the information found, two symmetric generators return to [1212]. The symmetric generators mentioned above cause those double cosets to collapse. Since there are no more possible new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_2t_1t_2t_1N}$

Consider the double coset $[12121] = Mt_1t_2t_1t_2t_1N = \{M(t_1t_2t_1t_2t_1)^n | n \in N\}$
$= \{Mt_1t_2t_1t_2t_1, Mt_1t_4t_1t_4t_1, Mt_6t_7t_6t_7t_6, Mt_1t_6t_1t_6t_1, Mt_6t_9t_6t_9t_6, Mt_8t_7t_8t_7t_8,$
$Mt_1t_8t_1t_8t_1, Mt_6t_1t_6t_1t_6, Mt_8t_9t_8t_9t_8, Mt_{10}t_7t_{10}t_7t_{10}, Mt_3t_2t_3t_2t_3, Mt_1t_{10}t_1t_{10}t_1,$
$Mt_6t_3t_6t_3t_6, Mt_8t_1t_8t_1t_8, Mt_{10}t_9t_{10}t_9t_{10}, Mt_3t_4t_3t_4t_3, Mt_2t_7t_2t_7t_2, Mt_5t_2t_5t_2t_5,$
$Mt_6t_5t_6t_5t_6, Mt_8t_3t_8t_3t_8, Mt_{10}t_1t_{10}t_1t_{10}, Mt_3t_6t_3t_6t_3, Mt_2t_9t_2t_9t_2, Mt_5t_4t_5t_4t_5,$
$Mt_4t_7t_4t_7t_4, Mt_7t_2t_7t_2t_7, Mt_8t_5t_8t_5t_8, Mt_{10}t_3t_{10}t_3t_{10}, Mt_3t_8t_3t_8t_3, Mt_2t_1t_2t_1t_2,$
$Mt_5t_6t_5t_6t_5, Mt_4t_9t_4t_9t_4, Mt_7t_4t_7t_4t_7, Mt_9t_2t_9t_2t_9, Mt_{10}t_5t_{10}t_5t_{10}, Mt_3t_{10}t_3t_{10}t_3,$

$Mt_2t_3t_2t_3t_2, Mt_5t_8t_5t_8t_5, Mt_4t_1t_4t_1t_4, Mt_7t_6t_7t_6t_7, Mt_9t_4t_9t_4t_9, Mt_2t_5t_2t_5t_2,$
$Mt_5t_{10}t_5t_{10}t_5, Mt_4t_3t_4t_3t_4, Mt_7t_8t_7t_8t_7, Mt_9t_6t_9t_6t_9, Mt_4t_5t_4t_5t_4, Mt_7t_{10}t_7t_{10}t_7,$
$Mt_9t_8t_9t_8t_9, Mt_9t_{10}t_9t_{10}t_9\}$.

We find that $Mt_1t_2t_1t_2t_1 = Mt_2t_1t_2t_1t_2$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_1t_2t_1$ is : $N^{12121} \geq N^{(12121)} = \{e, (1,2)(3,4)(5,6)(7,8)(9,10)\}$, since this element in N only sends $t_1 \to t_2$. Thus the number of single cosets of $Mt_1t_2t_1t_2t_1N$ is at most $\frac{|N|}{|N^{(12121)}|} = \frac{50}{2} = 25$. Now the orbits of [12121] are:

$$\mathcal{O} = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_2t_1$, we have five possible new double cosets, but we find the following:

$$Mt_1t_2t_1t_2t_1t_1 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_1t_2t_1t_3 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_1t_2t_1t_5 = Mt_1t_2t_7t_2 \in [1272]$$
$$Mt_1t_2t_1t_2t_1t_7 = Mt_1t_2t_5t_2 \in [1252]$$
$$Mt_1t_2t_1t_2t_1t_9 = Mt_1t_2t_7 \in [127]$$

Based on the information found, two symmetric generators return to [1212]. The symmetric generators mentioned above cause those double cosets to collapse. Since there are no more possible new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_2t_1t_9N}$

Consider the double coset $[1219] = Mt_1t_2t_1t_9N = \{M(t_1t_2t_1t_9)^n | n \in N\} =$
$\{Mt_1t_2t_1t_9, Mt_1t_4t_1t_9, Mt_6t_7t_6t_4, Mt_1t_6t_1t_9, Mt_6t_9t_6t_4, Mt_8t_7t_8t_6, Mt_1t_8t_1t_9,$
$Mt_6t_1t_6t_4, Mt_8t_9t_8t_6, Mt_{10}t_7t_{10}t_8, Mt_3t_2t_3t_1, Mt_1t_{10}t_1t_9, Mt_6t_3t_6t_4, Mt_8t_1t_8t_6,$
$Mt_{10}t_9t_{10}t_8, Mt_3t_4t_3t_1, Mt_2t_7t_2t_{10}, Mt_5t_2t_5t_3, Mt_6t_5t_6t_4, Mt_8t_3t_8t_6, Mt_{10}t_1t_{10}t_8,$
$Mt_3t_6t_3t_1, Mt_2t_9t_2t_{10}, Mt_5t_4t_5t_3, Mt_4t_7t_4t_2, Mt_7t_2t_7t_5, Mt_8t_5t_8t_6, Mt_{10}t_3t_{10}t_8,$
$Mt_3t_8t_3t_1, Mt_2t_1t_2t_{10}, Mt_5t_6t_5t_3, Mt_4t_9t_4t_2, Mt_7t_4t_7t_5, Mt_9t_2t_9t_7, Mt_{10}t_5t_{10}t_8,$
$Mt_3t_{10}t_3t_1, Mt_2t_3t_2t_{10}, Mt_5t_8t_5t_3, Mt_4t_1t_4t_2, Mt_7t_6t_7t_5, Mt_9t_4t_9t_7, Mt_2t_5t_2t_{10},$

$Mt_5t_{10}t_5t_3, Mt_4t_3t_4t_2, Mt_7t_8t_7t_5, Mt_9t_6t_9t_7, Mt_4t_5t_4t_2, Mt_7t_{10}t_7t_5, Mt_9t_8t_9t_7,$ $Mt_9t_{10}t_9t_7\}$. The coset stabiliser of the double coset $Mt_1t_2t_1t_9N$ is : $N^{1219} \geq N^{(1219)} = \{e)\}$. Thus the number of single cosets of $Mt_1t_2t_1t_9N$ is at most $\frac{|N|}{|N^{(1219)}|} = \frac{50}{1} = 50$. Now the orbits of [1219] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_9$, we have 10 possible new double cosets, but we find the following:

$$Mt_1t_2t_1t_9t_1 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_1t_9t_3 = Mt_1t_2t_7 \in [127]$$
$$Mt_1t_2t_1t_9t_4 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_2t_1t_9t_5 = Mt_1t_2t_1t_9 \in [1219]$$
$$Mt_1t_2t_1t_9t_6 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_2t_1t_9t_7 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_1t_9t_8 = Mt_1t_2t_1t_9t_1 \in [12191]$$
$$Mt_1t_2t_1t_9t_9 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_1t_9t_{10} = Mt_1t_2t_1t_9 \in [1219]$$

Based on the information found, three symmetric generators loop back to [1219] and two return to [121]. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbit forms one new double coset which is: $Mt_1t_2t_1t_9t_2$.

$\mathbf{Mt_1t_2t_1t_9t_2N}$

Consider the double coset $[12192] = Mt_1t_2t_1t_9t_2N = \{M(t_1t_2t_1t_9t_2)^n | n \in N\}$ $= \{Mt_1t_2t_1t_9t_2, Mt_1t_4t_1t_9t_4, Mt_6t_7t_6t_4t_7, Mt_1t_6t_1t_9t_6, Mt_6t_9t_6t_4t_9, Mt_8t_7t_8t_6t_7,$ $Mt_1t_8t_1t_9t_8, Mt_6t_1t_6t_4t_1, Mt_8t_9t_8t_6t_9, Mt_{10}t_7t_{10}t_8t_7, Mt_3t_2t_3t_1t_2, Mt_1t_{10}t_1t_9t_{10},$ $Mt_6t_3t_6t_4t_3, Mt_8t_1t_8t_6t_1, Mt_{10}t_9t_{10}t_8t_9, Mt_3t_4t_3t_1t_4, Mt_2t_7t_2t_{10}t_7, Mt_5t_2t_5t_3t_2,$

$Mt_6t_5t_6t_4t_5, Mt_8t_3t_8t_6t_3, Mt_{10}t_1t_{10}t_8t_1, Mt_3t_6t_3t_1t_6, Mt_2t_9t_2t_{10}t_9, Mt_5t_4t_5t_3t_4,$ $Mt_4t_7t_4t_2t_7, Mt_7t_2t_7t_5t_2, Mt_8t_5t_8t_6t_5, Mt_{10}t_3t_{10}t_8t_3, Mt_3t_8t_3t_1t_8, Mt_2t_1t_2t_{10}t_1,$ $Mt_5t_6t_5t_3t_6, Mt_4t_9t_4t_2t_9, Mt_7t_4t_7t_5t_4, Mt_9t_2t_9t_7t_2, Mt_{10}t_5t_{10}t_8t_5, Mt_3t_{10}t_3t_1t_{10},$ $Mt_2t_3t_2t_{10}t_3, Mt_5t_8t_5t_3t_8, Mt_4t_1t_4t_2t_1, Mt_7t_6t_7t_5t_6, Mt_9t_4t_9t_7t_4, Mt_2t_5t_2t_{10}t_5,$ $Mt_5t_{10}t_5t_3t_{10}, Mt_4t_3t_4t_2t_3, Mt_7t_8t_7t_5t_8, Mt_9t_6t_9t_7t_6, Mt_4t_5t_4t_2t_5, Mt_7t_{10}t_7t_5t_{10},$ $Mt_9t_8t_9t_7t_8, Mt_9t_{10}t_9t_7t_{10}\}$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_1t_9t_2N$ is : $N^{12192} \geq N^{(12192)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_1t_9t_2N$ is at most $\frac{|N|}{|N^{(12192)}|} = \frac{50}{1} = 50$. Now the orbits of [12192] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_9t_2$, we have ten possible new double cosets, but we find the following:

$$\begin{aligned}
Mt_1t_2t_1t_9t_2t_1 &= Mt_1t_2t_1t_{10}t_7t_{10} \in [12110710] \\
Mt_1t_2t_1t_9t_2t_2 &= Mt_1t_2t_1t_9 \in [1219] \\
Mt_1t_2t_1t_9t_2t_3 &= Mt_1t_9t_2t_{10} \in [19210] \\
Mt_1t_2t_1t_9t_2t_4 &= Mt_1t_2t_1t_9t_2 \in [12192] \\
Mt_1t_2t_1t_9t_2t_5 &= Mt_1t_2t_5t_8 \in [1258] \\
Mt_1t_2t_1t_9t_2t_6 &= Mt_1t_2t_1t_9 \in [1219] \\
Mt_1t_2t_1t_9t_2t_7 &= Mt_1t_9t_2t_{10} \in [19210] \\
Mt_1t_2t_1t_9t_2t_8 &= Mt_1t_2t_1t_9t_2 \in [12192] \\
Mt_1t_2t_1t_9t_2t_9 &= Mt_1t_2t_5t_2 \in [1252] \\
Mt_1t_2t_1t_9t_2t_{10} &= Mt_1t_2t_1t_9 \in [1219]
\end{aligned}$$

Based on the information found, three symmetric generators return to [1219], and two loop back into [12192]. The remaining symmetric generators mentioned above cause those double cosets to collapse. Since there are no more possible new double cosets to investigate, this branch ends here.

$\mathbf{Mt_1t_2t_1t_{10}N}$

Consider the double coset $[12110] = Mt_1t_2t_1t_{10}N = \{M(t_1t_2t_1t_{10})^n | n \in N\}$ $= \{Mt_1t_2t_1t_{10}, Mt_1t_4t_1t_2, Mt_6t_7t_6t_5, Mt_1t_6t_1t_4, Mt_6t_9t_6t_7, Mt_8t_7t_8t_5, Mt_1t_8t_1t_6, Mt_6t_1t_6t_9, Mt_8t_9t_8t_7, Mt_{10}t_7t_{10}t_5, Mt_3t_2t_3t_{10}, Mt_1t_{10}t_1t_8, Mt_6t_3t_6t_1, Mt_8t_1t_8t_9, Mt_{10}t_9t_{10}t_7, Mt_3t_4t_3t_2, Mt_2t_7t_2t_5, Mt_5t_2t_5t_{10}, Mt_6t_5t_6t_3, Mt_8t_3t_8t_1, Mt_{10}t_1t_{10}t_9, Mt_3t_6t_3t_4, Mt_2t_9t_2t_7, Mt_5t_4t_5t_2, Mt_4t_7t_4t_5, Mt_7t_2t_7t_{10}, Mt_8t_5t_8t_3, Mt_{10}t_3t_{10}t_1, Mt_3t_8t_3t_6, Mt_2t_1t_2t_9, Mt_5t_6t_5t_4, Mt_4t_9t_4t_7, Mt_7t_4t_7t_2, Mt_9t_2t_9t_{10}, Mt_{10}t_5t_{10}t_3, Mt_3t_{10}t_3t_8, Mt_2t_3t_2t_1, Mt_5t_8t_5t_6, Mt_4t_1t_4t_9, Mt_7t_6t_7t_4, Mt_9t_4t_9t_2, Mt_2t_5t_2t_3, Mt_5t_{10}t_5t_8, Mt_4t_3t_4t_1, Mt_7t_8t_7t_6, Mt_9t_6t_9t_4, Mt_4t_5t_4t_3, Mt_7t_{10}t_7t_8, Mt_9t_8t_9t_6, Mt_9t_{10}t_9t_8\}$. The coset stabiliser of the double coset $Mt_1t_2t_1t_{10}$ is : $N^{12110} \geq N^{(12110)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_1t_{10}N$ is at most $\frac{|N|}{|N^{(12110)}|} = \frac{50}{1} = 50$. Now the orbits of [12110] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_{10}$, we have 10 possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_1t_{10}t_1 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_1t_{10}t_2 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_2t_1t_{10}t_3 = Mt_1t_9t_2t_1[1921]$$
$$Mt_1t_2t_1t_{10}t_4 = Mt_1t_2t_1t_8 \in [1218]$$
$$Mt_1t_2t_1t_{10}t_5 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_2t_1t_{10}t_6 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_2t_1t_{10}t_8 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_1t_{10}t_9 = Mt_1t_9t_2 \in [192]$$
$$Mt_1t_2t_1t_{10}t_{10} = Mt_1t_2t_1 \in [121]$$

Based on the information found, two symmetric generators return to [121] and two loop back into [12110]. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbit forms one new double coset which is: $Mt_1t_2t_1t_{10}t_7$.

$\mathbf{Mt_1t_2t_1t_{10}t_7N}$

Consider the double coset $[121107] = Mt_1t_2t_1t_{10}t_7N = \{M(t_1t_2t_1t_{10}t_7)^n | n \in N\}$ $= \{Mt_1t_2t_1t_{10}t_7, Mt_1t_4t_1t_2t_7, Mt_6t_7t_6t_5t_2, Mt_1t_6t_1t_4t_7, Mt_6t_9t_6t_7t_2, Mt_8t_7t_8t_5t_4,$ $Mt_1t_8t_1t_6t_7, Mt_6t_1t_6t_9t_2, Mt_8t_9t_8t_7t_4, Mt_{10}t_7t_{10}t_5t_6, Mt_3t_2t_3t_{10}t_9, Mt_1t_{10}t_1t_8t_7,$ $Mt_6t_3t_6t_1t_7, Mt_8t_1t_8t_9t_2, Mt_{10}t_9t_{10}t_7t_4, Mt_3t_4t_3t_2t_6, Mt_2t_7t_2t_5t_8, Mt_5t_2t_5t_{10}t_1,$ $Mt_6t_5t_6t_3t_2, Mt_8t_3t_8t_1t_4, Mt_{10}t_1t_{10}t_9t_6, Mt_3t_6t_3t_4t_9, Mt_2t_9t_2t_7t_8, Mt_5t_4t_5t_2t_1,$ $Mt_4t_7t_4t_5t_{10}, Mt_7t_2t_7t_{10}t_3, Mt_8t_5t_8t_3t_4, Mt_{10}t_3t_{10}t_1t_6, Mt_3t_8t_3t_6t_9, Mt_2t_1t_2t_9t_8,$ $Mt_5t_6t_5t_4, Mt_4t_9t_4t_7, Mt_7t_4t_7t_2t_3, Mt_9t_2t_9t_{10}t_5, Mt_{10}t_5t_{10}t_3t_6, Mt_3t_{10}t_3t_8t_9,$ $Mt_2t_3t_2t_1t_8, Mt_5t_8t_5t_6t_1, Mt_4t_1t_4t_9t_{10}, Mt_7t_6t_7t_4t_3, Mt_9t_4t_9t_2t_5, Mt_2t_5t_2t_3t_8,$ $Mt_5t_{10}t_5t_8t_1, Mt_4t_3t_4t_1t_{10}, Mt_7t_8t_7t_6t_3, Mt_9t_6t_9t_4t_5, Mt_4t_5t_4t_3t_{10}, Mt_7t_{10}t_7t_8t_3,$ $Mt_9t_8t_9t_6t_5, Mt_9t_{10}t_9t_8t_5\}$. The coset stabiliser of the double coset $Mt_1t_2t_1t_{10}t_7$ is : $N^{121107} \geq N^{(121107)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_1t_{10}t_7N$ is at most $\frac{|N|}{|N^{(121107)}|} = \frac{50}{1} = 50$. Now the orbits of [121107] are:

$$\mathcal{O} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_{10}t_7$, we have ten possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_1t_{10}t_7t_1 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_1t_{10}t_7t_2 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_2t_1t_{10}t_7t_3 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_1t_{10}t_7t_4 = Mt_1t_2t_1t_{10}t_7 \in [12110]$$
$$Mt_1t_2t_1t_{10}t_7t_5 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$
$$Mt_1t_2t_1t_{10}t_7t_6 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_2t_1t_{10}t_7t_7 = Mt_1t_2t_1t_{10} \in [12110]$$
$$Mt_1t_2t_1t_{10}t_7t_8 = Mt_1t_2t_1t_{10}t_7t_{10} \in [12110710]$$
$$Mt_1t_2t_1t_{10}t_7t_9 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$

Based on the information found, one symmetric generator returns to [12110] and three loop back into [121107]. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbit forms one new double coset which is: $Mt_1t_2t_1t_{10}t_7t_{10}$, but since $Mt_1t_2t_1t_{10}t_7t_8 = Mt_1t_2t_1t_{10}t_7t_{10}$, two symmetric generators advance to this new double coset.

$\mathbf{Mt_1t_2t_1t_{10}t_7t_{10}N}$

Consider the double coset $[12110710] = Mt_1t_2t_1t_{10}t_7t_{10}N = \{M(t_1t_2t_1t_{10}t_7t_{10})^n | n \in N\} = \{Mt_1t_2t_1t_{10}t_7t_{10}, Mt_1t_4t_1t_2t_7t_2, Mt_6t_7t_6t_5t_2t_5, Mt_1t_6t_1t_4t_7t_4, Mt_6t_9t_6t_7t_2t_7, Mt_8t_7t_8t_5t_4t_5, Mt_1t_8t_1t_6t_7t_6, Mt_6t_1t_6t_9t_2t_9, Mt_8t_9t_8t_7t_4t_7, Mt_{10}t_7t_{10}t_5t_6t_5, Mt_3t_2t_3t_{10}t_9t_{10}, Mt_1t_{10}t_1t_8t_7t_8, Mt_6t_3t_6t_1t_7t_1, Mt_8t_1t_8t_9t_2t_9, Mt_{10}t_9t_{10}t_7t_4t_7, Mt_3t_4t_3t_2t_9t_2, Mt_2t_7t_2t_5t_8t_5, Mt_5t_2t_5t_{10}t_1t_{10}, Mt_6t_5t_6t_3t_2t_3, Mt_8t_3t_8t_1t_4t_1, Mt_{10}t_1t_{10}t_9t_6t_9, Mt_3t_6t_3t_4t_9t_4, Mt_2t_9t_2t_7t_8t_7, Mt_5t_4t_5t_2t_1t_2, Mt_4t_7t_4t_5t_{10}t_5, Mt_7t_2t_7t_{10}t_3t_{10}, Mt_8t_5t_8t_3t_4t_3, Mt_{10}t_3t_{10}t_1t_6t_1, Mt_3t_8t_3t_6t_9t_6, Mt_2t_1t_2t_9t_8t_9, Mt_5t_6t_5t_4t_1t_4, Mt_4t_9t_4t_7t_{10}t_7, Mt_7t_4t_7t_2t_3t_2, Mt_9t_2t_9t_{10}t_5t_{10}, Mt_{10}t_5t_{10}t_3t_6t_3, Mt_3t_{10}t_3t_8t_9t_8, Mt_2t_3t_2t_1t_8t_1, Mt_5t_8t_5t_6t_1t_6, Mt_4t_1t_4t_9t_{10}t_9, Mt_7t_6t_7t_4t_3t_4, Mt_9t_4t_9t_2t_5t_2, Mt_2t_5t_2t_3t_8t_3, Mt_5t_{10}t_5t_8t_1t_8, Mt_4t_3t_4t_1t_{10}t_1, Mt_7t_8t_7t_6t_3t_6, Mt_9t_6t_9t_4t_5t_4, Mt_4t_5t_4t_3t_{10}t_3, Mt_7t_{10}t_7t_8t_3t_8, Mt_9t_8t_9t_6t_5t_6, Mt_9t_{10}t_9t_8t_5t_8\}$. We find that $Mt_1t_2t_1t_{10}t_7t_{10} = Mt_{10}t_3t_{10}t_1t_6t_1$, thus the coset stabiliser of the double coset $Mt_1t_2t_1t_{10}t_7t_{10}$ is : $N^{12110710} \geq N^{(12110710)} = \{e, (1,10)$ $(2,3)(4,5)$
$(6,7)(8,9)\}$, since this element in N sends $t_1 \to t_{10}, t_2 \to t_3, t_1 \to t_{10}, t_{10} \to t_1, t_7 \to t_6$ and $t_{10} \to t_1$. Thus the number of single cosets of $Mt_1t_2t_1t_{10}t_7t_{10}N$ is at most $\frac{|N|}{|N^{(12110710)}|} = \frac{50}{2} = 25$. Now the orbits of $[12110710]$ are:

$$\mathcal{O} = \{\{1,10\},\{2,3\},\{4,5\},\{6,7\},\{8,9\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_{10}t_7$, we have 10 possible new double cosets, but we find the following:

$$Mt_1t_2t_1t_{10}t_7t_{10}t_1 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$
$$Mt_1t_2t_1t_{10}t_7t_{10}t_2 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_2t_1t_{10}t_7t_{10}t_4 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_2t_1t_{10}t_7t_{10}t_6 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$
$$Mt_1t_2t_1t_{10}t_7t_{10}t_8 = Mt_1t_2t_1t_9t_2 \in [12192]$$

Based on the information found, four symmetric generators return to [121107] and the remaining collapse. Since there are no possible new double cosets, this branch ends here.

$\mathbf{Mt_1t_2t_1t_8N}$

Consider the double coset $[1218] = Mt_1t_2t_1t_8N = \{M(t_1t_2t_1t_8)^n | n \in N\}$ $= \{Mt_1t_2t_1t_8, Mt_1t_4t_1t_{10}, Mt_6t_7t_6t_3, Mt_1t_6t_1t_2, Mt_6t_9t_6t_5, Mt_8t_7t_8t_3, Mt_1t_8t_1t_4,$ $Mt_6t_1t_6t_7, Mt_8t_9t_8t_5, Mt_{10}t_7t_{10}t_3, Mt_3t_2t_3t_8, Mt_1t_{10}t_1t_6, Mt_6t_3t_6t_9, Mt_8t_1t_8t_7,$ $Mt_{10}t_9t_{10}t_5, Mt_3t_4t_3t_{10}, Mt_2t_7t_2t_3, Mt_5t_2t_5t_8, Mt_6t_5t_6t_1, Mt_8t_3t_8t_9, Mt_{10}t_1t_{10}t_7,$ $Mt_3t_6t_3t_2, Mt_2t_9t_2t_5, Mt_5t_4t_5t_{10}, Mt_4t_7t_4t_3, Mt_7t_2t_7t_8, Mt_8t_5t_8t_1, Mt_{10}t_3t_{10}t_9,$ $Mt_3t_8t_3t_4, Mt_2t_1t_2t_7, Mt_5t_6t_5t_2, Mt_4t_9t_4t_5, Mt_7t_4t_7t_{10}, Mt_9t_2t_9t_8, Mt_{10}t_5t_{10}t_1,$ $Mt_3t_{10}t_3t_6, Mt_2t_3t_2t_9, Mt_5t_8t_5t_4, Mt_4t_1t_4t_7, Mt_7t_6t_7t_2, Mt_9t_4t_9t_{10}, Mt_2t_5t_2t_1,$ $Mt_5t_{10}t_5t_6, Mt_4t_3t_4t_9, Mt_7t_8t_7t_4, Mt_9t_6t_9t_2, Mt_4t_5t_4t_1, Mt_7t_{10}t_7t_6,$ $Mt_9t_8t_9t_4, Mt_9t_{10}t_9t_6\}$. The coset stabiliser of the double coset $Mt_1t_2t_1t_8$ is : $N^{1218} \geq N^{(1218)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_1t_8N$ is at most $\frac{|N|}{|N^{(1218)}|} = \frac{50}{1} = 50$. Now the orbits of [1218] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_8$, we

have 10 possible new double cosets, but we find the following:

$$\begin{aligned}
Mt_1t_2t_1t_8t_2 &= Mt_1t_2t_1t_{10} \in [12110]\\
Mt_1t_2t_1t_8t_3 &= Mt_1t_2 \in [12]\\
Mt_1t_2t_1t_8t_4 &= Mt_1t_2t_1t_2 \in [1212]\\
Mt_1t_2t_1t_8t_5 &= Mt_1t_2t_1t_8 \in [1218]\\
Mt_1t_2t_1t_8t_6 &= Mt_1t_2t_1 \in [121]\\
Mt_1t_2t_1t_8t_7 &= Mt_1t_2t_1t_8 \in [1218]\\
Mt_1t_2t_1t_8t_8 &= Mt_1t_2t_1 \in [121]\\
Mt_1t_2t_1t_8t_9 &= Mt_1t_2 \in [12]\\
Mt_1t_2t_1t_8t_{10} &= Mt_1t_2t_1t_2 \in [1212]
\end{aligned}$$

Based on the information found, two symmetric generators loop back to [1218] and two return to [121]. The symmetric generators mentioned above cause those double cosets to collapse. Now, the remaining orbit forms one new double coset which is: $Mt_1t_2t_1t_8t_1$.

$\mathbf{Mt_1t_2t_1t_8t_1N}$

Consider the double coset $[12181] = Mt_1t_2t_1t_8t_1N = \{M(t_1t_2t_1t_8t_1)^n | n \in N\}$ $= \{Mt_1t_2t_1t_8t_1, Mt_1t_4t_1t_{10}t_1, Mt_6t_7t_6t_3t_6, Mt_1t_6t_1t_2t_1, Mt_6t_9t_6t_5t_6, Mt_8t_7t_8t_3t_8,$ $Mt_1t_8t_1t_4t_1, Mt_6t_1t_6t_7t_6, Mt_8t_9t_8t_5t_8, Mt_{10}t_7t_{10}t_3t_{10}, Mt_3t_2t_3t_8t_3, Mt_1t_{10}t_1t_6t_1,$ $Mt_6t_3t_6t_9t_6, Mt_8t_1t_8t_7t_8, Mt_{10}t_9t_{10}t_5t_{10}, Mt_3t_4t_3t_{10}t_3, Mt_2t_7t_2t_3t_2, Mt_5t_2t_5t_8t_5,$ $Mt_6t_5t_6t_1t_6, Mt_8t_3t_8t_9t_8, Mt_{10}t_1t_{10}t_7t_{10}, Mt_3t_6t_3t_2t_3, Mt_2t_9t_2t_5t_2, Mt_5t_4t_5t_{10}t_5,$ $Mt_4t_7t_4t_3t_4, Mt_7t_2t_7t_8t_7, Mt_8t_5t_8t_1t_8, Mt_{10}t_3t_{10}t_9t_{10}, Mt_3t_8t_3t_4t_3, Mt_2t_1t_2t_7t_2,$ $Mt_5t_6t_5t_2t_5, Mt_4t_9t_4t_5t_4, Mt_7t_4t_7t_{10}t_7, Mt_9t_2t_9t_8t_9, Mt_{10}t_5t_{10}t_1t_{10}, Mt_3t_{10}t_3t_6t_3,$ $Mt_2t_3t_2t_9t_2, Mt_5t_8t_5t_4t_5, Mt_4t_1t_4t_7t_4, Mt_7t_6t_7t_2t_7, Mt_9t_4t_9t_{10}t_9, Mt_2t_5t_2t_1t_2,$ $Mt_5t_{10}t_5t_6t_5, Mt_4t_3t_4t_9t_4, Mt_7t_8t_7t_4t_7, Mt_9t_6t_9t_2t_9, Mt_4t_5t_4t_1t_4, Mt_7t_{10}t_7t_6t_7,$ $Mt_9t_8t_9t_4t_9, Mt_9t_{10}t_9t_6t_9\}$. We find that $Mt_1t_2t_1t_8t_1 = Mt_3t_4t_3t_{10}t_3$ $= Mt_5t_6t_5t_2t_5 = Mt_7t_8t_7t_4t_7 = Mt_9t_{10}t_9t_6t_9$. Thus the coset stabiliser of the double coset $Mt_1t_2t_1t_8t_1$ is : $N^{12181} \geq N^{(12181)} = \{e, (1,3,5,7,9)(2,4,6,8,10),$ $(1,5,9,3,7)(2,6,10,4,8), (1,7,3,9,5)(2,8,4,10,6), (1,9,7,5,3)(2,10,8,6,4)\}$. Thus the number of single cosets of $Mt_1t_2t_1t_8t_1N$ is at most $\frac{|N|}{|N^{(12181)}|} = \frac{50}{5} = 10$. Now the orbits of [12181] are:

$$\mathcal{O} = \{\{1,3,5,7,9\}, \{2,4,6,8,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_1t_8t_1$, we have two possible new double cosets, but we find the following:

$$Mt_1t_2t_1t_8t_1t_1 = Mt_1t_2t_1t_8 \in [1218]$$
$$Mt_1t_2t_1t_8t_1t_1 = Mt_1t_9t_2t_{10} \in [19210]$$

Based on the information found, five symmetric generators return to [1218]. The other symmetric generator collapses, thus this branch ends here.

$\mathbf{Mt_1t_2t_7N}$

Consider the double coset $[127] = Mt_1t_2t_7N = \{M(t_1t_2t_7)^n | n \in N\} = \{Mt_1t_2t_7,$ $Mt_1t_4t_7, Mt_6t_7t_2, Mt_1t_6t_7, Mt_6t_9t_2, Mt_8t_7t_4, Mt_1t_8t_7, Mt_6t_1t_2, Mt_8t_9t_4, Mt_{10}t_7t_6,$ $Mt_3t_2t_9, Mt_1t_{10}t_7, Mt_6t_3t_2, Mt_8t_1t_4, Mt_{10}t_9t_6, Mt_3t_4t_9, Mt_2t_7t_8, Mt_5t_2t_1, Mt_6t_5t_2,$ $Mt_8t_3t_4, Mt_{10}t_1t_6, Mt_3t_6t_9, Mt_2t_9t_8, Mt_5t_4t_1, Mt_4t_7t_{10}, Mt_7t_2t_3, Mt_8t_5t_4, Mt_{10}t_3t_6,$ $Mt_3t_8t_9, Mt_2t_1t_8, Mt_5t_6t_1, Mt_4t_9t_{10}, Mt_7t_4t_3, Mt_9t_2t_5, Mt_{10}t_5t_6, Mt_3t_{10}t_9, Mt_2t_3t_8,$ $Mt_5t_8t_1, Mt_4t_1t_{10}, Mt_7t_6t_3, Mt_9t_4t_5, Mt_2t_5t_8, Mt_5t_{10}t_1, Mt_4t_3t_{10}, Mt_7t_8t_3,$ $Mt_9t_6t_5, Mt_4t_5t_{10}, Mt_7t_{10}t_7, Mt_9t_8t_9, Mt_9t_{10}t_9\}$. The coset stabiliser of the double coset $Mt_1t_2t_1$ is : $N^{127} \geq N^{(127)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_7N$ is at most $\frac{|N|}{|N^{(127)}|} = \frac{50}{1} = 50$. Now the orbits of [127] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_7$, we

have 10 possible new double cosets, but we find the following:

$$Mt_1t_2t_7t_1 = Mt_1t_2t_9 \in [129]$$
$$Mt_1t_2t_7t_3 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_7t_4 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_7t_5 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_7t_7 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_7t_8 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_7t_9 = Mt_1t_2t_1t_9 \in [1219]$$
$$Mt_1t_2t_7t_{10} = Mt_1t_9t_2t_1 \in [1921]$$

Based on the information found, one symmetric generator returns to [12] and two symmetric generators move forward to two new double cosets $Mt_1t_2t_7t_6$ and $Mt_1t_2t_7t_2$ respectively. The remaining double cosets collapse.

$\mathbf{Mt_1t_2t_7t_2N}$

Consider the double coset $[1272] = Mt_1t_2t_7t_2N = \{M(t_1t_2t_7t_2)^n | n \in N\}$ $= \{Mt_1t_2t_7t_2, Mt_1t_4t_7t_4, Mt_6t_7t_2t_7, Mt_1t_6t_7t_6, Mt_6t_9t_2t_9, Mt_8t_7t_4t_7, Mt_1t_8t_7t_8,$ $Mt_6t_1t_2t_1, Mt_8t_9t_4t_9, Mt_{10}t_7t_6t_7, Mt_3t_2t_9t_2, Mt_1t_{10}t_7t_{10}, Mt_6t_3t_2t_3, Mt_8t_1t_4t_8,$ $Mt_{10}t_9t_6t_9, Mt_3t_4t_9t_4, Mt_2t_7t_8t_7, Mt_5t_2t_1t_2, Mt_6t_5t_2t_5, Mt_8t_3t_4t_3, Mt_{10}t_1t_6t_1,$ $Mt_3t_6t_9t_6, Mt_2t_9t_8t_9, Mt_5t_4t_1t_4, Mt_4t_7t_{10}t_7, Mt_7t_2t_3t_2, Mt_8t_5t_4t_5, Mt_{10}t_3t_6t_3,$ $Mt_3t_8t_9t_8, Mt_2t_1t_8t_1, Mt_5t_6t_1t_6, Mt_4t_9t_{10}t_9, Mt_7t_4t_3t_4, Mt_9t_2t_5t_2, Mt_{10}t_5t_6t_5,$ $Mt_3t_{10}t_9t_{10}, Mt_2t_3t_8t_3, Mt_5t_8t_1t_8, Mt_4t_1t_{10}t_1, Mt_7t_6t_3t_6, Mt_9t_4t_5t_4, Mt_2t_5t_8t_5,$ $Mt_5t_{10}t_1t_{10}, Mt_4t_3t_{10}t_3, Mt_7t_8t_3t_8, Mt_9t_6t_5t_6, Mt_4t_5t_{10}t_5, Mt_7t_{10}t_7t_{10},$ $Mt_9t_8t_9t_8, Mt_9t_{10}t_9t_{10}\}$. We find through **MAGMA** that $Mt_1t_2t_7t_2 = Mt_8t_5t_4t_5$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_7t_2$ is : $N^{1272} \geq N^{(1272)} = \{e, (1,8)(2,5)(3,10)(4,7)(6,9)\}$. Since this element in N sends $t_1 \to t_8, t_2 \to t_5, t_7 \to t_4$, and $t_2 \to t_5$. Thus the number of single cosets of $Mt_1t_2t_7t_2N$ is at most $\frac{|N|}{|N^{(1272)}|} = \frac{50}{2} = 50$. Now the orbits of [1272] are:

$$\mathcal{O} = \{\{1,8\},\{2,5\},\{3,10\},\{4,7\},\{6,9\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_7t_2$, we have 5 possible new double cosets, but we find the following:

$$Mt_1t_2t_7t_2t_1 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_2t_7t_2t_2 = Mt_1t_2t_7 \in [127]$$
$$Mt_1t_2t_7t_2t_3 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_7t_2t_4 = Mt_1t_2t_1t_2t_3 \in [12123]$$
$$Mt_1t_2t_7t_2t_6 = Mt_1t_2t_1t_2t_1 \in [12121]$$

Based on the information found, two symmetric generators return to [127] and the remaining double cosets collapse. Thus, we research $Mt_1t_2t_7t_6$.

$\mathbf{Mt_1t_2t_7t_6N}$

Consider the double coset $[1276] = Mt_1t_2t_7t_6N = \{M(t_1t_2t_7t_6)^n | n \in N\}$ $= \{Mt_1t_2t_7t_6, Mt_1t_4t_7t_8, Mt_6t_7t_2t_1, Mt_1t_6t_7t_{10}, Mt_6t_9t_2t_3, Mt_8t_7t_4t_1, Mt_1t_8t_7t_2,$ $Mt_6t_1t_2t_5, Mt_8t_9t_4t_3, Mt_{10}t_7t_6t_1, Mt_3t_2t_9t_6, Mt_1t_{10}t_7t_4, Mt_6t_3t_2t_7, Mt_8t_1t_4t_5,$ $Mt_{10}t_9t_6t_3, Mt_3t_4t_9t_8, Mt_2t_7t_8t_1, Mt_5t_2t_1t_6, Mt_6t_5t_2t_9, Mt_8t_3t_4t_7, Mt_{10}t_1t_6t_5,$ $Mt_3t_6t_9t_{10}, Mt_2t_9t_8t_3, Mt_5t_4t_1t_8, Mt_4t_7t_{10}t_1, Mt_7t_2t_3t_6, Mt_8t_5t_4t_9, Mt_{10}t_3t_6t_7,$ $Mt_3t_8t_9t_2, Mt_2t_1t_8t_5, Mt_5t_6t_1t_{10}, Mt_4t_9t_{10}t_3, Mt_7t_4t_3t_8, Mt_9t_2t_5t_6, Mt_{10}t_5t_6t_9,$ $Mt_3t_{10}t_9t_4, Mt_2t_3t_8t_7, Mt_5t_8t_1t_2, Mt_4t_1t_{10}t_5, Mt_7t_6t_3t_{10}, Mt_9t_4t_5t_8, Mt_2t_5t_8t_9,$ $Mt_5t_{10}t_1t_4, Mt_4t_3t_{10}t_7, Mt_7t_8t_3t_2, Mt_9t_6t_5t_{10}, Mt_4t_5t_{10}t_9, Mt_7t_{10}t_3t_4, Mt_9t_8t_5t_2,$ $Mt_9t_{10}t_5t_4\}$. The coset stabiliser of the double coset $Mt_1t_2t_7t_6N$ is : $N^{1276} \geq N^{(1276)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_7t_6N$ is at most $\frac{|N|}{|N^{(1276)}|} = \frac{50}{1} = 50$. Now the orbits of [1276] are:

$$\mathcal{O} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_7t_6$, we

have ten possible new double cosets, but we find the following:

$$Mt_1t_2t_7t_6t_1 = Mt_1t_9t_2t_{10} \in [19210]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_1t_{10}t_7t_{10} \in [12110710]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_1t_2t_3 \in [12123]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_7 \in [127]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_9t_2t_1t_2 \in [19212]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$
$$Mt_1t_2t_7t_6t_1 = Mt_1t_2t_7t_2 \in [1272]$$

Based on the information found, one symmetric generator returns to [127] and the remaining double cosets collapse. Thus this branch ends here. We next investigate $Mt_1t_2t_5N$

$\mathbf{Mt_1t_2t_5N}$

Consider the double coset $[125] = Mt_1t_2t_5N = \{M(t_1t_2t_5)^n | n \in N\} = \{Mt_1t_2t_5,$ $Mt_1t_4t_5, Mt_6t_7t_{10}, Mt_1t_6t_5, Mt_6t_9t_{10}, Mt_8t_7t_2, Mt_1t_8t_5, Mt_6t_1t_{10}, Mt_8t_9t_2, Mt_{10}t_7t_4,$ $Mt_3t_2t_7, Mt_1t_{10}t_5, Mt_6t_3t_{10}, Mt_8t_1t_2, Mt_{10}t_9t_4, Mt_3t_4t_7, Mt_2t_7t_6, Mt_5t_2t_9, Mt_6t_5t_{10},$ $Mt_8t_3t_2, Mt_{10}t_1t_4, Mt_3t_6t_7, Mt_2t_9t_6, Mt_5t_4t_9, Mt_4t_7t_8, Mt_7t_2t_1, Mt_8t_5t_2, Mt_{10}t_3t_4,$ $Mt_3t_8t_7, Mt_2t_1t_6, Mt_5t_6t_9, Mt_4t_9t_8, Mt_7t_4t_1, Mt_9t_2t_3, Mt_{10}t_5t_4, Mt_3t_{10}t_7, Mt_2t_3t_6,$ $Mt_5t_8t_9, Mt_4t_1t_8, Mt_7t_6t_1, Mt_9t_4t_3, Mt_2t_5t_6, Mt_5t_{10}t_9, Mt_4t_3t_8, Mt_7t_8t_1, Mt_9t_6t_3,$ $Mt_4t_5t_8, Mt_7t_{10}t_1, Mt_9t_8t_3, Mt_9t_{10}t_3\}$. The coset stabiliser of the double coset $Mt_1t_2t_5$ is : $N^{125} \geq N^{(125)} = \{e\}$. Thus the number of single cosets of $Mt_1t_2t_5N$ is at most $\frac{|N|}{|N^{(125)}|} = \frac{50}{1} = 50$. Now the orbits of [125] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_5$, we have ten possible new double cosets, but we find the following:

$$Mt_1t_2t_5t_1 = Mt_1t_2t_9 \in [129]$$
$$Mt_1t_2t_5t_3 = Mt_1t_2t_1t_9 \in [1219]$$
$$Mt_1t_2t_5t_4 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_5t_5 = Mt_1t_2 \in [12]$$
$$Mt_1t_2t_5t_6 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_5t_7 = Mt_1t_2t_7 \in [127]$$
$$Mt_1t_2t_5t_9 = Mt_1t_2t_1 \in [121]$$
$$Mt_1t_2t_5t_{10} = Mt_1t_2t_7 \in [127]$$

Based on the information found, one symmetric generator returns to [12] and two symmetric generators move forward to two new double cosets $Mt_1t_2t_5t_2$ and $Mt_1t_2t_5t_8$ respectively. The remaining double cosets collapse.

$\mathbf{Mt_1t_2t_5t_2N}$

Consider the double coset $[1252] = Mt_1t_2t_5t_2N = \{M(t_1t_2t_5t_2)^n | n \in N\}$
$= \{Mt_1t_2t_5t_2, Mt_1t_4t_5t_4, Mt_6t_7t_{10}t_7, Mt_1t_6t_5t_6, Mt_6t_9t_{10}t_9, Mt_8t_7t_2t_7, Mt_1t_8t_5t_8, Mt_6t_1t_{10}t_1, Mt_8t_9t_2t_9, Mt_{10}t_7t_4t_7, Mt_3t_2t_7t_2, Mt_1t_{10}t_5t_{10}, Mt_6t_3t_{10}t_3, Mt_8t_1t_2t_1, Mt_{10}t_9t_4t_9, Mt_3t_4t_7t_4, Mt_2t_7t_6t_7, Mt_5t_2t_9t_2, Mt_6t_5t_{10}t_5, Mt_8t_3t_2t_3, Mt_{10}t_1t_4t_1, Mt_3t_6t_7t_6, Mt_2t_9t_6t_9, Mt_5t_4t_9t_4, Mt_4t_7t_8t_7, Mt_7t_2t_1t_2, Mt_8t_5t_2t_5, Mt_{10}t_3t_4t_3, Mt_3t_8t_7t_8, Mt_2t_1t_6t_1, Mt_5t_6t_9t_6, Mt_4t_9t_8t_9, Mt_7t_4t_1t_4, Mt_9t_2t_3t_2, Mt_{10}t_5t_4t_5, Mt_3t_{10}t_7t_{10}, Mt_2t_3t_6t_3, Mt_5t_8t_9t_8, Mt_4t_1t_8t_1, Mt_7t_6t_1t_6, Mt_9t_4t_3t_4, Mt_2t_5t_6t_5, Mt_5t_{10}t_9t_{10}, Mt_4t_3t_8t_3, Mt_7t_8t_1t_8, Mt_9t_6t_3t_6, Mt_4t_5t_8t_5, Mt_7t_{10}t_1t_{10}, Mt_9t_8t_3t_8, Mt_9t_{10}t_3t_{10}\}$. We find that $Mt_1t_2t_5t_2 = Mt_6t_7t_{10}t_7$. Therefore the coset stabiliser of the double coset $Mt_1t_2t_5t_2N$ is : $N^{1252} \geq N^{(1252)} = \{e, (1,6)(2,7)(3,8)(4,9)(5,10)\}$, since this element in N sends $t_1 \to t_6, t_2 \to t_7, t_5 \to t_{10}$, and $t_2 \to t_7$. Thus the number of single cosets of $Mt_1t_2t_5t_2N$ is at most $\frac{|N|}{|N^{(1252)}|} = \frac{50}{2} = 25$. Now the orbits of [1252] are:

$$\mathcal{O} = \{\{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_5t_2$, we have five possible new double cosets, but we find the following:

$$Mt_1t_2t_5t_2t_1 = Mt_1t_2t_5t_8 \in [1258]$$
$$Mt_1t_2t_5t_2t_2 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_5t_2t_3 = Mt_1t_2t_1t_2t_1 \in [12121]$$
$$Mt_1t_2t_5t_2t_4 = Mt_1t_2t_1t_2 \in [1212]$$
$$Mt_1t_2t_5t_2t_5 = Mt_1t_2t_1t_2t_9 \in [12129]$$

Based on the information found, two symmetric generators return to [125] and the remaining double cosets collapse. Thus this branch ends here. We next investigate $Mt_1t_2t_5t_8N$

$\mathbf{Mt_1t_2t_5t_8N}$

Consider the double coset $[1258] = Mt_1t_2t_5t_8N = \{M(t_1t_2t_5t_8)^n|n \in N\} = \{Mt_1t_2t_5t_8, Mt_1t_4t_5t_{10}, Mt_6t_7t_{10}t_3, Mt_1t_6t_5t_2, Mt_6t_9t_{10}t_5, Mt_8t_7t_2t_3, Mt_1t_8t_5t_4, Mt_6t_1t_{10}t_7, Mt_8t_9t_2t_5, Mt_{10}t_7t_4t_3, Mt_3t_2t_7t_8, Mt_1t_{10}t_5t_6, Mt_6t_3t_{10}t_9, Mt_8t_1t_2t_7, Mt_{10}t_9t_4t_5, Mt_3t_4t_7t_{10}, Mt_2t_7t_6t_3, Mt_5t_2t_9t_8, Mt_6t_5t_{10}t_1, Mt_8t_3t_2t_9, Mt_{10}t_1t_4t_7, Mt_3t_6t_7t_2, Mt_2t_9t_6t_5, Mt_5t_4t_9t_{10}, Mt_4t_7t_8t_3, Mt_7t_2t_1t_8, Mt_8t_5t_2t_1, Mt_{10}t_3t_4t_9, Mt_3t_8t_7t_4, Mt_2t_1t_6t_7, Mt_5t_6t_9t_2, Mt_4t_9t_8t_5, Mt_7t_4t_1t_{10}, Mt_9t_2t_3t_8, Mt_{10}t_5t_4t_1, Mt_3t_{10}t_7t_6, Mt_2t_3t_6t_9, Mt_5t_8t_9t_4, Mt_4t_1t_8t_7, Mt_7t_6t_1t_2, Mt_9t_4t_3t_{10}, Mt_2t_5t_6t_1, Mt_5t_{10}t_9t_6, Mt_4t_3t_8t_9, Mt_7t_8t_1t_4, Mt_9t_6t_3t_2, Mt_4t_5t_8t_1, Mt_7t_{10}t_1t_6, Mt_9t_8t_3t_4, Mt_9t_{10}t_3t_6\}$. The coset stabiliser of the double coset $Mt_1t_2t_5t_8N$ is : $N^{1258} \geq N^{(1258)} = \{e\}$, since this element The number of single cosets of $Mt_1t_2t_5t_8N$ is at most $\frac{|N|}{|N^{(1258)}|} = \frac{50}{1} = 50$. Now the orbits of [1258] are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_5t_8$, we

have ten possible new double cosets, but we find the following using **MAGMA** :

$$Mt_1t_2t_5t_8t_1 = Mt_1t_9t_2t_{10} \in [19210]$$
$$Mt_1t_2t_5t_8t_2 = Mt_1t_9t_2t_1 \in [1921]$$
$$Mt_1t_2t_5t_8t_3 = Mt_1t_2t_1t_{10}t_7 \in [121107]$$
$$Mt_1t_2t_5t_8t_4 = Mt_1t_2t_5t_2 \in [1252]$$
$$Mt_1t_2t_5t_8t_5 = Mt_1t_2t_1t_9t_2 \in [12192]$$
$$Mt_1t_2t_5t_8t_6 = Mt_1t_9t_2t_1t_2 \in [19212]$$
$$Mt_1t_2t_5t_8t_7 = Mt_1t_2t_7t_6 \in [1276]$$
$$Mt_1t_2t_5t_8t_8 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_5t_8t_9 = Mt_1t_2t_1t_{10}t_7t_{10} \in [12110710]$$
$$Mt_1t_2t_5t_8t_4 = Mt_1t_2t_1t_2t_9 \in [12129]$$

Based on the information found, one symmetric generator returns to [125] and the remaining double cosets collapse. Thus this branch ends here. We next investigate $Mt_1t_2t_9N$

$\mathbf{Mt_1t_2t_9N}$

Consider the double coset $[129] = Mt_1t_2t_9N = \{M(t_1t_2t_9)^n | n \in N\} = \{Mt_1t_2t_9,$ $Mt_1t_4t_9, Mt_6t_7t_4, Mt_1t_6t_9, Mt_6t_9t_4, Mt_8t_7t_6, Mt_1t_8t_9, Mt_6t_1t_4, Mt_8t_9t_6, Mt_{10}t_7t_8,$ $Mt_3t_2t_1, Mt_1t_{10}t_9, Mt_6t_3t_4, Mt_8t_1t_6, Mt_{10}t_9t_8, Mt_3t_4t_1, Mt_2t_7t_{10}, Mt_5t_2t_3, Mt_6t_5t_4,$ $Mt_8t_3t_6, Mt_{10}t_1t_8, Mt_3t_6t_1, Mt_2t_9t_{10}, Mt_5t_4t_3, Mt_4t_7t_2, Mt_7t_2t_5, Mt_8t_5t_6, Mt_{10}t_3t_8,$ $Mt_3t_8t_1, Mt_2t_1t_{10}, Mt_5t_6t_3, Mt_4t_9t_2, Mt_7t_4t_5, Mt_9t_2t_7, Mt_{10}t_5t_8, Mt_3t_{10}t_1,$ $Mt_2t_3t_{10}, Mt_5t_8t_3, Mt_4t_1t_2, Mt_7t_6t_5, Mt_9t_4t_7, Mt_2t_5t_{10}, Mt_5t_{10}t_3, Mt_4t_3t_2, Mt_7t_8t_5,$ $Mt_9t_6t_7, Mt_4t_5t_2, Mt_7t_{10}t_5, Mt_9t_8t_7, Mt_9t_{10}t_7\}$. We find that $Mt_1t_2t_9 = Mt_{10}t_3t_8$, thus the coset stabiliser of the double coset $Mt_1t_2t_9N$ is : $N^{129} \geq N^{(129)} = \{e, (1,10)(2,3)(4,5)(6,7)(8,9)\}$. Thus, the number of single cosets of $Mt_1t_2t_9N$ is at most $\frac{|N|}{|N^{(129)}|} = \frac{50}{2} = 25$. Now, the orbits of [129] are:

$$\mathcal{O} = \{\{1,10\}, \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}\}.$$

Choosing a representative from each orbit and right multiplying to $Mt_1t_2t_9$, we have five possible new double cosets, but we find the following:

$$Mt_1t_2t_9t_1 = Mt_1t_2t_9 \in [129]$$
$$Mt_1t_2t_9t_2 = Mt_1t_2t_5 \in [125]$$
$$Mt_1t_2t_9t_4 = Mt_1t_2t_9 \in [129]$$
$$Mt_1t_2t_9t_6 = Mt_1t_2t_7 \in [127]$$
$$Mt_1t_2t_9t_8 = Mt_1t_2 \in [12]$$

Based on the information found, two symmetric generators return to [12] and two symmetric generators loop back into [129]. The remaining collapse, therefore this branch ends here.

We have the index of $\frac{|G|}{|M|} = 1008$. Therefore we have that G is the union of the 28 double cosets found above, illustrated as follows:

$$G = MeN \cup Mt_1N \cup Mt_1t_2N \cup Mt_1t_9N \cup Mt_1t_2t_7N \cup Mt_1t_2t_7t_2N \cup Mt_1t_2t_7t_6N$$

$$\cup Mt_1t_2t_5N \cup Mt_1t_2t_5t_2N \cup Mt_1t_2t_5t_8N \cup Mt_1t_9t_1N \cup Mt_1t_9t_2N$$

$$\cup Mt_1t_9t_2t_{10}N \cup Mt_1t_9t_2t_1N \cup Mt_1t_9t_2t_1t_2N \cup Mt_1t_2t_1N$$

$$\cup Mt_1t_2t_1t_8N \cup Mt_1t_2t_1t_8t_1N \cup Mt_1t_2t_1t_{10}N \cup Mt_1t_2t_1t_{10}t_7N$$

$$\cup Mt_1t_2t_1t_{10}t_7t_{10}N \cup Mt_1t_2t_1t_2N \cup Mt_1t_2t_1t_2t_1N \cup Mt_1t_2t_1t_2t_3N$$

$$\cup Mt_1t_2t_1t_2t_7t_9N \cup Mt_1t_2t_1t_9N \cup Mt_1t_2t_1t_9t_2N cup Mt_1t_2t_1t_2t_9N$$

Adding the number of singles cosets in each double cosets yields the desiired result.

$$|G| \leq |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(19)}|} + \frac{|N|}{|N^{(127)}|} + \frac{|N|}{|N^{(1272)}|} + \frac{|N|}{|N^{(1276)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(1252)}|}$$

$$+\frac{|N|}{|N^{(1258)}|}+\frac{|N|}{|N^{(191)}|}+\frac{|N|}{|N^{(192)}|}+\frac{|N|}{|N^{(19210)}|}+\frac{|N|}{|N^{(1921)}|}+\frac{|N|}{|N^{(19212)}|}+\frac{|N|}{|N^{(121)}|}+\frac{|N|}{|N^{(1218)}|}$$

$$+\frac{|N|}{|N^{(12181)}|}+\frac{|N|}{|N^{(12110)}|}+\frac{|N|}{|N^{(121107)}|}+\frac{|N|}{|N^{(12110710)}|}+\frac{|N|}{|N^{(1212)}|}+\frac{|N|}{|N^{(12121)}|}+$$

$$\frac{|N|}{N^{(12123)}|}+\frac{|N|}{|N^{(121279)}|}+\frac{|N|}{|N^{(1219)}|}+\frac{|N|}{|N^{(12192)}|}+\frac{|N|}{|N^{(12129)}|}$$

which equals:

$$|G| \leq (1+10+50+10+50+25+50+50+25+50+2+50+50+50+50+50+50+10+50+50+25+50+25+25+25+50+50+25)\times 1200 = |G| = 1209600.$$

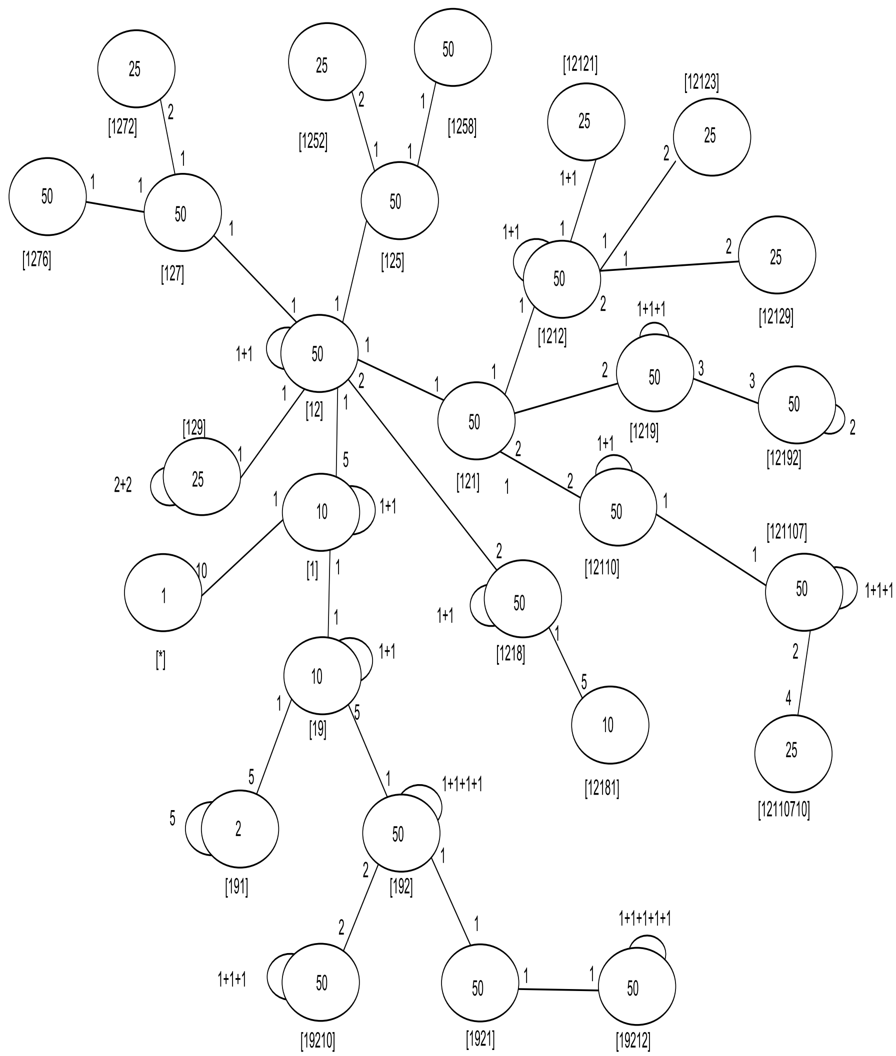

Figure 5.2: Cayley graph of J_2 over $(10:2):A_5$

Next we will briefly show the process of factoring a large group by the center. We do this to attempt to perform double coset enumeration of G over N. In this particular case, the group was still too large.

5.2.1 Factoring $S(4,4)$ by the Center $Z(G)$

A presentation for $S(4,4)$ is given by:

```
G=<v,w,x,y,z,t|v^2,w^4,x^2,y^3,z^3,w^-2*x,
(w^-1*v)^2,(x*y^-1)^2,
v*z^-1*v*z,(x*z^-1)^2,(y,z),w*y^-1*w^-1*y*z^-1,
(t,v*x*z^-1),t^2,(v*w^-1*t)^2, (v*w*z*t)^5>
```

We verify that the order of G is correct and check if this group has a center, with the following code:

```
#G;
/*3916800*/
f, G1, k:=CosetAction(G,sub<G|v,w,x,y,z>);
#k;
CompositionFactors(G1);
 G
    |  C(2, 4)                    = S(4, 4)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
Center(G1);
/*Permutation group acting on a set of cardinality 54400
Order = 4 = 2^2*/
C:=Center(G1);
/*Order 4*/
D:=C.1;
E:=C.2;
```

Here the center is generated by two permutations, which are of order two, but too large to show here. Thus we divide our center into two parts, D and E from above.

Now to write these permutations into words, we use the Schrierer System. We need to write these permutations into words to include these relations into our progenitor.

```
N:=G1;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
 ArrayP:=[Id(N): i in [1..#G]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(v); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(w^-1); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=f(y^-1); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=f(z^-1); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;
if Eltseq(Sch[i])[j] eq -6 then P[j]:=f(t^-1); end if;
end for;
 PP:=Id(N);
 for k in [1..#P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq D then Sch[i]; end if;
end for;
for i in [1..#N] do if ArrayP[i] eq E then Sch[i]; end if;
 end for;

MAGMA gives us these relations:

(v * t * w * t * x * t * x * t)^3,
(w * t * w * t * x * t * z * t * x * t * x * t *
 x * t * w * t * y^-1 * t * x * t* x * z * t * x * t)
```

We can now include these relations into our progenitor and check the new order of G:

```
G<v,w,x,y,z,t>:=Group<v,w,x,y,z,t|v^2,w^4,x^2,y^3,z^3,
w^-2*x,(w^-1*v)^2,(x*y^-1)^2,
 v*z^-1*v*z,(x*z^-1)^2,(y,z),w*y^-1*w^-1*y*z^-1,
 (t,v*x*z^-1),t^2,(v*w^-1*t)^2,
 (v*y^-1*z^-1*t)^0,(v*w*t)^0,(x*t)^0,(y*t)^0,
(z*t)^4,(w*t)^0,(y*v*t)^0,
 (v*w*z*t)^5, (v * t * w * t * x * t * x * t)^3,
```

```
(w * t * w * t * x * t * z * t * x * t * x * t *
 x * t * w * t * y^-1 * t * x * t * x * z * t * x * t)>;
 #G;
979200
```

Thus, we obtain the desired result, though the order of G is still to large to perfrom double coset enumeration.

Chapter 6

Monomial Progenitors: Creating Character Table of G from H and Monomial Progenitor Produces Sporadic Group M_{11}

Definition 6.1. *[Rot95]* **Kernel** *of* $\chi = \{g \in G | \chi(g) = \chi(1)\}$

Theorem 6.2. *[Led87] Let N be a normal saubgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N_x)$ defines a representation of G LIFTED from G/N. If $\phi_0(N_x)$ is the character of $A_0(N_x)$, then $\phi(x) = \phi_0(N_x)$ is the LIFTED character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. The LIFTING PROCESS preserves irreducibility.*

Construction of Character Table of $C_5 : C_4$

We will demonstrate how to construct the character table of $C_5 : C_4$ from a normal subgroup H.

To construct the character table of G we first need to find a normal subgroup H.

```
Gh:=DerivedGroup(G);
 Gh;
```

```
Permutation group Gh acting on a set of cardinality 10
Order = 5
H:=(1, 3, 5, 7, 9)(2, 4, 6, 8, 10);
```

Thus $H =< (1,3,5,7,9)(2,4,6,8,10) >$, and $H \cong D_5$. We will let
$H =< (2,10)(3,9)(4,8)(5,7),(1,3,5,7,9)(2,4,6,8,10) >$, such that $\frac{|G|}{|H|} = 4$. This group $G/H \cong C_4$. We now investigate the structure of the group $C_4 =< e, a^4, b >$. We know that there will be four transversals of C_4 in G, which create the set:

```
{H(Id),H(1, 2, 9, 8)(3, 6, 7, 4)(5, 10),
H(1, 9)(2, 8)(3, 7)(4, 6),H(1, 8, 9, 2)(3, 4, 7, 6)(5, 10)}
```

We will need to construct the character table of C_4, which we begin by obtaining the conjugacy classes. The conjugacy classes of C_4 are labeled as follows:

Table 6.1: Conjugacy Classes of C_4

Class	Perm	Rep
D_1	e	a^0b^0
D_2	(1, 3)(2, 4)	a^2b^0
D_3	(1, 2, 3, 4)	ab^0
D_4	(1, 4, 3, 2)	a^3b^0

Since we are working with C_4, we know that we most likely have a fourth root of unity. To verify, we use the formula: $z = 1^{\frac{1}{4}} = [cos(\frac{2\pi k}{4}) + isin(\frac{2\pi k}{4})] = [cos(\frac{\pi k}{2}) + isin(\frac{\pi k}{2})]$. If we look at the unit circle, the values of z will be found at every $\frac{\pi}{2}$ interval. This means that $z = 1, I, -1$ and $-I$. We have $i^0 = 1, i = I, i^2 = -1$, and $i^3 = -I$ for our character table of C_4. We will now compute:

for $x_{.1}^{.}$
$i^0 = (i^0)^0(i^0)^0 = 1 \cdot 1 = 1$
$i = (i^1)^0(i^1)^0 = 1 \cdot 1 = 1$
$i^2 = (i^2)^0(i^2)^0 = 1 \cdot 1 = 1$
$i^3 = (i^3)^0(i^3)^0 = 1 \cdot 1 = 1$

for $x_{.2}^{.}$
$i = (i)^0(i)^0 = 1 \cdot 1 = 1$

$i = (i)^2(i^2)^0 = -1 \cdot 1 = -1$
$i = (i)(i)^0 = I \cdot 1 = I$
$i = (i)^3(i)^0 = -I \cdot 1 = -I$

for $x^{\cdot}_{.3}$
$i^2 = (i^2)^0(i)^0 = 1 \cdot 1 = 1$
$i^2 = (i^2)^2(i^2)^0 = i^4 \cdot = 1$
$i^2 = (i^2)(i)^0 = -1 \cdot 1 = -1$
$i^2 = (i^2)^3(i)^0 = i^6 \cdot 1 = -1$

for $x^{\cdot}_{.4}$
$i^3 = (i^3)^0(i)^0 = 1 \cdot 1 = 1$
$i^3 = (i^3)^2(i^2)^0 = i^6 \cdot 1 = -1$
$i^3 = (i^3)(i)^0 = i^3 \cdot 1 = -I$
$i^3 = (i^3)^3(i)^0 = i^9 \cdot 1 = I$

The complete character table of C_4 is as follows:

Table 6.2: Character Table of C_4

Class	D_1	D_2	D_3	D_4
Rep	a^0b^0	a^2b^0	ab^0	a^3b^0
$x^{\cdot}_{.1}$	1	1	1	1
$x^{\cdot}_{.2}$	1	-1	I	-I
$x^{\cdot}_{.3}$	1	1	-1	-1
$x^{\cdot}_{.4}$	1	-1	-I	I

The same process will be applied to the construction of the character table of G. The generators of $G =< e, A^4, B^{-5}, (A^{-1} * B^{-2} * A * B^{-1}) >$. We need to find all the class representatives of G to find which class in C_4 they live in to be able to perfrom the lifting process.

```
S:=Set(Gh);
q:=[{Id(G)},{},{},{}];
for i in [1..#T] do for g in S do
q[i]:=q[i] join {g*T[i]}; end for; end for;
```

```
q;

 {
      B   (1, 3, 5, 7, 9)(2, 4, 6, 8, 10),
     B^4    (1, 9, 7, 5, 3)(2, 10, 8, 6, 4),
     B^3    (1, 7, 3, 9, 5)(2, 8, 4, 10, 6),
     B^2    (1, 5, 9, 3, 7)(2, 6, 10, 4, 8),
        Id(G)
    },
    {
     A*B^3    (1, 8, 7, 10)(2, 5, 6, 3)(4, 9),
    A*B (1, 4, 5, 2)(3, 8)(6, 9, 10, 7),
    A*B^2  (1, 6)(2, 3, 10, 9)(4, 7, 8, 5),
    A*B^4  (1, 10, 3, 4)(2, 7)(5, 8, 9, 6),
      A   (1, 2, 9, 8)(3, 6, 7, 4)(5, 10)


    A^2*B^2     (1, 3)(4, 10)(5, 9)(6, 8),
    A^2*B^4     (1, 7)(2, 6)(3, 5)(8, 10),
    A^2*B    (2, 10)(3, 9)(4, 8)(5, 7),
    A^2*B^3     (1, 5)(2, 4)(6, 10)(7, 9),
      A^2  (1, 9)(2, 8)(3, 7)(4, 6)

 {
   A^3*B      (1, 10, 7, 8)(2, 3, 6, 5)(4, 9),
   A^3    (1, 8, 9, 2)(3, 4, 7, 6)(5, 10),
 A^3*B^3;    (1, 4, 3, 10)(2, 7)(5, 6, 9, 8),
   A^3*B^4      (1, 6)(2, 9, 10, 3)(4, 5, 8, 7),
   A^3*B^2       (1, 2, 5, 4)(3, 8)(6, 7, 10, 9)
```

For example, any element of B, B^4, B^3, orB^2 will live in $Id(G)$ of the character table of C_4. Likewise all elements $A * B^3, ..., A$ will live in the class representative of A in the character table of C_4, etc.

The conjugacy classes and their representatives are given in the following table for an easier read:

We begin the lifting process as follows:

for $x_{.1}$

$x_{.1}(e) = x^{\cdot}_{.1}(e) = 1$

$x_{.1}(a^2) = x^{\cdot}_{.1}(a^2) = 1$

$x_{.1}(a) = x^{\cdot}_{.1}(a) = 1$

Table 6.3: Character Table of C_4

Class	Rep	Permutation	Length
C_1	$Id(G)$	e	1
C_2	A^2	(1, 9)(2, 8)(3, 7)(4, 6)	5
C_3	A	(1, 9)(2, 8)(3, 7)(4, 6)	5
C_4	A^3	(1, 8, 9, 2)(3, 4, 7, 6)(5, 10)	5
C_5	B	(1, 8, 9, 2)(3, 4, 7, 6)(5, 10)	5

$x_{.1}(a^3) = x^{\cdot}_{.1}(a^3) = 1$

$x_{.1}(b) = x^{\cdot}_{.1}(b) = 1$

for $x_{.2}$

$x_{.2}(e) = x^{\cdot}_{.2}(e) = 1$

$x_{.2}(a^2) = x^{\cdot}_{.2}(a^2) = -1$

$x_{.2}(a) = x^{\cdot}_{.2}(a) = I$

$x_{.2}(a^3) = x^{\cdot}_{.2}(a^3) = -I$

$x_{.2}(b) = x^{\cdot}_{.2}(b) = 1$

for $x_{.3}$

$x_{.3}(e) = x^{\cdot}_{.3}(e) = 1$

$x_{.3}(a^2) = x^{\cdot}_{.3}(a^2) = 1$

$x_{.3}(a) = x^{\cdot}_{.3}(a) = -1$

$x_{.3}(a^3) = x^{\cdot}_{.3}(a^3) = -1$

$x_{.3}(b) = x^{\cdot}_{.3}(b) = 1$

for $x_{.4}$

$x_{.4}(e) = x^{\cdot}_{.4}(e) = 1$

$x_{.4}(a^2) = x^{\cdot}_{.4}(a^2) = -1$

$x_{.4}(a) = x^{\cdot}_{.4}(a) = -I$

$x_{.4}(a^3) = x^{\cdot}_{.4}(a^3) = I$

$x_{.4}(b) = x^{\cdot}_{.4}(b) = 1$

The character table of G is the following.

Table 6.4: Character Table of G

Class	C_1	C_2	C_3	C_4	C_5
Rep	e	a^2	a	a^3	b
$x_{.1}$	1	1	1	1	1
$x_{.2}$	1	-1	I	-I	1
$x_{.3}$	1	1	-1	-1	1
$x_{.4}$	1	-1	-I	I	1
$x_{.5}$	α	β	γ	δ	ϵ

Notice that we have an unknown row, namely $x_{.5}$. We will now demonstrate how to complete the table of the unknown values $\alpha, \beta, \gamma, \delta$, and ϵ. We know by Theorem 1.87 the sum of the squares of degrees of our irreducible linear characters is equal to the $|G|$ of the character $x_{.1}$. In this case, we have $1^2 + 1^2 + 1^2 + 1^2 = 4$. Therefore $\alpha = 4$. To obtain the remaining unknown values, we will use definition 1.61, in which we take the dot product of two columns. The condition is

$$\sum_{i=1}^{k} X_{\alpha}^{(i)} \overline{X_{\beta}^{(j)}} = 0.$$

$$\sum_{i=1}^{k} X_{.5}^{(1)} \overline{X_{.5}^{(2)}} = 1 * \overline{1} + 1 * \overline{-1} + 1 * \overline{1} + 1 * \overline{-1} + 4 * \overline{\beta} = 0 \implies \beta = 0.$$
$$\sum_{i=1}^{k} X_{.5}^{(1)} \overline{X_{.5}^{(3)}} = 1 * \overline{1} + 1 * \overline{I} + 1 * \overline{-1} + 1 * \overline{-I} + 4 * \overline{\gamma} = 0 \implies \gamma = 0.$$
$$\sum_{i=1}^{k} X_{.5}^{(1)} \overline{X_{.5}^{(4)}} = 1 * \overline{1} + 1 * \overline{-I} + 1 * \overline{-1} + 1 * \overline{I} + 4 * \overline{\delta} = 0 \implies \delta = 0.$$
$$\sum_{i=1}^{k} X_{.5}^{(1)} \overline{X_{.5}^{(5)}} = 1 * \overline{1} + 1 * \overline{1} + 1 * \overline{1} + 1 * \overline{1} + 4 * \overline{\epsilon} = 0 \implies \epsilon = 1.$$

Therefore we have successfully completed the character table of G from H.

This process can be applied to all monomial representatives, but the simple case is illustrated above for an easier read.

Table 6.5: Character Table of G

Class	C_1	C_2	C_3	C_4	C_5
Rep	e	a^2	a	a^3	b
$x_{.1}$	1	1	1	1	1
$x_{.2}$	1	-1	I	-I	1
$x_{.3}$	1	1	-1	-1	1
$x_{.4}$	1	-1	-I	I	1
$x_{.5}$	4	0	0	0	1

6.0.2 Monomial Progenitor $11^{*4} :_m (C_5 : C_4)$

We will demonstrate how to construct a monomial presentation of $11^{*4} :_m (C_5 : C_4)$. A presentation for $(C_5 : C_4)$ is given by the following:

$$G < a, b, c | a^5, b^4, c, a^b = a^2 >. \text{ Here the order } G \text{ is 20.}$$

To construct a monomial presentation we first must induce an irreducible linear character from a subgroup H of G. To obtain an irreducible character we choose a subgroup H of G with an index equal to the degree of an irreducible character of G. Consider the character table of $G = (C_5 : C_4)$ in Table 1 and note G has characters $\chi_{.1}, \chi_{.2}, ..., \chi_{.5}$. We proceed using $\chi_{.5}$ which has a degree of four and look for a subgroup of order 5 so that $\frac{|G|}{|H|} = 4$. Thus we get the following index:

$$[G : H] = [(C_5 : C_4) : C_5] = 4$$

Since the index of the two groups is 4, if a matrix representation exists it will be represented by 4×4 matrices.

Verifying the Induction

We produce a character table for C_5 in table 2. We will verify the induction $\chi_{.2}$ of C_5 to $\chi_{.5}$ of $(C_5 : C_4)$ by considering the irreducible characters ϕ (of H) and ϕ^G (of G). $G = (C_5 : C_4)$ is generated by xx and yy where $xx = (1, 3, 5, 7, 9)(2, 4, 6, 8, 10)$ and $yy = (1, 2, 9, 8)(3, 6, 7, 4)(5, 10)$. Using our definition of induction along with the following equivalencies: $1 = 1$, $Z_1\#1 = 4$, $Z_1\#2 = 5$, $Z_1\#3 = 9$, $Z_1\#4 = 3$, we can reproduce ϕ^G using ϕ (of H).

$\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{20}{5} = 4$.

$\phi^G_1 = \frac{4}{1} \sum_{w \in H \cap C_1} \phi(w)$

which implies $\phi^G_1 = \frac{4}{1}(\phi(1)) = 4(1) = 4$.

$\phi^G_2 = \frac{4}{5} \sum_{w \in H \cap C_2} \phi(w)$

$\phi^G_2 = \frac{4}{5} \sum_{w \in H \cap C_2} \phi$

which implies $\phi^G_2 = \frac{4}{5}(\phi(0)) = \frac{4}{5}(0) = 0$.

$\phi^G_3 = \frac{4}{5} \sum_{w \in H \cap C_3} \phi(w)$

$\phi^G_3 = \frac{4}{5} \sum_{w \in H \cap C_3} \phi$

which implies $\phi^G_3 = \frac{4}{5}(\phi(0)) = \frac{4}{5}(0) = 0$.

$\phi^G_4 = \frac{4}{5} \sum_{w \in H \cap C_4} \phi(w)$

which implies $\phi^G_4 = \frac{4}{5}(\phi(0)) = \frac{4}{5}(0) = 0$.

$\phi^G_5 = \frac{4}{4} \sum_{w \in H \cap C_5} \phi(w)$

$\phi^G_5 = \frac{4}{4} \sum_{w \in H \cap C_2} \phi$

which implies $\phi^G_4 = \frac{4}{5}(\phi(1,3,5,7,9)(2,4,6,8,10) = \frac{4}{5}(-1) = -1$.

Therefore, $\phi \uparrow^G_H =\ 4, 0, 0, 0, -1$ and we have verified that $\chi_{.3}$ of C_5 induces $\chi_{.5}$ of $(5 : C_4)$.

Table 6.6: Character Table of G

χ	C_1	C_2	C_3	C_4	C_5
$\chi.1$	1	1	1	1	1
$\chi.2$	1	1	-1	-1	1
$\chi.3$	1	-1	$-I$	I	1
$\chi.4$	1	-1	I	$-I$	1
$\chi.5$	4	0	0	0	-1

# denotes algebraic conjugation.
Z_1 is the primitive fifth root of unity.

Table 6.7: Character Table of H

χ	D_1	D_2	D_3	D_4	D_5
$\chi.1$	1	1	1	1	1
$\chi.2$	1	Z_1	$Z_1\#2$	$Z_1\#3$	$Z_1\#4$
$\chi.3$	1	$Z_1\#2$	$Z_1\#4$	Z_1	$Z_1\#3$
$\chi.4$	1	$Z_1\#3$	Z_1	$Z_1\#4$	$Z_1\#2$
$\chi.5$	1	$Z_1\#4$	$Z_1\#3$	$Z_1\#2$	Z_1

# denotes algebraic conjugation.
Z_1 is the primitive fifth root of unity.

Table 6.8: $\chi_{.5}$ of G

ϕ^G	Class	Size	Class Representative
4	C_1	1	Id(G)
0	C_2	5	(1,9)(2,8)(3,7)(4,6)
0	C_3	5	(1,2,9,8)(3,6,7,4)(5,10)
0	C_4	5	(1,8,2,9)(2,4,7,6)(5,10)
-1	C_5	4	(1,3,5,7,9)(2,4,6,8,10)

Through induction, we now verify the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

Table 6.9: $\chi_{.2}$ of H

ϕ	Class	Size	Class Representative
1	D_1	1	Id(H)
4	D_2	1	(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)
5	D_3	1	(1, 5, 9, 3, 7)(2, 6, 10, 4, 8)
9	D_4	1	(1, 7, 3, 9, 5)(2, 8, 4, 10, 6)
3	D_5	1	(1, 9, 7, 5, 3)(2, 10, 8, 6, 4)

$$A(yy) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Verifying the Monomial Representation

$G =< e, (1,3,5,7,9)(2,4,6,8,10), (1,2,9,8)(3,6,7,4)(5,10) >$ and $H =< e, (1,3,5,7,9)(2,4,6,8,10) >$. Since H is a subgroup of G whose index is equal to the degree of G, we have that: $G = H \cup Ht_1 \cup Ht_2H \cup Ht_3H \cup Ht_4$, where the $t_i's$ are transversals of G acting on H. The transversals of G are labeled as follows:

$t_1 = e$,

$t_2 = (1,2,9,8)(3,6,7,4)(5,10)$,

$t_3 = (1,9)(2,8)(3,7)(4,6)$,

$t_4 = (1,8,9,2)(3,4,7,6)(5,10)$. We will now use the following formula to verify the matrices: Recall that G is generated by $x \sim (1,3,5,7,9)(2,4,6,8,10)$ and $y \sim (1,2,9,8)(3,6,7,4)(5,10)$. Here, ϕ of a permutation results in 0 when that permutation does not live in H.

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \phi(t_1xt_3^{-1}) & \phi(t_1xt_4^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \phi(t_2xt_3^{-1}) & \phi(t_2xt_4^{-1}) \\ \phi(t_3xt_1^{-1}) & \phi(t_3xt_2^{-1}) & \phi(t_3xt_3^{-1}) & \phi(t_3xt_4^{-1}) \\ \phi(t_4xt_1^{-1}) & \phi(t_4xt_2^{-1}) & \phi(t_4xt_3^{-1}) & \phi(t_4xt_4^{-1}) \end{bmatrix}$$

$a_{11} : \phi(t_1 x t_1^{-1}) = \phi(x^{t_1}) = \phi(x) = \phi((1,3,5,7,9)(2,4,6,8,10)) = 4$

$a_{12} : \phi(t_1 x t_2^{-1}) = \phi(ext_2^{-1}) =$
$\phi((1,3,5,7,9)(2,4,6,8,10) * (1,8,9,2)(3,4,7,6)(5,10)) =$
$\phi((1,4,3,10)(2,7)(5,6,9,8)) = 0$

$a_{13} : \phi(t_1 x t_3^{-1}) = \phi(ext_3^{-1}) = \phi((1,3,5,7,9)(2,4,6,8,10) *$
$(1,9)(2,8)(3,7)(4,6)) = \phi(1,7)(2,6)(3,5)(8,10) = 0$

$a_{14} : \phi(t_1 x t_4^{-1}) = \phi(ext_4^{-1}) = \phi((1,3,5,7,9)(2,4,6,8,10) *$
$(1,2,9,8)(3,6,7,4)(5,10) = \phi((1,6)(2,3,10,9)(4,7,8,5)) = 0$

$a_{21} : \phi(t_2 x t_1^{-1}) = \phi((1,2,9,8)(3,6,7,4)(5,10)xe) =$
$\phi((1,2,9,8)(3,6,7,4)(5,10) * (1,3,5,7,9)(2,4,6,8,10)) =$
$\phi((1,4,5,2)(3,8)(6,9,10,7)) = 0$

$a_{22} : \phi(t_2 x t_2^{-1}) = \phi(x^{(1,8,9,2)(3,4,7,6)(5,10)}) = \phi((1,7,3,9,5)(2,8,4,10,6)) = 9$

$a_{23} : \phi(t_2 x t_3^{-1}) = \phi((1,2,9,8)(3,6,7,4)(5,10) * (1,3,5,7,9)(2,4,6,8,10) *$
$(1,9)(2,8)(3,7)(4,6) = \phi((1,6)(2,9,10,3)(4,5,8,7)) = 0$

$a_{24} : \phi(t_2 x t_4^{-1}) = \phi((1,2,9,8)(3,6,7,4)(5,10) *$
$(1,3,5,7,9)(2,4,6,8,10) * (1,2,9,8)(3,6,7,4)(5,10) =$
$\phi((1,3)(4,10)(5,9)(6,8)) = 0$

$a_{31} : \phi(t_3 x t_1^{-1}) = \phi((1,9)(2,8)(3,7)(4,6) * (1,3,5,7,9)(2,4,6,8,10)e) =$
$\phi((2,10)(3,9)(4,8)(5,7)) = 0$

$a_{32} : \phi(t_3 x t_2^{-1}) = \phi((1,9)(2,8)(3,7)(4,6) * (1,3,5,7,9)(2,4,6,8,10) *$
$(1,8,9,2)(3,4,7,6)(5,10)) = \phi((1,8,7,10)(2,5,6,3)(4,9)) = 0$

$a_{33}:\phi(t_3xt_3^{-1})=\phi(x^{(1,9)(2,8)(3,7)(4,6)})=\phi((1,9,7,5,3)(2,10,8,6,4))=3$

$a_{34}:\phi(t_3xt_4^{-1})=\phi((1,9)(2,8)(3,7)(4,6)*(1,3,5,7,9)(2,4,6,8,10)*$
$(1,2,9,8)(3,6,7,4)(5,10))=\phi((1,2,5,4)(3,8)(6,7,10,9))=0$

$a_{41}:\phi(t_4xt_1^{-1})=\phi((1,8,9,2)(3,4,7,6)(5,10)*(1,3,5,7,9)(2,4,6,8,10)*e)=$
$\phi((1,10,7,8)(2,3,6,5)(4,9))=0$

$a_{42}:\phi(t_4xt_2^{-1})=\phi((1,8,9,2)(3,4,7,6)(5,10)*(1,3,5,7,9)(2,4,6,8,10)*$
$(1,8,9,2)(3,4,7,6)(5,10))=\phi((1,5)(2,4)(6,10)(7,9))=0$

$a_{43}:\phi(t_4xt_3^{-1})=\phi((1,8,9,2)(3,4,7,6)(5,10)*(1,3,5,7,9)(2,4,6,8,10)*$
$(1,9)(2,8)(3,7)(4,6))=\phi((1,10,3,4)(2,7)(5,8,9,6))=0$

$a_{44}:\phi(t_4xt_4^{-1})=\phi((1,3,5,7,9)(2,4,6,8,10)^{(1,8,9,2)(3,4,7,6)(5,10)})=$
$\phi((1,5,9,3,7)(2,6,10,4,8))=5$

We then follow the same procedure for $A(yy)$ and find that the matrix is correct. Therefore the matrix representation of $A(xx)$ and $A(yy)$ respectively are as follows:

$$A(xx)=\begin{bmatrix}4&0&0&0\\0&9&0&0\\0&0&3&0\\0&0&0&5\end{bmatrix}$$
$$A(yy)=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

To prove the faithful representation of $(C_5:C_4)$ generated by $<x^5,y^4,z,x^y=x^2>$, where $|(C_5:C_4|=20$, we simply check the order of each matrix representation: $|A(xx)|=5$, and $|A(yy)|=4$, then $|A(x)||A(y)|=20$. which is the order of our index.

We can now conclude that $G = < x, y > \cong < A(x), A(y) >$. Now, to finalize the process, we factor our progenitor by necessary relations. We verify we have the correct progenitor by using the Grindstaff Lemma which verifies the index of our progenitor is the order of 11^{*4}.

```
G<x,y,t>:=Group<x,y,t|y^4,(x^-5),(y^-1*x^-2*y*x^-1),t^11,
 t^x=t^4,(t,t^y),(t,t^(y^2)),(t,t^(y^3))>;
 #G;
292820
 Index(G,sub<G|x,y>);
14641
11^4;
14641
14641*20 = 292820 = |G|.
```

The homomorphic images obtained from this progenitor can be found in chapter 8.

Constructing a Permutation Representation

We worked in Z_{11} on matrices of degree 4×4, which implies we are working with 4 $t_{i's}$ of order 11. Since we have a semi-direct product in our progenitor, the elements of $C_5 : C_4$ will act as an automorphism on $< t_1 > * < t_2 > * < t_3 > * < t_4 >$. So, $a_{i,j} = a \iff t_i \to t_j^a$, since this is an automorphism. Therefore, for our $A(xx)$ we have:

$$A(xx) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

where t_1 corresponds to column 1, t_2 to column 2, and so on. We will label the entries of the matrix as follows: $a_{12} = a, a_{22} = b, a_{21} = c$, and $a_{22} = d$. Then,

$$a_{11} = a \iff t_1 \to t_1^a$$

$$a_{22} = a \iff t_1 \to t_2^b$$

$$a_{33} = a \iff t_1 \to t_3^c$$

$$a_{44} = a \iff t_1 \to t_4^d$$

$$a_{11} = a \iff t_1 \to t_1^c \qquad\qquad a_{22} = a \iff t_1 \to t_2^d$$

We can now construct a table with our $t_{i's}$ with nonzero entries to obtain the permutation representation. Keep in mind we are working in $\mathbb{Z}_{11}$. We will have a total of 40 $t_{i's}$ for $A(xx)$.

For $\mathbf{a_{11}}$

$$t_1^1 \to t_1^4$$
$$t_1^2 \to (t_1^4)^2 = t_1^8$$
$$t_1^3 \to (t_1^4)^3 = t_1^{12} = t_1$$
$$t_1^4 \to (t_1^4)^4 = t_1^{16} = t_1^5$$
$$t_1^5 \to (t_1^4)^5 = t_1^{20} = t_1^9$$
$$t_1^6 \to (t_1^4)^6 = t_1^{24} = t_1^2$$
$$t_1^7 \to (t_1^4)^7 = t_1^{28} = t_1^6$$
$$t_1^8 \to (t_1^4)^8 = t_1^{48} = t_1^{10}$$
$$t_1^9 \to (t_1^4)^9 = t_1^{36} = t_1^3$$
$$t_1^{10} \to (t_1^4)^{10} = t_1^{40} = t_1^7$$

For $\mathbf{a_{22}}$

$t_2^1 \to t_2^9$

$t_2^2 \to (t_2^9)^2 = t_2^{18} = t_2^7$

$t_2^3 \to (t_2^9)^3 = t_2^{27} = t_2^5$

$t_2^4 \to (t_2^9)^4 = t_2^{36} = t_2^3$

$t_2^5 \to (t_2^9)^5 = t_2^{45} = t_2^1$

$t_2^6 \to (t_2^9)^6 = t_2^{54} = t_2^{10}$

$t_2^7 \to (t_2^9)^7 = t_2^{63} = t_2^8$

$t_2^8 \to (t_2^9)^8 = t_2^{72} = t_2^6$

$t_2^9 \to (t_2^9)^9 = t_2^{81} = t_2^4$

$t_2^{10} \to (t_2^9)^{10} = t_2^{90} = t_2^2$

For $\mathbf{a_{33}}$

$t_3^1 \to t_3^3$

$t_3^2 \to (t_3^3)^2 = t_3^6 = t_3^6$

$t_3^3 \to (t_3^3)^3 = t_3^9 = t_3^9$

$t_3^4 \to (t_3^3)^4 = t_3^{12} = t_3^1$

$t_3^5 \to (t_3^3)^5 = t_3^{15} = t_3^4$

$t_3^6 \to (t_3^3)^6 = t_3^{18} = t_3^7$

$t_3^7 \to (t_3^3)^7 = t_3^{21} = t_3^{10}$

$t_3^8 \to (t_3^3)^8 = t_3^{24} = t_3^2$

$t_3^9 \to (t_3^3)^9 = t_3^{27} = t_3^5$

$t_3^{10} \to (t_3^3)^{10} = t_3^{30} = t_3^8$

For $\mathbf{a_{44}}$

$$t_4^1 \to t_4^5$$
$$t_4^2 \to (t_4^5)^2 = t_4^{10} = t_4^10$$
$$t_4^3 \to (t_4^5)^3 = t_4^{15} = t_4^4$$
$$t_4^4 \to (t_4^5)^4 = t_4^{20} = t_4^9$$
$$t_4^5 \to (t_4^5)^5 = t_4^{25} = t_4^3$$
$$t_4^6 \to (t_4^5)^6 = t_4^{30} = t_4^8$$
$$t_4^7 \to (t_4^5)^7 = t_4^{35} = t_4^2$$
$$t_4^8 \to (t_4^5)^8 = t_4^{40} = t_4^7$$
$$t_4^9 \to (t_4^5)^9 = t_4^{45} = t_4^1$$
$$t_4^{10} \to (t_4^5)^{10} = t_4^{50} = t_4^6$$

To find our permutations, we used tables 6.1 and 6.2:

Table 6.10: Permutation Table of A(xx)

#	t_i	Mapping to t_j^a	Element of Permutation
1	t_1	t_1^4	13
2	t_2	t_2^9	34
3	t_3	t_3^3	11
4	t_4	t_4^5	20
5	t_1^2	t_1^8	29
6	t_2^2	t_2^7	26
7	t_3^2	t_3^6	23
8	t_4^2	t_4^{10}	40
9	t_1^3	t_1^1	1
10	t_2^3	t_2^5	18
11	t_3^3	t_3^9	35
12	t_4^3	t_4^4	16
13	t_1^4	t_1^5	17
14	t_2^4	t_2^3	10
15	t_3^4	t_3^1	3
16	t_4^4	t_4^9	36
17	t_1^5	t_1^9	33
18	t_2^5	t_2^1	2
19	t_3^5	t_3^4	15
20	t_4^5	t_4^3	12

Therefore, our permutation representation is the following:
$A(xx) = <(1, 13, 17, 33, 9)(2, 34, 14, 10, 18)(3, 11, 35, 19, 15)(4, 20, 12, 16, 36)$
$(5, 29, 37, 25, 21)(6, 26, 30, 22, 38)(7, 23, 27, 39, 31)(8, 40, 24, 32, 28) >$.

Table 6.10: Permutation Table of A(xx)

#	t_i	Mapping to t_j^a	Element of Permutation
21	t_1^6	t_1^2	5
22	t_2^6	t_2^{10}	38
23	t_3^6	t_3^7	27
24	t_4^6	t_4^8	32
25	t_1^7	t_1^6	21
26	t_2^7	t_2^8	30
27	t_3^7	t_3^{10}	39
28	t_4^7	t_4^2	8
29	t_1^8	t_1^{10}	37
30	t_2^8	t_2^6	22
31	t_3^8	t_3^2	7
32	t_4^8	t_4^7	28
33	t_1^9	t_1^3	9
34	t_2^9	t_2^4	14
35	t_3^9	t_3^5	19
36	t_4^9	t_4^1	4
37	t_1^{10}	t_1^7	25
38	t_2^{10}	t_2^2	6
39	t_3^{10}	t_3^8	31
40	t_4^{10}	t_4^6	24

For our $A(yy)$ we would have:

$$t_1 \to t_2^1$$
$$t_2 \to t_3^1$$
$$t_3 \to t_4^1$$
$$t_4 \to t_1^1$$

Thus, we would apply the same process and our permutation representation would be: $A(yy) = <(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)$ $(21,22,23,24)(25,26,27,28)(29,30,31,32)(33,34,35,36)(37,38,39,40)>$.

This demonstrates that our presentation is correct since we have $|A(xx) \times A(yy)| = 20 = |G|$.

The Monomial Progenitor:

To build the monomial progenitor, we simply need to compute the stabiliser $(N, t_1, t_1^2, t_1^3, t_1^4)$

We are looking for what element in N fixes our $t_{1's}$. The work is as follows:

```
S:=Sym(40);
yy:=S!(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)
(17,18,19,20)(21,22,23,24)(25,26,27,28)(29,30,31,32)
(33,34,35,36)(37,38,39,40);
xx:=S!(1,13,17,33,9)(2,34,14,10,18)(3,11,35,19,15)
(4,20,12,16,36)(5,29,37,25,21)(6,26,30,22,38)(7,23,
27,39,31)(8,40,24,32,28);
N<x,y>:=Group<x,y|y^4,(x^-5),(y^-1*x^-2*y*x^-1)>;
Normaliser:=Stabiliser(N,{1,5,9,13,17,21,25,29,33,37});
Stabiliser(N,{1,5,9,13,17,21,25,29,33,37});

NN<x,y>:=Group<x,y|y^4,(x^-5),(y^-1*x^-2
*y*x^-1)>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..40]];
for i in [2..20] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
```

```
ArrayP[i]:=PP;
end for;

Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,
17,19});

Stabiliser(N,{1,3,5,7,9,11,13,15,17,19});

A:=N! (1, 13, 17, 33, 9)(2, 34, 14, 10, 18)(3,
 11, 35, 19, 15)(4, 20, 12, 16,36)(5, 29, 37,
25, 21)(6, 26, 30, 22, 38)(7, 23, 27, 39, 31)
(8, 40, 24, 32, 28);

Normaliser eq sub<N|A>;

/*true*/

for i in [1..#N] do if ArrayP[i] eq A then Sch[i];
 end if; end for;

/*x*/
```

The original progenitor for G was:

```
G<x,y>:=Group<x,y|y^4,(x^-5),(y^-1*
x^-2*y*x^-1)>;
```

The new monomial progenitor:

```
G<x,y,t>:=Group<x,y,t|y^4,(x^-5),
(y^-1*x^-2*y*x^-1),t^11,t^x=t^4,
(t,t^y),(t,t^(y^2)),(t,t^(y^3))>;
#G;
/*292820*/
```

This is verfied by the Grindstaff Lemma as follows:

```
Index(G,sub<G|x,y>);
14641
```

Since we are working with 11^{*4}, which equals 14641, and the index of $G \times |G| =$ 292820, as desired, we have proved we have correctly constructed a monomial progenitor for G.

Chapter 7

Finding Generators $PGL_2(13):2$

Consider:

$G \cong \frac{2^{*12}:S_4}{<x,y,z,t|t^2=1,[t^{(x*y)}]=1,[(x*y^-1*(z^(y^-2))*t)]^7=1,[y*t]^2=1>}$, where our control group $N = 2^{*12}:S_4$ and G is the homomorphic image of N factored by the relations: $[x*y^{-1}*(z^{(y^{-2})}*t)]^7, [y*t]^2$ and the action of $N = 2^{*12}:S_4$ on the 12 symmetric generators given by:

$x \sim (1,4)(2,5)(3,6)(8,9)(10,\infty), y \sim (1,7,4)(2,8,6)(3,9,5)(10,\infty,0)$ and

$z \sim (1,6)(2,5)(3,4)(7,0)(8,10)(9,\infty)$.We will show that $G \cong PGL_2(13)$.

The $PGL_2(13):2$ group is composed of 2×2 matrices over a field q such that $q = p^n$. Every finite field is of order p^n where p is a prime. $L_2(13) = \{x \mapsto \frac{a+b(x)}{c+d(x)}, where\ a,b,c,d \in F_{13}, x \in F_{13} \cup \{\infty\} | ad-bc = 1\ or\ a\ nonzero\ square.\} =< \alpha, \beta, \gamma, \delta >$. Note: α, β, γ and δ (will represent an automorphism) are our generators of $PGL_2(13)$ we will be defining. Therefore we will be working on a field of order 13 where

$F = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and $F_{13} \cup \{\infty\}$ over 13 letters. Note: $0 = 13$, and $\infty = 14$. To begin the process, let us first define our maps:

$\alpha : x \mapsto x+1$.

$\beta : x \mapsto kx$, where k is a generator of all nonzero squares.

$\gamma : x \mapsto -\frac{1}{x}$.

$aut : x \mapsto \frac{2}{x}$

Let us begin with $\alpha : x \mapsto x+1$ We begin with the element 0.

$0 \mapsto 0 + 1 = 1$

$1 \mapsto 1 + 1 = 2$

$2 \mapsto 2 + 1 = 3$

$3 \mapsto 3 + 1 = 4$

$4 \mapsto 4 + 1 = 5$

$5 \mapsto 5 + 1 = 6$

$6 \mapsto 6 + 1 = 7$

$7 \mapsto 7 + 1 = 8$

$8 \mapsto 8 + 1 = 9$

$9 \mapsto 9 + 1 = 10$

$10 \mapsto 10 + 1 = 11$

$11 \mapsto 11 + 1 = 12$

$\infty \mapsto \infty + 1 = \infty$

Therefore, we obtain the following permutation:
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)(\infty)$. For $\beta : x \mapsto kx$. To find k, we need to find an element that produces all nonzero squares using modulus 13. The non zero squares of $F_{13} = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11^2, 12^2\}$. If we take every square in the set mod 13, we will obtain the nonzero squares: $\{1, 3, 4, 9, 10, 12\}$. Therefore, we must find a k such that it produces this set of non zero squares. We find that 4 works as follows. $4^1 = 4, 4^2 = 3, 4^3 = 12, 4^4 = 9, 4^5 = 10, 4^6 = 1, 4^7 = 4, 4^8 = 3, 4^9 = 12, 4^{10} = 9, 4^{11} = 10, 4^{12} = 1$. Then our mapping for β is $\beta : x \mapsto 4x$.

$0 \mapsto 4(0) = 0$

$1 \mapsto 4(1) = 4$

$4 \mapsto 4(4) = 16 = 3 mod 13$

$3 \mapsto 4(3) = 12$

$12 \mapsto 4(12) = 48 = 9 mod 13$

$9 \mapsto 4(9) = 36 = 10 mod 13$

$10 \mapsto 4(10) = 40 = 1 mod 13$

$2 \mapsto 4(2) = 8$

$8 \mapsto 4(8) = 32 = 6 mod 13$

$6 \mapsto 4(6) = 24 = 11 mod 13$

$11 \mapsto 4(11) = 44 = 5 mod 13$

$5 \mapsto 4(5) = 20 = 7 mod 13$

$7 \mapsto 4(7) = 28 = 2 mod 13$

$\infty \mapsto 4(\infty) = \infty$

Then our permutation is: $(\infty)(1, 4, 3, 12, 9, 10)(2, 8, 6, 11, 5, 7)$.

Now, for γ we have:

$\gamma : x \mapsto -\frac{1}{x}$ $0 \mapsto -\frac{1}{0} = \infty$

$\infty \mapsto -\frac{1}{\infty} = 0$

$1 \mapsto -\frac{1}{1} = -1 = 12 mod 13$

$12 \mapsto -\frac{1}{12} = -1(12^{-1})$

To find the inverse of 12, we need to find a number such that the product of it and 12 is 1 mod 13. Thus, the inverse of 12 is 12. Then:

$12 \mapsto -1(12) = -12 = 1 mod 13$

. Likewise, we find the other mappings:

$2 \mapsto -\frac{1}{2} = -1(2^{-}1) = -1(7) = -7 = 6 mod 13$

$6 \mapsto -\frac{1}{6} = -1(6^{-}1) = -1(11) = -11 = 2 mod 13$

$3 \mapsto -\frac{1}{3} = -1(3^{-1}) = -1(9) = -9 = 4 mod 13$

$4 \mapsto -\frac{1}{4} = -1(4^{-1}) = -1(10) = -10 = 3 mod 13$

$5 \mapsto -\frac{1}{5} = -1(5^{-1}) = -1(8) = -8 = 5 mod 13$

$7 \mapsto -\frac{1}{7} = -1(7^{-1}) = -1(2) = -2 = 11 mod 13$

$11 \mapsto -\frac{1}{11} = -1(11^{-1}) = -1(6) = -6 = 7 mod 13$

$8 \mapsto -\frac{1}{8} = -1(8^{-1}) = -1(5) = -5 = 8 mod 13$

$9 \mapsto -\frac{1}{9} = -1(9^{-1}) = -1(3) = -3 = 10 mod 13$

$10 \mapsto -\frac{1}{10} = -1(10^{-1}) = -1(4) = -4 = 9 mod 13$

Since we have $PGL_2(13) : 2$ we must create an automorphism for the element of order two not normal in our group. Note: If we did not have this element of order two, we would simply have a $PSL_2(13)$. For the automorphism we find a map that produces a nonzero entry that is not a perfect square. We find the following mapping: $\delta : x \mapsto \frac{2}{x}$. Since $\frac{a(x)+b}{c(x)+d} = \frac{2+0x}{0+1x} = 2$. Thus the equation $ad - bc = 2$ produces a nonzero square. $\delta : x \mapsto \frac{2}{x}$

Like before, we work with finding inverses of our elements and obtain the following permutation for δ: $(1,2)(3,5)(4,7)(6,9)(8,10)(11,12)(0,\infty)$.
We verify in $MAGMA$ that our permutations are correct:

```
S:=Sym(14);
 alpha:=S!(13,1,2,3,4,5,6,7,8,9,10,11,12);
  beta:=S!(1,4,3,12,9,10)(2,8,6,11,5,7);
 gamma:=S!(13,14)(1,12)(2,6)(3,4)(7,11)(9,10);
 #sub<S|alpha,beta,gamma>;
/*1092*/
 aut:=S!(1,2)(3,5)(4,7)(6,9)(8,10)(11,12)(13,14);
 PGL:=sub<S|alpha,beta,gamma,aut>;
s,t:=IsIsomorphic(G1,PGL);
 s;
/*true*/
```

We obtain the mapping from $MAGMA$ that show how elements of $G1$ of cardinality 91 are mapped to elements of $PGL_2(13):2$ of cardinaliy 14:

```
Homomorphism of GrpPerm: G1, Degree 91,
 Order 2^3 * 3 * 7 * 13 into
GrpPerm: PGL, Degree 14, Order 2^3 * 3 * 7 * 13 induced by
(4, 6)(5, 8)(9, 11)(12, 20)(13, 22)(14, 18)(15, 17)(16, 26)(19,
23)(21, 28)(24, 31)(27, 37)(29, 32)(30, 41)(33, 49)(34, 53)(35,
47)(38, 59)(39, 61)(40, 57)(42, 51)(43, 50)(44, 65)(45, 67)(46,
63)(48, 54)(52, 69)(55, 56)(58, 62)(60, 72)(64, 68)(66, 76)(70,
71)(73, 75)(74, 80)(77, 79)(78, 82)(81, 85)(83, 86)(84, 91)(87,
90)(88, 89)
|--> (1, 14)(2, 12)(3, 13)(4, 9)(5, 7)(6, 11)(8, 10)
(3, 4, 6)(5, 8, 7)(9, 12, 14)(10, 15, 17)(11, 18, 20)(13, 23,
21)(16, 26, 25)(19, 22, 28)(24, 33, 35)(27, 38, 40)(29, 42,
43)(30, 44, 46)(31, 47, 49)(32, 50, 51)(34, 54, 52)(36, 55,
56)(37, 57, 59)(39, 62, 60)(41, 63, 65)(45, 68, 66)(48, 53,
 69)(58, 61, 72)(64, 67, 76)(73, 81, 82)(74, 83, 79)(75, 78,
 85)(77, 86, 80)(84, 88, 90)(87, 89, 91)
 |--> (1, 7, 6)(2, 10, 9)(4, 8, 12)(5, 14, 11),
(2, 3)(4, 6)(5, 9)(7, 10)(8, 11)(12, 17)(13, 24)(14, 18)(15, 20)
(16,27)(19, 29)(21, 30)(22, 31)(23, 32)(25, 36)(26, 37)(28, 41)
(33,46)(34, 52)(35, 50)(38, 56)(39, 45)(40, 57)(42, 65)(43, 47)
(44,51)(48, 70)(49, 63)(53, 69)(54, 71)(55, 59)(58, 73)(60, 74)
(61,67)(62, 75)(64, 77)(66, 78)(68, 79)(72, 80)(76, 82)(81, 86)
(83, 85)(84, 87)(90, 91)
|--> (1, 13)(2, 12)(3, 11)(4, 10)(5, 9)(6, 8)
```

```
(1, 2)(3, 5)(4, 7)(6, 8)(9, 13)(10, 16)(11, 19)(12, 21)(14, 23)
(15,25)(17, 26)(18, 28)(20, 22)(24, 34)(27, 39)(29, 43)(30, 45)
(31,48)(33, 52)(35, 54)(36, 55)(37, 58)(38, 60)(40, 62)(41, 64)
(44,66)(46, 68)(47, 69)(49, 53)(50, 51)(57, 72)(59, 61)(63, 76)
(65,67)(73, 82)(74, 84)(77, 87)(78, 85)(79, 88)(80, 89)(83, 90)
(86,91)
 |--> (1, 12)(2, 5)(3, 13)(4, 6)(7, 8)(9, 14)(10, 11)
```

Now we will construct the homomorphic map which proves that $G \cong PGL_2(13) : 2$ from our progenitor N. Note: the automorphism was only needed for our element of order 2 not normal in $PGL_2(13) : 2$. Since the permutations above are the homomorphic images of $PGL_2(13) : 2$, we will have:

$\phi(x) = (1, \infty)(2, 12)(3, 0)(4, 9)(5, 7)(6, 11)(8, 10)$,

$\phi(y) = (1, 7, 6)(2, 10, 9)(4, 8, 12)(5, \infty, 11)$,

$\phi(z) = (1, 0)(2, 12)(3, 11)(4, 10)(5, 9)(6, 8)$, and

$\phi(t) = (1, 12)(2, 5)(3, 0)(4, 6)(7, 8)(9, \infty)(10, 11)$. Recall: we replaced 0 for 13 and ∞ for14 in our permutations defined above. For the mappings, we must find variables $\frac{a(x)+b}{c(x)+d}$ that satisfy each set of homomorphic mappings from above. For $\phi(x) = (1, \infty)(2, 12)(3, 0)(4, 9)(5, 7)(6, 11)(8, 10)$, we construct the following map:

$1 \mapsto \infty$

$\frac{a(x)+b}{c(x)+d} = \infty$

$\frac{a(1)+b}{c(1)+d} = \infty$

$\frac{a+b}{c+d} = \infty$

$\implies c + d = 0$

$\implies c = -d.$

$\infty \mapsto 1$

$\frac{a(\infty)+b}{c(\infty)+d} = 1$

$\frac{a}{c} = 1$

$\implies a = c$

$0 \mapsto 3$

$\implies \frac{b}{d} = 3$

$\implies b = 3d.$

Here, since we already have two variables in terms of c we will attempt to write b in terms of c.

$3 \mapsto 0$

$\frac{3a+b}{3c+d} = 0$

$\implies b = -3a$

$\implies b = -3c$ since a = c

$\implies b = 10c(mod 13)$.

Now we have all variables in terms of c, therefore we write the following map.

$\frac{c(x)+10c}{c(x)-c}$

$\implies \frac{c(x+10)}{c(x-1)}$.

We now verify we have the correct map:

$\frac{x+10}{x-1}$

Does $2 \mapsto 12$?

$\frac{(2+10)}{(2-1)} = \frac{12}{1} = 12$.

$\frac{12+10}{(12-1)} = \frac{22}{11} = 2$.

Likewise, we verify the remaining elements in $\phi(x)$. Similarly, we construct the remaining maps for $\phi(y) = \frac{11x}{x+8}$, $\phi(z) = \frac{6x+4}{x+7}$ and $t_\infty = \frac{9x+12}{x+4}$.

Now, $|N| = 12$ implies we have 12 symmetric generators which must be defined in terms of our homomorphism. To do this we conjugate $\phi(t)$ denoted $t = (1,12)(2,5)(3,0)(4,6)$ $(7,8)(9,14)(10,11)$ (where $t \sim t_1$) by an element of our homomorphism we will denote as:

$\phi(x) = X := S!(1,14)(2,12)(3,13)(4,9)(5,7)(6,11)(8,10)$,

$\phi(y) = Y := S!(1,7,6)(2,10,9)(4,8,12)(5,14,11)$,

$\phi(z) = Z := S!(1,11)(2,9)(3,10)(4,12)(6,14)(8,13)$ which will produce the twelve symmetric generators. Recall from our N we have

$x \sim (1,4)(2,5)(3,6)(8,9)(10,12)$,

$y \sim (1,7,4)(2,8,6)(3,9,5)(10,12,11)$

and $z \sim (1,6)(2,5)(3,4)(7,11)(8,10)(9,12)$.

$t^X = (1,4)(2,14)(3,13)(5,10)(6,8)(7,12)(9,11) = t_4$

$t^{(Y^2)*Z)^{-1}} = (1,10)(2,14)(3,7)(4,11)(5,8)(6,13)(9,12) = t_2$

$t = (1,12)(2,5)(3,13)(4,6)(7,8)(9,14)(10,11) = t_1$

$t^Z = (1,3)(2,6)(4,11)(5,9)(7,13)(8,10)(12,14) = t_6$

$t^{Y} = (1,8)(2,11)(3,13)(4,7)(5,9)(6,12)(10,14) = t_7$
$t^{(X*Z)} = (1,2)(3,5)(4,7)(6,9)(8,10)(11,12)(13,14) = t_3$
$t^{Z*Y^{-1}} = (1,13)(2,4)(3,6)(5,8)(7,9)(10,11)(12,14) = t_8$
$t_2^{X} = (1,12)(2,4)(3,11)(5,13)(6,9)(7,10)(8,14) = t_5$
$t_3^{Y} = (1,2)(3,14)(4,5)(6,8)(7,10)(9,12)(11,13) = t_9$
$t_8^{Z} = (1,3)(2,7)(4,6)(5,13)(8,11)(9,12)(10,14) = t_{10}$
$t_7^{Z} = (1,9)(2,5)(3,6)(4,14)(7,12)(8,10)(11,13) = t_{11}$
$t_9^{Z} = (1,8)(2,4)(3,7)(5,12)(6,10)(9,11)(13,14) = t_{12}$

Therefore, we have defined our 12 $t_{i's}$ in terms of our progenitor N by conjugation as the group $L_2(13)$ given by:
$x \sim (t_1, t_4)(t_2, t_5)(t_3, t_6)(t_8, t_9)(t_{10}, t_{12})$
$y \sim (t_1, t_7, t_4)(t_2, t_8, t_6)(t_3, t_9, t_5)(t_{10}, t_{12}, t_{11})$
$z \sim (t_1, t_6)(t_2, t_5)(t_3, t_4)(t_7, t_0)(t_8, t_{10})(t_9, t_{12})$

Finally, the additional relations given by: $(x * y^{-1} * (z^{(y^{-2})}) * t)^7 = 1$ and $(y * t)^2 = 1$ hold, since $|X * Y^{-1} * (Z^{(Y^{-2})} * T| = 7$ and $|Y * T| = 2$ as desired. Therefore we have shown that $|G| \geq |PGL_2(13)|$, but $|G| \leq |PGL_2(13)|$ by double coset enumeration. Thus, $G \cong PGL_2(13)$.

7.0.3 Double Coset Enumeration of $PGL_2(13)$

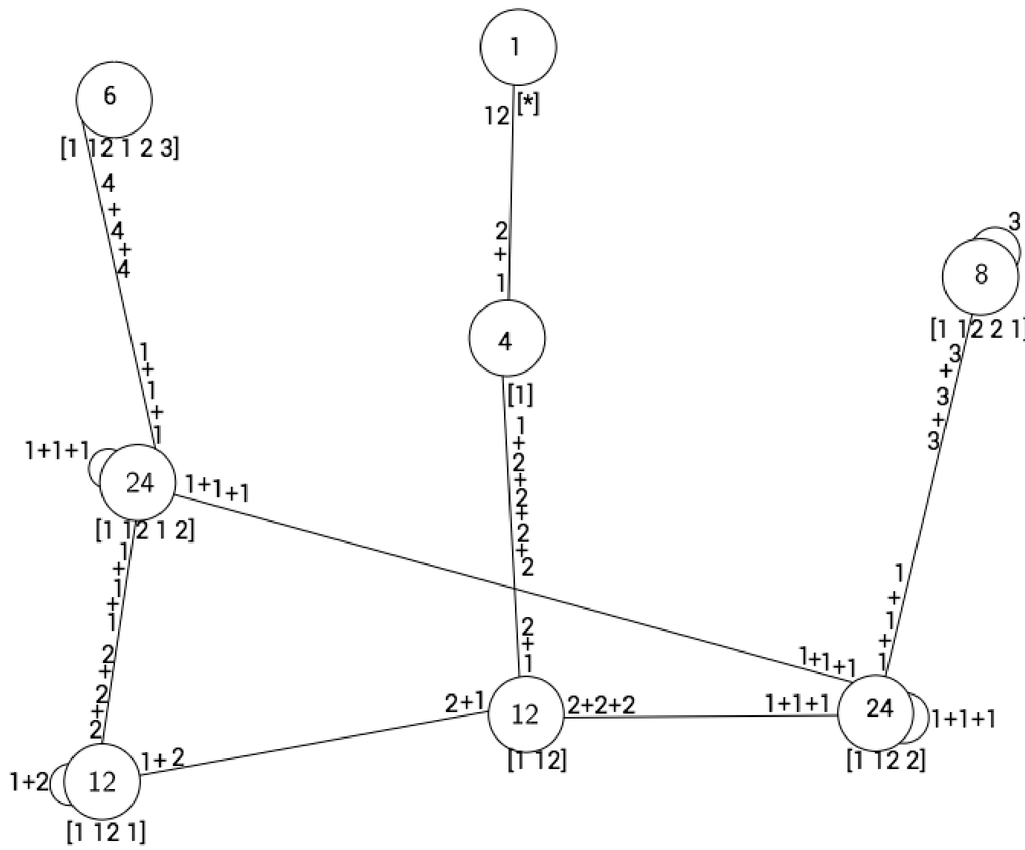

Figure 7.1: DCE of $PGL_2(13)$ [Lun18]

Chapter 8

Progenitors and Their Homomorphic Images

Table 7.1: $2^{*10} : (5^2 : C_2)$

Note: For the following table, we have relations labeled $a, ..., s$, and will only include the relations used to find each group G. For Progenitor:

$G =< a, b, t | b^4, b^{-2} * a^{-1} * b^2 * a^{-1}, b^{-1} * a^3 * b * a^{-1}, t^2, (t, a^{-1} * b^{-1} * a), (a^5 * t)^c, ((a * b)^2 * t)^d, (b * a * b * t)^e, (a * b * t)^f, (b^{-1} * a^{-1} * t)^g, (a^2 * b * t)^h, (a^{-1} * b^{-1} * a * t)^i, (a^2 * t)^j, (a * t)^k >;$

n	o	p	q	r	s	G $\cong$	#G
0	3	0	0	0	6	$2 \times J_2$	1209600
0	5	0	6	18	8	$10 \times J_2$	6048000
0	0	0	0	5	5	$5^4 : D_{10}$	6250
0	0	0	0	0	4	$5^3 : D_8$	1000
0	0	0	0	0	5	$5^3 : (5^2 : D_{10})$	31250
0	4	4	0	6	6	$2 \times (2^{12} : (5^2 : C_6))$	1228800

Table 7.2: $2^{*10} : (5^2 : C_2)$

Note: For the following table, we used the same progenitor as above, but here our $t_{i's}$ were of order 3.

n	o	p	q	r	s	G $\cong$	#G
0	3	5	0	5	5	A_5	60
0	2	0	0	5	0	$U(3,4)$	62400
0	0	0	0	5	5	$5^4 : D_{10}$	6250
0	0	0	0	0	4	$5^8 : A_5$	23437500
0	0	0	0	0	5	$5^3 : (5^2 : D_{10})$	31250
0	4	4	0	6	6	$2 \times (2^{12} : (5^2 : C_6))$	1228800

Table 7.3: $2^{*60} : (S_5)$ Famous Lemma

For progenitor:

$G < x, y, t | x^2, y^6, (y * x * y^-1 * x)^2, (x * y^-1)^5, (t, x^y), t^2, (y^2 * x * y^-2)^m, (y^3 * t)^a, ((y * x * y)^2 * t)^b, (y^2 * t)^c, (y * x * y * t)^d, (y * x * t)^e, (y * t)^f >$

a	b	c	d	e	f	m	G $\cong$	#G
0	0	0	2	5	0	4	$2^4 : S_5$	1920
0	0	0	2	0	5	6	S_6	720
0	0	2	0	0	4	4	$2^5 : S_5$	3840
0	0	2	0	6	6	2	$3^4 \because S_5 \times 2$	19440
0	0	2	0	7	0	2	$2 : L_2(49)$	11760
0	2	0	0	0	3	4	$3 : S_5$	2160
0	0	4	3	6	5	6	$2^6 : S_5$	7680
0	3	0	0	0	3	2	$L_2(25)$	7800
2	4	0	0	0	5	2	$2 : M_{12}$	190080
0	0	0	4	0	3	0	$2^5 : S_6$	23040

Table 7.4: $2^{*36} : (3^2) :^{\cdot} D_8$ Famous Lemma

For Progenitor:
$G < v, w, x, y, z, t | v^2, w^4, x^2, y^3, z^3, w^{-}2 * x, (w^{-1} * v)^2, (x * y^{-1})^2, v * z^{-1} * v * z, (x * z^{-1})^2, (y, z), w * y^{-1} * w^{-1} * y * z^{-1}, (t, v * x * z^{-1}), t^2, (v * w^{-1} * t)^m, (v * y^{-1} * z^{-1} * t)^a, (v * w * t)^b, (x * t)^c, (y * t)^d, (z * t)^e, (w * t)^f, (y * v * t)^g, (v * w * z * t)^h >$

a	b	c	d	e	f	g	h	m	G $\cong$	#G
0	0	0	0	4	0	0	5	2	$4^{*}S(4,4)$	3916800
0	0	0	0	0	0	0	3	2	$2^{*}S_6$	1440
0	0	0	0	0	0	3	5	3	$3^4 :^{*} S_6$	58320
0	0	0	0	0	0	5	3	4	$4 : PGL(3,4)$	368640
0	0	0	0	0	0	4	3	4	$4^{*}PGL(3,4)$	161280
0	0	0	0	0	0	5	3	4	$2^9 : S_6$	368640
0	0	0	0	6	3	4	0	0	$2^6 : 6 \times S_6$	276480
0	0	0	0	6	6	0	4	2	$2^{12} : S_6$	1474560
0	0	0	0	6	8	5	0	2	$PSL(4,3) : 2$	24261120
0	0	0	0	6	6	6	8	2	$4^{*}(U(4,3) : 4)$	52254720
0	0	0	0	6	8	5	0	2	$2^3 : (U(4,3) : 4)$	104509440

Table 7.5: $2^{*10} : Alt_5$

For Progenitor: $G < x, y, t | x^2, y^5, (x * y^{-1})^3, t^2, (t, y^{-1} * x), (y * x * y^{-1} * t)^a, (y * x * t)^b, (y * t)^c, (y^2 * t)^d >$

a	b	c	d	G $\cong$	#G
0	4	6	5	$6 : (Alt_6 : S_6)$	1555200
0	5	5	6	$2^{10} : L(2,11)$	675840
0	6	4	0	$2^{\cdot}((Alt_6 \times Alt_6) : 2)$	14400
0	6	5	5	$2^{14} : Alt_5$	983040
4	0	4	0	$2 : L(2,16)$	8160
4	0	5	5	$2^8 : Alt_5$	15360
6	4	6	6	$3 : (Alt_6 : Sym_6)$	777600
8	0	5	5	$2^{18} : Alt_5$	15728640

Table 7.6: $2^{*10} : (2^4 : 5)$

For Progenitor:
$G < x, y, t | x^5, y^2, (x * y * x^{-1} * y)^2, (y * x^{-1})^5, (t, y), (x^2 * y * x^{-2} * t)^a, (x^2 * y * x^{-2} * y * t)^b, (x^2 * y * x^{-1} * y * x^{-1} * t)^c, (x * t)^d, (x^2 * t)^e, (x^-2 * t)^f, (x^{-1} * t)^g >$

a	b	c	d	e	f	g	G ≅	#G
0	0	0	0	0	3	2	$2^9 : Alt_5$	30720
0	0	2	0	3	9	10	$2^5 : Alt_5$	1920
0	0	2	3	0	5	3	$2^8 : Alt_5$	15360
0	0	2	3	5	5	6	$2^{13} : Alt_5$	491520
0	0	2	4	4	4	4	$2^{10} : (2^4 : 10)$	163840
0	0	0	0	0	3	2	$3^4 : (2^4 : 10)$	38880
0	0	0	0	0	3	2	$2^{11} : Alt_5$	245760
0	0	0	0	0	3	2	$2^{10} : Alt_6$	368640
0	0	0	0	0	3	2	$2^9 : Alt_6$	184320

Table 7.7: $2^{*10} : (2 \times A_5)$

For Progenitor:
$G < x, y, z, t | x^3, y^2, z^5, x^{-1} * y * x * y, y * z^{-1} * y * z, (z^{-1} * x)^3, (x^{-1} * z^-2)^2, t^2, (t, x), (y * t)^a, (x * z^2 * t)^b, (x * y * z^2 * t)^c, (x * t)^d, (z * t)^e, (z^2 * t)^f, (x * y * t)^g, (y * z * t)^h, (y * z^{-2} * t)^i >$

a	b	c	d	e	f	g	h	i	G ≅	#G
0	0	0	0	0	0	0	4	4	$2^2 : PSL(2, 16)$	32640
0	0	0	0	3	0	6	8	8	$2 : Alt_{12}$	479001600
0	0	0	0	4	4	3	0	0	$6 : S(4, 4)$	5875200
0	0	0	6	0	8	6	4	4	$PGL(2, 16)$	16320
0	0	3	0	0	0	3	0	10	$2 : M_{12}$	190080
0	0	3	0	0	4	0	0	0	$3 : PSL(2, 16)$	24480
0	0	4	0	0	4	6	6	10	$2^* : (2 : S(4, 4))$	3916800
0	0	4	6	4	4	0	0	0	$6^* S(4, 4)$	7833600
0	0	5	0	0	8	3	0	5	$4 : PSL(3, 4)$	161280
0	0	5	0	3	5	0	0	0	J_1	175560

Table 7.8: $11^{*2} : D_{10}$

For Progenitor:
$G < x, y, t >:= Group < x, y, t | y^2, (x^-1 * y)^2, x^-5, t^11, t^(x^-2) = t^3, (y * t)^a, (y * t^2)^b, (y * t^3)^c, (y * t^4)^d, (y * t^5)^e, (y * t^6)^f, (y * t^8)^g, (y * t^9)^h, (y * t^10)^i >$

a	b	c	d	e	f	g	h	i	G ≅	#G
0	0	0	0	0	0	0	0	3	$PSL(2, 11)$	660
0	0	0	0	0	0	0	0	4	$5 : PSL(2, 11)$	6600
0	0	0	0	0	0	0	5	0	$PSL(2, 11) \times PSL(2, 11)$	435600
0	0	0	6	0	8	6	4	4	$2^{11} : PSL(2, 11)$	1351680
0	0	3	0	0	0	3	0	10	$2^{10} : PSL(2, 11)$	675840

Table 7.9: $11^{*4} : C_5 : C_4$

For Progenitor:
$G < x, y, t >:= Group < x, y, t | y^4, x^-5, y^-1 * x^-2 * y * x^-1, t^11, t^(x^2) = t^5, (y * t)^a, (y * t^2)^b, (y * t^3)^c, (y * t^4)^d, (y * t^5)^e, (y * t^6)^f, (y * t^8)^g, (y * t^9)^h, (y * t^10)^i >$

a	b	c	d	e	f	g	h	i	G ≅	#G
0	0	0	0	0	0	0	0	3	M_{11}	7920

Table 7.10: $31^{*2} : (3 \times 5) : 2$

For Progenitor:
$G < x, y, t >:= Group < x, y, t | y^2, (x^-1 * y)^2, x^15, t^31, t^(x^4) = t^28, (y * t^21)^a, (y * t^22)^b, (y * t^23)^c, (y * t^24)^d, (y * t^25)^e, (y * t^26)^f, (y * t^27)^g, (y * t^28)^h, (y * t^29)^i, (y * t^30)^j >;$

a	b	c	d	e	f	g	h	i	j	G ≅	#G
0	0	0	0	0	0	0	0	0	3	$PSL_2(31)$	14880
0	0	0	0	0	0	0	0	4	0	$15 : (2 \times L_2(31)$	446400

Appendix A

MAGMA Code

A.1 Building a Progenitor for $2^{*10} : 2 \times A_5$

```
/*NumberOfTransitiveGroups(10);
N:=TransitiveGroup(10,11);
#N;
/*60*/
Generators(N);
/*    (2, 4, 10)(5, 7, 9),
    (1, 6)(2, 7)(3, 8)(4, 9)(5, 10),
    (1, 3, 5, 7, 9)(2, 4, 6, 8, 10)

*/

S:=Sym(10);
xx:=S!(2, 4, 10)(5, 7, 9);
yy:=S!(1, 6)(2, 7)(3, 8)(4, 9)(5, 10);
zz:=S!(1, 3, 5, 7, 9)(2, 4, 6, 8, 10);

N:=sub<S|xx,yy,zz>;
#N;
/*120*/
FPGroup(N);

NN<x,y,z>:=Group<x,y,z|x^3,y^2,z^5,
x^-1*y*x*y,y*z^-1*y*z,(z^-1*x)^3,
(x^-1*z^-2)^2>;
#NN;
```

```
/*120*/
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
 ArrayP:=[Id(N): i in [1..120]];
for i in [2..120] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
end for;
 PP:=Id(N);
 for k in [1..#P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
end for;
N1:=Stabiliser(N,1);
#N1;
/*10*/
N1;

/* Permutation group N1 acting
on a set of cardinality 10
Order = 12 = 2^2 * 3
    (2, 4, 10)(5, 7, 9)
    (2, 8, 10)(3, 5, 7)
*/

for i in [1..120] do if ArrayP[i] eq
N!(2, 4, 10)(5, 7, 9)
then Sch[i]; end if; end for;
/* x */
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^5,
x^-1*y*x*y,y*z^-1*y*z,(z^-1*x)^3,
(x^-1*z^-2)^2,t^2,(t,x)>;

#G;
/*0*/
N12:=Stabiliser(N,[1,2]);
Cent:=Centraliser(N,N12);
Cent;

/* Permutation group Cent acting on\
```

```
 a set of cardinality 10
Order = 6 = 2 * 3
  (3, 9, 5)(4, 10, 8)
 (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)

*/
C:=Classes(N);
#C;
C;
/*10*/
for i in [2..10] do
i, Orbits(Centraliser(N,C[i][3]));
end for;

for j in [2..10] do
C[j][3];
for i in [1..120] do if ArrayP[i] eq
C[j][3] then Sch[i]; end if; end for;
end for;

/*(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)
y
(1, 5)(2, 8)(3, 7)(6, 10)
x * z^2
(1, 10)(2, 3)(4, 9)(5, 6)(7, 8)
x * y * z^2
(2, 4, 10)(5, 7, 9)
x
(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)
z
(1, 5, 9, 3, 7)(2, 6, 10, 4, 8)
z^2
(1, 6)(2, 9, 10, 7, 4, 5)(3, 8)
x * y
(1, 8, 5, 2, 9, 6, 3, 10, 7, 4)
y * z
(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
y * z^-2

*/
/*FIRST ORDER RELATIONS:*/
for a,b,c,d,e,f,g,h,i in [0..10] do
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^5,
x^-1*y*x*y,y*z^-1*y*z,(z^-1*x)^3,
```

```
(x^-1*z^-2)^2,t^2,(t,x),
(y*t)^a,
(x*z^2*t)^b,
(x * y * z^2*t)^c,
(x*t)^d,
(z*t)^e,
(z^2*t)^f,
(x*y*t)^g,
(y*z*t)^h,
(y*z^-2*t)^i
>;
if #G gt 100 then a,b,c,d,e,f,g,h,i;
#G;
end if;
end for;
```

A.2 MAGMA Code for Building Monomial Progenitor $11^{*4} :_m C_5 : C_4$

```
G:=TransitiveGroup(10,4);
IsAbelian(G);
G;
xx:=G! (1, 3, 5, 7, 9)(2, 4, 6, 8, 10);
yy:=G! (1, 2, 9, 8)(3, 6, 7, 4)(5, 10);
S:=Subgroups(G);
CG:=CharacterTable(G);
/* Class |   1  2  3  4  5
Size  |   1  5  5  5  4
Order |   1  2  4  4  5
-----------------------
p  =  2   1  1  2  2  5
p  =  5   1  2  3  4  1
-----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1 -1  1
X.3   0   1 -1 -I  I  1
X.4   0   1 -1  I -I  1
X.5   +   4  0  0  0 -1


Explanation of Character Value Symbols
```

```
--------------------------------------

I = RootOfUnity(4)*/
for i in [1..#S] do if Index (G,S[i]`subgroup) eq 4 then i;
 end if; end for;
/*3 Pick SB 3 and label those generators for your H group*/

x1:=G!(1, 3, 5, 7, 9)(2, 4, 6, 8, 10);
H:=sub<G|x1>;
CH:=CharacterTable(H);
I:=Induction(CH[2],G);
I eq CG[5];
CH;
/*
Character Table of Group H
--------------------------


-------------------------------
Class |   1    2    3    4    5
Size  |   1    1    1    1    1
Order |   1    5    5    5    5
-------------------------------
p  =  5   1    1    1    1    1
-------------------------------
X.1   +   1    1    1    1    1
X.2   0   1   Z1 Z1#2 Z1#3 Z1#4
X.3   0   1 Z1#2 Z1#4   Z1 Z1#3
X.4   0   1 Z1#3   Z1 Z1#4 Z1#2
X.5   0   1 Z1#4 Z1#3 Z1#2   Z1


Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

Z1     = (CyclotomicField(5: Sparse := true)) ! [ RationalField()
 | 0, 1, 0, 0 ]*/
CH[2];
T:=Transversal(G,H);
C:=CyclotomicField(5);
GG:=GL(4,C);
```

```
A:=[[C.1,0,0,0]: i in [1..4]];
for i,j in [1..4] do A[i,j]:=0; end for;
B:=[[C.1,0,0,0]: i in [1..4]];
for i,j in [1..4] do B[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2]( T[i]*xx*T[j]^-1); end if; end for;
for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2]( T[i]*yy*T[j]^-1); end if; end for;
Order(GG!A);
Order(GG!B);



GG!A;
/*
[zeta_5 0 0 0]
[0 zeta_5^3 0 0]
[0 0 -zeta_5^3 - zeta_5^2 - zeta_5 - 1 0]
[0 0 0 zeta_5^2]*/
GG!B;
/*
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]*/
/* We notice that the highest power of Zeta used is 3
 in this case. We are in Cyclotomic field 5 which mean
s there are values for C.1, C.1^2, ..., c.1^5.
However, we are not required to label each C in this
 instance because we only use 3 values of C.1.
 Namely, C.1, C.1^2, and C.1^3 (Look at the 2
 matrices and notice the powers of Zeta). So when
 we do the mat function in a second, rather than
 putting all 5 elements, only label C.1, C.1^2,
and C.1^3 and their opposites. */
T:=Transversal(G,H);
C:=CyclotomicField(5);
/*To find your C.1...C.1^n, all you do is pick
 your bease, it will always be 2 or 3,
then in magma you do 2 mod your field,
 which here we are using 11 since 5|11-1.
Then you do 2^1 mod 11 = 2,
then 2^2 mod 11 = 4, ect*/
mat := function(n,p,D,k)
```

```
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1
 then D[i,j]:=4; end if;
if CH[n](T[i]*p*T[j]^-1) eq -C.1
 then D[i,j]:=-4; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2
 then D[i,j]:=5; end if;
if CH[n](T[i]*p*T[j]^-1) eq -C.1^2
 then D[i,j]:=-5; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^3
 then D[i,j]:=9; end if;
if CH[n](T[i]*p*T[j]^-1) eq -C.1^3
 then D[i,j]:=-9; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^4
 then D[i,j]:=3; end if;
if CH[n](T[i]*p*T[j]^-1) eq -C.1^4
 then D[i,j]:=-3; end if;
if CH[n](T[i]*p*T[j]^-1) in {1}
 then D[i,j]:=CH[n](T[i]*p*T[j]^-1); end if;
end if; end for;
return D;
end function;
GG:=GL(4,11);
A:=[[0,0,0,0]: i in [1..4]];
mat(2,xx,A,4);
AA:=GG!mat(2,xx,A,4);
Order(GG!AA);
/*5*/
B:=[[0,0,0,0]: i in [1..4]];
mat(2,yy,B,4);
BB:=GG!mat(2,yy,B,4);
Order(GG!BB);
/*4*/
HH:=sub<GG|AA,BB>;
IsIsomorphic(HH,G);
/*true*/
C:=CyclotomicField(10);
A:=[[C.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do A[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1
 in H then A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do B[i,j]:=0; end for;
```

```
for i,j in [1..4] do if T[i]*yy*T[j]^-1 in
 H then B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
 end if; end for;

perm := function(n, p, mat)
/* Return the matrix converted to
 permutation of S_{n*p}.
*/
C<u>:=CyclotomicField(p);
Z:=Integers ();
 s:=[];
for i in [1..n] do
 s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do
j:=0; done:=0;
repeat
if z[1,i]/u^j in Z then
 if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
 done:=1;
end if; end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do
for a in [1..n] do
 w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for; end for;
 S:=Sym(n*p);
 w:=S!w;
return w;
end function;
GG:=GL(4,C);
AA:=GG!A;
AA;
/* [ 4  0  0  0]
[ 0  9  0  0]
[ 0  0  3  0]
[ 0  0  0  5]
t1, t1^5, t1^10, t1^4, */
```

```
BB:=GG!B;
/* BB;
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]*/
perm(4,10,AA);
/* (1, 9, 17, 25, 33)(2, 26, 10, 34, 18
)(3, 35, 27, 19, 11)(4, 20, 36, 12, 28)
(5,13, 21, 29, 37)(6, 30, 14, 38, 22)
(7, 39, 31, 23, 15)(8, 24, 40, 16, 32)*/
 perm(4,10,BB);
/*(1, 4, 3, 2)(5, 8, 7, 6)(9, 12, 11, 10)
(13, 16, 15, 14)(17, 20, 19, 18)(21, 24,
    23, 22)(25, 28, 27, 26)(29, 32,
 31, 30)(33, 36, 35, 34)(37, 40, 39, 38)*/
G;
FPGroup(G);
Finitely presented group on 2 generators
Relations
    $.2^4 = Id($)
    $.1^-5 = Id($)
    $.2^-1 * $.1^-2 * $.2 * $.1^-1 = Id($)
G<x,y>:=Group<x,y|y^4,x^-5,
y^-1*x^-2*y*x^-1>;
S:=Sym(40);
xx:=S! (1, 9, 17, 25, 33)(2, 26, 10, 34, 18)
(3, 35, 27, 19, 11)(4, 20, 36, 12, 28)(5,13,
 21, 29, 37)(6, 30, 14, 38, 22)(7, 39, 31,
 23, 15)(8, 24, 40, 16, 32);
yy:=S! (1, 4, 3, 2)(5, 8, 7, 6)(9, 12, 11,
10)(13, 16, 15, 14)(17, 20, 19, 18)(21, 24,
    23, 22)(25, 28, 27, 26)(29, 32, 31, 30)
(33, 36, 35, 34)(37, 40, 39, 38);
N:=sub<S|xx,yy>;

Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
```

```
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;


Normaliser:=Stabiliser(N,{1, 9, 17, 25, 33});
Generators(Normaliser);
/*  (1, 17, 33, 9, 25)(2, 10, 18, 26, 34)(3, 27, 11, 35, 19)
(4, 36, 28, 20, 12)(5, 21, 37, 13, 29)(6, 14, 22, 30, 38)
(7, 31, 15, 39, 23)(8, 40, 32, 24, 16)*/
Stabiliser(N, {1, 9, 17, 25, 33});
A:=Normaliser! (1, 17, 33, 9, 25)(2, 10, 18, 26, 34)
(3, 27, 11, 35, 19)(4, 36, 28, 20, 12)(5, 21, 37, 13, 29)
(6, 14, 22, 30, 38)(7, 31, 15, 39, 23)(8, 40, 32, 24, 16);
Normaliser eq sub<N|A>;
for i in [1..#N] do if ArrayP[i] eq A then Sch[i]; end if; end for;
/*x^2
t^(x^2);
*/
Normaliser eq sub<N|xx,xx^2>;

/*this is my old progenitor for group G:*/

G<x,y>:=Group<x,y|y^2,(x^-1*y)^2,x^5>;

/*Now check Progenitor for mon presentation*/
G<x,y,t>:=Group<x,y,t|y^4,x^-5,y^-1*x^-2*y*x^-1,
t^11,t^(x^2)=t^5,(t,t^(y)),(t,t^(y^2)),(t,t^(y^3))>;
#G;
/* 292820*/
Index(G,sub<G|x,y>);
/*14641*/
C:=Classes(N);

#C;
/*4*/
 for i in [2..5] do
 i,Orbits(Centralizer(N,C[i][3]));
 end for;
```

```
for j in [2..5] do
 C[j][3];
 for i in [1..20] do if ArrayP[i] eq C[j][3]
 then Sch[i]; end if;
 end for; end for;
/* y^2
(1, 4, 3, 2)(5, 8, 7, 6)(9, 12, 11, 10)
(13, 16, 15, 14)(17, 20, 19, 18)(21, 24,
    23, 22)(25, 28, 27, 26)(29, 32,
31, 30)(33, 36, 35, 34)(37, 40, 39, 38)
y
(1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)
(13, 14, 15, 16)(17, 18, 19, 20)(21, 22,
    23, 24)(25, 26, 27, 28)(29, 30, 31,
 32)(33, 34, 35, 36)(37, 38, 39, 40)
y^-1
(1, 9, 17, 25, 33)(2, 26, 10, 34, 18)
(3, 35, 27, 19, 11)(4, 20, 36, 12, 28)
(5, 13, 21, 29, 37)(6, 30, 14, 38, 22)
(7, 39, 31, 23, 15)(8, 24, 40, 16, 32)
x*/
for a,b,c,d,e,f,g,h,i in [0..10] do
G<x,y,t>:=Group<x,y,t|y^4,x^-5,
y^-1*x^-2*y*x^-1,t^11,t^(x^2)=t^5,
(y^2*t)^a,
(y^2*t^2)^b,
(y^2*t^3)^c,
(y^2*t^4)^d,
(y^2*t^5)^e,
(y^2*t^6)^f,
(y^2*t^8)^g,
(y^2*t^9)^h,
(y^2*t^10)^i>;
if #G gt 10 then a,b,c,d,e,f,g,h,i;
#G;
end if;
end for;
Mon1041
for a,b,c,d,e,f,g,h,i in [0..10] do
G<x,y,t>:=Group<x,y,t|y^4,x^-5,
y^-1*x^-2*y*x^-1,t^11,t^(x^2)=t^5,
(y*t)^a,
(y*t^2)^b,
(y*t^3)^c,
```

```
(y*t^4)^d,
(y*t^5)^e,
(y*t^6)^f,
(y*t^8)^g,
(y*t^9)^h,
(y*t^10)^i>;
if #G gt 10 then a,b,c,d,e,f,g,h,i;
#G;
end if;
end for;
Mon1042
/*0 0 0 0 0 0 0 0 3    7920*/
for a,b,c,d,e,f,g,h,i in [0..10] do
G<x,y,t>:=Group<x,y,t|y^4,x^-5,
y^-1*x^-2*y*x^-1,t^11,
t^(x^2)=t^5, (y*t^10)^3>;
/*CompositionFactors(G1);
    G
    |  M11
    1*/
```

A.3 Double Coset of J_2 over $M = A_5 : C_5$

```
G<x,y,t>:=Group<x,y,t| x^5,y^2,
x^-1*y*x^-1*y*x*y*x*y,t^3,(t,x),
 (y * x^-2 * y * x^-1*t)^2,
(x^-2 * y * x^-1*t)^5>;
#G;
S:=Sym(10);
xx:=S!(2, 4, 6, 8, 10);
yy:=S!(1, 6)(2, 7)(3, 8)(4, 9)(5, 10);
N:=sub<S|xx,yy>;
#N;
Set(N);
HH:=sub<G| x,y,y * x^-1 * t * y * x * y * t * y^2
* x^2 * y * t * y^3 * x^2 * y * x * y * t *  y^4 *
 x^2 * y^3 * x^2 * y * x * y * t * y^4 * x^2 * y
* x * y * t * y^2 * t *  y^3 * x^2 * y^3 * x^2 *
 y^3 * x^2 * y * x * y * t * y^2 * t * y^4 * x^2 * y
* x * y * t * y^3 * x^2 * y^3 * x^2 * y * x * y *
 t * y^4 * x^2 * y * x * y * t *  y^2 * t * y^4 *
 x^2 * y * x * y * t * y^3 * x^2 * y^3 * x^2 *
```

```
y * x^-1>;
#HH;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
IM:=sub<G1|f(x),f(y),f(y * x^-1 * t * y * x * y *
 t * y^2 * x^2 * y * t * y^3 * x^2 * y * x * y * t
*  y^4 * x^2 * y^3 * x^2 * y * x * y * t * y^4 *
 x^2 * y * x * y * t * y^2 * t *  y^3 * x^2 * y^3
 * x^2 * y^3 * x^2 * y * x * y * t * y^2 * t * y^4 *
x^2 * y * x * y * t * y^3 * x^2 * y^3 * x^2 * y * x
* y * t * y^4 * x^2 * y * x * y * t *  y^2 * t * y^4 *
x^2 * y * x * y * t * y^3 * x^2 * y^3 * x^2 * y *
 x^-1)>;
#IM/#IN;
ts := [ Id(G1): i in [1 .. 10] ];
ts[1]:=f(t); ts[2]:=f(t^(y^x^3)); ts[3]:=f(t^(x^y));
 ts[4]:=f(t^(y^x^4));ts[5]:=f(t^(y*x^2*y*x^-2));
 ts[6]:=f(t^(y^x^5)); ts[7]:=f(t^(y*x^-2)^2);
ts[8]:=f(t^(y^x)); ts[9]:=f(t^(y*x^-1*y*x^-2));
ts[10]:=f(t^(y^x^2));
/*This tells me how many of each type of DC I will have.
 In other words how many double cosets of 1 t, of two ts,
three ts, ect. */
#DoubleCosets(G, sub<G|x,y,y * x^-1 * t * y * x * y * t *
 y^2 * x^2 * y * t * y^3 * x^2 * y * x * y * t *  y^4 * x^2
 * y^3 * x^2 * y * x * y * t * y^4 * x^2 * y * x * y * t *
 y^2 * t *  y^3 * x^2 * y^3 * x^2 * y^3 * x^2 * y * x
* y * t * y^2 * t * y^4 * x^2 * y * x * y * t * y^3 * x^2
 * y^3 * x^2 * y * x * y * t * y^4 * x^2 * y * x * y * t *
 y^2 * t * y^4 * x^2 * y * x * y * t * y^3 * x^2 * y^3 *
 x^2 * y * x^-1>, sub<G|x,y>);
DoubleCosets(G, sub<G|x,y,y * x^-1 * t * y * x * y * t
 * y^2 * x^2 * y * t * y^3 * x^2 * y * x * y * t *  y^4
 * x^2 * y^3 * x^2 * y * x * y * t * y^4 * x^2 * y * x *
 y * t * y^2 * t *  y^3 * x^2 * y^3 * x^2 * y^3 * x^2 *
 y * x * y * t * y^2 * t * y^4 * x^2 * y * x * y * t * y^3
 * x^2 * y^3 * x^2 * y * x * y * t * y^4 * x^2 * y * x *
 y * t *  y^2 * t * y^4 * x^2 * y * x * y * t * y^3 * x^2
* y^3 * x^2 * y * x^-1>, sub<G|x,y>);
/*
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * x * y * t^-1 *
 y * x * y * t^-1,GrpFP>, <GrpFP, t * y * t * y * t^-1,
 GrpFP>, <GrpFP, t * y * t * y * t, GrpFP>,<GrpFP, t *
 y * t, GrpFP>, <GrpFP, t, GrpFP>, <GrpFP, t * y * x *
```

```
y * t^-1,GrpFP>, <GrpFP, t * y * x * y * t^-1 * y * t,
 GrpFP>, <GrpFP, t * y * t * y * x* y * t^-1, GrpFP>,
 <GrpFP, t * y * t * y * t^-1 * y * t^-1, GrpFP> }*/
Index(G,HH); /*208*/
#G/#IN; /*1248*/
prodim := function(pt, Q, I)
 v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
    for i := 1 to 10 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;
for i in [1..1248] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
/*10*/
Orbits(N);
/*  GSet{@ 1, 6, 8, 10, 3, 2, 5, 4, 7, 9 @}*/

N1:=Stabiliser(N,1);
SSS:={[1]}; SSS:=SSS^N;
Seqq:=Setseq(SSS);
for i in [1..#SSS] do for n in IM do if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1;
/* Permutation group N1 acting on a set of
 cardinality 10
Order = 5
    (2, 4, 6, 8, 10)*/
#N1;
#N/#N1;
T1:=Transversal(N,N1);
T1; /* These are the transversals,
for which you conjugate Mt1N by:
    Id(N),
    (1, 6)(2, 7)(3, 8)(4, 9)(5, 10),
    (1, 8, 3, 10, 5, 2, 7, 4, 9, 6),
```

```
    (1, 4, 9, 2, 7, 10, 5, 8, 3, 6),
    (1, 10, 5, 4, 9, 8, 3, 2, 7, 6),
    (1, 3, 5, 7, 9),
    (1, 9, 7, 5, 3),
    (1, 2, 7, 8, 3, 4, 9, 10, 5, 6),
    (1, 5, 9, 3, 7),
    (1, 7, 3, 9, 5) since t1 goes to all elements 1..10*/
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1);
/* GSet{@ 1 @},
    GSet{@ 3 @},
    GSet{@ 5 @},
    GSet{@ 7 @},
    GSet{@ 9 @},
    GSet{@ 2, 4, 6, 8, 10 @}*/
for g in IM do for h in IN do if ts[1]*ts[9] eq
 g*(ts[1])^h
 then g,h; break; end if; end for; end for;

N12:=Stabiliser(N,[1,2]);
 SSS:={[1,2]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;

N12; #N12;
#N/#N12;
T12:=Transversal(N,N12);
 for i in [1..#T12] do ss:=[1,2]^T12[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
```

```
N13:=Stabiliser(N,[1,3]);
 SSS:={[1,3]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;

N13;
/* (2, 4, 6, 8, 10)*/
 #N13;
#N/#N13;
T13:=Transversal(N,N13);
 for i in [1..#T13] do ss:=[1,3]^T13[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

N15:=Stabiliser(N,[1,5]);
 SSS:={[1,5]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[5] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 5 ]
[ 6, 10 ]
[ 8, 2 ]
[ 10, 4 ]
[ 3, 7 ]
[ 2, 6 ]
[ 5, 9 ]
[ 4, 8 ]
[ 7, 1 ]
```

```
[ 9, 3 ]*/
for g in N do if [1,5]^g eq [6,10] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [8,2] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [10,4] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [3,7] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [2,6] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [5,9] then
N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [4,8] then
 N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [7,1] then
N15:=sub<N|N15,g>; end if; end for;
for g in N do if [1,5]^g eq [9,3] then
 N15:=sub<N|N15,g>; end if; end for;
N15; #N15;
#N/#N15;
T15:=Transversal(N,N15);
 for i in [1..#T15] do ss:=[1,5]^T15[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

N17:=Stabiliser(N,[1,7]);
 SSS:={[1,7]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[7] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
N17; #N17;
#N/#N17;
/*10*/
T17:=Transversal(N,N17);
 for i in [1..#T17] do ss:=[1,7]^T17[i];
```

```
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IM do for h in IN do if ts[1]
*ts[7] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;

*****************[1,2]*********************************
Orbits(N12); /*1 .. 10*/

for g in IM do for h in IN do if ts[1]*ts[2]
*ts[2] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[3] eq g*(ts[1]*ts[2]*ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4] eq g*(ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5] eq g*(ts[1]*ts[2]*ts[4])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[6] eq g*(ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[7] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[8] eq g*(ts[1]*ts[2]*ts[4])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[9] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[10] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;

N121:=Stabiliser(N,[1,2,1]);
 SSS:={[1,2,1]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
```

```
 if ts[1]*ts[2]*ts[1] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 1 ]
[ 10, 3, 10 ]*/
for g in N do if [1,2,1]^g eq [10,3,10] then
 N121:=sub<N|N121,g>; end if; end for;
N121; #N121;
#N/#N121;
/*25*/
T121:=Transversal(N,N121);
 for i in [1..#T121] do ss:=[1,2,1]^T121[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
N124:=Stabiliser(N,[1,2,4]);
 SSS:={[1,2,4]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[4] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 4 ]
[6, 7, 9]*/
for g in N do if [1,2,4]^g eq [6,7,9] then
 N124:=sub<N|N124,g>; end if; end for;
N124; #N124;
#N/#N124;
/*25*/
T124:=Transversal(N,N124);
 for i in [1..#T124] do ss:=[1,2,4]^T124[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
N125:=Stabiliser(N,[1,2,5]);
```

```
 SSS:={[1,2,5]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[5] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 5] */
N125; #N125;
#N/#N125;
/*50*/
T125:=Transversal(N,N125);
 for i in [1..#T125] do ss:=[1,2,5]^T125[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
***************************[1,3]***
Orbits(N13);
/* GSet{@ 1 @},
    GSet{@ 3 @},
    GSet{@ 5 @},
    GSet{@ 7 @},
    GSet{@ 9 @},
    GSet{@ 2, 4, 6, 8, 10 @}*/
for g in IM do for h in IN do if ts[1]*
ts[3]*ts[1] eq g*(ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[3]*ts[2] eq g*(ts[1]*ts[3]*ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[3]*ts[3] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[3]*ts[5] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[3]*ts[7] eq g*(ts[1])^h
 then g,h; break; end if; end for;
end for;
```

```
for g in IM do for h in IN do if ts[1]
*ts[3]*ts[9] eq g*(ts[1]*ts[3]*ts[2])^h
 then g,h; break; end if; end for; end for;
N132:=Stabiliser(N,[1,3,2]);
 SSS:={[1,3,2]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[3]*ts[2] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 3, 2]
[6,8,7] */
for g in N do if [1,3,2]^g eq [6,8,7] then
 N132:=sub<N|N132,g>; end if; end for;

N132; #N132;
#N/#N132;
/*25*/
T132:=Transversal(N,N132);
 for i in [1..#T132] do ss:=[1,3,2]^T132[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
N139:=Stabiliser(N,[1,3,9]);
 SSS:={[1,3,9]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[3]*ts[9] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 3, 9 ]
[ 3, 5, 1 ]
[ 5, 7, 3 ]
[ 7, 9, 5 ]
[ 9, 1, 7 ]*/
```

```
for g in N do if [1,3,9]^g eq [3,5,1]
then N139:=sub<N|N139,g>; end if; end for;
for g in N do if [1,3,9]^g eq [5,7,3]
 then N139:=sub<N|N139,g>; end if; end for;
for g in N do if [1,3,9]^g eq [7,9,5]
 then N139:=sub<N|N139,g>; end if; end for;
for g in N do if [1,3,9]^g eq [9,1,7]
 then N139:=sub<N|N139,g>; end if; end for;

N139; #N139;
#N/#N139;
/*2*/
T139:=Transversal(N,N139);
 for i in [1..#T139] do ss:=[1,3,9]^T139[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
*********************[1,5]*************

Orbits(N15);
/*  GSet{@ 1, 6, 8, 10, 3, 2, 5, 4, 7, 9 @}*/

for g in IM do for h in IN do if ts[1]
*ts[5]*ts[1] eq g*(ts[1])^h
 then g,h; break; end if; end for; end for;

********************[1,3,2]*********
Orbits(N132);
/*  GSet{@ 1, 6 @},
    GSet{@ 2, 7 @},
    GSet{@ 3, 8 @},
    GSet{@ 4, 9 @},
    GSet{@ 5, 10 @}*/
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[2]*ts[1] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[2]*ts[2] eq g*(ts[1]*ts[3]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[2]*ts[3] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]
```

```
*ts[2]*ts[4] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[2]*ts[5] eq g*(ts[1]*ts[3]*ts[2])^h
 then g,h; break; end if; end for; end for;



**************************[1,3,9]****
Orbits(N139);
/* GSet{@ 1, 3, 5, 7, 9 @},
    GSet{@ 2, 4, 10, 6, 8 @}*/
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[9]*ts[1] eq g*(ts[1]*ts[3])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[3]
*ts[9]*ts[2] eq g*(ts[1]*ts[3]*ts[9])^h
 then g,h; break; end if; end for; end for;

**************************[121]*******
Orbits(N121);
for g in IM do for h in IN do if ts[1]*ts[3]*
ts[9]*ts[2] eq g*(ts[1]*ts[3]*ts[9])^h
 then g,h; break; end if; end for; end for;
**********************************

N121:=Stabiliser(N,[1,2,1]);
 SSS:={[1,2,1]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[1] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 5][10,3,10] */
for g in N do if [1,2,1]^g eq [10,3,10] then
 N121:=sub<N|N121,g>; end if; end for;
N121; #N121;
#N/#N121;
/*50*/
T121:=Transversal(N,N121);
```

```
 for i in [1..#T121] do ss:=[1,2,1]^T121[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
for g in IM do for h in IN do if ts[1]*
ts[2]*ts[1]*ts[1] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[2]*ts[1]*ts[2] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[2]*ts[1]*ts[4] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[2]*ts[1]*ts[6] eq g*(ts[1]*ts[2]*ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]
*ts[2]*ts[1]*ts[8] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
**************************[124]***
Orbits(N124);
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[1] eq g*(ts[1]*ts[2]*ts[4])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[2] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;

for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[3] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;

for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[4] eq g*(ts[1]*ts[2]*ts[4])^h
 then g,h; break; end if; end for; end for;


for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[5] eq g*(ts[1]*ts[5])^h
 then g,h; break; end if; end for; end for;
********************[125*********
N125:=Stabiliser(N,[1,2,5]);
 SSS:={[1,2,5]};
```

```
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[5] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 5] */
N125; #N125;
#N/#N125;
/*50*/
T125:=Transversal(N,N125);
 for i in [1..#T125] do ss:=[1,2,5]^T125[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N125);
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[1] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[2] eq g*(ts[1]*ts[3]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[3] eq g*(ts[1]*ts[2]*ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[4] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[5] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[6] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[7] eq g*(ts[1]*ts[2]*ts[1])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[8] eq g*(ts[1]*ts[3]*ts[2])^h
 then g,h; break; end if; end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[9] eq g*(ts[1]*ts[2])^h
 then g,h; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[5]*ts[10] eq g*(ts[1]*ts[2]*ts[4]*ts[5])^h
 then g,h; break; end if; end for; end for;
************************[1245]******
N1245:=Stabiliser(N,[1,2,4,5]);
 SSS:={[1,2,4,5]};
 SSS:=SSS^N;
Seqq:=Setseq(SSS);
 for i in [1..#SSS] do for n in IM do
 if ts[1]*ts[2]*ts[4]*ts[5] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]] *ts[Rep(Seqq[i])[3]] *
ts[Rep(Seqq[i])[4]]
 then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/* [ 1, 2, 4, 5 ]
[ 3, 4, 6, 7 ]
[ 5, 6, 8, 9 ]
[ 7, 8, 10, 1 ]
[ 9, 10, 2, 3 ]*/
for g in N do if [1,2,4,5]^g eq [3,4,6,7]
 then N1245:=sub<N|N1245,g>; end if; end for;
for g in N do if [1,2,4,5]^g eq [5,6,8,9]
 then N1245:=sub<N|N1245,g>; end if; end for;
for g in N do if [1,2,4,5]^g eq [7,8,10,1]
 then N1245:=sub<N|N1245,g>; end if; end for;
for g in N do if [1,2,4,5]^g eq [9,10,2,3]
 then N1245:=sub<N|N1245,g>; end if; end for;

N1245; #N1245;
#N/#N1245;
/*10*/
T1245:=Transversal(N,N1245);
 for i in [1..#T1245] do ss:=[1,2,4,5]
^T1245[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..1248] do if cst[i] ne
[] then m:=m+1; end if; end for; m;
Orbits(N1245);
```

```
for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[5]*ts[1] eq g*(ts[1]*ts[2]*ts[4])^h
 then g,h; break; end if; end for; end for;

for g in IM do for h in IN do if ts[1]*ts[2]
*ts[4]*ts[5]*ts[2] eq g*(ts[1]*ts[2]*ts[5])^h
 then g,h; break; end if; end for; end for;



G<x,y>:=Group<x,y,t|x^2,y^6,(y*x*y^-1*x)^2
,(x*y^-1)^5,
(t,x^y),t^2,(y^2*x*y^-2)^2,
(y^3*t)^0,((y * x * y)^2*t)^0,(y^2*t)^2
,(y*x*y*t)^0,(y*x*t)^6,(y*t)^6>;
#G;
/*19440*/
 f, G1, k:=CosetAction(G,sub<G|x,y>);
 #k;
/*1*/
CompositionFactors(G1);
   /* G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1*/
NL:=NormalLattice(G1);
NL;

IsAbelian(NL[2]);
/*true*/

H:=NL[2];
 q,ff:=quo<G1|NL[2]>;
```

```
 q;
/*Permutation group q acting on a set of cardinality 24
Order = 240 = 2^4 * 3 * 5
 (1, 2)(3, 5)(4, 6)(7, 11)(8, 13)(9, 15)(10, 17)(12, 18)(14, 21)(16,
  22)(19, 24)(20, 23)
 (1, 3, 4, 7, 12, 2)(5, 8, 14, 21, 24, 18)(6, 9, 15, 11, 16, 10)(13,
  19, 22, 20, 23, 17)
 (1, 3)(2, 4)(5, 8)(6, 10)(7, 12)(9, 16)(11, 15)(13, 20)(14, 18)(17,
        23)(19, 22)(21, 24)*/

X:=[3,3,3,3];

 IsIsomorphic(NL[2],AbelianGroup(GrpPerm,(X)));
/*true Mapping from: GrpPerm: H to GrpPerm: $, Degree 12,
Order 3^4
Composition of Mapping from: GrpPerm: H to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 12, Order 3^4*/


nl:=NormalLattice(q);
nl;

E:=DirectProduct(nl[2],nl[4]);
 IsIsomorphic(E,q);
/*true Homomorphism of GrpPerm: E, Degree 48, Orde
r 2^4 * 3 * 5 into
GrpPerm: q, Degree 24, Order 2^4 * 3 * 5 induced by
    Id(E) |--> Id(q)
 (1, 23)(2, 20)(3, 17)(4, 13)(5, 10)(6, 8)(7, 19)(9, 14)(11, 24)(12,
  22)(15, 21)(16, 18) |--> (1, 23)(2, 20)(3, 17)(4, 13)(5, 10)(6,
 8)(7, 19)(9, 14)(11, 24)(12, 22)(15, 21)(16, 18)
 (25, 44)(26, 47)(27, 34)(28, 32)(29, 41)(30, 37)(31, 48)(33, 45)(35,
 43)(36, 40)(38, 39)(42, 46) |--> (1, 18)(2, 12)(3, 10)(4, 15)(5,
 17)(6, 9)(7, 24)(8, 14)(11, 19)(13, 21)(16, 23)(20, 22)
(25, 30, 33, 42, 29, 41)(26, 37, 43, 35, 45, 36)(27, 47, 32, 38, 40,
34)(28, 31, 48, 39, 46, 44) |--> (1, 3, 4, 7, 12, 2)(5, 8, 14, 21,
 24, 18)(6, 9, 15, 11, 16, 10)(13, 19, 22, 20, 23, 17)*/

 IsIsomorphic(nl[2],CyclicGroup(2));

/*true Mapping from: GrpPerm: $, Degree 24, Order 2
 to GrpPerm: $, Degree 2,
Order 2
```

```
Composition of Mapping from: GrpPerm: $, Degree 24,
Order 2 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 2, Order 2*/

IsIsomorphic(nl[4],SymmetricGroup(5));
/*true Homomorphism of GrpPerm: $, Degree 24,
Order 2^3 * 3 * 5 into
GrpPerm: $, Degree 5, Order 2^3 * 3 * 5 induced by
    (1, 20)(2, 23)(3, 10)(4, 8)(5, 17)(6, 13)(7, 24)
(9, 21)(11, 19)(12,
        16)(14, 15)(18, 22) |--> (1, 2)
    (1, 6, 9, 18, 5, 17)(2, 13, 19, 11, 21, 12)
(3, 23, 8, 14, 16, 10)(4,
        7, 24, 15, 22, 20) |--> (1, 3, 5)(2, 4)*/

FPGroup(SymmetricGroup(5));

S<a,b>:=Group<a,b|a^5,b^2,(a^-1*b)^4,
(a*b*a^-2*b*a)^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,nl[4]);
s;
/*true*/

FPGroup(nl[2]);
C<c,d>:=Group<c,d|d^2,c>;

ff1,cc,kk1:=CosetAction(C,sub<C|Id(C)>);

s,t:=IsIsomorphic(nl[2],cc);

s;
/*true*/

/*Here e and f will be a and b from S respectively
 and g,h will be c,d from
C respectively*/


D<e,f,g,h>:=Group<e,f,g,h|e^5,f^2,(e^-1*f)^4,
(e*f*e^-2*f*e)^2,h^2,g, (g,e),
(g,f),(h,e),(h,f)>;
```

```
ff2,dd,kk2:=CosetAction(D,sub<D|Id(D)>);
s,t:=IsIsomorphic(q,dd);

s;

T:=Transversal(G1,NL[2]);
 ff(T[2]) eq q.1;
/*true*/
 ff(T[3]) eq q.2;
/*true*/
 ff(T[4]) eq q.3;
/*true*/

 Order(T[2]);
2
 Order(T[3]);
6
 Order(T[4]);
2
Order(q.1);
2
 Order(q.2);
6
 Order(q.3);
2


/*T[2] = q.1, T[3] = q.2, T[4] = q.3.
The transversals of NL[2] and the generators of q match up,
therefore we might have
a semidirect product or a mixed extension if we can write
the elements of q as products of elements of K*/

 Generators(NL[2]);

A:=G1!
(1, 161, 155)(2, 143, 162)(3, 144, 146)(4, 149, 147)(5, 145,
123)(6,119, 156)(7, 134, 140)(8, 122, 158)(9, 96, 157)(10,
80, 56)(11,
 111, 74)(12, 101, 77)(13, 120, 151)(14, 108, 63)(15, 154,
114)(16,
105, 76)(17, 81, 98)(18, 82, 70)(19, 33, 110)(20, 79, 57)
(21, 51,
 138)(22, 55, 127)(23, 153, 115)(24, 159, 87)(25, 39, 136)
```

```
(26, 160,
88)(27, 141, 139)(28, 102, 75)(29, 53, 131)(30, 103, 97)
(31, 73,
112)(32, 45, 113)(34, 109, 35)(36, 148, 47)(37, 126, 92)
(38, 125,
 66)(40, 118, 68)(41, 124, 59)(42, 142, 137)(43, 106, 85)
(44, 116,
 150)(46, 83, 133)(48, 117, 152)(49, 60, 130)(50, 52, 128)
(54, 72,
 129)(58, 135, 69)(61, 121, 100)(62, 91, 99)(64, 107, 67)
(65, 94, 89)(71, 104, 86)(78, 90, 93)(84, 95, 132);

 B:=G1!(1, 153, 151)(2, 144, 152)(3, 117, 162)(4, 126, 133)
(5, 135, 109)(6,
80, 78)(7, 107, 129)(8, 108, 89)(9, 105, 104)(10, 93, 156)(11,
154, 84)(12, 75, 106)(13, 161, 115)(14, 94, 158)(15, 132,
74)(16,
        71, 157)(17, 57, 124)(18, 159, 61)(19, 38, 137)(20, 59,
 81)(21,
        148, 53)(22, 136, 45)(23, 120, 155)(24, 100, 70)(25,
113, 55)(26,
        99, 97)(27, 73, 130)(28, 85, 101)(29, 138, 36)(30,
 160, 62)(31,
        60, 139)(32, 127, 39)(33, 125, 42)(34, 123, 58)(35
, 145, 69)(37,
        83, 147)(40, 150, 50)(41, 98, 79)(43, 77, 102)(44,
 52, 118)(46,
        149, 92)(47, 131, 51)(48, 143, 146)(49, 141, 112)
(54, 134, 67)(56,
        90, 119)(63, 65, 122)(64, 72, 140)(66, 142, 110)
(68, 116, 128)(76,
        86, 96)(82, 87, 121)(88, 91, 103)(95, 111, 114);
    C:=G1!(1, 154, 158)(2, 149, 139)(3, 126, 112)(4,
141, 162)(5, 125, 127)(6,
        81, 100)(7, 118, 138)(8, 161, 114)(9, 101, 99)(10,
57, 82)(11, 94,
        151)(12, 91, 157)(13, 111, 89)(14, 153, 84)(15,
122, 155)(16, 75,
        103)(17, 121, 156)(18, 56, 79)(19, 113, 34)(20,
70, 80)(21, 134,
        68)(22, 145, 66)(23, 132, 63)(24, 78, 59)(25, 58
, 137)(26, 104,
        85)(27, 143, 147)(28, 97, 105)(29, 129, 52)(30,
 76, 102)(31, 144,
```

```
        92)(32, 109, 33)(35, 110, 45)(36, 107, 44)(37,
73, 146)(38, 55,
        123)(39, 135, 42)(40, 51, 140)(41, 159, 90)(43,
 160, 86)(46, 60,
        152)(47, 64, 150)(48, 83, 130)(49, 117, 133)(50,
 131, 72)(53, 54,
        128)(61, 119, 98)(62, 96, 77)(65, 120, 74)(67,
116, 148)(69, 142,
        136)(71, 106, 88)(87, 93, 124)(95, 108, 115);
    D:=G1!(1, 159, 157)(2, 145, 150)(3, 135, 128)(4,
 125, 148)(5, 116, 162)(6,
        102, 132)(7, 130, 113)(8, 124, 99)(9, 161, 87)
(10, 85, 111)(11,
        56, 106)(12, 154, 90)(13, 82, 104)(14, 98, 103)
(15, 78, 77)(16,
        153, 61)(17, 97, 108)(18, 71, 151)(19, 138, 83)
(20, 160, 65)(21,
        133, 33)(22, 64, 139)(23, 100, 76)(24, 96, 155)
(25, 129, 73)(26,
        89, 57)(27, 55, 107)(28, 95, 156)(29, 37, 137)
(30, 63, 81)(31,
        136, 72)(32, 134, 49)(34, 118, 48)(35, 40,
152)(36, 147, 38)(39,
        54, 112)(41, 91, 158)(42, 53, 126)(43, 74, 80)
(44, 143, 123)(45,
        140, 60)(46, 110, 51)(47, 149, 66)(50, 144,
69)(52, 146, 58)(59,
        62, 122)(67, 141, 127)(68, 117, 109)(70, 86,
120)(75, 84, 119)(79,
        88, 94)(92, 142, 131)(93, 101, 114)(105, 115,
121);

NL[2] eq sub<G1|A,B,C,D>;
/*q,ff:=quo<G1|NL[2]>;
 q;*/
/*Write the transversals as elements of q*/
 FPGroup(q);
/*Finitely presented group on 3 generators
Relations
     $.1^2 = Id($)
    $.2^6 = Id($)
    $.3^2 = Id($)
    ($.2^-1 * $.3)^2 = Id($)
    $.2^-2 * $.1 * $.2^2 * $.3 * $.1 * $.3 = Id($)
```

```
    ($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)
    $.1 * $.2^-3 * $.1 * $.2^-1 * $.3 * $.1 *
 $.2^-1 * $.3 = Id($)
    ($.1 * $.2^-1)^5 = Id($)*/

/*The above presentation of q can be given by
Q<e,f,g>:=Group<e,f,g|e^2,f^6,g^2,(f^-1*g)^2,
f^-2*e*f^2*g*e*g, (f*e*f^-1*e)^2,
e*f^-3*e*f^-1*g*e*f^-1*g,(e*f^-1)^5>;
ff,Q1,kk:=CosetAction(Q,sub<Q|Id(Q)>); */

Generators(q);

    EE:=q! (1, 2)(3, 6)(4, 9)(5, 10)(7, 11)(8, 12);

    FF:=q! (1, 3, 7, 6, 11, 2)(4, 5, 9, 10, 12, 8);

    GG:=q! (1, 4)(2, 5)(3, 8)(6, 10)(7, 12)(9, 11);


q eq sub<q|EE,FF,GG>;

/*true*/
 T[2];

 H:=G1!(2, 3)(6, 10)(8, 14)(9, 16)(11, 15)(13, 23)
(18, 24)(21, 36)(22, 39)(25,
    45)(26, 30)(27, 49)(29, 51)(32, 55)(33, 38)(35,
 58)(37, 46)(40, 52)(42,
    66)(44, 68)(47, 53)(54, 64)(56, 78)(57, 81)(59,
 79)(60, 73)(61, 87)(62,
    88)(63, 89)(71, 96)(75, 101)(76, 104)(77, 106)
(82, 100)(84, 114)(85,
    102)(93, 119)(94, 122)(98, 124)(99, 103)(107,
 134)(109, 123)(110,
    137)(111, 132)(112, 139)(117, 143)(126, 149)
(128, 150)(129, 140)(133,
    147)(135, 145)(146, 152)(151, 155)(153, 161);
 T[3];

 I:=G1!(2, 4)(3, 5, 7)(6, 11, 9, 17, 13, 12)(8, 15)
(10, 18, 20)(14, 24, 43, 23, 41,
    26)(16, 28, 30)(19, 33, 32, 45, 35, 34)(21, 37,
 22, 40, 31, 38)(25, 46,
```

```
    47)(29, 52, 50, 72, 54, 53)(36, 58, 83, 64, 39,
 60)(42, 49, 44, 69, 48,
    67)(51, 73, 55)(56, 79, 57, 82, 70, 80)(59, 85,
 63)(61, 65, 62)(66, 68,
    92)(71, 95, 93)(74, 99, 98, 89, 77, 100)(75, 102,
76, 105, 97, 103)(78,
    106, 132, 124, 104, 108)(81, 111, 101)(84, 87,
86, 115, 90, 88)(91, 119,
    94, 96, 121, 120)(107, 135, 130)(109, 113, 110)
(112, 123, 138, 146, 127,
    140)(116, 142, 117)(118, 144, 125, 134, 126,
145)(128, 131, 129)(133,
    150, 136, 152, 148, 137)(139, 147)(141, 143)
(151, 157, 156)(153, 159,
    160)(154, 161)(155, 158);
 T[4];

 J:=G1!(1, 2)(3, 6)(4, 8)(5, 9)(7, 13)(10, 19)(11,
21)(12, 22)(14, 25)(15, 27)(16,
    29)(17, 31)(18, 32)(20, 35)(23, 42)(24, 44)(26,
 48)(28, 50)(30, 54)(33,
    56)(34, 57)(36, 59)(37, 61)(38, 62)(39, 63)(40,
 65)(41, 47)(43, 46)(45,
    70)(49, 71)(51, 74)(52, 75)(53, 76)(55, 77)(58,
 84)(60, 86)(64, 90)(66,
    91)(67, 93)(68, 94)(69, 95)(72, 97)(73, 98)(78,
 107)(79, 109)(80,
    110)(81, 112)(82, 113)(83, 85)(87, 116)(88,
 117)(89, 118)(92, 121)(96,
    123)(99, 125)(100, 126)(101, 127)(102, 128)
(103, 129)(104, 130)(105,
    131)(106, 133)(108, 136)(111, 138)(114, 141)
(115, 142)(119, 146)(120,
    140)(122, 147)(124, 148)(132, 135)(134, 151)
(137, 153)(139, 154)(143,
    155)(144, 156)(145, 157)(149, 158)(150, 159)
(152, 160)(161, 162);


/*Now, a presentation of NL[2]=<A,B,C,D> is
<w,x,y,z|w^3,x^3,y^3,z^3,(w,x),(w,y),(x,z),
(w,z),(x,z),(y,z)> becuase of
/*FPGroup(NL[2]);
Finitely presented group on 5 generators
```

```
Relations
    $.2^3 = Id($)
    $.3^3 = Id($)
    $.4^3 = Id($)
    $.5^3 = Id($)
    ($.2, $.3) = Id($)
    ($.2, $.4) = Id($)
    ($.3, $.4) = Id($)
    ($.2, $.5) = Id($)
    ($.3, $.5) = Id($)
    ($.4, $.5) = Id($)
    $.1 = Id($)*/

 and a presentation of q=<EE,FF,GG> is
<e,f,g|e^2,f^6,g^2,(f^-1*g)^2,f^-2*e*
f^2*g*e*g, (f*e*f^-1*e)^2,
e*f^-3*e*f^-1*g*e*f^-1*g,(e*f^-1)^5>;
 from above*/

/*Need to find the action of q on NL[2], since
 this is the semi direct part*/
for i,j,k,l in [1..3] do if A^H eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
3 1 3 3

for i,j,k,l in [1..3] do if A^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
3 3 1 3

for i,j,k,l in [1..3] do if A^J eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end
 for;
2 3 3 3

for i,j,k,l in [1..3] do if B^H eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
1 3 3 3

for i,j,k,l in [1..3] do if B^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
```

```
3 3 3 1

for i,j,k,l in [1..3] do if B^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
1 1 1 1

for i,j,k,l in [1..3] do if C^H eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end
 for;
3 3 1 3

for i,j,k,l in [1..3] do if C^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
1 3 3 3

for i,j,k,l in [1..3] do if C^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
3 3 2 3

for i,j,k,l in [1..3] do if D^H eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
3 3 3 1

for i,j,k,l in [1..3] do if D^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
2 2 2 2

for i,j,k,l in [1..3] do if D^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end
 for;
3 3 3 2


/* Writing q=<EE,FF,GG> in terms of NL[2]=
<A,B,C,D>, by pulling up:
> FPGroup(q);
Finitely presented group on 3 generators
Relations
    $.1^2 = Id($)
```

```
    $.2^6 = Id($)
    $.3^2 = Id($)
    ($.2^-1 * $.3)^2 = Id($)
    $.2^-2 * $.1 * $.2^2 * $.3 * $.1 * $.3 =
 Id($)
    ($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)
    $.1 * $.2^-3 * $.1 * $.2^-1 * $.3 * $.1 *
 $.2^-1 * $.3 = Id($)
    ($.1 * $.2^-1)^5 = Id($) */

T:=Transversal(G1,NL[2]);
ff(T[2]) eq EE;
/* true */
ff(T[3]) eq FF;
/* true */
ff(T[4]) eq GG;
/* true */
Order(T[2]);
/* 2 */
Order(T[3]);
/* 6 */
Order(T[4]);
/* 2 */
Order(T[2]^-1*T[3]); /*this is (FF^-1*EE)^5,
($.1 * $.2^-1)^5 = Id($)*/
/* 5 */
/*Order is 5, does not change, so leave alone*/

 Order(T[3]^-1*T[4])^2;
/*36, changes so run code*/
/*T[3]^-1*T[4] in NL[2]; true*/

for i,j,k,l in [1..3] do
if (T[3]^-1*T[4])^2 eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
   /*(FF^-1*GG)^2*/
    /* 1 3 2 3*/

/* Thus, (f^-1*g)^2=w*x^3*y^2*z^3=
w*y^2 */

Order(T[3]*T[2]*T[3]^-1*T[2]);
/*($.2 * $.1 * $.2^-1 * $.1)^2 = Id($),
 leave alone*/
```

```
/* 2 */

/* Thus, (f*e*f^-1*e)^2=identity
 (does not change)) */

Order(T[3]^-2*T[2]*T[3]^2*T[4]*T[2]*T[4]);
    /*$.2^-2 * $.1 * $.2^2 * $.3 * $.1 * $.3 =
Id($)*/

    /* 3, changes, so run code */
 T[3]^-2*T[2]*T[3]^2*T[4]*T[2]*T[4] in NL[2];
/*true*/

for i,j,k,l in [1..3] do
if (T[3]^-2*T[2]*T[3]^2*T[4]*T[2]*T[4]) eq
 A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;

/* 2 1 1 1*/

/* Thus, f^-2*e*f^2*g*e*g=w^2*x*y*z */

Order((T[3]*T[2]*T[3]^-1*T[2])^2);

/* 1 */

/* Thus, (f*e*f^-1*e)^2=identity remains
unchanged. */

Order(T[2]*T[3]^-3*T[2]*T[3]^-1
*T[4]*T[2]*T[3]^-1*T[4]);

/* 3 */

for i,j,k,l in [1..3] do
if T[2]*T[3]^-3*T[2]*T[3]^-1*T[4]*T[2]*
T[3]^-1*T[4] eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;

/* 1 2 3 3 */

/* Thus, e*f^-3*e*f^-1*g*e*f^-1*g=w*x^2 */
```

```
Order((T[2]*T[3]^-1)^5);

/* 1 */
/* Thus,  (e*f^-1)^5 =
 identity remains unchanged. */

/* The following shows that G1 is isomorphic
 to 3^4:^{\cdot}(S_5x2) */

/*w,x,y,z is for 3^4, and e, f, g is q*/

H<w,x,y,z,e,f,g>:=Group<w,x,y,z,e,f,g|
w^3,x^3,y^3,z^3,(w,x),(w,y),(x,z),(w,z),(x,z),(y,z),
e^2,f^6,g^2,(f^-1*g)^2=w*y^2,f^-2*e*f^2*g*e*g=
w^2*x*y*z, (f*e*f^-1*e)^2,
e*f^-3*e*f^-1*g*e*f^-1*g=w*x^2,(e*f^-1)^5,
w^e=x,w^f=y,w^g=w^2,x^e=w,x^f=z,x^g=w*x*y*z,
y^e=y,y^f=w,y^g=y^2,z^e=z,z^f=w^2*x^2*y^2*z^2
,z^g=z^2>;
#H;
/* 19440 */
#G1;
/* 19440 */
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
/* true */
```

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