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Monomial Progenitors and Related Topics

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MONOMIAL PROGENITORS AND RELATED TOPICS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Madai Obaid Alnominy

March 2018

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ABSTRACT

The main objective of this project is to find the original symmetric presentations of some very important finite groups and to give our constructions of some of these groups. We have found the Mathieu sporadic group M_{11}, $HS \times D_5$, where HS is the sporadic group Higman-Sim group, the projective special unitary group $U(3,5)$ and the projective special linear group $L_2(149)$ as homomorphic images of the monomial progenitors $11^{*4} :_m (5:4)$, $5^{*6} :_m S_5$ and $149^{*2} :_m D_{37}$. We have also discovered $2^4 : S_3 \times C_2$, $2^4 : A_5$, $(2^5 : S_4)$, $2^5 : S_3 \times S_3$, $3^3 : S_4 \times C_2$, S_6, $2^9 : PGL(2,7)$, $2^2 \bullet (S_6 : S_6)$, $PGL(2,19)$, $((A_5 : A_5 \times A_5) : D_6)$, $6 \bullet (U_4(3) : 2)$, $2 \bullet PGL(2,13)$, S_7, $PGL(2,8)$, $PSL(2,19)$, $2 \times PGL(2,81)$, $2^5 : (S_6 \times A_5)$, $2^6 : S_4 \times D_3$, $U(4,3)$, $3^4 : S_4$, $3^2 : D_6$, $2 \bullet (PGL(2,7) : PSL(2,7))$, $2^2 : (S_5 : S_5)$ and $2^3 : (PSL_3(4) : 2)$ as homomorphic images of the permutation progenitors $2^{*8} : (2 \times 4 : 2)$, $2^{*16} : (2 \times 4 : C_2 \times C_2)$, $2^{*9} : (S_3 \times S_3)$, $2^{*9} : (S_3 \times A_3)$, $2^{*9} : (3^2 : 2^3)$ and $2^{*9} : (3^3 : A_3)$. We have also constructed $2^4 : S_3 \times C_2$, $2^4 : A_5$, $(2^5 : S_4)$, $2^5 : S_3 \times S_3$, $3^3 : S_4 \times C_2$, S_6, M_{11} and $U(3,5)$ by using the technique of double coset enumeration. We have determined the isomorphism types of the most of the images mentioned in this thesis. We demonstrate our work for the following examples: $3^4 : (3^2 * 2^3) \times 2$, $2^9 : PGL(2,7)$, $2^{\bullet}S_6$, $(5^4 : (D_4 \times S_3))$, and $3 :^{\bullet} PSL(2,19) \times 2$.

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Introduction

Group theory is the significant topic in mathematics and abstract algebra. Group theory allows to study the algebraic structures known as group. In this thesis, we show symmetric presentations of finite groups. Also, we are interested to show some of groups which we could get them from our progenitors through our research for example Mathieu groups, alternating groups, linear groups, symmetric groups and Unitary groups. We show shortly in the following some of the concepts which we will discuss them in this thesis. **In chapter** 1, we give some important definitions and theorem which are used in this thesis. **In chapter 2**, we provide some of relations lemma which help us to find relations to factor our progenitors $m^n : N$ by them. Thus, we apply the factoring lemma, the famous lemma, or the first order relation lemma to factor our progenitors. **In chapter 3**, we describe character table construction. So, we build table of D_4, and we will induce a linear character of a proper subgroups. **In chapter 4**, we will see how we can compute an extension problem to find isomorphic type of our group by looking to the composition factor of group. **In chapter 5**, we will show progenitors and their homomorphic images. **In chapter 6**, we give transitive group on 9 letters with some of their homomorphic images. **In chapter 7**, we clarify how we can induce and verify a linear character of the monomial progenitors. Also, we verify the monomial representation and construct a permutation representation by hand. Moreover, we find a representation of the monomial progenitor. **In chapter 8**, we will demonstrate the technique of double coset enumeration and we will construct Caylay diagram of double coset enumeration. **In chapter 9**, we apply the technique of double coset enumeration over maximal subgroup. So, we will show double coset enumeration of M_{11} over maximal subgroup S_5 and double coset enumeration of $U(3,5)$ over maximal subgroup A_7.

Chapter 1

Group Theory Preliminaries

1.1 Definitions

Definition 1.1. *A* **group** G $(G, *)$ *is a nonempty collection of elements with an associative operation* $*$*, such that:*

- *there exists an identity element,* $e \in G$ *such that* $e * a = a * e$ *for all* $a \in G$*;*
- *for every* $a \in G$*, there exists an element* $b \in G$ *such that* $a * b = e = b * a$*. [Rot95]*

Definition 1.2. *For group* G*, a* **subgroup** S *of* G *is a nonempty subset where* $s \in G$ *implies* $s^{-1} \in G$ *and* $s, t \in G$ *imply* $st \in G$*. We denote subgroup* S *of* G *as* $S \leq G$*. [Rot95]*

Definition 1.3. *Let* H *be a subgroup of group* G*.* H *is a* **proper** *subgroup of* G *if* $H \neq G$*. We denote this as* $H < G$*. [Rot95]*

Definition 1.4. *Let* G *be a group and* $H \leq G$*.* H *is a* **maximal subgroup** *of* G *if there is no normal subgroup* $N \leq G$ *such that* $H < N < G$*. [Rot95]*

Definition 1.5. *A* **symmetric group***,* S_X*, is the group of all permutations of* X*, where* $X \in \mathbb{N}$*.* S_X *is a group under compositions. [Rot95]*

Definition 1.6. *If* X *is a nonempty set, a* **permutation** *of* X *is a bijection* $\phi : X \longrightarrow X$*. [Rot95]*

Definition 1.7. *If $x \in X$ and $\phi \in S_X$, then ϕ* **fixes** *x if $\phi(x) = x$ and ϕ* **moves** *x if $\phi(x) \neq x$. [Rot95]*

Definition 1.8. *For permutations $\alpha, \beta \in S_X$, α and β are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\textit{if } \alpha(x) \neq x, \textit{ then } \beta(a) = a \textit{ and if } \alpha(y) = y, \textit{ then } \beta(y) \neq y. \textit{ [Rot95]}$$

Definition 1.9. *A permutation which interchanges a pair of elements is a* **transposition***. [Rot95]*

Definition 1.10. *In group G, if a,$b \in G$, a and b* **commute** *if $a * b = b * a$. [Rot95]*

Definition 1.11. *A group G is* **abelian** *if every pair of elements in G commutes with one another. [Rot95]*

Definition 1.12. *Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$. [Rot95]*

Definition 1.13. *Let G be a group. G is* **simple** *if the only normal subgroups of G are 1 and G. [Rot95]*

Definition 1.14. *Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we say G is* **elementary abelian***. [Rot95]*

Definition 1.15. *Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \to H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all a,$b \in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to H, $G \cong H$, if there is exists an isomorphism $f : G \to H$. [Rot95]*

Definition 1.16. *Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *. [Rot95]*

Definition 1.17. *Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X. [Rot95]*

Definition 1.18. *Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \lhd G$ to denote K as being normal in G. [Rot95]*

Definition 1.19. *Let* a, $b \in G$. *We denote the* **commutator** *of* a *and* b *by* $[a,b]$, *where* $[a,b] = aba^{-1}b^{-1}$. *[Rot95]*

Definition 1.20. *Let* G *be a group. The* **Derived Group** *of* G, *denoted* G', *is the subgroup of* G *formed by all the commutators of* G. *[Rot95]*

Definition 1.21. *Let* G *be a group. The* **index** *of* $H \leq G$, *denoted* $[G : H]$, *is the number of right cosets of* H *in* G. *[Rot95]*

Definition 1.22. *Let* X *be a set and* G *be a group. We say* X *is a* **G-set** *if there exists a function* $\phi : G \times X \to X$ *(which we call an* **action***) and the following hold for* ϕ*:*

- $1x = x$, *for all* $x \in X$.
- $g(hx) = (gh)x$, *for* g,$h \in G$ *and* $x \in X$. *[Rot95]*

Definition 1.23. *Let* G *be a group. The* **center** *of* G, $Z(G)$, *is the set of all elements in* G *that commute with all elements of* G. *[Rot95]*

Definition 1.24. *Let* G *be a group. If* $H \leq G$, *the* **normalizer** *of* H *in* G *is defined by* $N_G(H) = \{a \in G | aHa^{-1} = H\}$. *[Rot95]*

Definition 1.25. *Let* G *be a group. If* $H \leq G$, *the* **centralizer** *of* H *in* G *is:*

$$C_G(H) = \{x \in G : [x,h] = 1 \text{ for all } h \in H\}. \text{ [Rot95]}$$

Definition 1.26. *Let* G *be a group and* X *be a G-set. For* $x \in X$, *the set* $x^G = \{x^g | g \in G\}$ *is a* **G-Orbit**. *[Rot95]*

Definition 1.27. *Let* X *be a G-set. Let* α *be an action of* G *on* X. *If* $\tilde{\alpha} : G \to S_X$ *is injective, we say* X *is* **faithful**. *[Rot95]*

Definition 1.28. *Let* G *be a group and* X *be a G-set.* X *is* **transitive** *if for all* x,$y \in X$ *there exists a* $g \in G$ *such that* $y = gx$. *[Rot95]*

Definition 1.29. *Let* G *be a group. A* **normal series** G *is a sequence of subgroups*

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

with $G_{i+1} \lhd G_i$. *Furthermore, the* **factor groups** *of* G *are given by* G_i/G_{i+1} *for* $i = 0, 1, \ldots, n-1$. *[Rot95]*

Definition 1.30. *Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G. [Rot95]*

Definition 1.31. *Let X be a G-set. Then for $B \subseteq X$, B is a* **block** *if for every $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$. [Rot95]*

Definition 1.32. *Let X and Y be G-sets. The function $f : X \to Y$ is a* **G-map** *if $f(gx) = gf(x)$, for all $x \in X$ and $g \in G$. [Rot95]*

Definition 1.33. *Let X be a G-set. X is* **primitive** *if X has no nontrivial blocks. If X is primitive, the only blocks of X are $B = X$ and $B = \emptyset$. [Rot95]*

1.2 Theorems

Theorem 1.34. *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles. [Rot95]*

Theorem 1.35. *Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*
- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*
- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$. [Rot95]*

Theorem 1.36. *The intersection of any family of subgroups of a group G is again a subgroup of G. [Rot95]*

Theorem 1.37. *If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 1.38. *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. [Rot95]*

Theorem 1.39. *If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|. \text{ [Rot95]}$$

Theorem 1.40. *If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$. [Rot95]*

Theorem 1.41. *The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$. [Rot95]*

Theorem 1.42. *Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$. [Rot95]*

Theorem 1.43. *Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]*

Theorem 1.44. *Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$. [Rot95]*

Theorem 1.45. *If $a \in G$, the number of conjugates of a is equal to the index of its centralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite. [Rot95]

Theorem 1.46. *If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$. [Rot95]*

Theorem 1.47. *If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker\rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H. [Rot95]*

Theorem 1.48. *If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set. [Rot95]*

Theorem 1.49. *Every two composition series of a group G are equivalent. We will refer to this Theorem as the* **Jordan-Hölder Theorem**. *[Rot95]*

Theorem 1.50. *Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$. [Rot95]*

Chapter 2

Writing Relations

Now we give important lemmas which assist us to find appropriate relations to factor our progenitors.

2.1 The Factoring Lemma

Factoring the progenitor $m^{*n} : N$ by (t_i, t_j) for $1 \leq i \leq j \leq n$ gives the group $m^n : N$. [Gri15]
This factoring lemma is used to check if the progenitor was written correctly or not.

2.1.1 Factoring Lemma Preformed on the Progenitor $11^{*2} : D_5$

Consider the progenitor $11^{*2} : D_5$. Now, if we factor the progenitor $11^{*2} : D_5$ by the relation $t_1t_2 = t_2t_1$, we must get $|\frac{11^{*2}:D_5}{(t_1,t_2)=1}|$=$11^2 \times 10 = 1210$. Where $t_1 \sim t$ and $t_2 \sim t^y$, the progenitor must generate our group $11^{*2} : D_5$. We will show this by using MAGMA.

```
S:=Sym(20);
xx:=S!(1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)(4,14,16,12,20);
yy:=S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)
(17, 18)(19, 20);
Nm:=sub<S|xx,yy>;
#Nm;
10

FPGroup(Nm);
```

```
Finitely presented group on 2 generators
Relations
$.2^2 = Id($)
($.1^-1 * $.2)^2 = Id($)
$.1^-5 = Id($)

NN<x,y>:=Group<x,y|y^2,(x^-1*y)^2,x^-5>;
#NN;
10

Stabiliser(Nm,{1,3,5,7,9,11,13,15,17,19});
Permutation group acting on a set of cardinality 20
Order = 5
(1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)
(4, 14, 16, 12, 20)

G<x,y,t>:=Group<x,y,t|y^2,(x^-1 * y)^2,x^-5,t^11,
t^x=t^5,(t,t^y)>;
#G;
1210

Index(G,sub<G|x,y>);
121

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
G
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(11)
*
|  Cyclic(11)
1

#Conjugates(G1,sub<G1|f(t)>);
2
```

Therefore, our progenitor is written correctly.

2.2 The Famous Lemma

Theorem 2.1. (Famous Lemma). *$N \cap < t_i, t_j > \leq C_N(N_{ij})$, where N_{ij} denotes the stabilizer in N of the two points i and j. [Cur07]*

2.2.1 The Famous Lemma Performed on the Progenitor $2^{*8} : (2 \times 4 : 2)$

We have given another way to produce important relations by a lemma is called the famous lemma. Through the following example, we will describe steps to apply the famous lemma. First, we need to store these generators of the permutation group $N =< ww, xx, yy, zz >$ in Magma. We will look for the highest digit in the permutations. Since 8 is the highest number, we will use sym(8).

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>
```

We check the order of N, so it is equal 16. Next, we want to ask Magma for a presentation for N.

```
FPGroup(N);
Finitely presented group on 4 generators
Relations
$.1^2 = Id($)
$.2^2 = Id($)
$.3^4 = Id($)
$.4^2 = Id($)
($.1 * $.3^-1)^2 = Id($)
$.2 * $.3^-1 * $.2 * $.3 = Id($)
$.3^-2 * $.4 = Id($)
$.1 * $.2 * $.1 * $.2 * $.4 = Id($)
```

This will be our presentation for N, where $.1 = w$, $.2 = x$, $.3 = y$, and $.4 = z$ type in NN=$Group < w, x, y, z | w^2, x^2, y^4, z^2, (w * y^-1)^2, x * y^-1 * x * y, y^-2 * z, w * x * w * x * z >$. Now, let us check the order of NN. This should be equal to the order of N, which was 16.

```
#NN;
16
```

We have 16, then we can move on to the next step. We now have a presentation for NN, now we will add a $t_i^{'}s$ and make them commute with the stabilizer.

```
N1:=Stabiliser(N,1);
Generators(N1);
(2, 5)(3, 7)
```

So, we will need to add $(t, (25)(37))$ to our presentation which means that t commutes with (25)(37). Also, we know that t is of order 2, since the progenitor given is 2^{*8}, so we will add t^2 to our presentation as well. To write (25)(37) in the presentation, we need to convert it into terms of w, x, y, and z. So, we will add this to our presentation that (t,w) since (2, 5)(3, 7) is equal w. Therefore, our presentation will look like this so far $G = Group < w, x, y, z, t | w^2, x^2, y^4, z^2, (w * y^-1)^2, x * y^-1 * x * y, y^-2 * z, w * x * w * x * z, t^2, (t, w) >$. We will now factor this presentation by the famous lemma. Famous lemma states that relations will look like as follow:
$(xt_i)^m = 1$ where m is odd and x sends 1 to 2 and $(t_i t_j)^m = x$ where m is even and x fixes 1 and 2. [Cur07]
Now, we need to find the centralizer of the point stabilizer of 1,2.

```
N12:=Stabiliser(N,[1,2]);
C:=Centraliser(N,N12);
C;
Permutation group N acting on a set of cardinality 8
Order = 16 = 2^4
(2, 5)(3, 7)
(1, 2)(3, 6)(4, 5)(7, 8)
(1, 3, 4, 7)(2, 6, 5, 8)
(1, 4)(2, 5)(3, 7)(6, 8)
```

We consider permutation (2,5)(3,7). The lemma tells us that if 1 and 2 are fixed then we use the following relation ,$(t_1 * t^2)^m = (t * t^x)^m = w$, where m is even. We also see that the permutation (1, 2)(3, 6)(4, 5)(7, 8), so we have this relation $((1, 2)(3, 6)(4, 5)(7, 8) * t_1)^m = (xt)^k = 1$ since 1 send to 2 and k is odd. Finally, we will add these relations to our presentation.

```
for k in [0..10] do for m in [0,1,3,5,7,9] do;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(t*x)^k=1,(t*t^x)^m=w>;
if #G gt 16 then k,m,#G;
end if; end for; end for;
```

2.3 The First Order Relation

A relation of the form $(x * t_i)^a$, where $x \in N$, and $t_i \in \{t_1, t_2, t_3, ...\}$ and $a \geq 2$. To apply the first order relation, we need to do these steps which are:

- We need to find a presentation of the progenitor $G = m^{*n} : N$.
- We need to find classes of N and factor our progenitor by these classes.
- We have to compute the centralizer of all classes and orbits of all classes. Then, we can develop the first order relation from those orbits.[Why06] In the following example, we apply those steps to factor our progenitor by the first order relation to find the homomorphism images.

2.3.1 The Progenitor $2^{*16} : (2 \times 4 : C_2 \times C_2)$ Factored by the First Order Relation

```
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
32
NN<x,y,z,w,h>:=Group<x,y,z,w,h|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,
(y*w^-1)^2,z*w^-1*z*w,w^-2*h,x*y*x*y*w^-1,x*z*x*z*h>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..32]];
for i in [2..32] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=ww^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=hh; end if;
```

```
if Eltseq(Sch[i])[j] eq -5 then P[j]:=hh^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,w^-2*h,x*y*x*y*w^-1,
x*z*x*z*h,t^2,(t,x)>;
C:=Classes(N);
C;

Conjugacy Classes of group N
----------------------------
[1]     Order 1       Length 1
Rep Id(N)

[2]     Order 2       Length 1
Rep (1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16)

[3]     Order 2       Length 2
Rep (1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15)

[4]     Order 2       Length 4
Rep (1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15)

[5]     Order 2       Length 4
Rep (1, 9)(2, 3)(4, 16)(5, 11)(6, 15)(7, 10)(8, 12)(13, 14)

[6]     Order 2       Length 4
Rep (2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16)

[7]     Order 4       Length 2
Rep (1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16)

[8]     Order 4       Length 2
Rep (1, 12, 5, 15)(2, 13, 10, 16)(3, 4, 7, 14)(6, 11, 8, 9)

[9]     Order 4       Length 4
Rep (1, 7, 5, 3)(2, 13, 10, 16)(4, 12, 14, 15)(6, 9, 8, 11)

[10]    Order 8       Length 4
Rep (1, 8, 14, 10, 5, 6, 4, 2)(3, 13, 15, 9, 7, 16, 12, 11)
```

```
[11]    Order 8       Length 4
Rep (1, 16, 4, 9, 5, 13, 14, 11)(2, 3, 6, 12, 10, 7, 8, 15)

for i in [2..#C] do
i, Orbits(Centraliser(N,C[i][3]));
end for;

[2]
GSet{@ 1, 2, 3, 4, 5, 8, 9, 6, 10, 7, 12, 14, 16, 13, 11, 15 @}
[3]
GSet{@ 1, 3, 16, 2, 8, 9, 12, 15, 4, 14, 10, 13, 11, 6, 5, 7 @}
[4]
GSet{@ 1, 2, 3, 11, 9, 7, 10, 5 @},
GSet{@ 4, 8, 12, 13, 16, 15, 6, 14 @}
[5]
GSet{@ 1, 9, 2, 11, 3, 5, 7, 10 @},
GSet{@ 4, 16, 8, 13, 12, 14, 15, 6 @}
[6]
GSet{@ 1, 15, 5, 12 @},
GSet{@ 3, 7, 14, 4 @},
GSet{@ 2, 8, 16, 11, 10, 6, 13, 9 @}
[7]
GSet{@ 1, 4, 16, 2, 5, 9, 6, 7, 3, 14, 13, 10, 15, 12, 8, 11 @}
[8]
GSet{@ 1, 12, 14, 5, 3, 4, 15, 7 @},
GSet{@ 2, 8, 13, 9, 10, 6, 16, 11 @}
[9]
GSet{@ 2, 13, 16, 10 @},
GSet{@ 6, 9, 8, 11 @},
GSet{@ 1, 7, 15, 5, 4, 3, 12, 14 @}
[10]
GSet{@ 1, 8, 14, 10, 5, 6, 4, 2 @},
GSet{@ 3, 13, 15, 9, 7, 16, 12, 11 @}
[11]
GSet{@ 1, 16, 4, 9, 5, 13, 14, 11 @},
GSet{@ 2, 3, 6, 12, 10, 7, 8, 15 @}

for j in [2..#C] do
C[j][3];

for i in [1..32] do if ArrayP[i] eq C[j][3] then Sch[i];
end if; end for;
(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16)
```

```
(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15)
(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15)
(1, 9)(2, 3)(4, 16)(5, 11)(6, 15)(7, 10)(8, 12)(13, 14)
(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16)
(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16)
(1, 12, 5, 15)(2, 13, 10, 16)(3, 4, 7, 14)(6, 11, 8, 9)
(1, 7, 5, 3)(2, 13, 10, 16)(4, 12, 14, 15)(6, 9, 8, 11)
(1, 8, 14, 10, 5, 6, 4, 2)(3, 13, 15, 9, 7, 16, 12, 11)
(1, 16, 4, 9, 5, 13, 14, 11)(2, 3, 6, 12, 10, 7, 8, 15)
h
z
y
y * z
x
w
z * w
z * x
y * x
y * x * z
```

Now, we multiply each word in above by t. $(h*t), (z*t), (y*t), (y*t^w), (y*z*t), (y*t^w), (x*t), (x*t^z), (x*t^y), (w*t), (z*w*t), (z*w*t^y), (z*x*t^y), (z*x*t^{(y*w)}), (z*x*t), (y*x*t), (y*x*t^z), (y*x*z*t), (y*x*z*t^y)$, where $t_1 = t$, $t_2 = t^y$, $t_3 = t^z$, $t_4 = t^w$, $t_5 = t^h$, $t_6 = t^{(y*w)}$ and we can raise this to some power such as a, b, c, d, f, ff, g, i, ii, j, jj, e, u, r. Then, we add these relations to our presentation and run it in the background on Magma to find some homomorphic images.

```
for a,b,c,d,f,ff,g,i,ii,j,jj,e,u,r in [0..10] do
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(h*t)^a,
(z*t)^b,(y*t)^c,(y*z*t)^d,(x*t)^f,(x*t^y)^ff,(w*t)^g,
(z*w*t)^i,(z*w*t^y)^ii,(z*x*t^y)^j,(z*x*t^(y*w))^jj,
(y*x*t)^e,(y*x*z*t)^u,(y*x*z*t^x)^r>;
a,b,c,d,f,ff,g,i,ii,j,jj,e,u,r,#G;
end for;
```

Some homomorphic images are obtained by factoring our presentation by the first order relation lemma:

0 0 0 0 0 0 0 0 0 4 3 8 141557760

0 0 0 0 0 0 0 0 0 0 3 4 2211840

```
0 0 0 0 0 0 0 3 8 6 4 196608
0 0 0 0 0 0 0 3 10 5 5 1920
0 0 0 0 0 0 0 3 10 6 4 30000
0 0 0 0 0 0 0 4 4 6 4 36864
0 0 0 0 0 0 0 4 6 4 4 5184
0 0 0 0 0 0 0 4 8 4 4 2097152
0 0 0 0 0 0 0 4 8 9 3 39191040
0 0 0 0 0 0 0 4 10 4 4 40000
0 0 0 0 0 0 0 5 4 5 4 327680
0 0 0 0 0 0 0 6 4 4 4 2304
0 0 0 0 0 0 0 8 3 8 9 672
0 0 0 0 0 0 0 8 3 10 9 4320
0 0 0 0 0 0 0 8 4 4 4 32768
0 0 0 0 0 0 2 0 5 5 8 1440
0 0 0 0 0 0 2 0 5 5 9 6840
0 0 0 0 0 0 2 8 5 5 10 10240
0 0 0 0 0 0 3 0 0 5 4 368640
0 0 0 0 0 0 3 0 4 0 4 3072
```

Chapter 3

Character Table Construction

3.1 Definitions and Theorem

Definition 3.1. *Let* $\chi_1, ..., \chi_k$ *be the irreducible characters and conjugacy classes of* G *and let* $g_1, ..., g_k$ *be representatives of the conjugacy classes of* G*. The* $k \times k$ *matrix whose ij-entry is* $\chi_i(g_i)$ *(for all i, j with* $1 \leq i \leq k, 1 \leq j \leq k$*) is* **called the character table of G**.

Definition 3.2. *The degree of a character* χ *is* $\chi_{(1)}$*. Note that a character whose degree is 1 is* **called a linear character.** *[Led87]*

Theorem 3.3. *Let* G *be a finite group having the distinct irreducible* $\chi^1, \chi^2, ..., \chi^k$*. Let* $1 \leq i, j \leq k$*. Then* $\sum_{i=1}^{k} \chi_\alpha^{(i)} \chi_\beta^{(\bar{i})} = \frac{|G|}{h} \delta_{\alpha\beta}$

- *In a character table, the dot product of any column with conjugate of any other column is 0.*
- *In a character table, the dot product of the column* α *with its own conjugate is* $\frac{|G|}{h_\alpha}$*. [GL93]*

Definition 3.4. In a character table *the sum of squares of the degrees of the distinct irreducible characters of* G *is equal to* $|G|$*. The degree of a character* χ *is* $\chi^{(1)}$*.[GL93]*

Definition 3.5. The number of irreducible character of G *is equal to the number of conjugacy classes of G.[Led87]*

Definition 3.6. Character table of a cyclic group
Let G be a cyclic group of order n. Then $G = < z >$*,and* $|z| = n$. *Let* $\epsilon_r = e^{\frac{2\pi ir}{n}}$, *where r =0,1,2,...,n, be the n nth roots of unity. For any* $z^s \in G$, *s=0,1,2,...,n, the values of the n irreducible characters* $\chi^{(r)}$ *are given by* $\chi^{(r)}(z^s) = e^{\frac{2\pi ir}{n}}$, *where r=0,1,2,...,n. [GL93]*

Definition 3.7. *Let* $a \in G$, *where G is a group. The* **conjugacy class** *of a is given by* $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$. *[Rot95]*

Definition 3.8. Character table of an abelian group*: Let G be a finite abelian group say G=*$< z_1 > \times < z_2 > \times < z_3 > \times .... \times < z_m >$, *where each* z_μ*is order of order* n_μ, *and* $|G| = n_1 n_2 ... n_m$. *Let* $x = z_1^{a_1} z_2^{a_2} ... z_m^{a_m}$, *where* $0 \leq a_\mu \leq n_\mu$ *be an arbitrary element of G.Now let, for each* $\mu, \epsilon_\mu = e^{\frac{2\pi ir_\mu}{n_\mu}}$, *where* $r_\mu = 0, 1, 2, ..., n_\mu$, *be the* n_μ *th roots of unity. Here* $\mu = 1, 2, .., m$. *Then corresponding to each m-tuple* $r = [r_1, r_2, ..., r_m]$, $\chi^{(r)}$ *are given by* $\chi^{(r)}(x) = e^{2\pi i \sum_{\mu=1}^{m} \frac{a_\mu r_\mu}{n_\mu}}$ *is an irreducible character of G. [GL93]*

Definition 3.9. Lifting process*: Let N be a normal subgroup of G and suppose that* $A_0(Nx)$ *is a representation of degree m of the group G/N. Then* $A(x) = A_0(Nx)$ *defines a representation of G/N LIFTED from G/N. If* $\phi_0(Nx)$ *is the character of* $A_0(Nx)$, *then* $\phi(x) = \phi_0(Nx)$ *is the lifted character of A(x). Also, if* $u \in N$*,then* $A(u) = I_m, \phi(u) = m = \phi(1)$. *The lifting process preserves irreducibility. [Led87]*

Definition 3.10. The group from a character table of the finite group*:*
(a) All normal subgroups of a finite group G can be obtained from its character table as follows:
i. ker $\chi^{(i)}$ *is the union of conjugacy classes* C_α *for which* $\chi_\alpha^{(i)}$ *=*$\chi_\alpha^{(i)}(1)$.
ii.a collection of conjugacy classes from a normal subgroup $\Longleftrightarrow$ *it is an intersection of kernels of its characters.*
(b) $|G\acute{}|$*= the number of linear characters of G.*
(c) G is simple if the kernel of each character is 1.
(d) The center is cyclic if the table contains a faithful character. [Led87]

Theorem 3.11. *Let* $\chi_1, ..., \chi_k$ *be the irreducible characters of G, and let* $g_1, ..., g_k$ *be representatives of the conjugacy classes of G. Then the following relations hold for any* $r, s \in 1, ..., k$.

(1) **The row orthogonality relations:**

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\bar{\chi_s}(\bar{g_i})}{|C_G(g_i)|} = \delta_{rs}.$$

(2) **The column orthogonality relations:**

$$\sum_{i=1}^{k} \chi_i(g_r)\bar{\chi_i}(\bar{g_s}) = \delta_{rs}|C_G(g_i)|.\text{[Led87]}$$

Definition 3.12. The inner product of characters of G, *say ϕ and ψ can be found as follows:*
$< \phi, \psi > = \frac{1}{|G|} h_\alpha \phi^\alpha \bar{\psi}^\alpha$*, where h_α is the number of elements in the class α, ϕ_α is the value of ϕ in the class α, ψ_α is the value of ψ in the class α. Also, note that χ^α is the value of χ at class α. [Led87]*

3.2 Building the Character Table of D_4

In this example, we will show how we can use the lifting process to find the character table of the dihedral group D_4. In order to build character table of the dihedral group D_4 of order 8. We will use the conjugacy classes.[Has17]
Let G=$D_4 = \{e, a, a^2, a^3b, a^2b, a^3b, ab\}$
Let $a = (1, 2, 3, 4)$, $b = (1, 4)(2, 3)$, $a^2 = (1, 3)(2, 4)$, $a^3 = (1, 4, 3, 2)$, $ab = (1, 3)$, $a^3b = (2, 4)$, and $a^2b = (1, 2)(3, 4)$.

```
Conjugacy Classes of group G
----------------------------
[1]     Order 1       Length 1
Rep Id(G)

[2]     Order 2       Length 1
Rep (1, 3)(2, 4)

[3]     Order 2       Length 2
Rep (1, 4)(2, 3)

[4]     Order 2       Length 2
Rep (2, 4)
```

```
[5]     Order 4       Length 2
Rep (1, 2, 3, 4)
```

Next, we will find a normal subgroup of G. Now a^2 lies in the center of G, as it commutes with every elements of D_4. Therefore, $N = \{1, a^2\}$ the center of D_4. Thus, the conjugacy classes being $\{1, a^2\}, \{a, a^3\}, \{b, a^2b\}, and\{ab, a^3b\}$. As a result, D_4 has 5 irreducible representations and $N = \{1\} \cup \{a^2\}$ is normal in D_4.
The quotient group $\frac{|D_4|}{|N|}=\frac{8}{2} = 4$ and it is isomorphic to $Z/2 \oplus Z/2$. The group $Z/2 \oplus Z/2$ has four classes $\{Id, a, b, ab\}$.

Table 3.1: Character Table of $Z/2 \oplus Z/2$

Classes	1	2	3	4
size	1	1	1	1
Rep	1	a	b	ab
x.1	1	1	1	1
x.2	1	-1	-1	1
x.3	1	-1	1	-1
x.4	1	1	-1	-1

Now, we will start to fill character table of D_4.

Table 3.2: Initial Character Table of D_4

Classes	1	2	3	4	5
Length	1	1	2	2	2
Rep	1	(1,3)(2,4)	(1,4)(2,3)	(2,4)	(1,2,3,4)
$\mathcal{X}^1$					
$\mathcal{X}^2$					
$\mathcal{X}^3$					
$\mathcal{X}^4$					
$\mathcal{X}^5$	f^5	α	β	γ	δ

We know that the trivial representation is a representation that takes every element in D_4 to the identity, so the first row will be the trivial character $\mathcal{X}^1$ which is the identity character.

1) First, we now lift $\mathcal{X}^1$ to D_4:

- $\mathcal{X}^1$ is identity character. So,the elements of the first row will be 1, 1, 1, 1, and 1.

2) Next, we lift $\mathcal{X}^2$ to D_4:

- $\mathcal{X}^2$(e)=x.2(Ne)=1.
- $\mathcal{X}^2$((1,3)(2,4))= x.2(N(1,3)(2,4))= 1.

(since (1,3)(2,4)$\in$ N)

- $\mathcal{X}^2$((1,4)(2,3))= x.2(N(1,4)(2,3)= -1.
- $\mathcal{X}^2$((2,4))= x.2(N(2,4))= 1 (since ab=(1,3) and x.2(N(2,4))=x.2(N(1,3)).
- $\mathcal{X}^2$(1,2,3,4)= x.2(N(1,2,3,4))= -1.

Thus, the elements of the second row will be 1, 1, -1, 1,and -1.

3) Also, we lift $\mathcal{X}^3$ to D_4:

- $\mathcal{X}^3$(e)= x.3(Ne) = 1.
- $\mathcal{X}^3$((1,3)(2,4))= x.3(N(1,3)(2,4)) =1.
- $\mathcal{X}^3$((1,4)(2,3))= x.3(N(1,4)(2,3) = 1.
- $\mathcal{X}^3$((2,4))= x.3(N(2,4)) = -1.
- $\mathcal{X}^3$(1,2,3,4)= x.3(N(1,2,3,4)) = -1.

So, the elements of the third row will be 1, 1, 1,-1, and -1.

4) Finally, we lift $\mathcal{X}^4$ to D_4:

- $\mathcal{X}^4(e)$= x.4(Ne)= 1.
- $\mathcal{X}^4((1,3)(2,4))$= x.4(N(1,3)(2,4))= 1.
- $\mathcal{X}^4((1,4)(2,3))$= x.4(N(1,4)(2,3))= -1.
- $\mathcal{X}^4((2,4))$= x.4(N(2,4))= -1.
- $\mathcal{X}^4(1,2,3,4)$= x.4(N(1,2,3,4))= 1.

So, the elements of the fourth row will be 1, 1, 1, -1, and -1.

We could lift the four characters of $Z/2 \oplus Z/2$ to obtain four characters $\mathcal{X}^1$, $\mathcal{X}^2$, $\mathcal{X}^3$, $\mathcal{X}^4$ of D_4 it remains to find $\mathcal{X}^5$. Then, we use the fact that the sum of squares of the degrees of the distinct irreducible characters of G which is equal to $|D_4|$ to find the first element of $\mathcal{X}^5$, it will be as follows:

$$(f^1)^2+(f^2)^2+(f^3)^2+(f^4)^2+(f^5)^2 = 1^2+1^2+1^2+1^2+(f^5)^2=|G|=8. \text{ So, } f^5=2.$$

For the moment denote the values of $\mathcal{X}^5$ for C2, C3, C4, and C5 by $\alpha, \beta, \gamma, \delta$ respectively. To find last row values we will use the orthogonality relations. By theorem, in a character table, the dot product of any column with the conjugate of any other column is 0.

The column 1 and column 2 of character table of D_4:

- $(1).(\bar{1})+(1).(\bar{1})+(1).(\bar{1})+(1).(\bar{1})+(2).(\bar{\alpha})=0 \Longrightarrow \bar{\alpha}=\frac{-4}{2}=-2$. Thus, α =-2.

The column 1 and column 3:

- $(1).(\bar{1})+(1).(\bar{-1})+(1).(\bar{1})+(1).(\bar{-1})+(2).(\bar{\beta})=0 \Longrightarrow \bar{\beta}=0$. So, $\beta=0$.

The column 1 and column 4:

- $(1).(\bar{1})+(1).(\bar{1})+(1).(\bar{-1})+(1).(\bar{-1})+(2).(\bar{\gamma})=0 \Longrightarrow \bar{\gamma}=0$. So, γ=0.

The column 1 and column 5:

- $(1).(\bar{1})+(1).(\bar{-1})+(1).(\bar{-1})+(1).(\bar{1})+(2).(\bar{\delta})=0. \Longrightarrow \bar{\delta}=0$. Thus, δ=0.

Therefore, the elements of the last row will be 2,-2, 0, 0, and 0. Then, character table of D_4 will be.

Table 3.3: Final Character Table of D_4

Classes	1	2	3	4	5
Length	1	1	2	2	2
Rep	1	(1,3)(2,4)	(1,4)(2,3)	(2,4)	(1,2,3,4)
$\mathcal{X}^1$	1	1	1	1	1
$\mathcal{X}^2$	1	1	-1	1	-1
$\mathcal{X}^3$	1	1	1	-1	-1
$\mathcal{X}^4$	1	1	-1	-1	1
$\mathcal{X}^5$	$f^5=2$	$\alpha=-2$	$\beta=0$	$\gamma=0$	$\delta=0$

3.3 Inducing a Linear Character of a Proper Subgroup up $H \cong 5:2^2$ of S_5

Let G be a group such that $G \cong S_5$. We want to give a faithful and irreducible monomial representation of G of degree n. Thus, we must induce a linear character of a proper subgroup up $H \cong 5:2^2$ of S_5. In order to induce up to a character of G of degree n the subgroup H must be of index n. In the following example, G has the following character table:

```
The character table of G
-----------------------------
Class |   1  2  3  4  5  6  7
Size  |   1 10 15 20 30 24 20
Order |   1  2  2  3  4  5  6
-----------------------------
p  =  2   1  1  1  4  3  6  4
p  =  3   1  2  3  1  5  6  2
p  =  5   1  2  3  4  5  1  7
-----------------------------
X.1   +   1  1  1  1  1  1  1
X.2   +   1 -1  1  1 -1  1 -1
X.3   +   4 -2  0  1  0 -1  1
X.4   +   4  2  0  1  0 -1 -1
X.5   +   5  1  1 -1 -1  0  1
X.6   +   5 -1  1 -1  1  0 -1
X.7   +   6  0 -2  0  0  1  0
```

Now, we want to induce a linear character of H up to a character of G of degree 6. So, we have taken a subgroup H of index 6 in G.

```
Index(G,H);
6
```

Now, the proper subgroup $H \cong 5:2^2$ of S_5 all the linear of H are lift to H of irreducible characters H/H' where H' denotes the derived subgroup of H.

```
G:=PrimitiveGroup(5,5);
IsAbelian(G);
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 6 then i;
end if; end for;
H:=S[16]`subgroup;
dH:=DerivedGroup(H);
dH;
(1, 4, 5, 3, 2)
```

Thus, $H' =< (1,4,5,3,2) >\cong 5$. The number of distinct linear characters of H is equal to $H/H' = \frac{20}{5} = 4$, so we will have 4 transversals.

Thus, $H/H' = \{H'e, H'(1,5,3,4), H'(1,3)(4,5), H'(1,4,3,5)\}$.

```
S:=Set(dH);q:=[{Id(H)},{},{},{}];
for i in [1..#T] do for g in S do
q[i]:=q[i] join {g*T[i]}; end for; end for;
q;
{
(1, 4, 5, 3, 2),
(1, 3, 4, 2, 5),
Id(H),
(1, 2, 3, 5, 4),
(1, 5, 2, 4, 3)
},
{
(1, 4, 2, 3),
(1, 3, 5, 2),
(2, 5, 4, 3),
(1, 5, 3, 4),
(1, 2, 4, 5)
},
{
(1, 2)(3, 4),
(1, 4)(2, 5),
(1, 3)(4, 5),
(2, 4)(3, 5),
(1, 5)(2, 3)
},
{
(1, 2, 5, 3),
(2, 3, 4, 5),
(1, 4, 3, 5),
(1, 3, 2, 4),
(1, 5, 4, 2)
}
```

So, the conjugacy classes of H/H' are $\{Id(H)\}$, $\{H'(1,4,2,3)\}$, $\{H'(1,2)(3,4)\}$, and $\{H'(1,2,5,3)\}$. Now, we will construct the character table of H/H'. The character table of cyclic group has primitive 4 the root of unity =i as follows:

Table 3.4: Initial Character Table of Cyclic Group Z_4

Classes	e	a	a^2	a^3
$\mathcal{X}^\bullet.1$	1	1	1	1
$\mathcal{X}^\bullet.2$	1	i	i^2	i^3
$\mathcal{X}^\bullet.3$	1	i^2	$(i^2)^2$	$(i^2)^3$
$\mathcal{X}^\bullet.4$	1	i^3	$(i^3)^2$	$(i^3)^3$

Where $i^2 = -1$, $i^3 = -i$, and $(i^3)^3 = i^9 = (i^4)^2 i = i$. Therefore, this is equivalent to character table of $H/H' \cong Z_4$.

Table 3.5: Final Character Table of Cyclic Group Z_4

Classes	e	a	a^2	a^3
$\mathcal{X}^\bullet.1$	1	1	1	1
$\mathcal{X}^\bullet.2$	1	i	-1	-i
$\mathcal{X}^\bullet.3$	1	-1	1	-1
$\mathcal{X}^\bullet.4$	1	-i	-1	i

Table 3.6: Character Table of $H/H' \cong Z_4$

Classes	1	2	3	4
size	1	1	1	1
Rep	H’	H’(1,2)(3,4)	H’(1,4,2,3)	H’(1,2,5,3)
Order	1	2	4	4
$\mathcal{X}^\bullet.1$	1	1	1	1
$\mathcal{X}^\bullet.2$	1	i	-1	-i
$\mathcal{X}^\bullet.3$	1	-1	1	-1
$\mathcal{X}^\bullet.4$	1	-i	-1	i

Thus, we could build the character table of H/H'. Now, we can construct the character table of H by lifting the character table of H/H' to H.

To build the character table of H, we first need to figure out the conjugacy classes of H. The following set are the conjugacy classes of H are $\{e\}$, $\{(1,2)(3,4)\}$, $\{(1,4,2,3)\}$, $\{(1,2,5,3)\}$, and $\{(1,4,5,3,2)\}$. So, we will construct the character table of H with five rows and column since there are five conjugacy classes in H. Then, we will use the definition of lifting character, we will lift each irreducible character from H/H' to H. We note that, $\mathcal{X}(h) = \mathcal{X}^\bullet(h)$ for $h \in H$.

First, the first row will be identity character.
Next, we find the second row by lifting $\mathcal{X}^{\bullet}.2$ to H.
($\mathcal{X}.2e$)=$\mathcal{X}^{\bullet}$.2(H'e)=1.
($\mathcal{X}.2(1,2)(3,4)$)=$\mathcal{X}^{\bullet}$.2(H'(1,2)(3,4))= i.
($\mathcal{X}.2(1,4,2,3)$)=$\mathcal{X}^{\bullet}$.2(H'(1,4,2,3))= -1.
($\mathcal{X}.2(1,2,5,3)$)=$\mathcal{X}^{\bullet}$.2(H'(1,2,5,3))= -i.
($\mathcal{X}.2(1,4,5,3,2)$)=$\mathcal{X}^{\bullet}$.2(H'(1,4,5,3,2))= 1.
(since=$\mathcal{X}^{\bullet}$.2(H'(1,4,5,3,2))=$\mathcal{X}^{\bullet}$.2(H'e)).

Now, we find the third row by lifting $\mathcal{X}^{\bullet}.3$ to H.
($\mathcal{X}.3e$)=$\mathcal{X}^{\bullet}$.3(H'e)=1.
($\mathcal{X}.3(1,2)(3,4)$)=$\mathcal{X}^{\bullet}$.3(H'(1,2)(3,4))= -1.
($\mathcal{X}.3(1,4,2,3)$)=$\mathcal{X}^{\bullet}$.3(H'(1,4,2,3))= 1.
($\mathcal{X}.3(1,2,5,3)$)=$\mathcal{X}^{\bullet}$.3(H'(1,2,5,3))= -1.
($\mathcal{X}.3(1,4,5,3,2)$)=$\mathcal{X}^{\bullet}$.3(H'(1,4,5,3,2))= 1.
(since=$\mathcal{X}^{\bullet}$.3(H'(1,4,5,3,2))=$\mathcal{X}^{\bullet}$.3(H'e)).

We will now find the fourth row by lifting $\mathcal{X}^{\bullet}.4$ to H.
($\mathcal{X}.4e$)=$\mathcal{X}^{\bullet}$.4(H'e)=1.
($\mathcal{X}.4(1,2)(3,4)$)=$\mathcal{X}^{\bullet}$.4(H'(1,2)(3,4))= -i.
($\mathcal{X}.4(1,4,2,3)$)=$\mathcal{X}^{\bullet}$.4(H'(1,4,2,3))= -1.
($\mathcal{X}.4(1,2,5,3)$)=$\mathcal{X}^{\bullet}$.4(H'(1,2,5,3))= i.
($\mathcal{X}.4(1,4,5,3,2)$)=$\mathcal{X}^{\bullet}$.4(H'(1,4,5,3,2))= 1.
(since=$\mathcal{X}^{\bullet}$.4(H'(1,4,5,3,2))=$\mathcal{X}^{\bullet}$.4(H'e)).

Lastly, we will fill the last row by using the orthogonality relations. We will suppose in the last row, we will have a, b, c, d, and f, respectively. So, we need to compute these values. We use the fact that the sum of squares of the degrees of the distinct irreducible characters of H which is equal to $|H|$ to find the first element of $\mathcal{X}^5$, it will be as follows:

- $1^2+1^2+1^2+1^2+a^2$=$|H|$=20. So, $a^2=16$ then $a=4$.

We first have to note that if we collect different columns, they will equal zero by using this definition:

$$\sum_k^{i=1} X^i_\alpha \bar{X}^j_\beta{=}0$$

- The column 1 and column 2 of the character table of H:
- $(1).(\bar{1}) + (1).(\bar{i}) + (1).(\bar{-1}) + (1).(\bar{-i}) + (4).(\bar{b}) = 0 \Longrightarrow \bar{b} = 0$. So, $b = 0$.
- The column 1 and column 3 of the character table of H:
- $(1).(\bar{1}) + (1).(\bar{-1}) + (1).(\bar{1}) + (1).(\bar{-1}) + (4).(\bar{c}) = 0 \Longrightarrow \bar{c} = 0$. So, $c = 0$.
- The column 1 and column 4 of the character table of H:
- $(1).(\bar{1}) + (1).(\bar{-i}) + (1).(\bar{-1}) + (1).(\bar{i}) + (4).(\bar{d}) = 0 \Longrightarrow \bar{d} = 0$. So, $d = 0$.
- The column 1 and column 5 of the character table of H:
- $(1).(\bar{1}) + (1).(\bar{1}) + (1).(\bar{1}) + (1).(\bar{1}) + (4).(\bar{f}) = 0 \Longrightarrow \bar{f} = -1$. So, f =-1.

Finally, we could fill our character table of H by lifting H/H' to H.

Table 3.7: Character Table of H

Classes	1	2	3	4	5
Size	1	5	5	5	4
Rep	1	(1,2)(3,4)	(1,4,2,3)	(1,2,5,3)	(1,4,5,3,2)
$\mathcal{X}^1$	1	1	1	1	1
$\mathcal{X}^2$	1	i	-1	-i	1
$\mathcal{X}^3$	1	-1	1	-1	1
$\mathcal{X}^4$	1	-i	-1	i	1
$\mathcal{X}^5$	$a = 4$	$b = 0$	$c = 0$	$d = 0$	$f = -1$

Chapter 4

Isomorphism Type of Groups

4.1 Extensions and Related Definitions

Definition 4.1. *[Rot95] G is a* **semi-direct product** *of two subgroups H and K if:*

1. $K \trianglelefteq G$, $Q \leq G$.
2. $G = KQ$.
3. $K \cap Q = 1$.

Definition 4.2. *[Rot95] Let G be a group. We say G is a* **direct product** *of two subgroups H and K if:*

1. $H \trianglelefteq G$, $K \trianglelefteq G$.
2. $G = HK$.
3. $H \cap K = 1$.

Definition 4.3. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **central extension** *by H if N is the center of G. We denote this by $G \cong N^{\bullet}H$.*

Definition 4.4. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **mixed extension** *by H if it is a combination of both central extensions and semi-direct products, where N is the normal subgroup of G but not central. We denote this by $G \cong N^{\bullet} : H$.*

Definition 4.5. *[Rot95]Let G be a group. A* **composition series** *of G given by:*

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

is a normal series where, for all i, either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_i$.

Definition 4.6. *[Rot95] If group G has a composition series, the factor groups of its series are the* **composition factors** *of G.*

Definition 4.7. *[Rot95] The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$.*

Definition 4.8. *[Rot95] A* **general linear group***, $GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$.*

Definition 4.9. *[Rot95] A* **special linear group***, $SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$.*

Definition 4.10. *[Rot95] A* **projective special linear group***, $PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}.$$

Definition 4.11. *[Rot95] A* **projective general linear group***, $PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}.$$

4.2 Extensions Examples:

We can find the isomorphism types of the progenitors by solving the extension problem. To solve the extension problem we need to know the composition factors of groups. We will recognize in the following on four types of extensions which are direct product, a semi direct, central extension ,and mixed extension. Moreover, we will provide an example of each one.

4.3 Semi Direct Product:

4.3.1 Verifications of $3^4 : ((3^2 * 2^3) \times 2)$

We found the following finite homomorphic image from the symmetric group 8 6 3 6 4 0 11664 which produces the following group:

```
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^6,
(z*w*t)^3,(z*x*t)^6,(y*x*t)^4,(y*x*z*t)^0>;
```

Our order group is equal 11664, and this group has the following composition factors:

```
G
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(3)
*
|  Cyclic(3)
*
```

```
|  Cyclic(3)
1
```

Now, we will prove the isomorphism type of G.

The composition series for G is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq G_8 \supseteq G_9 \supseteq G_{10}$ where $G_{10} = 1$.

The composition factors are:

$G = (G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash G_7)(G_7 \backslash G_8)(G_8 \backslash G_9)(G_9 \backslash G_{10})$

$=(G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash G_7)(G_7 \backslash G_8)(G_8 \backslash G_9)(G_9 \backslash 1)$

$=(G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash G_7)(G_7 \backslash G_8)(G_8 \backslash G_9)G_9$

$=C_3C_3C_3C_3C_2C_3C_3C_2C_2C_2$

The normal lattice of G is as follows:

```
Normal subgroup lattice
-----------------------
[30]  Order 11664  Length 1  Maximal Subgroups: 23 24 25
26 27 28 29
---
[29]  Order 5832   Length 1  Maximal Subgroups: 16 17 20
[28]  Order 5832   Length 1  Maximal Subgroups: 17 18 21
[27]  Order 5832   Length 1  Maximal Subgroups: 16 19 21
[26]  Order 5832   Length 1  Maximal Subgroups: 18 19 20
[25]  Order 5832   Length 1  Maximal Subgroups: 20 21 22
[24]  Order 5832   Length 1  Maximal Subgroups: 16 18 22
[23]  Order 5832   Length 1  Maximal Subgroups: 15 17 19 22
---
[22]  Order 2916   Length 1  Maximal Subgroups: 11 13
[21]  Order 2916   Length 1  Maximal Subgroups: 13
[20]  Order 2916   Length 1  Maximal Subgroups: 13
[19]  Order 2916   Length 1  Maximal Subgroups: 9 13
[18]  Order 2916   Length 1  Maximal Subgroups: 12 13 14
[17]  Order 2916   Length 1  Maximal Subgroups: 10 13
[16]  Order 2916   Length 1  Maximal Subgroups: 13
[15]  Order 1944   Length 1  Maximal Subgroups: 9 10 11
---
[14]  Order 1458   Length 1  Maximal Subgroups: 8
[13]  Order 1458   Length 1  Maximal Subgroups: 7 8
[12]  Order 1458   Length 1  Maximal Subgroups: 5 8
[11]  Order 972    Length 1  Maximal Subgroups: 7
[10]  Order 972    Length 1  Maximal Subgroups: 7
[ 9]  Order 972    Length 1  Maximal Subgroups: 7
```

```
---
[ 8]  Order 729    Length 1  Maximal Subgroups: 4 6
[ 7]  Order 486    Length 1  Maximal Subgroups: 6
---
[ 6]  Order 243    Length 1  Maximal Subgroups: 3
[ 5]  Order 162    Length 1  Maximal Subgroups: 4
---
[ 4]  Order 81     Length 1  Maximal Subgroups: 3
---
[ 3]  Order 27     Length 1  Maximal Subgroups: 2
---
[ 2]  Order 3      Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1      Length 1  Maximal Subgroups:
```

Next, we will ask Magma about the largest abelian group by this loop:

```
for i in [1..#NL]do if IsAbelian(NL[i]) then i;end if;end for;
1
2
3
4
```

So, we see that the largest abelian group is NL[4] of order 81 from our normal lattice of G1. We first need to find the isomorphism type of NL[4], which has 3 possibilities, such as $3 \times 3 \times 3 \times 3$, 3^4, and $3^3 \times 3$.
We check and find the following:

```
Permutation group acting on a set of cardinality 729
Order = 81 = 3^4
X:=[3,3,3,3];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
true
```

As a result, we found that $NL[4] = 3^4$. Now, we can produce a presentation for NL[4] as follows:

```
FPGroup(NL[4]);
Finitely presented group on 4 generators
Relations
$.1^3 = Id($)
$.2^3 = Id($)
$.3^3 = Id($)
$.4^3 = Id($)
```

```
($.1, $.2) = Id($)
($.1, $.3) = Id($)
($.2, $.3) = Id($)
($.1, $.4) = Id($)
($.2, $.4) = Id($)
($.3, $.4) = Id($)
S<a,b,c,d>:=Group<a,b,c,d|a^3,b^3,c^3,d^3,(a,b),(a,c),(b,c),
(a,d),(b,d),(c,d)>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[4]);
s;
true
```

Now, we will factor G1 by NL[4] resulting q which is the quotient group.

```
H:=NL[4];
q,ff:=quo<G1|NL[4]>;q;
Permutation group q acting on a set of cardinality 12
Order = 144 = 2^4 * 3^2
```

Consider the composition factor of q as follows:

```
CompositionFactors(q);
G
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(3)
1
```

We will now find the normal lattice for q by this Magma loop:

```
  nl:=NormalLattice(q);
  nl;
Normal subgroup lattice of q
----------------------
[21]  Order 144  Length 1  Maximal Subgroups: 14 15 16
17 18 19 20
```

```
---
[20]  Order 72   Length 1  Maximal Subgroups: 7 9 12
[19]  Order 72   Length 1  Maximal Subgroups: 9 10 13
[18]  Order 72   Length 1  Maximal Subgroups: 7 8 13
[17]  Order 72   Length 1  Maximal Subgroups: 8 10 12
[16]  Order 72   Length 1  Maximal Subgroups: 11 12 13
[15]  Order 72   Length 1  Maximal Subgroups: 7 10 11
[14]  Order 72   Length 1  Maximal Subgroups: 8 9 11
---
[13]  Order 36   Length 1  Maximal Subgroups: 5
[12]  Order 36   Length 1  Maximal Subgroups: 5
[11]  Order 36   Length 1  Maximal Subgroups: 4 5 6
[10]  Order 36   Length 1  Maximal Subgroups: 5
[ 9]  Order 36   Length 1  Maximal Subgroups: 5
[ 8]  Order 36   Length 1  Maximal Subgroups: 5
[ 7]  Order 36   Length 1  Maximal Subgroups: 5
---
[ 6]  Order 18   Length 1  Maximal Subgroups: 3
[ 5]  Order 18   Length 1  Maximal Subgroups: 3
[ 4]  Order 18   Length 1  Maximal Subgroups: 2 3
---
[ 3]  Order 9    Length 1  Maximal Subgroups: 1
---
[ 2]  Order 2    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:
```

We now check if we have a direct product or not, so we want to see if q is collected of a direct product. As a result, we could find the direct product between two normal subgroups which are nl[17] and nl[2]. The product of their order give me the order of q=144. To make sure by this Magma code:

```
E:=DirectProduct(nl[17],nl[2]);
IsIsomorphic(E,q);
true
```

We find that q is isomorphic to nl[17] and nl[2], where $nl[17] \cong 3^2 * 2^3$ and $nl[2] \cong 2$. The presentation of $q \cong nl[17] \times nl[2]$ which is obtained from FPGroup(q):

```
H<e,f,g,h,o>:=Group<e,f,g,h,o|e^2,f^2,g^2,h^2,o^2,(e*g)^2,
(f*g)^2,(e*h)^2,(f * h)^2,(g * h)^2,(e* o)^2,
(g *o)^2,e*f*e*f*h,h*o*h*g*o*h*o,(f*o*f*o* h)^2>;
```

```
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
true
```

Now, we will check the relation between NL[4] and q. Since our q is not the center of G, so we do not have a central extension. Also, we do not have direct product between NL[4] and q, so we do not have direct product. Then, we only have two options which are a semi direct product or mixed extension.
We will label our generators of q as follows:

```
EE:=q!(2, 4)(5, 6)(7, 11)(8, 10)(9, 12);
FF:=q!(1, 2)(3, 4)(5, 7)(6, 9)(8, 12)(10, 11);
GG:=q!(1, 3)(2, 4)(5, 6)(7, 9)(8, 10)(11, 12);
DD:=q!(1, 3)(2, 4)(5, 8)(6, 10)(7, 12)(9, 11);
KK:=q!(2, 5)(4, 6)(7, 11)(9, 12);
```

Also, we will label our generators of NL[4] as follows:

```
A:=G1!NL[4].1;
B:=G1!NL[4].2;
C:=G1!NL[4].3;
D:=G1!NL[4].4;
H:=G1!T[2];
J:=G1!T[3];
I:=G1!T[4];
M:=G1!T[5];
O:=G1!T[6];
```

Here a, b, c, and d represent elements of NL[4], and e, f, g, h, and o represent elements of q. Where A, B, C, and D ≅ a, b, c, and d, respectively and H, J, I, M, and O ≅ e, f, g, h, and o, respectively. Now, we need to check the action q on NL[4] by the following Magma code:

```
for i,j,k,l in [1..3] do if A^H eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
1 3 3 3 ----> a^e=a
for i,j,k,l in [1..3] do if A^J eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
1 3 1 3 ---> a^f=a*c
```

```
for i,j,k,l in [1..3] do if A^I eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
2 3 3 3 --->a^g=a^2
for i,j,k,l in [1..3] do if A^M eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
1 1 3 3 ---> a^h=a*b
for i,j,k,l in [1..3] do if A^O eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
2 3 3 3 ----> a^o=a^2
for i,j,k,l in [1..3] do if B^H eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 1 3 3 --->b^e=b
for i,j,k,l in [1..3] do if B^J eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 1 1 3 ---> b^f=b*c
for i,j,k,l in [1..3] do if B^I eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 2 3 3 ----> b^g=b^2
for i,j,k,l in [1..3] do if B^M eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 2 3 3 ----> b^h=b^2
for i,j,k,l in [1..3] do if B^O eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 1 3 3 ----> b^o=b
for i,j,k,l in [1..3] do if C^H eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 1 2 3 ----> c^e=b*c^2
for i,j,k,l in [1..3] do if C^J eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 2 3 ----> c^f=c^2
for i,j,k,l in [1..3] do if C^I eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 2 3 --->c^g=c^2
```

```
for i,j,k,l in [1..3] do if C^M eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 2 3 ----> c^h=c^2
for i,j,k,l in [1..3] do if C^O eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 1 2 ----> c^o=c*d^2
for i,j,k,l in [1..3] do if D^H eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 3 2 ----> d^e=d^2
for i,j,k,l in [1..3] do if D^J eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 3 1 ----> d^f=d
for i,j,k,l in [1..3] do if D^I eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 3 2 ----> d^g=d^2
for i,j,k,l in [1..3] do if D^M eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 3 1 ---->d^h=d
for i,j,k,l in [1..3] do if D^O eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
3 3 3 2 ---->d^o=d^2
```

Finally, we put presentations of q and NL[4] together with $a^e = a, a^f = a * c$, $a^g = a^2, a^h = a * b, a^o = a^2$,$b^e = b, b^f = b * c, b^g = b^2, b^h = b^2, b^o = b$, $c^e = b * c^2, c^f = c^2, c^g = c^2, c^h = c^2, c^o = c * d^2, d^e = d^2, d^f = d, d^g = d^2$, $d^h = d, d^o = d^2$.

Thus,

```
M<a,b,c,d,e,f,g,h,o>:=Group<a,b,c,d,e,f,g,h,o|a^3,b^3,
c^3,d^3,(a,b),(a,c),(b,c),(a,d),(b,d),(c,d),
e^2,f^2,g^2,h^2,o^2,(e*g)^2,(f*g)^2,(e*h)^2,
(f * h)^2,(g * h)^2,(e* o)^2,(g *o)^2,
e*f*e*f*h,h*o*h*g*o*h*o,(f *o * f *o* h)^2,
a^e=a,a^f=a*c,a^g=a^2,a^h=a*b,a^o=a^2,
b^e=b,b^f=b*c,b^g=b^2,b^h=b^2,
b^o=b,c^e=b*c^2,c^f=c^2,c^g=c^2,c^h=c^2,c^o=c*d^2,
```

```
d^e=d^2,d^f=d,d^g=d^2,d^h=d,d^o=d^2>;
```

We will now check if M is given above and G1 are isomorphic ?

```
#M;
11664
#G1;
11664
f1,M1,k1:=CosetAction(M,sub<M|Id(M)>);
s:=IsIsomorphic(G1,M1);
s;
true
```

Therefore, M and G1 are isomorphic. The isomorphism type of our group G is $3^4 : (3^2 * 2^3) \times 2$.

4.4 Verification of $2^9 : PGL(2,7)$

We begin the process with the presentation of G given as follows:

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,
y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^8,(y*t)^3,(x*w*t)^0,
(w*x*y*t)^0,(w*x*y*t)^7>;
```

The order of our group is equal 172032. We will now prove the isomorphism type of G. To prove that we have to know the compositions factors of G1.

```
CompositionFactors(G1);
G
|  Cyclic(2)
*
|  A(1, 7)                     = L(2, 7)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
```

The composition series for G is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq G_8 \supseteq G_9 \supseteq G_{10} \supseteq G_{11}$

where $G_{11} = 1$.

The composition factors are:

$G = (G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash G_7)(G_7 \backslash G_8)$

$(G_8\backslash G_9)(G_9\backslash G_{10})(G_{10}\backslash G_{11})$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)$

$(G_6\backslash G_7)(G_7\backslash G_8)(G_8\backslash G_9)(G_9\backslash G_{10})(G_{10}\backslash 1)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)$

$(G_8\backslash G_9)(G_9\backslash G_{10})G_{10}$

$=C_2C_2C_2C_2C_2C_2C_2C_2C_2PGL(2,7)C_2$.

The normal lattice of G1 is as follows:

```
Normal subgroup lattice
-----------------------
[9]  Order 172032  Length 1  Maximal Subgroups: 6 7 8
---
[8]  Order 86016   Length 1  Maximal Subgroups: 5
[7]  Order 86016   Length 1  Maximal Subgroups: 5
[6]  Order 86016   Length 1  Maximal Subgroups: 4 5
---
[5]  Order 43008   Length 1  Maximal Subgroups: 3
---
[4]  Order 512     Length 1  Maximal Subgroups: 2 3
---
[3]  Order 256     Length 1  Maximal Subgroups: 1
---
[2]  Order 2       Length 1  Maximal Subgroups: 1
---
[1]  Order 1       Length 1  Maximal Subgroups:
```

Looking at the normal lattice of G we note the largest abelian group NL[4] of order 512.

```
NL[4];
Permutation group acting on a set of cardinality 10752
Order = 512 = 2^9
```

We can write the isomorphism type of NL[4] such as 2^9. We will use this loop to make sure $NL[4] \cong 2^9$.

```
X:=[2,2,2,2,2,2,2,2,2];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
true
```

Now, we will write the presentation of NL[4] by FPGroup(NL[4]).

```
S<w,x,y,z,h,o,p,u,r>:=Group<w,x,y,z,h,o,p,u,r|
w^2,x^2,y^2,z^2,h^2,o^2,p^2,u^2,r^2,(w * x)^2,
```

```
(w * y)^2,(x * y)^2,(w * z)^2,(x * z)^2,(y * z)^2,
(w * h)^2,(x * h)^2,(y* h)^2,(z * h)^2,(w * o)^2,
(x* o)^2,(y * o)^2,(z * o)^2,(h * o)^2,(w * p)^2,
(x * p)^2,(y * p)^2,(z* p)^2,(h * p)^2,(o * p)^2,(w * u)^2,
(x * u)^2,(y * u)^2,(z * u)^2,(h * u)^2,(o * u)^2,(p * u)^2,
(w * r)^2,(x* r)^2,(y * r)^2,(z * r)^2,(h * r)^2,
(o * r)^2,(p* r)^2,(u * r)^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[4]);
s;
true
```

Next, we will factor G1 by NL[4] resulting q which is the quotient group.

```
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
Permutation group q acting on a set of cardinality 8
Order = 336 = $2^4 * 3 * 7$
```

So, we found the order of q=336 and the composition factors of q as follows:

```
CompositionFactors(q);
G
|  Cyclic(2)
*
|  A(1, 7)                    = L(2, 7)
1
IsIsomorphic(PGL(2,7),q);
true
```

We can see that q is isomorphic to $PGL(2,7)$.

Now, we will find our presentation for q by using FPGroup(q). It gives us the following presentation for q:

```
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a*b)^2,(c*b)^3,
(a*c)^8,b*a*c*b*a*c*b*a*c*b*a*c*a*b*c*a*b*c*a*b*c>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
true
```

From previous steps, we could know NL[4] and q what are. Now, we want to figure out what the isomorphic types between them since our q is not the center of G, so we do not have a central extension. Also, there is no normal subgroup of order 336 in G that intersects with NL[4] of order 512. Therefore, we do not have a direct product extension. So, it remains two possibilities which are a semi direct or mixed extension. If we can write a presentation with the action of the generators of H on the generators of NL[4], so G is a semi-direct product.

```
T:=Transversal(G1,NL[4]);
EE:=q!(1, 2)(3, 4)(5, 6)(7, 8);
FF:=q! (1, 2)(5, 7)(6, 8);
GG:=q!(2, 3)(4, 5)(6, 8);
ff(T[2]) eq EE;
true
ff(T[3]) eq FF;
true
ff(T[4]) eq GG;
true
q eq sub<q|EE,FF,GG>;
true
A:=G1!NL[4].1;
B:=G1!NL[4].2;
C:=G1!NL[4].3;
D:=G1!NL[4].4;
F:=G1!NL[4].5;
H:=G1!NL[4].6;
P:=G1!NL[4].7;
O:=G1!NL[4].8;
V:=G1!NL[4].9;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
```

Next, we need to store the generators of NL[4]. We will indicate to the generators of NL[4] as A, B, C, D, F, H, P, O and V and the generators of q as EE, FF, and GG. Now, we can use the following code to determine the action of q on NL[4].

```
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then
i,j,k,l,m,mm,ii,jj,kk; end if; end for;
1 2 1 2 2 2 2 2 2
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then
i,j,k,l,m,mm,ii,jj,kk; end if; end for;
```

```
1 2 1 2 2 2 2 2 2
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk
then i,j,k,l,m,mm,ii,jj,kk; end if; end for;
1 2 2 2 2 2 2 2 2
```

We indicated here only the generator A of NL[4] which be raised by J, I and M. All the elements in NL[4] are order 2. We will repeat this process for each of remaining generators of NL[4] (B,C,...,V) which can be found in the appendix. Now, we can find the action of q on NL[4] which are $w^a = w*y, w^b = w*y, w^c = w, x^a = x*y, x^b = x,$ $x^c = x*y*z, y^a = y, y^b = y, y^c = z, z^a = y*z*u, z^b = y*z, z^c = y, h^a = h,$ $h^b = y*h, h^c = y*h*o*u, o^a = y*h*o*p*u, o^b = x*o*u, o^c = y*u, p^a = p,$ $p^b = p, p^c = y*h*o*p*r, u^a = u, u^b = u, u^c = z*o, r^a = p*r, r^b = p*u*r, r^c = y*z*r$. Lastly, we will collect the presentation NL[4] and q with the action of q on NL[4] words into one presentation as follows.

```
H<w,x,y,z,h,o,p,u,r,a,b,c>:=Group<w,x,y,z,h,o,p,u,r,a,b,c| w^2,
x^2,y^2,z^2,h^2,o^2,p^2,u^2,r^2,(w * x)^2,
(w * y)^2,(x * y)^2,(w * z)^2,(x * z)^2,(y * z)^2,
(w * h)^2,(x * h)^2,(y* h)^2,(z * h)^2,(w * o)^2,
(x* o)^2,(y * o)^2,(z * o)^2,(h * o)^2,(w * p)^2,(x * p)^2,
(y * p)^2,(z* p)^2,(h * p)^2,(o * p)^2,(w * u)^2,
(x * u)^2,(y * u)^2,(z * u)^2,(h * u)^2,
(o * u)^2,(p * u)^2,(w * r)^2,(x* r)^2,(y * r)^2,(z * r)^2,
(h *r)^2,(o*r)^2,(p*r)^2,(u*r)^2,w^a=w*y,w^b=w*y,w^c=w,
x^a=x*y,x^b=x,x^c=x*y*z,y^a=y,y^b=y,y^c=z,
z^a=y*z*u,z^b=y*z,z^c=y,h^a=h,h^b=y*h,h^c=y*h*o*u,
o^a=y*h*o*p*u,o^b=x*o*u,o^c=y*u,p^a=p,p^b=p,p^c=y*h*o*p*r,
u^a=u,u^b=u,u^c=z*o,r^a=p*r,r^b=p*u*r,r^c=y*z*r,a^2,b^2,
c^2,(a*b)^2,(c*b)^3,(a*c)^8,b*a*c*b*a*c*b*a*c*b*a*c*a
*b*c*a*b*c*a*b*c>;
#H;#G1;
172032
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);s;
true
```

Therefore, we have semi direct $G \cong 2^9 : PGL(2,7)$

4.5 Central Extension

To investigate an extension is a central extension, or not we first need to compute the center of our group. If it is larger more than one and equal the largest abelian group, so we will have a central extension.

4.5.1 Verification of $2^{\bullet}S_6$

We begin the process with the presentation of G given as follows:

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,x*y^-1*x*y,
y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^8,(x*y^-1*t)^5,(y*t)^4,(x*w*t)^0,
(w*x*y*t)^5,(w*x*y*t)^0>;
```

The order of our group is equal 1440. We will now find the compositions factors of our group to recognize if we have a central extension or not.

```
CompositionFactors(G1);
G
|  Cyclic(2)
*
|  Alternating(6)
*
|  Cyclic(2)
1
```

The composition series for G is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3$ where $G_3 = 1$.

The composition factors are:

$G = (G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash 1)$

$=(G_0\backslash G_1)(G_1\backslash G_2)G_2$

$=C_2A_6C_2$

```
Normal subgroup lattice
-----------------------

[7]  Order 1440  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 720   Length 1  Maximal Subgroups: 3
[5]  Order 720   Length 1  Maximal Subgroups: 2 3
[4]  Order 720   Length 1  Maximal Subgroups: 3
---
[3]  Order 360   Length 1  Maximal Subgroups: 1
---
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

Now, Magma will help us to determine if G1 has a center. If we have a center then, we are looking for which element in the normal lattice is the center.

```
Center(G1);
Permutation group acting on a set of cardinality 360
Order = 2
Center(G1) eq NL[2];
true
The presentation of NL[2] is
S<e>:=Group<e|e^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[2]);
s;
true
```

Next, we can factor G1 by the center and define the isomorphism type of the resulting factor group q. We can see the compositions factors of q and normal lattice as follows:

```
CompositionFactors(q);
G
|  Cyclic(2)
*
|  Alternating(6)
1

nl:=NormalLattice(q);
nl;
Normal subgroup lattice
-----------------------
```

```
[3]  Order 720  Length 1  Maximal Subgroups: 2
---
[2]  Order 360  Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

We could figure out from the compositions factors of q that $q \cong S_6$ where the order of S_6 is equal 720.

```
IsIsomorphic(SymmetricGroup(6),q);
true
```

We find a presentation of S_6 by [WB99] as a reference, then we use Magma to make sure if our presentation is correct or not as follows:

```
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a*b)^2,
(b *c)^4,b*a*c*b*a*c*b*a*c*a*b*c*a*b*c,
(a*c)^8,c*a*c*b*c*a*c*b*c*b*a*c*b*c*a*c*b*c*a*c*b>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
true
```

Thus, our presentation of S_6 is correct. Then, we have to write the generators of S_6 in terms of the generators of the center. Now, we need to find transversals.

```
T:=Transversal(G1,NL[2]);
EE:=q!q.2;
FF:=q!q.3;
GG:=q!q.5;
q,ff:=quo<G1|NL[2]>;
ff(T[3]) eq FF;
true
ff(T[2]) eq EE;
true
ff(T[4]) eq GG;
true
q eq sub<q|EE,FF,GG>;
true
A:=G1!NL[2].2;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
```

J, I, and M represent the generators of S_6 and A represents NL[2] which is the center. Now, we can use the following code to determine the action of q on the center.

```
for i in [1..2] do if A^J eq A^i then i; end if; end for;
1 ---> e^J=e
for i in [1..2] do if A^I eq A^i then i; end if; end for;
1 ---> e^I=e
for i in [1..2] do if A^M eq A^i then i; end if; end for;
1 ---> e^M=e
```

Now, we add these relations $e^J = e$, $e^I = e$, and $e^M = e$ with the presentation of center to the presentation of q and check if it is isomorphic to G.

```
H<a,b,c,e>:=Group<a,b,c,e|a^2,b^2,c^2,(a*b)^2,(b*c)^4,
b*a*c*b*a*c*b*a*c*a\*b*c*a*b*c,(a * c)^8,
c * a * c * b * c *a * c *b * c* b * a * c * b* c * a* c
*b * c * a * c * b,e^2,e^a=e,e^b=e,e^c=e>;
#H;
1440
#G1;
1440
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
true
```

Therefor, we have a central extension $2^{\bullet}S_6$.

4.6 Isomorphism Type of $G = 2^{*16} : (5^4 : (D_4 \times S_3))$

Our presentation is given by:
G=$< x, y, z, w, h, t | x^2, y^2, z^2, w^4, h^2, (y*z)^2, (x*w^{-1})^2, (y*w^{-1})^2, z*w^{-1}*z*w, w^{-2}*h, x*y*x*y*w^{-1}, x*z*x*z*h, t^2, (t, x), (y*t)^4, (z*w*t)^0, (z*x*t)^3, (y*x*t)^{10}, (y*x*z*t)^6 >$. Now, we will use a few Magma commands to find a minimal faithful permutation representation of G1. To Create an image of G, we will use this code.

```
f,G1,k:=CosetAction(G,sub<G|x,y>);G1;
30000
```

To find all subgroups of G, and gather all subgroups which are found in SL, we will use the following codes.

```
SL:=Subgroups(G1);
T :={X`subgroup: X in SL};
#T;
724
```

To determines faithful permutation representations of G.

```
TrivCore := {H:H in T| #Core(G1,H) eq 1};
#TrivCore;
670
```

This code gives us permutation representations of the smallest degree.

```
mdeg := Min({Index(G1,H):H in TrivCore});
```

To determines how many faithful permutation representations have a minimal number of letters by the following code.

```
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
6
```

To picks a representative from Good.

```
H := Rep(Good);
#H;
1000
```

To create a permutation representation of the chosen representative from Good.

```
f,G1,K := CosetAction(G1,H);
G1;
Originally Permutation group G1 acting on a set of
cardinality 3750
Order = 30000 = 2^4 * 3 * 5^4
now, our Permutation group G1 acting on a set of
 cardinality 30
Order = 30000 = 2^4 * 3 * 5^4
```

Now, we will take the permutation representation of G1 which is found above and find its isomorphism type. Our composition factors of our group as follows:

```
CompositionFactors(G1);
G
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
1
```

Now, we will prove the isomorphism type of G1.

The composition series for G1 is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq G_8 \supseteq G_9$ where $G_9 = 1$.

The composition factors are:

$G = (G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)(G_8\backslash G_9)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)(G_8\backslash 1)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)G_8$

$=C_5C_5C_5C_5C_2C_2C_2C_3C_2$

Next, we have to find the normal lattice of G and ask Magma about the largest abelian subgroup in the normal lattice.

```
NL:=NormalLattice(G1); NL;
[26]  Order 30000  Length 1  Maximal Subgroups: 19 20 21 22 23
 24 25
---
[25]  Order 15000  Length 1  Maximal Subgroups: 14 15 16
[24]  Order 15000  Length 1  Maximal Subgroups: 12 14 17
[23]  Order 15000  Length 1  Maximal Subgroups: 15 17 18
[22]  Order 15000  Length 1  Maximal Subgroups: 12 16 18
```

```
[21]  Order 15000  Length 1  Maximal Subgroups: 13 16 17
[20]  Order 15000  Length 1  Maximal Subgroups: 12 13 15
[19]  Order 15000  Length 1  Maximal Subgroups: 11 13 14 18
---
[18]  Order 7500   Length 1  Maximal Subgroups: 7 10
[17]  Order 7500   Length 1  Maximal Subgroups: 10
[16]  Order 7500   Length 1  Maximal Subgroups: 10
[15]  Order 7500   Length 1  Maximal Subgroups: 10
[14]  Order 7500   Length 1  Maximal Subgroups: 6 10
[13]  Order 7500   Length 1  Maximal Subgroups: 5 10
[12]  Order 7500   Length 1  Maximal Subgroups: 8 9 10
[11]  Order 5000   Length 1  Maximal Subgroups: 5 6 7
---
[10]  Order 3750   Length 1  Maximal Subgroups: 3 4
[ 9]  Order 3750   Length 1  Maximal Subgroups: 4
[ 8]  Order 3750   Length 1  Maximal Subgroups: 4
[ 7]  Order 2500   Length 1  Maximal Subgroups: 3
[ 6]  Order 2500   Length 1  Maximal Subgroups: 3
[ 5]  Order 2500   Length 1  Maximal Subgroups: 3
---
[ 4]  Order 1875   Length 1  Maximal Subgroups: 2
[ 3]  Order 1250   Length 1  Maximal Subgroups: 2
---
[ 2]  Order 625    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1      Length 1  Maximal Subgroups:
```

So, we see that the largest abelian group NL[2] of order 625 by the normal lattice of G1. Now, we check if NL[2] isomorphic to 5^4 by the following Magma code.

```
X:=[5,5,5,5];
IsIsomorphic(NL[2],AbelianGroup(GrpPerm,(X)));
true
```

Therefore, we could see that $NL[2] = 5^4$.

Next, we have to create a presentation for NL[2] as follows:

$S =< w, x, y, z | w^5, x^5, y^5, z^5, (w, x), (w, y), (x, y), (w, z), (x, z), (y, z) >$.

```
nl:=NormalLattice(q);
[25]  Order 48  Length 1  Maximal Subgroups: 18 19 20 21 22
 23 24
---
[24]  Order 24  Length 1  Maximal Subgroups: 11 14 15
[23]  Order 24  Length 1  Maximal Subgroups: 12 14 16
[22]  Order 24  Length 1  Maximal Subgroups: 11 13 16
[21]  Order 24  Length 1  Maximal Subgroups: 10 12 13 15
[20]  Order 24  Length 1  Maximal Subgroups: 13 14 17
[19]  Order 24  Length 1  Maximal Subgroups: 15 16 17
[18]  Order 24  Length 1  Maximal Subgroups: 11 12 17
---
[17]  Order 12  Length 1  Maximal Subgroups: 7 8 9
[16]  Order 12  Length 1  Maximal Subgroups: 8
[15]  Order 12  Length 1  Maximal Subgroups: 6 8
[14]  Order 12  Length 1  Maximal Subgroups: 8
[13]  Order 12  Length 1  Maximal Subgroups: 4 8
[12]  Order 12  Length 1  Maximal Subgroups: 5 8
[11]  Order 12  Length 1  Maximal Subgroups: 8
[10]  Order 8   Length 1  Maximal Subgroups: 4 5 6
---
[ 9]  Order 6   Length 1  Maximal Subgroups: 3
[ 8]  Order 6   Length 1  Maximal Subgroups: 2 3
[ 7]  Order 6   Length 1  Maximal Subgroups: 3
[ 6]  Order 4   Length 1  Maximal Subgroups: 2
[ 5]  Order 4   Length 1  Maximal Subgroups: 2
[ 4]  Order 4   Length 1  Maximal Subgroups: 2
---
[ 3]  Order 3   Length 1  Maximal Subgroups: 1
[ 2]  Order 2   Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1   Length 1  Maximal Subgroups:
```

We could create a presentation for q as follows:
H=$< e, f, g, h, v | e^2, f^2, g^2, h^2, v^2, (e*g)^2, (f*g)^2, (e*h)^2, (f*h)^2, (g*h)^2, (e*v)^2, (h*v)^2, e*f*e*f*h, v*e*f*e*v*f, g*v*g*e*v*g*v >$.
Now, we want to check if q which has order of 144 is gathered of a direct product or not. Thus, we could see that the direct product between two normal subgroups which are nl[10] and nl[7].

```
E:=DirectProduct(nl[10],nl[7]);
IsIsomorphic(E,q);
```

```
true
```

Note, we found that nl[10] is isomorphic to dihedral group which is D_4 and nl[7] is isomorphic to symmetric group which is S_3 by this Magma loop.

```
nl[10];
Permutation group acting on a set of cardinality 12
Order = 8 = 2^3
(2, 5)(6, 8)(10, 12)
(1, 2)(3, 5)(4, 8)(6, 7)(9, 12)(10, 11)
(1, 3)(2, 5)(4, 7)(6, 8)(9, 11)(10, 12)
IsIsomorphic(DihedralGroup(4),nl[10]);
true
nl[7];
Permutation group acting on a set of cardinality 12
Order = 6 = 2 * 3
(1, 3)(2, 5)(4, 11)(6, 12)(7, 9)(8, 10)
(1, 4, 9)(2, 8, 12)(3, 7, 11)(5, 6, 10)
IsIsomorphic(nl[7],SymmetricGroup(3));
true
```

From the previous steps, we find q= $D_4 \times S_3$ and NL[2]=5^4. We do not have a central extension between Nl[2] and q because the center of this group equal 1, so we have only two options to explain the relation between NL[2] and q which are a semi direct or mixed extension. Now, we will label our generators of q as the follows:

```
EE=q!q.1
FF=q!q.2
GG=q!q.3
DD=q!q.4
CC=q!q.6
```

Also, we will label our generators of NL[2] as follows:

```
A=G1!NL[2].1
B=G1!NL[2].2
C=G1!NL[2].3
D=G1!NL[2].4
```

Also,

```
J=G1!T[2]
I=G1!T[3]
M=G1!T[4]
O=G1!T[5]
Q=G1!T[6].
```

Here w, x, y, and z represent elements of NL[2], and e, f, g, h, and v represent elements of q. Where A, B, C,and D ≅ w, x, y, and z, respectively and J, I, M, O, and Q ≅ e, f, g, h, and v, respectively. Now, we need to check the action q on NL[2]. I will write here only action q on A, but B, C, and D will be in appendix. The Magma code:

```
for i,j,k,l in [1..5] do if A^J eq A^i*B^j*C^k*D^l then
 i,j,k,l;
end if; end for;
4 5 5 5 ---> w^e=w^4
for i,j,k,l in [1..5] do if A^I eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
4 4 5 5 ---> w^f=w^4*x^4
for i,j,k,l in [1..5] do if A^M eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
5 2 3 1 ---> w^g=x^2*y^3*z
for i,j,k,l in [1..5] do if A^O eq A^i*B^j*C^k*D^l then
i,j,k,l;
end if; end for;
4 5 5 5 ---> w^h=w^4
for i,j,k,l in [1..5] do if A^Q eq A^i*B^j*C^k*D^l then
 i,j,k,l;
end if; end for;
1 5 5 5 ---> w^v=w
```

Finally, the presentation of $G = 2^{*16} : (5^4 : (D_4 \times S_3))$ as follows:

```
 M<w,x,y,z,e,f,g,h,v|w^5,x^5,y^5,z^5,(w,x),(w,y),(x,y),
 (w,z),(x,z),(y,z),e^2,f^2,g^2,h^2,v^2,(e*g)^2,(f*g)^2,
 (e*h)^2,(f * h)^2,(g * h)^2,(e * v)^2,(h * v)^2,
 e*f*e*f*h,v*e*f*e*v*f,g*v*g*e*v*g*v,
 w^e=w^4,w^f=w^4*x^4,w^g=x^2*y^3*z,
 w^h=w^4,w^v=w,x^e=w^2*x,x^f=x,x^g=x^3*y^2,
 x^h=x^4,x^v=w^3*x^4,y^e=w^2*x^2*y^4*z,y^f=y,
 y^g=x*y^2,y^h=y^4,y^v=w^2*x^2*y^4*z,
z^e=z,z^f=x^3*y^2*z^4,z^g=w*x,z^h=z^4,z^v=w^4*x^4*z>
#M;
30000
#G1;
30000
f1,M1,k1:=CosetAction(M,sub<M|Id(M)>);
s:=IsIsomorphic(G1,M1);s;
true
```

4.7 Mixed Extensions

A mixed extension is an incorporation of a semi-direct product and a central extension. When we have a center which is not the largest group, so we will have a mixed extension denoted $:^{\bullet}$.

4.7.1 Verification of $G \cong 3 :^{\bullet} PSL(2,19) \times 2$

Our presentation is given by:
$G =< y, w, t | y^2, w^3, (y * w^-1)^2, t^2, (t, y), (y * t)^2, (w * t)^{10}, (y * w^-1 * t)^9 >$.
Also, the composition factors of this group is as follows:

```
CompositionFactors(G1);
G
|  A(1, 19)                    = L(2, 19)
*
|  Cyclic(2)
*
|  Cyclic(3)
1
```

The composition series for G is:
$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3$ where $G_3 = 1$.
The composition factors are:
$G = (G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)$
$=(G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash 1)$
$=(G_0 \backslash G_1)(G_1 \backslash G_2)G_2$
$=C_3 C_2 PSL(2,19)$.
Now, we can see from the normal lattice of G that NL[2] is the largest abelian subgroup.

```
Normal subgroup lattice
-----------------------
[6]  Order 20520  Length 1  Maximal Subgroups: 3 5
---
[5]  Order 10260  Length 1  Maximal Subgroups: 2 4
---
[4]  Order 3420   Length 1  Maximal Subgroups: 1
---
[3]  Order 6      Length 1  Maximal Subgroups: 2
---
```

```
[2]  Order 3      Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

We find that $NL[2] \cong 3$ and has the given presentation: $S < d >=< c|d^-3 >$.

Now, we use Magma code to find the quotient q and the normal lattice of q.

```
H:=NL[2];
q,ff:=quo<G1|NL[2]>;
q;
Permutation group q acting on a set of cardinality 40
Order = 6840 = 2^3 * 3^2 * 5 * 19

Normal subgroup lattice
-----------------------
[4]  Order 6840  Length 1  Maximal Subgroups: 2 3
---
[3]  Order 3420  Length 1  Maximal Subgroups: 1
---
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

Now, we will find the composition factors of q.

```
CompositionFactors(q);
G
|  A(1, 19)                  = L(2, 19)
*
|  Cyclic(2)
1
```

We will now check if q isomorphic to $PSL(2,19) \times 2$ or not.

```
s:=IsIsomorphic(q,DirectProduct(PSL(2,19),CyclicGroup(2)));
s;
true
```

As a result, we can say that q is isomorphic to $PSL(2,19) \times 2$.

Now, we will write our presentation of q by applying the FPGroup command in Magma.

It gives us the following presentation:

H=$Group < a,b,c|a^2, b^3, c^2, (b^{-1}*a)^2, (a*c)^2, b*c*b^{-1}*c*b*c*b^{-1}*c*a*b^{-1}*c*$

$b*c*b^{-1}*c*b*c*b^{-1}*c, (b^{-1}*c)^{10}, c*b^{-1}*c*b*c*b*c*b^{-1}*c*b^{-1}*c*b*c*b^{-1}*c*b^{-1}*c*b^{-1}*c*b*a*c*b^{-1}*c*b^{-1}*c*b*c*b*c*b*c*b^{-1}*c*b*c*b*c*b^{-1} >$. Now, we will check the relation between NL[2] and q. Since NL[2] is not the center of G, there is no normal subgroup of order q which is 6840 so that is not direct product between q and NL[2]. Then, we only have two options which are a semi direct product or mixed extension. So, we must determine the transversals of NL[2].

```
T:=Transversal(G1,NL[2]);
ff(T[2]) eq q.1;
true
ff(T[3]) eq q.2;
true
ff(T[4]) eq q.3;
true
```

Now, we will label our generators of NL[2] which is A. Also, we have these elements d represents an element of NL[2], and a, b, and c represent elements of $q \cong PSL(2,19) \times 2$. Where J, I, and M represent a, b, and c respectively.

```
A:=NL[2].2;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
```

Next, we will investigate the semi-direct product part of our extension, so we need to check the action of the generators of q on the generators of NL[2].

```
for i in [1..3] do if A^J eq A^i then i;
 end if; end for;
2
for i in [1..3] do if A^I eq A^i then i;
end if; end for;
1
for i in [1..3] do if A^M eq A^i then i;
 end if; end for;
2
```

So, we could get these relations $d^a = d^2$, $d^b = d$, and $d^c = d^2$ from the code above and add them to our presentation. Now, we have completed the semi-direct part of our presentation, so we will check a mixed extension part of our presentation. We have to check the presentation of q.

```
FPGroup(q);
Finitely presented group on 3 generators
Relations
$.1^2 = Id($)
$.2^3 = Id($)
$.3^2 = Id($)
($.2^-1 * $.1)^2 = Id($)
($.1 * $.3)^2 = Id($)
$.2 * $.3 * $.2^-1 * $.3 * $.2 * $.3 *
 $.2^-1 * $.3 * $.1 *$.2^-1 * $.3 *$.2 * $.3 *
  $.2^-1 * $.3 * $.2 *
$.3 * $.2^-1 * $.3 = Id($)
($.2^-1 * $.3)^10 = Id($)
$.3 * $.2^-1 * $.3 * $.2 * $.3 * $.2 * $.3 * $.2^-1 * $.3 *
$.2^-1 * $.3 *$.2 * $.3 * $.2^-1 * $.3 * $.2^-1 * $.3
* $.2^-1 * $.3 * $.2 *$.1 * $.3 *$.2^-1 * $.3 * $.2^-1 *
$.3 * $.2 * $.3 * $.2 * $.3 * $.2 * $.3 * $.2^-1 *
$.3 * $.2 * $.3 * $.2 * $.3 * $.2^-1 = Id($)
```

Recall that T[2]=.1, T[3]=.2, and T[4]=.3. We use the following Magma code to investigate which the order of elements of q changed. We found that.

```
for i in [1..3] do
if  T[2]^2 eq A^i then i;
end if; end for;
3
for i in [1..3] do
if T[3]^3 eq A^i then i;
end if; end for;
3
for i in [1..3] do
if T[4]^2 eq A^i then i;
end if; end for;
3
for i in [1..3] do
if (T[3]^-1 * T[2])^2 eq A^i then i;
end if; end for;
3
for i in [1..3] do
if (T[2] * T[4])^2 eq A^i then i;
end if; end for;
3
for i in [1..3] do
if T[3] * T[4] * T[3]^-1 * T[4] * T[3] * T[4]
```

```
*T[3]^-1 *T[4] * T[2] * T[3]^-1 * T[4] *T[3]
*T[4] * T[3]^-1* T[4] * T[3]*T[4] * T[3]^-1 * T[4] eq A^i
then i;end if; end for;
3
for i in [1..3] do
if (T[3] ^-1 * T[4] )^10  eq A^i then i;
end if; end for;
3
for i in [1..3] do
if T[4]* T[3] ^-1*T[4]*T[3]*T[4]*T[3]*T[4]
* T[3] ^-1 *T[4] * T[3] ^-1 *T[4] *T[3]*T[4]
*T[3] ^-1 * T[4] * T[3] ^-1 *T[4] *T[3] ^-1 *
T[4] * T[3] *T[2] * T[4] *T[3] ^-1 *T[4] *
T[3] ^-1 *  T[4] * T[3]* T[4]*T[3]*T[4]* T[3]*T[4]
*T[3] ^-1 *T[4] * T[3] *T[4] * T[3]  * T[4] *T[3] ^-1
 eq A^i then i;
end if; end for;
1
```

From above, we can see $c*b^{-1}*c*b*c*b*c*b^{-1}*c*b^{-1}*c*b*c*b^{-1}*c*b^{-1}*c*b^{-1}*c*b*a*c*b^{-1}*c*b^{-1}*c*b*c*b*c*b*c*b^{-1}*c*b*c*b*c*b^{-1}$ is the only one that changed, where T[2]=a, T[3]=b, and T[4]=c.
We can check by using Magma.

```
c * b^-1 * c * b * c * b *
c * b^-1* c * b^-1 * c *
b * c * b^-1 * c * b^-1 * c * b^-1* c * b * a * c *
b^-1 * c * b^-1 * c * b * c * b * c * b * c * b^-1 *
c * b * c * b * c * b^-1 eq d;
true
```

Now, we collect our generators of q and NL[2] with relations to our presentation, then we can check our presentation in Magma.

```
H<a,b,c,d>:=
Group<d,a,b,c|
a^2 ,b^3 ,c^2 ,(b^-1 * a)^2 ,
(a * c)^2 ,b * c * b^-1 * c * b * c * b^-1 * c * a * b^-1 * c *
b * c * b^-1 * c * b * c * b^-1 * c ,(b^-1 * c)^10 ,
c * b^-1 * c * b * c * b * c * b^-1 * c * b^-1 * c *
b * c * b^-1 * c * b^-1 * c * b^-1 * c * b * a * c *
b^-1 * c * b^-1 * c * b * c * b * c * b * c * b^-1 *
c * b * c * b * c * b^-1=d,d^3,d^a=d^2,d^b=d,d^c=d^2>;
```

```
#H;
20520
#G1;
20520
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
true
```

Therefore, G is a mixed extension of 3 by $PSL(2,19) \times 2$. $G \cong 3 :^{\bullet} PSL(2,19) \times 2$.

Chapter 5

Progenitors and Their Homomorphic Images

5.1 Homomorphic Images of Progenitor $2^{*8} : (2 \times 4 : 2)$

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,
y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^a,(y*t)^b,(x*w*t)^c,
(w*x*y*t)^d,(w*x*y*t)^e,(z*t)^f,(x*y^-1*t)^m>;
f,G1,k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
IN:=sub<G1|f(w),f(x),f(y),f(z)>;
CompositionFactors(G1);
```

Table 5.1: $2^{*8} : (2 \times 4 : 2)$

a	b	c	d	e	f	m	G ≅	#G
8	3	0	0	7	0	0	$2^9 : PGL(2,7)$	172032
9	3	0	0	7	0	0	$2^8 : L_8$	129024
10	3	5	0	5	0	0	$2^5 : A_5$	1920
10	3	5	0	10	0	0	$2^5 : S_5$	3840
10	3	8	0	8	0	0	$2^2 \bullet (S_6 : S_6)$	2073600
7	3	7	0	9	0	0	$PGL(2,8)$	504
7	3	7	0	8	0	0	$PSL(2,7) : 2$	336
5	4	5	0	9	0	0	$PGL(2,19)$	6840
5	5	5	5	0	0	0	$PSL(2,11)$	660
6	7	3	9	0	0	0	$2^7 : PSL(2,8)$	64512
3	8	3	0	10	0	0	$2 \times 3 : S_6$	4320
7	5	0	0	4	0	0	$2 \times 4 : (PSL(3,4) : 2)$	322560
10	3	0	0	5	0	0	$2^6 : S_5$	7680
8	4	0	5	0	0	5	$2 \bullet S_6$	1440
9	10	9	6	9	0	3	$3 : PGL(2,19)$	20520
3	9	3	0	9	0	9	$PSL(2,19)$	3420
3	3	0	0	0	0	0	$2^4 : S_3 \times 2$	192
5	6	6	4	0	0	0	$2^2 : (S_5 : S_5)$	57600
6	10	9	0	6	0	3	$(A_5 : (A_5 \times A_5)) : D_6$	2592000

5.2 Homomorphic Images of Progenitor $2^{*16} : (2{\times}4 : C_2{\times}C_2)$

```
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),
(y*t)^a,
(z*w*t)^b,
(z*x*t)^c,
(y*x*t)^d,
(y*x*z*t)^e,
(y*x*z*t^x)^f,
(h*t)^m>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
CompositionFactors(G1);
```

Table 5.2: $2^{*16} : (2 \times 4 : C_2 \times C_2)$

a	b	c	d	e	f	m	G ≅	#G
9	0	8	3	8	0	0	$2 \bullet PGL(2,7)$	672
5	3	0	5	5	0	0	$2^4 : A_5$	960
8	4	5	0	3	0	0	$2^2 \bullet (PSL(3,4) : 2)$	161280
3	5	4	0	0	0	0	$3 \times 2 : (S_6 : A_6)$	1555200
3	6	10	6	7	0	0	$2 \bullet S_7$	10080
4	0	3	10	6	0	0	$5^4 : (D_4 \times S_3)$	30000
8	8	0	3	4	0	0	$8 : (PSL(3,4) : 2) : 2$	645120
3	0	4	8	9	0	0	$6 \bullet (U_4(3) : 2)$	39191040
6	3	6	4	0	0	0	$3^4 : (3^2 \times 2^3) \times 2$	11664
6	3	10	4	0	0	0	$5^4 : (2^3 \times 3^2) \times 2$	90000
10	4	5	0	6	0	0	$2^7 : S_5$	15360
8	4	0	3	5	0	0	$2 : (PSL(3,4) : 2) \bullet 2$	161280
7	0	0	4	3	0	0	$(PSL(3,4) : 2) : 2$	80640
4	0	0	6	3	0	0	$3 : (S_6 \times 2) \times 2$	8640
6	0	2	7	7	0	0	$2 \bullet PGL(2,13)$	4368
3	0	5	6	7	0	0	S_7	5040
3	0	0	10	4	0	0	$2 \bullet (S_6 : 2)$	2880
3	0	4	8	6	0	0	$2^2 \bullet (PGL(2,7) : PSL(2,7))$	225792
3	7	4	0	6	0	0	$2 \bullet (PGL(2,7) : PSL(2,7))$	112896

5.3 Homomorphic Images of Progenitor $11^{*4} :_m (5:4)$

```
G<x,y,t>:=Group<x,y,t|x^5,y^4,y^3*x^3*y*x,t^11,t^x=t^5,
(y*(t^6))^m,(y*(t^7))^a,(y*(t^8))^s,(y*(t^9))^d,(y*(t^10))^f,
(x*t)^g,(x*t^y)^h,(x*t^2)^j,(x*t^(y^2))^k,(x^2*t)^l,(x^2*t^y)^q,
(x^2*t^2)^w,(x^2*t^(y^2))^e>;
```

Table 5.3: $11^{*4} :_m (5:4)$

m	a	s	d	f	g	h	G ≅	#G
0	0	0	3	6	0	0	M_{11}	7920

5.4 Homomorphic Images of Progenitor $5^{*6} :_m S_5$

```
primitivegroup(5,5);
xx:=G.1;
yy:=G.2;
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,
t^5,(t,y*x^-1*y),t^(x*y*x^-2)=t^4,((y*x*y)*t^(x^4*x^-1)^2)^a,
(y*x^2*t)^b,(y*x^2*t^(x^-1))^c,(y*x^2*t^(y*x^2))^d,(y*x^2*t^2)^f>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
```

Table 5.4: $5^{*6} :_m S_5$

a	b	c	d	f	G ≅	#G
0	4	8	0	6	$HS \times D_5$	443520000
7	0	8	4	0	$2 \bullet U_3(5)$	252000
0	0	0	4	7	$U_3(5)$	126000

5.5 Homomorphic Images of Progenitor $149^{*2} :_m D_{37}$

```
primitivegroup(37,2);
xx:=G.1;
yy:=G.2;
G<x,y,t>:=Group<x,y,t|y^2,(x^-1*y)^2,x^-37,t^149,t^(x^13)=t^33,
(y*t)^m>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
```

Table 5.5: $149^{*2} :_m D_{37}$

$\#k$	m	G $\cong$	#G
1	3	$L_2(149)$	1653900

Chapter 6

Transitive Groups on 9 Letters

6.1 Transitive Group (9, 4)

We will write a progenitor on a transitive group on 9 letters. We investigate the number of transitive groups that exist on 9 letters by the following Magma code:

```
NumberOfTransitiveGroups(9);
34
```

Now, we will take the fourth group in the sequence.

```
N:=TransitiveGroup(9,4);
S:=Sym(9);
xx:= S!(1, 2, 9)(3, 4, 5)(6, 7, 8);
yy:=S!(1, 2)(4, 5)(7, 8);
zz:=S!(1, 4, 7)(2, 5, 8)(3, 6, 9);
N:=sub<S|xx,yy,zz>;
#N;
for c,v,b,n,m,a,s in [0..10] do
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^3,(x^-1*y)^2,(x,z),
y*z^-1*y*z,t^2,(t,x*y),(t,x*y*z^-1),
(y*t)^c,((y * z^-1)*t)^v,
(z*t)^b,(x*t)^n,
((x * z^-1)*t)^m,((x * z)*t)^a,((y * z)*t)^s>;
if #G gt 10 then c,v,b,n,m,a,s,#G;
end if;
end for;
```

Table 6.1: $2^{*9} : (S_3 \times A_3)$

c	v	b	n	m	a	s	G ≅	#G
0	0	4	6	6	6	10	$5^2 : S_3 \times 2$	300
0	0	4	6	6	6	6	$3^2 : S_3 \times 2$	108
0	0	4	9	0	9	7	$PGL(2,8)$	504
0	0	10	10	10	0	5	$\bullet S_5$	120
9	0	10	9	0	9	0	$PSL(2,19)$	3420
0	0	0	0	0	8	7	$PGL(2,7)$	336
0	0	0	8	0	8	8	$L_2 \times 2$	672
0	0	2	6	0	6	6	$3^2 : D_6$	108

6.2 Transitive Group (9, 8)

```
N:=TransitiveGroup(9,8);
S:=Sym(9);
xx:=S! (1, 2, 9)(3, 4, 5)(6, 7, 8);
yy:=S!(3, 6)(4, 7)(5, 8);
zz:=S! (1, 2)(4, 5)(7, 8);
ww:=S!(1, 4, 7)(2, 5, 8)(3, 6, 9);
N:=sub<S|xx,yy,zz,ww>;
#N;
for c,v,b,n,m,a,s,d,f,g,h,j,o in [0..10] do
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^2,z^2,w^3,
x^-1*y*x*y,(x^-1*z)^2,(y*z)^2,
(x,w),(y*w^-1)^2,z*w^-1*z*w,t^2,(t,y),(t,x * z),
(z*t)^c,((z * w^-1)*t)^v,(y*t)^b,
((y * w^-1)*t)^n,((y * z)*t)^m,
((y * z * w^-1)*t)^a,(x*t)^s,
((x * w^-1)*t)^d,(w*t)^f,
((x * w)*t)^g,((z * w)*t)^h,
((x * y)*t)^j,((x * y * w^-1)*t)^o>;
if #G gt 10 then c,v,b,n,m,a,s,d,f,g,h,j,o ,#G;
end if; end for;
```

Table 6.2: $2^{*9} : (S_3 \times S_3)$

c	v	b	n	m	a	s	d	f	g	h	j	o	G ≅	#G
0	0	0	0	0	0	0	0	0	0	4	4	6	$2^3 : (2 \times 3 : S_6)$	34560
0	0	0	0	0	0	0	0	0	4	6	4	0	$2^7 : (S_3 \times S_3)$	4608
0	0	0	0	0	0	0	0	0	4	6	4	6	$2^5 : (S_3 \times S_3)$	1152
0	0	0	0	0	0	0	0	0	5	0	0	5	$3^4 : (S_5 \times A_5)$	583200
0	0	0	0	0	0	0	0	0	0	5	4	5	$2^5 : (S_6 \times A_5)$	1382400
0	0	0	0	0	0	0	0	0	6	6	4	8	$2^9 : (S_3 \times S_4)$	73728
0	0	0	0	0	0	0	0	5	0	6	0	4	$2 \times PGL(2, 81)$	1062720
0	0	0	0	0	0	0	0	5	5	8	0	8	$2^3 : (PSL(3, 4) : 2)$	322560
0	0	0	0	0	0	0	0	0	8	5	5	5	$2^2 : (PSL(3, 4) : 2)$	161280
0	0	0	0	0	0	0	0	4	0	5	4	5	$2^5 : S_6$	23040
0	0	0	0	0	0	0	0	4	0	8	3	6	$2^6 : S_4 \times D_3$	9216

6.3 Transitive Group (9, 14)

```
N:=TransitiveGroup(9,14);
S:=Sym(9);
xx:=S!(1, 2, 9)(3, 4, 5)(6, 7, 8);
yy:=S!(1, 8, 2, 4)(3, 5, 6, 7);
zz:=S! (1, 6, 2, 3)(4, 7, 8, 5);
hh:=S!(1, 4, 7)(2, 5, 8)(3, 6, 9);
N:=sub<S|xx,yy,zz,hh>;
#N;
for c,v,b,n,m,a,s,d,f,g in [0..10] do
G<x,y,z,h,t>:=Group<x,y,z,h,t|x^3,y^4,z^4,h^3,
z^-1 * y^2 * z^-1,y^-1 *z^-1 * y * z^-1,
h^-1 * z^-1 * x^-1 * z,h^-1 * z * x * z^-1,
(x, h),x^-1 * y^-1 *x^-1 * y* h^-1,t^2,(t,h * y),
(x*t)^c,((x * y^-1)*t)^v,
((x * z^-1)*t)^b,((x * h^-1)*t)^n,
((y^-1 * x)*t)^m,((z^-1 * x)*t)^a,
((x * y)*t)^s,((x * y * z^-1)*t)^d,
(z*t)^f,((z * x * y)*t)^g>;
if #G gt 10 then c,v,b,n,m,a,s,d,f,g,#G;
end if;
end for;
```

Table 6.3: $2^{*9} : (3^2 : 2^3)$

c	v	b	n	m	a	s	d	f	g	G ≅	#G
0	0	0	0	0	0	7	0	5	5	$U(4,3)$	3265920
0	0	0	0	0	0	6	4	0	8	$2^2 : (S_6 : 2)$	5760
0	0	0	0	0	0	0	0	4	4	$(2 \times 3^4 : (2^2 : 2))$	2592
0	0	0	0	4	0	4	6	6	0	25509168	25509168
0	0	0	0	4	0	4	8	0	4	$(2 \times 3^3 : (2^3 \times 3^2 : 2))$	7776

6.4 Transitive Group (9, 17)

```
for s,d,f,g,h,j,k,l,q,w in [0..10] do
G<x,y,t>:=Group<x,y,t|x^3,y^3,x^-1 * y * x^-1 * y^-1 * x*
y * x* y^-1,t^2,(t,y * x^-1 * y^-1),
(t,y * x^-1 * y * x^-1 * y),
((x^-1, y)*t)^s,
((x, y)*t)^d,
((x * y * x * y * x^-1 * y)*t)^f,
((y * x^-1 * y * x * y * x^-1)*t)^g,
(y*t)^h,
((y^-1)*t)^j,
((y^-1 * x)*t)^k,
(((y^-1 * x)^2)*t)^l,
((y^-1 * x^-1)*t)^q,
(((y^-1 * x^-1)^2)*t)^w>;
if #G gt 10 then s,d,f,g,h,j,k,l,q,w,#G;
end if;
end for;
```

Table 6.4: $2^{*9} : (3^3 : A_3)$

s	d	f	g	h	j	k	l	q	w	G $\cong$	#G
0	0	0	0	0	0	0	0	0	4	$3^4 : S_4$	1944
0	0	0	0	0	0	0	0	0	5	$3^5 : A_5$	14580

Chapter 7

Monomial Progenitors

7.1 Definitions

Definition 7.1. *We denote monomial progenitors as* $p^{*n} :_m N$.

Definition 7.2. Monomial Matrix *is A square matrix that has exactly one non-zero entry in each row and each column. [HK06]*

Definition 7.3. Monomial Representations *[HK06] Let G be a group. A monomial representation is a map* $A \longrightarrow GL(n : F)$, *since G is homomorphism which provided that A(x) and A(y) are monomial matrices. Then, the monomial representation of G is given by this formula :*

$$A(xx)=\begin{bmatrix} \phi(t_1xt_1^{-}1) & \phi(t_1xt_2^{-}1) & \cdots & \phi(t_1xt_n^{-}1) \\ \phi(t_2xt_1^{-}1) & \phi(t_2xt_2^{-}1) & \cdots & \phi(t_2xt_n^{-}1) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(t_nxt_1^{-}1) & \phi(t_nxt_2^{-}1) & \cdots & \phi(t_nxt_n^{-}1) \end{bmatrix}$$

Definition 7.4. Monomial Character *A character* ϕ *of G is monomial if* ϕ *is induced by a linear character of a subgroup H (not necessarily proper) of G.[HK06]*

Definition 7.5. Induced Character

Let $H \leq G$ *and* ϕ *be a character of G. The formula for induced character is*

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w).$$

Where

ϕ^G_α *is the value of* ϕ^G *at each element of the class* C_α*.*

n=[G : H].

h_α *is the number of elements in the class* C_α *of G.[HK06]*

7.2 The Monomial Progenitors $11^{*2} :_m D_5$

We will construct a monomial presentation of $11^{*2} :_m D_5$. Let G be generated by xx and yy where $xx = (1,2,3,4,5)$ and $yy = (1,4)(2,3)$. Also, Consider the subgroup $H = \{Id, (1,4,2,5,3), (1,2,3,4,5), (1,5,4,3,2), (1,3,5,2,4)\}$. To achieve a monomial presentation we must induce a linear character from a subgroup H of G. We choose a subgroup with index matching the degree of an irreducible character of G. Note, G has characters $X.1$, $X.2$, $X.3$, and $X.4$. We proceed using $x.3$ and look for a subgroup of order 5. So that $\frac{|G|}{|H|} = \frac{10}{5} = 2$ and $[G : H] = 2$. Since the index of the two groups is 2. If a matrix representation exists it will be represented by a 2×2 matrices.

Conjugacy Classes Representative of The Group G are:

$C_1 = Id(G)$

$C_2 = (1,4)(2,3)$

$C_3 = (1,2,3,4,5)$

$C_4 = (1,3,5,2,4)$

Conjugacy Classes Representative of The Group H are:

$D_1 = Id(H)$

$D_2 = (1,4,2,5,3)$

$D_3 = (1,2,3,4,5)$

$D_4 = (1,5,4,3,2)$

$D_5 = (1,3,5,2,4)$

Consider the irreducible characters of H and G given below:

Table 7.1: Character Table of G

Class	C_1	C_2	C_3	C_4
Size	1	5	2	2
Representative	Id(G)	(1,4)(2,3)	(1,2,3,4,5)	(1,3,5,2,4)
X.3	2	0	$Z1 = 14$	$Z1\#2 = 7$
X.4	2	0	$Z1\#2 = 7$	$Z1 = 14$

Before we move to next step, let us to explain how we can compute Z1, and $Z1\#2$ of the character table of G. We will depend on Magma to do that.

```
Z1 :=(CyclotomicField(5: Sparse := true)) !
 [ RationalField() | 0, 0, 1, 1 ];
Z1;
zeta(5)_5^3 + zeta(5)_5^2
PrimitiveRoot(11);
2
|G|/|H|=10/5;
2
2^2;
so, zeta(5)=4
Z1=zeta(5)_5^3 + zeta(5)_5^2=4^3 mod 11 +4^2 mod 11; 14
Z1#2=((zeta(5)_5)^2)^3 +((zeta(5)_5)^2)^2
=4^6 mod 11 + 4^4 mod 11; 7
```

Table 7.2: Character Table of H

Class	D_1	D_2	D_3	D_4	D_5
Size	1	1	1	1	1
Representative	Id(H)	(1,4,2,5,3)	(1,2,3,4,5)	(1,5,4,3,2)	(1,3,5,2,4)
X.2	1	Z1=4	$Z1\#2 = 5$	$Z1\#3 = 9$	$Z1\#4 = 3$

We can compute Z1, $Z1\#2$, $Z1\#3$ and $Z1\#4$ of the character table of H by Magma program:

```
Z1:=(CyclotomicField(5: Sparse := true))
! [ RationalField() | 0, 1, 0, 0 ];
```

```
Z1;
zeta(5)_5
Z1=zeta(5)_5=4
Z1#2=(zeta(5)_5)^2= 4^2 mod 11; 5
Z1#3=(zeta(5)_5)^3= 4^3 mod 11; 9
Z1#4=(zeta(5)_5)^4= 4^4 mod 11; 3
```

7.2.1 The Induction Process

Now, we will induce the character X.2 of H up to X.3 of G to obtain the character of G by using the tables above and using our definition of induction as follows:

$$\phi_\alpha^G= \frac{n}{h_\alpha} \sum_{w\in C_\alpha\cap H} \phi(w)$$

- $\phi_1^G= \frac{2}{h_1=1} \sum_{w\in C_1\cap H} \phi(e)$ =2

- $\phi_2^G = \frac{2}{h_2=5} \sum_{w\in C_2\cap H} \phi(0)$=0

- $\phi_3^G = \frac{2}{h_3=2} \sum_{w\in C_3\cap H} \phi((1,2,3,4,5)(1,5,4,3,2))$
$= \phi((1,2,3,4,5)) + \phi((1,5,4,3,2)) = Z1 = 14$

- $\phi_4^G = \frac{2}{h_4=2} \sum_{w\in C_4\cap H} \phi((1,3,5,2,4)(1,4,2,5,3))$
$= \phi((1,3,5,2,4) + \phi((1,4,2,5,3) = 7.$

Therefore, we have verified that ϕ_H^G=2, 0, 14, and 7. So, the character X.2 of H induces the character X.3 of G.

7.2.2 Verifying the Monomial Representation

We have a linear character of the subgroup H of index n in G .
Thus, we will let $G = Ht_1 \cup Ht_2 = He \cup H(1,4)(2,3)$. Where, $t_1 = e$ and $t_2 = (1,4)(2,3)$. We are now in a position to give the monomial representation of the progenitor $11^{*2} :_m D_5$.

$$A(xx)=\begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \end{bmatrix}$$

We now investigate the monomial representation has the following generators:

$$A(xx)=\begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} \text{and } A(yy)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A[xx]: Row 1

- $\phi(t_1xt_1^{-1}) = \phi((1,2,3,4,5))$=5
- $\phi(t_1xt_2^{-1}) = \phi(e(1,2,3,4,5)((1,4)(2,3))^{-1})$=0

A[xx]: Row 2

- $\phi(t_2xt_1^{-1}) = \phi((1,4)(2,3)(1,2,3,4,5))$=0
- $\phi(t_2xt_2^{-1}) = \phi((1,4)(2,3)(1,2,3,4,5)((1,4)(2,3))^{-1})$=9

A[yy]: Row 1

- $\phi(t_1yt_1^{-1}) = \phi((1,4)(2,3)) = 0$
- $\phi(t_1yt_2^{-1}) = \phi(e(1,4)(2,3)((1,4)(2,3))^{-1})$=1

A[yy]: Row 2

- $\phi(t_2yt_1^{-1}) = \phi((1,4)(2,3)(1,4)(2,3)) = 1$
- $\phi(t_2yt_2^{-1}) = \phi((1,4)(2,3)((1,4)(2,3))^{-1})$=0

Therefore, we could verify the monomial representation has the following generators:

$$A(xx)=\begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} \text{ and } A(yy)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

7.2.3 Construction a Permutation Representation:

We worked in Z_{11} on matrices of degree 2×2 which implies we are producing a $11^{*2} :_m D_5$ progenitor permutation representation. We want to represent x and y as permutation to do so we see that $a_{i,j} = n$ if the automorphism takes $t_i \Longrightarrow t_j^n$. There are 11^{*2} is a free product of 2 cyclic groups of order 11. Thus, we will construct a table with 2 $t_i's$ of order 11 labeled from 1.....20.

$A(xx)=\begin{bmatrix}5 & 0\\ 0 & 9\end{bmatrix}$ (where a_{11}=5, and a_{22}=9).Therefore, $t_1 \longrightarrow t_1^5$,and $t_2 \longrightarrow t_2^9$.

$A(yy)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (where a_{12}=1, and a_{21}=1).Therefore, $t_1 \longrightarrow t_2$,and $t_2 \longrightarrow t_1$.

Table 7.3: Automorphism of A(xx)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}
t_1^5	t_2^9	t_1^{10}	t_2^7	t_1^4	t_2^5	t_1^9	t_2^3	t_1^3	t_2	t_1^8	t_2^{10}	t_1^2	t_2^8	t_1^7	t_2^6	t_1	t_2^4	t_1^6	t_2^2
9	18	19	14	7	10	17	6	5	2	15	20	3	16	13	12	1	8	11	4

Table 7.4: Automorphism of A(yy)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6	t_2^7	t_1^7	t_2^8	t_1^8	t_2^9	t_1^9	t_2^{10}	t_1^{10}
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19

Therefore, consider the automorphism of xx and automorphism of yy as follows:

A(xx)=(1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)(4, 14, 16, 12, 20).

A(yy)=(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20).

7.2.4 Creating a Representation of the Progenitor:

Now, we need to find a presentation for our progenitor.

We must choose at t to normalizer from our two choices$\langle t_1\rangle$*$\langle t_2\rangle$.

Let t$\sim t_1$ and we must find permutations which normalize $\langle t_1\rangle$ fix the following set $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_2, t_2^2, t_2^3, t_2^4, t_2^5, t_2^6, t_2^7, t_2^8, t_2^9, t_2^{10}\}$.

Therefore, we need to compute the set stabilizer of the set $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}\}$ $=\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

```
S:=Sym(20);
xx:=G.1;
yy:=G.2;
xx:=S!(1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)(4,14,16,12,20);
yy:=S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)
(17, 18)(19, 20);
Nm:=sub<S|xx,yy>;
FPGroup(N);
Finitely presented group on 2 generators
Relations
$.2^2 = Id($)
($.1^-1 * $.2)^2 = Id($)
$.1^-5 = Id($)
```

Now, we will convert FPGroup(N) to our presentation.

$G < x, y > := Group < x, y | y^2, (x^-1 * y)^2, x^{-5} >;$

Next, we will find the stabilizer of $< t_1 >$.

```
Stabilizer$(Nm,{1,3,5,7,9,11,13,15,17,19});
(1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)
(4, 14, 16, 12, 20)
```

Thus, we discover the stabilizer of $< t_1 >$ in D_5 is generated by one element which we will label A.

A=(1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)(4, 14, 16, 12, 20). For a presentation we must convert this permutation into words which we find to be:

(1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)(4, 14, 16, 12, 20)=x.

As a result, we find that A sends 1 to 9 or t_1 to t_1^5. We will add this $t^x = t^5$ to our presentation. Therefore, we obtain the monomial progenitor as follows:

$$11^{*2} :_m D_5 =< x, y, t|y^2, (x^-1 * y)^2, x^{-5}, t^{11}, t^x = t^5, (t, t^y) > .$$

7.2.5 The Monomial Progenitors $59^{*2} :_m D_{29}$

We first establish a monomial presentation of $59^{*2} :_m D_{29}$.

G has order 58 and it is generated by xx and yy where xx=(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,22, 23, 24, 25, 26, 27, 28, 29)

yy=(1, 28)(2, 27)(3, 26)(4, 25)(5, 24)(6, 23)(7, 22)(8, 21)(9, 20)(10, 19)(11,18)(12, 17)(13, 16)(14, 15).

We must induce a linear character from a subgroup H of G to create a monomial presentation. To insure we get an irreducible character, we must choose a subgroup with index matching the degree of an irreducible character of G. The characters of G are $\chi.1, \chi.2, \chi.3, \chi.4, \chi.5, \chi.6, \chi.7, \chi.8, \chi.9, \chi.10, ...and\chi.16.$

We have a subgroup H of order 29, where $H = \{1, 28, 26, 24, 22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3\}$. So that $\frac{|G|}{|H|} = \frac{58}{29} = 2$ and $[G : H] = 2$. Since the index of the group is 2. If a matrix representation exists, it will be represented by a 2×2 matrices.

Conjugacy Classes Representative of The Group G are:

$C_1 = Id(G)$

$C_2 = (1, 20)(2, 19)(3, 18)(4, 17)(5, 16)(6, 15)(7, 14)(8, 13)(9, 12)(10, 11)(21, 29)(22, 28)(23, 27)(24, 26)$

$C_3 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29)$

$C_4 = (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28)$

$C_5 = (1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 3, 6, 9, 12, 15, 18, 21, 24, 27)$

$C_6 = (1, 5, 9, 13, 17, 21, 25, 29, 4, 8, 12, 16, 20, 24, 28, 3, 7, 11, 15, 19, 23, 27, 2, 6, 10, 14, 18, 22, 26)$

$C_7 = (1, 6, 11, 16, 21, 26, 2, 7, 12, 17, 22, 27, 3, 8, 13, 18, 23, 28, 4, 9, 14, 19, 24, 29, 5, 10, 15, 20, 25)$

$C_8 = (1, 7, 13, 19, 25, 2, 8, 14, 20, 26, 3, 9, 15, 21, 27, 4, 10, 16, 22, 28, 5, 11, 17, 23, 29, 6, 12, 18, 24)$

$C_9 = (1, 8, 15, 22, 29, 7, 14, 21, 28, 6, 13, 20, 27, 5, 12, 19, 26, 4, 11, 18, 25, 3, 10, 17, 24, 2, 9, 16, 23)$

$C_{10} = (1, 9, 17, 25, 4, 12, 20, 28, 7, 15, 23, 2, 10, 18, 26, 5, 13, 21, 29, 8, 16, 24, 3, 11, 19, 27, 6, 14, 22)$

$C_{11} = (1, 10, 19, 28, 8, 17, 26, 6, 15, 24, 4, 13, 22, 2, 11, 20, 29, 9, 18, 27, 7, 16, 25, 5, 14, 23, 3, 12, 21)$

$C_{12} = (1, 11, 21, 2, 12, 22, 3, 13, 23, 4, 14, 24, 5, 15, 25, 6, 16, 26, 7, 17, 27, 8, 18, 28, 9, 19, 29, 10, 20)$

$C_{13} = (1, 12, 23, 5, 16, 27, 9, 20, 2, 13, 24, 6, 17, 28, 10, 21, 3, 14, 25, 7, 18, 29, 11, 22, 4, 15, 26, 8, 19)$

$C_{14} = (1, 13, 25, 8, 20, 3, 15, 27, 10, 22, 5, 17, 29, 12, 24, 7, 19, 2, 14, 26, 9, 21, 4, 16, 28, 11, 23, 6, 18)$

$C_{15} = (1, 14, 27, 11, 24, 8, 21, 5, 18, 2, 15, 28, 12, 25, 9, 22, 6, 19, 3, 16, 29, 13, 26, 10, 23, 7, 20, 4, 17)$

$C_{16} = (1, 15, 29, 14, 28, 13, 27, 12, 26, 11, 25, 10, 24, 9, 23, 8, 22, 7, 21, 6, 20, 5, 19, 4, 18, 3, 17, 2, 16)$

Conjugacy Classes Representative of The Group H are:

$D_1 = Id(H)$

$D_2 = (1, 28, 26, 24, 22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3)$

$D_3 = (1, 26, 22, 18, 14, 10, 6, 2, 27, 23, 19, 15, 11, 7, 3, 28, 24, 20, 16, 12, 8, 4, 29, 25, 21, 17,$

$13, 9, 5)$

$D_4 = (1, 24, 18, 12, 6, 29, 23, 17, 11, 5, 28, 22, 16, 10, 4, 27, 21, 15, 9, 3, 26, 20, 14, 8, 2, 25, 19, 13, 7)$

$D_5 = (1, 22, 14, 6, 27, 19, 11, 3, 24, 16, 8, 29, 21, 13, 5, 26, 18, 10, 2, 23, 15, 7, 28, 20, 12, 4, 25, 17, 9)$

$D_6 = (1, 20, 10, 29, 19, 9, 28, 18, 8, 27, 17, 7, 26, 16, 6, 25, 15, 5, 24, 14, 4, 23, 13, 3, 22, 12, 2, 21, 11)$

$D_7 = (1, 18, 6, 23, 11, 28, 16, 4, 21, 9, 26, 14, 2, 19, 7, 24, 12, 29, 17, 5, 22, 10, 27, 15, 3, 20, 8, 25, 13)$

$D_8 = (1, 16, 2, 17, 3, 18, 4, 19, 5, 20, 6, 21, 7, 22, 8, 23, 9, 24, 10, 25, 11, 26, 12, 27, 13, 28, 14, 29, 15)$

$D_9 = (1, 14, 27, 11, 24, 8, 21, 5, 18, 2, 15, 28, 12, 25, 9, 22, 6, 19, 3, 16, 29, 13, 26, 10, 23, 7, 20, 4, 17)$

$D_{10} = (1, 12, 23, 5, 16, 27, 9, 20, 2, 13, 24, 6, 17, 28, 10, 21, 3, 14, 25, 7, 18, 29, 11, 22, 4, 15, 26, 8, 19)$

$D_{11} = (1, 10, 19, 28, 8, 17, 26, 6, 15, 24, 4, 13, 22, 2, 11, 20, 29, 9, 18, 27, 7, 16, 25, 5, 14, 23, 3, 12, 21)$

$D_{12} = (1, 8, 15, 22, 29, 7, 14, 21, 28, 6, 13, 20, 27, 5, 12, 19, 26, 4, 11, 18, 25, 3, 10, 17, 24, 2, 9, 16, 23)$

$D_{13} = (1, 6, 11, 16, 21, 26, 2, 7, 12, 17, 22, 27, 3, 8, 13, 18, 23, 28, 4, 9, 14, 19, 24, 29, 5, 10, 15, 20, 25)$

$D_{14} = (1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 3, 6, 9, 12, 15, 18, 21, 24, 27)$

$D_{15} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29)$

$D_{16} = (1, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$

$D_{17} = (1, 27, 24, 21, 18, 15, 12, 9, 6, 3, 29, 26, 23, 20, 17, 14, 11, 8, 5, 2, 28, 25, 22, 19, 16, 13, 10, 7, 4)$

$D_{18} = (1, 25, 20, 15, 10, 5, 29, 24, 19, 14, 9, 4, 28, 23, 18, 13, 8, 3, 27, 22, 17, 12, 7, 2, 26, 21, 16, 11, 6)$

$D_{19} = (1, 23, 16, 9, 2, 24, 17, 10, 3, 25, 18, 11, 4, 26, 19, 12, 5, 27, 20, 13, 6, 28, 21, 14, 7, 29,$

$22, 15, 8)$

$D_{20} = (1, 21, 12, 3, 23, 14, 5, 25, 16, 7, 27, 18, 9, 29, 20, 11, 2, 22, 13, 4, 24, 15, 6, 26, 17, 8, 28, 19, 10)$

$D_{21} = (1, 19, 8, 26, 15, 4, 22, 11, 29, 18, 7, 25, 14, 3, 21, 10, 28, 17, 6, 24, 13, 2, 20, 9, 27, 16, 5, 23, 12)$

$D_{22} = (1, 17, 4, 20, 7, 23, 10, 26, 13, 29, 16, 3, 19, 6, 22, 9, 25, 12, 28, 15, 2, 18, 5, 21, 8, 24, 11, 27, 14)$

$D_{23} = (1, 15, 29, 14, 28, 13, 27, 12, 26, 11, 25, 10, 24, 9, 23, 8, 22, 7, 21, 6, 20, 5, 19, 4, 18, 3, 17, 2, 16)$

$D_{24} = (1, 13, 25, 8, 20, 3, 15, 27, 10, 22, 5, 17, 29, 12, 24, 7, 19, 2, 14, 26, 9, 21, 4, 16, 28, 11, 23, 6, 18)$

$D_{25} = (1, 11, 21, 2, 12, 22, 3, 13, 23, 4, 14, 24, 5, 15, 25, 6, 16, 26, 7, 17, 27, 8, 18, 28, 9, 19, 29, 10, 20)$

$D_{26} = (1, 9, 17, 25, 4, 12, 20, 28, 7, 15, 23, 2, 10, 18, 26, 5, 13, 21, 29, 8, 16, 24, 3, 11, 19, 27, 6, 14, 22)$

$D_{27} = (1, 7, 13, 19, 25, 2, 8, 14, 20, 26, 3, 9, 15, 21, 27, 4, 10, 16, 22, 28, 5, 11, 17, 23, 29, 6, 12, 18, 24)$

$D_{28} = (1, 5, 9, 13, 17, 21, 25, 29, 4, 8, 12, 16, 20, 24, 28, 3, 7, 11, 15, 19, 23, 27, 2, 6, 10, 14, 18, 22, 26)$

$D_{29} = (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28)$

7.2.6 The Induction Process

Now, we will induce the character $\chi.2$ of H up to $\chi.5$ of G to obtain the character of G by using the tables $\chi.5$ of G and $\chi.2$ of H using our definition of induction as follows:

$$\phi_{\alpha}^{G} = \tfrac{n}{h_{\alpha}} \textstyle\sum_{w \in C_{\alpha} \cap H} \phi(w)$$

$\phi_1^G = \frac{2}{h_1=1} \sum_{w \in C_1 \cap H} \phi(e)$ =2.

$\phi_2^G = \frac{2}{h_2=29} \sum_{w \in C_2 \cap H} \phi(0)$=0.

$\phi_3^G = \frac{2}{h_3=2}\sum_{w\in C_3\cap H}\phi(1,2,3,4,5,6,7,8,9,10,11,12,13,$ $14,15,16,17,18,19,$ $20,21,22,23,24,25,26,27,28,29)+\phi(1,29,28,27,26,25,24,23,22,21,20,19,18,17,16,$ $15,14,13,12,11,10,9,8,7,6,5,4,3,2)$=($Z1\#14+Z1\#15$)=29+57=86=$Z1\#7$.

$\phi_4^G = \frac{2}{h_4=2}\sum_{w\in C_4\cap H}\phi(1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,2,4,6,8,$ $10,12,14,16,18,20,22,24,26,28)$+$\phi(1,28,26,24,22,20,18,16,14,12,10,8,6,4,2,29,27,$ $25,23,21,19,17,15,13,11,9,7,5,3))$=($Z1\#28$+Z1)=15+4=19=$Z1\#14$.

$\phi_5^G = \frac{2}{h_5=2}\sum_{w\in C_5\cap H}\phi(1,4,7,10,13,16,19,22,25,28,2,5,8,11,14,17,20,23,26,$ $29,3,6,9,12,15,18,21,24,27)+\phi(1,27,24,21,18,15,12,9,6,3,29,26,23,20,17,14,$ $11,8,5,2,28,25,22,19,16,13,10,7,4)$=($Z1\#13+Z1\#16$)=22+51=73=$Z1\#8$.

$\phi_6^G = \frac{2}{h_6=2}\sum_{w\in C_6\cap H}\phi(1,29,28,27,26,25,24,23,22,21,20,19,18,17,16,15,14,$ $13,12,11,10,9,8,7,6,5,4,3,2)$=$Z1$.

$\phi_7^G = \frac{2}{h_7=2}\sum_{w\in C_7\cap H}\phi(1,6,11,16,21,26,2,7,12,17,22,27,3,8,13,18,23,28,$ $4,9,14,1924,29,5,10,15,20,25)+\phi(1,25,20,15,10,5,29,24,19,14,9,4,28,23,18,13,$ $8,3,27,22,17,12,7,2,26,21,16,11,6)$ =$Z1\#12$+$Z1\#17$=35+27=62=$Z1\#6$.

$\phi_8^G = \frac{2}{h_8=2}\sum_{w\in C_8\cap H}\phi(1,7,13,19,25,2,8,14,20,26,3,9,15,21,27,4,10,16,22,$ $28,5,11,17,23,29,6,12,18,24)+\phi(1,24,18,12,6,29,23,17,11,5,28,22,16,10,4,27,$ $21,15,9,3,26,20,14,8,2,25,19,13,7)$=$Z1\#26$+$Z1\#3$=12+5=17=$Z1\#13$.

$\phi_9^G = \frac{2}{h_9=2}\sum_{w\in C_9\cap H}\phi(1,8,15,22,29,7,14,21,28,6,13,20,27,5,12,19,26,4,$ $11,18,25,3,10,17,24,2,9,16,23)+\phi(1,23,16,9,2,24,17,10,3,25,18,11,4,26,19,12,$ $5,27,20,13,6,28,21,14,7,29,22,15,8)$=$Z1\#11$+$Z1\#18$=53+49=102=$Z1\#9$.

$\phi_{10}^G = \frac{2}{h_{10}=2}\sum_{w\in C_{10}\cap H}\phi(1,9,17,25,4,12,20,28,7,15,23,2,10,18,26,5,$ $13,21,29,8,16,24,3,11,19,27,6,14,22)+\phi(1,22,14,6,27,19,11,3,24,16,8,29,$ $21,13,5,26,18,10,2,23,15,7,28,20,12,4,25,17,9)$=$Z1\#25$+$Z1\#4$=3+20=23=

$Z1\#2$.

$\phi_{11}^{G} = \frac{2}{h_{11}=2} \sum_{w\in C_{11}\cap H} \phi(1, 10, 19, 28, 8, 17, 26, 6, 15, 24, 4, 13, 22, 2, 11, 20, 29, 9,$
$18, 27, 7, 16, 25, 5, 14, 23, 3, 12, 21) + \phi(1, 21, 12, 3, 23, 14, 5, 25, 16, 7, 27, 18, 9, 29, 20, 11,$
$2, 22, 13, 4, 24, 15, 6, 26, 17, 8, 28, 19, 10)$=$Z1\#10$+$Z1\#19$=28+19=47=$Z1\#5$.

$\phi_{12}^{G} = \frac{2}{h_{12}=2} \sum_{w\in C_{12}\cap H} \phi(1, 11, 21, 2, 12, 22, 3, 13, 23, 4, 14, 24, 5, 15, 25, 6, 16, 26,$
$7, 17, 27, 8, 18, 28, 9, 19, 29, 10, 20) + \phi(1, 20, 10, 29, 19, 9, 28, 18, 8, 27, 17, 7, 26, 16, 6, 25,$
$15, 5, 24, 14, 4, 23, 13, 3, 22, 12, 2, 21, 11)$=$Z1\#24$+$Z1\#5$=45+21=66=$Z1\#12$.

$\phi_{13}^{G} = \frac{2}{h_{13}=2} \sum_{w\in C_{13}\cap H} \phi(1, 12, 23, 5, 16, 27, 9, 20, 2, 13, 24, 6, 17, 28, 10, 21, 3, 14,$
$25, 7, 18, 29, 11, 22, 4, 15, 26, 8, 19) + \phi(1, 19, 8, 26, 15, 4, 22, 11, 29, 18, 7, 25, 14, 3, 21, 10,$
$28, 17, 6, 24, 13, 2, 20, 9, 27, 16, 5, 23, 12)$=$Z1\#9$+$Z1\#20$=7+17=24=$Z1\#10$.

$\phi_{14}^{G} = \frac{2}{h_{14}=2} \sum_{w\in C_{14}\cap H} \phi(1, 13, 25, 8, 20, 3, 15, 27, 10, 22, 5, 17, 29, 12, 24, 7, 19, 2,$
$14, 26, 9, 21, 4, 16, 28, 11, 23, 6, 18) + \phi(1, 18, 6, 23, 11, 28, 16, 4, 21, 9, 26, 14, 2, 19, 7, 24,$
$12, 29, 17, 5, 22, 10, 27, 15, 3, 20, 8, 25, 13)$=$Z1\#23$+$Z1\#6$=26+25=51=$Z1\#3$.

$\phi_{15}^{G} = \frac{2}{h_{15}=2} \sum_{w\in C_{15}\cap H} \phi(1, 14, 27, 11, 24, 8, 21, 5, 18, 2, 15, 28, 12, 25, 9, 22, 6, 19,$
$3, 16, 29, 13, 26, 10, 23, 7, 20, 4, 17) + \phi(1, 17, 4, 20, 7, 23, 10, 26, 13, 29, 16, 3, 19, 6, 22, 9,$
$25, 12, 28, 15, 2, 18, 5, 21, 8, 24, 11, 27, 14)$=$Z1\#8$+$Z1\#21$=46+9=55=$Z1\#4$.

$\phi_{16}^{G} = \frac{2}{h_{16}=2} \sum_{w\in C_{16}\cap H} \phi(1, 15, 29, 14, 28, 13, 27, 12, 26, 11, 25, 10, 24, 9, 23, 8, 22, 7,$
$21, 6, 20, 5, 19, 4, 18, 3, 17, 2, 16) + \phi(1, 16, 2, 17, 3, 18, 4, 19, 5, 20, 6, 21, 7, 22, 8, 23, 9,$
$24, 10, 25, 11, 26, 12, 27, 13, 28, 14, 29, 15)$=$Z1\#22$+$Z1\#7$=36+41=77=$Z1\#11$.
Therefore, we have verified that ϕ_{H}^{G}=2, 0, $Z1\#7$, $Z1\#14$, $Z1\#8$,$Z1$, $Z1\#6$, $Z1\#13$, $Z1\#9$, $Z1\#2$, $Z1\#5$, $Z1\#12$, $Z1\#10$, $Z1\#3$, $Z1\#4$, and $Z1\#11$.
So, χ.2 of H induces χ.5 of G.

Table 7.5: $X.2$ of H

Class	Size	Order	ϕ^H
D.1	1	1	1
D.2	1	29	$Z1 = 4$
D.3	1	29	$Z1\#2 = 16$
D.4	1	29	$Z1\#3 = 5$
D.5	1	29	$Z1\#4 = 20$
D.6	1	29	$Z1\#5 = 21$
D.7	1	29	$Z1\#6 = 25$
D.8	1	29	$Z1\#7 = 41$
D.9	1	29	$Z1\#8 = 46$
D.10	1	29	$Z1\#9 = 7$
D.11	1	29	$Z1\#10 = 28$
D.12	1	29	$Z1\#11 = 53$
D.13	1	29	$Z1\#12 = 35$
D.14	1	29	$Z1\#13 = 22$
D.15	1	29	$Z1\#14 = 29$
D.16	1	29	$Z1\#15 = 57$
D.17	1	29	$Z1\#16 = 51$
D.18	1	29	$Z1\#17 = 27$
D.19	1	29	$Z1\#18 = 49$
D.20	1	29	$Z1\#19 = 19$
D.21	1	29	$Z1\#20 = 17$
D.22	1	29	$Z1\#21 = 9$
D.23	1	29	$Z1\#22 = 36$
D.24	1	29	$Z1\#23 = 26$
D.25	1	29	$Z1\#24 = 45$
D.26	1	29	$Z1\#25 = 3$
D.27	1	29	$Z1\#26 = 12$
D.28	1	29	$Z1\#27 = 48$
D.29	1	29	$Z1\#28 = 15$

7.2.7 Verifying the Monomial Representation

We have a linear character of the subgroup H of index n in G .

Thus, we will let $G = Ht_1 \cup Ht_2$, Where, $t_1 = e$ and $t_2 = (1,28)(2,27)(3,26)(4,25)$ $(5,24)(6,23)(7,22)(8,21)(9,20)(10,19)(11,18)(12,17)(13,16)(14,15)$. We are now in a position to give the monomial representation of the progenitor $59^{*2} :_m D_{29}$.

Table 7.5: $X.5$ of G

Class	Size	Order	ϕ^G
C.1	1	1	2
C.2	29	2	0
C.3	2	29	$Z1\#7 = 86$
C.4	2	29	$Z1\#14 = 19$
C.5	2	29	$Z1\#8 = 73$
C.6	2	29	$Z1 = 57$
C.7	2	29	$Z1\#6 = 62$
C.8	2	29	$Z1\#13 = 17$
C.9	2	29	$Z1\#9 = 102$
C.10	2	29	$Z1\#2 = 23$
C.11	2	29	$Z1\#5 = 47$
C.12	2	29	$Z1\#12 = 66$
C.13	2	29	$Z1\#10 = 24$
C.14	2	29	$Z1\#3 = 51$
C.15	2	29	$Z1\#4 = 55$
C.16	2	29	$Z1\#11 = 77$

$$A(xx)=\begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) \end{bmatrix}$$

We now investigate the monomial representation has the following generators:

$$A(xx)=\begin{bmatrix} 29 & 0 \\ 0 & 57 \end{bmatrix} \text{and } A(yy)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A[xx]: Row 1

•$\phi(t_1xt_1^{-1}) = \phi((1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29))$=29.

• $\phi(t_1xt_2^{-1}) = \phi((1,27)(2,26)(3,25)(4,24)(5,23)(6,22)(7,21)(8,20)(9,19)(10,18)(11,17)(12,16)(13,15)(28,29))$=0.

A[xx]: Row 2

•$\phi(t_2xt_1^{-1}) = \phi((1,29)(2,28)(3,27)(4,26)(5,25)(6,24)(7,23)(8,22)(9,21)(10,20)(11,19)(12,18)(13,17)(14,16))$=0.

•$\phi(t_2xt_2^{-1}) = \phi((1,28)(2,27)(3,26)(4,25)(5,24)(6,23)(7,22)(8,21)(9,20)(10,19)(11,18)$
$(12,17)(13,16)(14,15)(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,$
$23,24,25,26,27,28,29)((1,28)(2,27)(3,26)(4,25)(5,24)(6,23)(7,22)(8,21)(9,20)(10,19)$
$(11,18)(12,17)(13,16)(14,15))^{-1})$=57.

$\Rightarrow A(xx)=\begin{bmatrix} 29 & 0 \\ 0 & 57 \end{bmatrix}$

A[yy]: Row 1

•$\phi(t_1yt_1^{-1}) = \phi((1,28)(2,27)(3,26)(4,25)(5,24)(6,23)(7,22)(8,21)(9,20)(10,19)$
$(11,18)(12,17)(13,16)(14,15))$=0.

• $\phi(t_1yt_2^{-1}) = \phi(e)$=1.

A[yy]: Row 2

• $\phi(t_2yt_1^{-1}) = \phi(e) = 1.$

• $\phi(t_2yt_2^{-1}) = \phi((1,28)(2,27)(3,26)(4,25)(5,24)(6,23)(7,22)(8,21)(9,20)$
$(10,19)(11,18)(12,17)$
$(13,16)(14,15))$=0.

$\Rightarrow A(yy)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

7.2.8 Construction a Permutation Representation:

We worked in Z_{59} on matrices of degree 2×2 which implies we are producing a $59^{*2} :_m D_{29}$ progenitor permutation representation. We want to represent x and y as permutation to do so we see that $a_{i,j} = n$ if the automorphism takes $t_i \Longrightarrow t_j^n$. There are 59^{*2} is a free product of 2 cyclic groups of order 59. Thus, we will construct a table with 2 $t_i's$ of order 59 labeled from 1.....116.

$$A(xx)=\begin{bmatrix} 29 & 0 \\ 0 & 57 \end{bmatrix}$$

(Where a_{11}=29, and a_{22}=57). Therefore, $t_1 \longrightarrow t_1^{29}$,and $t_2 \longrightarrow t_2^{57}$.

Table 7.7: Automorphism of A(xx)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7
t_1^{29}	t_2^{57}	t_1^{58}	t_2^{55}	t_1^{28}	t_2^{53}	t_1^{57}	t_2^{51}	t_1^{27}	t_2^{49}	t_1^{56}	t_2^{47}	t_1^{26}	t_2^{45}
57	114	115	110	55	106	113	102	53	98	111	94	51	90

15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}	t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}
t_1^{55}	t_2^{43}	t_1^{25}	t_2^{41}	t_1^{54}	t_2^{39}	t_1^{24}	t_2^{37}	t_1^{53}	t_2^{35}	t_1^{23}	t_2^{33}	t_1^{52}	t_2^{31}	t_1^{22}
109	86	49	82	107	78	47	74	105	70	45	66	103	62	43

30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}
t_2^{29}	t_1^{51}	t_2^{27}	t_1^{21}	t_2^{25}	t_1^{50}	t_2^{23}	t_1^{20}	t_2^{21}	t_1^{49}	t_2^{19}	t_1^{19}	t_2^{17}	t_1^{48}	t_2^{15}
58	101	54	41	50	99	46	39	42	97	38	37	34	95	30

45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}	t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}
t_1^{18}	t_2^{13}	t_1^{47}	t_2^{11}	t_1^{17}	t_2^{9}	t_1^{46}	t_2^{7}	t_1^{16}	t_2^{5}	t_1^{45}	t_2^{3}	t_1^{15}	t_2	t_1^{44}
35	26	93	22	33	18	91	14	31	10	89	6	29	2	87

	60	61	62	63	64	65	66	67	68	69	70	71	72	73
	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}	t_1^{37}
	t_2^{58}	t_1^{14}	t_2^{56}	t_1^{43}	t_2^{54}	t_1^{13}	t_2^{52}	t_1^{42}	t_2^{50}	t_1^{12}	t_2^{48}	t_1^{41}	t_2^{46}	t_1^{11}
116	27	112	85	108	25	104	83	100	23	96	81	92	21	

74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89
t_2^{37}	t_1^{38}	t_2^{38}	t_1^{39}	t_2^{39}	t_1^{40}	t_2^{40}	t_1^{41}	t_2^{41}	t_1^{42}	t_2^{42}	t_1^{43}	t_2^{43}	t_1^{44}	t_2^{44}	t_1^{45}
t_2^{44}	t_1^{40}	t_2^{42}	t_1^{10}	t_2^{40}	t_1^{39}	t_2^{38}	t_1^{9}	t_2^{36}	t_1^{38}	t_2^{34}	t_1^{8}	t_2^{32}	t_1^{37}	t_2^{30}	t_1^{7}
88	79	84	19	80	77	76	17	72	75	68	15	64	73	60	13

90	91	92	93	94	95	96	97	98	99	100	101	102	103	104
t_2^{45}	t_1^{46}	t_2^{46}	t_1^{47}	t_2^{47}	t_1^{48}	t_2^{48}	t_1^{49}	t_2^{49}	t_1^{50}	t_2^{50}	t_1^{51}	t_2^{51}	t_1^{52}	t_2^{52}
t_2^{28}	t_1^{36}	t_2^{26}	t_1^{6}	t_2^{24}	t_1^{35}	t_2^{22}	t_1^{5}	t_2^{20}	t_1^{34}	t_2^{18}	t_1^{4}	t_2^{16}	t_1^{33}	t_2^{14}
56	71	52	11	48	69	44	9	40	67	36	7	32	65	28

105	106	107	108	109	110	111	112	113	114	115	116
t_1^{53}	t_2^{53}	t_1^{54}	t_2^{54}	t_1^{55}	t_2^{55}	t_1^{56}	t_2^{56}	t_1^{57}	t_2^{57}	t_1^{58}	t_2^{58}
t_1^{3}	t_2^{12}	t_1^{32}	t_2^{10}	t_1^{2}	t_2^{8}	t_1^{31}	t_2^{6}	t_1	t_2^{4}	t_1^{30}	t_2^{2}
5	24	63	20	3	16	61	12	1	8	59	4

A(xx)=(1,57,29,43,95,69,23,105,5,55,89,13,51,91,71,81,17,49,33,41,37,39,97, 9,53,31,101,7,113) (2,114,8,102,32,54,10,98,40,38,42,34,50,18,82,72,92,52,14,90,56,6, 106,24,70,96 ,44,30,58) (3,115,59,87,73,21,47,93,11,111,61,27,103,65,25,45,35,99,67,83, 75,79,77,19,107,63,85,15,109) (4,110,16,86,64,108,20,78,80,76,84,68,100,36,46,26,66,104, 28,62,112,12,94,48,22,74,88,60,116).

$$A(yy)=\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \end{bmatrix}$$

(Where a_{12}=1, and a_{21}=1). Therefore, $t_1 \longrightarrow t_2$,and $t_2 \longrightarrow t_1$.

Table 7.8: Automorphism of A(yy)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6	t_2^7	t_1^7	t_2^8	t_1^8	t_2^9
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}	t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}
t_1^9	t_2^{10}	t_1^{10}	t_2^{11}	t_1^{11}	t_2^{12}	t_1^{12}	t_2^{13}	t_1^{13}	t_2^{14}	t_1^{14}	t_2^{15}	t_1^{15}	t_2^{16}	t_1^{16}
17	20	19	22	21	24	23	26	25	28	27	30	29	32	31

33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}
t_2^{17}	t_1^{17}	t_2^{18}	t_1^{18}	t_2^{19}	t_1^{19}	t_2^{20}	t_1^{20}	t_2^{21}	t_1^{21}	t_2^{22}	t_1^{22}	t_2^{23}	t_1^{23}	t_2^{24}
34	33	36	35	38	37	40	39	42	41	44	43	46	45	48

48	49	50	51	52	53	54	55	56	57	58	59	60	61	62
t_2^{24}	t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}
t_1^{24}	t_2^{25}	t_1^{25}	t_2^{26}	t_1^{26}	t_2^{27}	t_1^{27}	t_2^{28}	t_1^{28}	t_2^{29}	t_1^{29}	t_2^{30}	t_1^{30}	t_2^{31}	t_1^{31}
47	50	49	52	51	54	53	56	55	58	57	60	59	62	61

63	64	65	66	67	68	69	70	71	72	73	74	75	76	77
t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}	t_1^{37}	t_2^{37}	t_1^{38}	t_2^{38}	t_1^{39}
t_2^{32}	t_1^{32}	t_2^{33}	t_1^{33}	t_2^{34}	t_1^{34}	t_2^{35}	t_1^{35}	t_2^{36}	t_1^{36}	t_2^{37}	t_1^{37}	t_2^{38}	t_1^{38}	t_2^{39}
64	63	66	65	68	67	70	69	72	71	74	73	76	75	78

78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93
t_2^{39}	t_1^{40}	t_2^{40}	t_1^{41}	t_2^{41}	t_1^{42}	t_2^{42}	t_1^{43}	t_2^{43}	t_1^{44}	t_2^{44}	t_1^{45}	t_2^{45}	t_1^{46}	t_2^{46}	t_1^{47}
t_1^{39}	t_2^{40}	t_1^{40}	t_2^{41}	t_1^{41}	t_2^{42}	t_1^{42}	t_2^{43}	t_1^{43}	t_2^{44}	t_1^{44}	t_2^{45}	t_1^{45}	t_2^{46}	t_1^{46}	t_2^{47}
77	80	79	82	81	84	83	86	85	88	87	90	89	92	91	94

94	95	96	97	98	99	100	101	102	103	104	105	106	107
t_2^{47}	t_1^{48}	t_2^{48}	t_1^{49}	t_2^{49}	t_1^{50}	t_2^{50}	t_1^{51}	t_2^{51}	t_1^{52}	t_2^{52}	t_1^{53}	t_2^{53}	t_1^{54}
t_1^{47}	t_2^{48}	t_1^{48}	t_2^{49}	t_1^{49}	t_2^{50}	t_1^{50}	t_2^{51}	t_1^{51}	t_2^{52}	t_1^{52}	t_2^{53}	t_1^{53}	t_2^{54}
93	96	95	98	97	100	99	102	101	104	103	106	105	108

108	109	110	111	112	113	114	115	116
t_2^{54}	t_1^{55}	t_2^{55}	t_1^{56}	t_2^{56}	t_1^{57}	t_2^{57}	t_1^{58}	t_2^{58}
t_1^{54}	t_2^{55}	t_1^{55}	t_2^{56}	t_1^{56}	t_2^{57}	t_1^{57}	t_2^{58}	t_1^{58}
107	110	109	112	111	114	113	116	115

A(yy)=(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)
(21,22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39,40)
(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)(51, 52)(53, 54)(55, 56)(57,58)
(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)(73, 74)(75, 76)(77, 78)
(79, 80)(81, 82)(83, 84)(85, 86)(87, 88)(89, 90)(91, 92)(93, 94)(95, 96)
(97, 98)(99, 100)(101, 102)(103, 104)(105, 106)(107, 108)(109, 110)(111, 112)
(113, 114)(115, 116).

Now, we can create permutations by using our labels for each automorphism.
We will Consider t1 by element 1 from the table.
This produces the permutation (1,57,29,43,95,69,23,105,5,55,89,13,51,91,71,81,17,49,33,41,37,39,97,9,53,31,101,7,113) if we follow each element and its corresponding automorphism number labeling. We can construct our automorphism by repeating the process for x, and y we produce the following permutations:
A(xx)=(1,57,29,43,95,69,23,105,5,55,89,13,51,91,71,81,17,49,33,41,37,39,97,

9,53,31,101,7,113) (2,114,8,102,32,54,10,98,40,38,42,34,50,18,82,72,92,52,14,90,56,6,
106,24,70,96 ,44,30,58) (3,115,59,87,73,21,47,93,11,111,61,27,103,65,25,45,35,99,67,83,
75,79,77,19,107,63,85,15,109) (4,110,16,86,64,108,20,78,80,76,84,68,100,36,46,26,66,104,
28,62,112,12,94,48,22,74,88,60,116).
A(yy)=(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)
(21,22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39,40)
(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)(51, 52)(53, 54)(55, 56)(57,58)
(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)(73, 74)(75, 76)(77, 78)
(79, 80)(81, 82)(83, 84)(85, 86)(87, 88)(89, 90)(91, 92)(93, 94)(95, 96)
(97, 98)(99, 100)(101, 102)(103, 104)(105, 106)(107, 108)(109, 110)(111, 112)
(113, 114)(115, 116).

So, we have completed the process of construction a permutation representation from our matrices.

7.2.9 Creating a Representation of the Progenitor:

Now, we need to find a presentation for our progenitor. We must choose at t to normalizer from our two choices$\langle t_1\rangle * \langle t_2\rangle$. Let t$\sim t_1$ and we must find permutations which normalize $\langle t_1\rangle$ fix the following set
$\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, ..., t_1^{116}, t_2, t_2^2, t_2^3, t_2^4, t_2^5, t_2^6, t_2^7, t_2^8, t_2^9, t_2^{10}, t_2^{11}, t_2^{12}, ..., t_2^{116}\}$.
Therefore, we need to compute the set stabilizer of the set
$\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, ..., t_1^{116}\}$.

```
n1:=Stabiliser(Nm,{ 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21,
23, 25, 27, 29,31, 33, 35,37, 39, 41, 43, 45, 47, 49,
 51, 53, 55,57, 59, 61, 63, 65,67, 69, 71, 73, 75, 77,
 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103,
 105, 107, 109, 111, 113, 115 });

Generators(n1);
(1, 95, 5, 51, 17, 37, 53, 113, 43, 105, 13, 81, 41,
9, 7, 29, 23,89, 71, 33, 97, 101, 57, 69, 55, 91,
49, 39, 31)(2, 32, 40, 50, 92,56, 70, 58, 102, 98,
```

```
34, 72, 90, 24,30, 8, 10, 42, 82, 14, 106, 44,114, 54,
38, 18, 52, 6, 96)(3, 73, 11, 103, 35, 75,107, 109, 87,
93, 27, 45, 83, 19, 15, 59, 47, 61, 25, 67, 77, 85, 115,
21, 111, 65,99, 79, 63)(4, 64, 80, 100, 66, 112, 22,
116, 86, 78, 68, 26, 62, 48, 60, 16, 20,84, 46, 28, 94,
 88, 110, 108, 76, 36, 104, 12, 74)

FPGroup(N);
Finitely presented group on 2 generators
Relations
$.2^2 = Id($)
($.1^-1 * $.2)^2 = Id($)
$.1^-29 = Id($)
```

So we will convert FPGroup(N) to our presentation where .2=y and .1=x . Also, we discover the Stabilizer of $< t_1 >$ in D_{29} is generated by one element which we will label A.

A=(1, 95, 5, 51, 17, 37, 53, 113, 43, 105, 13, 81, 41, 9, 7, 29, 23, 89, 71, 33,
97, 101, 57, 69, 55, 91, 49, 39, 31)(2, 32, 40, 50, 92, 56, 70, 58, 102, 98, 34,
72, 90, 24, 30, 8, 10, 42, 82, 14, 106, 44, 114, 54, 38, 18, 52, 6, 96)
(3, 73, 11, 103, 35, 75, 107, 109, 87, 93, 27, 45, 83, 19, 15, 59, 47, 61, 25, 67,
77, 85, 115, 21, 111, 65, 99, 79, 63)(4, 64, 80, 100, 66, 112, 22, 116, 86,
78, 68, 26, 62, 48, 60, 16, 20, 84, 46, 28, 94, 88, 110, 108, 76, 36, 104, 12, 74).

For a presentation we must convert these permutations into words which we find to be:
x^4=(1, 95, 5, 51, 17, 37, 53, 113, 43, 105, 13, 81, 41, 9, 7, 29, 23, 89, 71, 33,
97, 101, 57, 69, 55, 91, 49, 39, 31)(2, 32, 40, 50, 92, 56, 70, 58, 102, 98, 34, 72, 90, 24,
30, 8, 10, 42, 82, 14, 106, 44, 114, 54, 38, 18, 52, 6, 96)
(3, 73, 11, 103, 35, 75, 107, 109, 87, 93, 27, 45, 83, 19, 15, 59, 47, 61, 25, 67, 77, 85,
115, 21, 111, 65, 99, 79, 63)(4, 64, 80, 100, 66, 112, 22, 116, 86, 78, 68, 26, 62, 48, 60,
16, 20, 84, 46, 28, 94, 88, 110, 108, 76, 36, 104, 12, 74).

As a result, we find that A sends 1 to 95 or t_1 to t_1^{48}. We will add this $t^{x^4} = t^{48}$ to our presentation. Therefore, we obtain the monomial progenitor as follows:

$$59^{*2} :_m D_{29} =< x, y, t | y^2, (x^-1 * y)^2, x^{-29}, t^{59}, t^{x^4} = t^{48}, (t, t^y) >.$$

Chapter 8

Double Coset Enumeration

8.1 Definition for Double Coset Enumeration

Definition 8.1. *Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$.[Rot95]*

Definition 8.2. *Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St. [Rot95]*

Definition 8.3. *Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G. [Rot95]*

Definition 8.4. *Let N be a group. The* **point stabilizer** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s. [Rot95]}$$

Definition 8.5. *Let N be a group. The* **coset stabiliser** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s. [Rot95]}$$

Theorem 8.6. *The* **Number of single cosets in** *NwN The above definition gives, $N^w = \{n \in N | Nwn = Nw\}$ =$\{n \in N | (Nw)^n = Nw\}$ $= N \cap N^w$. The number of single cosets in NwN is given by $[N : N^{(w)}]$. [Rot95]*

Definition 8.7. *Let G be a group and X be a G-set. For $x \in X$, the set $x^G = \{x^g | g \in G\}$ is a* **G-Orbit***. [Rot95]*

Definition 8.8. *Let G be a group and X be a G-set. X is* **transitive** *if for all x,$y \in X$ there exists a $g \in G$ such that $y = gx$. [Rot95]*

Definition 8.9. *Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$. [Rot95]*

8.2 Double Coset Enumeration of $2^4 : S_3 \times C_2$ over $(2 \times 4 : 2)$

Consider the group G= $\frac{2^{*8}:(2\times 4:2)}{(xt)^3[wxyt]^3=e} \cong (2^4 : S_3 \times C_2)$ where N=$(2 \times 4 : 2)$. A symmetric presentation of G is $G = Group < w, x, y, z, t | w^2, x^2, y^4, z^2, (w * y^{-1})^2, x * y^{-1} * x * y, y^{-2} * z, w * x * w * x * z, t^2, (t, w), (x * t)^3, (w * x * y * t)^3 >$.

Where $w \sim (2, 5)(3, 7)$, $x \sim (1, 2)(3, 6)(4, 5)(7, 8)$, $y \sim (1, 3, 4, 7)(2, 6, 5, 8)$ and $z \sim (1, 4)(2, 5)(3, 7)(6, 8)$

First, we are going to expand our relations.

The first relation is

$$(x * t)^3 = e$$
$$(x * t_1)^3 = e \text{ since } t \sim t_1$$
$$x^3 * t_1^{x^2} * t_1^{x} * t_1 = e$$
$$x^3 * t_1^{e} * t_2 * t_1 = e$$
$$x^3 * t_1 * t_2 = t_1$$

So, we have $t_1t_2 \sim t_1$.

The second relation is

$$(w * x * y * t)^3 = e$$
$$(w * x * y * t_1)^3 = e$$
$$(w * x * y)^3 * t_1^{(w*x*y)^2} * t_1^{w*x*y} * t_1 = e$$
$$(w * x * y)^3 * t_4 * t_6 * t_1 = e$$
$$(w * x * y)^3 * t_4 * t_6 = t_1$$

So, we have $t_4t_6 \sim t_1$.

Moreover, if we conjugate the previous relations by all elements in N=$\{e, (2,5)(3,7), (1,7)(2,6)(3,4)(5,8), (1,6,4,8)(2,3,5,7),$
$(1,7,4,3)(2,8,5,6), (1,5)(2,4)(3,8)(6,7),$
$(1,8,4,6)(2,3,5,7), (1,5,4,2)(3,6,7,8), (1,2,4,5)(3,8,7,6),$
$(1,6,4,8)(2,7,5,3), (1,4)(2,5)(3,7)(6,8), (1,3,4,7)(2,6,5,8),$
$(1,8,4,6)(2,7,5,3), (1,2)(3,6)(4,5)(7,8), (1,3)(2,8)(4,7)(5,6), (1,4)(6,8)\}$.
So, we get theses relations by conjugate the first relation by all elements of N

$$12 \cong 1,\ 15 \cong 1,\ 76 \cong 7,\ 63 \cong 6,\ 78 \cong 7,\ 54 \cong 5,\ 83 \cong 8,\ 51 \cong 5,\ 24 \cong 2,\ 67 \cong 6,\ 45 \cong 4,\ 36 \cong 3,\ 87 \cong 8,\ 21 \cong 2,\ 38 \cong 3,\ 42 \cong 4.$$

Also, we will get theses relations by conjugate the second relation by all elements of N

$$46 \cong 1,\ 32 \cong 7,\ 84 \cong 6,\ 27 \cong 5,\ 61 \cong 8,\ 53 \cong 2,\ 18 \cong 4,\ 75 \cong 3.$$

We have to find the total number of cosets of N in G. This is the index of N in G. The index will be the order of G divided by the order of N. $\frac{|G|}{|N|}$=$\frac{192}{16}$=12 single cosets and we have 4 Double cosets.

```
#DoubleCosets(G,sub<G|x,y>,sub<G|w,x,y,z>);
4
```

Double Cosets

Double Coset [*] : We start with the double coset NeN, where e is the word of length zero denote by [*]. We have $NeN = \{n \in N|Nen\}$=$\{Ne\}$=$\{N\}$. So, the double coset NeN consist of single coset N. Thus, the number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{16}{16}$=1. Since N is transitive on $\{1,2,3,4,5,6,7,8\}$ and the orbit of N on $\{1,2,3,4,5,6,7,8\}$ is $\{1,2,3,4,5,6,7,8\}$. Now, we will determine the double coset to which Nt_i belongs, where $i \in \{1,2,3,4,5,6,7,8\}$. We consider i=1, so Nt_1 is a representative coset, and hence we have a new double coset Nt_1N which is denote by [1].

Double Coset [1] : We consider the double coset NwN, where w is a word of length one. $Nt_1N = \{n \in N|Nt_1n\}$=$\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8\}$.
We consider the coset stabilizer $N^{(1)}$, note that the coset stabilizer of Nt_1 is equal to the point stabilizer N^1. $N^{(1)} = \{n \in N|Nt_1^n = t_1\} \geq < (2,5)(3,7) >$.
Since $|N^{(1)}|$=2 then the number of single cosets in [1] is $\frac{|N|}{|N^{(1)}|}$=$\frac{16}{2}$=8.

Now, the orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are $\{1\}$,$\{4\}$,$\{6\}$,$\{8\}$,$\{2, 5\}$, and $\{3, 7\}$.
We choose a representative from each orbit.
If we select t_1 from the orbit $\{1\}$, t_4 from the orbit $\{4\}$, t_6 from the orbit $\{6\}$, t_8 from the orbit $\{8\}$, t_2 from the orbit $\{2, 5\}$, and t_7 from the orbit $\{3, 7\}$, we notice the following:

- Nt_1t_1=Nt_1^2=N $\in$ [*]

This will collapse and hence it goes back to the double coset[*] since t_1^2=e.

- Nt_1t_4=Nt_1t_4 $\in$ [14]

This is a new double coset, which will extend the Cayley diagram from [1] to [14]. Since there is one element in this orbit, there is one t_i that extend [1] to [14].

- Nt_1t_6=Nt_4 $\in$ [1]

By the second relation $(w * x * y)^3 * t_4 * t_6 = t_1$ if we add t_6 to both sides, we will get that

$$(w * x * y)^3 * t_4 * t_6 * t_6 = t_1 * t_6$$
$$(w * x * y)^3 * t_4 * e = t_1 * t_6$$
$$(w * x * y)^3 * t_4 = t_1 * t_6.$$
$$\text{So, } t_1t_6 \sim t_1 \in [1].$$

Since there is one element in the orbit $\{6\}$, so one element will loop back into the double coset [1].

- Nt_1t_8=Nt_4 $\in$ [1]

Because if we conjugate the second relation $t_4 * t_6 = t_1$ by (1,4)(2,5)(3,7)(6,8)$\in N$, we will get on $t_1 * t_8 = t_4$.
Since there is one element in the orbit $\{8\}$, so one element will loop back into the double coset [1].

- Nt_1t_2=Nt_1 $\in$ [1]

By the first relation t_1t_2=x^3t_1. Since there are two elements in the orbit $\{2, 5\}$, so two elements will loop back into the double coset [1].

- Nt_1t_7=Nt_1t_7 $\in$ [17]

This is a new double coset, which will extend the Cayley diagram from [1] to [17]. Since there is two elements in this orbit $\{3, 7\}$, there are two t_is that extend [1] to [17].
Thus, t_1 takes [1] to [*] and t_2, t_8, t_6, t_5 take [1] to itself.

Double Coset [14] : We are a new double coset [14]. $Nt_1t_4N = \{n \in N | N(t_1t_4)n\}$. Now, we will determine the coset stabilizer $N^{(14)}$, so the coset stabilizer of Nt_1t_4 is given by: $N^{(14)} = \{n \in N | N(t_1t_4)^n = Nt_1t_4\} \geq < (2,5)(3,7), (1,3,4,7)(2,6,5,3) >$.
If we conjugate t_1t_4 by all elements of N, we notice that we have these single cosets are equal to each other $14 \sim 73 \sim 68 \sim 52 \sim 86 \sim 25 \sim 41 \sim 37$. The number of single cosets in the double coset [14] is $\frac{|N|}{|N^{(14)}|}$=$\frac{16}{16}$=1 a single coset in [14].
We can see that the orbits of $N^{(14)}$ on $\{1,2,3,4,5,6,7,8\}$ is $\{1,2,3,4,5,6,7,8\}$. We now take the representative 4 of the orbit $\{1,2,3,4,5,6,7,8\}$ and determine that $Nt_1t_4t_4 = Nt_1 \in [1]$. As a result, all of eight t_i take [14] to 1.

Double Coset [17] : We are a new double coset [17]. $Nt_1t_7N = \{n \in N | N(t_1t_7)n\}$. Now, we will determine the coset stabilizer $N^{(17)}$, so the coset stabilizer of Nt_1t_7 is given by: $N^{(17)} = \{n \in N | N(t_1t_7)^n = Nt_1t_7\} \geq < (1,4)(2,5)(3,7)(6,8) >$.
When we conjugate t_1t_7 by all elements of N, It gives us a list of 8 single cosets, they have equal names $17 \sim 43 \sim 26 \sim 34 \sim 58 \sim 62 \sim 71 \sim 85$. The number of single cosets in the double coset [17] is $\frac{|N|}{|N^{(17)}|}$=$\frac{16}{8}$=2 a single coset in [17]. Next, we can see that the orbits of $N^{(17)}$ on $\{1,2,3,4,5,6,7,8\}$ is $\{1,2,3,4,5,6,7,8\}$. We now take the representative 7 of the orbit $\{1,2,3,4,5,6,7,8\}$ and determine that $Nt_1t_7t_7 = Nt_1 \in [1]$. As a result, all of eight t_i take [17] to 1.
Finally, we can construct the Cayley diagram. Since the set of right cosets are closed under right multiplication by t_i where i=1,2,3,4,5,6,7,8. We can find the index of N in G. We conclude that

$$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(17)}|}) \times |N|$$

$$|G| \leq (1 + 8 + 1 + 2) \times 16$$

$$|G| \leq (12 \times 16) \leq 192$$

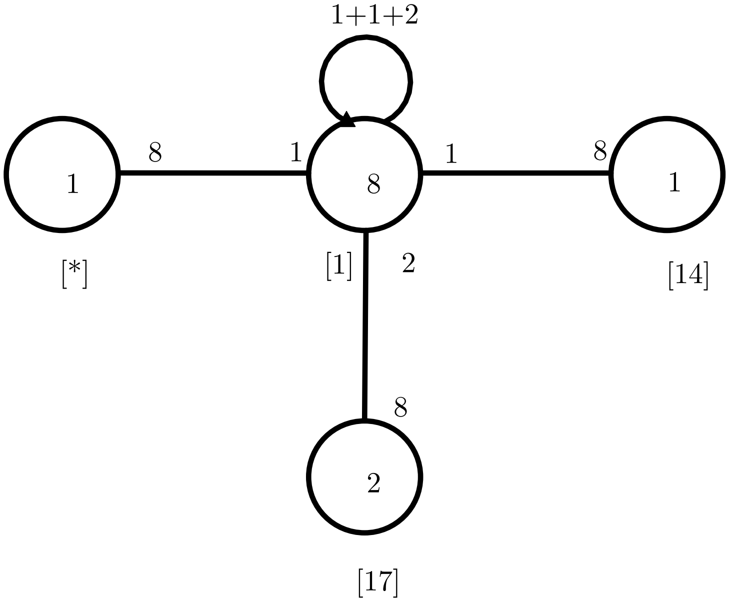

Figure 8.1: Cayley Graph of $2^4 : S_3 \times C_2$ over $(2 \times 4 : 2)$

8.2.1 Verification of $2^4 : S_3 \times C_2$

Consider the progenitor $2^{*8} : (2 \times 4 : 2)$ where N is given by :

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
```

We will prove the isomorphism type of G. The composition factors of G are:

```
G
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(2)
```

```
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
```

The composition series for G is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7$ where $G_7 = 1$.

The composition factors are:

$G = (G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash G_7)$

$=(G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)(G_6 \backslash 1)$

$=(G_0 \backslash G_1)(G_1 \backslash G_2)(G_2 \backslash G_3)(G_3 \backslash G_4)(G_4 \backslash G_5)(G_5 \backslash G_6)G_7$

$=G_2 G_2 G_2 G_2 G_2 G_3 G_2$.

The normal lattice of G is as follows:

```
Normal subgroup lattice
-----------------------
[10]  Order 192  Length 1  Maximal Subgroups: 7 8 9
---
[ 9]  Order 96   Length 1  Maximal Subgroups: 6
[ 8]  Order 96   Length 1  Maximal Subgroups: 6
[ 7]  Order 96   Length 1  Maximal Subgroups: 5 6
---
[ 6]  Order 48   Length 1  Maximal Subgroups: 4
[ 5]  Order 32   Length 1  Maximal Subgroups: 3 4
---
[ 4]  Order 16   Length 1  Maximal Subgroups: 2
---
[ 3]  Order 8    Length 1  Maximal Subgroups: 2
---
[ 2]  Order 4    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:
```

We are looking for the largest abelian group if possible or the center which factor G. In this case, G does not have a center, so we will find the largest abelian group by the following loop.

```
for i in [1..#NL]do if IsAbelian(NL[i]) then i;end if;end for;
1
2
3
4
IsAbelian(NL[4]);
true
```

As we see NL[4] is the largest abelian group. Now, we have to verify if $NL[4] \cong 2^4$ by this Magma code.

```
X:=[2,2,2,2];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
true
```

So, Magma tells us that NL[4] isomorphic to the abelian group 2^4. Therefore, we have that $G_2 = 2^4$. Next, we will factor G1 by NL[4] resulting q which is the quotient group.

```
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
Permutation group q acting on a set of cardinality 6
Order = 12 = 2^2 * 3
```

We will find the normal lattice for q by this Magma loop.

```
nl:=NormalLattice(q);
nl;
Normal subgroup lattice
-----------------------
[7]  Order 12  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 6   Length 1  Maximal Subgroups: 3
[5]  Order 6   Length 1  Maximal Subgroups: 2 3
[4]  Order 6   Length 1  Maximal Subgroups: 3
---
[3]  Order 3   Length 1  Maximal Subgroups: 1
[2]  Order 2   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
```

First, we will fulfill if we have a direct product or not, so we want to see if q is collected of a direct product. Since we posses a normal subgroup of order 6 which is nl[4], and a subgroup of order 2 which is nl[2]. So, the product of two a normal subgroups orders give us the order of q=12. We will check if q is direct product between nl[4] and nl[2] by this loop:

```
E:=DirectProduct(nl[4],nl[2]);
IsIsomorphic(E,q);
\*
true
*\
IsIsomorphic(nl[4],SymmetricGroup(3));
\*
true
*\
IsIsomorphic(nl[2],CyclicGroup(2));
\*
true
*\
```

As a result, we find that $q \cong S_3 \times C_2$ where $S_3 = nL[4]$ and $C_2 = nL[2]$. Now, we have to write a presentation of q. We will write a presentation for NL[4].

```
FPGroup(NL[4]);
Finitely presented group on 4 generators
Relations
$.1^2 = Id($)
$.2^2 = Id($)
$.3^2 = Id($)
$.4^2 = Id($)
($.1 * $.2)^2 = Id($)
($.1 * $.3)^2 = Id($)
($.2 * $.3)^2 = Id($)
($.1 * $.4)^2 = Id($)
($.2 * $.4)^2 = Id($)
($.3 * $.4)^2 = Id($)
```

Our presentation is NL[4]=$< w, x, y, z | w^2, x^2, y^2, z^2, (w*x)^2, (w*y)^2, (x*y)^2, (w*z)^2, (x*z)^2, (y*z)^2 >$. Now, we have to find the action $S_3 \times 2$ on 2^4. We determine the isomorphism type of this group since we had no center in G, so it is not a central extension. Also, we did not find a direct product as well since we have no normal subgroup of order 12. Therefore, the isomorphism type of this group must be a semi-direct product or a mixed extension. We will find transversal because it will help us to find a form to connect q to $NL[4]$ to make an isomorphic representation of G.

```
T:=Transversal(G1,NL[4]);
ff(T[2]) eq q.1;
true
ff(T[3]) eq q.2;
true
ff(T[4]) eq q.5;
true
```

We will label our generators of q, where T[2] is q.1, T[3] is q.2, and T[4] is q.5.

```
Generators(NL[4]);
A:=G1!(2, 6)(3, 8)(4, 5)(7, 9);
B:=G1!(1, 11)(3, 7)(8, 9)(10, 12);
C:=G1!(2, 5)(3, 7)(4, 6)(8, 9);
D:=G1!(1, 10)(3, 8)(7, 9)(11, 12);
H:=G1!T[2];
I:=G1!T[3];
J:=G1!T[4];
```

The following Magma code gives us the action of the transversals on the generators NL[4] are of order 2. A, B, C, D are represented by w, x, y, z and H, I, J are e, f, g.

```
for i,j,k,l in [1..2] do if A^H eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 2 1 2 -->  w^e=w*y
for i,j,k,l in [1..2] do if A^I eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
1 2 1 2   --> w^f=w*y
for i,j,k,l in [1..2] do if A^J eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
2 1 2 1 ..> w^g=x*z
for i,j,k,l in [1..2] do if B^H eq A^i*B^j*C^k*D^l
 then i,j,k,l;end if; end for;
2 1 2 2 --> x^e=x
for i,j,k,l in [1..2] do if B^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 1 1 2 --> x^f=x*y
for i,j,k,l in [1..2] do if B^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 2 1 2 --> x^g=y
for i,j,k,l in [1..2] do if C^H eq A^i*B^j*C^k*D^l
 then i,j,k,l; end if; end for;
2 2 1 2 --> y^e=y
for i,j,k,l in [1..2] do if C^I eq A^i*B^j*C^k*D^l
```

```
then i,j,k,l;end if; end for;
2 2 1 2 --> y^f=y
for i,j,k,l in [1..2] do if C^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
2 1 2 2 ---> y^g=x
for i,j,k,l in [1..2] do if D^H eq A^i*B^j*C^k*D^l
 then i,j,k,l;end if; end for;
2 1 2 1 --> z^e=x*z
for i,j,k,l in [1..2] do if D^I eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
1 1 1 1 --> z^f=w*x*y*z
for i,j,k,l in [1..2] do if D^J eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
1 2 1 2 --> z^g=w*y
FPGroup(q);
Finitely presented group on 5 generators
Relations
$.1^2 = Id($)
$.2^2 = Id($)
$.5^2 = Id($)
($.1 * $.2)^2 = Id($)
($.1 * $.5)^2 = Id($)
($.5 * $.2)^3 = Id($)
$.3 = $.1
$.4 = Id($)
```

We will suppose T[2]=e, T[3]=f and T[4]=g. Now, we will test the elements of q and their relations. If we write the elements of q as the transversals of $G \backslash NL[4]$ we might observe the order of the relations changes.

```
Order(T[2]);
2 doesnt change
Order(T[3]);
2 doesnt change
Order(T[4]);
2 doesnt change
(T[2]*T[3])^2;
(2, 5)(3, 7)(4, 6)(8, 9) the same order does not change
(T[2]*T[4])^2;
Id(G1) is equal id so leave alone
(T[4]*T[3])^3;
Id(G1)
```

Finally, we have the complete presentation of G, which we verify by using Magma:

```
H<w,x,y,z,e,f,g>:=Group<w,x,y,z,e,f,g|w^2,x^2,y^2,z^2,
(w*x)^2,(w*y)^2,(x*y)^2,(w*z)^2,(x*z)^2,
(y*z)^2,e^2,f^2,g^2,(e*f)^2,(e*g)^2,(g*f)^3,
w^e=w*y,w^f=w*y,w^g=x*z,x^e=x,x^f=x*y,x^g=y,
y^e=y,y^f=y,y^g=x,z^e=x*z,
z^f=w*x*y*z,z^g=w*y>;
#H;
192
#G1;
192
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
true
```

We solved the extension for G, so we could obtain that $G \cong 2^4 : S_3 \times C_2$.

8.3 Double Coset Enumeration of $(2^5 : S_4)$ over $(2 \times 4 : C_2 \times C_2)$

Consider the group G $\cong \frac{2^{*16}:(2\times 4:C_2\times C_2)}{[y*x*z*t]^3[y*x*t]^6[y*t]^3} \cong (2^5 : S_4)$ where N=$(2 \times 4 : C_2 \times C_2)$. The presentation of G is given by:

```
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,w^-2*h,x*y*x*y*w^-1,
x*z*x*z*h,t^2,(t,x),(y*t)^3,(z*w*t)^0,(z*x*t)^0,
(y*x*t)^6,(y*x*z*t)^3>;
```

Where

$x \sim (2,8)(3,7)(4,14)(6,10)(9,13)(11,16)$

$y \sim (1,2)(3,9)(4,8)(5,10)(6,14)(7,11)(12,16)(13,15)$

$z \sim (1,3)(2,9)(4,12)(5,7)(6,13)(8,16)(10,11)(14,15)$

$w \sim (1,4,5,14)(2,6,10,8)(3,12,7,15)(9,13,11,16)$

$h \sim (1,5)(2,10)(3,7)(4,14)(6,8)(9,11)(12,15)(13,16)$

Now, we check the number of single cosets that will appear in our Cayley diagram. $\frac{|G|}{|N|}$=$\frac{768}{32}$=24, this means the sum of all single cosets should equal 24 in our Cayley diagram. Next, we will figure out the number of double cosets of our group by helping this Magma code.

```
#DoubleCosets(G,sub<G|x,y,z,w,h>,sub<G|x,y,z,w,h>);
5
```

This means that our Cayley diagram has 5 circles. The next step is that we are going to expand our relations.

The first relation is

$$(y * x * z * t)^3 = e$$
$$(y * x * z * t_1)^3 = e \text{ since } t \sim t_1$$
$$(y * x * z)^3 * t_1^{(y*x*z)^2} * t_1^{y*x*z} * t_1 = e$$
$$(y * x * z)^3 * t_4 * t_{16} * t_1 = e$$
$$(y * x * z)^3 * t_4 * t_{16} = t_1$$

So, we have $t_4t_{16} \sim t_1$.

The second relation is

$$\begin{gathered}(y*x*t)^6 = e\\ (y*x*t_1)^6 = e \text{ since } t \sim t_1\\ (y*x)^6 * t_1^{(y*x)^5} * t_1^{(y*x)^4} * t_1^{(y*x)^3} * t_1^{(y*x)^2} * t_1^{y*x} * t_1 = e\\ (y*x)^6 * t_6 * t_5 * t_{10} * t_{14} * t_8 * t_1 = e\\ (y*x)^6 * t_6 * t_5 * t_{10} * t_{14} * t_8 = t_1\end{gathered}$$

So, we have $t_6t_5t_{10}t_{14}t_8 \sim t_1$.

The last relation is

$$\begin{gathered}(y*t)^3 = e\\ (y*t_1)^3 = e \text{ since } t \sim t_1\\ y^3 * t_1^{y^2} * t_1^{y} * t_1 = e\\ y^3 * t_1 * t_2 * t_1 = e\\ y^3 * t_1 * t_2 = t_1\end{gathered}$$

So, we have $t_1t_2 \sim t_1$.

Constructing the Cayley Diagram:

Circle One: First Double Coset :

We will use this definition of a double coset $NwN = \{n \in N | Nw^n\}$. We consider our first double coset $NeN = \{n \in N | Nen\}$=$\{Ne\}$=$\{N\}$ which we denote by [*]. Our control group N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$, so it has a single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. The number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{32}{32}$=1. So, we will put 1 into the first circle. Next, we will determine the double coset to which Nt_i belongs, where $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. We consider i=1, so Nt_1 is a representative coset, and hence we have a new double coset Nt_1N.

Circle two: Second Double Coset:

We consider the double coset NwN, where w is a word of length one.

$Nt_1N = \{n \in N | Nt_1n\}$. We consider the coset stabilizer $N^{(1)}$, note that the coset stabilizer of Nt_1 is equal to the point stabilizer N^1. $N^{(1)} = \{n \in N | Nt_1^n = t_1\} \geq$ $< (2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16) >$. Since $|N^{(1)}|$=2 then the number of single cosets in [1] is $\frac{|N|}{|N^{(1)}|}=\frac{32}{2}=16$.

Now, the orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ are $\{1\}$, $\{5\}$, $\{12\}$, $\{15\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 14\}$, $\{6, 10\}$, $\{9, 13\}$ and $\{11, 16\}$.

We take representative from each orbit and determine to which double coset Nt_1t_1, Nt_1t_5, Nt_1t_{12}, Nt_1t_{15}, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_6, Nt_1t_9, and Nt_1t_{11} belongs.

We notice the following:

- $Nt_1t_1=Nt_1^2=N \in [*]$.

This will collapse and hence it goes back to the double coset [*] since t_1^2=e.

- $Nt_1t_{12}=Nt_1t_{12} \in [112]$.

This is a new double coset, which will extend the Cayley diagram from [1] to [112].

- $Nt_1t_5=Nt_1t_5 \in [15]$.

This is a new double coset, which will extend the Cayley diagram from [1] to [15]. Since there is one element in this orbit, there is one symmetric generator which will extend to the double coset [15].

- $Nt_1t_{15}=Nt_1t_{15} \in [115]$.

This is a new double coset, which will extend the Cayley diagram from [1] to [115]. Since there is one element in this orbit, there is one t_i that extend [1] to [115].

- $Nt_1t_3=Nt_1t_3 \in [13]$.

This is a new double coset, which will extend the Cayley diagram from [1] to [13]. Since there is two elements in this orbit $\{3, 7\}$, there is two t_i that extend [1] to [13].

- $Nt_1t_4=Nt_1t_4 \in [14]$.

This is a new double coset, which will extend the Cayley diagram from [1] to [14]. Since there is two elements in this orbit $\{4, 14\}$, there is two t_i that extend [1] to [14].

- $Nt_1t_2=Nt_1 \in [1]$.

By the last relation $y^3 * t_1 * t_2 = t_1$, there are two elements in the orbit $\{2, 8\}$, so two elements will loop back into the double coset [1].

- $Nt_1t_6=Nt_1 \in [1]$.

From this relation $t_1 * t_6 = y * x * w * t_1$, there are two elements in the orbit $\{6, 10\}$, so two elements will loop back into the double coset [1].

- $Nt_1t_9=Nt_{14} \in [1]$.

By this relation $t_1 * t_9 = y * x * z * t_{14}$, there are two elements in the orbit $\{9, 13\}$, so two elements will loop back into the double coset [1].

- Nt_1t_{11}=$Nt_{14} \in [1]$.

By this relation $t_1 * t_{11} = y * z * x * t_{14}$, there are two elements in the orbit $\{11, 16\}$, so two elements will loop back into the double coset [1].

As a result, we observe that from the previous steps eight elements loop back into the double coset [1] and 5 new double cosets which are [15], [112], [115], [13], and [14]. Now, we will check by this Magma code if some of these new double cosets are equal.

```
for n in IN do if ts[1]*ts[15] eq n*ts[1]*ts[12]then n;
end if; end for;
for n in IN do if ts[1]*ts[15] eq n*ts[1]*ts[3]then n;
end if; end for;
for n in IN do if ts[1]*ts[3] eq n*ts[1]*ts[12]then n;
 end if; end for;
```

So, [112] = [115] = [13] the same double cosets. Therefore, we have 3 new double cosets [115], [15], and [14].

Circle Three: Third Double Coset :

We are a new double coset [15]. $Nt_1t_5N = \{n \in N | Nt_1t_5n\}$. The coset stabilizer of Nt_1t_5N is given by $N^{(15)} = \{n \in N | Nt_1t_5^n = t_1t_5\} \geq <(2,8)(3,7)(4,14)(6,10)(9,13)$ $(11,16),(1,2)(3,9)(4,8)(5,10)(6,14)(7,11)(12,16)(13,15)>$.

When we conjugate t_1t_5 by all elements of N, we will observe that there are cosets have equal names and the relation increases our $|N^{(15)}|$. The cosets have equal names as follows: $15 \sim 210 \sim 37 \sim 414 \sim 51 \sim 86 \sim 911 \sim 68 \sim 102 \sim 73 \sim 1215 \sim 144 \sim 1613 \sim 1316 \sim 119 \sim 1512$.

Since $|N^{(15)}|$=32 then the number of single cosets in [15] is $\frac{|N|}{|N^{(15)}|}$=$\frac{32}{32}$=1 a single coset in [15]. Now, we can see the orbits of $N^{(15)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ are $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$.

We take 5 representative from the orbit and determine that $Nt_1t_5t_5 = Nt_1 \in [1]$.

As a result, all sixteen t_i take [15] to [1].

Circle four: Fourth Double Coset:

We are a new double coset [112]. $Nt_1t_{12}N = \{n \in N | Nt_1t_{12}n\}$. The coset stabilizer of $Nt_1t_{12}N$ is given by $N^{(112)} = \{n \in N | Nt_1t_{12}^n = t_1t_{12}\} \geq <(2,8)(3,7)(4,14)(6,10)$

$(9,13)(11,16),(1,3)(2,9)(4,12)(5,7)(6,13)(8,16)(10,11)(14,15) >$. When we conjugate t_1t_{12} by all elements of N, we can find these cosets have equal names $112 \sim 34 \sim 515 \sim 714$. Since $|N^{(112)}|$=8 then the number of single cosets in [112] is $\frac{|N|}{|N^{(112)}|}$=$\frac{32}{8}$=4 a single coset in [112].

Now, we can see the orbits of $N^{(112)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$ are $\{1,3,5,7\}$, $\{4,14,12,15\}$, and $\{2,8,9,16,10,6,13,11\}$. We choose a representative from each orbit. If we select t_3 from the orbit $\{1,3,5,7\}$, t_{12} from the orbit $\{4,14,12,15\}$ and t_2 from the orbit $\{2,8,9,16,10,6,13,11\}$.

Then, we notice that.

- $Nt_1t_{12}t_3 \in [1]$.

From this relation:

$t_1t_{12} = h * t_3t_4$

$t_1t_{12}t_3 = h * t_4 \in [1]$.

Therefore, 4 symmetric generators will send to the double coset [1].

- $Nt_1t_{12}t_{12}$=$Nt_1 \in [1]$.

Therefore, 4 symmetric generators will send to the double coset [1].

- $Nt_1t_{12}t_2$=$Nt_{11} \in [1]$.

From this relation:

$t_1t_{12} = z * x * t_2t_{11}$

$t_1t_{12}t_2 = z * x * t_{11} \in [1]$.

Therefore, 8 symmetric generators will extend to the double coset [1].

Finally, all of sixteen t_i take [112] to [1].

Circle five: Fifth Double Coset:

We label the fifth double coset [14] and compute the coset stabilizer $N^{(14)}$. The coset stabilizer of Nt_1t_4N is given by $N^{(14)} = \{n \in N | Nt_1t_4^n = t_1t_4\} \geq < (1,2)(3,9)(4,8)(5,10)$ $(6,14)(7,11)(12,16)(13,15) >$. Since $|N^{(14)}|$=16 then the number of single cosets in [14] is $\frac{|N|}{|N^{(14)}|}$=$\frac{32}{16}$=2 a single coset in [14].

Now, we can see the orbits of $N^{(14)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$ are $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$.

We choose a representative 4 from the orbit $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$ and determine that $Nt_1t_4t_4 = Nt_1 \in [1]$.

As a result, all sixteen t_i take [14] to [1]. Finally, we could construct our Cayley diagram.

$$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(112)}|} + \frac{|N|}{|N^{(15)}|}) \times |N|$$
$$|G| \leq (1 + 16 + 2 + 4 + 1) \times 32$$
$$|G| \leq (24 \times 32) \leq 768$$

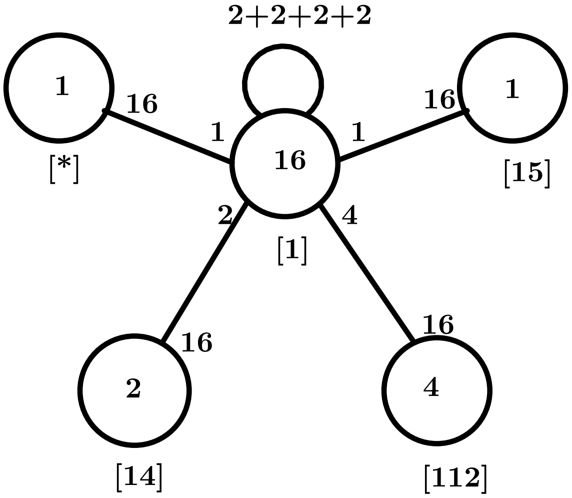

Figure 8.2: Cayley Graph of $(2^5 : S_4)$ over $(2 \times 4 : C_2 \times C_2)$

8.3.1 Verification of $(2^5 : S_4)$

Consider the progenitor $2^{*16} : N$ where N is given by:

```
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,
h^2,(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^3,
(z*w*t)^0,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^3>;
```

We will prove the isomorphism type of G. The composition factors of G is given by:

```
G
|  Cyclic(2)
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
```

The composition series for G is:

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq G_8 \supseteq G_9$ where $G_9 = 1$

The composition factors are:

$G = (G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)(G_8\backslash G_9)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)(G_8\backslash 1)$

$=(G_0\backslash G_1)(G_1\backslash G_2)(G_2\backslash G_3)(G_3\backslash G_4)(G_4\backslash G_5)(G_5\backslash G_6)(G_6\backslash G_7)(G_7\backslash G_8)G_9$

$=G_2G_2G_2G_2G_2G_2G_2G_3G_2$.

The normal lattice of G is as follows:

```
NL:=NormalLattice(G1);
NL;
Normal subgroup lattice
-----------------------

[16]  Order 768  Length 1  Maximal Subgroups: 13 14 15
---
[15]  Order 384  Length 1  Maximal Subgroups: 12
[14]  Order 384  Length 1  Maximal Subgroups: 12
[13]  Order 384  Length 1  Maximal Subgroups: 11 12
---
[12]  Order 192  Length 1  Maximal Subgroups: 10
[11]  Order 128  Length 1  Maximal Subgroups: 7 8 9 10
```

```
---
[10]  Order 64   Length 1  Maximal Subgroups: 4 5 6
---
[ 9]  Order 32   Length 1  Maximal Subgroups: 3 5
[ 8]  Order 32   Length 1  Maximal Subgroups: 3 4
[ 7]  Order 32   Length 1  Maximal Subgroups: 3 6
---
[ 6]  Order 16   Length 1  Maximal Subgroups: 2
[ 5]  Order 16   Length 1  Maximal Subgroups: 2
[ 4]  Order 16   Length 1  Maximal Subgroups: 2
---
[ 3]  Order 8    Length 1  Maximal Subgroups: 2
---
[ 2]  Order 4    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:
```

We are looking for the largest abelian group if possible or the center which factor G. In this case, G does not have a center, so we will find the largest abelian group by the following loop.

```
for i in [1..#NL]do if IsAbelian(NL[i]) then i;end if;end for;
\*
1
2
3
4
5
6
9
*\
IsAbelian(NL[9]);
\*
true
```

As we see NL[9] is the largest abelian group. Now, we check $NL[9] \cong 2^5$ by Magma code.

```
X:=[2,2,2,2,2];
```

```
IsIsomorphic(NL[9],AbelianGroup(GrpPerm,(X)));
true
```

So, Magma tells us that NL[9] isomorphic to the abelian group 2^5. Therefore, we have that $G_2 = 2^5$. Now, we can produce a presentation for NL[9] as:

```
S<w,x,y,z,v>:=Group<w,x,y,z,v|w^2,x^2,y^2,z^2,v^2,
(w*x)^2,(w*y)^2,(x*y)^2,(w*z)^2,(x*z)^2,(y*z)^2,
(w*v)^2,(x*v)^2,(y*v)^2,(z*v)^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[9]);
s;
true
```

Next, we will factor G1 by NL[9] resulting q which is the quotient group.

```
H:=NL[9];
q,ff:=quo<G1|NL[9]>;
q;
Permutation group q acting on a set of cardinality 4
Order = 24 = 2^3 * 3
(1, 2)(3, 4)
(2, 4)
Id(q)
(1, 3)(2, 4)
Id(q)
(3, 4)
```

We will now find the normal lattice for q by this Magma loop.

```
nl:=NormalLattice(q);
nl;
Normal subgroup lattice
-----------------------
[4]  Order 24  Length 1  Maximal Subgroups: 3
---
[3]  Order 12  Length 1  Maximal Subgroups: 2
---
[2]  Order 4   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
```

We could figure out q isomorphic to S_4 as follows:

```
IsIsomorphic(SymmetricGroup(4),q);
true
FPGroup(q);
Finitely presented group on 6 generators
Relations
$.1^2 = Id($)
$.2^2 = Id($)
$.4^2 = Id($)
$.6^2 = Id($)
($.1 * $.4)^2 = Id($)
($.2 * $.4)^2 = Id($)
($.1 * $.6)^2 = Id($)
$.1 * $.2 * $.1 * $.2 * $.4 = Id($)
$.6 * $.4 * $.1 * $.6 * $.4 = Id($)
($.6 * $.2)^3 = Id($)
$.3 = Id($)
$.5 = Id($)
```

The presentation of $q \cong S_4$ as follows:

```
H<e,f,g,h>:=Group<e,f,g,h|e^2,f^2,g^2,h^2,(e*g)^2,(f*g)^2,
(e*h)^2,e*f*e*f*g,h*g*e*h*g,(h*f)^3>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
true
```

Now, we begin the approach of defining the action of Q on K by labeling the generators of NL[9] and the generators of q as follows:

```
H:=NL[9];
q,ff:=quo<G1|NL[9]>;
q;
Permutation group q acting on a set of cardinality 4
Order = 24 = 2^3 * 3
(1, 2)(3, 4)
(2, 4)
Id(q)
(1, 3)(2, 4)
Id(q)
(3, 4)
EE:=q!(1, 2)(3, 4);
FF:=q!(2, 4);
GG:=q!(1, 3)(2, 4);
```

```
DD:=q!(3, 4);
T:=Transversal(G1,NL[9]);
ff(T[2]);
ff(T[3]);
ff(T[4]);
ff(T[5]);
ff(T[3]) eq FF;
true
ff(T[2]) eq EE;
true
ff(T[4]) eq GG;
true
ff(T[5]) eq DD;
true
q eq sub<q|EE,FF,GG,DD>;
true
```

We will label our generators of q, where T[2], T[3], T[4], and T[5] are represented by q.2, q.3, q.4, and q.6, respectively.

```
Generators(NL[9]);
A:=G1!(1, 24)(2, 7)(3, 9)(4, 15)(5, 6)(8, 19)(10, 20)
(11, 21)(12, 13)(14, 18)(16, 22)(17, 23);
B:=G1!(1, 16)(2, 15)(3, 19)(4, 7)(5, 12)(6, 13)(8, 9)
(10, 21)(11, 20)(14, 23)(17, 18)(22, 24);
C:=G1!(1, 17)(2, 6)(4, 12)(5, 7)(13, 15)(14, 22)
(16, 18)(23, 24);
D:=G1! (2, 6)(3, 11)(4, 12)(5, 7)(8, 10)(9, 21)
(13, 15)(19, 20);
F:=G1! (2, 4)(3, 9)(5, 13)(6, 12)(7, 15)(8, 19)
(10, 20)(11, 21);
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
O:=G1!T[5];
```

Now, the following Magma code gives us the action of the transversal on the generators NL[9] which are of order 2. A, B, C, D, and F are represented by w, x, y, z, and v and J, I, M, and O are represented by e, f, g, and h.

```
for i,j,k,l,m in [1..2] do if A^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
1 2 2 1 2 ---> w^e=w*z
for i,j,k,l,m in [1..2] do if A^I eq A^i*B^j*C^k*D^l*F^m
```

```
then i,j,k,l,m; end if; end for;
1 1 1 1 2 ---> w^f=w*x*y*z
for i,j,k,l,m in [1..2] do if A^M eq A^i*B^j*C^k*D^l*F^m
 then i,j,k,l,m;end if; end for;
1 2 1 2 2 ---> w^g=w*y
for i,j,k,l,m in [1..2] do if A^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 1 1 1 2 ---> w^h=x*y*z
for i,j,k,l,m in [1..2] do if B^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 1 1 1 2 ---> x^e=x*y*z
for i,j,k,l,m in [1..2] do if B^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 1 2 1 2 ---> x^f=x*z
for i,j,k,l,m in [1..2] do if B^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 1 2 1 2 ---> x^g=x*z
for i,j,k,l,m in [1..2] do if B^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
1 2 2 1 2 ---> x^h=w*z
for i,j,k,l,m in [1..2] do if C^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 2 1 2 2 ---> y^e=y
for i,j,k,l,m in [1..2] do if C^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 1 1 2 ---> y^f=y*z
for i,j,k,l,m in [1..2] do if C^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 2 1 2 2 ---> y^g=y
for i,j,k,l,m in [1..2] do if C^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 1 2 2 ---> y^h=y
for i,j,k,l,m in [1..2] do if D^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 2 2 1 2 ---> z^e=z
for i,j,k,l,m in [1..2] do if D^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 2 1 2 ---> z^f=z
for i,j,k,l,m in [1..2] do if D^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 2 1 2 ---> z^g=z
for i,j,k,l,m in [1..2] do if D^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m; end if; end for;
2 2 1 1 2 ---> z^h=y*z
```

```
for i,j,k,l,m in [1..2] do if F^J eq A^i*B^j*C^k*D^l*F^m
 then i,j,k,l,m;end if; end for;
2 2 2 1 1 ---> v^e=z*v
for i,j,k,l,m in [1..2] do if F^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 2 2 1 ---> v^f=v
for i,j,k,l,m in [1..2] do if F^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
2 2 2 2 1 ---> v^g=v
for i,j,k,l,m in [1..2] do if F^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;end if; end for;
1 1 1 1 1 ---> v^h=w*x*y*z*v
```

Now, we put together presentations of NL[9] and q with words are given above. Thus, M=$Group < w, x, y, z, v, e, f, g, h | w^2, x^2, y^2, z^2, v^2, (w*x)^2, (w*y)^2, (x*y)^2, (w*z)^2, (x*z)^2, (y*z)^2, (w*v)^2, (x*v)^2, (y*v)^2, (z*v)^2, w^e = w*z, w^f = w*x*y*z, w^g = w*y, w^h = x*y*z, x^e = x*y*z, x^f = x*z, x^g = x*z, x^h = w*z, y^e = y, y^f = y*z, y^g = y, y^h = y, z^e = z, z^f = z, z^g = z, z^h = y*z, v^e = z*v, v^f = v, v^g = v, v^h = w*x*y*z*v, e^2, f^2, g^2, h^2, (e*g)^2, (f*g)^2, (e*h)^2, e*f*e*f*g, h*g*e*h*g, (h*f)^3 >$.

```
#M;#G1;
768
f1,M1,k1:=CosetAction(M,sub<M|Id(M)>);
s:=IsIsomorphic(G1,M1);
s;
true
```

As we see by asking Magma if this is isomorphic to G1, we determine this is true, so we have a semi direct extension. We solved the extension for G, so we could obtain that $G \cong 2^5 : S_4$.

8.4 The Double Coset Enumeration of $2^{*4} : A_5$ over 2^2

We take the progenitor $2^{*4} : A_5$ where 2^{*4} is the free product of 4 copies of cyclic groups of order 2 and we will factor our group by these relations $(y*t)^3$, $(x*t)^5$, and $(x*y*t)^5$. let $G \cong 2^{*4} : A_5$ be a symmetric presentation of G given by:

```
G<x,y,t>:=Group<x,y,t|x^2,y^2,x*y^-1*x*y,t^2,(x*t)^10,(y*t)^3,
(x*t)^5,(x*y*t)^5>;
```

Where $x = (1,2)(3,4)$ and $y = (1,3)(2,4)$.

Using computer-based program-MAGMA:

1. The order of our group is equal 60.

2. There are 6 double coset in this double coset enumeration of G over N.

3. the number of single cosets that will appear in our Cayley diagram,it is $\frac{|G|}{|N|}$= $\frac{60}{4}$=15, this means the sum of all single cosets should equal 15 in our Cayley diagram.

Relations:

The next step is that we are going to expand our relations.

The first relation is:

$$(y * t)^3 = e$$
$$(y * t_1)^3 = e \text{ since } t \sim t_1$$
$$y^3 * t_1^{y^2} * t_1^{y} * t_1 = e$$
$$y^3 * t_1 * t_3 * t_1 = e$$
$$y^3 * t_1 * t_3 = t_1$$

So, we have $t_1 t_3 \sim t_1$.

Moreover, if we conjugate the previous relation by all elements of N=$\{Id(N), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. We have

- $((1,3)(2,4)t_1t_3 = t_1)^e \Rightarrow (1,3)(2,4)t_1t_3 = t_1$.
- $((1,3)(2,4)t_1t_3 = t_1)^{(1,2)(3,4)} \Rightarrow (2,4)(1,3)t_2t_4 = t_2$.
- $((1,3)(2,4)t_1t_3 = t_1)^{(1,3)(2,4)} \Rightarrow (3,1)(4,2)t_3t_1 = t_3$.
- $((1,3)(2,4)t_1t_3 = t_1)^{(1,4)(2,3)} \Rightarrow (4,2)(3,1)t_4t_2 = t_4$.

The second relation is:

$$(x * t)^5 = e$$
$$(x * t_1)^5 = e \text{ since } t \sim t_1$$
$$x^5 * t_1^{x^4} * t_1^{x^3} * t_1^{x^2} * t_1^{x} * t_1 = e$$
$$x^5 * t_1 * t_2 * t_1 * t_2 * t_1 = e$$
$$x^5 * t_1 * t_2 * t_1 = t_1 * t_2$$

So, we have $t_1 t_2 t_1 \sim t_1 t_2$.

Moreover, if we conjugate the previous relation by all elements of N, we have

- $((1,2)(3,4)t_1t_2t_1 = t_1t_2)^e \Rightarrow (1,2)(3,4)t_1t_2t_1 = t_1t_2$
- $((1,2)(3,4)t_1t_2t_1 = t_1t_2)^{(1,2)(3,4)} \Rightarrow (2,1)(4,3)t_2t_1t_2 = t_2t_1$
- $((1,2)(3,4)t_1t_2t_1 = t_1t_2)^{(1,3)(2,4)} \Rightarrow (3,4)(1,2)t_3t_4t_3 = t_3t_4$
- $((1,2)(3,4)t_1t_2t_1 = t_1t_2)^{(1,4)(2,3)} \Rightarrow (4,3)(2,1)t_4t_3t_4 = t_4t_3$.

The last relation is:

$$\begin{gathered}(x*y*t)^5 = e\\ (x*y*t_1)^5 = e \text{ since } t \sim t_1\\ (x*y)^5 * t_1^{(x*y)^4} * t_1^{(x*y)^3} * t_1^{(x*y)^2} * t_1^{x*y} * t_1 = e\\ (x*y)^5 * t_1 * t_4 * t_1 * t_4 * t_1 = e\\ (x*y)^5 * t_1 * t_4 * t_1 = t_1t_4\end{gathered}$$

So, we have $t_1t_4t_1 \sim t_1t_4$.

Also, if we conjugate the previous relation by all elements of N, we have

- $((1,4)(2,3)t_1t_4t_1 = t_1t_4)^e \Rightarrow (1,4)(2,3)t_1t_4t_1 = t_1t_4$
- $((1,4)(2,3)t_1t_4t_1 = t_1t_4)^{(1,2)(3,4)} \Rightarrow (2,3)(1,4)t_2t_3t_2 = t_2t_3$
- $((1,4)(2,3)t_1t_4t_1 = t_1t_4)^{(1,3)(2,4)} \Rightarrow (3,2)(4,1)t_3t_2t_3 = t_3t_2$
- $((1,4)(2,3)t_1t_4t_1 = t_1t_4)^{(1,4)(2,3)} \Rightarrow (4,1)(3,2)t_4t_1t_4 = t_4t_1$.

8.4.1 Constructing the Cayley Diagram:

We will begin the manual double coset enumeration by looking at our first double coset.

Double Coset [*] :

We will use this definition of a double coset $NwN = \{n \in N | Nw^n\}$. We consider our first double coset $NeN = \{n \in N | Nen\}$=$\{Ne\}$=$\{N\}$ which we denote by [*]. Our control group N is transitive on $\{1,2,3,4\}$ so, it has a single orbit $\{1,2,3,4\}$. The number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{4}{4}$=1. Now, we will determine the double coset to which Nt_i belongs, where $i \in \{1,2,3,4\}$. We consider i=1, so Nt_1 is a representative coset, and hence we have a new double coset Nt_1N which is denote by [1].

Double Coset [1] :

We consider the double coset NwN, where w is a word of length one. $NwN = \{n \in$

$N|Nw^n\}$ $Nt_1N = \{n \in N|Nt_1n\}$ $=\{Nt_1, Nt_2, Nt_3, Nt_4\}$.

We consider the coset stabilizer $N^{(1)}$, note that the coset stabilizer of Nt_1 is equal to the point stabilizer N^1. $N^{(1)} = \{n \in N|Nt_1^n = t_1\} \geq < Id(N) >$. Since $|N^{(1)}|$=1 then the number of single cosets in [1] is $\frac{|N|}{|N^{(1)}|}$=$\frac{4}{1}$=4. Now, the orbits of $N^{(1)}$ on $\{1, 2, 3, 4\}$ are {1},{2},{3}, and {4}. We take representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4$ belong.

We observe as follows:

- Nt_1t_1=Nt_1^2=$N \in$ [*]

This will collapse and hence it goes back to the double coset [*] since t_1^2=e.

- Nt_1t_2=$Nt_1t_2 \in$ [12].

One element will move forward to a new double coset represented as [12].

- Nt_1t_3=$Nt_1 \in$ [1].

By this relation $t_1t_3 = yt_1$, there is one element in the orbit so one element will loop back into the double coset [1].

- Nt_1t_4=$Nt_1t_4 \in$ [14].

One element will move forward to a new double coset represented as [14].

Double Coset [12] :

We are a new double coset [12]. $Nt_1t_2N = \{n \in N|Nt_1t_2n\}$. The coset stabilizer of Nt_1t_2N is given by $N^{(12)} = \{n \in N|Nt_1t_2^n = t_1t_2\} \geq < e >$.

Since $|N^{(12)}|$=1 then the number of single cosets in [12] is $\frac{|N|}{|N^{(12)}|}$=$\frac{4}{1}$=4 a single coset in [12].

Next, we conjugate Nt_1t_2 by all elements of N=$\{Id(N), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$.

We have

- $(t_1t_2)^e = t_1t_2$.
- $(t_1t_2)^{(1,2)(3,4)} = t_2t_1$.
- $(t_1t_2)^{(1,3)(2,4)} = t_3t_4$.
- $(t_1t_2)^{(1,4)(2,3)} = t_4t_3$.

Now, the orbits of $N^{(12)}$ on $\{1, 2, 3, 4\}$ are {1}, {2}, {3}, and {4}. We take representative from each orbit and determine to which double coset

$\{Nt_1t_2t_1, Nt_1t_2t_2, Nt_1t_2t_3, Nt_1t_2t_4\}$ belong. We notice the following:

- $t_1t_2t_1 = x * t_1t_2 \in$ [12].

so,one element will loop back into the double coset [12].

- $t_1t_2t_2 = t_1 \in [1]$. So, one element will extend from [12] to the double coset [1].
- $t_1t_2t_3 = t_1t_2t_3 \in [123]$.

This is a new double coset, which will extend the Cayley diagram from [12] to [123]. Since there is one element in this orbit, there is one t_i that extend [12] to [123].

- $t_1t_2t_4 = t_1t_4 \in [14]$.

So, one element will extend from [12] to the double coset [14].

Double Coset [14] :

We are a new double coset [14]. $Nt_1t_4N = \{n \in N|Nt_1t_4n\}$. The coset stabilizer of Nt_1t_4N is given by $N^{(14)} = \{n \in N|Nt_1t_4^n = t_1t_4\} \geq < Id(N) >$.
Since $|N^{(14)}|$=1 then the number of single cosets in [12] is $\frac{|N|}{|N^{(12)}|}$=$\frac{4}{1}$=4 a single coset in [14]. Now, the orbits of $N^{(14)}$ on $\{1, 2, 3, 4\}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$. We take representative from each orbit and determine to which double coset
$\{Nt_1t_4t_1, Nt_1t_4t_2, Nt_1t_4t_3, Nt_1t_4t_4\}$ belong. We notice the following:

- $t_1t_4t_1 = x * y * t_1t_4 \in [14]$.

So, one element will loop back into the double coset [14].

- $t_1t_4t_4 = t_1 \in [1]$. So, one element will extend from [12] to the double coset [1].
- $t_1t_4t_3 = t_1t_4t_3 \in [143]$.

This is a new double coset, which will extend the Cayley diagram from [14] to [143]. Since there is one element in this orbit, there is one t_i that extends [14] to [143].

- $t_1t_4t_2 = t_1t_2 \in [12]$. So, one element will extend from [14] to the double coset [12].

Double Coset [123] :

We are a new double coset [123]. $Nt_1t_2t_3N = \{n \in N|Nt_1t_2t_3n\}$. The coset stabilizer of $Nt_1t_2t_3N$ is given by $N^{(123)} = \{n \in N|Nt_1t_2t_3^n = t_1t_2t_3\} \geq < (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) >$. When we conjugate $t_1t_2t_3$ by all elements of N, we observed that there are cosets have equal names $123 \sim 214 \sim 341 \sim 432$ and the relation increases our $|N^{(123)}|$. Since $|N^{(123)}|$=4 then the number of single cosets in [123] is $\frac{|N|}{|N^{(123)}|}$=$\frac{4}{4}$=1 a single coset in [123]. Now, the orbits of $N^{(123)}$ on $\{1, 2, 3, 4\}$ are $\{1, 2, 3, 4\}$. We take representative from the orbit and determine to which double coset belongs. We will pick 3 from the orbit $\{1, 2, 3, 4\}$. As a result, four symmetric generators are going to the double coset [12] because $t_1t_2t_3t_3 = t_1t_2 \in [12]$.

Double Coset [143] :

We are a new double coset [143]. $Nt_1t_4t_3N = \{n \in N | Nt_1t_4t_3n\}$. The coset stabilizer of $Nt_1t_4t_3N$ is given by $N^{(143)} = \{n \in N | Nt_1t_4t_3^n = t_1t_4t_3\} \geq < (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) >$. Since $|N^{(143)}|$=4 then the number of single cosets in [143] is $\frac{|N|}{|N^{(143)}|}$=$\frac{4}{4}$=1 a single coset in [143]. Now, the orbits of $N^{(143)}$ on $\{1,2,3,4\}$ are$\{1,2,3,4\}$. We take representative from the orbit and determine to which double coset belong. We will pick 3 from the orbit $\{1,2,3,4\}$. As a result, 4 symmetric generators are going from the double coset [143] to [14] because $t_1t_4t_3t_3 = t_1t_4 \in [14]$.

Conclusion:

The double coset enumeration gives that:

$$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(143)}|}) \times |N|$$
$$|G| \leq (1 + 4 + 4 + 1 + 4 + 1) \times 4$$
$$|G| \leq (15 \times 4) \leq 60.$$

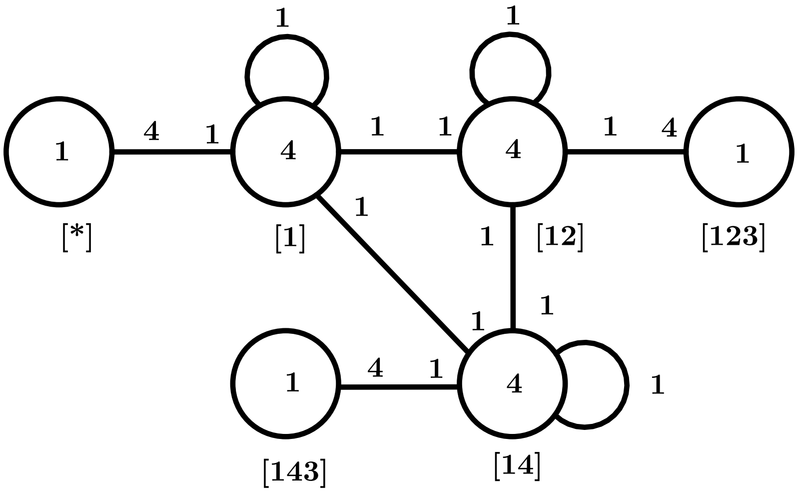

Figure 8.3: Cayley Graph of $2^{*4} : A_5$ over 2^2

8.5 Manual Double Coset Enumeration of S_6 over $(2\times 4:2)$

In the beginning, we can do some steps which help us to figure out our Cayley diagram how it will be. First, we have to calculate that $\frac{|G|}{|N|}$= $\frac{720}{16}$=45. This means the sum of all single cosets should equal 45 in our Cayley diagram. Then, we need to know how many double cosets we will have, we can check that by Magma code as follows:

```
#DoubleCosets(G,sub<G|w,x,y,z>,sub<G|w,x,y,z>);
8
```

So, our Cayley diagram has 8 double cosets. Now, our group has this presentation $G=<x,y,z,w,t|x^2,y^2,z^2,w^4,(y*z)^2,(x*w^{-1})^2,(y*w^{-1})^2,z*w^{-1}*z*w,x*y*x*y*w^{-1},t^2,(t,x),(y{*}t)^4,(z{*}w{*}t)^3,(y{*}x{*}t)^6,(y{*}x{*}z{*}t)^5,x{*}t{*}w^{-1}{*}t{*}w^{-1}{*}t{*}w^{-1},(z{*}t)^4>$. Where x=(2, 5)(6, 8), y=(1, 2)(3, 6)(4, 5)(7, 8), z=(1, 3)(2, 6)(4, 7)(5, 8), and w=(1, 4)(2, 5)(3, 7)(6, 8).

Next, we factor our progenitor $\frac{2^{*8}:(2\times 4:2)}{(y*t)^4(z*w*t)^3(y*x*t)^6(y*x*z*t)^5(z*t)^4}\cong S_6$ by these relations:

The first relation is:

$(y*t)^4=e.$

$(y*t_1)^4=e$ since $t\sim t_1$.

$y^4*t_1^{y^3}*t_1^{y^2}*t_1^{y}*t_1=e.$

$t_2*t_1*t_2*t_1=e.$

$t_2*t_1=t_1*t_2.$

So, we have $t_2t_1\sim t_1t_2$.

The second relation is:

$(z*w*t)^3=e.$

$(z*w*t_1)^3=e$ since $t\sim t_1$.

$(z*w)^3*t_1^{(z*w)^2}*t_1^{z*w}*t_1=e.$

$(z*w)^3*t_1*t_7*t_1=e.$

$(z*w)^3*t_1=t_1*t_7.$

So, we have $t_1\sim t_1t_7$.

The third relation is:

$(y*x*t)^6=e.$

$(y*x*t_1)^6=e$ since $t\sim t_1$.

$(y*x)^6*t_1^{(y*x)^5}*t_1^{(y*x)^4}*t_1^{(y*x)^3}*t_1^{(y*x)^2}*t_1^{y*x}*t_1=e.$

$(y*x)^6 * t_5 * t_1 * t_2 * t_4 * t_5 * t_1 = e$

$(y*x)^6 * t_5 * t_1 * t_2 = t_1 * t_5 * t_4$.

So, we have $t_5t_1t_2 \sim t_1t_5t_4$.

The forth relation is:

$(y*x*z*t)^5 = e$

$(y*x*z*t_1)^5 = e$ since $t \sim t_1$

$(y*x*z)^5 * t_1^{(y*x*z)^4} * t_1^{(y*x*z)^3} * t_1^{(y*x*z)^2} * t_1^{y*x*z} * t_1 = e$

$(y*x*z)^5 * t_1 * t_6 * t_4 * t_8 * t_1 = e$

$(y*x*z)^5 * t_1 * t_6 = t_1 * t_8 * t_4$

So, we have $t_1t_6 \sim t_1t_8t_4$.

The last relation is:

$(z*t)^4 = e$

$(z*t_1)^4 = e$ since $t \sim t_1$

$z^4 * t_1^{z^3} * t_1^{z^2} * t_1^{z} * t_1 = e$

$z^4 * t_3 * t_1 * t_3 * t_1 = e$

$z^4 * t_3 * t_1 = t_1 * t_3$

So, we have $t_3t_1 \sim t_1t_3$.

Constructing the Cayley Diagram:

NeN :

We will use this definition of a double coset $NwN = \{n \in N|Nw^n\}$.

We consider our first double coset $NeN = \{n \in N|Nen\}$=$\{Ne\}$=$\{N\}$ which we denote by [*]. Our control group N is transitive on $\{1,2,3,4,5,6,7,8\}$, so it has a single orbit $\{1,2,3,4,5,6,7,8\}$. The number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{16}{16}$=1. So, we have one single double coset in a circle [*]. We take a representative from the orbit and determine to which double coset it belongs. Therefore, we get a new double coset Nt_1N.

$\mathbf{Nt_1N}$:

We consider the double coset NwN, where w is a word of length one. $Nt_1N = \{n \in N|Nt_1n\}$ =$\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8\}$. The coset stabilizer is $N^{(1)}$, and the coset stabilizer of Nt_1 is equal to the point stabilizer N^1.

$N^{(1)} = \{n \in N | Nt_1^n = t_1\} \geq <(2,5)(6,8)>$.

We compute that $\frac{|N|}{|N^{(1)}|}$=$\frac{16}{2}$=8 single cosets in [1]. Now, the orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8\}$ are $\{1\}$, $\{3\}$, $\{4\}$, $\{7\}$, $\{2,5\}$, and $\{6,8\}$. We take a representative from each orbit and determine to which double coset belong.

We observe that Nt_1t_1=Nt_1^2=$N \in$ [*], so it goes to the double coset [*] since t_1^2=e. Nt_1t_3=$Nt_1t_3 \in$ [13], so this is a new double coset, which will extend the Cayley diagram from [1] to [13]. Nt_1t_4=$Nt_1 \in$ [1], so one symmetric generator loops back into [1]. Nt_1t_7=$Nt_1 \in$ [1], so one symmetric generator loops back into [1]. Nt_1t_2=$Nt_1t_2 \in$ [12]. This is a new double coset, which will extend the Cayley diagram from [1] to [12]. Since there is two elements in this orbit $\{2,5\}$, there is two t_i that extend from [1] to [12]. Nt_1t_6=$Nt_1t_6 \in$ [16]. This is a new double coset, which will extend the Cayley diagram from [1] to [16]. Since there is two elements in this orbit $\{6,8\}$, there is two t_i that extend from [1] to [16].

$\mathbf{Nt_1t_2N}$:

We have a new double coset which is called by [12]. We use this definition $Nt_1t_2N = \{n \in N | Nt_1t_2n\}$. The coset stabilizer of Nt_1t_2N is given by $N^{(12)} = \{n \in N | Nt_1t_2^n = t_1t_2\} \geq <e, (1,2)(3,6)(4,5)(7,8)>$.

When we conjugate t_1t_2 by all elements of N, we have these cosets which have equal name $12 \sim 21$. Since $|N^{(12)}|$=2 then the number of single cosets in [12] is $\frac{|N|}{|N^{(12)}|}$=$\frac{16}{2}$=8 single coset in [12]. Now, we can see the orbits of $N^{(12)}$ on $\{1,2,3,4,5,6,7,8\}$ are $\{1,2\}$, $\{3,6\}$, $\{4,5\}$, and $\{7,8\}$. We take a representative from each orbit and determine which are belong. We notice that $Nt_1t_2t_2$=$Nt_1 \in$ [1], so two elements send from [12] to [1]. $Nt_1t_2t_3$=$Nt_1t_2t_3 \in$ [123], so it is a new double coset. $Nt_1t_2t_4$=$Nt_1t_2 \in$ [12], so two symmetric generators loop back into [12]. $Nt_1t_2t_7$=$Nt_1t_6 \in$ [16], so two symmetric generators send from the double coset [12] to [16].

$\mathbf{Nt_1t_3N}$:

We have a new double coset which is denoted by [13]. $Nt_1t_3N = \{n \in N | Nt_1t_3n\}$ and the coset stabilizer of Nt_1t_3N is given by $N^{(13)} = \{n \in N | Nt_1t_3^n = t_1t_3\} \geq <(2,5)(6,8)(1,3)(2,6)(4,7)(5,8)>$. Since $|N^{(13)}|$=8 then the number of single cosets in [13] is $\frac{|N|}{|N^{(13)}|}$=$\frac{16}{8}$=2 single coset in [13]. Now, we can see the orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8\}$ are $\{1,3,4,7\}$, and $\{2,5,6,8\}$. We take a representative from each orbit and determine which are belong. We notice that $Nt_1t_3t_3$=$Nt_1 \in$ [1], so 4 symmetric generators send from the double coset [13] to [1]. $Nt_1t_3t_2$=$Nt_1t_3t_2 \in [132]$, so it is a new double coset.

$\mathbf{Nt_1t_6N}$:

We label the fifth double coset [16] and compute the coset stabilizer $N^{(16)}$. The coset stabilizer of Nt_1t_6N is given by $N^{(16)} = \{n \in N|Nt_1t_6^n = t_1t_6\} \geq < e >$. Since $|N^{(16)}|$=1 then the number of single cosets in [16] is $\frac{|N|}{|N^{(16)}|}$=$\frac{16}{1}$=16 single coset in [16]. Now, we can see the orbits of $N^{(16)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, and $\{8\}$. We take a representative from each orbit and determine which are belong. We observe that $Nt_1t_6t_1$=$Nt_1t_6t_1 \in [161]$, so this is new double coset. $Nt_1t_6t_2$=$Nt_1t_2t_3 \in [123]$. $Nt_1t_6t_3$=$Nt_1t_3t_2 \in [132]$. $Nt_1t_6t_4$=$Nt_1t_6 \in [16]$. $Nt_1t_6t_5$=$Nt_1t_2 \in [12]$. $Nt_1t_6t_6$=$Nt_1 \in [1]$. $Nt_1t_6t_7$=$Nt_1t_6 \in [16]$. $Nt_1t_6t_8$=$Nt_1t_6 \in$ [16].

$\mathbf{Nt_1t_2t_3N}$:

Now, we label the new double coset [123] and compute the coset stabilizer $N^{(123)}$. The coset stabilizer of $Nt_1t_2t_3N$ is given by $N^{(123)} = \{n \in N|Nt_1t_2t_3^n = t_1t_2t_3\} \geq <(1, 2)(3, 6)(4, 5)(7, 8), (1, 7)(2, 8)(3, 4)(5, 6) >$. When we conjugate $t_1t_2t_3$ by all elements of N, we could investigate that these cosets have equal name $123 \sim 216 \sim 786 \sim 875$. Since $|N^{(123)}|$=4 then the number of single cosets in [123] is $\frac{|N|}{|N^{(123)}|}$=$\frac{16}{4}$=4 single coset in [123]. Now, the orbits of $N^{(123)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are $\{1, 2, 7, 8\}$, and $\{3, 6, 4, 5\}$. We take a representative from each orbit and determine which are belong. We could see that $Nt_1t_2t_3t_1$=$Nt_1t_6 \in [16]$. $Nt_1t_2t_3t_3$=$Nt_1t_2 \in [12]$.

$\mathbf{Nt_1t_3t_2N}$:

We label the new double coset [132] and calculate the coset stabilizer $N^{(132)}$. The coset stabilizer of $Nt_1t_3t_2N$ is given by $N^{(132)} = \{n \in N|Nt_1t_3t_2^n = t_1t_3t_2\} \geq <(1, 3)(2, 6)(4, 7)(5, 8), (1, 4)(3, 7) >$. Also, we find that these cosets $132 \sim 316 \sim 472 \sim 746$ have equal name after we conjugate $Nt_1t_3t_2$ by all elements of N. Since $|N^{(132)}|$=4 then the number of single cosets in [132] is $\frac{|N|}{|N^{(132)}|}$=$\frac{16}{4}$=4 single coset in [132]. Now, the orbits of $N^{(132)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are $\{2, 6\}$, $\{5, 8\}$, and $\{1, 3, 4, 7\}$. We take a representative from each orbit and determine which are belong. We could see that $Nt_1t_3t_2t_2$=$Nt_1t_3 \in [13]$. $Nt_1t_3t_2t_5$=$Nt_1t_3t_2 \in [132]$. $Nt_1t_3t_2t_1$=$Nt_1t_6 \in [16]$.

$\mathbf{Nt_1t_6t_1N}$:

We label the last double coset [161] and calculate the coset stabilizer $N^{(161)}$. The coset stabilizer of $Nt_1t_6t_1N$ is given by $N^{(161)} = \{n \in N | Nt_1t_6t_1^n = t_1t_6t_1\} \geq <(1,2)(3,6)(4,5)(7,8),(1,3)(2,6)(4,7)(5,8)>$. Since $|N^{(161)}|$=8 then the number of single cosets in [161] is $\frac{|N|}{|N^{(161)}|}$=$\frac{16}{8}$=2 single coset in [161].

Now, the orbits of $N^{(161)}$ on $\{1,2,3,4,5,6,7,8\}$ are $\{1,2,3,4,5,6,7,8\}$. We take a representative from the orbit and determine which is belongs. So, we take 1 from the orbit $\{1,2,3,4,5,6,7,8\}$. $Nt_1t_6t_1t_1$=$Nt_1t_6 \in$ [16], so 8 symmetric generators go to [16].

Finally , we could build our Cayley diagram.

$$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(16)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(132)}|} + + \frac{|N|}{|N^{(161)}|}) \times |N|$$

$$|G| \leq (1 + 8 + 8 + 2 + +16 + 4 + 4 + 2) \times 16$$

$$|G| \leq (45 \times 16) \leq 720$$

.

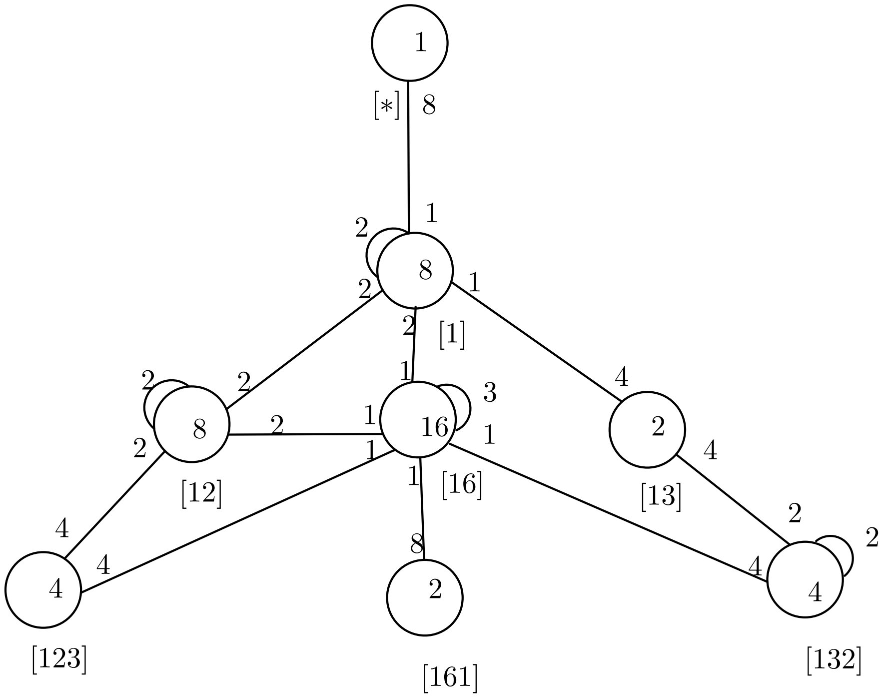

Figure 8.4: Cayley Graph of S_6 over $(2 \times 4) : 2$

8.6 Double Coset Enumeration of $2^5 : S_3 \times S_3$ over $S_3 \times S_3$

Consider the group $\frac{2^{*9}:S_3\times S_3}{((x*w)*t)^4((z*w)*t)^6((x*y)*t)^4((x*y*w^{-1})*t)^6} \cong 2^5 : S_3 \times S_3$, where $x \sim (1,2,9)(3,4,5)(6,7,8)$, $y \sim (3,6)(4,7)(5,8)$, $z \sim (1,2)(4,5)(7,8)$, and $w \sim (1,4,7)(2,5,8)(3,6,9)$.

First, we are going to expand our relations.

The first relation is

$((x*w)*t)^4 = e$

$((x*w)*t_1)^4 = e$ since $t \sim t_1$

$(x*w)^4 * t_1^{(x*w)^3} * t_1^{(x*w)^2} * t_1^{(x*w)} * t_1 = e$

$(x*w)^4 * t_1 * t_6 * t_5 * t_1 = e$

$(x*w)^4 * t_1 * t_6 = t_1 * t_5$

So, we have $t_1 t_6 \sim t_1 t_5$.

The second relation is

$((z*w)*t)^6 = e$

$((z*w)*t_1)^6 = e$ since $t \sim t_1$

$(z*w)^6 * t_1^{(z*w)^5} * t_1^{(z*w)^4} * *t_1^{(z*w)^3} * t_1^{(z*w)^2} * t_1^{(z*w)} * t_1 = e$

$(z*w)^6 * t_8 * t_4 * t_2 * t_7 * t_5 * t_1 = e$

$(z*w)^6 * t_8 * t_4 * t_2 = t_1 * t_5 * t_7$

So, we have $t_8 t_4 t_2 \sim t_1 t_5 t_7$.

The third relation is

$((x*y)*t)^4 = e$

$((x*y)*t_1)^4 = e$ since $t \sim t_1$

$(x*y)^4 * t_1^{(x*y)^3} * t_1^{(x*y)^2} * t_1^{(x*y)} * t_1 = e$

$(x*w)^4 * t_1^{(x*y)^3} * t_9 * t_2 * t_1 = e$

$(x*w)^4 * (3,6)(4,7)(5,8) * t_9 = t_1 * t_2$.

So, we have $N(3,6)(4,7)(5,8)t_9 = N t_1 t_2$.

The last relation is

$((x*y*w^{-1})*t)^6 = e$

$((x*y*w^{-1})*t_1)^6 = e$ since $t \sim t_1$

$(x*y*w^{-1})^6 * t_1^{(x*y*w^{-1})^5} * t_1^{(x*y*w^{-1})^4} * *t_1^{(x*y*w^{-1})^3} * t_1^{(x*y*w^{-1})^2} * t_1^{(x*y*w^{-1})} * t_1 = e$

$t_6 * t_2 * t_7 * t_9 * t_8 * t_1 = e$

So, we have $Nt_6t_2t_7 = Nt_1t_8t_9$.

First, $NeN = \{n \in N|Nen\}$=$\{Ne\}$=$\{N\}$ which we denote by [*]. Our control group N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ so, it has a single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{36}{36}$=1. So, we have one single double coset in a circle [*]. Now, we right multiply our coset representative N by t_1 and find to which double coset Nt_1 belongs. Therefore, we get a new double coset Nt_1N.

Next, we consider Nt_1 be the coset representative of the new double coset Nt_1N. $Nt_1N = \{n \in N|Nt_1n\}$ =$\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9\}$. The coset stabilizer is $N^{(1)}$, and the coset stabilizer of Nt_1 is equal to the point stabilizer N^1. $N^{(1)} = \{n \in N|Nt_1^n = t_1\} \geq < e >$. We compute that $\frac{|N|}{|N^{(1)}|}$=$\frac{36}{4}$=9 single cosets in [1].

Now, the orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are $\{1\}, \{2, 9\}, \{4, 7\}, and\{3, 6, 5, 8\}$. We take a representative from each orbit and determine to which double coset belongs. We observe that Nt_1t_1=Nt_1^2=$N \in$ [*], so it goes to the double coset [*] since t_1^2=e. Nt_1t_2=Nt_1t_4 $\in$[14], so this is a new double coset, 4 symmetric generators extend from [1] to the new double coset [14]. Nt_1t_3=Nt_1t_3 $\in$[13], so this is a new double coset, 4 symmetric generators extend from [1] to the new double coset [13]. Now, we have a new double coset which is called by [14]. We use this definition $Nt_1t_4N = \{n \in N|Nt_1t_4n\}$. The coset stabilizer of Nt_1t_4N is given by $N^{(14)} = \{n \in N|Nt_1t_4^n = t_1t_4\} \geq < (2, 9)(3, 5)(6, 8), (3, 6)(4, 7)(5, 8) >$. Since $|N^{(14)}|$=4 then the number of single cosets in [14] is $\frac{|N|}{|N^{(14)}|}$=$\frac{36}{4}$=9 single coset in [14]. The orbits of $N^{(14)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are $\{1\}, \{2, 9\}, \{4, 7\}, and\{3, 6, 5, 8\}$. Now, we choose a representative from each orbit. $Nt_1t_4t_1$=$Nt_1t_4t_1 \in$ [141], so this is a new double coset, one symmetric generators extends from [1] to the new double coset [141]. $Nt_1t_4t_2$=Nt_1 $\in$[1], so 2 symmetric generators go from [14] to [1]. $Nt_1t_4t_4$=Nt_1 $\in$[1], so 2 symmetric generators extend from [14] to [1]. $Nt_1t_4t_3$=$Nt_1t_3t_1$ $\in$[1], so 4 symmetric generators extend from [14] to the new double coset [131]. Now, we consider $Nt_1t_4t_1N$. $Nt_1t_4t_1N = \{n \in N|Nt_1t_4t_1n\}$. The coset stabilizer of $Nt_1t_4t_1N$ is given by $N^{(141)} = \{n \in N|Nt_1t_4t_1^n = t_1t_4t_1\} \geq < (2, 9)(3, 5)(6, 8), (1, 2, 9)(3, 4, 5)(6, 7, 8), (1, 2)(4, 5)(7, 8) >$. Since $|N^{(141)}|$=36 then the number of single cosets in [141] is $\frac{|N|}{|N^{(141)}|}$=$\frac{36}{36}$=1 single coset in [141]. The orbit of $N^{(141)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We pick a representative from the orbit, we select t_1, and determine to which double coset $Nt_1t_4t_1t_1$ belongs.

Therefore, $Nt_1t_4t_1t_1 = Nt_1t_4e = Nt_1t_4 \in [14]$, so 9 symmetric generators go from [141] to [14]. Now, we have a new double coset which is called by [13]. We use this definition $Nt_1t_3N = \{n \in N | Nt_1t_3n\}$. The coset stabilizer of Nt_1t_3N is given by $N^{(13)} = \{n \in N | Nt_1t_3^n = t_1t_3\} \geq < (1,5,6)(2,3,7)(4,8,9), (2,9)(3,8)(4,7)(5,6) >$. Since $|N^{(13)}|$=6 then the number of single cosets in [13] is $\frac{|N|}{|N^{(13)}|}$=$\frac{36}{6}$=6 single coset in [13]. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9\}$ are $\{1,5,6\}, and\{2,3,9,4,7,8\}$. We take a representative from the orbit, so we select t_3, and t_1 and determine to which double coset $Nt_1t_3t_3$ and $Nt_1t_3t_1$ belongs. Therefore, $Nt_1t_3t_3 = Nt_1e = Nt_1 \in [1]$, so 6 symmetric generators are going to the double coset [1]. $Nt_1t_3t_1 = Nt_1t_3t_1 \in [131]$ this is new double coset denoted by [131]. Thus, 3 symmetric generators are going to the new double coset [131].

We consider $Nt_1t_3t_1N$. $Nt_1t_3t_1N = \{n \in N | Nt_1t_3t_1n\}$. The coset stabilizer of $Nt_1t_3t_1N$ is given by $N^{(131)} = \{n \in N | Nt_1t_3t_1^n = t_1t_3t_1\} \geq$
$< (1,3,8)(2,4,6)(5,7,9), (2,9)(3,8)(4,7)(5,6) >$. Since $|N^{(131)}|$=6 then the number of single cosets in [131] is $\frac{|N|}{|N^{(131)}|}$=$\frac{36}{6}$=6 single coset in [131]. The orbits of $N^{(131)}$ on $\{1,2,3,4,5,6,7,8,9\}$ are $\{1,3,8\}, and\{2,4,9,6,7,5\}$. We take a representative from the orbit, so we select t_2, and t_1 and determine to which double coset $Nt_1t_3t_1t_1$ and $Nt_1t_3t_1t_2$ belongs. $Nt_1t_3t_1t_1 = Nt_1t_3e = Nt_1t_3 \in [13]$, so 3 symmetric generators are going to the double coset [13]. $Nt_1t_3t_1t_2 = Nt_1t_4 \in [14]$. Thus, 6 symmetric generators send from [131] to the double coset [14].

Finally , we could build our Cayley diagram.

$$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(141)}|} + \frac{|N|}{|N^{(131)}|}) \times |N|$$

$$|G| \leq (1 + 9 + 9 + 6 + 1 + 6) \times 36$$

$$|G| \leq (32 \times 36) \leq 1152.$$

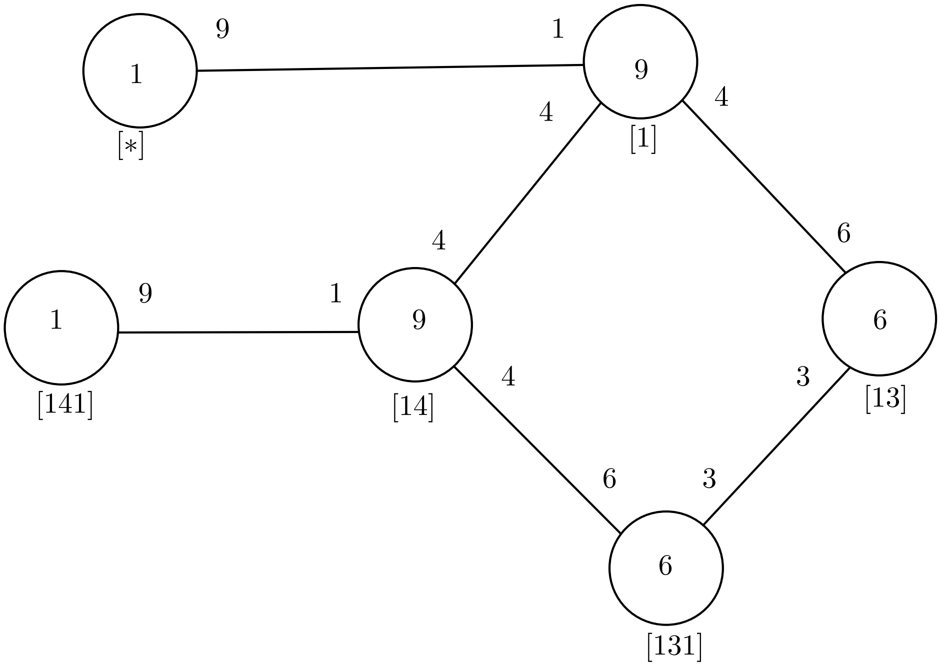

Figure 8.5: Cayley Graph of $2^5 : S_3 \times S_3$ over $S_3 \times S_3$

8.7 Double Coset Enumeration of $3^3 : S_4 \times C_2$ over $S_3 \times S_3$

We factor our group $G = Group < x, y, z, w, t | x^3, y^2, z^2, w^3, x^{-1}*y*x*y, (x^{-1}*z)^2, (y*z)^2, (x, w), (y*w^{-1})^2, z*w^{-1}*z*w, t^2, (t, y), (t, x*z), (w*t)^3, ((x*y)*t)^6, ((x*y*w^{-1})*t)^4 >$ by $(w*t)^3$, $((x*y)*t)^6$, and $((x*y*w^{-1})*t)^4$.
Note, $x \sim (1, 2, 9)(3, 4, 5)(6, 7, 8)$, $y \sim (3, 6)(4, 7)(5, 8)$, $z \sim (1, 2)(4, 5)(7, 8)$, and $w \sim (1, 4, 7)(2, 5, 8)(3, 6, 9)$.
Now, we compute the number of single cosets, so $\frac{|G|}{|N|} = \frac{1296}{36} = 36$ this means the sum of all single cosets should equal 36 in our Caylay diagram. We will now check the number of double cosets of our group by using Magma.

```
#DoubleCosets(G,sub<G|x,y,z,w>,sub<G|x,y,z,w>);
5
```

This indicates that our Cayley diagram has 5 circles.

Now, we will begin to construct our Cayley diagram. We will start with double coset NeN, where e is the word of length zero denote by [*]. $NeN = \{n \in N|Nen\}$=$\{Ne\}$=$\{N\}$ which we denote by [*]. Our control group N is transitive on $\{1,2,3,4,5,6,7,8,9\}$ so, it has a single orbit $\{1,2,3,4,5,6,7,8,9\}$. The number of right cosets in [*] is equal to $\frac{|N|}{|N|}$=$\frac{36}{36}$=1. So, we have one single double coset in a circle [*]. Now, we right multiply our coset representative N by t_1 and find to which double coset Nt_1 belongs. Therefore, we get a new double coset Nt_1N.

Next, we are a new double coset which is labeled by [1]. $Nt_1N = \{n \in N|Nt_1n\}$ =$\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9\}$. The coset stabilizer is $N^{(1)}$, and the coset stabilizer of Nt_1 is equal to the point stabilizer N^1.

$N^{(1)} = \{n \in N|Nt_1^n = t_1\} \geq < e >$. We determine that $\frac{|N|}{|N^{(1)}|}$=$\frac{36}{4}$=9 single cosets in [1]. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9\}$ are $\{1\},\{2,9\},\{4,7\}, and\{3,6,5,8\}$. We pick a representative from each orbit and determine to which double coset belongs. We see that Nt_1t_1=Nt_1^2=$N \in$ [*], so it goes to the double coset [*] since t_1^2=e. Nt_1t_2=Nt_1t_2 $\in$[12]. This is a new double coset, so 2 symmetric generators extend from [1] to the new double coset [12]. Nt_1t_4=Nt_1 $\in$[1], so 2 symmetric generators loop back into [1]. Nt_1t_3 $\in$[13]. This is a new double coset, so 4 symmetric generators extend from [1] to the new double coset [13]. Now, we have a new double coset which is called by [12]. We use this definition $Nt_1t_2N = \{n \in N|Nt_1t_2n\}$. The coset stabilizer of Nt_1t_2N is given by $N^{(12)} = \{n \in N|Nt_1t_2^n = t_1t_2\} \geq < (3,6)(4,7)(5,8), (1,2,9)(3,4,5)(6,7,8) >$. Since $|N^{(12)}|$=6 then the number of single cosets in [12] is $\frac{|N|}{|N^{(12)}|}$=$\frac{36}{6}$=6 single coset in [12]. The orbits of $N^{(12)}$ on $\{1,2,3,4,5,6,7,8,9\}$ are $\{1,2,9\}, and\{3,6,4,7,5,8\}$. We take a representative from each orbit, so we select t_2, and t_3 and determine to which double coset $Nt_1t_2t_2$ and $Nt_1t_2t_3$ belongs. Therefore, $Nt_1t_2t_2 = Nt_1e = Nt_1 \in$ [1], so 3 symmetric generators are going to the double coset [1]. $Nt_1t_2t_3 = Nt_1t_3 \in$ [13]. Thus, 6 symmetric generators are going to the new double coset [13]. We have a new double coset which is called by [13]. $Nt_1t_3N = \{n \in N|Nt_1t_3n\}$. The coset stabilizer of Nt_1t_3N is given by $N^{(13)} = \{n \in N|Nt_1t_3^n = t_1t_3\} \geq < (1,4)(2,3)(5,9)(6,8) >$. Since $|N^{(13)}|$=2 then the number of single cosets in [13] is $\frac{|N|}{|N^{(13)}|}$=$\frac{36}{2}$=18 single coset in [13]. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9\}$ are $\{7\},\{1,4\},\{2,3\},\{5,9\} and\{6,8\}$. We take a representative from each orbit, and determine to which double coset belongs. Therefore, $Nt_1t_3t_7 = Nt_1t_3t_7 \in [137]$ this is a new double coset. $Nt_1t_3t_1 = Nt_1t_3 \in [13]$. Thus, 2 symmetric generators loop back into the double coset [13]. $Nt_1t_3t_3 = Nt_1e = Nt_1 \in [1]$ since $t_3^2 = e$, so 2 symmetric generators send to the double coset [1]. $Nt_1t_3t_5 = Nt_1t_3 \in [13]$, so 2 symmetric generators loop back into [13]. $Nt_1t_3t_6 = Nt_1t_2 \in [12]$, so 2 symmetric generators send to [12]. We have a new double coset which is called by [137]. $Nt_1t_3t_7N = \{n \in N|Nt_1t_3t_7n\}$. The coset stabilizer of $Nt_1t_3t_7N$ is given by $N^{(137)} = \{n \in N|Nt_1t_3t_7^n = t_1t_3t_7\} \geq < (1,2,9)(3,4,5)(6,7,8),(1,4,7)(2,5,8)(3,6,9) >$. Since $|N^{(137)}|$=18 then the number of single cosets in [137] is $\frac{|N|}{|N^{(137)}|}$=$\frac{36}{18}$=2 single coset in [137]. The orbit of $N^{(137)}$ on $\{1,2,3,4,5,6,7,8,9\}$ is $\{1,2,4,9,5,7,3,8,6\}$. We take a representative from the orbit, and determine to which double coset belongs. Therefore, $Nt_1t_3t_7t_7 = Nt_1t_3 \in [13]$, so 9 symmetric generators are going to the double coset [13]. Finally , we could create our Cayley diagram.

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(137)}|}) \times |N|$

$|G| \leq (1+9+6+18+2) \times 36$

$|G| \leq (36 \times 36) \leq 1296.$

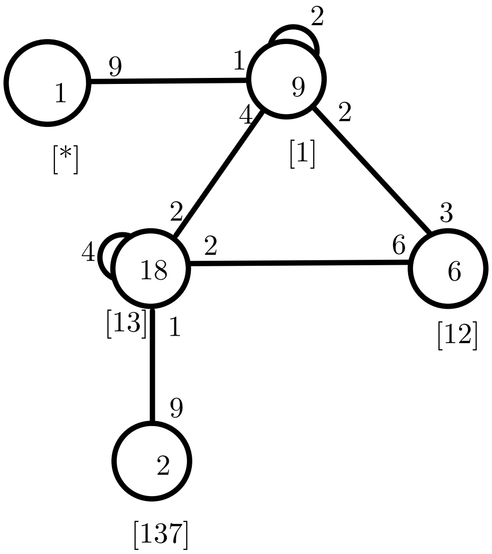

Figure 8.6: Cayley Graph of $3^3 : S_4 \times C_2$ over $S_3 \times S_3$

Chapter 9

Double Coset Enumeration over a Maximal Subgroup

Before we will start to find the double coset enumeration over maximal subgroup for M_{11} over S_5, we have to know why we need to find the coset enumeration over maximal subgroup. Typically, double coset enumeration is performed over the control N as the previous examples. At the time this process can be very complicated and boring. As a result, to make this process is a much easier process, we can achieve the same what we want by doing the process of double coset enumeration over a maximal subgroup of our progenitor. Therefore, to create the double coset enumeration of M over N, we will take $N \leq M \leq G$ and accomplish the single coset enumeration of $G = \cup M t_i^{'} s N$.

9.1 Double Coset Enumeration of M_{11} over Maximal Subgroup S_5

Consider the group G=$\frac{11^{*4}:_m(5:4)}{(y*(t^9))^3(y*(t^{10})^6}$, where $N \cong 5:4$. A symmetric presentation of G is G=$Group < x, y, t | x^5, y^4, y^3 * x^3 * y * x, t^{11}, t^x = t^5, (y * (t^9))^3, (y * (t^{10}))^6 >$. When we computed how many double coset of our group over N , so we get 25 double coset of M_{11} over N. Also, we could fulfill from that, we have a suitable maximal subgroup in our case that gives opartanity to apply double coset enumeration over M. First of all, we ask Magma for maximal subgroups of M as follows:

```
>  M:=MaximalSubgroups(G1);
> M;
Conjugacy classes of subgroups
------------------------------

[1]     Order 48            Length 165
Permutation group acting on a set of cardinality 396
Order = 48 = 2^4 * 3
[2]     Order 120           Length 66
Permutation group acting on a set of cardinality 396
Order = 120 = 2^3 * 3 * 5
[3]     Order 660           Length 12
Permutation group acting on a set of cardinality 396
Order = 660 = 2^2 * 3 * 5 * 11
[4]     Order 144           Length 55
Permutation group acting on a set of cardinality 396
Order = 144 = 2^4 * 3^2
[5]     Order 720           Length 11
Permutation group acting on a set of cardinality 396
Order = 720 = 2^4 * 3^2 * 5
```

Now, we ask Magma to find how many suitable candidates which are divisible by N.

```
>for i in [1..5] do #M[i]`subgroup; end for;
48
120
660
144
720
```

Then, we will conjugate and determine M[2]‘subgroup of order 120 is the maximal subgroup which contains N.

```
> D:=Conjugates(G1,M[2]`subgroup);
> DD:=Setseq(D);
> for i in [1..#DD] do if f(x) in DD[i] and f(y) in DD[i]
then i;end if; end for;
61
> for g in DD[61] do if Order(g) eq 2 and
sub<DD[61]|f(x),f(y),g> eq DD[61] then g@rho;
end if; end for;
g:=WordGroup(G1);
rho:=InverseWordMap(G1);
g2:=function w9 := g.3 * g.1; w10 := w9 * g.2;
```

```
w18 := w10 * g.2; w3 := g.3^-1; w19 := w18* w3;
return w19;end function;
> g2(G);
t * x * y^2 * t^-1
> H:=sub<G|x,y,t * x * y^2 * t^-1>;
> #H;
120
> #DoubleCosets(G,sub<G|x,y>,H);
7
> f, G1, k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(f(H));
G
|  Cyclic(2)
*
|  Alternating(5)
1
```

Now, we can begin our process to find our double coset of M_{11} over $M =< f(x), f(y), f(t * x * y^2 * t^-1) >$ which is isomorphic to S_5.

We need to find the total number of unique cosets of M in G. This is the index of G in M. The index will be the order of G divided by the order of M.
$\frac{|G|}{|M|}$=$\frac{7920}{120}$=66 single cosets and we have 7 Double cosets.

9.1.1 Construction Cayley Graph

Circle[$*$]: First, we look for the first circle which is celled by [*] of our Cayley diagram. We note that our double coset definition changes to we have $MwN = \{n \in N | Mw^n\}$. Thus, our first double coset we have, $MeN = \{n \in N | Me^n\} = \{e \in N | Me\} = \{M\}$. The number of right cosets in [*] is equal to 1, where $|G| = 7920$,and $|N| = 20$. Since N is transitive on
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$
$21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$, so N has two orbits which are $\{9, 4, 17, 12, 20, 3, 25, 28, 11, 36, 19, 35, 2, 33, 27, 10, 18, 34, 26\}$, and
$\{5, 13, 8, 21, 16, 24, 7, 29, 32, 15, 40, 23, 39, 6, 37, 31, 14, 22, 38, 30\}$.
Now, we will pick element from each orbit, so we take the representative 1 of the orbit $\{9, 4, 17, 12, 20, 3, 25, 28, 11, 36, 19, 35, 2, 33, 27, 10, 18, 34, 26\}$, and the representative 5 of

the orbit $\{5, 13, 8, 21, 16, 24, 7, 29, 32, 15, 40, 23, 39, 6, 37, 31, 14, 22, 38, 30\}$.

Therefore, we will have two new double cosets which are Mt_1N and Mt_5N.

Circle[1]: We get a new double coset Mt_1N, so we have the second circle which is called by [1]. We note that $Mt_1N = \{n \in N | Mt_1^n\}$. Now we consider the coset stabilizer $M^{(1)}$, which is equal to the point stabilizer M^1.

$M^{(1)} = \{e, (1, 19)(2, 12)(3, 25)(4, 34)(5, 23)(6, 16)(7, 29)(8, 38)(9, 27)(10, 20)(11, 33)(13, 31)$ $(14, 24)(15, 37)(17, 35)(18, 28)(21, 39)(22, 32)(26, 36)(30, 40)\}$, then the number of single cosets of Mt_1N is $\dfrac{|N|}{|M^{(1)}|}=\dfrac{20}{2}$=10 a single coset in [1]. Now, we are looking at the generator $M^{(1)}$.

So, we can figure out the orbits on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$ $19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ are

$\{1, 19\}, \{2, 12\}, \{3, 25\}, \{4, 34\},$
$\{5, 23\}, \{6, 16\}, \{7, 29\}, \{8, 38\},$
$\{9, 27\}, \{10, 20\}, \{11, 33\}, \{13, 31\},$
$\{14, 24\}, \{15, 37\}, \{17, 35\}, \{18, 28\},$
$\{21, 39\}, \{22, 32\}, \{26, 36\}, \{30, 40\}$

Next, we will take a representative from each orbit. Then, we will determine $\{Mt_1t_1, Mt_1t_{12}, Mt_1t_3, Mt_1t_4, Mt_1t_5, Mt_1t_6, Mt_1t_7, Mt_1t_8, Mt_1t_9, Mt_1t_{10}, Mt_1t_{11},$ $Mt_1t_{13}, Mt_1t_{14}, Mt_1t_{15}, Mt_1t_{17}, Mt_1t_{18}, Mt_1t_{21}, Mt_1t_{22}, Mt_1t_{26}, Mt_1t_{30}\}$ which double coset belong. As a result, $\bullet Mt_1t_1, Mt_1t_3, Mt_1t_4, Mt_1t_7, Mt_1t_{15}, Mt_1t_{17}$ belong to the double coset [5], so the double coset [1] sends 12 elements to [5] since each orbit contains two elements.

$\bullet Mt_1t_{12}$ is new double coset. Also, since Mt_1t_{12} is equal Mt_1t_{22}, so we will consider them as one new double coset which we will denote by [112]. Thus, the double coset [1] sends 4 symmetric generators to [112] , since $\{Mt_1t_2, Mt_1t_{12}\}$, and $\{Mt_1t_{22}, Mt_1t_{32}\}$ contain two elements. Note, we could know that Mt_1t_{12} is equal Mt_1t_{22} by using Magma code as follows:

```
for g in IM do for h in IN do if ts[1]*ts[12] eq
g*(ts[1]*ts[22])^h then "true"; break;
end if; end for; end for;
true
```

•$Mt_1t_5, Mt_1t_9, Mt_1t_{11}, Mt_1t_{13}$ are back to [1], so 8 elements will loop back into the double coset [1].

•Mt_1t_6 is a new double coset which is called [16]. Also, since Mt_1t_6 is equal Mt_1t_{10}, so we will consider them as one new double coset which we will denote by [16]. Thus, the double coset [1] sends 4 symmetric generators to [16], since $\{Mt_1t_6, Mt_1t_{16}\}$, and$\{Mt_1t_{10}, Mt_1t_{20}\}$ contain two elements.

```
for g in IM do for h in IN do if ts[1]*ts[6] eq
g*(ts[1]*ts[10])^h then "true"; break;
end if; end for; end for;
true
```

•Mt_1t_8 is a new double coset which is called [18]. Since Mt_1t_8 is equal $Mt_1t_{14}, Mt_1t_{26}, and$ Mt_1t_{30}, so we will consider them as one new double coset which we will denote by [18]. Thus, the double coset [1] sends 8 elements to [18].

•Mt_1t_{18} is a new double coset which is called [118]. Then, the double coset [1] sends 2 symmetric generators to [118].

•Mt_1t_{21} will go to the double coset M=[*] Then, the double coset [1] sends 2 symmetric generators to the double coset [*].

Circle [5] : Now taking a new double coset Mt_5N, so we have the second circle which we call it by [5]. We note that $Mt_5N = \{n \in N | Mt_5^n\}$. Now we consider the coset stabilizer $M^{(5)}$, which is equal to the point stabilizer M^5.

$M^{(5)} = \{e, (1, 27)(2, 36)(3, 17)(4, 10)(5, 31)(6, 40)(7, 21)(8, 14)(9, 35)(11, 25)(12, 18)(13, 39)$ $(15, 29)(16, 22)(19, 33)(20, 26)(23, 37)(24, 30)(28, 34)(32, 38)\}$, then the number of single cosets of Mt_5N is $\frac{|N|}{|M^{(5)}|}$=$\frac{20}{2}$=10 a single coset in [5]. Now, we are looking at the generator $M^{(5)}$. So, we can figure out the orbits on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ are

{ 1, 27 }, { 2, 36 }, { 3, 17 }, { 4, 10 },
{5, 31 }, { 6, 40 }, {7, 21 }, { 8, 14},
{9, 35 }, { 11, 25 }, { 12, 18}, { 13, 39},
{ 15, 29 }, { 16, 22}, {19, 33 }, { 20, 26 },
{ 23, 37 }, {24, 30 }, { 28, 34}, { 32, 38 }.

Now, we choose a representative from each orbit.

If we select$\{Mt_5t_1, Mt_5t_2, Mt_5t_3, Mt_5t_4, Mt_5t_5, Mt_5t_6, Mt_5t_7, Mt_5t_8, Mt_5t_9, Mt_5t_{12},$

$Mt_5t_{11}, Mt_5t_{13}, Mt_5t_{16}, Mt_5t_{15}, Mt_5t_{19}, Mt_5t_{20}, Mt_5t_{23}, Mt_5t_{24}, Mt_5t_{28}, Mt_5t_{32}\}$ belong. Therefore, we could know that.

•$Mt_5t_1, Mt_5t_5, Mt_5t_7, Mt_5t_8, Mt_5t_{15}, Mt_5t_{19}$ belong to the double coset [1].

So, the double coset [5] sends 12 symmetric generators to [1] since each orbit contains two elements.

•Mt_5t_2 and $Mt_5t_{16} \in$ the double coset [16].

So, the double coset [5] sends 4 symmetric generators to the double coset [16].

•Mt_5t_3, Mt_5t_9,Mt_5t_{13}, and $Mt_5t_{23} \in$ the double coset [5].

So, 8 elements loop back into the double coset [5].

•Mt_5t_4 ,Mt_5t_{12}, Mt_5t_{28}, and $Mt_5t_{32} \in$ the double coset [18].

So, the double coset [5] sends 8 symmetric generators to the double coset [18].

•$Mt_5t_6 \in$ the double coset [118].

So, the double coset [5] sends 2 symmetric generators to the double coset [118].

•$Mt_5t_{11} \in$ the double coset [*].

So, the double coset [5] sends 2 symmetric generators to the double coset [*]=M.

•Mt_5t_{20} and $Mt_5t_{24} \in$ the double coset [112].

So, the double coset [5] sends 4 symmetric generators to the double coset [112].

Circle [16] : Next, we continue the same our previous processes in the new double coset which is denote by [16]. We use the definition $Mt_1t_6N = \{n \in N | Mt_1t_6^n\}$. Now we consider the coset stabilizer $M^{(16)}$, which is equal to the point stabilizer M^{16}. so the coset stabilizer of Mt_1t_6 is given by: $M^{(16)} = \{n \in N | M(t_1t_6)^n = Mt_1t_6\} \geq < e, (1,19)(2,12)(3,25)(4,34)(5,23)(6,16)(7,29)(8,38)(9,27)(10,20)(11,33)(13,31)(14,24) (15,37)(17,35)(18,28)(21,39)(22,32)(26,36)(30,40) >$.

The number of single cosets in the double coset [16] is $\frac{|N|}{|M^{(16)}|}$=$\frac{20}{2}$=10 a single coset in [16]. Now, we are looking at the generator $M^{(16)}$. So, we can see the orbits on $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$
$21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40\}$ are

$\{1,19\}, \{2,12\}, \{3,25\}, \{4,34\},$
$\{5,23\}, \{6,16\}, \{7,29\}, \{8,38\},$
$\{9,27\}, \{10,20\}, \{11,33\}, \{13,31\},$
$\{14,24\}, \{15,37\}, \{17,35\}, \{18,28\},$
$\{21,39\}, \{22,32\}, \{26,36\}, \{30,40\}$

Next, we will take a representative from each orbit.Then,we will determine $\{Mt_1t_6t_1, Mt_1t_6t_{12}, Mt_1t_6t_3, Mt_1t_6t_4, Mt_1t_6t_5, Mt_1t_6t_6, Mt_1t_6t_7, Mt_1t_6t_8, Mt_1t_6t_9,$ $Mt_1t_6t_{10}, Mt_1t_6t_{11}, Mt_1t_6t_{13}, Mt_1t_6t_{14}, Mt_1t_6t_{15}, Mt_1t_6t_{17}, Mt_1t_6t_{18}, Mt_1t_6t_{21}, Mt_1t_6t_{22},$ $Mt_1t_6t_{26}, Mt_1t_6t_{30}\}$which double coset belong.

As a result, •$Mt_1t_6t_1, Mt_1t_3, Mt_1t_6t_6$ belong to the double coset [16], so 4 elements loop back into the double coset[16].

•$Mt_1t_6t_4$ and $Mt_1t_6t_{17} \in$ the double coset [118].

So, the double coset [16] sends 4 elements to the double coset [118].

•$Mt_1t_6t_3, Mt_1t_6t_8, Mt_1t_6t_9, Mt_1t_6t_{10}, Mt_1t_6t_{11}, Mt_1t_6t_{18}, Mt_1t_6t_{21}, Mt_1t_6t_{30}$ belong to the double coset [18], the double coset [16] sends 16 elements to the double coset [18].

•$Mt_1t_6t_7, Mt_1t_6t_{13}, Mt_1t_6t_{14}, Mt_1t_6t_{22}$ belong to the double coset [112], the double coset [16] sends 8 elements to the double coset [112].

Circle[18]: The fifth circle which is represented by a new double coset $Mt_{18}N$. We note that $Mt_{18}N = \{n \in N | Mt_{18}^n\}$. Now we consider the coset stabilizer $M^{(18)}$, which is equal to the point stabilizer M^{18}.

$M^{(18)} = \{e\}$, then the number of single cosets of $Mt_{18}N$ is $\frac{|N|}{|M^{(18)}|}$=$\frac{20}{1}$=20 a single coset in [18]. Now, we will see at the generator $M^{(18)}$. So, we can find the orbits on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$ $21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, ..., \{40\}$. Next, we will determinate every orbit which double coset is belong. Then, we found that by helping Magma.

•$Mt_1t_8t_1, Mt_1t_8t_8, Mt_1t_8t_{15}, Mt_1t_8t_{17}, Mt_1t_8t_{19}, Mt_1t_8t_{20}, Mt_1t_8t_{26}, Mt_1t_8t_{34}$ belong to the double coset [112], so 8 symmetric generators extend from the double coset [18] to [112].

•$Mt_1t_8t_2, Mt_1t_8t_{21}, Mt_1t_8t_{27}, Mt_1t_8t_{30}, Mt_1t_8t_{36}, Mt_1t_8t_{16}, Mt_1t_8t_{39}, Mt_1t_8t_{40}$ belong to the double coset [16], so 8 symmetric generators extend from the double coset [18] to [16].

•$Mt_1t_8t_4, Mt_1t_8t_5, Mt_1t_8t_6, Mt_1t_8t_{11}, Mt_1t_8t_{12}, Mt_1t_8t_{14}, Mt_1t_8t_{16}, Mt_1t_8t_{18}, Mt_1t_8t_{25},$ $Mt_1t_8t_{29}, Mt_1t_8t_{31}, Mt_1t_8t_{35} \in [18]$, so 12 symmetric generators loop back into the double coset [18].

•$Mt_1t_8t_3, Mt_1t_8t_{28}, Mt_1t_8t_{33}, Mt_1t_8t_{38} \in [1]$, so 4 symmetric generators send from the double coset [18] to the double coset [1].

•$Mt_1t_8t_7, Mt_1t_8t_{10}, Mt_1t_8t_{13}, Mt_1t_8t_{24} \in [5]$, so 4 symmetric generators send from the double coset [18] to the double coset [5].

•$Mt_1t_8t_9, Mt_1t_8t_{22}, Mt_1t_8t_{23}, Mt_1t_8t_{32} \in [118]$, so 4 symmetric generators send from the double coset [18] to the double coset [118].

Circle[112]: The sixth double coset which labels by [112]. $Mt_1t_{12}N = \{n \in N | M(t_1t_{12})n\}$. Now, we will determine the coset stabilizer $M^{(112)}$, so the coset stabilizer of Mt_1t_{12} is given by:

$M^{(112)} = \{n \in N | M(t_1t_{12})^n = Mt_1t_{12}\} \geq < e, (1, 11)(2, 28)(3, 33)(4, 18)(5, 15)(6, 32)$
$(7, 37)(8, 22)(9, 19)(10, 36)(12, 26)(13, 23)(14, 40)(16, 30)(17, 27)(20, 34)(21, 31)(24, 38)$
$(25, 35)(29, 39) >$.

• The number of single cosets in the double coset [112] is $\dfrac{|N|}{|M^{(112)}|}=\dfrac{20}{2}=10$ a single coset in [112].

We can see that the orbits of $M^{(112)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ are
$\{1, 11\}, \{2, 28\}, \{3, 33\}, \{4, 18\}, \{5, 15\}$,
$\{6, 32\}, \{7, 37\}, \{8, 22\}, \{9, 19\}, \{10, 36\}$,
$\{12, 26\}, \{13, 23\}, \{14, 40\}, \{16, 30\}, \{17, 27\}$,
$\{20, 34\}, \{21, 31\}, \{24, 38\}, \{25, 35\}, \{29, 39\}$.

We now take the representative from each orbit and we see where it goes. Therefore, we could know that.

•$Mt_1t_{12}t_1, Mt_1t_{12}t_4, Mt_1t_{12}t_{12}, Mt_1t_{12}t_{25}$ belong to the double coset [16].

So, the double coset [112] sends 8 elements to [16] since each orbit contains two elements.

•$Mt_1t_{12}t_2, Mt_1t_{12}t_3, Mt_1t_{12}t_7, Mt_1t_{12}t_8, Mt_1t_{12}t_{14}, Mt_1t_{12}t_{21}, Mt_1t_{12}t_{24}, Mt_1t_{12}t_{29}$ belong to the double coset [18].

So, the double coset [112] sends 16 symmetric generators to [18] since each orbit contains two elements.

•$Mt_1t_{12}t_5, Mt_1t_{12}t_{20}$ belong to the double coset [118].

So, the double coset [112] sends 4 symmetric generators to [118] since each orbit contains two elements.

•$Mt_1t_{12}t_6, Mt_1t_{12}t_9$ belong to the double coset [1].

So, the double coset [112] sends 4 symmetric generators to [1] since each orbit contains two elements.

•$Mt_1t_{12}t_{16}, Mt_1t_{12}t_{17}$ belong to the double coset [5].

So, the double coset [112] sends 4 symmetric generators to [5] since each orbit contains two elements.

•$Mt_1t_{12}t_{10}, Mt_1t_{12}t_{13} \in$ [112] So, 4 symmetric generators loop back into [112].

Circle[118]: The last double coset which labels by [118]. $Mt_1t_{18}N = \{n \in N | M(t_1t_{18})n\}$. Now, we will determine the coset stabilizer $M^{(118)}$, so the coset stabilizer of Mt_1t_{18} is given by:

$M^{(118)} = \{n \in N | M(t_1t_{18})^n = Mt_1t_{18}\} \geq < e, (1, 34, 11, 20)(2, 19, 28, 9)(3, 12, 33, 26)$ $(4, 25, 18, 35)(5, 38, 15, 24)(6, 23, 32, 13)(7, 16, 37, 30)(8, 29, 22, 39)(10, 27, 36, 17)(14, 31,$ $40, 21) >$.

• The number of single cosets in the double coset [118] is $\frac{|N|}{|M^{(118)}|}$=$\frac{20}{4}$=5 a single coset in [118].

We can see that the orbits of $M^{(118)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$ $19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ are

$\{1, 34, 11, 20\}, \{2, 19, 28, 9\}, \{3, 12, 33, 26\}, \{4, 25, 18, 35\},$
$\{5, 38, 15, 24\}, \{6, 23, 32, 13\}, \{7, 16, 37, 30\}, \{8, 29, 22, 39\},$
$\{10, 27, 36, 17\}, \{14, 31, 40, 21\}$.

We now take the representative from each orbit and we see where it goes. Therefore, we could know that.

•$Mt_1t_{18}t_1, Mt_1t_{18}t_2, Mt_1t_{18}t_6, Mt_1t_{18}t_{10}$ belong to the double coset [18].

So, the double coset [118] sends 16 symmetric generators to [18] since each orbit contains 4 elements.

•$Mt_1t_{18}t_3$ belong to the double coset [5].

So, the double coset [118] sends 4 symmetric generators to the double coset [5] since each orbit contains 4 elements.

•$Mt_1t_{18}t_5$ belong to the double coset [1].

So, the double coset [118] sends 4 symmetric generators to [1] since each orbit contains 4 elements.

•$Mt_1t_{18}t_8, Mt_1t_{18}t_{14}$ belong to the double coset [16].

So, the double coset [118] sends 8 symmetric generators to [16] since each orbit contains 4 elements.

•$Mt_1t_{18}t_4, Mt_1t_{18}t_7$ belong to the double coset [112].

So, the double coset [118] sends 8 symmetric generators to [112] since each orbit contains 4 elements.

Finally, we could complete our Caylay diagram of M_{11} over Maximal Subgroup S_5.

$$|G| \leq (|M| + \frac{|N|}{|M^{(1)}|} + \frac{|N|}{|M^{(5)}|} + \frac{|N|}{|M^{(16)}|} + \frac{|N|}{|M^{(18)}|} + \frac{|N|}{|M^{(112)}|} + \frac{|N|}{|M^{(118)}|}) \times 120$$
$$|G| \leq (1 + 10 + 10 + 10 + 20 + 10 + 5) \times 120$$
$$|G| \leq (66 \times 120) \leq 7920$$

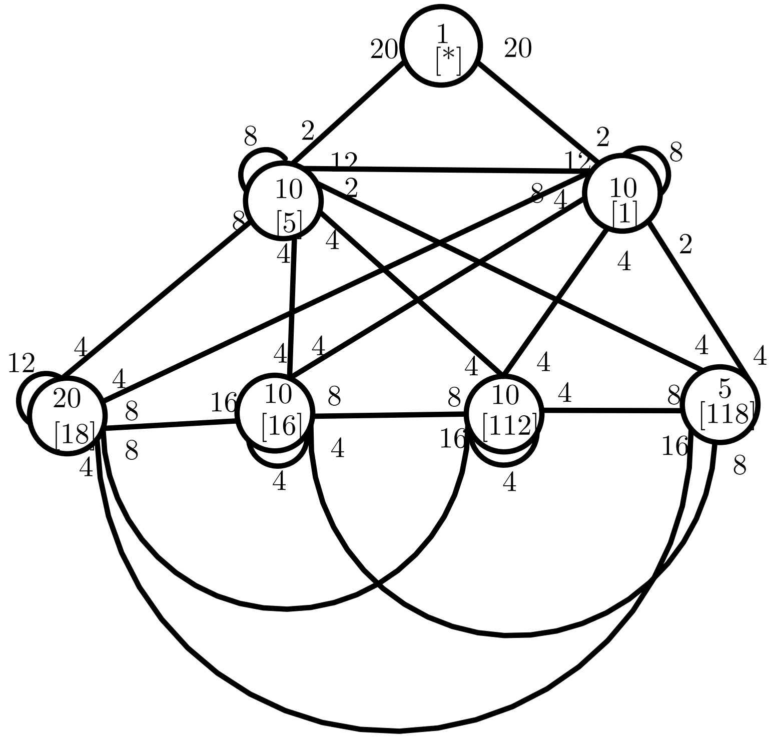

Figure 9.1: Cayley Graph of M_{11} over Maximal Subgroup S_5

9.2 Double Coset Enumeration of $U(3,5)$ over Maximal Subgroup A_7

We will apply manual double coset enumeration of $U(3,5)$ over $M \cong A_7$. Consider the group $G = \frac{5^{*6}:_m S_5}{(y*x^2*t^{(y*x^2)})^4,(y*x^2*t^2)^7}$. A symmetric presentation of G is $G =< x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t^5,(t,y*x^-1*y),t^{(x*y*x^-2)} = t^4,(y*x^2*t^{(y*x^2)})^4,(y*x^2*t^2)^7 >$. Where the order of the group $|G| \cong |U(3,5)| = 126000$ and $|N \cong S_5|$=120.

We now find a suitable maximal subgroup that contains N for performing the maximal double coset enumeration of $U(3,5)$ over M and N. First, we have to find the maximal subgroup as follows:

```
M:=MaximalSubgroups(G1);
M;
Conjugacy classes of subgroups
------------------------------

[1]     Order 240          Length 525
Permutation group acting on a set of cardinality 1050
Order = 240 = 2^4 * 3 * 5
[2]     Order 1000         Length 126
Permutation group acting on a set of cardinality 1050
Order = 1000 = 2^3 * 5^3
[3]     Order 720          Length 175
Permutation group acting on a set of cardinality 1050
Order = 720 = 2^4 * 3^2 * 5
[4]     Order 720          Length 175
Permutation group acting on a set of cardinality 1050
Order = 720 = 2^4 * 3^2 * 5
[5]     Order 720          Length 175
Permutation group acting on a set of cardinality 1050
Order = 720 = 2^4 * 3^2 * 5
[6]     Order 2520         Length 50
Permutation group acting on a set of cardinality 1050
Order = 2520 = 2^3 * 3^2 * 5 * 7
[7]     Order 2520         Length 50
Permutation group acting on a set of cardinality 1050
Order = 2520 = 2^3 * 3^2 * 5 * 7
[8]     Order 2520         Length 50
Permutation group acting on a set of cardinality 1050
Order = 2520 = 2^3 * 3^2 * 5 * 7
```

To find all maximal subgroup of G by $\frac{|M|}{|N|}$. In order to determine which subgroup contains N, we will use this following Magma code:

```
C:=Conjugates(G1,M[8]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i] then i;
end if; end for;
28
```

So, we will use this M[8]'subgroup. In appendix, we can see how we could find the relation which will create M.

Thus, our $M \cong A_7$ and its order is 2520. Also, we have 5 double cosets and 50 single cosets.

```
M:=sub<G|x,y,t * x * t^-1>;
#M;
2520
#DoubleCosets(G,M,sub<G|x,y>);
5
IM:=sub<G1|f(x),f(y),f(t * x * t^-1)>;
CompositionFactors(IM);
G
|  Alternating(7)
1
#G/#M;
50
```

Now, we can begin our method to find the double coset enumeration of $U(3,5)$ over the maximal subgroup A_7 as our previous example.

The first double coset which is labeling by [*] in our Cayley diagram. We have that our double coset definition is $MwN = \{n \in N|Mw^n\}$.Thus, our first double coset we have, $MeN = \{n \in N|Me^n\} = \{e \in N|Me\} = \{M\}$=[*].

The number of right cosets in [*] is equal to 1, where $|G| = 126000$, and $|N| = 120$. Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$ $21, 22, 23, 24\}$. So, N has one orbit which is $\{1, 6, 3, 11, 5, 4, 12, 22, 2, 8, 17, 20, 7, 10, 18, 19, 16, 9, 14, 23, 24, 21, 13, 15\}$.

We now take element from the orbit, so we take the representative 1 of the orbit $\{1, 6, 3, 11, 5, 4, 12, 22, 2, 8, 17, 20, 7, 10, 18, 19, 16, 9, 14, 23, 24, 21, 13, 15\}$ Therefore, we will have one new double coset which is Mt_1N.

The second double coset which is labeling by [1] in our Cayley diagram. We note that $Mt_1N = \{n \in N | Mt_1^n\}$. Now we consider the coset stabilizer $M^{(1)}$, which is equal to the point stabilizer M^1.

$M^{(1)} = \{e, (2,21,6,16,23)(3,12,22,5,8)(4,11,14,9,18)(10,17,20,15,24),$
$(1,6,16)(2,11,9)(3,20,5)(4,13,18)(7,12,22)(8,17,15)(10,19,24)(14,23,21)\}$. Then, the number of single cosets of Mt_1N is $\dfrac{|N|}{|M^{(1)}|}$=$\dfrac{120}{60}$=2 a single coset in [1].

Now, we are looking at the generator $M^{(1)}$.

So, we can see the orbits on$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,$
$18,19,20,21,22,23,24\}$ are $\{1,6,11,4,2,18,16,9,14,23,21,13\}$ and $\{3,12,20,$
$17,22,7,19,10,24,8,5,15\}$.

Next, we will take a representative from each orbit. We could determine that.

•$Mt_1t_1 \in [11]$ is the new double coset.

So, the double coset [1] sends 12 symmetric generators to the new double coset [11].

•$Mt_1t_3 \in$ [*].

So, the double coset [1] sends 12 symmetric generators to the double coset [*].

The third double coset which is labeling by [11] in our Cayley diagram. We want to indicate that $Mt_1t_1N = \{n \in N | Mt_1{t_1}^n\}$ and the coset stabilizer $M^{(11)}$, which is equal to the point stabilizer M^{11}.

$M^{(11)} = \{e, (2,16,21,23,6)(3,5,12,8,22)(4,9,11,18,14)(10,15,17,24,20)\}$, then the number of single cosets of Mt_1t_1N is $\dfrac{|N|}{|M^{(11)}|}$=$\dfrac{120}{10}$=12 a single coset in [11].

Now, we can see the orbits on$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$
$21,22,23,24\}$ are $\{1,13\}$, $\{7,19\}$, $\{2,6,4,14,18,9,11,23,16,21\}$,and $\{3,22,24,17,15,20,$
$10,8,5,12\}$.

Next, we will take a representative from each orbit, so we will pick 13 from the orbit $\{1,13\}$, 7 from the orbit $\{7,19\}$, 2 from the orbit$\{2,6,4,14,18,9,11,23,16,21\}$, and 3 from $\{3,22,24,17,15,20,10,8,5,12\}$. Then, we can determine that.

•$Mt_1t_1t_{13} \in [11]$.

So, 2 symmetric generators loop back into the double coset [11]. We could figure this out by the following Magma code:

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[13]
eq g*(ts[1]*ts[1])^h then "true"; break;
```

```
end if; end for; end for;
true
```

•$Mt_1t_1t_7 \in [11]$.

So, 2 symmetric generators go to the double coset [1].

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[7]
eq g*(ts[1])^h then "true"; break;
end if; end for; end for;
true
```

•$Mt_1t_1t_2 \in [112]$ this is a new double coset.

So, 10 symmetric generators send from the double coset [11] to the new double coset [112].

•$Mt_1t_1t_3 \in [113]$ this is a new double coset.

So, 10 symmetric generators send from the double coset [11] to the new double coset [113].

The forth double coset which is labeling by [112] in our Cayley diagram. We can see that $Mt_1t_1t_2N = \{n \in N | Mt_1t_1t_2^n\}$ and the coset stabilizer $M^{(112)}$, which is equal to the point stabilizer M^{112}. $M^{(112)}$=$\{e, (1,20)(2,7)(3,6)(4,17)(5,16)(8,13)(9,12)$ $(10,23)(11,22)(14,19)(15,18)(21,24)\}$, then the number of single cosets of $Mt_1t_1t_2N$ is $\frac{|N|}{|M^{(112)}|}$=$\frac{120}{4}$=30 a single coset in [112].

Now, we can see the orbits on$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$ $21,22,23,24\}$ are $\{1,20,8,13\}, \{2,7,19,14\}, \{3,6,4,17\}, \{5,16,18,15\}, \{9,12,10,23\},$ $\{11,22,24,21\}$. We will pick a representative from each orbit, so we determine that.

•$Mt_1t_1t_2t_1 \in [11]$.

So, 4 symmetric generators send from the double coset [112] to the double coset [11].

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[1]
eq g*(ts[1]*ts[1])^h then "true";
break; end if; end for; end for;
true
```

•$Mt_1t_1t_2t_2$, $Mt_1t_1t_2t_3$, $Mt_1t_1t_2t_5$, $Mt_1t_1t_2t_9$, and $Mt_1t_1t_2t_{11} \in [112]$.

So, 20 symmetric generators loop back into the double coset [112].

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[2]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[3]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
true
```

The last double coset which is labeling by [113] in our Cayley diagram. We can see that $Mt_1t_1t_3N = \{n \in N|Mt_1t_1t_3^n\}$ and the coset stabilizer $M^{(113)}$, which is equal to the point stabilizer M^{113}.
$M^{(113)} = \{(1,3)(2,22)(4,8)(5,6)(7,9)(10,14)(11,12)(13,15)(16,20)(17,18)(19,21)(23,24),(1,20)(2,7)(3,6)(4,17)(5,16)(8,13)(9,12)(10,23)(11,22)(14,19)(15,18)(21,24)\}$, then the number of single cosets of $Mt_1t_1t_3N$ is $\frac{|N|}{|M^{(113)}|}$=$\frac{120}{24}$=5 a single coset in [113].
Now, the orbit on$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\}$ is $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\}$.
We will pick a representative from the orbit, so we determine that.
•$Mt_1t_1t_3t_1 \in [11]$. So, 24 symmetric generators send from the double coset [113] to the double coset [11].

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[3]*ts[1]
eq g*(ts[1]*ts[1])^h then "true"; break;
end if; end for; end for;
true
```

Finally, we conclude that our Caylay diagram of $U(3,5)$ over maximal subgroup A_7.

$$|G| \leq (|M| + \frac{|N|}{|M^{(1)}|} + \frac{|N|}{|M^{(11)}|} + \frac{|N|}{|M^{(112)}|} + \frac{|N|}{|M^{(113)}|}) \times 2520$$
$$|G| \leq (1+2+12+30+5) \times 2520$$
$$|G| \leq (50 \times 2520) \leq 126000.$$

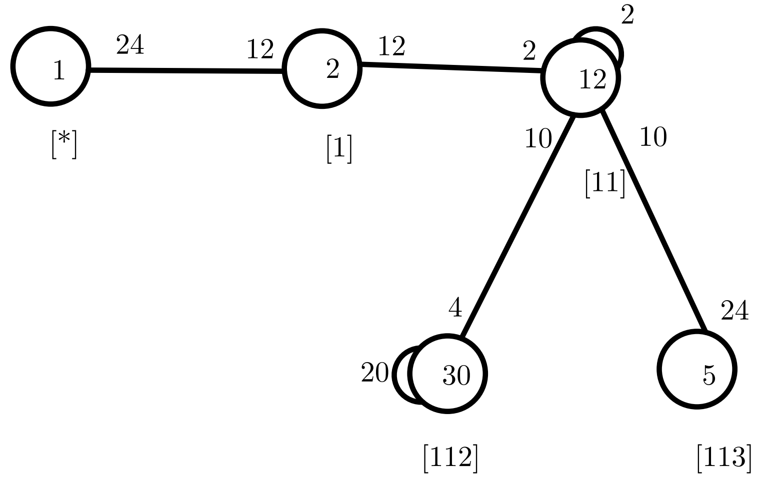

Figure 9.2: Cayley Graph of $U(3,5)$ over Maximal Subgroup A_7

Appendix A

MAGMA Code for Extension Types

A.1 $G \cong 2^9 : PGL(2,7)$

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,x*y^
-1*x*y,
y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^8,(y*t)^3,(x*w*t)^0,
(w*x*y*t)^0,(w*x*y*t)^7>;
#G;
f,G1,k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
N:=sub<G1|f(w),f(x),f(y),f(z)>;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then i;end if;end for;
NL[4];
X:=[2,2,2,2,2,2,2,2,2];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
FPGroup(NL[4]);
S<w,x,y,z,h,o,p,u,r>:=Group<w,x,y,z,h,o,p,u,r| w^2,x^2,y^2,z^2,
h^2,o^2,p^2,u^2,r^2,(w * x)^2,(w * y)^2,(x * y)^2,(w * z)^2,
(x * z)^2,(y * z)^2,(w * h)^2,
(x * h)^2,(y* h)^2,(z * h)^2,(w * o)^2,(x* o)^2,(y * o)^2,
```

```
(z * o)^2,(h * o)^2,(w * p)^2,(x * p)^2,
(y * p)^2,(z* p)^2,(h * p)^2,(o * p)^2,(w * u)^2,(x * u)^2,
(y * u)^2, (z * u)^2,(h * u)^2,(o * u)^2,(p * u)^2,
(w * r)^2,(x* r)^2,(y * r)^2,
(z * r)^2,(h * r)^2,(o * r)^2,(p* r)^2,(u * r)^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[4]);
s;
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
FPGroup(q);
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a*b)^2,(c*b)^3,(a*c)^8,
b*a*c*b*a*c*b*a*c*b*a*c*a*b*c*a*b*c*a*b*c>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
CompositionFactors(q);
NL1:=NormalLattice(q);
NL1;
#PGL(2,7);
EE:=q!(1, 2)(3, 4)(5, 6)(7, 8);
FF:=q! (1, 2)(5, 7)(6, 8);
GG:=q!(2, 3)(4, 5)(6, 8);
ff(T[2]);
ff(T[3]);
ff(T[4]);
ff(T[5]);
ff(T[6]);
ff(T[2]) eq EE;
ff(T[3]) eq FF;
ff(T[4]) eq GG;
q eq sub<q|EE,FF,GG>;
A:=G1!NL[4].1;
B:=G1!NL[4].2;
C:=G1!NL[4].3;
D:=G1!NL[4].4;
F:=G1!NL[4].5;
H:=G1!NL[4].6;
P:=G1!NL[4].7;
O:=G1!NL[4].8;
V:=G1!NL[4].9;
J:=G1!T[2];
I:=G1!T[3];
```

```
M:=G1!T[4];
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if A^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if B^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if B^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if B^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if C^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if C^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if C^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if D^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if D^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
```

```
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if D^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if F^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if F^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if F^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if H^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if H^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if H^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if P^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if P^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if P^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if O^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
```

```
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if O^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if O^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if V^J eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if V^I eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
for i,j,k,l,m,mm,ii,jj,kk in [1..2] do if V^M eq
A^i*B^j*C^k*D^l*F^m*H^mm*P^ii*O^jj*V^kk then i,j,k,l,m,mm,ii,
jj,kk;
end if; end for;
H<w,x,y,z,h,o,p,u,r,a,b,c>:=Group<w,x,y,z,h,o,p,u,r,a,b,c|
 w^2,x^2,y^2,
z^2,h^2,o^2,p^2,u^2,r^2,(w * x)^2,(w * y)^2,(x * y)^2,
(w * z)^2,(x * z)^2,(y * z)^2,(w * h)^2,(x * h)^2,(y* h)^2,
(z * h)^2,
(w * o)^2,(x* o)^2,(y * o)^2,(z * o)^2,(h * o)^2,(w * p)^2,
(x * p)^2,
(y * p)^2,(z* p)^2,(h * p)^2,(o * p)^2,(w * u)^2,(x * u)^2,
(y * u)^2,
(z * u)^2,
(h * u)^2,(o * u)^2,(p * u)^2,(w * r)^2,(x* r)^2,(y * r)^2,
(z * r)^2,
(h * r)^2,
(o * r)^2,(p* r)^2,(u*r)^2,w^a=w*y,w^b=w*y,w^c=w,x^a=x*y,
x^b=x,x^c=x*y*z,
y^a=y,y^b=y,y^c=z,z^a=y*z*u,z^b=y*z,z^c=y,
h^a=h,h^b=y*h,h^c=y*h*o*u,o^a=y*h*o*p*u,o^b=x*o*u,o^c=y*u,
p^a=p,p^b=p,p^c=y*h*o*p*r,
u^a=u,u^b=u,u^c=z*o,r^a=p*r,r^b=p*u*r,r^c=y*z*r,a^2,b^2,
c^2,(a*b)^2,(c*b)^3,(a*c)^8,b*a*c*b*a*c*b*a*c*b*a*c*a*b*c*
a*b*c*a*b*c>;
#H;
#G1;
```

```
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);s;
```

A.2 $G \cong 3^4 : (3^2 * 2^3) \times 2$

```
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^6,(z*w*t)^3,
(z*x*t)^6,(y*x*t)^4,(y*x*z*t)^0>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then
i;end if;end for;
NL[4];
X:=[3,3,3,3];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
CompositionFactors(q);
Center(G1);
nl:=NormalLattice(q);
nl;
E:=DirectProduct(nl[17],nl[2]);
IsIsomorphic(E,q);
FPGroup(q);
H<e,f,g,h,o>:=Group<e,f,g,h,o|e^2,f^2,g^2,h^2,o^2,(e*g)^2,
(f*g)^2,
(e*h)^2,(f * h)^2,(g * h)^2,(e* o)^2,(g *o)^2,
e * f* e * f * h,h * o * h * g * o * h * o,(f *o * f *o* h)
^2>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
FPGroup(NL[4]);
S<a,b,c,d>:=Group<a,b,c,d|a^3,b^3,c^3,d^3,(a,b),(a,c),(b,c),
(a,d),(b,d),
(c,d)>;
```

```
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[4]);
s;
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
EE:=q!(2, 4)(5, 6)(7, 11)(8, 10)(9, 12);
FF:=q!(1, 2)(3, 4)(5, 7)(6, 9)(8, 12)(10, 11);
GG:=q!(1, 3)(2, 4)(5, 6)(7, 9)(8, 10)(11, 12);
DD:=q!(1, 3)(2, 4)(5, 8)(6, 10)(7, 12)(9, 11);
KK:=q!(2, 5)(4, 6)(7, 11)(9, 12);
T:=Transversal(G1,NL[4]);
ff(T[2]) eq EE;
ff(T[3]) eq FF;
ff(T[4]) eq GG;
ff(T[5]) eq  DD;
ff(T[6]) eq  KK;
Generators(NL[4]);
A:=G1!NL[4].1;
B:=G1!NL[4].2;
C:=G1!NL[4].3;
D:=G1!NL[4].4;
H:=G1!T[2];
J:=G1!T[3];
I:=G1!T[4];
M:=G1!T[5];
O:=G1!T[6];
for i,j,k,l in [1..3] do if A^H eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if A^J eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if A^I eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if A^M eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if A^O eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if B^H eq A^i*B^j*C^k*D^l then i,
j,k,l;
```

```
end if; end for;
for i,j,k,l in [1..3] do if B^J eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if B^I eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if B^M eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if B^O eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if C^H eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if C^J eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if C^I eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if C^M eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if C^O eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if D^H eq A^i*B^j*C^k*D^l then i,
j,k,l; end if; end for;
for i,j,k,l in [1..3] do if D^J eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if D^I eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if D^M eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
for i,j,k,l in [1..3] do if D^O eq A^i*B^j*C^k*D^l then i,
j,k,l;
end if; end for;
M<a,b,c,d,e,f,g,h,o>:=Group<a,b,c,d,e,f,g,h,o|a^3,b^3,c^3,
d^3,
```

```
(a,b),(a,c),(b,c),(a,d),(b,d),(c,d),e^2,f^2,g^2,h^2,o^2,
(e*g)^2,(f*g)^2,(e*h)^2,(f * h)^2,(g * h)^2,(e* o)^2,(g *o)^2,
e*f*e*f*h,h*o*h*g*o*h*o,(f *o * f *o* h)^2,
a^e=a,a^f=a*c,a^g=a^2,a^h=a*b,a^o=a^2,b^e=b,b^f=b*c,b^g=b^2,
b^h=b^2,
b^o=b,c^e=b*c^2,c^f=c^2,c^g=c^2,c^h=c^2,c^o=c*d^2,
d^e=d^2,d^f=d,d^g=d^2,d^h=d,d^o=d^2>;
#M;
#G1;
f1,M1,k1:=CosetAction(M,sub<M|Id(M)>);
s:=IsIsomorphic(G1,M1);s;
```

A.3 $5^4 : D_4 \times S_3$

```
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^4,(z*w*t)^0,
(z*x*t)^3,(y*x*t)^10,(y*x*z*t)^6>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
NL[2];
Center(G1);
X:=[5,5,5,5];
IsIsomorphic(NL[2],AbelianGroup(GrpPerm,(X)));
H:=NL[2];
q,ff:=quo<G1|NL[2]>;
q;
IsAbelian(q);
FPGroup(NL[2]);
S<w,x,y,z>:=Group<w,x,y,z|w^5,x^5,y^5,z^5,(w,x),(w,y),(x,y),
(w,z),(x,z),(y,z)>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[2]);
s;
FPGroup(q);
H<e,f,g,h,v>:=Group<e,f,g,h,v|e^2,f^2,g^2,h^2,v^2,(e*g)^2,
```

```
(f*g)^2,(e*h)^2,(f * h)^2,(g * h)^2,
(e * v)^2,(h * v)^2,e * f * e * f * h,
v * e * f * e * v * f,g * v * g * e * v * g* v >;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
nl:=NormalLattice(q);
nl;
E:=DirectProduct(nl[10],nl[7]);
IsIsomorphic(E,q);
IsIsomorphic(nl[7],SymmetricGroup(3));
nl[7];
nl[10];
IsIsomorphic(DihedralGroup(4),nl[10]);
E:=DirectProduct(DihedralGroup(4),SymmetricGroup(3));
IsIsomorphic(E,q);
T:=Transversal(G1,NL[2]);
A:=G1!NL[2].1;
B:=G1!NL[2].2;
C:=G1!NL[2].3;
D:=G1!NL[2].4;
q;
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
ff(T[4]) eq q.3;
ff(T[5]) eq q.4;
ff(T[6]) eq q.6;
EE:=q!q.1;
FF:=q!q.2;
GG:=q!q.3;
DD:=q!q.4;
CC:=q!q.6;
ff(T[3]) eq FF;
ff(T[2]) eq EE;
ff(T[4]) eq GG;
ff(T[5]) eq DD;
ff(T[6]) eq CC;
q eq sub<q|EE,FF,GG,DD,CC>;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
O:=G1!T[5];
Q:=G1!T[6];
for i,j,k,l in [1..5] do if A^J eq
```

```
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if A^I eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if A^M eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
A^M eq A^4*B^2*C^3;
for i,j,k,l in [1..5] do if A^O eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
A^O eq A^4;
for i,j,k,l in [1..5] do if A^Q eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
A^Q eq A*B^3*D^3;
for i,j,k,l in [1..5] do if B^J eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
B^J eq B;
for i,j,k,l in [1..5] do if B^I eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
B^I eq A^4*B*C^3;
for i,j,k,l in [1..5] do if B^M eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
B^M eq B*D;
for i,j,k,l in [1..5] do if B^O eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
B^O eq B^4;
for i,j,k,l in [1..5] do if B^Q eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
B^Q eq D^4;
for i,j,k,l in [1..5] do if C^J eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
C^J eq C^4*D^4;
for i,j,k,l in [1..5] do if C^I eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
C^I eq A^4*C^4;
for i,j,k,l in [1..5] do if C^M eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
C^M eq C*D;
for i,j,k,l in [1..5] do if C^O eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
C^O eq C^4;
for i,j,k,l in [1..5] do if C^Q eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
C^Q eq A*B^2*C^4*D;
for i,j,k,l in [1..5] do if D^J eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
```

```
D^J eq D;
for i,j,k,l in [1..5] do if D^I eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
D^I eq A^2*B^3*C^2*D^4;
for i,j,k,l in [1..5] do if D^M eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
D^M eq D^4;
for i,j,k,l in [1..5] do if D^O eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
D^O eq D^4;
for i,j,k,l in [1..5] do if D^Q eq
A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
D^Q eq B^4;
S<w,x,y,z,e,f,g,h,v>:=Group<w,x,y,z,e,f,g,h,v|w^5,x^5,y^5,
z^5,(w,x),(w,y),(x,y),(w,z),(x,z),(y,z),e^2,f^2,g^2,h^2,v^2,
(e*g)^2,(f*g)^2,(e*h)^2,(f * h)^2,(g * h)^2,
(e * v)^2,(h * v)^2,e * f * e * f * h,
v * e * f * e * v * f,g * v * g * e * v * g* v,
w^e=w^4*x^2*z^2,w^f=w,w^g=w^4*x^2*y^3,w^h=w^4,
w^v=w*x^3*z^3,x^e=x,x^f=w^4*x*y^3,x^g=x*z,x^h=x^4,x^v=z^4,
y^e=y^4*z^4,y^f=w^4*y^4,y^g=y*z,y^h=y^4,y^v=w*x^2*y^4*z,
z^e=z,z^f=w^2*x^3*y^2*z^4,
z^g=z^4,z^h=z^4,z^v=x^4>;
#S;
f,g,k:=CosetAction(S,sub<S|Id(S)>);
s:=IsIsomorphic(G1,g);
s;
```

A.4 $2 \bullet S_6$

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^8,(x*y^-1*t)^5,
(y*t)^4,(x*w*t)^0,(w*x*y*t)^5,(w*x*y*t)^0>;
#G;
f,G1,k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
NL[2];
H:=NL[2];
```

```
q,ff:=quo<G1|NL[2]>;
q;
CompositionFactors(q);
Center(G1) eq NL[2];
nl:=NormalLattice(q);
nl;

H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a * b)^2,(b *c)^4,
b*a*c*b*a*c*b*a*c*a*b*c*a*b*c,
(a * c)^8, c * a * c * b * c *a * c *b * c* b * a * c *
b* c * a * c *b * c * a * c * b>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
S<e>:=Group<e|e^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[2]);
s;
T:=Transversal(G1,NL[2]);
EE:=q!q.2;
FF:=q!q.3;
GG:=q!q.5;
q,ff:=quo<G1|NL[2]>;
ff(T[3]) eq FF;
ff(T[2]) eq EE;
ff(T[4]) eq GG;
q eq sub<q|EE,FF,GG>;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
A:=G1!NL[2].2;
for i in [1..2] do if A^J eq A^i then i; end if; end for;
for i in [1..2] do if A^I eq A^i then i; end if; end for;
for i in [1..2] do if A^M eq A^i then i; end if; end for;
H<a,b,c,e>:=Group<a,b,c,e|a^2,b^2,c^2,(a * b)^2,(b *c)^4,
b*a*c*b*a*c*b*a*c*a*b*c*a*b*c,(a * c)^8, c * a * c * b *
c *a * c *b * c* b * a * c * b* c * a* c *b * c * a * c
*b,e^2,e^a=e,e^b=e,e^c=e>;
#H;
#G1;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
```

A.5 $G = 2^{*16} \cong 5^4 : ((2^3 \times 3^2) \times 2)$

```
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^6,(z*w*t)^3,
(z*x*t)^10,(y*x*t)^4,(y*x*z*t)^0>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then
i;end if;end for;
NL[2];
Center(G1);
X:=[5,5,5,5];
H:=NL[2];
q,ff:=quo<G1|NL[2]>;
q;
IsIsomorphic(DihedralGroup(72),q);
N:=NormalLattice(q);
N;
E:=DirectProduct(N[17],N[2]);
IsIsomorphic(E,q);
N[17];
FPGroup(q);
H<e,f,g,h,o>:=Group<e,f,g,h,o|e^2,f^2,g^2,h^2,o^2,
(e*g)^2,(f*g)^2,(e*h)^2,(f * h)^2, (g * h)^2,(e* o)^2
,(g *o)^2,e * f* e * f * h,h * o * h * g * o * h * o,
(f *o * f *o* h)^2>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
FPGroup(NL[2]);
S<a,b,c,d>:=Group<a,b,c,d|a^5,b^5,c^5,d^5,(a,b),(a,c),
(b,c),(a,d),(b,d),(c,d)>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[2]);
s;
```

```
EE:=q!q.1;
FF:=q!q.2;
GG:=q!q.3;
DD:=q!q.4;
NN:=q!q.6;
T:=Transversal(G1,NL[2]);
q,ff:=quo<G1|NL[2]>;
ff(T[2]) eq EE;
ff(T[3]) eq FF;
ff(T[4]) eq GG;
ff(T[5]) eq DD;
ff(T[6]) eq NN;
A:=G1!NL[2].1;
B:=G1!NL[2].2;
C:=G1!NL[2].3;
D:=G1!NL[2].4;
H:=G1!T[2];
J:=G1!T[3];
I:=G1!T[4];
M:=G1!T[5];
O:=G1!T[6];
for i,j,k,l in [1..5] do if A^M eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if A^O eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if B^H eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if B^J eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if B^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if B^M eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if B^O eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if C^H eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if C^J eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if C^I eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if C^M eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if C^O eq A^i*B^j*C^k*D^l
```

```
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if D^H eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if D^J eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if D^I eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;
for i,j,k,l in [1..5] do if D^M eq A^i*B^j*C^k*D^l
then i,j,k,l;end if; end for;
for i,j,k,l in [1..5] do if D^O eq A^i*B^j*C^k*D^l
then i,j,k,l; end if; end for;

S<a,b,c,d,e,f,g,h,o>:=Group<a,b,c,d,e,f,g,h,o|a^5,b^5,c^5,
d^5,
(a,b),(a,c),(b,c),(a,d),(b,d),(c,d),e^2,f^2,g^2,h^2,o^2,
(e*g)^2,
(f*g)^2,(e*h)^2,(f * h)^2,
(g * h)^2,(e* o)^2,(g *o)^2,e * f* e * f * h,
h * o * h * g * o * h * o,(f *o * f *o* h)^2,a^e=a^4*b^4,
a^f=a^4*c^4,
a^g=a^4,a^h=a^4*c^4,a^o=a^4
,b^e=b,b^f=a^2*b*c,b^g=b^4,b^h=b^4*d^4,b^o=b^4,c^e=c^4*d^4,
c^f=c,c^g=c^4,c^h=c,c^o=a*c,d^e=d,d^f=d,
d^g=d^4,d^h=d,d^o=b*d>;
#S;
#G1;
f1,S1,k1:=CosetAction(S,sub<S|Id(S)>);
s:=IsIsomorphic(G1,S1);
s;
```

A.6 $G = 2^{*8} : 2^4 : (S_3 \times C_2)$

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>;
#N;
Set(N);
```

```
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
#G;
f, G1, k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then
i;end if;end for;
IsAbelian(NL[4]);
X:=[2,2,2,2];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,(X)));
H:=NL[4];
q,ff:=quo<G1|NL[4]>;
q;
EE:=q!(1, 2)(3, 4)(5, 6);
FF:=q!(3, 5)(4, 6);
GG:=q!(1, 3)(2, 4);
T:=Transversal(G1,NL[4]);
ff(T[3]) eq FF;
ff(T[2]) eq EE;
ff(T[4]) eq GG;
q eq sub<q|EE,FF,GG>;
FPGroup(SymmetricGroup(3));
FPGroup(nl[2]);
S<a,b>:=Group<a,b|b^2,a^-3,(a^-1*b)^2>;
C<c>:=Group<c|c^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,nl[4]);
s;
ff1,cc,kk1:=CosetAction(C,sub<C|Id(C)>);
s,t:=IsIsomorphic(nl[2],cc);
s;
Generators(NL[4]);
A:=G1!(2, 6)(3, 8)(4, 5)(7, 9);
B:=G1!(1, 11)(3, 7)(8, 9)(10, 12);
C:=G1!(2, 5)(3, 7)(4, 6)(8, 9);
D:=G1!(1, 10)(3, 8)(7, 9)(11, 12);
nl:=NormalLattice(q);
nl;
E:=DirectProduct(nl[4],nl[2]);
IsIsomorphic(E,q);
IsIsomorphic(nl[4],SymmetricGroup(3));
```

```
T:=Transversal(G1,NL[4]);
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
ff(T[4]) eq q.5;
A:=G1!(2, 6)(3, 8)(4, 5)(7, 9);
B:=G1!(1, 11)(3, 7)(8, 9)(10, 12);
C:=G1!(2, 5)(3, 7)(4, 6)(8, 9);
D:=G1!(1, 10)(3, 8)(7, 9)(11, 12);
H:=G1!(3, 7)(4, 6)(10, 12);
I:=G1!(2, 3)(4, 8)(5, 7)(6, 9)(10, 12);
J:=G1!(1, 2)(4, 10)(5, 11)(6, 12)(8, 9);
IsIsomorphic(nl[2],CyclicGroup(2));
T[2];
H:=G1!(3, 7)(4, 6)(10, 12);
T[3];
I:=G1!(2, 3)(4, 8)(5, 7)(6, 9)(10, 12);
T[4];
J:=G1!(1, 2)(4, 10)(5, 11)(6, 12)(8, 9);
Generators(q);
FPGroup(NL[4]);
for i,j,k,l in [1..2] do if A^H eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if A^I eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if A^J eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if B^H eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if B^I eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if B^J eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if C^H eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if C^I eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
```

```
for i,j,k,l in [1..2] do if C^J eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if D^H eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if D^I eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do if D^J eq A^i*B^j*C^k*D^l
then i,j,k,l;
end if; end for;
FPGroup(q);
Order(T[2]);
Order(T[3]);
Order(T[4]);
Order(T[2]);
(T[2]*T[3])^2;
(T[2]*T[4])^2;
(T[4]*T[3])^3;
for i,j,k,l in [1..2] do
if (T[2]^-1*T[3])^2 eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do
if (T[2]^-1*T[4])^2 eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do
if (T[4]^-1*T[3])^3 eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
for i,j,k,l in [1..2] do
if A^i*B^j*C^k*D^l eq G1!(1, 2, 3)(4, 9, 12)(5, 7, 11)
(6, 8, 10) then i,j,k,l;
end if; end for;
H<w,x,y,z,e,f,g>:=Group<w,x,y,z,e,f,g|w^2,x^2,y^2,z^2,(w*x)^2,
(w*y)^2,(x*y)^2,(w*z)^2,(x*z)^2,(y*z)^2,e^2,f^2,g^2,(e*f)^2,
(e*g)^2,(g*f)^3,w^e=w*y,w^f=w*y,w^g=x*z,x^e=x,x^f=x*y,x^g=
y,y^e=y,y^f=y,
y^g=x,z^e=x*z,z^f=w*x*y*z,z^g=w*y>;
#H;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
```

A.7 $G = 2^{*16} : (2^5 : S_4)$

```
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^3,(z*w*t)^0,
(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
#sub<G1|f(x),f(y),f(z),f(w),f(h)>;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|w,x,y,z,h>,sub<G|w,x,y,z,h>);
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then i;end if;end for;
IsAbelian(NL[9]);
NL[9];
X:=[2,2,2,2,2];
IsIsomorphic(NL[9],AbelianGroup(GrpPerm,(X)));
H:=NL[9];
q,ff:=quo<G1|NL[9]>;
q;
EE:=q!(1, 2)(3, 4);
FF:=q!(2, 4);
GG:=q!(1, 3)(2, 4);
DD:=q!(3, 4);
T:=Transversal(G1,NL[9]);
ff(T[2]);
ff(T[3]);
ff(T[4]);
ff(T[5]);
ff(T[3]) eq FF;
ff(T[2]) eq EE;
ff(T[4]) eq GG;
ff(T[5]) eq DD;
```

```
q eq sub<q|EE,FF,GG,DD>;
Center(G1);
Center(G1) eq NL[9];
nl:=NormalLattice(q);
nl;
FPGroup(q);
H<e,f,g,h>:=Group<e,f,g,h|e^2,f^2,g^2,h^2,(e*g)^2,(f*g)^2,
(e*h)^2,e*f*e*f*g,h*g*e*h*g,(h*f)^3>;
ff,ss,kk:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(q,ss);
s;
FPGroup(NL[9]);
S<w,x,y,z,v>:=Group<w,x,y,z,v|w^2,x^2,y^2,z^2,v^2,
(w*x)^2,(w*y)^2,(x*y)^2,(w*z)^2,(x*z)^2,(y*z)^2,(w*v)^2,
(x*v)^2,(y*v)^2,(z*v)^2>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[9]);
s;
Generators(NL[9]);
A:=G1!(1, 24)(2, 7)(3, 9)(4, 15)(5, 6)(8, 19)(10, 20)(11, 21)
(12, 13)(14, 18)(16, 22)(17, 23);
B:=G1!(1, 16)(2, 15)(3, 19)(4, 7)(5, 12)(6, 13)(8, 9)(10, 21)
(11, 20)(14, 23)(17, 18)(22, 24);
C:=G1!(1, 17)(2, 6)(4, 12)(5, 7)(13, 15)(14, 22)(16, 18)
(23, 24);
D:=G1! (2, 6)(3, 11)(4, 12)(5, 7)(8, 10)(9, 21)(13, 15)
(19, 20);
F:=G1! (2, 4)(3, 9)(5, 13)(6, 12)(7, 15)(8, 19)(10, 20)
(11, 21);
T[2]; T[3];T[4]; T[5];
J:=G1!(3, 8)(4, 12)(5, 7)(9, 20)(10, 11)(14, 22)(16, 18)
(19, 21);
I:=G1!(2, 3)(4, 9)(5, 8)(6, 11)(7, 10)(12, 21)(13, 19)(14, 24)
(15, 20)(22, 23);
M:=G1!(2, 5, 6, 7)(3, 10, 11, 8)(4, 13, 12, 15)(9, 20, 21, 19)
(14, 22)(23, 24);
O:=G1!(1, 2)(4, 14)(5, 16)(6, 17)(7, 18)(9, 20)(12, 22)
(13, 23)(15, 24)(19, 21);
for i,j,k,l,m in [1..2] do if A^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if A^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
```

```
for i,j,k,l,m in [1..2] do if A^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if A^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if B^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if B^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if B^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if B^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if C^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if C^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if C^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if C^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if D^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if D^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if D^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if D^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if F^J eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
```

```
end if; end for;
for i,j,k,l,m in [1..2] do if F^I eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if F^M eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
for i,j,k,l,m in [1..2] do if F^O eq A^i*B^j*C^k*D^l*F^m
then i,j,k,l,m;
end if; end for;
S<w,x,y,z,v,e,f,g,h>:=Group<w,x,y,z,v,e,f,g,h|w^2,x^2,y^2,z^2,
v^2,(w*x)^2,(w*y)^2,(x*y)^2,(w*z)^2,(x*z)^2,(y*z)^2,(w*v)^2,
(x*v)^2,(y*v)^2,(z*v)^2,w^e=w*z,w^f=w*x*y*z,w^g=w*y
,w^h=x*y*z,x^e=x*y*z,x^f=x*z,x^g=x*z,x^h=w*z,y^e=y,y^f=y*z,y^g=y,
y^h=y,z^e=z,z^f=z,z^g=z,z^h=y*z,v^e=z*v,v^f=v,v^g=v,v^h=w*x*y*z*v,
e^2,f^2,g^2,h^2,(e*g)^2,(f*g)^2,(e*h)^2,
e*f*e*f*g,h*g*e*h*g,(h*f)^3>;
#S; #G1;
f1,S1,k1:=CosetAction(S,sub<S|Id(S)>);
s:=IsIsomorphic(G1,S1);
s;
```

A.8 $2^{*8} : 2^4 : (S_3 \times C_2)$

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>;
#N;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,x*y^
-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
#G;
f, G1, k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
CompositionFactors(G1);
IN:=sub<G1|f(w),f(x),f(y),f(z)>;
NN<w,x,y,z>:=Group<w,x,y,z|w^2,x^2,y^4,z^2,(w*y^-1)^2,x*y^
-1*x*y,y^-2*z,w*x*w*x*z>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..16]];
for i in [2..16] do
```

```
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=ww^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=zz^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
f,G1,k:=CosetAction(G,sub<G|w,x,y,z>);
#DoubleCosets(G,sub<G|x,y>,sub<G|w,x,y,z>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(w),f(x),f(y),f(z)>;
ts := [ Id(G1): i in [1 .. 8] ];
ts[1]:=f(t);
ts[2]:=f(t^(x^3));
ts[3]:=f(t^(w*y));
ts[4]:=f(t^(z^3));
ts[5]:=f(t^(x*w));
ts[6]:=f(t^(w*x*y));
ts[7]:=f(t^(z*y));
ts[8]:=f(t^(x*y^-1));
ts;
```

```
cst:= [null : i in [1 .. 12]]
where null is[Integers() |];
for i := 1 to 8 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N);
N1:=Stabiliser(N,1);
N1s:=N1;
Orbits(N1s);
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[7] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[8] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6] eq m*(ts[1])^n then m,n;
end if;end for;
#N/#N1;
#N/#N1s;
#N1s;
N1s;
N14:=Stabiliser(N,[1,4]);
N14s:=N14;
S:={[1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]]*ts
[Rep(SSS[i])[2]]then print SSS[i];
end if; end for; end for;
for g in N do if 1^g eq 2 and 4^g eq 5 then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if 1^g eq 3 and 4^g eq 7 then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if 1^g eq 4 and 4^g eq 1 then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if 1^g eq 5 and 4^g eq 2 then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if 1^g eq 6 and 4^g eq 8 then N14s:=sub<N|N14s,g>;
end if; end for;
```

```
for g in N do if 1^g eq 7 and 4^g eq 3 then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if 1^g eq 8 and 4^g eq 6 then N14s:=sub<N|N14s,g>;
end if; end for;
#N14s;
N14s;
#N/#N14s;
T:=Transversal(N,N14s);
for i := 1 to #T do
ss := [1,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N14s);
for n in IN do if ts[1]*ts[4] eq n*ts[2]*ts[5] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[3]*ts[7] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[4]*ts[1] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[5]*ts[2] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[6]*ts[8] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[7]*ts[3] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[8]*ts[6] then n;
end if; end for;
N17:=Stabiliser(N,[1,7]);
N17s:=N17;
S:={[1,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[7] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,7]^g eq [4,3] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [2,6] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [3,4] then N17s:=sub<N|N17s,
```

```
g>; end if; end for;
for g in N do if [1,7]^g eq [5,8] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [6,2] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [7,1] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [8,5] then N17s:=sub<N|N17s,
g>; end if; end for;
#N17s;
N17s;
#N/#N17s;
T:=Transversal(N,N17s);
for i := 1 to #T do
ss := [1,7]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N17s);
for n in IN do if ts[1]*ts[7] eq n*ts[4]*ts[3] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[2]*ts[6] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[3]*ts[4] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[5]*ts[8] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[6]*ts[2] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[7]*ts[1] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[8]*ts[5] then n;
end if; end for;
```

A.9 $3 :^{\bullet} PSL(2,19) \times 2$

```
S:=Sym(3);
yy:=S!(2, 3);
ww:=S!(1, 2, 3);
N:=sub<S|yy,ww>;
#N;
G<y,w,t>:=Group<y,w,t|y^2,w^3,
```

```
(y*w^-1)^2,t^2,(t,y),(y*t)^2,
(w*t)^10,(y*w^-1*t)^9>;
f,G1,k:=CosetAction(G,sub<G|y,w>);
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
S<d>:=Group<d|d^-3>;
ff,ss,kk:=CosetAction(S,sub<S|Id(S)>);
s,t:=IsIsomorphic(ss,NL[2]);
s;
H:=NL[2];
q,ff:=quo<G1|NL[2]>;
q;
T:=Transversal(G1,NL[2]);
q,ff:=quo<G1|NL[2]>;
s:=IsIsomorphic(q,DirectProduct(PSL(2,19),CyclicGroup(2)));
s;
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
ff(T[4]) eq q.3;
Order(T[3]);
Order(q.2);
Order(T[4]);
Order(q.3);
FPGroup(q);
H<a,b,c>:=Group<a,b,c|a^2 ,b^3 ,c^2 ,(b^-1 * a)^2 ,
(a * c)^2 ,b * c * b^-1 * c * b * c * b^-1 * c * a *
b^-1 * c * b * c * b^-1 * c * b * c * b^-1 * c ,
(b^-1 * c)^10 ,c * b^-1 * c * b * c * b * c * b^-1 *
c * b^-1 * c * b * c * b^-1 * c * b^-1 * c * b^-1 *
c * b * a * c * b^-1 * c * b^-1 * c * b * c * b * c *
b * c * b^-1 * c * b * c * b * c * b^-1>;
#H;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(q,DirectProduct(PSL(2,19),CyclicGroup(2)));
s;
A:=NL[2].2;
J:=G1!T[2];
I:=G1!T[3];
M:=G1!T[4];
for i in [1..3] do if A^J eq A^i then i; end if; end for;
for i in [1..3] do if A^I eq A^i then i; end if; end for;
for i in [1..3] do if A^M eq A^i then i; end if; end for;
for i in [1..3] do
```

```
if  T[2]^2 eq A^i then i;end if; end for;
for i in [1..3] do
if T[3]^3 eq A^i then i;end if; end for;
for i in [1..3] do
if T[4]^2 eq A^i then i;end if; end for;
for i in [1..3] do
if (T[3]^-1 * T[2])^2 eq A^i then i;end if; end for;
for i in [1..3] do
if (T[2] * T[4])^2 eq A^i then i;end if; end for;
for i in [1..3] do
if T[3] * T[4] * T[3]^-1 * T[4] * T[3] * T[4]
* T[3]^-1 * T[4] * T[2] * T[3]^-1 * T[4] *
T[3] *  T[4] * T[3]^-1* T[4] * T[3]*T[4] *
T[3]^-1 * T[4] eq A^i then i;end if; end for;
for i in [1..3] do
if    ( T[3] ^-1 * T[4] )^10  eq A^i then i;end if; end for;
for i in [1..3] do
if T[4]  * T[3] ^-1 * T[4]  * T[3]  *T[4]  *T[3]
* T[4]  * T[3] ^-1 *  T[4] * T[3] ^-1 *  T[4] *
T[3]  *  T[4] * T[3] ^-1 * T[4] * T[3] ^-1 *  T[4]
* T[3] ^-1 *  T[4] * T[3]  * T[2] * T[4] *
T[3] ^-1 *  T[4] * T[3] ^-1 *  T[4] * T[3]  *  T[4]
 * T[3]  *  T[4]* T[3]  *  T[4] * T[3] ^-1 *
T[4] * T[3]  *  T[4] * T[3]  * T[4] *T[3] ^-1 eq
A^i then i;end if; end for;
H<a,b,c,d>:=Group<d,a,b,c|a^2 ,b^3 ,c^2 ,(b^-1 * a)^2 ,
(a * c)^2 ,b * c * b^-1 * c * b * c * b^-1 * c
 * a * b^-1 * c * b * c * b^-1 * c * b * c * b^-1 * c ,
 (b^-1 * c)^10 ,  c * b^-1 * c * b * c * b * c * b^-1 *
  c * b^-1 * c * b * c * b^-1 * c * b^-1 * c * b^-1 * c
  * b * a * c * b^-1 * c * b^-1 * c * b * c * b * c * b
  * c * b^-1 * c * b * c * b * c * b^-1=d,d^3,d^a=d^2,
  d^b=d,d^c=d^2>;
#H;
#G1;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(G1,H1);
s;
```

Appendix B

MAGMA Code for DCE

B.1 $2^4 : S_3 \times C_2$ over $(2 \times 4 : 2)$

```
S:=Sym(8);
ww:=S!(2, 5)(3, 7);
xx:=S!(1,2)(3, 6)(4, 5)(7, 8);
yy:=S!(1, 3, 4, 7)(2, 6, 5, 8);
zz:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|ww,xx,yy,zz>;
#N;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,
y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
#G;
f, G1, k:=CosetAction(G,sub<G|w,x,y,z>);
#k;
CompositionFactors(G1);
IN:=sub<G1|f(w),f(x),f(y),f(z)>;
NN<w,x,y,z>:=Group<w,x,y,z|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..16]];
for i in [2..16] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=ww^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=xx^-1; end if;
```

```
if Eltseq(Sch[i])[j] eq 3 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=zz^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
G<w,x,y,z,t>:=Group<w,x,y,z,t|w^2,x^2,y^4,z^2,(w*y^-1)^2,
x*y^-1*x*y,y^-2*z,w*x*w*x*z,t^2,(t,w),(x*t)^3,(w*x*y*t)^3>;
f,G1,k:=CosetAction(G,sub<G|w,x,y,z>);
#DoubleCosets(G,sub<G|x,y>,sub<G|w,x,y,z>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(w),f(x),f(y),f(z)>;
ts := [ Id(G1): i in [1 .. 8] ];
ts[1]:=f(t); ts[2]:=f(t^(x^3)); ts[3]:=f(t^(w*y)); ts[4]:=
f(t^(z^3));
ts[5]:=f(t^(x*w)); ts[6]:=f(t^(w*x*y)); ts[7]:=f(t^(z*y));
ts[8]:=f(t^(x*y^-1));
ts;
cst:= [null : i in [1 .. 12]]
where null is[Integers() |];
for i := 1 to 8 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
Orbits(N);
N1:=Stabiliser(N,1);
N1s:=N1;
```

```
Orbits(N1s);
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[7] eq m*(ts[1])^n then m,n;
 end if;end for;
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[8] eq m*(ts[1])^n then m,n;
 end if;end for;
for m,n in IN do if ts[1]*ts[6] eq m*(ts[1])^n then m,n;
 end if;end for;
#N/#N1;
N14:=Stabiliser(N,[1,4]);
N14s:=N14;
S:={[1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]]*
ts[Rep(SSS[i])[2]]then print SSS[i];
end if; end for; end for;
for g in N do if 1^g eq 2 and 4^g eq 5 then N14s:=sub<N|N14s,
g>; end if; end for;
for g in N do if 1^g eq 3 and 4^g eq 7 then N14s:=sub<N|N14s,
g>; end if; end for;
for g in N do if 1^g eq 4 and 4^g eq 1 then N14s:=sub<N|N14s
,g>; end if; end for;
for g in N do if 1^g eq 5 and 4^g eq 2 then N14s:=sub<N|N14s,
g>; end if; end for;
for g in N do if 1^g eq 6 and 4^g eq 8 then N14s:=sub<N|N14s,
g>; end if; end for;
for g in N do if 1^g eq 7 and 4^g eq 3 then N14s:=sub<N|N14s,
g>; end if; end for;
for g in N do if 1^g eq 8 and 4^g eq 6 then N14s:=sub<N|N14s,
g>; end if; end for;
#N14s;
N14s;
#N/#N14s;
T:=Transversal(N,N14s);
for i := 1 to #T do
ss := [1,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
```

```
end if; end for;m;
Orbits(N14s);
for n in IN do if ts[1]*ts[4] eq n*ts[2]*ts[5] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[3]*ts[7] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[4]*ts[1] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[5]*ts[2] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[6]*ts[8] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[7]*ts[3] then n;
end if; end for;
for n in IN do if ts[1]*ts[4] eq n*ts[8]*ts[6] then n;
end if; end for;
N17:=Stabiliser(N,[1,7]);
N17s:=N17;
S:={[1,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[7] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];end if; end for; end for;
for g in N do if [1,7]^g eq [4,3] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [2,6] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [3,4] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [5,8] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [6,2] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [7,1] then N17s:=sub<N|N17s,
g>; end if; end for;
for g in N do if [1,7]^g eq [8,5] then N17s:=sub<N|N17s,
g>; end if; end for;
#N17s;
N17s;
#N/#N17s;
T:=Transversal(N,N17s);
for i := 1 to #T do
```

```
ss := [1,7]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N17s);
for n in IN do if ts[1]*ts[7] eq n*ts[4]*ts[3] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[2]*ts[6] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[3]*ts[4] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[5]*ts[8] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[6]*ts[2] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[7]*ts[1] then n;
end if; end for;
for n in IN do if ts[1]*ts[7] eq n*ts[8]*ts[5] then n;
end if; end for;
```

B.2 $2^{*16} : (2^5 : S_4)$over $(2 \times 4 : C_2 \times C_2)$

```
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)
(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)
(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)
(9, 13, 11, 16);
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)
(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
FPGroup(N);
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^3,
(z*w*t)^0,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^3>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#DoubleCosets(G,sub<G|x,y,z,w,h>,sub<G|x,y,z,w,h>);
#G;
```

```
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#k;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
CompositionFactors(G1);
Set(N);
NN<x,y,z,w,h>:=Group<x,y,z,w,h|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,w^-2*h,
x*y*x*y*w^-1,x*z*x*z*h>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..32]];
for i in [2..32] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=ww^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=hh; end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=hh^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do 1^ArrayP[i],Sch[i];end for;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,h^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^3,
(z*w*t)^0,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^3>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#DoubleCosets(G,sub<G|x,y,z,w,h>,sub<G|x,y,z,w,h>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
```

```
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(x),f(y),f(z),f(w),f(h)>;
ts := [ Id(G1): i in [1 .. 16] ];
ts[1]:=f(t);
ts[2]:=f(t^y);
ts[3]:=f(t^z);
ts[4]:=f(t^w);
ts[5]:=f(t^h);
ts[6]:=f(t^(y * w));
ts[7]:=f(t^(z * x));
ts[8]:=f(t^(y * x));
ts[9]:=f(t^(y * z));
ts[10]:=f(t^(y * h));
ts[11]:=f(t^( y * z * h));
ts[12]:=f(t^(z*w));
ts[13]:=f(t^(y*z*x));
ts[14]:=f(t^(x*w^-1));
ts[15]:=f(t^(z*w^-1));
ts[16]:=f(t^(y*z*w^-1));
ts;
cst:= [null : i in [1 .. 24]]
where null is[Integers() |];
for i := 1 to 16 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..24] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
for i in [1..24] do i, cst[i]; end for;
N1:=Stabiliser(N,1);
N1s:=N1;
SSS:={[1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq n*ts[Rep(Seqq[i])[1]]then print Rep(Seqq[i]);
end if; end for; end for;
T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..24] do if cst[i] ne []then m:=m+1;
end if; end for; m;
Orbits(N1s);
for m,n in IN do if ts[1]*ts[5] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[12] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[15] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[3] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[9] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[11] eq m*(ts[1])^n then m,n;
end if;end for;
#N/#N1;
#N/#N1s;
#N1s;
N1s;
Orbits(N1s);
m:=N!(1, 2)(3, 9)(4, 8)(5, 10)(14, 6)(7, 11)(12, 16)(15, 13);
n:=N!(2, 8)(3, 7)(4, 14)(9, 13)(6, 10)(16, 11);
[1]^n;
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[2] eq f(y)*ts[1];
for m,n in IN do if ts[1]*ts[6] eq m*(ts[1])^n then m,n;
end if;end for;
m:=N!(1, 6)(2, 4)(3, 13)(5, 8)(14, 10)(9, 12)(7, 16)(15, 11);
n:=N!(2, 8)(3, 7)(4, 14)(9, 13)(6, 10)(16, 11);
[1]^n;
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[6] eq f(y*w)*ts[1];
for m,n in IN do if ts[1]*ts[9] eq m*(ts[1])^n then m,n;
end if;end for;
```

```
m:=N!(1, 16, 4, 9, 5, 13, 14, 11)(2, 3, 6, 12, 10, 7, 8, 15);
n:=N!(1, 14)(2, 10)(3, 12)(4, 5)(7, 15)(16, 13);
[1]^n;
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[9] eq f(y*x*z)*ts[14];
for m,n in IN do if ts[1]*ts[11] eq m*(ts[1])^n then m,n;
end if;end for;
m:=N!(1, 13, 4, 11, 5, 16, 14, 9)(2, 7, 6, 15, 10, 3, 8, 12);
n:=N!(1, 14)(2, 10)(3, 12)(4, 5)(7, 15)(16, 13);
[1]^n;
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[11] eq f(y*z*x)*ts[14];
for n in IN do if ts[1]*ts[15] eq n*ts[1]*ts[12]then n;
end if; end for;
for n in IN do if ts[1]*ts[15] eq n*ts[1]*ts[3]then n;
end if; end for;
for n in IN do if ts[1]*ts[3] eq n*ts[1]*ts[12]then n;
end if; end for;
N15:=Stabiliser(N,[1,5]);
N15s:=N15;
S:={[1,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[5]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,5]^g eq [2,10] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [3,7] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [4,14] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [14,4] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [5,1] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [8,6] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [9,11] then N15s:=sub<N|N15s,g>;
end if; end for;
```

```
for g in N do if [1,5]^g eq [11,9] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [10,2] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [16,13] then N15s:=sub<N|N15s,g>;
end if; end for;
for g in N do if [1,5]^g eq [15,12] then N15s:=sub<N|N15s,g>;
end if; end for;
#N15s;
N15s;
#N/#N15;
#N/#N15s;
T:=Transversal(N,N15s);
for i := 1 to #T do
ss := [1,5]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..24] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N15s);
ts[1]*ts[5] eq ts[3]*ts[7];
ts[1]*ts[5] eq ts[4]*ts[14];
ts[1]*ts[5] eq ts[14]*ts[4];
ts[1]*ts[5] eq ts[5]*ts[1];
ts[1]*ts[5] eq ts[15]*ts[12];
N112:=Stabiliser(N,[1,12]);
N112s:=N112;
S:={[1,12]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[12] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,12]^g eq [3,4] then N112s:=sub<N|N112s,g>;
end if; end for;
for g in N do if [1,12]^g eq [5,15] then N112s:=sub<N|N112s,g>;
end if; end for;
for g in N do if [1,12]^g eq [7,14] then N112s:=sub<N|N112s,g>;
end if; end for;
#N112s;
N112s;
#N/#N112;
```

```
#N/#N112s;
T:=Transversal(N,N112s);
for i := 1 to #T do
ss := [1,12]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..24] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
for n in IN do if ts[1]*ts[12] eq n*ts[3]*ts[4] then n;
end if; end for;
for n in IN do if ts[1]*ts[12] eq n*ts[5]*ts[15] then n;
end if; end for;
Orbits(N112s);
for n in IN do if ts[1]*ts[12] eq n*ts[7]*ts[14] then n;
end if; end for;
m:=N!(1, 5)(2, 10)(3, 7)(4, 14)(8, 6)(9, 11)(12, 15)(16, 13);
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[12] eq f(h)*ts[3]*ts[4];
ts[1]*ts[12] eq f(h)*ts[7]*ts[14];
ts[1]*ts[12] eq ts[5]*ts[15];
N14:=Stabiliser(N,[1,4]);
N14s:=N14;
S:={[1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for i in [1..24] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
for g in N do if [1,4]^g eq [2,8] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [3,12] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [4,5] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [5,14] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [9,16] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [6,2] then N14s:=sub<N|N14s,g>;
```

```
end if; end for;
for g in N do if [1,4]^g eq [10,6] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [12,7] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [7,15] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [8,10] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [14,1] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [13,9] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [11,13] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [16,11] then N14s:=sub<N|N14s,g>;
end if; end for;
for g in N do if [1,4]^g eq [5,3] then N14s:=sub<N|N14s,g>;
end if; end for;
#N14s;
#N/#N14;
#N/#N14s;
T14:=Transversal(N,N14s);
#T14;
for i := 1 to #T14 do
ss := [1,4]^T14[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
Orbits(N14s);
ts[1]*ts[4] eq ts[4]*ts[5];
ts[1]*ts[4] eq ts[5]*ts[14];
ts[1]*ts[4] eq ts[14]*ts[1];
for n in IN do if ts[1]*ts[4] eq n*ts[2]*ts[8] then n;
end if; end for;
m:=N!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
for i in [1 .. 32] do if ArrayP[i] eq m then Sch[i];
end if; end for;
ts[1]*ts[4] eq f(w)*ts[2]*ts[8];
ts[1]*ts[4] eq f(w)*ts[6]*ts[2];
ts[1]*ts[4] eq f(w)*ts[10]*ts[6];
ts[1]*ts[4] eq f(w)*ts[8]*ts[10];
```

B.3 $2^{*4} : A_5$ over 2^2

```
S:=Sym(4);
xx:=S!(1,2)(3,4);
yy:=S!(1,3)(2,4);
N:=sub<S|xx,yy>;
#N;
G<x,y,t>:=Group<x,y,t|x^2,y^2,x*y^-1*x*y,t^2,(y*t)^3,
(x*t)^5,(x*y*t)^5>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
Set(N);
G<x,y,t>:=Group<x,y,t|x^2,y^2,x*y^-1*x*y,t^2,(y*t)^3,
(x*t)^5,(x*y*t)^5>;
#G;
NN<x,y>:=Group<x,y|x^2,y^2,x*y^-1*x*y>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..4]];
for i in [2..4] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
G<x,y,t>:=Group<x,y,t|x^2,y^2,x*y^-1*x*y,t^2,(y*t)^3,
(x*t)^5,(x*y*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
```

```
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 4] ];
ts[1]:=f(t);
ts[2]:=f(t^x);
ts[3]:=f(t^y);
ts[4]:=f(t^(x*y));
ts;
cst:= [null : i in [1 .. 15]]
where null is[Integers() |];
for i := 1 to 4 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
for i in [1..15] do i, cst[i]; end for;
N1:=Stabiliser(N,1);
N1s:=N1;
SSS:={[1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq n*ts[Rep(Seqq[i])[1]]then print Rep(Seqq[i]);
end if; end for; end for;
T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne []then m:=m+1;
end if; end for; m;
Orbits(N1s);
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[3] eq m*(ts[1])^n then m,n;
end if;end for;
```

```
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n then m,n;
end if;end for;
#N/#N1;
#N/#N1s;
#N1s;
N1s;
N12:=Stabiliser(N,[1,2]);
N12s:=N12;
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[2]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
#N12s;
N12s;
#N/#N12;
#N/#N12s;
T:=Transversal(N,N12s);
for i := 1 to #T do
ss := [1,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N12s);
N14:=Stabiliser(N,[1,4]);
N14s:=N14;
S:={[1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
#N14s;
N14s;
#N/#N14;
#N/#N14s;
T:=Transversal(N,N14s);
for i := 1 to #T do
```

```
ss := [1,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N14s);
N123:=Stabiliser(N,[1,2,3]);
N123s:=N123;
S:={[1,2,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]*ts[3]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,2,3]^g eq [2,1,4] then N123s:=
sub<N|N123s,g>;
end if; end for;
for g in N do if [1,2,3]^g eq [3,4,1] then N123s:=
sub<N|N123s,g>;
end if; end for;
for g in N do if [1,2,3]^g eq [4,3,2] then N123s:=
sub<N|N123s,g>;
end if; end for;
#N123s;
N123s;
#N/#N123s;
T:=Transversal(N,N123s);
for i := 1 to #T do
ss := [1,2,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N123s);
N143:=Stabiliser(N,[1,4,3]);
N143s:=N143;
S:={[1,4,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for> for g in IN do if ts[1]*ts[4]*ts[3]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
```

```
then print SSS[i];
end if; end for; end for;
for g in N do if [1,4,3]^g eq [2,3,4] then
N143s:=sub<N|N143s,g>; end if; end for;
for g in N do if [1,4,3]^g eq [3,2,1] then
 N143s:=sub<N|N143s,g>;end if; end for;
for g in N do if [1,4,3]^g eq [4,1,2] then
 N143s:=sub<N|N143s,g>; end if; end for;
#N143s;
N143s;
#N/#N143s;
T:=Transversal(N,N143s);
for i := 1 to #T do
ss := [1,4,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N143s);
for m,n in IN do if ts[1]*ts[2]*ts[4] eq
m*(ts[1]*ts[4])^n then m,n;end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[3] eq
 m*(ts[1]*ts[4])^n then m,n;end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[1] eq
 m*(ts[1])^n then m,n;end if;end for;
for m,n in IN do if ts[1]*ts[4]*ts[2] eq
m*(ts[1]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[4]*ts[3] eq
 m*(ts[1]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[4]*ts[1] eq
 m*(ts[1])^n then m,n;end if;end for;
```

B.4 S_6 over $(2 \times 4) : 2$

```
% from here we will factor S6 by NL[4]=32 to decrease
the number of DCE %
S:=Sym(16);
xx:=S!(2, 8)(3, 7)(4, 14)(6, 10)(9, 13)(11, 16);
yy:=S!(1, 2)(3, 9)(4, 8)(5, 10)(6, 14)(7, 11)(12, 16)(13, 15);
zz:=S!(1, 3)(2, 9)(4, 12)(5, 7)(6, 13)(8, 16)(10, 11)(14, 15);
ww:= S!(1, 4, 5, 14)(2, 6, 10, 8)(3, 12, 7, 15)(9, 13, 11, 16);
```

```
hh:=S!(1, 5)(2, 10)(3, 7)(4, 14)(6, 8)(9, 11)(12, 15)(13, 16);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,w^4,
h^2,(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^4,
(z*w*t)^3,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^5>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
CompositionFactors(G1);
N:=G1;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#G]];
for i in [2..#G] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=f(w)^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(h); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
NL:=NormalLattice(G1);
NL;
for i in [1..#NL]do if IsAbelian(NL[i]) then i;
end if;end for;
#Generators(NL[4]);
for j in [1..6] do
for i in [1..#Sch] do if ArrayP[i] eq NL[4].j then Sch[i];
xdsssssssssend if; end for; end for;
G<x,y,z,w,h,t>:=Group<x,y,z,w,h,t|x^2,y^2,z^2,
w^4,h^2,(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,
w^-2*h,x*y*x*y*w^-1,x*z*x*z*h,t^2,(t,x),(y*t)^4,(z*w*t)^3,
(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^5,h,t * h * t,
x * t * w^-1 * t * w^-1 * t * w^-1,(z * t)^4,
y * x * t * h * t * y * t * w * t * w^-1 * t>;
```

```
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,h>);
#G1;
#k;
#sub<G|x,y,z,w,h>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|w,x,y,z,h>,sub<G|w,x,y,z,h>);
N:=sub<S|xx,yy,zz,ww,hh>;
#N;
N<x,y,z,w>:=Group<x,y,z,w|x^2,y^2,z^2,w^2,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,w^-2,x*y*x*y*w^-1,x*z*x*z>;
#N;
16
H:=sub<N|x>;
f,G1,k:=CosetAction(N,H);
G1;
%Start from here  DCE of G over the group N of order 16 %
------------------------------------------------------------
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^2,z^2,w^4,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,x*y*x*y*w^-1,t^2,
(t,x),(y*t)^4,
(z*w*t)^3,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^5,
x * t * w^-1 * t * w^-1 * t * w^-1,(z * t)^4>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
#G1;
#sub<G|x,y,z,w>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|w,x,y,z>,sub<G|w,x,y,z>);
S:=Sym(8);
xx:=S!(2, 5)(6, 8);
yy:=S!(1, 2)(3, 6)(4, 5)(7, 8);
zz:=S!(1, 3)(2, 6)(4, 7)(5, 8);
ww:=S!(1, 4)(2, 5)(3, 7)(6, 8);
N:=sub<S|xx,yy,zz,ww>;
#N;
NN<x,y,z,w>:=Group<x,y,z,w|x^2,y^2,z^2,w^2,
(y*z)^2,(x*w^-1)^2,(y*w^-1)^2,
z*w^-1*z*w,w^-2,x*y*x*y*w^-1,x*z*x*z>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..16]];
for i in [2..16] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=ww^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do 1^ArrayP[i],Sch[i];end for;
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^2,z^2,w^4,(y*z)^2,
(x*w^-1)^2,(y*w^-1)^2,z*w^-1*z*w,x*y*x*y*w^-1,t^2,(t,x),
(y*t)^4,(z*w*t)^3,(z*x*t)^0,(y*x*t)^6,(y*x*z*t)^5,x * t
 * w^-1 * t * w^-1 * t * w^-1,
(z * t)^4>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(x),f(y),f(z),f(w)>;
ts := [ Id(G1): i in [1 .. 8] ];
ts[1]:=f(t);
ts[2]:=f(t^y);
ts[3]:=f(t^z);ts[4]:=f(t^w);
ts[5]:=f(t^(y*x));ts[6]:=f(t^(y*z));
ts[7]:=f(t^(z*w));ts[8]:=f(t^(y*x*z));
ts;
cst:= [null : i in [1 .. 45]]
where null is[Integers() |];
```

```
for i := 1 to 8 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
for i in [1..45] do i, cst[i]; end for;
N1:=Stabiliser(N,1);
N1:=Stabiliser(N,[1]);
S:={[1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1] eq g*ts[Rep(SSS[i])[1]]then print SSS[i];
end if; end for; end for;
N1s:=N1;
#N1s;
T1:=Transversal(N,N1s);
#T1;
T1:=Transversal(N,N1s);
for i := 1 to #T1 do
ss := [1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;for i in [1..45] do if cst[i] ne []
then m:=m+1; end if; end for; m;
T1:=Transversal(N,N1s);
#T1;
T1:=Transversal(N,N1s);
for i := 1 to #T1 do
ss := [1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;for i in [1..45] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1s);
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[3] eq m*(ts[1])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6] eq m*(ts[1])^n then m,n;
end if;end for;
```

```
#N/#N1s;

N1s;

N12:=Stabiliser(N,[1,2]);
N12s:=N12;
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[2]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
#N12s;
N12s;
for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>;
end if; end for;
#N/#N12s;
T:=Transversal(N,N12s);
for i := 1 to #T do
ss := [1,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N12s);
for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1])^n
then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[2])^n
then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[7] eq m*(ts[1]*ts[6])^n
then m,n;
end if;end for;
ts[1]*ts[2]*ts[2] eq ts[1];
for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(ts[1]*ts[2])^n
then m,n;
end if; end for;

N13:=Stabiliser(N,[1,3]);
N13s:=N13;
S:={[1,3]};
```

```
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[3]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,3]^g eq [3,1] then N13s:=sub<N|N13s,g>;
end if; end for;
for g in N do if [1,3]^g eq [4,7] then N13s:=sub<N|N13s,g>;
end if; end for;
for g in N do if [1,3]^g eq [7,4] then N13s:=sub<N|N13s,g>;
end if; end for;
N13s;
# N13s;
#N/#N13s;
T:=Transversal(N,N13s);
for i := 1 to #T do
ss := [1,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;

Orbits(N13s);
for m,n in IN do if ts[1]*ts[3]*ts[1] eq
 m*(ts[1])^n then m,n;end if; end for;m;
for m,n in IN do if ts[1]*ts[3]*ts[2] eq
 m*(ts[1]*ts[2])^n then m,n;
end if; end for;
#N13s;

N16:=Stabiliser(N,[1,6]);
N16s:=N16;
S:={[1,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[6]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
#N16s;
#N/#N16s;
```

```
T:=Transversal(N,N16s);
for i := 1 to #T do
ss := [1,6]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N16s);
for m,n in IN do if ts[1]*ts[6]*ts[6] eq
 m*(ts[1])^n then m,n;end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[1] eq
 m*(ts[1]*ts[3]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[2] eq
 m*(ts[1]*ts[2]*ts[3])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[3] eq
 m*(ts[1]*ts[3]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[4] eq
m*(ts[1]*ts[6])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[5] eq
 m*(ts[1]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[7] eq
 m*(ts[1]*ts[6])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[6]*ts[8] eq
 m*(ts[1]*ts[6])^n then m,n;
end if;end for;
for i in [1 .. 16] do if ArrayP[i] eq
N!(2,5)(6,8) then Sch[i]; end if;end for;
ts[1]*ts[6]*ts[8] eq f(x)*ts[1]*ts[6];
N123:=Stabiliser(N,[1,2,3]);
N123s:=N123;
S:={[1,2,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]*ts[3]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*
ts[Rep(SSS[i])[3]]then print SSS[i];
end if; end for; end for;
```

```
for g in N do if [1,2,3]^g eq [2,1,6] then
N123s:=sub<N|N123s,g>;end if; end for;
for g in N do if [1,2,3]^g eq [7,8,4] then
N123s:=sub<N|N123s,g>;end if; end for;
for g in N do if [1,2,3]^g eq [8,7,5] then
N123s:=sub<N|N123s,g>;end if; end for;
#N123s;
N123s;
#N/#N123s;
T:=Transversal(N,N123s);
for i := 1 to #T do
ss := [1,2,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N123s);
for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[1] eq
 m*(ts[1]*ts[6])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[3] eq
 m*(ts[1]*ts[2])^n then m,n;
end if;end for;
ts[1]*ts[2]*ts[3]*ts[3] eq ts[1]*ts[2];

N132:=Stabiliser(N,[1,3,2]);
N132s:=N132;
S:={[1,3,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]*ts[2]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,3,2]^g eq [3,1,6] then
N132s:=sub<N|N132s,g>;end if; end for;
for g in N do if [1,3,2]^g eq [4,7,2] then
 N132s:=sub<N|N132s,g>;end if; end for;
for g in N do if [1,3,2]^g eq [7,4,6] then
 N132s:=sub<N|N132s,g>;end if; end for;
#N132s;
#N/#N132s;
T:=Transversal(N,N132s);
```

```
for i := 1 to #T do
ss := [1,3,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N132s);
for m,n in IN do if ts[1]*ts[3]*ts[2]*ts[2] eq
 m*(ts[1]*ts[3])^n then m,n;end if;end for;
ts[1]*ts[3]*ts[2]*ts[2] eq ts[1]*ts[3];
for m,n in IN do if ts[1]*ts[3]*ts[2]*ts[5] eq
m*(ts[1]*ts[3]*ts[2])^n then m,n;
end if;end for;
for m,n in IN do if ts[1]*ts[3]*ts[2]*ts[1] eq
m*(ts[1]*ts[6])^n then m,n;
end if;end for;
N161:=Stabiliser(N,[1,6,1]);
N161s:=N161;
S:={[1,6,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[6]*ts[1]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,6,1]^g eq [2,3,2] then
 N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [3,2,3] then
 N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [4,8,4] then
N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [6,1,6] then
N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [5,7,5] then
 N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [7,5,7] then
N161s:=sub<N|N161s,g>;end if; end for;
for g in N do if [1,6,1]^g eq [8,4,8] then
 N161s:=sub<N|N161s,g>;end if; end for;
#N161s;
N161s;
#N/#N161s;
T:=Transversal(N,N161s);
```

```
for i := 1 to #T do
ss := [1,6,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..45] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N161s);
for m,n in IN do if ts[1]*ts[6]*ts[1]*ts[1] eq m*(ts[1]*ts[6])
^n then m,n;
end if;end for;
```

B.5 $2^4 : S_3 \times S_3$ over $S_3 \times S_3$

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^2,z^2,w^3,x^-1*y*x*y,
(x^-1*z)^2,(y*z)^2,(x,w),(y*w^-1)^2,z*w^-1*z*w,t^2,
(t,y),(t,x * z),((x * w)*t)^4,((z * w)*t)^6,((x * y)*t)^4,
((x * y * w^-1)*t)^6>;
Index(G,sub<G|x,y,z,w>);
f, G1, k:=CosetAction(G,sub<G|x,y,z,w>);
#k;
CompositionFactors(G1);
S:=Sym(9);
xx:=S! (1, 2, 9)(3, 4, 5)(6, 7, 8);
yy:=S!(3, 6)(4, 7)(5, 8);
zz:=S! (1, 2)(4, 5)(7, 8);
ww:=S!(1, 4, 7)(2, 5, 8)(3, 6, 9);
N:=sub<S|xx,yy,zz,ww>;
#N;
NN<x,y,z,w>:=Group<x,y,z,w|x^3,y^2,z^2,w^3,x^-1*y*x*y,
(x^-1*z)^2,(y*z)^2,(x,w),(y*w^-1)^2,z*w^-1*z*w>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..36]];
for i in [2..36] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
end for;
PP:=Id(N);
```

```
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do 1^ArrayP[i],Sch[i];end for;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
 W,phi:=WordGroup(G1);
 rho:=InverseWordMap(G1);
 IN:=sub<G1|f(x),f(y),f(z),f(w)>;
 ts := [ Id(G1): i in [1 .. 9] ];
 ts[1]:=f(t);
 ts[2]:=f(t^z);
 ts[3]:=f(t^( w * x^-1));
 ts[4]:=f(t^( y * w));
 ts[5]:=f(t^(  y * z * w));
 ts[6]:=f(t^( (w*x*y)^-1 ));
 ts[7]:=f(t^( w^5));
 ts[8]:=f(t^( x*w^2));
 ts[9]:=f(t^( y * x^-1));
  #G/#N;
 cst:= [null : i in [1 .. 32]]
 where null is[Integers() |];
 for i := 1 to 9 do
 cst[prodim(1, ts, [i])]:=[i];
 end for;
 m:=0; for i in [1..32] do if cst[i] ne [] then
  m:=m+1; end if; end for; m;
 for i in [1..32] do i, cst[i]; end for;
N1:=Stabiliser(N,1);
N1:=Stabiliser(N,[1]);
S:={[1]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1] eq g*ts[Rep(SSS[i])[1]]then print SSS[i];
end if; end for; end for;
N1s:=N1;
#N1s;
 T1:=Transversal(N,N1s);
 #T1;
 T1:=Transversal(N,N1s);
 for i := 1 to #T1 do
 ss := [1]^T1[i];
 cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;for i in [1..32] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
 T1:=Transversal(N,N1s);
 #T1;

 Orbits(N1s);
 for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n
  then m,n; end if;end for;
 for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n
 then m,n; end if;end for;
 for m,n in IN do if ts[1]*ts[3] eq m*(ts[1])^n
 then m,n; end if;end for;
 #N/#N1s;
  N13:=Stabiliser(N,[1,3]);
  N13s:=N13;
  S:={[1,3]};
  SS:=S^N;
  SSS:=Setseq(SS);
  for i in [1..#SS] do
  for g in IN do
  if ts[1]*ts[3]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
  then print SSS[i];
  end if; end for; end for;
   for g in N do if [1,3]^g eq [5,7] then
   N13s:=sub<N|N13s,g>; end if; end for;
   for g in N do if [1,3]^g eq [1,8] then
    N13s:=sub<N|N13s,g>; end if; end for;
   for g in N do if [1,3]^g eq [5,7] then
    N13s:=sub<N|N13s,g>; end if; end for;
   for g in N do if [1,3]^g eq [5,9] then
```

```
   N13s:=sub<N|N13s,g>; end if; end for;
   for g in N do if [1,3]^g eq [6,2] then
 N13s:=sub<N|N13s,g>; end if; end for;
for g in N do if [1,3]^g eq [6,4] then
N13s:=sub<N|N13s,g>; end if; end for;
N13s;
# N13s;
#N/#N13s;
T:=Transversal(N,N13s);
for i := 1 to #T do
ss := [1,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
  end for;
 m:=0; for i in [1..32] do if cst[i] ne [] then m:=m+1;
   end if; end for;m;
 Orbits(N13s);
for m,n in IN do if ts[1]*ts[3]*ts[1] eq
 m*(ts[1])^n then m,n;end if; end for;m;
for m,n in IN do if ts[1]*ts[3]*ts[2] eq
 m*(ts[1])^n then m,n;end if; end for;m;
 for m,n in IN do if ts[1]*ts[2] eq
 m*(ts[1]*ts[4])^n then m,n;end if; end for;m;
 for m,n in IN do if ts[1]*ts[2] eq
 m*(ts[1]*ts[4])^n then m,n;end if; end for;m;
N14:=Stabiliser(N,[1,4]);
 N14s:=N14;
 S:={[1,4]};
 SS:=S^N;
 SSS:=Setseq(SS);
 for i in [1..#SS] do
 for g in IN do
 if ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
 end if; end for; end for;
for g in N do if [1,4]^g eq [1,7] then
N14s:=sub<N|N14s,g>; end if; end for;
 # N14s;N14s;
 #N/#N14s;
 T:=Transversal(N,N14s);
 for i := 1 to #T do
 ss := [1,4]^T[i];
 cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0; for i in [1..32] do if cst[i] ne [] then m:=m+1;
```

```
 end if; end for;m;
Orbits(N14s);
 for m,n in IN do if ts[1]*ts[4]*ts[2] eq
 m*(ts[1])^n then m,n; end if; end for;m;
 for m,n in IN do if ts[1]*ts[4]*ts[4] eq
  m*(ts[1])^n then m,n; end if; end for;m;
 for m,n in IN do if ts[1]*ts[4]*ts[3] eq
 m*(ts[1]*ts[3]*ts[1])^n then m,n;end if; end for;m;
 N141:=Stabiliser(N,[1,4,1]);
 N141s:=N141;
 S:={[1,4,1]};
 SS:=S^N;
 SSS:=Setseq(SS);
 for i in [1..#SS] do
 for g in IN do if ts[1]*ts[4]*ts[1]eq
 g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 *ts[Rep(SSS[i])[3]]then print SSS[i];
 end if; end for; end for;
 for g in N do if [1,4,1]^g eq [2,5,2] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [1,7,1] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [4,7,4] then
 N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [9,3,9] then
 N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [2,8,2] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [5,8,5] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [4,1,4] then
 N141s:=sub<N|N141s,g>; end if; end for;
 for g in N do if [1,4,1]^g eq [7,4,7] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [7,1,7] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [9,6,9] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [3,6,3] then
  N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [5,2,5] then
 N141s:=sub<N|N141s,g>;end if; end for;
 for g in N do if [1,4,1]^g eq [8,5,8] then
 N141s:=sub<N|N141s,g>;end if; end for;
```

```
for g in N do if [1,4,1]^g eq [8,2,8] then
 N141s:=sub<N|N141s,g>;end if; end for;
for g in N do if [1,4,1]^g eq [3,9,3] then
N141s:=sub<N|N141s,g>;end if; end for;
for g in N do if [1,4,1]^g eq [6,3,6] then
 N141s:=sub<N|N141s,g>;end if; end for;
for g in N do if [1,4,1]^g eq [6,9,6] then
 N141s:=sub<N|N141s,g>;end if; end for;
#N141s;
#N/#N141s;
T:=Transversal(N,N141s);
for i := 1 to #T do
ss := [1,4,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..32] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
Orbits(N141s);
for m,n in IN do if ts[1]*ts[4]*ts[1]*ts[1] eq
m*(ts[1]*ts[4])^n then m,n;end if;end for;
N131:=Stabiliser(N,[1,3,1]);
N131s:=N131;
S:={[1,3,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]*ts[1]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
 for g in N do if [1,3,1]^g eq [3,8,3] then
 N131s:=sub<N|N131s,g>;end if; end for;
for g in N do if [1,3,1]^g eq [1,8,1] then
 N131s:=sub<N|N131s,g>; end if; end for;
for g in N do if [1,3,1]^g eq [8,1,8] then
N131s:=sub<N|N131s,g>;end if; end for;
for g in N do if [1,3,1]^g eq [8,3,8] then
N131s:=sub<N|N131s,g>;end if; end for;
for g in N do if [1,3,1]^g eq [3,1,3] then
 N131s:=sub<N|N131s,g>; end if; end for;
#N131s;
N131s;
#N/#N131s;
T:=Transversal(N,N131s);
```

```
 for i := 1 to #T do
 ss := [1,3,1]^T[i];
 cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0; for i in [1..32] do if cst[i] ne [] then m:=m+1;
 end if; end for;m;
  Orbits(N131s);
 for m,n in IN do if ts[1]*ts[3]*ts[1]*ts[1] eq
  m*(ts[1]*ts[3])^n then m,n;end if;end for;
 for m,n in IN do if ts[1]*ts[3]*ts[1]*ts[2] eq
  m*(ts[1]*ts[4])^n then m,n; end if;end for;
```

B.6 $3^3 : S_4 \times 2$ over $S_3 \times S_3$

```
S:=Sym(9);
xx:=S! (1, 2, 9)(3, 4, 5)(6, 7, 8);
yy:=S!(3, 6)(4, 7)(5, 8);
zz:=S! (1, 2)(4, 5)(7, 8);
ww:=S!(1, 4, 7)(2, 5, 8)(3, 6, 9);
N:=sub<S|xx,yy,zz,ww>;
#N;
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^2,z^2,w^3,
x^-1*y*x*y,(x^-1*z)^2,(y*z)^2,(x,w),(y*w^-1)^2,
z*w^-1*z*w,t^2,(t,y),(t,x * z),(w*t)^3,((x * w)*t)^0,
((z * w)*t)^0,((x * y)*t)^6,((x * y * w^-1)*t)^4>;
#G;
#G/#N;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
CompositionFactors(G1);
NN<x,y,z,w>:=Group<x,y,z,w|x^3,y^2,z^2,w^3,x^-1*y*x*y,
(x^-1*z)^2,(y*z)^2,(x,w),(y*w^-1)^2,z*w^-1*z*w>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..36]];
for i in [2..36] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
end for;
PP:=Id(N);
```

```
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do 1^ArrayP[i],Sch[i];end for;

G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3,y^2,z^2,w^3,x^-1*y*x*y,
(x^-1*z)^2,(y*z)^2,(x,w),(y*w^-1)^2,z*w^-1*z*w,
t^2,(t,y),(t,x * z),(w*t)^3,
((x * w)*t)^0,((z * w)*t)^0,((x * y)*t)^6,
((x * y * w^-1)*t)^4>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
IN:=sub<G1|f(x),f(y),f(z),f(w)>;
ts := [ Id(G1): i in [1 .. 9] ];
ts[1]:=f(t);
ts[2]:=f(t^z);
ts[3]:=f(t^( w * x^-1));
ts[4]:=f(t^( y * w));
ts[5]:=f(t^(  y * z * w));
ts[6]:=f(t^( (w*x*y)^-1 ));
ts[7]:=f(t^( w^5));
ts[8]:=f(t^( x*w^2));
ts[9]:=f(t^( y * x^-1));
#G/#N;
cst:= [null : i in [1 .. 36]]
where null is[Integers() |];
for i := 1 to 9 do
cst[prodim(1, ts, [i])]:=[i];
end for;
m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
```

```
for i in [1..36] do i, cst[i]; end for;
N1:=Stabiliser(N,1);
N1:=Stabiliser(N,[1]);
S:={[1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1] eq g*ts[Rep(SSS[i])[1]]then print SSS[i];
end if; end for; end for;
N1s:=N1;
#N1s;
T1:=Transversal(N,N1s);
#T1;
for i := 1 to #T1 do
ss := [1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1s);
for m,n in IN do if ts[1]*ts[2] eq m*(ts[1])^n
then m,n; end if;end for;
for m,n in IN do if ts[1]*ts[4] eq m*(ts[1])^n
then m,n; end if;end for;
for m,n in IN do if ts[1]*ts[3] eq m*(ts[1])^n
then m,n; end if;end for;
for m,n in IN do if ts[1]*ts[3] eq m*(ts[1]*ts[2])^n
then m,n; end if;end for;
#N/#N1s;
N13:=Stabiliser(N,[1,3]);
N13s:=N13;
S:={[1,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[3]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,3]^g eq [4,2] then
N13s:=sub<N|N13s,g>; end if; end for;
N13s;
# N13s;
```

```
#N/#N13s;
T:=Transversal(N,N13s);
for i := 1 to #T do
ss := [1,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1;
end if; end for;m;

Orbits(N13s);
for m,n in IN do if ts[1]*ts[3]*ts[2] eq
m*(ts[1])^n then m,n;end if; end for;
for m,n in IN do if ts[1]*ts[3]*ts[1] eq
m*(ts[1]*ts[3])^n then m,n;end if; end for;
for m,n in IN do if ts[1]*ts[3]*ts[7] eq
m*(ts[1]*ts[2])^n then m,n;end if; end for;
for m,n in IN do if ts[1]*ts[3]*ts[5] eq
m*(ts[1]*ts[3])^n then m,n;end if; end for;
for m,n in IN do if ts[1]*ts[3]*ts[6] eq
m*(ts[1]*ts[2])^n then m,n;end if; end for;

N12:=Stabiliser(N,[1,2]);
N12s:=N12;
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[2]eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];end if; end for; end for;
for g in N do if [1,2]^g eq [2,9] then
N12s:=sub<N|N12s,g>; end if; end for;
for g in N do if [1,2]^g eq [9,1] then
N12s:=sub<N|N12s,g>; end if; end for;
#N12s;
N12s;
#N/#N12s;
T:=Transversal(N,N12s);
for i := 1 to #T do
ss := [1,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1;
end if; end for;m;
```

```
Orbits(N12s);
for m,n in IN do if ts[1]*ts[2]*ts[2] eq
m*(ts[1])^n then m,n;end if;end for;
for m,n in IN do if ts[1]*ts[2]*ts[3] eq
m*(ts[1]*ts[3])^n then m,n;end if;end for;
N137:=Stabiliser(N,[1,3,7]);
N137s:=N137;
S:={[1,3,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]*ts[7]eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
for g in N do if [1,3,7]^g eq [ 2, 4, 8 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq [ 4, 6, 1 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq [ 9, 5, 6 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 5, 7, 2 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 2, 6, 5 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 7, 9, 4 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 3, 8, 9 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 1, 8, 4 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 8, 1, 5 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 9, 7, 3 ]
then N137s:=sub<N|N137s,g>;
```

```
end if; end for;
for g in N do if [1,3,7]^g eq  [ 5, 9, 8 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 8, 3, 2 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 6, 2, 3 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 4, 2, 7 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq   [ 7, 5, 1 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 3, 1, 6 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
for g in N do if [1,3,7]^g eq  [ 6, 4, 9 ]
then N137s:=sub<N|N137s,g>;
end if; end for;
#N137s;
N137s;
#N/#N137s;
T:=Transversal(N,N137s);
for i := 1 to #T do
ss := [1,3,7]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for; m:=0; for i in [1..36] do if cst[i]
ne [] then m:=m+1;end if; end for;m;
Orbits(N137s);
for m,n in IN do if ts[1]*ts[3]*ts[7]*ts[7] eq
m*(ts[1]*ts[3])^n then m,n;end if; end for;
```

Appendix C

MAGMA Code for Maximal DCE

C.1 M_{11} over Maximal Subgroup S_5

```
S:=Sym(40);
xx:=S!(1, 9, 17, 25, 33)(2, 26, 10, 34, 18)
(3, 35, 27, 19, 11)(4, 20, 36, 12, 28)(5,13, 21, 29, 37)
(6, 30, 14, 38, 22)
(7, 39, 31, 23, 15)(8, 24, 40, 16, 32);
yy:=S!(1, 4, 3, 2)(5, 8, 7, 6)(9, 12, 11, 10)
(13, 16, 15, 14)(17, 20, 19, 18)(21, 24,23, 22)
(25, 28, 27, 26)(29, 32, 31, 30)(33, 36, 35, 34)
(37, 40, 39, 38);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^5,y^4,y^3*x^3*y*x,t^11,t^x=t^5,
(y*(t^9))^3,(y*(t^10))^6>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
HH2:=sub<G|x,y>;
M:=sub<G|x,y,t * x * y^2 * t^-1>;
DoubleCosets(G,M,HH2);
#DoubleCosets(G,M,HH2);
IM:=sub<G1|f(x),f(y),f(t * x * y^2 * t^-1)>;
IN:=sub<G1|f(x),f(y)>;
NN<a,b>:=Group<a,b|a^5,b^4,b^3*a^3*b*a>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..20]];
```

```
for i in [2..20] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
ts := [ Id(G1): i in [1 .. 40] ];
ts[1]:=f(t);
ts[2]:=f(t^(y^-1));
ts[3]:=f(t^(y^2));
ts[4]:=f(t^y);
ts[5]:=(ts[1])^2;
ts[6]:=(ts[2])^2;
ts[7]:=(ts[3])^2;
ts[8]:=(ts[4])^2;
ts[9]:=(ts[1])^4;
ts[10]:=(ts[2])^4;
ts[11]:=(ts[3])^4;
ts[12]:=(ts[4])^4;
ts[13]:=(ts[1])^8;
ts[14]:=(ts[2])^8;
ts[15]:=(ts[3])^8;
ts[16]:=(ts[4])^8;
ts[17]:=(ts[1])^5;
ts[18]:=(ts[2])^5;
ts[19]:=(ts[3])^5;
ts[20]:=(ts[4])^5;
ts[21]:=(ts[1])^10;
ts[22]:=(ts[2])^10;
ts[23]:=(ts[3])^10;
ts[24]:=(ts[4])^10;
ts[25]:=(ts[1])^9;
ts[26]:=(ts[2])^9;
ts[27]:=(ts[3])^9;
ts[28]:=(ts[4])^9;
ts[29]:=(ts[1])^7;
ts[30]:=(ts[2])^7;
```

```
ts[31]:=(ts[3])^7;
ts[32]:=(ts[4])^7;
ts[33]:=(ts[1])^3;
ts[34]:=(ts[2])^3;
ts[35]:=(ts[3])^3;
ts[36]:=(ts[4])^3;
ts[37]:=(ts[1])^6;
ts[38]:=(ts[2])^6;
ts[39]:=(ts[3])^6;
ts[40]:=(ts[4])^6;
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
cst := [null : i in
[1 .. Index(G,sub<G|x,y,t * x * y^2 * t^-1>)]]
where null is
[Integers() | ];
Orbits(N);
N1:=Stabiliser (N,[1]);
SSS:={[1]};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IM do
if ts[1]eq n*ts[Rep(Seqq[i])[1]] then print Rep(Seqq[i]);
end if; end for; end for;
N1s:=N1;
for n in N do if 1^n eq 19 then N1s:=sub<N|N1s,n>;
end if; end for;
N1; #N1;
T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..66] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for i in [1..40] do for n in IM do if ts[1]
eq n*ts[i] then i; end if; end for; end for;
```

```
Orbits(N1s);
for g in IM do for h in IN do if ts[1]*ts[5] eq
g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[9] eq
g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[11] eq
g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[13] eq
g*(ts[1])^h then "true"; break; end if;
end for; end for;
N1s;
#N1s;
for g in IM do for h in IN do if ts[1]*ts[3] eq
g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[4] eq
g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[7] eq
g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[15] eq
g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[17] eq
g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6] eq
g*(ts[1]*ts[10])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8] eq
g*(ts[1]*ts[14])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8] eq
g*(ts[1]*ts[26])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8] eq
g*(ts[1]*ts[30])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[12] eq
g*(ts[1]*ts[22])^h then "true"; break; end if;
```

```
end for; end for;
N5:=Stabiliser (N,[5]);
SSS:={[5]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[5]eq n*ts[Rep(Seqq[i])[1]]then print Rep(Seqq[i]);
end if; end for; end for;
N5s:=N5;
for n in N do if 5^n eq 31 then N5s:=sub<N|N5s,n>;
end if; end for;
N5; #N5;
T5:=Transversal(N,N5s);
for i in [1..#T5] do
ss:=[5]^T5[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..66] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for i in [1..40] do for n in IM do if ts[5] eq
 n*ts[i] then i;
end if; end for; end for;
{5,31}^N;
Orbits(N5s);
#N5s;
N5s;
#N/#N5s;
for g in IM do for h in IN do if ts[5]*ts[3]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[9]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[13]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[23]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[1]
```

```
eq g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[7]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[5]*ts[8]
eq g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[15]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[5]*ts[19]
eq g*(ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[2]
eq g*(ts[5]*ts[16])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[5]*ts[4]
eq g*(ts[5]*ts[12])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[4]
eq g*(ts[5]*ts[28])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[5]*ts[4]
eq g*(ts[5]*ts[32])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[5]*ts[20]
eq g*(ts[5]*ts[24])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]
eq g*(ts[5]*ts[2])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]
eq g*(ts[5]*ts[32])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]
eq g*(ts[5]*ts[4])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]
eq g*(ts[5]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]
eq g*(ts[5]*ts[28])^h then "true"; break; end if;
end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[12]
eq g*(ts[5]*ts[20])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[21]
eq g*(ts[5]*ts[11])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]
eq g*(ts[5]*ts[6])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]
eq g*(ts[5]*ts[6])^h then "true"; break; end if;
 end for; end for;
N16:=Stabiliser (N,[1,6]);
SSS:={[1,6]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1] *ts[6]eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N16s:=N16;
for n in N do if 1^n eq 19 and 6^n eq 16 then
 N16s:=sub<N|N16s,n>;
end if; end for;
N16; #N16;
T16:=Transversal(N,N16s);
for i in [1..#T16] do
ss:=[1,6]^T16[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..66] do if cst[i] ne []
then m:=m+1; end if; end for; m;
[1,6]^N16s;
for n in IM do if ts[1]*ts[6] eq n*ts[19]*ts[16] then n;
end if; end for;
for i in [1..#T16] do ([1,6]^N16s)^T16[i]; end for;
Orbits(N16s);
#N/#N16s;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[5]
eq g*(ts[1])^h then "true"; break; end if;
```

```
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[26]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[2]
eq g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[15]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[1]
eq g*(ts[5]*ts[2])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[1]
eq g*(ts[5]*ts[16])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[1]
eq g*(ts[1]*ts[10])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[3]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[3]
eq g*(ts[1]*ts[14])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[4]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[6]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[7]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[8]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[9]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[10]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[11]
```

```
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[13]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[14]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[17]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[18]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[21]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[22]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[26]
eq g*(ts[1]*ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[26]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[30]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[6]*ts[1]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[12]*ts[1]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[12]*ts[2]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[12]*ts[3]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[12]*ts[4]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
 end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[12]*ts[5]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
 end for; end for;
N18:=Stabiliser (N,[1,8]);
SSS:={[1,8]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1] *ts[8]eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N18s:=N18;
N18; #N18;
T18:=Transversal(N,N18s);
for i in [1..#T18] do
ss:=[1,8]^T18[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..66] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N18s);
[1,8]^N18s;
for i in [1..#T18] do ([1,8]^N18s)^T18[i]; end for;
Orbits(N18s);
#N/#N18s;
#N/#N18;
#N18s;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[1]
eq g*(ts[1]*ts[12])^h then "true"; break;
end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[2]
eq g*(ts[1]*ts[6])^h then "true"; break;
end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[3]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[4]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[5]
```

```
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[6]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[7]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[8]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[9]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[10]
eq g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[11]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[12]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[13]
eq g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[14]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[15]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[16]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[17]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[18]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[19]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
 end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[8]*ts[20]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[21]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[22]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[23]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[24]
eq g*(ts[5])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[25]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[26]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[27]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[28]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[29]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[30]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[31]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[32]
eq g*(ts[1]*ts[18])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[33]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[34]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
```

```
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[35]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[36]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[37]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[38]
eq g*(ts[1])^h then "true"; break; end if; end for;
end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[39]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[8]*ts[40]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
N118:=Stabiliser (N,[1,18]);
SSS:={[1,18]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1] *ts[18]eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N118s:=N118;
for n in IM do if 1^n eq 20 and 18^n eq 25 then
N118s:=sub<N|N118s,n>;end if; end for;
for n in IM do if 1^n eq 20 and 18^n eq 25 then
 N118s:=sub<N|N118s,n>; end if; end for;
for n in IM do if 1^n eq 11 and 18^n eq 4 then
 N118s:=sub<N|N118s,n>; end if; end for;
for n in IM do if 1^n eq 34 and 18^n eq 35 then
 N118s:=sub<N|N118s,n>;end if; end for;
for n in N do if 1^n eq 34 and 18^n eq 35 then
 N118s:=sub<N|N118s,n>;
end if; end for;
#N118;
```

```
#N118s;
N118s;
T118:=Transversal(N,N118s);
for i in [1..#T118] do
ss:=[1,18]^T118[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..66] do if cst[i] ne []
then m:=m+1; end if; end for; m;
#N118s;
#N/#N118s;
Orbits(N118s);
for g in IM do for h in IN do if ts[1]*ts[18]*ts[1]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[2]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[3]
eq g*(ts[5])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[4]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[5]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[6]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[7]
eq g*(ts[1]*ts[12])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[10]
eq g*(ts[1]*ts[8])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[18]*ts[14]
eq g*(ts[1]*ts[6])^h then "true"; break; end if;
end for; end for;
```

C.2 $U(3,5)$ over Maximal Subgroup A_7

```
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,
(x*y*x^-2*y*x)^2,t^5,(t,y*x^-1*y),t^(x*y*x^-2)=t^4,
(y*x^2*t^(y*x^2))^4,(y*x^2*t^2)^7>;
#G;
#sub<G|x,y>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
M:=MaximalSubgroups(G1);
#M;
#PrimitiveGroup(5,5);
for i in [1..8] do #M[i]`subgroup/120; end for;
for i in [1..8] do if f(x) in M[i]`subgroup and
f(y) in M[i]`subgroup then i; end if; end for;
C:=Conjugates(G1,M[3]`subgroup);
CC:=Setseq(C);
#CC;
for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i]
then i; end if; end for;
C:=Conjugates(G1,M[4]`subgroup);
CC:=Setseq(C);
#CC;
for i in [1..#CC] do f(x) in CC[i] and f(y) in
CC[i] then i; end if; end for;
for i in [1..#CC] do if f(x) in CC[i] and f(y) in
CC[i] then i; end if; end for;
C:=Conjugates(G1,M[5]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in
CC[i] then i; end if; end for;
C:=Conjugates(G1,M[6]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in
CC[i] then i;end if; end for;
C:=Conjugates(G1,M[7]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in
CC[i] then i;
end if; end for;
C:=Conjugates(G1,M[8]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in
 CC[i] then i;
end if; end for;
```

```
28
M8:=CC[28];
Generators(M8);
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,
t^5,(t,y*x^-1*y),t^(x*y*x^-2)=t^4,(y*x^2*t^(y*x^2))^4,
(y*x^2*t^2)^7>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
for g in G1 do if sub<G1|f(x),f(y),g> eq M8 then
A:=g; break;
end if; end for;
Order(A);
W:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
function(W)
w4 := W.3 * W.1; w3 := W.3^-1; w5 := w4 * w3;
 return w5;
end function
AA:=function(W)
w4 := W.3 * W.1; w3 := W.3^-1; w5 := w4 * w3;
return w5;
end function;
AA(G);
t * x * t^-1
M:=sub<G|x,y,t * x * t^-1>;
#M;
#DoubleCosets(G,M,sub<G|x,y>);
IM:=sub<G1|f(x),f(y),f(t * x * t^-1)>;
CompositionFactors(IM);
...............start DCE..............
S:=Sym(24);
xx:=S!(1, 6, 11, 4, 2)(5, 22, 20, 19, 24)
(7, 12, 17, 10, 8)(13, 18, 23, 16, 14);
yy:=S!(1, 3)(2, 22)(4, 8)(5, 6)(7, 9)(10, 14)(11, 12)
(13, 15)(16, 20)(17, 18)(19,21)(23,24);
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,
(x*y*x^-2*y*x)^2,t^5,(t,y*x^-1*y),
t^(x*y*x^-2)=t^4,(y*x^2*t^(y*x^2))^4,
(y*x^2*t^2)^7>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
N:=sub<S|xx,yy>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
```

```
HH2:=sub<G|x,y>;
M:=sub<G|x,y,t * x * t^-1>;
DoubleCosets(G,M,HH2);
#DoubleCosets(G,M,HH2);
IM:=sub<G1|f(x),f(y),f(t * x * t^-1)>;
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 24] ];
ts[1]:=f(t);
ts[2]:=f(t^(x^-1));
ts[3]:=f(t^y);
ts[4]:=f(t^(x^-2));
ts[5]:=f(t^( x * y));
ts[6]:=f(t^((y * x)^2));
ts[7]:=f(t^(x^2 * y * x^-1));
ts[8]:=f(t^(x^2 * y * x^-2));
ts[9]:=f(t^(x^2 * y * x^-1 * y));
ts[10]:=f(t^(x^-2 * y * x^-1));
ts[11]:=f(t^(x^2));
ts[12]:=f(t^(x^2 * y));
ts[13]:=f(t^((x^2 * y * x^-1)^2));
ts[14]:=f(t^((x^2 * y)^2));
ts[15]:=f(t^(x^-1 * y * x * y * x^2 * y));
ts[16]:=f(t^(x^-1 * y * x * y));
ts[17]:=f(t^(x^2 * y * x));
ts[18]:=f(t^(x^2 * y * x * y));
ts[19]:=f(t^(x * y * x^-2));
ts[20]:=f(t^(y^x));
ts[21]:=f(t^(x^-1 * y * x^2 * y * x));
ts[22]:=f(t^(x^-1 * y));
ts[23]:=f(t^(x * y * x^-1 * y));
ts[24]:=f(t^(x * y * x^-1));
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y,t * x * t^-1>)]]
where null is [Integers() | ];
Orbits(N);
N1:=Stabiliser (N,[1]);
SSS:={[1]};
SSS:=SSS^N;
```

```
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IM do
if ts[1]eq n*ts[Rep(Seqq[i])[1]] then print Rep(Seqq[i]);
end if; end for; end for;
N1:=Stabiliser (N,[1]);
SSS:={[1]};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IM do
if ts[1]eq n*ts[Rep(Seqq[i])[1]] then print
 Rep(Seqq[i]);end if; end for; end for;
N1s:=N1;
for n in N do if 1^n eq 6 then
 N1s:=sub<N|N1s,n>; end if; end for;
for n in N do if 1^n eq 11 then
 N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 4 then
 N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 2 then
 N1s:=sub<N|N1s,n>; end if; end for;
for n in N do if 1^n eq 18 then
 N1s:=sub<N|N1s,n>; end if; end for;
for n in N do if 1^n eq 16 then
 N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 9 then
 N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 14 then
 N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 23 then
N1s:=sub<N|N1s,n>;end if; end for;
for n in N do if 1^n eq 21 then
 N1s:=sub<N|N1s,n>;
end if; end for;
for n in N do if 1^n eq 13 then
 N1s:=sub<N|N1s,n>;
end if; end for;
N1; #N1s;
N1s;
T1:=Transversal(N,N1s);
for i in [1..#T1] do
```

```
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..50] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N1; #N1s;
N1; #N1s;
Orbits(N1s);
#N/#N1s;
N11:=Stabiliser (N,[1,1]);
SSS:={[1,1]};
SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IM do
if ts[1] *ts[1]eq n*ts[Rep(Seqq[i])[1]]*ts
[Rep(Seqq[i])[2]]then print Rep(Seqq[i]);
end if; end for; end for;
N11s:=N11;
N11; #N11;
for n in N do if 1^n eq 13 and  1^n eq 13 then
 N11s:=sub<N|N11s,n>;
end if; end for;
T11:=Transversal(N,N11s);
for i in [1..#T11] do
ss:=[1,1]^T11[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..50] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N11s);
for g in IM do for h in IN do if ts[1]*ts[1]*ts[13]
eq g*(ts[1]*ts[1])^h then "true";
break; end if; end for; end for;
#N/#N11s;
N11s;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[7]
eq g*(ts[1])^h then "true"; break;
end if; end for; end for;
N112:=Stabiliser(N,[1,1,2]);
SSS:={[1,1,2]};
SSS:=SSS^N;
#SSS;
```

```
Seqq:=Setseq(SSS);
for i in [1..#SSS] do for n in IM do
if ts[1]*ts[1]*ts[2] eq n*ts
[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]then print Rep(Seqq[i]);
end if; end for; end for;
for g in N do if [1,1,2]^g eq [20,20,7] then
 N112:=sub<N|N112,g>;
end if; end for;
for g in N do if [1,1,2]^g eq [8,8,19]
then N112:=sub<N|N112,g>; end if; end for;
for g in N do if [1,1,2]^g eq [13,13,19] then
 N112:=sub<N|N112,g>;
end if; end for;
N112s:=N112;
N112; #N112;
#N112s;
#N/#N112s;
T112:=Transversal(N,N112s);
for i in [1..#T112] do
ss:=[1,1,2]^T112[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..50] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N112s);
N112s;
#N/#N112s;
for g in IM do for h in IN do if
ts[1]*ts[1]*ts[2]*ts[1]
eq g*(ts[1])^h then "true"; break; end if;
end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[1]
eq g*(ts[1]*ts[1])^h then "true"; break; end if;
 end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[2]
eq g*(ts[1]*ts[1])^h then "true"; break;
end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[2]
eq g*(ts[1])^h then "true"; break;
end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[2]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
```

```
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[3]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[5]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[9]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[1]*ts[2]*ts[11]
eq g*(ts[1]*ts[1]*ts[2])^h then "true";
break; end if; end for; end for;
N113:=Stabiliser(N,[1,1,3]);
SSS:={[1,1,3]};
SSS:=SSS^N;
Seqq:=Setseq(SSS);
for i in [1..#SSS] do for n in IM do
if ts[1]*ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i]
[3]]then print Rep(Seqq[i]);
end if; end for; end for;
for g in N do if [1,1,3]^g eq [3,3,1]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [10,10,11] then
N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [20,20,6] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [14,14,12] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [23,23,22] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [24,24,2] then
N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [6,6,20] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [11,11,10] then
 N113:=sub<N|N113,g>;
```

```
end if; end for;
for g in N do if [1,1,3]^g eq [16,16,5] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [17,17,4] then
N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [21,21,7] then
N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [5,5,16]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [12,12,14] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [22,22,23] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [2,2,24]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [18,18,8] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [4,4,17]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [19,19,9] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [7,7,21]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [8,8,18]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [9,9,19]
then N113:=sub<N|N113,g>; end if; end for;
for g in N do if [1,1,3]^g eq [15,15,13] then
 N113:=sub<N|N113,g>;
end if; end for;
for g in N do if [1,1,3]^g eq [13,13,15] then
 N113:=sub<N|N113,g>;
end if; end for;
N113s:=N113;
N113s;
#N/#N113s;
T113:=Transversal(N,N113s);
```

```
for i in [1..#T113] do
ss:=[1,1,3]^T113[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..50] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N113s);
for g in IM do for h in IN do if ts[1]*ts[1]*ts[3]*ts[1]
eq g*(ts[1]*ts[1])^h then "true";break; end if;
end for; end for;
```

Appendix D

MAGMA Code for Monomial Progenitors

D.1 PrimitiveGroup(37,2)

```
G:=PrimitiveGroup(37,2);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq
2 then i; end if; end for;
H:=S[3]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37] do for j in [3..20] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[2],G);
I eq CG[11];
CH[2];
C:=CyclotomicField(37);
A:=[[C.1,0] : i in [1..2]];
```

```
for i ,j in [1..2] do A[i,j]:=0; end for;
GG:=GL(2,C);
T:=Transversal(G,H);
#T;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0] : i in [1..2]];
for i ,j in [1..2] do B[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 18 then D[i,j]:=37; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^19 then D[i,j]:=145; end if;
if CH[n](T[i]*p*T[j]^-1) eq  1 then D[i,j]:=1; end if;
end if; end for;
return D;
end function;
GG:=GL(2,149);
A:=[[0,0] : i in [1..2]];
AA:=GG!mat(2,xx,A,2);
BB:=GG!mat(2,yy,A,2);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
A:=[[C.1,0] : i in [1..2]]; for i ,j in [1..2] do A[i,j]:=0;
end for;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
```

```
end if; end for;
B:=[[C.1,0] : i in [1..2]]; for i ,j in [1..2]
do B[i,j]:=0;
end for; for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H
then B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge
0 then w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(2,C);
HH:=sub<Sym(2*148)|perm(2,148,GG!A),perm(2,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(2,148,GG!A);
perm(2,148,GG!B);
```

D.2 PrimitiveGroup(37,3)

```
G:=PrimitiveGroup(37,3);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq  3
then i; end if; end for;
H:=S[3]`subgroup;
CH:=CharacterTable(H);
CH;
#H;#G;
for i in [2..37] do for j in [4..15] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[2],G);
I eq CG[10];
CH[2];
C:=CyclotomicField(37);
A:=[[C.1,0,0] : i in [1..3]];
for i ,j in [1..3] do A[i,j]:=0; end for;
GG:=GL(3,C);
T:=Transversal(G,H);
#T;
for i,j in [1..3] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0] : i in [1..3]];
for i ,j in [1..3] do B[i,j]:=0; end for;
for i,j in [1..3] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^3 then D[i,j]:=73; end if;
```

```
if CH[n](T[i]*p*T[j]^-1) eq C.1^30 then D[i,j]:=29; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^4 then D[i,j]:=125; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
GG:=GL(3,149);
A:=[[0,0,0] : i in [1..3]];
AA:=GG!mat(2,xx,A,3);
BB:=GG!mat(2,yy,A,3);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
A:=[[C.1,0,0] : i in [1..3]]; for i ,j in [1..3] do
A[i,j]:=0;
end for;
for i,j in [1..3] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0] : i in [1..3]]; for i ,j in [1..3] do
B[i,j]:=0; end for; for i,j in [1..3] do if T[i]*yy*T
[j]^-1 in H then  B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
```

```
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(3,C);
HH:=sub<Sym(3*148)|perm(3,148,GG!A),perm(3,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(3,148,GG!A);
perm(3,148,GG!B);
```

D.3 PrimitiveGroup(37,4)

```
G:=PrimitiveGroup(37,4);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 4
then i; end if; end for;
H:=S[3]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37] do for j in [5..13] do if
 Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[2],G);
```

```
I eq CG[9];
CH[2];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do A[i,j]:=0; end for;
GG:=GL(4,C);
T:=Transversal(G,H);
#T;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do B[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^21 then D[i,j]:=19; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^16 then D[i,j]:=102;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^22 then D[i,j]:=6;end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
GG:=GL(4,149);
A:=[[0,0,0,0] : i in [1..4]];
AA:=GG!mat(2,xx,A,4);
BB:=GG!mat(2,yy,A,4);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
```

```
BB;
C:=CyclotomicField(148);
A:=[[C.1,0,0,0] : i in [1..4]]; for i ,j in [1..4] do
A[i,j]:=0;
end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0] : i in [1..4]]; for i ,j in [1..4] do
B[i,j]:=0;
end for; for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1)
mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(4,C);
HH:=sub<Sym(4*148)|perm(4,148,GG!A),perm(4,148,GG!B)>;
#HH;
```

```
IsIsomorphic(G,HH);
perm(4,148,GG!A);
perm(4,148,GG!B);
```

D.4 PrimitiveGroup(37,5)

```
G:=PrimitiveGroup(37,5);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 6
then i; end if; end for;
H:=S[4]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37] do for j in [7..12] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[7],G);
I eq CG[12];
CH[7];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0,0,0] : i in [1..6]];
for i ,j in [1..6] do A[i,j]:=0; end for;
GG:=GL(6,C);
T:=Transversal(G,H);
#T;
for i,j in [1..6] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[7](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0,0,0] : i in [1..6]];
for i ,j in [1..6] do B[i,j]:=0; end for;
for i,j in [1..6] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[7](T[i]*yy*T[j]^-1);
```

```
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 23 then D[i,j]:=96; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^8 then D[i,j]:=129;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 14 then D[i,j]:=104;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 6 then D[i,j]:=114;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 29then D[i,j]:=67;end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
GG:=GL(6,149);
A:=[[0,0,0,0,0,0] : i in [1..6]];
AA:=GG!mat(7,xx,A,6);
BB:=GG!mat(7,yy,A,6);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
A:=[[C.1,0,0,0,0,0] : i in [1..6]]; for i ,j in
 [1..6] do A[i,j]:=0;
end for;
for i,j in [1..6] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[7](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0] : i in [1..6]]; for i ,j in
 [1..6] do B[i,j]:=0;
end for; for i,j in [1..6] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[7](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
```

```
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
 w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
 w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(6,C);
HH:=sub<Sym(6*148)|perm(6,148,GG!A),perm(6,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(6,148,GG!A);
perm(6,148,GG!B);
```

D.5 PrimitiveGroup(37,6)

```
G:=PrimitiveGroup(37,6);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq  9
then i; end if; end for;
H:=S[3]`subgroup;
```

```
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37] do for j in [10..13] do if
 Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[12],G);
I eq CG[10];
CH[12];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0,0,0,0,0,0] : i in [1..9]];
for i ,j in [1..9] do A[i,j]:=0; end for;
GG:=GL(9,C);
T:=Transversal(G,H);
#T;
for i,j in [1..9] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[12](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0,0,0,0,0,0] : i in [1..9]];
for i ,j in [1..9] do B[i,j]:=0; end for;
for i,j in [1..9] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[12](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 18 then D[i,j]:=37; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 32 then D[i,j]:=123;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=107;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 14 then D[i,j]:=104;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 24 then D[i,j]:=46;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 20 then D[i,j]:=85;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 29 then D[i,j]:=67;end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
```

```
end function;
GG:=GL(9,149);
A:=[[0,0,0,0,0,0,0,0,0] : i in [1..9]];
AA:=GG!mat(12,xx,A,9);
BB:=GG!mat(12,yy,A,9);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
A:=[[C.1,0,0,0,0,0,0,0,0] : i in [1..9]]; for i ,j
 in [1..9] do A[i,j]:=0;
end for;
for i,j in [1..9] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[12](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0,0,0,0] : i in [1..9]]; for i ,j
 in [1..9] do B[i,j]:=0;
end for; for i,j in [1..9] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[12](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
 w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
```

```
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(9,C);
HH:=sub<Sym(9*148)|perm(9,148,GG!A),perm(9,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(9,148,GG!A);
perm(9,148,GG!B);
```

D.6 PrimitiveGroup(37,7)

```
G:=PrimitiveGroup (37,7);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq  12
then i; end if; end for;
H:=S[4]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37]  do for j in [13..15] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[14],G);
I eq CG[15];
CH[14];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0] : i in [1..12]];
```

```
for i ,j in [1..12] do A[i,j]:=0; end for;
GG:=GL(12,C);
T:=Transversal(G,H);
#T;
for i,j in [1..12] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[14](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0] : i in [1..12]];
for i ,j in [1..12] do B[i,j]:=0; end for;
for i,j in [1..12] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[14](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 23 then D[i,j]:=96; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^26 then D[i,j]:=5;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 27 then D[i,j]:=80;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^8 then D[i,j]:=129;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1   then D[i,j]:=16;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 14 then D[i,j]:=104;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 11 then D[i,j]:=30;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 6 then D[i,j]:=114;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 10 then D[i,j]:=95;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 29 then D[i,j]:=67;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^36 then D[i,j]:= 28;end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
GG:=GL(12,149);
A:=[[0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..12]];
AA:=GG!mat(14,xx,A,12);
BB:=GG!mat(14,yy,A,12);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
```

```
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0] : i in [1..12]];
for i ,j in [1..12] do A[i,j]:=0;
end for;
for i,j in [1..12] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[14](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0] : i in [1..12]];
 for i ,j in [1..12] do B[i,j]:=0;
end for; for i,j in [1..12] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[14](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
 w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
```

```
GG:=GL(12,C);
HH:=sub<Sym(12*148)|perm(12,148,GG!A),perm(12,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(12,148,GG!A);
perm(12,148,GG!B);
```

D.7 PrimitiveGroup(37,8)

```
G:=PrimitiveGroup(37,8);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq  18
then i; end if; end for;
H:=S[4]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..37] do for j in [19..20] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[14],G);
I eq CG[20];
CH[14];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..18]];
for i ,j in [1..18] do A[i,j]:=0; end for;
GG:=GL(18,C);
T:=Transversal(G,H);
#T;
for i,j in [1..18] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[14](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
```

```
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..18]];
for i ,j in [1..18] do B[i,j]:=0; end for;
for i,j in [1..18] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[14](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^ 29 then D[i,j]:=67;end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^35 then D[i,j]:=39; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^18 then D[i,j]:=37; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^23 then D[i,j]:=96; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^13 then D[i,j]:=81; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^17 then D[i,j]:=142; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^32 then D[i,j]:=123; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^8 then D[i,j]:=129; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=107; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^19 then D[i,j]:=145; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^14 then D[i,j]:=104; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^22 then D[i,j]:=6; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^24 then D[i,j]:=46; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^6 then D[i,j]:=114; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^20 then D[i,j]:=85; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^5 then D[i,j]:=63; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;end if;
end if; end for;
return D;
end function;
GG:=GL(18,149);
A:=[[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..18]];
AA:=GG!mat(14,xx,A,18);
BB:=GG!mat(14,yy,A,18);
Order(AA);
Order(xx);
Order(BB);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(148);
```

```
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] :
 i in [1..18]]; for i ,j in [1..18] do A[i,j]:=0;
end for;
for i,j in [1..18] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[20](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] :
 i in [1..18]]; for i ,j in [1..18] do B[i,j]:=0;
end for; for i,j in [1..18] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[20](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
 w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
 w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(18,C);
HH:=sub<Sym(18*148)|perm(18,148,GG!A),perm(18,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(18,148,GG!A);
perm(18,148,GG!B);
```

D.8 PrimitiveGroup(37,9)

```
G:=PrimitiveGroup(37,9);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq  36
then i; end if; end for;
H:=S[4]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
I:=Induction(CH[2],G);
I eq CG[37];
C:=CyclotomicField(37);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..36]];
for i ,j in [1..36] do A[i,j]:=0; end for;
GG:=GL(36,C);
T:=Transversal(G,H);
#T;
for i,j in [1..36] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..36]];
for i ,j in [1..36] do B[i,j]:=0; end for;
for i,j in [1..36] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1 then D[i,j]:=16; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^19 then D[i,j]:=145; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^28 then D[i,j]:=88; end if;
```

```
if CH[n](T[i]*p*T[j]^-1) eq C.1^14 then D[i,j]:=104; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^7 then D[i,j]:=36; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^22 then D[i,j]:=6; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^11 then D[i,j]:=30; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^24 then D[i,j]:=46; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^12 then D[i,j]:=33; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^6 then D[i,j]:=114; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^3 then D[i,j]:=73; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^20 then D[i,j]:=85; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^10 then D[i,j]:=95; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^5 then D[i,j]:=63; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^21 then D[i,j]:=19; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^29 then D[i,j]:=67; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^33 then D[i,j]:=31; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^35 then D[i,j]:=39; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^36then D[i,j]:=28; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^18 then D[i,j]:=37; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^9 then D[i,j]:=127; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^23 then D[i,j]:=96; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^30 then D[i,j]:=29; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^26 then D[i,j]:=5; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^13 then D[i,j]:=81; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^25 then D[i,j]:=140; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^34 then D[i,j]:=49; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^17 then D[i,j]:=142; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^27 then D[i,j]:=80; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^32 then D[i,j]:=123; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^16 then D[i,j]:=102; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^8 then D[i,j]:=129; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^4 then D[i,j]:=125; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=107; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=107; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^36 then D[i,j]:=28; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^18 then D[i,j]:=37; end if;
```

```
if CH[n](T[i]*p*T[j]^-1) eq C.1^9 then D[i,j]:=127; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^23 then D[i,j]:=96; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^30 then D[i,j]:=29; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^7 then D[i,j]:=36; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^30 then D[i,j]:=29; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^26 then D[i,j]:=5; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^13 then D[i,j]:=81; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^20 then D[i,j]:=85; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^15 then D[i,j]:=25; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^13 then D[i,j]:=81; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^25 then D[i,j]:=140; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^31 then D[i,j]:=17; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^34 then D[i,j]:=49; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^17 then D[i,j]:=142; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^27 then D[i,j]:=80; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^27 then D[i,j]:=80; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
end if; end for;
return D;
end function;
GG:=GL(36,149);
A:=[[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..36]];
AA:=GG!mat(2,xx,A,36);
BB:=GG!mat(2,yy,A,36);
HH:=sub<GG|AA,BB>;
#HH;
IsIsomorphic(HH,G);
AA;
BB;
Order(AA);
Order(xx);
Order(BB);
C:=CyclotomicField(148);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..36]]; for i ,j
in [1..36] do A[i,j]:=0;
```

```
end for;
for i,j in [1..36] do if T[i]*xx*T[j]^-1 in H then
 A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] : i in [1..36]]; for i ,j
in [1..36] do B[i,j]:=0;
end for; for i,j in [1..36] do if T[i]*yy*T[j]^-1 in H then
 B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j) ge 0 then
 w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
 w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(36,C);
HH:=sub<Sym(36*148)|perm(36,148,GG!A),perm(36,148,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(36,148,GG!A);
perm(36,148,GG!B);
```

D.9 PrimitiveGroup(5,2)

```
G:=PrimitiveGroup(5,2);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
G;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq
2 then i; end if; end for;
H:=S[3]`subgroup;
H;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..5] do for j in [3..4] do if
Induction(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[2],G);
I eq CG[3];
CH[2];
C:=CyclotomicField(5);
T:=Transversal(G,H);
#T;
A:=[[C.1,0] : i in [1..2]];
for i ,j in [1..2] do A[i,j]:=0; end for;
GG:=GL(2,C);
T:=Transversal(G,H);
#T;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0] : i in [1..2]];
for i ,j in [1..2] do B[i,j]:=0; end for;
for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
```

```
end if; end for;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=5; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^3 then D[i,j]:=9; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
end if; end for;
return D;
end function;
GG:=GL(2,11);
A:=[[0,0] : i in [1..2]];
AA:=GG!mat(2,xx,A,2);
BB:=GG!mat(2,yy,A,2);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(10);
A:=[[C.1,0] : i in [1..2]]; for i ,j in [1..2] do A[i,j]:=0;
end for;
for i,j in [1..2] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0] : i in [1..2]]; for i ,j in [1..2] do B[i,j]:=0;
for> end for;
for i,j in [1..2] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
```

```
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j)
ge 0 then w[i]:
=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(2,C);
HH:=sub<Sym(2*10)|perm(2,10,GG!A),perm(2,10,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(2,10,GG!A);
perm(2,10,GG!B);
```

D.10 PrimitiveGroup(5,3)

```
G:=PrimitiveGroup(5,3);
IsAbelian(G);
CG:=CharacterTable(G);
CG;xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 4
then i; end if; end for;
H:=S[3]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
for i in [2..5] do for j in [5] do if Induction
```

```
(CH[i],G) eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[2],G);
I eq CG[5];
CH[2];
C:=CyclotomicField(5);
T:=Transversal(G,H);
#T;
A:=[[C.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do A[i,j]:=0; end for;
GG:=GL(4,C);
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG!A;
GG!B;
Order(GG!B);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1 then D[i,j]:=4;
end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^3 then D[i,j]:=9;
end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=5;
end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^4 then D[i,j]:=3;
end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1;
end if;
end if; end for;
return D;
end function;
GG:=GL(4,11);
A:=[[0,0,0,0] : i in [1..4]];
AA:=GG!mat(2,xx,A,4);
BB:=GG!mat(2,yy,A,4);
Order(AA);
Order(xx);
Order(BB);
Order(yy);
HH:=sub<GG|AA,BB>;
#HH;
#G;
```

```
IsIsomorphic(HH,G);
AA;
BB;
C:=CyclotomicField(10);
A:=[[C.1,0,0,0] : i in [1..4]]; for i ,j in [1..4]
do A[i,j]:=0;
end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0] : i in [1..4]]; for i ,j in
[1..4] do B[i,j]:=0;
end for;
for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation
of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j)
ge 0 then w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
```

```
GG:=GL(4,C);
HH:=sub<Sym(4*10)|perm(4,10,GG!A),perm(4,10,GG!B)>;
#HH;
IsIsomorphic(G,HH);
perm(4,10,GG!A);
perm(4,10,GG!B);
```

D.11 PrimitiveGroup(5,4)

```
G:=PrimitiveGroup(5,4);
IsAbelian(G);
false
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq
5 then i; end if; end for;
H:=S[8]`subgroup;
CH:=CharacterTable(H);
CH;
#H;
#G;
I:=Induction(CH[2],G);
I eq CG[5];
true
CH[2];
T:=Transversal(G,H);
T;
C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
for i ,j in [1..5] do A[i,j]:=0; end for;
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG:=GL(5,C);
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0,0] : i in [1..5]];
for i ,j in [1..5] do B[i,j]:=0; end for;
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
```

```
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
Order(GG!B);
GG:=GL(5,7);
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1 then D[i,j]:=4; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=2; end if;
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq -1 then D[i,j]:=-1; end if;
end if; end for;
return D;
end function;
A:=[[0,0,0,0,0]: i in [1..5]];
mat(2,xx,A,5);
mat(2,yy,A,5);
AA:=GG!mat(2,xx,A,5);
BB:=GG!mat(2,yy,A,5);
HH:=sub<GG|AA,BB>;
IsIsomorphic(HH,G);
C:=CyclotomicField(6);
A:=[[C.1,0,0,0,0]: i in [1..5]];
for i,j in [1..5] do A[i,j]:=0; end for;
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1);
end if; end for;
GG:=GL(5,C);
GG!A;
B:=[[C.1,0,0,0,0]: i in [1..5]];
for i,j in [1..5] do B[i,j]:=0; end for;
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1);
end if; end for;
GG!B;
perm := function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/
C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do
s[i]:=i;
end for;
```

```
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do
j:=0; done:=0;
repeat
if z[1,i]/u^j in Z then
if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if; end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do
for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for; end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
perm(5,6,GG!A);
perm(5,6,GG!B);
HH:=sub<Sym(30)|perm(5,6,GG!A),perm(5,6,GG!B)>;
#HH;
IsIsomorphic(HH,G);
```

D.12 PrimitiveGroup(5,5)

```
G:=PrimitiveGroup(5,5);
IsAbelian(G);
CG:=CharacterTable(G);
CG;
xx:=G.1;
yy:=G.2;
S:=Subgroups(G);
for i in [1..#S] do if Index(G,S[i]`subgroup) eq 6
then i; end if; end for;

H:=S[16]`subgroup;
CH:=CharacterTable(H);
CH;
```

```
#H;
#G;
for i in [2..5] do for j in [3..7] do if Induction(CH[i],G)
eq CG[j] then i,j;
end if; end for;
end for;
I:=Induction(CH[3],G);
I eq CG[7];
T:=Transversal(G,H);
#T;
C:=CyclotomicField(4);
A:=[[C.1,0,0,0,0,0] : i in [1..6]];
for i ,j in [1..6] do A[i,j]:=0; end for;
for i,j in [1..6] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[3](T[i]*xx*T[j]^-1);
end if; end for;
GG:=GL(6,C);
GG!A;
Order(GG!A);
Order(xx);
B:=[[C.1,0,0,0,0,0] : i in [1..6]];
for i ,j in [1..6] do B[i,j]:=0; end for;
for i,j in [1..6] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[3](T[i]*yy*T[j]^-1);
end if; end for;
GG!

mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq 1 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1 then D[i,j]:=2; end if;
if CH[n](T[i]*p*T[j]^-1) eq -C.1 then D[i,j]:=3; end if;
end if; end for;
return D;
end function;
GG:=GL(6,5);
A:=[[0,0,0,0,0,0] : i in [1..6]];
AA:=GG!mat(3,xx,A,6);
BB:=GG!mat(3,yy,A,6);
HH:=sub<GG|AA,BB>;
#HH;
#G;
sIsomorphic(HH,G);
AA;
```

```
BB;
C:=CyclotomicField(4);
A:=[[C.1,0,0,0,0,0] : i in [1..6]]; for i ,j
in [1..6] do A[i,j]:=0;end for;
for i,j in [1..6] do if T[i]*xx*T[j]^-1 in H
then A[i,j]:=CH[3](T[i]*xx*T[j]^-1);
end if; end for;
B:=[[C.1,0,0,0,0,0] : i in [1..6]]; for i ,j
in [1..6] do B[i,j]:=0;
end for;
for i,j in [1..6] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[3](T[i]*yy*T[j]^-1);
end if; end for;
perm:= function(n, p, mat)
/* Return the matrix converted to permutation of S_{n*p}.
*/ C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do j:=0;
done:=0;
repeat if z[1,i]/u^j in Z then if Z!(z[1,i]/u^j)
ge 0 then w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if;
end if;
j:=j+1;   until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for;
end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;
GG:=GL(6,C);
IsIsomorphic(G,HH);
perm(6,4,GG!A);
perm(6,4,GG!B);
```

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