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## Investigation of Finite Groups Through Progenitors

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INVESTIGATION OF FINITE GROUPS THROUGH PROGENITORS

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Charles Anthony Baccari

December 2017

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## ABSTRACT

The goal of this presentation is to find original symmetric presentations of finite groups. It is frequently the case, that progenitors factored by appropriate relations produce simple and even sporadic groups as homomorphic images. We have discovered two of the twenty-six sporadic simple groups namely,  $M_{12}$  and  $J_1$  and the Lie type group  $\text{Suz}(8)$ . In addition several linear and classical groups will also be presented. We will present several progenitors including:  $2^{*12} : 2^2 \times 3 : 2$ ,  $2^{*11} : PSL_2(11)$ , and  $2^{*5} : (5 : 4)$  which have produced the homomorphic images:  $M_{12} : 2$ ,  $J_1 \times 2$ , and  $\text{Suz}(8) \times 2$ . We will present the monomial progenitors:  $17^{*10} :_m PGL_2(9)$ ,  $3^{*4} :_m Z_2 \wr D_4$  and  $13^{*2} :_m (2^2 \times 3) : 2$  which produce the homomorphic images:  $13^2 : ((2 \times 13) : (2 \times 3))$ ,  $2 \times S_9$ , and  $(2^2)^\bullet PGL_4(3)$ . Once we have a presentation of a group we can verify the group's existence through double coset enumeration. We will give proofs of isomorphism types of the presented images:  $S_3 \times PGL_2(7) \times S_5$ ,  $2^8 : A_5$ , and  $2^\bullet U(2) : 2$ .

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# Introduction

Group theory is a topic spanning many areas of mathematics. Group theory allows us to study symmetry, determine the solvability of  $n^{th}$  degree polynomials as Galois did in the 19<sup>th</sup> century, provides new perspectives on geometry and analysis, and even serves a purpose in a more recently discovered field of mathematics, matroid theory. Group theory has very intricate machinery capable of solving some of the most perplexing problems in mathematics through this notion of groups.

A group is a set of elements which satisfy the four group axioms. The set of elements and group operation  $(G,*)$  must satisfy: closure under the operation among elements, every element in the group  $G$  must have an inverse also contained in  $G$ , the set must contain an identity element ( $e$ ), and associativity must hold for any elements in  $G$ . The order of this group of elements is simply the number of elements in the set.

There are also special types of groups with elements that act commutatively which have a special name, abelian groups. Certain collections of elements pulled from a group can form a subgroup of  $G$  and the order of this subgroup  $H$ , will divide the order of  $G$  as was proven by Lagrange. A subgroup of  $G$  is said to be normal in  $G$  if the elements of the subgroup commute with the elements of  $G$ . If a nontrivial group  $G$  contains only the trivial  $\{e\}$  and set  $G$  normal subgroups then  $G$  is said to be a simple group. These simple groups are special to our field of study as they encompass Lie type groups and the sporadic simple groups. These special simple groups are valuable components to our research. Sporadic groups typically have large orders but that huge size speaks very little about the complexity of the operations taking place among its elements.

The following are some of the concepts which will be investigated in this document. **Chapter 1** will give the reader proofs, lemmas, and definitions of discussed topics. **Chapter 2** will go through the process of constructing progenitors. **Chapter 3** will demonstrate the ability to find isomorphic images using the homomorphic properties of groups. **Chapter 4** will prove the existence of these groups through the process of double coset enumeration. **Chapter 5** will demonstrate our ability to induce a linear character and construct progenitors from their faithful monomial representations.

## Chapter 1

# Group Theory Prerequisites

### 1.1 Definitions

**Definition 1.1.** [Rot95] A group  $G$  is **abelian** if every pair of elements in  $G$  commutes with one another.

**Definition 1.2.** [Rot95] A **group**  $G$  ( $G, *$ ) is a nonempty collection of elements with an associative operation  $*$ , such that:

- there exists an identity element,  $e \in G$  such that  $e * a = a * e$  for all  $a \in G$ ;
- for every  $a \in G$ , there exists an element  $b \in G$  such that  $a * b = e = b * a$ .

**Definition 1.3.** [Rot95] For group  $G$ , a **subgroup**  $S$  of  $G$  is a nonempty subset where  $s \in G$  implies  $s^{-1} \in G$  and  $s, t \in G$  imply  $st \in G$ . We denote subgroup  $S$  of  $G$  as  $S \leq G$ .

**Definition 1.4.** [Rot95] Let  $H$  be a subgroup of group  $G$ .  $H$  is a **proper** subgroup of  $G$  if  $H \neq G$ . We denote this as  $H < G$ .

**Definition 1.5.** [Rot95] Let  $G$  be a group and  $H \leq G$ .  $H$  is a **maximal subgroup** of  $G$  if there is no normal subgroup  $N \leq G$  such that  $H < N < G$ .

**Definition 1.6.** [Rot95] A **symmetric group**,  $S_X$ , is the group of all permutations of  $X$ , where  $X \in \mathbb{N}$ .  $S_X$  is a group under compositions.

**Definition 1.7.** [Rot95] If  $X$  is a nonempty set, a **permutation** of  $X$  is a bijection  $\phi : X \rightarrow X$ .

**Definition 1.8.** [Rot95] If  $x \in X$  and  $\phi \in S_X$ , then  $\phi$  **fixes**  $x$  if  $\phi(x) = x$  and  $\phi$  **moves**  $x$  if  $\phi(x) \neq x$ .

**Definition 1.9.** [Rot95] For permutations  $\alpha, \beta \in S_X$ ,  $\alpha$  and  $\beta$  are **disjoint** if every element moved by one permutation is fixed by the other. Precisely,

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(x) = x \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y.$$

**Definition 1.10.** [Rot95] A permutation which interchanges a pair of elements is a **transposition**.

**Definition 1.11.** [Rot95] In group  $G$ , if  $a, b \in G$ ,  $a$  and  $b$  **commute** if  $a * b = b * a$ .

**Definition 1.12.** [Rot95] Let  $G$  be a group. The **order** of  $G$  is the number of elements contained in  $G$ . We denote the order of  $G$  by  $|G|$ .

**Definition 1.13.** [Rot95] Let  $G$  be a group.  $G$  is **simple** if the only normal subgroups of  $G$  are  $1$  and  $G$ .

**Definition 1.14.** [Rot95] Let  $p$  be prime. If  $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ , then we say  $G$  is **elementary abelian**.

**Definition 1.15.** [Rot95] Let  $(G, *)$  and  $(H, \circ)$  be groups. The function  $\phi : G \rightarrow H$  is a **homomorphism** if  $\phi(a * b) = \phi(a) \circ \phi(b)$ , for all  $a, b \in G$ . An **isomorphism** is a bijective homomorphism. We say  $G$  is **isomorphic** to  $H$ ,  $G \cong H$ , if there is exists an isomorphism  $f : G \rightarrow H$ .

**Definition 1.16.** [Rot95] Let  $f : G \rightarrow H$  be a homomorphism. The **kernel of a homomorphism** is the set  $\{x \in G \mid f(x) = 1\}$ , where  $1$  is the identity in  $H$ . We denote the kernel of  $f$  as **ker f**.

**Definition 1.17.** [Rot95] Let  $X$  be a nonempty subset of a group  $G$ . Let  $w \in G$  where  $w = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ , with  $x_i \in X$  and  $e_i = \pm 1$ . We say that  $w$  is a **word** on  $X$ .

**Definition 1.18.** [Rot95] Let  $G$  be a group such that  $K \leq G$ .  $K$  is **normal** in  $G$  if  $gKg^{-1} = K$ , for every  $g \in G$ . We will use  $K \triangleleft G$  to denote  $K$  as being normal in  $G$ .

**Definition 1.19.** [Rot95] Let  $G$  be a group. We say  $G$  is a **direct product** of two subgroups  $H$  and  $K$  if:

1.  $H \trianglelefteq G, K \trianglelefteq G$ ;
2.  $G = HK$ ;
3.  $H \cap K = 1$ ,

**Definition 1.20.** [Rot95]  $G$  is a **semi-direct product** of two subgroups  $H$  and  $K$  if:

1.  $K \trianglelefteq G, Q \leq G$ ;
2.  $G = KQ$ ;
3.  $K \cap Q = 1$ .

**Definition 1.21.** [Rot95] Let  $a, b \in G$ . We denote the **commutator** of  $a$  and  $b$  by  $[a, b]$ , where  $[a, b] = aba^{-1}b^{-1}$ .

**Definition 1.22.** [Rot95] Let  $G$  be a group. The **Derived Group** of  $G$ , denoted  $G'$ , is the subgroup of  $G$  formed by all the commutators of  $G$ .

**Definition 1.23.** [Rot95] Let  $G$  be a group and  $S \subseteq G$ . For  $t \in G$ , a **right coset** of  $S$  in  $G$  is the subset of  $G$  such that  $St = \{st : s \in S\}$ . We say  $t$  is a **representative** of the coset  $St$ .

**Definition 1.24.** [Rot95] Let  $G$  be a group. The **index** of  $H \leq G$ , denoted  $[G : H]$ , is the number of right cosets of  $H$  in  $G$ .

**Definition 1.25.** [Rot95] Let  $G$  be a group and  $H$  and  $K$  be subgroups of  $G$ . A **double coset** of  $H$  and  $K$  of the form  $HgK = \{Hgk | k \in K\}$  is determined by  $g \in G$ .

**Definition 1.26.** [Rot95] Let  $N$  be a group. The **point stabiliser** of  $w$  in  $N$  is given by:

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i \text{'s.}$$

**Definition 1.27.** [Rot95] Let  $N$  be a group. The **coset stabiliser** of  $Nw$  in  $N$  is given by:

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i \text{'s.}$$



**Definition 1.28.** [Rot95] Let  $X$  be a set and  $G$  be a group. We say  $X$  is a **G-set** if there exists a function  $\phi : G \times X \rightarrow X$  (which we call an **action**) and the following hold for  $\phi$ :

- $1x = x$ , for all  $x \in X$ .
- $g(hx) = (gh)x$ , for  $g, h \in G$  and  $x \in X$ .

**Definition 1.29.** [Rot95] Let  $G$  be a group. The **center** of  $G$ ,  $Z(G)$ , is the set of all elements in  $G$  that commute with all elements of  $G$ .

**Definition 1.30.** [Rot95] Let  $G$  be a group and  $H, N \leq G$  such that  $|G| = |N||H|$ .  $G$  is a **central extension** by  $H$  if  $N$  is the center of  $G$ . We denote this by  $G \cong N \bullet H$ .

**Definition 1.31.** [Rot95] Let  $G$  be a group and  $H, N \leq G$  such that  $|G| = |N||H|$ .  $G$  is a **mixed extension** by  $H$  if it is a combination of both central extensions and semi-direct products, where  $N$  is the normal subgroup of  $G$  but not central. We denote this by  $G \cong N \bullet H$ .

**Definition 1.32.** [Rot95] Let  $G$  be a group. If  $H \leq G$ , the **normalizer** of  $H$  in  $G$  is defined by  $N_G(H) = \{a \in G \mid aHa^{-1} = H\}$

**Definition 1.33.** [Rot95] Let  $G$  be a group. If  $H \leq G$ , the **centralizer** of  $H$  in  $G$  is:

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}.$$

**Definition 1.34.** [Rot95] Let  $a \in G$ , where  $G$  is a group. The **conjugacy class** of  $a$  is given by  $a^G = \{a^g \mid g \in G\} = \{g^{-1}ag \mid g \in G\}$

**Definition 1.35.** [Rot95] Let  $G$  be a group and  $X$  be a  $G$ -set. For  $x \in X$ , the set  $x^G = \{x^g \mid g \in G\}$  is a **G-Orbit**.

**Definition 1.36.** [Rot95] Let  $X$  be a  $G$ -set. Let  $\alpha$  be an action of  $G$  on  $X$ . If  $\tilde{\alpha} : G \rightarrow S_X$  is injective, we say  $X$  is **faithful**.

**Definition 1.37.** [Rot95] Let  $G$  be a group and  $X$  be a  $G$ -set.  $X$  is **transitive** if for all  $x, y \in X$  there exists a  $g \in G$  such that  $y = gx$ .

**Definition 1.38.** [Rot95] Let  $G$  be a group. A **normal series**  $G$  is a sequence of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

with  $G_{i+1} \triangleleft G_i$ . Furthermore, the **factors groups** of  $G$  are given by  $G_i/G_{i+1}$  for  $i = 0, 1, \dots, n-1$ .

**Definition 1.39.** [Rot95] Let  $G$  be a group. A **composition series** of  $G$  given by:

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

is a normal series where, for all  $i$ , either  $G_{i+1}$  is a maximal normal subgroup of  $G_i$  or  $G_{i+1} = G_i$ .

**Definition 1.40.** [Rot95] If group  $G$  has a composition series, the factor groups of its series are the **composition factors** of  $G$ .

**Definition 1.41.** [Rot95] Let  $X$  be a set and  $\Delta$  by a family of words on  $X$ . A group  $G$  has **generators**  $X$  and **relations**  $\Delta$  if  $G \cong F/R$ , where  $F$  is a free group with basis  $X$  and  $R$  is the normal subgroup of  $F$  generated by  $\Delta$ . We say  $\langle X | \Delta \rangle$  is a **presentation** of  $G$ .

**Definition 1.42.** [Rot95] The **Dihedral Group**  $D_n$ ,  $n$  even and greater than 2, groups are formed by two elements, one of order  $\frac{n}{2}$  and one of order 2. A presentation for a Dihedral Group is given by  $\langle a, b | a^{\frac{n}{2}}, b^2, (ab)^2 \rangle$ .

**Definition 1.43.** [Rot95] A **general linear group**,  $GL(n, \mathbb{F})$  is the set of all  $n \times n$  matrices with nonzero determinant over field  $\mathbb{F}$ .

**Definition 1.44.** [Rot95] A **special linear group**,  $SL(n, \mathbb{F})$  is the set of all  $n \times n$  matrices with determinant 1 over field  $\mathbb{F}$ .

**Definition 1.45.** [Rot95] A **projective special linear group**,  $PSL(n, \mathbb{F})$  is the set of all  $n \times n$  matrices with determinant 1 over field  $\mathbb{F}$  factored by its center:

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F}))}$$

**Definition 1.46.** [Rot95] A **projective general linear group**,  $PGL(n, \mathbb{F})$  is the set of all  $n \times n$  matrices with nonzero determinant over field  $\mathbb{F}$  factored by its center:

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F}))}$$

**Definition 1.47.** [Rot95] Let  $X$  be a  $G$ -set. Then for  $B \subseteq X$ ,  $B$  is a **block** if for every  $g \in G$ , either  $gB = B$  or  $gB \cap B = \emptyset$ .

**Definition 1.48.** [Rot95] Let  $X$  and  $Y$  be  $G$ -sets. The function  $f : X \rightarrow Y$  is a **G-map** if  $f(gx) = gf(x)$ , for all  $x \in X$  and  $g \in G$ .

**Definition 1.49.** [Rot95] Let  $X$  be a  $G$ -set.  $X$  is **primitive** if  $X$  has no nontrivial blocks. If  $X$  is primitive, the only blocks of  $X$  are  $B = X$  and  $B = \emptyset$ .

**Definition 1.50.** [Rot95] Let  $H$  and  $K$  be permutation groups on  $X$  and  $Y$  respectively then  $Z = X \times Y$  where  $Z = \{(x, y) | x \in X, y \in Y\}$  and  $X \cap Y = \emptyset$ . If a permutation group is defined on  $Z$  then we call this the Wreath Product of  $H$  by  $K$ , denoted  $H \wr K$ . Define  $H \leq S_X$  and  $K \leq S_Y$ .

**Definition 1.51.** [Rot95] Let  $\gamma \in H$  and  $y$  be a fixed element of  $Y$  define  $\gamma(y) : \{(x, y) \rightarrow ((x)\gamma, y), (x, y_1) \rightarrow (x, y_1) \text{ if } y_1 \neq y\}$

**Definition 1.52.** [Rot95] If  $k \in K$  then  $k^* : (x, y) \rightarrow (x, (y)k)$ .

**Definition 1.53.** [Rot95] Let  $\gamma(y)$  and  $k^*$  be permutations of  $S_Z$  then

$$\phi : H \rightarrow S_Z \text{ given by } \gamma \rightarrow \gamma(y),$$

where  $\phi$  is a 1-1 homomorphism. Then the image

$$\phi(H) = \{\gamma(y) | \gamma \in H\} = H(y).$$

$$\text{And } \psi : K \rightarrow S_Z \text{ given by } k \rightarrow k^*,$$

where  $\psi$  is also a 1-1 homomorphism. Then  $\psi(k) = k^* = \{k^* | k \in K\}$

**Definition 1.54.** [Cur07] Let  $G$  be a group and  $T = \{t_1, t_2, \dots, t_n\}$  be a symmetric generating group of  $G$  with  $|t_i| = m$ . We define the progenitor to be the semi direct product  $m^{*n} : N$ , where  $m^{*n}$  is a free product among  $t_i$ 's and  $N$  is a control group that acts transitively on  $n$  letters.

**Definition 1.55.** [Led87] The **degree of a character**  $\chi$  is  $\chi(1)$ . Note that a character whose degree is 1 is called a linear character.

**Definition 1.56.** [Led87] Let  $A(x) = (a_{ij}(x))$  be a matrix representation of  $G$  of degree  $m$ . We consider the characteristic polynomial of  $A(x)$ , namely

$$\det(\lambda - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \dots & -a_{1m}(x) \\ \lambda - a_{21}(x) & -a_{22}(x) & \dots & -a_{2m}(x) \\ \dots & \dots & \dots & \dots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \dots & \lambda - a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree  $m$  in  $\lambda$ , and inspection shows that the coefficient of  $-\lambda^{m-1}$  is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + \dots + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of  $A(x)$ , abbreviated to  $\text{tr}A(x)$ , so that

$$\phi(x) = \text{tr}A(x)$$

We regard  $\phi(x)$  as a function on  $G$  with values in  $K$ , we call it the **character** of  $A(x)$ .

**Definition 1.57.** [GL93] Let  $\rho: G \rightarrow GL(n, \mathbb{C})$  with each  $n \times n$  matrix  $\rho(g)$  ( $g \in G$ ) we associate the complex number given by adding all the diagonal entries of the matrix and call this number  $\chi(g)$ . The function  $\chi: G \rightarrow \mathbb{C}$  is called the character of the representation of rho.

**Definition 1.58.** [GL93] Suppose that  $\chi$  and  $\phi$  are functions from  $G$  to  $\mathbb{C}$ , Define

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\theta(g)}$$

**Definition 1.59.** [Led87] Let  $H \leq G$  and  $\phi(u)$  be a character of  $H$  and define  $\phi(x) = 0$  if  $x \notin H$ . Then

$$\phi^G(x) = \begin{cases} \phi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

**Definition 1.60.** [Led87] Let  $G$  be a finite group and  $H$  be a subgroup such that  $[G : H] = n$ . Let  $C_\alpha$ , where  $\alpha = 1, 2, \dots, m$  be the conjugacy classes of  $G$  with  $|C_\alpha| = h_\alpha$ ,  $\alpha = 1, 2, \dots, m$ . Let  $\phi$  be a character of  $H$  and  $\phi^G$  be the character of  $G$  induced from the character of  $\phi$  of  $H$  up to  $G$ . The values of  $\phi^G$  on the  $m$  classes of  $G$  are given by:

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w), \alpha = 1, 2, 3, \dots, m.$$

**Definition 1.61.** [GL93] If  $\chi$  and  $\phi$  are functions from  $G$  to  $\mathbb{C}$ , and  $\lambda \in \mathbb{C}$ , then we define  $\chi + \phi : G \rightarrow \mathbb{C}$  by

$$(\chi + \phi)(g) = \chi(g) + \phi(g) \quad (g \in G)$$

and we define  $\lambda\chi : G \rightarrow \mathbb{C}$  by

$$\lambda\chi(g) = \lambda(\chi(g)) \quad (g \in G)$$

**Definition 1.62.** [GL93] Let  $G$  be a finite group having the distinct irreducible characters  $\chi^{(1)}, \chi^{(2)}, \dots, \chi^{(k)}$ . let  $1 \leq i, j \leq k$ . Then  $\sum_{i=1}^k \chi_\alpha^{(i)} \overline{\chi_\beta^{(i)}} = \frac{|G|}{h_\alpha} \delta_{\alpha\beta}$ .

(a) In a character table, the dot product of any column with the conjugate of other column is 0.

(b) In a character table, the dot product of the column  $\alpha$  with its own conjugate is  $\frac{|G|}{h_\alpha}$

**Definition 1.63.** [GL93] Let  $G$  be a cyclic group of order  $n$ . Then  $G = \langle z \rangle$  and  $|z| = n$ . Let  $\epsilon_r = e^{\frac{2\pi ir}{n}}$ , where  $r = 0, 1, 2, \dots, n$ , be the  $n$ th roots of unity. For any  $z^s \in G$ ,  $s = 0, 1, 2, \dots, n$ , the values of the irreducible characters  $\chi^{(r)}$  are given by  $\chi^{(r)}(z^s) = e^{\frac{2\pi irs}{n}}$  where  $r = 0, 1, 2, \dots, n$ .

**Definition 1.64.** [GL93] Let  $G$  be a finite abelian group, say  $G = \langle z_1 \rangle \times \langle z_2 \rangle \times \dots \times \langle z_m \rangle$ , where each  $z_\mu$  is of order  $n_\mu$ , and  $|G| = n_1 n_2 \dots n_m$ . Let  $x = z_1^{a_1} z_2^{a_2} \dots z_m^{a_m}$ , where  $0 \leq a_\mu < n_\mu$  be an arbitrary element of  $G$ . Now let, for each  $\mu$ ,  $\epsilon_\mu = e^{\frac{2\pi ir_\mu}{n_\mu}}$ , where  $r_\mu = 0, 1, 2, \dots, n_\mu$ , be the  $n_\mu$ th roots of unity. Here  $\mu = 1, 2, \dots, m$ . Then corresponding to each  $m$ -tuple  $r = [r_1, r_2, \dots, r_m]$ ,  $\chi^{(r)}$ , given by,  $\chi^{(r)}(x) = e^{\frac{2\pi i}{n} \sum_{\mu=1}^m \frac{a_\mu r_\mu}{n_\mu}}$

**Definition 1.65.** [GL93] Lifting Process

Let  $N$  be a normal subgroup of  $G$  and suppose that  $A_0(Nx)$  is a representation of degree  $m$  of the group  $G/N$ . Then  $A(x) = A_0(Nx)$  defines a representation of  $G/N$  LIFTED from  $G/N$ . If  $\phi_0(Nx)$  is the character of  $A_0(Nx)$ , then  $\phi(x) = \phi_0(Nx)$  is the LIFTED character of  $A(x)$ . Also, if  $u \in N$ , then  $A(u) = I_m, \phi(u) = m = \phi(1)$ . The Lifting Process preserves irreducibility.

**Definition 1.66.** [GL93] If  $A$  is a representation of  $G$  with character  $\phi$ , then those elements  $u$  which satisfy  $\phi(u) = \phi(1)$ , form a normal subgroup of  $M$ , which is the *KERNEL* of  $A$ . Then  $A_0(Mx) = A(x)$  is a representation of  $A$ , and we may regard  $A$  as having been lifted from  $A$ .

## 1.2 Theorems

**Theorem 1.67.** [Rot95] Every permutation  $\alpha \in S_n$  is either a cycle or a product of disjoint cycles.

**Theorem 1.68.** [Rot95] Let  $f : (G, *) \rightarrow (G', \circ)$  be a homomorphism. The following hold true:

- $f(e) = e'$ , where  $e'$  is the identity in  $G'$ ,
- If  $a \in G$ , then  $f(a^{-1}) = f(a)^{-1}$ ,
- If  $a \in G$  and  $n \in \mathbb{Z}$ , then  $f(a^n) = f(a)^n$ .

**Theorem 1.69.** [Rot95] The intersection of any family of subgroups of a group  $G$  is again a subgroup of  $G$ .

**Theorem 1.70.** [Rot95] If  $S \leq G$ , then any two right (or left) cosets of  $S$  in  $G$  are either identical or disjoint.

**Theorem 1.71.** [Rot95] If  $G$  is a finite group and  $H \leq G$ , then  $|H|$  divides  $|G|$  and  $[G : H] = |G|/|H|$ .

**Theorem 1.72.** [Rot95] If  $S$  and  $T$  are subgroups of a finite group  $G$ , then

$$|ST||S \cap T| = |S||T|.$$

**Theorem 1.73.** [Rot95] If  $N \triangleleft G$ , then the cosets of  $N$  in  $G$  form a group, denoted by  $G/N$ , of order  $[G : N]$ .

**Theorem 1.74.** [Rot95] The commutator subgroup  $G'$  is a normal subgroup of  $G$ . Moreover, if  $H \triangleleft G$ , then  $G/H$  is abelian if and only if  $G' \leq H$ .

**Theorem 1.75.** [Rot95] Let  $\phi : G \rightarrow H$  be a homomorphism with kernel  $K$ . Then  $K$  is a normal subgroup of  $G$  and  $G/K \cong \text{im}\phi$ .

**Theorem 1.76.** [Rot95] Let  $N$  and  $T$  be subgroups of  $G$  with  $N$  normal. Then  $N \cap T$  is normal in  $T$  and  $T/(N \cap T) \cong NT/N$ .

**Theorem 1.77.** [Rot95] Let  $G$  be a group with normal subgroups  $H$  and  $K$ . If  $HK = G$  and  $H \cap K = 1$ , then  $G \cong H \times K$ .

**Theorem 1.78.** [Rot95] If  $a \in G$ , the number of conjugates of  $a$  is equal to the index of its centralizer:

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of  $|G|$  when  $G$  is finite.

**Theorem 1.79.** [Rot95] If  $H \leq G$ , then the number  $c$  of conjugates of  $H$  in  $G$  is equal to the index of its normalizer:  $c = [G : N_G(H)]$ , and  $c$  divides  $|G|$  when  $G$  is finite. Moreover,  $aHa^{-1} = bHb^{-1}$  if and only if  $b^{-1}a \in N_G(H)$ .

**Theorem 1.80.** [Rot95] Every group  $G$  can be imbedded as a subgroup of  $S_G$ . In particular, if  $|G| = n$ , then  $G$  can be imbedded in  $S_n$ .

**Theorem 1.81.** [Rot95] If  $H \leq G$  and  $[G : H] = n$ , then there is a homomorphism  $\rho : G \rightarrow S_n$  with  $\ker \rho \leq H$ . The homomorphism  $\rho$  is called the representation of  $G$  on the cosets of  $H$ .

**Theorem 1.82.** [Rot95] If  $X$  is a  $G$ -set with action  $\alpha$ , then there is a homomorphism  $\tilde{\alpha} : S_X$  given by  $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$ . Conversely, every homomorphism  $\varphi : G \rightarrow S_X$  defines an action, namely,  $gx = \varphi(g)x$ , which makes  $X$  into a  $G$ -set.

**Theorem 1.83.** [Rot95] Every two composition series of a group  $G$  are equivalent.

We will refer to this Theorem as the **Jordan-Hölder Theorem**.

**Theorem 1.84.** [Rot95] Let  $X$  be a faithful primitive  $G$ -set of degree  $n \geq 2$ . If  $H \triangleleft G$  and if  $H \neq 1$ , then  $X$  is a transitive  $H$ -set. Also,  $n$  divides  $|H|$ .

**Theorem 1.85.** [Rot95] The wreath product of the group  $H$  by  $K$ , denoted  $H \wr K$  is the semi-direct product  $H^n : K$ , where  $n$  is the number of letters on which  $K$  acts and  $H^n$  is the direct copies of  $H$ .  $K$  permutes the  $n$  isomorphic copies of  $H$ .

$$H \wr K = \{H(y), K^* | y \in Y\}$$

$$H \wr K = \Pi H(y) : K^*$$

**Theorem 1.86.** [Led87] The number of irreducible characters of  $G$  is equal to the number of conjugacy classes of  $G$ .

**Theorem 1.87.** [GL93] Let  $\rho: G \rightarrow GL(n, \mathbb{C})$  be a representation of  $G$ , and let  $\chi$  be the character of  $\rho$ .

(1) For  $g \in G$

$$|\chi(g)| = \chi(1) \iff g\rho = \lambda I_n \text{ for some } \lambda \text{ in } \mathbb{C}$$

(2)  $\text{Ker } \rho = \{g \in G; \chi(g) = \chi(1)\}$

**Theorem 1.88.** [GL93] The sum of squares of the degrees of the distinct irreducible characters of  $G$  is equal to  $|G|$ . The degree of a character  $\chi$  is  $\chi(1)$

**Theorem 1.89.** [GL93] All irreducible characters of a finite abelian group  $G$  have degree 1.

**Theorem 1.90.** [Has17] Let  $N \trianglelefteq G$

(a) If  $\chi$  is a character of  $G \Rightarrow \chi^\bullet(Ng) = \chi(g)$

(b) If  $\chi^\bullet$  is a character of  $G/N \Rightarrow \chi(g) = \chi^\bullet(Ng)$  is a character of  $G$ .

(c)  $\chi$  is irreducible  $\iff \chi^\bullet$  is irreducible.

### 1.3 Lemmas

**Lemma 1.91.** [Rot95] Let  $X$  be a  $G$ -set, and let  $xy \in X$ .

- If  $H \leq G$ , then  $Hx \cap Hy \neq \emptyset$  implies  $Hx = Hy$ .
- If  $H \triangleleft G$ , then the subsets  $Hx$  are blocks of  $X$ .

**Lemma 1.92.** [Cur07]  $N \cap \langle t_i, t_j \rangle \leq C_N(N_{ij})$  where  $N_{ij}$  denotes the stabilizer in  $N$  of the two points  $i$  and  $j$ .

**Lemma 1.93.** [Gri15] The Factoring Lemma: Factoring the progenitor  $m^{*n} : N$  by  $(t_i, t_j)$  for  $1 \leq i < j \leq n$  gives the group  $m^n : N$ .



## Chapter 2

# Progenitors, Relations, and Modifying Finite Presentations

### 2.1 Introduction

The word progenitor is synonymous for the word parent in the English language. The progenitor we work with is the "parent" of many of our discoveries and was developed by Robert T. Curtis. Curtis' progenitor takes a finite control group  $N$  and forms a semi-direct product correspondence with a free product  $m^{*n}$  which results in an infinite group that we can factor by relations to obtain a finite homomorphic image of a group. The progenitor is where most of our research originates and this section will explain in detail how to construct a progenitor using a group found in the Magma database as our control group.

### 2.2 Constructing Progenitors using the Magma Database

Curtis chose his progenitor to be of the form  $2^{*m} : N$  because he was able to provide proof that he could produce any non-abelian simple group as a homomorphic image. This is especially useful since the more interesting groups, sporadic groups, are of this type. There are a total of 26 sporadic groups that we hope to produce. The

Mathieu 12 group denoted  $M_{12}$  and Suzuki group  $Suz(8)$  are two sporadic simple groups that presentations have been found on 12 and 5 letters in this document.

Now, we wish to find a suitable  $N$  in the database that we can use. To do this we investigate Magma using the code `N:=TransitiveGroup(k,f)`; where  $k$  is the number of letters and  $f$  is the number of the storage file. For this example we will use `N:=TransitiveGroup(12,25)`; a transitive group on 12 letters in the, 25<sup>th</sup> storage space. Now that we have an  $N$  (which we determine to be isomorphic to  $2^2 : (2^2 \times 3)$  in chapter 2:

```
> N:=TransitiveGroup(12,25);
```

We label are "N" as  $G$  so we can run a test to insure the group we have produced is in fact the same `N:=TransitiveGroup(12,25)`. Now that we have our group, we wish to obtain a presentation. To do so, we ask Magma to produce a finite presentation using `FPGroup(N)`:

```
> FPGroup(G);
Finitely presented group on 5 generators
Relations
$.1^2=Id($)
$.2^2=Id($)
$.3^3=Id($)
$.4^2=Id($)
$.5^2=Id($)
$.2^$.1=$.2
$.3^$.1=$.3
$.3^$.2=$.3
$.4^$.1=$.4
$.4^$.2=$.4
$.4^$.3=$.5
$.5^$.1=$.5
$.5^$.2=$.5
```

```

$.5^$.3=$.4*$.5
$.5^$.4=$.5
Mapping from: GrpFP to GrpPC: G

```

\$.1 represents a word in the presentation which we will label  $x$  in this document, \$.2 represents  $y$ , and so on. With this information we construct the following presentation for our  $N$ .

```

ZZ< x, y, z, w, u >:= Group < x, y, z, w, u | x^2, y^2, z^3, w^2, u^2, y^x = y, z^x = z,
z^y = z, w^x = w, w^y = w, w^z = u, u^x = u, u^y = u, u^z = w * u, u^w = u >;

```

Now, we want to verify that our presentation for our  $N$  matches the one for TransitiveGroup(12,25) so we run the following code to construct an image of our  $ZZ$  which and then ask Magma to determine if the group  $N$  is Isomorphic to the image of  $ZZ$ .

```

f, G1, k:=CosetAction(ZZ, sub<ZZ | Id(ZZ)>);
> IsIsomorphic(N, G1);
true Mapping from: GrpPerm: N to GrpPerm: G1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G1

```

We still must construct a free product among our  $t$ 's. We do know that this presentation will have free product  $2^{*12}$  since we are working in TransitiveGroup(12) and the  $t$ 's are order 2. To produce the infinite progenitor, our method involves stabilizing one of our  $t$ 's using our permutations in  $N$  and converting those permutations to words using the generators of  $N$ . To begin, we want the most efficient presentation of  $N$ . Using the following code we obtain an efficient presentation:

```

> SL:=Subgroups(G1);
> T := X\subgroup: X in SL;
> #T;
39
> TrivCore := H:H in T | #Core(G1,H) eq 1;
> mdeg := Min(Index(G1,H):H in TrivCore);

```

```

> Good := H: H in TrivCore | Index(G1,H) eq mdeg;
> #Good;
5
> H := Rep(Good);
> #H;
4
> f,G1,K := CosetAction(G1,H);
> s:=IsIsomorphic(G1,N);
> G1;
Permutation group G1 acting on a set of cardinality 12
Order = 48 = 2^4 * 3
(1, 2) (3, 5) (4, 6) (7, 9) (8, 10) (11, 12)
(1, 3) (2, 5) (4, 7) (6, 9) (8, 11) (10, 12)
(1, 4, 8) (2, 6, 10) (3, 7, 11) (5, 9, 12)
(1, 2) (3, 5) (4, 7) (6, 9) (8, 12) (10, 11)
(1, 5) (2, 3) (4, 6) (7, 9) (8, 11) (10, 12)

```

We will now label and construct  $N$  using these permutations. To label a permutation in Magma we must store that permutation somewhere. We start by using **S:=Sym(12)**; so we can store the permutations in  $S$ . Our permutation for  $x$  will be input as **X:=S!(1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12)**; and the other generators will be labeled in similar fashion. We then declare  $N$  using the command:

```
N:=sub<S | X, Y, Z, W, U>;
```

Next we will stabilize the 12<sup>th</sup>  $t$  by pulling only the permutations in  $N$  which don't contain 12 as follows:

```

> N0:=Stabiliser(N,12);
> N0;
Permutation group N0 acting on a set of cardinality 12
Order = 4 = 2^2
(1, 5) (2, 3) (4, 7) (6, 9)
(1, 2) (3, 5) (4, 9) (6, 7)

```

From here we must determine what these elements in the stabilizer are in terms of  $x,y,z,w$ , and  $u$ . To do this we use:

```

> Sch:=SchreierSystem(ZZ, sub<ZZ|Id(ZZ)>);
> ArrayP:=[Id(N): i in [1..48]];
> for i in [2..48] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=X; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=Y; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=Z; end if;
for|for> if Eltseq(Sch[i])[j] eq -3 then P[j]:=Z^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 4 then P[j]:=W; end if;
for|for> if Eltseq(Sch[i])[j] eq 5 then P[j]:=U; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;

```

The code above must be modified to fit the permutations in  $N$ . Since  $x, y, w$  and  $u$  are order 2, they do not have an inverse so this is accounted for in the SchreierSystem. This command when paired with the following for loops using the stabilizing permutations from  $N_0$  will give words which can be introduced to the presentations of  $N$  and gives us the infinite progenitor we seek to produce. After the SchreierSystem has been accepted we use the following commands:

```

> for i in [1..48] do if ArrayP[i] eq N!(1,5)(2,3)(4,7)(6,9)
for|if> then print Sch[i]; end if; end for;
x * w * u
>for i in [1..48] do if ArrayP[i] eq N!(1,2)(3,5)(4,9)(6,7)
for|if> then print Sch[i]; end if; end for;
y * u

```

To complete the process and make our progenitor infinite we introduce  $t_{12}$  into our presentation and make  $t_{12}$  commute with the two words we just created leaving us with the new infinite progenitor,  $2^{*12} : 2^2 : (2^2 \times 3)$  given by the presentation:

```

P<x, y, z, w, u, t>:=Group<x, y, z, w, u, t|x^2, y^2, z^3, w^2, u^2, y^x=y,
z^x=z, z^y=z, w^x=w, w^y=w, w^z=u, u^x=u, u^y=u, u^z=w*u, u^w=u, t^2,
(t, x*w*u), (t, y*u)>

```

which we can verify is infinite by typing `#P` in Magma observing a return value 0. We have constructed an infinite progenitor which can be factored by relations to produce groups of finite order.

### 2.2.1 First Order Relations and Producing Finite Representations

The process of determining first order relations is one way of producing relations, which can be used to factor our infinite progenitor to produce finite images of other groups. Whenever a symmetric presentation of a progenitor is given we should produce the first order relations of the control group that was used.

To begin this process, we will require words that can be inserted into our presentation that represent the conjugacy classes of  $N$ . Using the same SchrierSystem from the previous section and declaring `CC:=Classes(N)` we will run a loop that will calculate the conjugacy classes effortlessly as follows:

```

for i in [2..16] do i, Orbits(Centraliser(N,CC[i][3]));
end for;
for j in [2..16] do
for i in [1..48] do if ArrayP[i] eq CC[j][3] then Sch[i];
end if; end for;
j;
end for;
x
2
y
3
x * y
4
u
5
x * w
6
x * y * w
7
y * u
8

```

```

z
9
z^-1
10
x * z
11
x * z^-1
12
y * z
13
y * z^-1
14
x * y * z * u
15
x * y * u * z^-1
16\\

```

Now using this list we multiply each word by  $t$  and taking the relation  $(x*t)$  for example, we can raise this to some power,  $a$  (in this case) and produce our final factorable relation. We produce the following list of relations:  $(x * t)^a, (y * t)^b, (z * t)^c, (z^{-1} * t)^d, (u * t)^e, (x * y * t)^f, (x * w * t)^g, (x * z * t)^h, (x * z^{-1} * t)^i, (y * z * t)^j, (y * z^{-1} * t)^k, (x * y * z * u * t)^l, (x * y * u * z^{-1} * t)^m,$  and  $(x * y * w * t)^n$ . These form our list of first order relations and can be run in the background on Magma which means Magma will substitute each variable for a number (determined by the user) and check if the relation produces a group of finite order. If it does Magma will store the powers for each relation used and the order of the group they produced. If the relations used produce an infinite group or one simply too large for Magma to calculate, Magma can be set to discard this information and will simply leave it out of the list it produces. We call this process, running progenitors in the background.

### 2.2.2 The Famous Lemma Performed on $2^{*10} : M_{10}$

Another way of producing relations is through a lemma known as the famous lemma. This method involves calculating the centralizer of the two point stabilizer and creating the relations with regard to this information. Curtis is responsible for constructing relations in this manner. Let  $N$  be given with the following permutation

representation:

```
S:=Sym(10);
X:=S!(1, 2)(4, 5)(6, 7)(8, 9);
Y:=S!(1, 3, 4, 6)(7, 8, 10, 9);
N:=sub<S|X,Y>;
```

We produce the point stabilizer  $N^{1,2} = \langle (3, 6, 10, 8)(4, 7, 5, 9), (3, 9, 10, 7)(4, 6, 5, 8) \rangle$  which contains 8 elements. We then calculate the centralizer of this group. Using the commands below, we determine the centralizer to be:

```
> N1:=Stabiliser(N, [1, 2]);
> Centralizer(N, N1);
Permutation group acting on a set of cardinality 10
Order = 2
(3, 10)(4, 5)(6, 8)(7, 9)
```

Now we must convert this to words using the SchrierSystem used to construct the progenitor. Magma determines the word to be  $y * x * y * x * y^{-1} * x * y^{-1} * x * y * x * y^{-1} * x * y^2$ .

We notice that the permutation  $(3, 10)(4, 5)(6, 8)(7, 9)$  fixes 1 and 2 by not including them rather than through a permutation containing  $(1, 2)$ . This is important as this indicates how to construct the relation. There are two ways we construct relations defined as follows:

$$Relation = \begin{cases} (xt_i)^m = 1 & \text{where } m \text{ is odd and } x \text{ sends } 1 \text{ to } 2 \\ (t_i t_j)^m = x & \text{where } m \text{ is even and } x \text{ fixes } 1 \text{ and } 2 \end{cases}$$

Now, since our permutation  $(3, 10)(4, 5)(6, 8)(7, 9)$  fixes 1 and 2 we construct the relation  $(t_1, t_2)^m = (3, 10)(4, 5)(6, 8)(7, 9)$  which reads as  $(t_1, t_1^x)^m = y * x * y * x * y^{-1} * x * y^{-1} * x * y * x * y^{-1} * x * y^2$  and can now be used in our progenitor to produce a finite presentation of an infinite progenitor. Unfortunately, the progenitor for  $2^{*10} : M_{10}$  did not produce any useful results with first order or famous lemma relations.



### 2.2.3 Changing the Control Group to $2^3 : S_3$

When finding finite groups using Magma, the control group may change to a proper subgroup of  $N$ . While this is not always the case, if this does happen it indicates that our progenitor has changed. This section aims to explain how to determine the new progenitor and control group being used. Once this process has been carried out, double coset enumeration will be possible to conduct.

```
> G<x, y, z, w, t>:=Group<x, y, z, w, t | x^2, y^3, z^3=w^2, w^3, y^x=y,
> z^x=z^2*w, z^y=z*w, w^x=w^2, w^y=w, w^z=w, t^2, (t, x*z^-1),
> (t, w*y^-1), (w^-1*t)^4, (z*y*t)^7, (x*t)^8>;
>
> f, G1, k:=CosetAction(G, sub<G | x, y, z, w>);
> CompositionFactors(G1);
  G
  |  Cyclic(2)
  *
  |  A(1, 7)                = L(2, 7)
  1
>#sub<G | x, y, z, w>;
6
```

The reader should note the original  $N$  is  $\cong 2^2 : (2^2 \times 3)$  which is order 48. Magma determined this  $\text{PGL}(2,7)$  on a subgroup of order 6 of the control group  $N$ . Before we can perform double coset enumeration on this  $N$  we must shrink our control group down to order 6 and eliminate unnecessary information. We proceed by checking the order of each individual element of  $\text{sub}\langle G | x, y, z, w \rangle$  under their image as follows:

```
> Order(f(x));
2
> Order(f(y));
1
> Order(f(z));
3
> Order(f(w));
1
```

This indicates that the elements  $y$  and  $w$  are no longer required in the presentation for  $N$  in this finite representation. So we begin eliminating  $w$  and  $y$  from the presentation of  $G$ . Any relation between  $w$  and  $y$  will clearly be deleted from the progenitor. We

start by deleting  $y^3$  and  $w^3$  from the progenitor and then notice  $w^x = w^2, w^y = w, w^z = w, (t, w * y^{-1})$ , and  $(w^{-1} * t)^4$ . These can all be deleted since  $w$  is no longer included in the presentation. The final relations we must consider are those such as  $z^3 = w^2$  and  $z^x = z^2 * w$ . In both cases deleting just the portion with  $w$  suffices and leaves  $z^3$  and  $z^x = z^2$  as relations. Once this has been completed for the entire presentation the following presentation is achieved.

```
> G<x, z, t>:=Group<x, z, t|x^2, z^3, z^x=z^2, t^2, (t, x*z^-1), (z*t)^7,
> (x*t)^8>;
> f, G1, k:=CosetAction(G, sub<G|x, z>);
> CompositionFactors(G1);
```

```

G
|  Cyclic(2)
*
|  A(1, 7)                = L(2, 7)
1
```

To obtain the permutations of  $x$  and  $z$  we must consider just the portion of this progenitor which does not include  $t$ . This leaves us with the following presentation:  $NN< X, Z >:=Group< X, Z|X^2, Z^3, Z^X = Z^2 >$ . From this point we construct a subgroup of  $NN$  using the stabilizer of our  $t$  (in our new progenitor) as follows:  $H :=sub< NN|X * Z^{-1} >$ . We find a permutation representation of the group as follows:

```
> NN<X, Z>:=Group<X, Z|X^2, Z^3, Z^X=Z^2>;
> H:=sub<NN|X*Z^-1>;
> ff, NN1, k1:=CosetAction(NN, H);
> ff(X);
(1, 2)
> ff(Z);
(1, 2, 3)
```

We then verify this was done correctly by using the following code:

```
> Stabiliser(NN1, 1) eq sub<NN1|ff(X*Z^-1)>;
true
```

Our new control group  $N$  given by the following permutations:

```
S:=Sym(3);
X:=S!(1, 2);
```

```
Y:=S!(1, 2, 3);
N:=sub<S|X,Y>;
```

is isomorphic to  $S_3$  leaving us with the progenitor factored by relations

$$\frac{2^*3:S_3}{(1,2,3)t_1t_3t_2t_1t_3t_2t_1,(e)t_2t_1t_2t_1t_2t_1t_2t_1} \cong PGL_2(7).$$

### 2.2.4 Factoring the Center $PGL_2(11)$ Factored by 3

In the case being presented, we are factoring by the center to decrease the amount of double cosets in our Cayley diagram. This condenses the Cayley diagram of  $G/Z(G)$  and shortens the amount of time required to complete a double coset enumeration. To begin the process, we consider the presentation:

```
> G<a,b,e,t>:=Group<a,b,e,t|a^2,b^2,e^3,b^a=b,e^a=e^2,e^b=e,
> t^2,(t,a*e),(b*t)^5,(e^-1*t)^3>;
> f,G1,k:=CosetAction(G,sub<G|a,b,e>);
> CompositionFactors(G1);
  G
  |  Cyclic(2)
  *
  |  A(1, 11)                = L(2, 11)
  *
  |  Cyclic(3)
  1
```

Magma will list anything that can be factored out below the linear group. This happens because typically we factor by a largest abelian group and determine direct products in this manner. In this case 3 can be factored out from the  $PGL_2(11)$ . The reader should note that we have verified using Magma that this is  $PGL_2(11)$  and not  $PSL_2(11) \times 2$ . How this was determined will be presented in the Chapter 3 on extension types.

To begin the process we will need to determine the generator for the center. Since we know the center is a normal subgroup of  $G$ , we will investigate the Normal Lattice to find an element of order 3 as follows:

```
NL:=NormalLattice(G1);
> NL;
Normal subgroup lattice
```

```
-----
[5] Order 3960 Length 1 Maximal Subgroups: 4
[4] Order 1980 Length 1 Maximal Subgroups: 2 3
[3] Order 660 Length 1 Maximal Subgroups: 1
[2] Order 3 Length 1 Maximal Subgroups: 1
[1] Order 1 Length 1 Maximal Subgroups:
```

The subgroup of order 3 turns out to be labeled NL[2] and we investigate the generators NL[2].1 and NL[2].2 and discover that  $\text{Order}(\text{NL}[2].2) = 3$  which is the element we seek. From here we store the element as  $F := \text{NL}[2].2$ ; in Magma. Since we cannot introduce permutations into presentations of groups we will need to convert this lengthy permutation found in the image of G into a word so that we can introduce it into the SchrierSystem to produce a word. This group was found on the progenitor  $2^{*12} : 2^2 : (2^2 \times 3)$  that was discussed earlier; because of this we will use the same SchrierSystem from earlier. Comparing our work from earlier, we will need to modify where the SchrierSystem obtains elements. Since our presentation was labeled NN and our N was simply N in the original case, our presentation is currently labeled G and our permutation is found in G1 so we will store  $N := G1$  and  $NN := G$ . We also must account for the order of G1 (3,960) in the SchrierSystem as well as the image of our elements. This will be reflected in the proceeding code:

```
> N:=G1;
> NN:=G;
> F:=NL[2].2;
>
> Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);
> ArrayP:=Id(N): i in [1..3960];
> for i in [2..3960] do
for> P:=Id(N): l in [1..#Sch[i]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(a); end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(b); end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(e); end if;
for|for> if Eltseq(Sch[i])[j] eq -3 then P[j]:=f(e)^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(t); end if;
for|for>
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
```

```

for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
>
> for i in [1..3960] do if ArrayP[i] eq F then print Sch[i];
end if; end for;
a*b*t*a*b*t*a*b*t*b*t*a*t*b*t*b*e^-1*t*b*e*t*b*e*t

```

The word generated from the SchrierSystem will then be introduced into the presentation of  $G$  giving us the final result:

```

> G<a,b,e,t>:=Group<a,b,e,t|a^2,b^2,e^3,b^a=b,
> e^a=e^2,e^b=e,t^2,(t,a*e),(b*t)^5,(e^-1*t)^3,
a*b*t*a*b*t*a*b*t*b*t*a*t*b*t*b*e^-1*t*b*e*t*b*e*t>;
> f,G1,k:=CosetAction(G,sub<G|a,b,e>);
> CompositionFactors(G1);
  G
  |  Cyclic(2)
  *
  |  A(1, 11)                = L(2, 11)
  1

```

This  $PGL_2(11)$  will be investigated more thoroughly in Chapter 4 if it turns out to be an original group found on the progenitor  $2^{*6} : D_{12}$ .

## 2.3 Wreath Product

Constructing a wreath product is another way we can create a control group for a progenitor. The process involves taking two known group, taking their generators and presentations and then finding the direct and semidirect product actions among the elements. One should note that the generators should be disjoint to complete this process and we can complete this process without requiring much assistance from Magma.

### 2.3.1 Constructing the Wreath Product of $2^{*8} : \mathbb{Z}_2 \wr D_4$

Let  $H = \mathbb{Z}_2 = \langle e, (1, 2) \rangle$  and  $K = D_4 = \langle (3, 4, 5, 6), (3, 5) \rangle$  with  $X = \{1, 2\}$  and  $Y = \{3, 4, 5, 6\}$  and let  $Z$  be defined  $Z = \{(x, y) | x \in X, y \in Y\}$ . We determine  $Z$

to be  $Z = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6)\}$ .

By the following definition we have,  $\gamma \in H$  and  $y$  is a fixed element of  $Y$ . Defined as  $\gamma(y) : \{(x, y) \rightarrow ((x)\gamma, y), (x, y_1) \rightarrow (x, y_1) \text{ if } y_1 \neq y\}$ . The following tables show these mappings. We label each element in  $Z$  from 1 to 8 and proceed as follows:

Table 2.1: Mapping of  $\gamma(3)$

1	(1, 3)	→	(2, 3)	5
2	(1, 4)	→	(1, 4)	2
3	(1, 5)	→	(1, 5)	3
4	(1, 6)	→	(1, 6)	4
5	(2, 3)	→	(1, 3)	1
6	(2, 4)	→	(2, 4)	6
7	(2, 5)	→	(2, 5)	7
8	(2, 6)	→	(2, 6)	8

This indicates that  $\gamma(3) = (1, 5)$ . We repeat the process for each remaining element in  $Y$  and obtain  $\gamma(4) = (2, 6)$ ,  $\gamma(5) = (3, 7)$  and  $\gamma(6) = (4, 8)$ . Now by definition let  $k \in K$  and define  $k^* : \{(x, y) \rightarrow (x, (y)k)\}$ . We construct the following tables for  $k_1^*$  and  $k_2^*$  using our permutations from  $D_4$ .

Table 2.2: Mapping of  $k_1^* = (3, 4, 5, 6)$

1	(1, 3)	→	(1, 4)	2
2	(1, 4)	→	(1, 5)	3
3	(1, 5)	→	(1, 6)	4
4	(1, 6)	→	(1, 3)	1
5	(2, 3)	→	(2, 4)	6
6	(2, 4)	→	(2, 5)	7
7	(2, 5)	→	(2, 6)	8
8	(2, 6)	→	(2, 3)	5

Table 2.3: Mapping of  $k_2^* = (3, 5)$ 

1	(1, 3)	→	(1, 5)	3
2	(1, 4)	→	(1, 4)	2
3	(1, 5)	→	(1, 3)	1
4	(1, 6)	→	(1, 6)	4
5	(2, 3)	→	(2, 5)	7
6	(2, 4)	→	(2, 4)	6
7	(2, 5)	→	(2, 3)	5
8	(2, 6)	→	(2, 6)	8

These tables indicate that  $k_1^* = (1, 2, 3, 4)(5, 6, 7, 8)$  and  $k_2^* = (1, 3)(5, 7)$ . Now, we know that  $Z_2^4 = \langle a|a^2 \rangle \times \langle b|b^2 \rangle \times \langle c|c^2 \rangle \times \langle d|d^2 \rangle$  where  $a = (1, 5)$ ,  $b = (2, 6)$ ,  $c = (3, 7)$ , and  $d = (4, 8)$  and the presentation for  $D_4 = \langle e, f|e^4, f^2, (ef)^2 \rangle$  where  $e = (1, 2, 3, 4)(5, 6, 7, 8)$  and  $f = (1, 3)(5, 7)$ . We know the elements of  $Z_2^4$  will commute in our presentation since they result in a direct product. We must now calculate the action of  $D_4$  on  $Z_2$  which gives us the following words to inject in our presentation:

$$\begin{aligned} a^e &= b, b^e = c, c^e = d, d^e = a, \\ a^f &= c, b^f = b, c^f = a, d^f = d \end{aligned}$$

Combining all the information we have gathered we construct the final presentation:

$$\begin{aligned} Z_2 \wr D_4 = \langle a, b, c, d, e, f|a^2, b^2, c^2, d^2, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), e^4, f^2, (ef)^2, \\ a^e = b, b^e = c, c^e = d, d^e = a, a^f = c, b^f = b, c^f = a, d^f = d \rangle \end{aligned}$$

which we verify is isomorphic to `TransitiveGroup(8,35)`, a group we know is isomorphic to  $Z_2^4 : D_4$  using the following Magma code:

```
> P<a, b, c, d, e, f>:=Group<a, b, c, d, e, f|a^2, b^2, c^2, d^2, (a, b),
> (a, c), (a, d), (b, c), (b, d), (c, d), e^4, f^2, (e*f)^2, a^e=b, b^e=c,
> c^e=d, d^e=a, a^f=c, b^f=b, c^f=a, d^f=d>;
> f3, P1, k3:=CosetAction(P, sub<P|Id(P)>);
> N:=TransitiveGroup(8, 35);
> IsIsomorphic(N, P1);
true
```

Thus far we have constructed a presentation of  $N$ , we must now introduce the free product,  $2^{*8}$  to finish the infinite progenitor. We proceed by calling  $t \sim t_8$  and determine the stabilizer of  $t_8$  using the following Magma code:

```

> S:=Sym(8);
> A:=S!(1, 5);
> B:=S!(2, 6);
> C:=S!(3, 7);
> D:=S!(4, 8);
> E:=S!(1, 2, 3, 4)(5, 6, 7, 8);
> F:=S!(1, 3)(5, 7);
> N:=sub<S|A,B,C,D,E,F>;
> N0:=Stabiliser(N,8);
> N0;
Permutation group N0 acting on a set of cardinality 8
Order = 16 = 2^4
(1, 5)
(2, 6)
(3, 7)
(1, 3)(5, 7)

```

The list produced at the bottom tells us  $t$  commutes with  $a$ ,  $b$ ,  $c$ , and  $f$  which is fortunate since we do not require the SchreierSystem in this example. Adding  $t^2, (t, a), \dots, (t, f)$  to our presentation of  $N$  creates our infinite progenitor  $2^{*8} : \mathbb{Z}_2 \wr D_4$  with the presentation:

```

P<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|t^2,a^2,b^2,c^2,d^2,
(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),e^4,f^2,(e*f)^2,a^e=b,
b^e=c,c^e=d,d^e=a,a^f=c,b^f=b,c^f=a,d^f=d,(t,a),(t,b),
(t,c),(t,f)>;

```

which can be verified to be infinite using Magma. Naturally, we would produce first order relations to factor our progenitor but for now we will investigate the irreducible monomial representations of our  $N$ . This group appears to have a more efficient presentation through Magma's WreathProduct() function, the wreath product function will be used later.

## 2.4 Symmetric Presentations of Groups

Once a progenitor is constructed and relations are created for that progenitor, we can begin to investigate finite representations. By inserting a relation into the presentation of the progenitor finite images emerge. Changing the power of the relation causes the relation to act differently upon the progenitor producing varying results.



### 2.4.1 $2^{*12} : 2^2 : (2^2 \times 3)$

The following table lists the findings found on the progenitor  $2^{*12} : 2^2 : (2^2 \times 3)$  with presentation:

$G \langle x, y, z, w, u, t \rangle := \text{Group} \langle x, y, z, w, u, t | x^2, y^2, z^3, w^2, u^2, y^x = y, z^x = z, z^y = z, w^x = w, w^y = w, w^z = u, u^x = u, u^y = u, u^z = w * u, u^w = u, t^2, (t, x * w * u), (t, y * u) \rangle$  factored by relations:  $(x * t)^a, (y * t)^b, (z * t)^c, (z^{-1} * t)^d, (u * t)^e, (x * y * t)^f, (x * w * t)^g, (x * z * t)^h, (x * z^{-1} * t)^i, (y * z * t)^j, (y * z^{-1} * t)^k, (x * y * z * u * t)^l, (x * y * u * z^{-1} * t)^m,$  and  $(x * y * w * t)^n$ .

Table 2.4: Finite Presentations of  $2^{*12} : 2^2 : (2^2 \times 3)$

a	b	c	d	e	f	g	h	i	j	k	l	m	n	Order of G	Name of G
3	0	5	0	0	0	0	0	0	0	0	0	0	0	190,080	$M_{12} : 2$
2	0	0	0	0	0	3	0	0	0	0	0	8	0	5,040	$S_7$
0	0	0	0	0	0	0	0	0	0	0	0	3	0	768	$2^6 : A_4$
0	0	0	0	0	0	0	5	0	0	5	0	5	0	15,360	$2^8 : A_5$
2	0	0	0	0	0	3	0	0	0	0	0	8	0	117,600	$PSL_2(49) : 2$

Note: By replacing the corresponding variable power of each relation the finite representation of each group can be achieved.

## 2.5 Other Symmetric Presentations Found on 12 Letters

This section is dedicated to finite presentations of groups found on transitive group 12. These progenitors all began of the form  $2^{*12} : N$  for some N in the Magma Database. The section will include a presentation of the progenitor with all relations included. The table will be similar to the table found in previous section.

### 2.5.1 $2^{*12} : 2^2 \times 3 : 2$

The progenitor  $2^{*12} : 2^2 \times 3 : 2$  factored by first order relations and the relations  $(y * z * t^x)^o, (z * t^x)^p$  given by the presentation:

$G \langle x, y, z, w, t \rangle := \text{Group} \langle x, y, z, w, t | x^2, y^2, z^3, w^2, y^x = y * w, z^x = z, z^y = z, w^x = w, w^y = w, w^z = w, t^2, (t, y * w), (w * t)^a, (x * t)^b, (y * t)^c, (z * t)^d, ((z^{-1}) * t)^e, (x * y * w * t)^f \rangle$

$t)^f, (z * w * t)^g, ((w * z^{-1}) * t)^h, (x * z * t)^i, ((x * z^{-1}) * t)^j, (y * z * t)^k, ((y * z^{-1}) * t)^l, (x * y * z * t)^m, ((x * y * z^{-1}) * t)^n, (y * z * t^x)^o, (z * t^x)^p >$ ;

Table 2.5: Finite Presentations Found on  $2^{*12} : 2^2 \times 3 : 2$

b	d	e	f	g	h	i	j	k	m	o	p	#G	Isomorphism
0	3	0	5	3	0	0	0	0	0	0	0	3,960	$3 \times PSL_2(11)^*$
0	0	4	0	0	3	9	0	0	0	0	0	6,840	$PGL_2(19)^*$
0	0	4	9	0	3	0	0	0	0	0	0	4,896	$PGL_2(17)^*$
0	0	4	0	0	3	8	0	0	0	0	0	1,344	$4 : \bullet PGL_2(7)^*$
0	0	0	0	3	0	0	0	0	7	3	0	1,092	$PSL_2(13)^*$
0	0	0	3	0	0	0	10	6	0	0	5	15,600	$PSL_2(25) : 2$
6	0	0	0	0	4	0	0	0	6	3	0	103,680	$2 \bullet U_4(2) : 2$

Note: An asterisk right of the isomorphism type indicates the control group changed.  
The relations with power a, c, l and n were unused and excluded.

### 2.5.2 $2^{*12} : D_3 \times D_4$

The progenitor  $2^{*12} : D_3 \times D_4$  factored by first order relations produces the following presentation:

$G \langle x, y, z, w, u, t \rangle := Group \langle x, y, z, w, u, t | x^2, y^2, z^2, w^2, u^3, y^x = y, z^x = z, z^y = z * w, w^x = w, w^y = w, w^z = w, u^x = u^2, u^y = u, u^z = u, u^w = u, t^2, (t, z * w), (t, x * w * u), (w * t)^a, (y * t)^b, (z * t)^c, (x * t)^d, (u * t)^e, (x * w * t)^f, (x * y * t)^g, (x * z * t)^h, (z * y * t)^i, (y * u * t)^j, (z * u * t)^k, (x * z * y * t)^l, (w * u^{-1} * t)^m, (z * y * u * t)^n >$ ;

Table 2.6: Finite Presentations Found on  $2^{*12} : D_3 \times D_4$

b	d	f	g	i	k	l	m	n	#G	Isomorphism
0	0	4	0	5	0	5	5	0	720	$PGL_2(9)^*$
0	0	0	0	0	3	5	0	0	14,400	$S_5^2$
0	3	6	0	0	0	0	0	6	241,920	$S_3 \times PGL_2(7) \times S_5$
5	0	0	0	0	0	6	3	0	120	$S_5$
5	0	0	8	0	0	0	3	0	967,680	$(2^2) \bullet PGL_3(4) : 2^2$
0	0	0	0	0	0	0	0	0	3,960	$3 : PGL_2(11)^*$
3	6	4	0	0	0	9	5	0	51,840	$U_4(2) : 2$

Note: Asterisk denotes change in original control group.  
All powers missing from table are equal to zero.

### 2.5.3 $2^{*12} : 2 \times 3 \times S_3$

The progenitor  $2^{*12} : (2 \times 3) \bullet S_3$  factored by relations:  $(y * z * w * t^x)^a$ ,  $(x * z^{-1} * t)^b$ ,  $(x * y * z * t)^c$ , and  $(x * y * z^{-1} * t)^d$  gives the following presentation:  
 $G \langle x, y, z, w, t \rangle := \text{Group} \langle x, y, z, w, t \mid x^2, y^2, z^3, w^3, t^2, y^x = y, z^x = z, z^y = z, w^x = w^2, w^y = w, w^z = w, (t, z^{-1} * w^{-1}), (y * z * w * t^x)^a, (x * z^{-1} * t)^b, (x * y * z * t)^c, (x * y * z^{-1} * t)^d \rangle$ ;

Table 2.7: Finite Presentations Found on  $2^{*12} : 2 \times 3 \times S_3$

a	b	c	d	#G	Isomorphism
0	3	3	0	24	$S_4$
3	5	0	5	60	$A_5^*$
3	5	0	10	9,720	$3^4 : (A_5 \times 2)$
9	10	3	12	20,520	$S_3 \times PSL(2, 19)^*$
7	15	3	6	12,180	$PSL_2(29)^*$
8	3	0	10	4,320	$6 : \bullet PGL_2(9)^*$
8	3	11	0	12,144	$PGL_2(23)^*$
7	13	0	3	1,092	$PSL_2(13)^*$
7	12	3	6	2,184	$PGL_2(13)^*$
7	9	3	0	504	$PSL_2(8)^*$
7	8	15	6	336	$PGL_2(7)^*$
5	5	0	5	660	$PSL_2(11)^*$

Note: Asterisk denotes change in original control group.

### 2.5.4 $2^{*12} : 3^2 \times 2 : 4$

The progenitor  $2^{*12} : 3^2 \times 2 : 4$  factored by relations:  $(x^{-1} * t)^a$ ,  $(y * x^{-1} * t)^b$ ,  $(u * w^{-1} * t)^c$ , and  $(y * u^{-1} * t)^d$  and given by the following presentation:  
 $G \langle x, y, z, w, u, t \rangle := \text{Group} \langle x, y, z, w, u, t \mid x^2 = z, y^2, z^2, w^3, u^3, y^x = y, z^x = z, z^y = z, w^x = w * u^2, w^y = w, w^z = w^2, u^x = w^2 * u^2, u^y = u, u^z = u^2, u^w = u, t^2, (t, u * w^{-1}), (t, y * z * w), (x^{-1} * t)^a, (y * x^{-1} * t)^b, (u * w^{-1} * t)^c, (y * u^{-1} * t)^d \rangle$ ;

Table 2.8: Finite Presentations Found on  $3^2 \times 2 : 4$ 

a	b	c	d	#G	Isomorphism
7	7	0	3	423,360	$A_7 \times PSL_2(7)$
0	9	2	3	2,448	$PSL_2(17)$
0	7	0	3	846,720	$2 \times A_7 \times PSL_2(7)$
0	5	0	5	360	$A_6$

## 2.6 Symmetric Presentations Found on Various Letters

The following charts are finite representations of groups which were found on various numbers of letters.

### 2.6.1 $2^{*5} : 5 : 4$

The progenitor  $2^{*5} : 5 : 4$  given by the following presentation factored by the first order relations:

$$G \langle x, y, z, t \rangle := \text{Group} \langle x, y, z, t \mid x^2 = y, y^2, z^5, y^x = y, z^x = z^2, z^y = z^4, t^2, (t, z * x), (y * z * t)^a, (z * t)^b, (z * x^{-1} * t)^c, (x^{-1} * z * t)^d \rangle$$

Table 2.9: Finite Presentations Found on  $2^{*5} : 5 : 4$ 

a	b	c	d	#G	Isomorphism
8	0	0	5	720	$S_6$
8	0	0	6	28,800	$2^{\bullet}A_5^2 : 2$
6	7	0	7	58,240	$Suz(8) \times 2$
6	8	0	7	80,640	$2^{\bullet}PSL_3(4) : 2$

The next presentation was discovered when we produced a progenitor using the maximal subgroup  $PSL_2(11)$  of  $M_{12}$ . The produced progenitor was  $2^{*11} : PSL_2(11)$  and when it was discovered this was a previously investigated group, work was discontinued.

```
> G<x,y,t>:=Group<x,y,t|x^5,y^3,t^2,
> y*x^2*y^-1*x^2*y*x^-1*y^-1*x^-1,(x*y*x*y*x)^2,
> (t,x*y*x^-1*y*x^2),(t,y*x^-1*y^-1*x*y*x^-1*y),
> (t,y*x^-2*y*x^-1*y*x),(((x*y^-1)^2)*t)^6,
> (((x*y)^2)*t)^6>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
```

```

G
| J1
*
| Cyclic(2)
1

```

## 2.7 The Factoring Lemma

The factoring lemma is a very simple process for ensuring that a progenitor that has been constructed is the one we set out to create. This lemma states that if the progenitor is factored by the relations which make all the  $t_i$ 's commute, that the progenitor  $m^{*n} : N$  gives the group  $m^n : N$ . The order of  $m^n : N = m^n * N$  which is what is used to verify our progenitor has been correctly produced. This was proved by Dustin Grindstaff.

### 2.7.1 Grindstaff Lemma on $3^{*4} : \mathbb{Z}_2 \wr D_4$

$N \cong \mathbb{Z}_2 \wr D_4$ , which created the progenitor  $2^{*8} : \mathbb{Z}_2 \wr D_4$  in the previous section will later be used to construct a monomial progenitor. That progenitor will be determined to be  $3^{*4} : \mathbb{Z}_2 \wr D_4$ . For now we will use Grindstaff's Lemma [Gri15] to verify our presentation for  $3^{*4} : \mathbb{Z}_2 \wr D_4$  is correct. Before we start using the Grindstaff lemma we compute the order of  $m^n : N$  to be  $3^4 * 128 = 10,368$ . We have the progenitor given as the following from earlier:

```

P<x,y,z,t>:=Group<x,y,z,t|x^4,y^2,z^2,t^3,(x^-1 * y)^2,
(x^-1*z*y)^2,(x*z*x^-1*z)^2,x^-1*z*x^-2*z*x^2*z*x^-1,
(t,x*z*y*z*x*z*x*z),(t,x^2*z*x^-1*z*x^-1),(t,x^2*z*x*z*x*z),
t^z=t^2>

```

When producing this progenitor the stabilized  $t$  was  $t_1$ . There are three  $t$ 's we need to make  $t_1$  commute with. To do this we simply use the permutations that were used to construct  $N$  which are:

```

> W:=WreathProduct(CyclicGroup(2),DihedralGroup(4));
> Generators(W);
> S:=Sym(8);
> x:=S!(1,4,3,2)(5,8,7,6);
> y:=S!(1,6)(2,5)(3,8)(4,7);
> z:=S!(1,5)(3,7)(4,8);

```

We are looking for a permutation or word that sends  $t_1$  to  $t_2$  which is easily identified as  $t_1^{x^{-1}}$ . We then also easily obtain  $t_3 = t_1^{x^2}$  and  $t_4 = t_1^x$ . To make  $t_1 \sim t$  commute with these elements we create the following relations:  $(t, t^{x^{-1}})$ ,  $(t, t^{x^2})$ , and  $(t, t^x)$ . We introduce these to the progenitor from earlier and determine the order of P to be:

```
> P<x, y, z, t>:=Group<x, y, z, t|x^4, y^2, z^2, t^3, (x^-1 * y)^2,
> (x^-1*z*y)^2, (x*z*x^-1*z)^2, x^-1*z*x^-2*z*x^2*z*x^2*z*x^-1,
> (t, x*z*y*z*x*z*x*z), (t, x^2*z*x^-1*z*x^-1), (t, x^2*z*x*z*x*z),
> t^z=t^2, (t, t^(x^-1)), (t, t^x^2), (t, t^x)>;
> #P;
10368
```

This of course verifies that our original progenitor was produced correctly and we should factor it by relations to find finite presentations of groups.

### 2.7.2 Other Progenitors Verified by the Factoring Lemma

The progenitor  $13^*2 :_m (2^2 \times 3) : 2$  factored by  $(t_1, t_2)$  given by the presentation:

```
> G<x, y, z, t>:=Group<x, y, z, t|x^4, y^2, z^3, t^13, (x^-1*y)^2,
> (x, z), y*z^-1*y*z, t^x^-1*t^-12, t^z^-1*t^-3, (x^2 * z^-1*t)^12,
> (t, t^y)>;
> #G;
4056
```

When we consider  $13^2 * 12 * 2$  we obtained  $169 * 24 = 4,056$ . This verifies our progenitor was constructed correctly and can be run in the background to obtain finite presentations.

The progenitor  $2^*8 : (2^3 : 2)$  factored by  $(t_1, t_2), (t_1, t_3), \dots, (t_1, t_8)$  given by the presentation:

```
> G<x, y, z, w, t>:=Group<x, y, z, w, t|x^4, y^2, z^2, w^2, t^2, x^-2*w,
> x^-1*y*x*y, (x^-1*z)^2, (y*z)^2, (t, y*z*x), (t, t^x), (t, t^y),
> (t, t^z), (t, t^w), (t, t^(y*x)), (t, t^(y*w)), (t, t^(y*z))>;
> #G;
4096
```

Which of course is equal to  $2^8 * 16 = 4,096$ .

The progenitor  $2^{*10} : M_{10}$  factored by  $(t_1, t_2), (t_1, t_3), \dots, (t_1, t_{10})$  given by the presentation:

```
> G<x, y, t>:=Group<x, y, t | x^2, y^4, y^-1*x*y^-2*x*y^2*x*y^2*x*y^-1,
> (y^-1*x)^8, y^-1*x*y^-2*x*y^-1*x*y^-2*x*y*x*y^2*x*y^-1*x*y^2*
> x, x*y^-2*x*y^-1*x*y^-1*x*y^-2*x*y*x*y*x*y^2*x*y^-1*x*y, t^2,
> (t, y^2*x*y*x*y^2), (t, y^2*x*y*x*y^-1*x*y^-1),
> (t, x*y^2*x*y*x*y^-1*x*y^-1*x), (t, t^x), (t, t^y), (t, t^(y^2)),
> (t, t^(y^2*x)), (t, t^(y^-1)), (t, t^(y^-1*x)), (t, t^(y^-1*x*y)),
> (t, t^(y^-1*x*y^-1)), (t, t^(y^-1*x*y^2))>;
>#G
737280
```

which is  $2^{10} * 720 = 1024 * 720 = 737,280$ .

The factoring lemma proves to be helpful in determining if a progenitor was constructed correctly. When we run progenitors in the background sometimes they do not produce finite presentations of groups that can be worked on. The factoring lemma reassures that the progenitor was constructed correctly and allows us to continue our research knowing that a useful finite representation was not produced for a progenitor.

## 2.8 Conclusion

Constructing an infinite progenitor and obtaining finite representations of groups is essential to the study of group theory. While we have several ways of constructing a progenitor, we must be able to distinguish between any two progenitors or groups being addressed. The idea of isomorphism type gives us the ability to say that a given group is exactly what we perceive it to be. This gives researchers the opportunity to work on groups and identify their findings in their research. Being able to distinguish isomorphism between groups also allows for researchers to make connections between why two structurally different groups are equivalent. Chapter 2 will aim to explain how to determine a group's isomorphism type.

## Chapter 3

# Isomorphism Type of Groups

### 3.1 Introduction

In mathematics we have a notion of equality that allows us to solve for unknown quantities. In algebra, people are typically asked to solve for an unknown variable. Group theory however, uses an approach more closely resembling that of geometry. In geometry when considering shapes; we determine if two shapes are congruent based on properties of the shape itself. Group theory has mechanisms which allow us to view permutations (of an unknown group) as geometric transformations and allow us to say two groups act similarly based on how those transformations match with known groups.

An isomorphism is a bijective homomorphism. If we can establish a one-to-one, onto correspondence between our unknown group  $G$  to a well-known group  $H$  we can determine this sense of equality, called an isomorphism existing between the two groups denoted  $G \cong H$ . This allows us to count sets of permutations (the order of the group) and give us a rough estimate of what the group we are working with might look like.

Different groups have different names based on how their symmetries interact. However, this does not mean that two different groups cannot have the same order. Consider  $D_{2(4)}$  and the group  $2 \times 2 \times 2$ , both have order 8 and both also have very different structure. In one set the group is abelian with every subgroup being abelian,



while in the other group there is a semidirect interaction among elements which causes some of the elements to not commute. How do we determine which set our unknown  $G$  is isomorphic to? This chapter aims to answer that question by investigating the group in question more deeply.

## 3.2 Extension Type

There are four types of extension types that take place in our research. Those four types are: direct products, semidirect products, central extensions, and mixed extensions. Each extension type is defined by the action being performed on the elements of the group. We will define and give examples of extension in the following sections.

## 3.3 Direct Products

Direct Products are the most frequently encountered problems we encounter. Consider group  $G$  with subgroups  $H$  and  $K$ , if  $H, K \trianglelefteq G$  with  $H \cap K = \{e\}$  such that  $G = H \times K$  then  $G$  is said to be a direct product of  $H$  and  $K$ .

### 3.3.1 Verification of Direct Products Using Magma

For this example we will consider  $G := \text{TransitiveGroup}(12, 19)$ . Our first step is to create a permutation representation of  $G$ . This process is done the same as in Chapter 2 and produces the following permutation representation of  $N$ :

```
S:=Sym(12);
A:=S!(1, 2, 4, 6)(3, 5, 9, 10)(7, 12, 11, 8);
B:=S!(1, 3, 8)(2, 5, 7)(4, 9, 12)(6, 10, 11);
C:=S!(1, 4)(2, 6)(3, 9)(5, 10)(7, 11)(8, 12);
D:=S!(1, 3, 8)(2, 7, 5)(4, 9, 12)(6, 11, 10);
N:=sub<S|A,B,C,D>;
```

Now, using the `#N` command, Magma tells us the order of  $N$  is 36. So we consider the normal lattice of  $N$ .

```
> NL:=NormalLattice(N);
> NL;
```

```
Normal subgroup lattice
-----
```

```
[10] Order 36 Length 1 Maximal Subgroups: 8 9
---
[ 9] Order 18 Length 1 Maximal Subgroups: 5 6 7
[ 8] Order 12 Length 1 Maximal Subgroups: 5
---
[ 7] Order 9 Length 1 Maximal Subgroups: 3 4
[ 6] Order 6 Length 1 Maximal Subgroups: 2 3
[ 5] Order 6 Length 1 Maximal Subgroups: 2 4
---
[ 4] Order 3 Length 1 Maximal Subgroups: 1
[ 3] Order 3 Length 1 Maximal Subgroups: 1
[ 2] Order 2 Length 1 Maximal Subgroups: 1
---
[ 1] Order 1 Length 1 Maximal Subgroups:
```

Now, we must determine which item on the list is our largest order abelian subgroup.

We use the command:

```
> for i in [1..#NL] do if IsAbelian(NL[i]) then i;
for|if> end if; end for;
```

which indicates that  $NL[1]$ ,  $NL[2]$ , ...,  $NL[9]$  are all abelian subgroups of  $N$ . Since  $NL[9]$  is the only subgroup of order 18, it is clearly the largest abelian subgroup.

Now, we must determine what  $NL[9]$  is. We know  $NL[9]$  is abelian but it could be several different groups. There are three possibilities for  $NL[9]$  to be isomorphic too.  $NL[9]$  is either  $3 \times 6$ ,  $2 \times 9$  or 18. If  $NL[9]$  is isomorphic to 18 then we do not have a direct product. Using the following Magma command we see that it is not.

```
> IsIsomorphic(CyclicGroup(18),NL[9]);
false
```

Now, we can construct a direct product using Magma and verify what  $NL[9]$  is isomorphic to. Using the following commands we determine  $NL[9] \cong 3 \times 6$

```

> E:=DirectProduct(CyclicGroup(3),CyclicGroup(6));
> IsIsomorphic(NL[9],E);
true Mapping from: GrpPerm: $, Degree 12, Order 2 * 3^2 to
GrpPerm: EComposition of Mapping from: GrpPerm: $, Degree 12,
Order 2 * 3^2 to GrpPC and Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: E

```

which concludes this example of determining a direct product.

### 3.3.2 Verifying $G \cong S_3 \times PGL_2(7) \times S_5$

The following finite presentation of  $G$  was obtained factoring the progenitor  $2^{*12} : D_3 \times D_4$ . This example of determining the isomorphism type was rather interesting because the investigation required backtracking throughout the process. First, we observe the composition factors of  $G$  as:

```

> G<x,y,z,w,u,t>:=Group<x,y,z,w,u,t|x^2,y^2,z^2,w^2,u^3,y^x=y,
> z^x=z,z^y=z*w,w^x=w,w^y=w,w^z=w,u^x=u^2,u^y=u,u^z=u,u^w=u,
> t^2,(t,z*w),(t,x*w*u),(x*w*t)^6,(z*y*u*t)^6,(x*t)^3>;
> f,G1,k:=CosetAction(G,sub<G|x,y,z,w,u>);
> CompositionFactors(G1);
G
| Cyclic(2)
*
| Alternating(5)
*
| Cyclic(2)
*
| A(1, 7) = L(2, 7)
*
| Cyclic(2)
*
| Cyclic(3)
1

```

To the experienced user there is evidence of  $S_3$  attached to this group. We proceed by investigating the normal lattice of  $G$  and note that  $NL[3]$  is a non-abelian order 6 subgroup. We can then verify that this is  $S_3$  using

```

>IsIsomorphic(Sym(3),NL[3]);
true

```

Now, we will create the quotient by breaking off  $NL[3]$  from  $G$  using the following command:

```
> q, ff:=quo<G1|NL[3]>;
> #q;
40,320
```

We continue while now ignoring the  $C_2$  and  $C_3$  from the bottom of our composition factors. Once again, experience indicates that  $C_2$  over  $PSL_2(7)$  could be  $PGL_2(7)$ . So we investigate the normal lattice of  $q$  knowing  $|PGL_2(7)| = 336$  as follows:

```
> NL1:=NormalLattice(q);
> NL1;
```

Normal subgroup lattice

```
-----
[10] Order 40320 Length 1 Maximal Subgroups: 7 8 9
----
[ 9] Order 20160 Length 1 Maximal Subgroups: 5 6
[ 8] Order 20160 Length 1 Maximal Subgroups: 6
[ 7] Order 20160 Length 1 Maximal Subgroups: 3 6
----
[ 6] Order 10080 Length 1 Maximal Subgroups: 2 4
----
[ 5] Order 336 Length 1 Maximal Subgroups: 4
----
[ 4] Order 168 Length 1 Maximal Subgroups: 1
[ 3] Order 120 Length 1 Maximal Subgroups: 2
----
[ 2] Order 60 Length 1 Maximal Subgroups: 1
----
[ 1] Order 1 Length 1 Maximal Subgroups:
```

Once again we use the argument in Magma, `IsIsomorphic(PGL(2,7),NL1[5])`; which turns out to be true. Now our job becomes to determine how  $PGL_2(7)$  and  $S_3$  interact amongst each other. We backtrack to our original normal lattice of  $G$  and look for a normal subgroup of order  $|H| = 3 * 336 = 2,016$ .

Luckily, the normal lattice of  $G$  has entry  $NL[17]$  of order 2,016. So we construct the following argument to verify that  $NL[17] \cong S_3 \times PGL_2(7)$ .

```
> E:=DirectProduct(PGL(2,7),Sym(3));
> if IsIsomorphic(E,NL[17]) then print true; end if;
true
```

Now since we know  $NL[17] \cong S_3 \times PGL_2(7)$  we can take the quotient  $G1$  by  $NL[17]$  to leave us with what appears to be  $S_5$ . Using the following commands we determine the following result:

```
> q1, ff:=quo<G1|NL[17]>;
> #q1;
120
> IsIsomorphic(q1,Sym(5));
true
```

From here we can verify that this is in fact a direct product using the following commands:

```
> E1:=DirectProduct(E,Sym(5));
> if IsIsomorphic(E1,G1) then true; end if;
true
```

This verifies that our finite  $G$  of order 241,920 is isomorphic to  $S_3 \times PGL_2(7) \times S_5$ .

### 3.4 Central Extensions

Another type of extension that takes place in our research is a central extension. In order for a central extension to take place, the group being studied must have a center. This process involves factoring out the center and then determining what remains.

#### 3.4.1 Verifying $G \cong 2^*U_4(2) : 2$

We begin the process with the finite  $G$  with composition factors

```
> CompositionFactors(G1);
G
| Cyclic(2)
*
| C(2, 3) = S(4, 3)
*
| Cyclic(2)
1
```

Now, after investigating the normal lattice of  $G_1$ , we determine that no direct product exists which will construct  $G$ . We do discover that  $NL[2]$  is the center of  $G$  and so we decide to produce a central extension. Now, if we factor  $G$  by  $NL[2]$  we produce the following  $q$  which we will need to determine before we can construct the central extension.

```
> q , ff:=quo<G1|NL[2]>;
> NLq:=NormalLattice(q);
> NLq;
```

Normal subgroup lattice

```
-----
[3]  Order 51840   Length 1   Maximal Subgroups: 2
----
[2]  Order 25920   Length 1   Maximal Subgroups: 1
----
[1]  Order 1       Length 1   Maximal Subgroups:
```

Since there is no normal subgroup of order 2 in  $q$ , this rules out the possibility of a direct product between  $U_4(2)$  and 2. We proceed as if a semidirect product exists. Now, we want a presentation for  $H$ . We could use  $NLq[2]$  as the presentation however, we will use  $H:=PGU(4,2)$  and  $FPGroup(H)$  gives

$$K \langle b, c \rangle := Group \langle b, c | b^3, c^6, (b^{-1} * c * b^{-1} * c^{-1})^2, (b * c^2 * b^{-1} * c^{-2})^2, c^{-3} * b * c^{-3} * b^{-1} * c^3 * b * c^3 * b^{-1}, (c^{-1} * b^{-1})^9, (b * c^{-2} * b^{-1} * c^{-1} * b * c^{-1} * b^{-1} * c^{-1})^2 \rangle.$$

Note: we labeled these using  $b$  and  $c$  since  $NL[2]$  is order 2 and has one generator  $a$ .

Now, we need to establish a connection between our presentation and  $NLq[2]$  since our presentation is more efficient. To do this we will compute the coset action of  $K$  and then establish a mapping by using  $s, t := IsIsomorphic$  where  $t$  is the mapping from our  $H$  to  $NLq[2]$ .

```
f2, K2, k2 := CosetAction(K, sub<K|Id(K)>);
s, t := IsIsomorphic(K2, NLq[2]);
```

The reader should note order does matter since we want our elements of  $K_2$  mapped to elements in  $NLq[2]$ . From here we can declare the transversal and then want to find the generators of  $K_2$  in  $NLq$ , since they are now in the form of transversals. To do this we

run the following code which takes  $b$  in  $K_2$ , determines its permutation representation in  $K$ , and then maps it through our mapping  $t$  to  $NLq[2]$ .

```
T:=Transversal(G1,NL[2]);
B:=t(f2(b));
C:=t(f2(c));
```

Now that  $B$  and  $C$  are permutations in  $NLq[2]$ , we want to determine which permutations they are. We run the following loops to determine which transversal is labeled for  $B$  and  $C$ .

```
> for i in [1..#T] do if ff(T[i]) eq B then i; break;
for|if> end if; end for;
2209
>for i in [1..#T] do if ff(T[i]) eq C then i; break;
for|if> end if; end for;
1082
```

Now, we need to find the element of order 2 in  $NLq[3]$  that doesn't exist in  $Nlq[2]$  as this will be the element that produces our extension. We find this element using:

```
> for d in NLq[3] do if Order(d) eq 2 and d not in NLq[2]
for|if> and NLq[3] eq sub<q|NLq[2],d>
for|if> then D:=d; break; end if; end for;
```

Now, we will need to use the Schreier to determine how  $d$  acts on  $b$  and  $c$ . We set the Schreier up as:

```
B:=ff(B);
C:=ff(C);
N:=sub<q|B,C>;
NN<b,c>:=Group<b,c|b^3,c^6,(b^-1*c*b^-1*c^-1)^2,
(b*c^2*b^-1*c^-2)^2,c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
(c^-1*b^-1)^9,(b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2>;
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=C; end if;
```

```

if Eltseq(Sch[i])[j] eq -2 then P[j]:=C^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

```

Once this has been set up, we use the following commands and obtain  $b^d$  and  $c^d$

```

> for i in [1..#N] do if ArrayP[i] eq B^D then print Sch[i];
for|if> end if; end for;
c * b * c * b^-1 * c * b * c^-2 * b^-1 * c^-1
> for i in [1..#N] do if ArrayP[i] eq C^D then print Sch[i];
for|if> end if; end for;
b^-1 * c * b * c * b^-1 * c^2 * b * c^-1 * b

```

adding the relations  $b^d = c * b * c * b^{-1} * c * b * c^{-2} * b^{-1} * c^{-1}$  and  $c^d = b^{-1} * c * b * c * b^{-1} * c^2 * b * c^{-1} * b$  we establish the semidirect product between  $U_4(2)$  and  $2$ , which we verify as isomorphic to  $q$ , leaving the following presentation for  $K$ :

$$K \langle b, c, d \rangle := \text{Group} \langle b, c, d | b^3, c^6, (b^{-1} * c * b^{-1} * c^{-1})^2, (b * c^2 * b^{-1} * c^{-2})^2, c^{-3} * b * c^{-3} * b^{-1} * c^3 * b * c^3 * b^{-1}, (c^{-1} * b^{-1})^9, (b * c^{-2} * b^{-1} * c^{-1} * b * c^{-1} * b^{-1} * c^{-1})^2, d^2, b^d = c * b * c * b^{-1} * c * b * c^{-2} * b^{-1} * c^{-1}, c^d = b^{-1} * c * b * c * b^{-1} * c^2 * b * c^{-1} * b \rangle.$$

Now that we have determined  $q \cong U_2(4) : 2$ , we will create a central extension with an element of order 2 with  $K$  and then show that presentation is isomorphic to our original  $G$  which will prove what we hope to show.

Continuing where we left off, we will use the presentation  $H \langle a \rangle := \langle a | a^2 \rangle$  for the order 2 group we intend to extend by. Now, we create a new mapping from  $K$  to  $q \cong U_4(2) : 2$  by using the  $s, t := \text{IsIsomorphic}(K1, q)$  command.

Once again we determine what  $B$ ,  $C$ , and  $D$  are stored as using the following commands:

```

> B:=t(f4(b));
> C:=t(f4(c));

```



```

> D:=t(f4(d));
> for i in [1..#T] do if ff(T[i]) eq B then i; break;
for|if> end if; end for;
T[2424]
> for i in [1..#T] do if ff(T[i]) eq C then i; break;
for|if> end if; end for;
T[20394]
> for i in [1..#T] do if ff(T[i]) eq D then i; break;
for|if> end if; end for;
T[7920]

```

We will store these values for  $T[2424]$ ,  $T[20394]$  and  $T[7920]$  but they will not be shown in this document as they are very large permutations. We will also store  $A:=NL[2].1$  which is the only element in  $H$ .

Now, we will check each of the elements in our presentation for  $K$  to see if they are equal to  $a$ , as that will indicate a central extension.

```

> for i in [0..1] do if (C^-1 * B^-1)^9 eq A^i then i;
for|if> break; end if; end for;
1

```

$B^3, C^6$  and all other relations in the presentation for  $K$  were not equal to  $a$  so we have left this code out. Now, adding  $a^2$ , and setting  $c^{-1} * b^{-1} = a$  to the presentation of  $K$ , we verify that our constructed  $H \cong 2^{\bullet}U_2(4) : 2$  with:

```

> f, G1, k:=CosetAction(G, sub<G|x, y, z, w>);
> H<a, b, c, d>:=Group<a, b, c, d|b^3, c^6, (b^-1*c*b^-1*c^-1)^2,
> (b*c^2*b^-1*c^-2)^2, c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
> (c^-1*b^-1)^9=a, (b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2,
> d^2, b^d=c*b*c*b^-1*c*b*c^-2*b^-1*c^-1,
> c^d= b^-1*c*b*c*b^-1*c^2*b*c^-1*b, a^2>;
> f4, H1, k4:=CosetAction(H, sub<H|Id(H)>);
> IsIsomorphic(H1, G1);
true

```

### 3.5 Semidirect Products

A semidirect product is very closely related to a direct product. Every requirement for a semidirect product is the same as the definition of a direct product except

that a direct product requires both subgroups to be normal while a semidirect product requires only one normal subgroup. This minor detail changes the action of the group considerably.

### 3.5.1 Verification of $N \cong 2^2 : (2^2 \times 3)$

To begin the process we consider  $N$  with the following generators:

```
S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 9)(8, 11)(10, 12);
Z:=S!(1, 4, 8)(2, 6, 10)(3, 7, 11)(5, 9, 12);
W:=S!(1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 11);
U:=S!(1, 5)(2, 3)(4, 6)(7, 9)(8, 11)(10, 12);
N:=sub<S|X,Y,Z,W,U>;
```

```
f,N1,k:=CosetAction(N,sub<N|Id(N)>);
NL:=NormalLattice(N1);
```

Considering the normal lattice of  $N$ , we determine  $NL[5]$  of order 4 (largest abelian) as a possible factor of  $N$ .

```
q, ff:=quo<N1|NL[5]>;
```

Next we want to know what  $NL[5]$  is isomorphic. To do this we construct the following argument and use Magma's presentation of  $NL[5]$ .

```
> X:=[2,2];
> IsIsomorphic(NL[5],AbelianGroup(GrpPerm,X));
true
```

$FPGroup(NL[5])$  gives us  $H \langle a, b \rangle := Group \langle a, b | a^2, b^2, (a * b)^2 \rangle$  as our presentation for  $H$ . We can choose to use any known presentation for  $H$  however, we will be required to make mappings if we choose to do so. For this reason, we will use Magma's  $FPGroup$ . From here we shall investigate  $q$ .

We have stored  $G/H \cong q$  with hope to determine what  $q$  is. We consider the normal lattice and realize  $NLq[10]$  is an order 12 abelian group. We quickly determine  $q \cong (2^2 \times 3)$ . Now, we want a presentation of  $q$  and Magma gives us the following

presentation.

$$Q \langle c, d, e \rangle := \text{Group} \langle c, d, e \mid c^2, d^2, e^3, (c * d)^2, c * e^{-1} * c * e, d * e^{-1} * d * e \rangle;$$

We want to produce a semidirect product so to do this we need the elements of  $q$  written in terms of  $N$ . We look to the transversals of  $N$  right multiplied by the elements of  $NL[5]$  which will give us the permutations in  $q$ . Specifically, we want to determine the generators of  $q$ ; as these 3 elements will give us the entire group. To do this, we run the following loop, once we determine what value  $q$  has stored its generators ( $q.1, q.2, \dots$ , or  $q.12$ )

```
for i in [1..#T] do if ff(T[i]) eq q.1 then i; end if; end for;
/* store as C */
for i in [1..#T] do if ff(T[i]) eq q.2 then i; end if; end for;
/*store as D */
for i in [1..#T] do if ff(T[i]) eq q.3 then i; end if; end for;
/*store as E */
```

Now, we can use the `Generators(NL[5])` command and store all 5 of our generators as:

```
A:=N1!(1, 7) (2, 8) (3, 5) (4, 6) (9, 20) (10, 19) (11, 18) (12, 17)
(13, 36) (14, 35) (15, 34) (16, 33) (21, 45) (22, 46) (23, 47)
(24, 48) (25, 43) (26, 44) (27, 41) (28, 42) (29, 38) (30, 37)
(31, 40) (32, 39);
```

```
B:=N1!(1, 12) (2, 11) (3, 10) (4, 9) (5, 19) (6, 20) (7, 17) (8, 18)
(13, 41) (14, 42) (15, 43) (16, 44) (21, 31) (22, 32) (23, 29)
(24, 30) (25, 34) (26, 33) (27, 36) (28, 35) (37, 48) (38, 47)
(39, 46) (40, 45);
```

```
C:=N1!(1, 5) (2, 6) (3, 7) (4, 8) (9, 18) (10, 17) (11, 20) (12, 19)
(13, 25) (14, 26) (15, 27) (16, 28) (21, 29) (22, 30) (23, 31)
(24, 32) (33, 42) (34, 41) (35, 44) (36, 43) (37, 46) (38, 45)
(39, 48) (40, 47);
```

```
D:=N1!(1, 9) (2, 10) (3, 11) (4, 12) (5, 18) (6, 17) (7, 20) (8, 19)
(13, 33) (14, 34) (15, 35) (16, 36) (21, 37) (22, 38) (23, 39)
(24, 40) (25, 42) (26, 41) (27, 44) (28, 43) (29, 46) (30, 45)
```

```
(31, 48) (32, 47);
```

```
E:=N1!(1, 13, 21) (2, 14, 22) (3, 15, 23) (4, 16, 24)
(5, 25, 29) (6, 26, 30) (7, 27, 31) (8, 28, 32) (9, 33, 37)
(10, 34, 38) (11, 35, 39) (12, 36, 40) (17, 41, 45)
(18, 42, 46) (19, 43, 47) (20, 44, 48);
```

Now, we check the action on A by the generators of q using the following code:

```
> for i,j in [1..2] do if A^C eq A^i*B^j then i,j;
for|if> break; end if; end for;
1 2
> for i,j in [1..2] do if A^D eq A^i*B^j then i,j;
for|if> break; end if; end for;
1 2
> for i,j in [1..2] do if A^E eq A^i*B^j then i,j;
for|if> break; end if; end for;
1 1
>
> for i,j in [1..2] do if B^C eq A^i*B^j then i,j;
for|if> break; end if; end for;
2 1
> for i,j in [1..2] do if B^D eq A^i*B^j then i,j;
for|if> break; end if; end for;
2 1
> for i,j in [1..2] do if B^E eq A^i*B^j then i,j;
for|if> break; end if; end for;
1 2
```

From the response, we determine  $a^c = a, a^d = a, a^e = a * b, b^c = b, b^d = b, b^e = a$  which will have constructed a semi direct product when inserted into our presentation containing the elements of both Q and H. After all is considered we have

$$H \langle a, b, c, d, e \rangle := \text{Group} \langle a, b, c, d, e \mid a^2, b^2, (a * b)^2, c^2, d^2, e^3, (c * d)^2, \\ c * e^{-1} * c * e, d * e^{-1} * d * e, a^c = a, a^d = a, a^e = a * b, b^c = b, b^d = b, b^e = a \rangle;$$

constructed to be  $2^2 : (2^2 \times 3)$  which turns out to be isomorphic to N which we verify through Magma.

### 3.5.2 Verification of $2^8 : A_5$

We begin the process with the presentation of  $G$  given as follows:

```
G<x,y,z,w,u,t>:=Group<x,y,z,w,u,t|x^2,y^2,z^3,w^2,u^2,y^x=y,
z^x=z,z^y=z,w^x=w,w^y=w,w^z=u,u^x=u,u^y=u,u^z=w*u,
u^w=u,t^2,(t,x*y*w),(t,y*u),(x*z*t)^5,(y*z^-1*t)^5,
(x*y*u*z^-1*t)^5>;
```

We use the coset action to convert this presentation into permutations and consider the composition factors of  $G$ :

```
> CompositionFactors(G1);
G
| Alternating(5)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
1
```

We note that this group has no center but based on the composition factors does have a largest abelian group which can be factored since it is clearly abelian up until  $A_5$ .

Looking at the normal lattice of  $G$  we note the normal subgroup of order 256 as  $2^8$ . Now, we verify the group is not a direct product by asking Magma, constructing the following direct product:

```
> X:=[2,2,2,2,2,2,2,2];
> E:=DirectProduct(Alt(5),AbelianGroup(GrpPerm,X));
```

```
> IsIsomorphic(E,G1);
false
```

so we can proceed to factor  $NL[5]$  from  $G1$  as follows:

```
q, ff:=quo<G1|NL[5]>;
```

Now, we need a presentation of both  $NL[5]$  and  $q$ .  $NL[5]$  is  $2^8$  which is easily constructed having 8 elements of order 2 which all commute. This is given by:

```
K<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^2,b^2,c^2,d^2,
e^2,f^2,g^2,h^2,(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),(a,h),
(b,c),(b,d),(b,e),(b,f),(b,g),(b,h),(c,d),(c,e),(c,f),(c,g),
(c,h),(d,e),(d,f),(d,g),(d,h),(e,f),(e,g),(e,h),(f,g),(f,h),
(g,h)>;
```

Now our presentation for  $q$  we can obtain using  $FPGroup(q)$ ; gives us the following presentation for  $q$ :

```
Q<i,j>:=Group<i,j|i^3,j^3,(i*j^-1*i^-1*j^-1)^2,
(i^-1*j*i^-1*j^-1)^2>;
```

The process conducted from here on out will determine the action of  $Q$  on  $K$ . We know that if we can express the elements of  $K$  in terms of  $Q$  this will demonstrate a semidirect correspondence since  $K$  is abelian and thus normal in  $G$  and  $Q$  is not normal in  $G$ . How we will do this is through the transversals of  $G$  with regard to  $NL[5]$ .

We construct the transversals and then determine which  $T[i]$  has been assigned to the generators of  $q$ .

```
> Generators(q);
{
  (1, 2) (3, 4),
  (1, 2, 3) (4, 5, 6)
}
> T:=Transversal(G1,NL[5]);
> for i in [1..#T] do if ff(T[i]) eq q.1 then i;
for|if> end if; end for;
3
> for i in [1..#T] do if ff(T[i]) eq q.2 then i;
for|if> end if; end for;
14
```

Now, we need to store these as well as the generators of NL[5]. These are very long permutations but can be found in the appendix. We will refer to the generators of NL[5] as A,B,C,D,E,F,G, and H and the generators of q (T[3] = q.1, and T[14] = q.2) as, I and K. Once we have stored these in Magma we can use the following code to determine the action of I and K on A,B,C,D,E,F,G, and H.

```
> for q,r,s,t,u,v,w,x in [1..2] do
for|if> if A^I eq A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x
for|if> then q,r,s,t,u,v,w,x;
for|if> end if; end for;
1 2 2 2 2 2 1 1
> for q,r,s,t,u,v,w,x in [1..2] do
for|if> if A^J eq A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x
for|if> then q,r,s,t,u,v,w,x;
for|if> end if;end for;
2 1 2 1 1 2 2 2
```

As indicated by the code, we are taking the generator A of NL[5] and conjugating by the generator q.1 (expressed in the transversal T[3]). What this does is generate the power to which each generator should be raised to. Since all the elements in NL[5] are order 2, any entry for q,r,s,t,u,v,w, and x can be left out of the presentation. This tells us  $a^i = a * g * h$  and  $a^j = b * d * e$  are part of our presentation. We will ultimately repeat this process for each of remaining generators of NL[5] (B,C,...,H), which will tell us the action of Q on K. The other relations are:  $b^i = b, b^j = a * h, c^i = d, c^j = f, d^i = c, d^j = d, e^i = b * c * d * e, e^j = a * b * d * f * h, f^i = c * d * f * h, f^j = h, g^i = g, g^j = c * d * e * g, h^i = h, \text{ and } h^j = c.$

Now, if we take the presentation of K and Q from earlier, and introduce the action of Q on K, we will have effectively produced a semidirect product among a known Q and K and proved it to be isomorphic to our original G. To finish off the process we construct the presentation and ask Magma to verify the isomorphism.

```
> Z<a,b,c,d,e,f,g,h,i,j>:=Group<a,b,c,d,e,f,g,h,i,j|i^2,j^3,
> a^2,b^2,c^2,d^2,e^2,f^2,g^2,h^2,(a,b),(a,c),(a,d),(a,e),
> (a,f),(a,g),(a,h),(b,c),(b,d),(b,e),(b,f),(b,g),(b,h),(c,d),
> (c,e),(c,f),(c,g),(c,h),(d,e),(d,f),(d,g),(d,h),(e,f),(e,g),
```

```

> (e, h), (f, g), (f, h), (g, h), (i*j^-1)^5, a^i=a*g*h, a^j=b*d*e,
> b^i=b, b^j=a*h, c^i=d, c^j=f, d^i=c, d^j=d, e^i=b*c*d*e,
> e^j=a*b*d*f*h, f^i=c*d*f*h, f^j=h, g^i=g, g^j=c*d*e*g,
> h^i=h, h^j=c>;

> f, Z1, k:=CosetAction(Z, sub<Z|Id(Z)>);
> if IsIsomorphic(Z1, G1) then true; end if;
true

```

This of course verifies  $G \cong 2^8 : A_5$

### 3.6 Mixed Extensions

Mixed extensions are a combination extension. When we complete a semidirect product there is a possibility that the group we have constructed is not isomorphic to the original  $G$ . If this happens, we must check to see if a central extension takes place which gives us the mixed extension denoted  $:\bullet$  indicating a mixed extension takes place.

#### 3.6.1 Verification of $G \cong 4 : \bullet PGL_2(7)$

We have obtained the finite presentation  $G$  with composition factors:

```

> CompositionFactors(G1);
G
|  Cyclic(2)
*
|  A(1, 7)          = L(2, 7)
*
|  Cyclic(2)
*
|  Cyclic(2)
1

```

Now, we consider the normal lattice of  $G$  and note that  $NL[3]$  is the largest abelian subgroup. We determine that  $NL[3] \cong 4$  and has the given presentation:

$$H \langle a, b \rangle := \text{Group} \langle a, b | b^2, a^{-2} = b \rangle$$



This of course is not the usual presentation of  $Z_4$  however we will use this to establish a systematic approach to determining mixed extensions. Now, considering  $G/\mathbb{Z}_4 \cong q$  we use Magma to determine the quotient and establish that  $q \cong PGL_2(7)$  as follows:

```
> q, ff:=quo<G1|NL[3]>;
> IsIsomorphic(q,PGL(2,7));
true
```

Now, taking the  $FPG$ roup( $q$ ) presentation we have  $Q$  given by the presentation:

$$Q \langle c, d, e, f \rangle := \text{Group} \langle c, d, e, f \mid c^2, d^2, e^2, f^2, (c*e)^2, (d*e)^2, c*d*c*d*e, (d*f*e)^2, f*d*f*d*f*e, f*c*e*f*c*f*e*c*f*c*f*c*d*f*c*e*f*c \rangle.$$

Since we can determine that the elements of  $q$  do not commute with the elements of  $k$ , we know that we do not have a direct product taking place. We proceed considering  $Q$ 's action on  $K$  as a semidirect product. We look for the generators of  $H$  ( $NL[3]$ ) and  $Q$  (Transversals) in  $G$  and determine them to be:

```
A:=G1!(1, 144, 168, 141)(2, 133, 167, 129)(3, 109, 165, 119)
(4, 117, 166, 134)(5, 118, 164, 98)(6, 93, 163, 110)
(7, 111, 162, 82)(8, 104, 159, 79)(9, 94, 160, 95)
(10, 77, 161, 112)(11, 64, 158, 123)(12, 122, 157, 61)
(13, 90, 154, 66)(14, 78, 156, 101)(15, 54, 153, 106)
(16, 107, 151, 50)(17, 85, 147, 76)(18, 55, 155, 92)
(19, 74, 145, 68)(20, 67, 150, 86)(21, 128, 142, 45)
(22, 91, 143, 56)(23, 43, 152, 108)(24, 38, 139, 127)
(25, 69, 138, 84)(26, 83, 135, 60)(27, 89, 149, 71)
(28, 75, 146, 65)(29, 126, 140, 32)(30, 131, 130, 40)
(31, 44, 132, 105)(33, 57, 125, 100)(34, 99, 120, 49)
(35, 58, 136, 88)(36, 87, 137, 73)(37, 72, 148, 70)
(39, 53, 115, 114)(41, 116, 113, 42)(46, 121, 103, 48)
(47, 102, 124, 59)(51, 96, 97, 52)(62, 80, 81, 63)
```

```
B:=G1!(1, 168)(2, 167)(3, 165)(4, 166)(5, 164)(6, 163)
(7, 162)(8, 159)(9, 160)(10, 161)(11, 158)(12, 157)(13, 154)
(14, 156)(15, 153)(16, 151)(17, 147)(18, 155)(19, 145)
(20, 150)(21, 142)(22, 143)(23, 152)(24, 139)(25, 138)
(26, 135)(27, 149)(28, 146)(29, 140)(30, 130)(31, 132)
(32, 126)(33, 125)(34, 120)(35, 136)(36, 137)(37, 148)
(38, 127)(39, 115)(40, 131)(41, 113)(42, 116)(43, 108)
(44, 105)(45, 128)(46, 103)(47, 124)(48, 121)(49, 99)
```

(50, 107) (51, 97) (52, 96) (53, 114) (54, 106) (55, 92) (56, 91)  
 (57, 100) (58, 88) (59, 102) (60, 83) (61, 122) (62, 81) (63, 80)  
 (64, 123) (65, 75) (66, 90) (67, 86) (68, 74) (69, 84) (70, 72)  
 (71, 89) (73, 87) (76, 85) (77, 112) (78, 101) (79, 104) (82, 111)  
 (93, 110) (94, 95) (98, 118) (109, 119) (117, 134) (129, 133)  
 (141, 144)

C:=G1!(2, 3) (4, 5) (6, 8) (7, 11) (9, 14) (10, 12) (13, 17) (15, 21)  
 (16, 24) (18, 28) (19, 26) (20, 25) (22, 31) (23, 29) (30, 39)  
 (32, 43) (33, 46) (34, 49) (35, 37) (38, 50) (40, 53) (41, 52)  
 (42, 51) (44, 56) (45, 54) (47, 59) (48, 57) (55, 65) (58, 70)  
 (60, 74) (61, 77) (62, 80) (63, 81) (64, 82) (66, 85) (67, 84)  
 (68, 83) (69, 86) (71, 89) (72, 88) (73, 87) (75, 92) (76, 90)  
 (78, 95) (79, 93) (91, 105) (94, 101) (96, 113) (97, 116) (98, 117)  
 (99, 120) (100, 121) (102, 124) (103, 125) (104, 110) (106, 128)  
 (107, 127) (108, 126) (109, 129) (111, 123) (112, 122) (114, 131)  
 (115, 130) (118, 134) (119, 133) (132, 143) (135, 145) (136, 148)  
 (138, 150) (139, 151) (140, 152) (141, 144) (142, 153) (146, 155)  
 (147, 154) (156, 160) (157, 161) (158, 162) (159, 163) (164, 166)  
 (165, 167)

D:=G1!(2, 4) (6, 9) (7, 10) (8, 11) (12, 14) (13, 18) (15, 22)  
 (16, 23) (17, 25) (19, 20) (21, 24) (26, 28) (27, 35) (29, 31)  
 (30, 40) (32, 44) (33, 47) (34, 48) (36, 37) (38, 45) (39, 51)  
 (41, 42) (43, 50) (46, 49) (52, 53) (54, 56) (55, 66) (57, 59)  
 (58, 71) (60, 75) (61, 78) (62, 63) (64, 79) (65, 83) (67, 68)  
 (69, 76) (70, 87) (72, 73) (74, 86) (77, 82) (80, 81) (84, 85)  
 (88, 89) (90, 92) (91, 106) (93, 95) (94, 110) (96, 114) (97, 115)  
 (98, 118) (99, 103) (100, 102) (101, 122) (104, 123) (105, 126)  
 (107, 108) (109, 119) (111, 112) (113, 116) (117, 129)  
 (120, 121) (124, 125) (127, 128) (130, 131) (132, 140)  
 (133, 134) (135, 146) (136, 149) (137, 148) (138, 147) (139, 142)  
 (141, 144) (143, 153) (145, 150) (151, 152) (154, 155) (156, 157)  
 (158, 159) (160, 163) (161, 162) (166, 167)

E:=G1!(2, 4) (3, 5) (6, 10) (7, 9) (8, 12) (11, 14) (13, 19) (15, 23)  
 (16, 22) (17, 26) (18, 20) (21, 29) (24, 31) (25, 28) (27, 36)  
 (30, 41) (32, 45) (33, 48) (34, 47) (35, 37) (38, 44) (39, 52)  
 (40, 42) (43, 54) (46, 57) (49, 59) (50, 56) (51, 53) (55, 67)  
 (58, 72) (60, 76) (61, 79) (64, 78) (65, 84) (66, 68) (69, 75)  
 (70, 88) (71, 73) (74, 90) (77, 93) (82, 95) (83, 85) (86, 92)  
 (87, 89) (91, 107) (94, 111) (96, 115) (97, 114) (98, 119)  
 (99, 102) (100, 103) (101, 123) (104, 122) (105, 127) (106, 108)  
 (109, 118) (110, 112) (113, 130) (116, 131) (117, 133) (120, 124)

```
(121, 125) (126, 128) (129, 134) (132, 139) (135, 147) (136, 148)
(137, 149) (138, 146) (140, 142) (143, 151) (145, 154) (150, 155)
(152, 153) (156, 158) (157, 159) (160, 162) (161, 163) (164, 165)
(166, 167)
```

```
F:=G1!(1, 2) (3, 6) (5, 7) (8, 13) (9, 10) (11, 15) (12, 16) (14, 20)
(17, 27) (18, 23) (19, 22) (21, 30) (24, 32) (25, 33) (26, 34)
(28, 37) (29, 38) (31, 42) (35, 48) (36, 47) (40, 45) (41, 44)
(43, 55) (46, 58) (49, 60) (50, 61) (51, 62) (52, 63) (54, 64)
(56, 68) (57, 69) (59, 73) (65, 70) (66, 79) (67, 78) (71, 76)
(72, 75) (74, 91) (77, 94) (80, 96) (81, 97) (82, 98) (83, 99)
(84, 100) (85, 89) (86, 101) (87, 102) (88, 103) (90, 104) (92, 108)
(93, 109) (95, 112) (105, 113) (106, 123) (107, 122) (110, 119)
(111, 118) (116, 132) (120, 135) (121, 136) (124, 137)
(125, 138) (126, 139) (127, 140) (128, 131) (129, 141) (130, 142)
(133, 144) (143, 145) (146, 148) (147, 149) (150, 156) (151, 157)
(152, 155) (153, 158) (154, 159) (160, 161) (162, 164) (163, 165)
(167, 168)
```

Now that are stored we will determine the action of  $Q$  on  $K$  using the code:

```
> for q,r in [0..1] do if A^C
for|if> eq A^q*B^r then q,r; end if;
for> end for;
1 1
> for q,r in [0..1] do if A^D
for|if> eq A^q*B^r then q,r; end if;
for> end for;
1 1
> for q,r in [0..1] do if A^E
for|if> eq A^q*B^r then q,r; end if;
for> end for;
1 0
> for q,r in [0..1] do if A^F
for|if> eq A^q*B^r then q,r; end if;
for> end for;
1 0
> for q,r in [0..1] do if B^C
for|if> eq A^q*B^r then q,r; end if;
for> end for;
0 1
> for q,r in [0..1] do if B^D
for|if> eq A^q*B^r then q,r; end if;
```

```

for> end for;
0 1
> for q,r in [0..1] do if B^E
for|if> eq A^q*B^r then q,r; end if;
for> end for;
0 1
> for q,r in [0..1] do if B^F
for|if> eq A^q*B^r then q,r; end if;
for> end for;
0 1

```

Adding  $a^c = a * b, a^d = a * b, a^e = a, a^f = a, b^c = b, b^d = b, b^e = b$ , and  $b^f = b$  to our presentation of  $PGL_2(7)$  and  $\mathbb{Z}_4$  we have the following presentation of the group  $Z$  we have constructed:

$$Z \langle a, b, c, d, e, f \rangle := \text{Group} \langle a, b, c, d, e, f | b^2, a^2 = b, c^2, d^2, e^2, f^2, (c*e)^2, (d*e)^2, c*d*c*d*e, (d*f*e)^2, f*d*f*d*f*e, f*c*e*f*c*f*e*c*f*c*f*c*d*f*c*e*f*c, a^c = a * b, a^d = a * b, a^e = a, a^f = a, b^c = b, b^d = b, b^e = b, b^f = b \rangle$$

Now, we attempt to determine the  $\#Z$  but it is still infinite. This indicates either an error in the semi-direct product we have constructed or that a central extension exists. We proceed knowing we have made no such errors. The central extension will appear in the form of writing the elements of  $Q$  in terms of  $K$ . We run a similar code but replace the conjugation of  $A^C$  from earlier with the relations from  $Q$ . Since only one of these relations produces a relation, we have:

```

> for q,r in [0..1] do if F*C*E*F*C*F*E*C*F*C*F*C*D*F*C*E*F*C
for|if> eq A^q*B^r then q,r; end if;
for> end for;
1 1

```

which implies  $f*c*e*f*c*f*e*c*f*c*f*c*d*f*c*e*f*c = a*b$ . Adding this to the presentation of  $Z$ , fixes the issue and we have thus constructed the mixed extension  $4 : \bullet PGL_2(7)$  which turns out to be isomorphic to  $G$ .

### 3.6.2 Verification of $M \cong 2^4 : \bullet S_4$

When we find a group such as  $G \cong M_{12} : 2$  on a relatively small  $N$  such as  $2^2 : (2^2 \times 3)$  it becomes necessary to determine a maximal subgroup of  $G$  so that we can perform the process of double coset enumeration over the maximal subgroup  $M$ . In this instance, we still require to know what group we are speaking of and fortunately this subgroup provides us with an example of a mixed extension.

We determine  $M \cong \frac{2^{*12}:(2^2)^{\bullet}A_4}{(x*t)^3,(z*t)^5,z*t*v*t*z*v*t*z^{-1}*t*z*t*z^{-1}*t}$  by considering the composition factors of  $M$ .

```
> M:=Ms[7] `subgroup;
> M;
Permutation group M acting on a set of cardinality 3960
Order = 384 = 2^7 * 3
> CompositionFactors(M);
G
| Cyclic(2)
*
| Cyclic(3)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
1
```

We have composition series  $M_1 \supseteq M_2 \supseteq M_3 \supseteq M_4 \supseteq M_5 \supseteq M_6 \supseteq M_7 \supseteq M_8 \supseteq 1$  where  $M = (M_1/M_2) \cdot (M_1/M_2) \cdot (M_2/M_3) \cdot (M_3/M_4) \cdot (M_4/M_5) \cdot (M_5/M_6) \cdot (M_6/M_7) \cdot (M_7/M_8) \cdot (M_8/1)$  or  $M = C_2 \cdot C_3 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2$ .

We will now look for the largest order abelian subgroup in the normal lattice as this will indicate a possible direct product taking place. One should note that multiple subgroups in the normal lattice can be abelian with equal highest degree. In the event that this happens choosing either subgroup will produce different isomorphism types however, the two isomorphism types are isomorphic to each other as well as  $M$  so either suffices.

```
> NL:=NormalLattice(M);
> NL;
Normal subgroup lattice
-----

[13] Order 384 Length 1 Maximal Subgroups: 10 11 12
---
[12] Order 192 Length 1 Maximal Subgroups: 9
[11] Order 192 Length 1 Maximal Subgroups: 9
[10] Order 192 Length 1 Maximal Subgroups: 8 9
---
[ 9] Order 96 Length 1 Maximal Subgroups: 6 7
[ 8] Order 64 Length 1 Maximal Subgroups: 4 6
---
[ 7] Order 48 Length 1 Maximal Subgroups: 5
[ 6] Order 32 Length 1 Maximal Subgroups: 3 5
---
[ 5] Order 16 Length 1 Maximal Subgroups: 2
[ 4] Order 16 Length 1 Maximal Subgroups: 3
---
[ 3] Order 8 Length 1 Maximal Subgroups: 2
---
[ 2] Order 4 Length 1 Maximal Subgroups: 1
---
[ 1] Order 1 Length 1 Maximal Subgroups:
```

We will proceed by determining what  $NL[4]$  is isomorphic too. We determine:

```
> X:=[2,2,2,2];
> IsIsomorphic(NL[4],AbelianGroup(GrpPerm,X));
true
```

Now, we create a presentation for  $NL[4]$  as:

$$K \langle a, b, c, d \rangle := \text{Group} \langle a, b, c, d | a^2, b^2, c^2, d^2, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d) \rangle$$

Factoring subgroup M by NL[4] we normally would look at the normal lattice of q and determine what q is but since q is order 24 we determine quickly that  $q \cong S_4$ . We determine we do not have a direct product between NL[4] and q so we must determine the action of Q on K. The presentation of  $q \cong S_4$  obtained from FPGGroup(q) is:

$$Q \langle e, f, g, h \rangle := \text{Group} \langle e, f, g, h | e^4, f^3, g^2, h^2, g * e^2 * h, (e * f^{-1})^2, e^{-1} * g * e * h, f^{-1} * g * f * h, f^{-1} * e^{-2} * f * g \rangle$$

Now, we begin the process of determining the action of Q on K by labeling the generators of NL[4] and the generators of q as follows:

```

> T:=Transversal(M,NL[4]);
> for i in [1..#T] do if ff(T[i]) eq q.1 then i;
for|if> end if; end for;
T[4]
> for i in [1..#T] do if ff(T[i]) eq q.2 then i;
for|if> end if; end for;
T[2]
> for i in [1..#T] do if ff(T[i]) eq q.3 then i;
for|if> end if; end for;
T[7]
> for i in [1..#T] do if ff(T[i]) eq q.4 then i;
for|if> end if; end for;
T[10]

> A:=NL[4].1; B:=NL[4].2; C:=NL[4].3; D:=NL[4].4;
> E:=T[4]; F:=T[2]; G:=T[7]; H:=T[10];

```

One should note that we have stored these permutations in a file. They are not shown here as they take up a lot of space. As before we use the code that will conjugate the generators of K by the generators of Q, and give us a word that is equal in terms of K. We will show only A, but the other generators B, C, and D have had this code run on them as well.

```

> for q,r,s,t in [0..1] do if A^E
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;

```

```

for> end for;
1 1 1 0
>
> for q,r,s,t in [0..1] do if A^F
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
1 0 0 1
>
> for q,r,s,t in [0..1] do if A^G
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
1 0 0 1
>
> for q,r,s,t in [0..1] do if A^H
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
1 0 1 0

```

Now, this indicates to us that the  $a^e = a * b * c$ ,  $a^f = a * f$ ,  $a^g = a * f$ , and  $a^h = a * c$  in our presentation. Once we have constructed these relations (for B,C, and D) we will add them to the overall presentation Z which contains all the elements from H and Q.

$$\begin{aligned}
Z \langle a, b, c, d, e, f, g, h \rangle := & \text{Group} \langle a, b, c, d, e, f, g, h \mid a^2, b^2, c^2, d^2, (a, b), (a, c), \\
& (a, d), (b, c), (b, d), (c, d), e^4, f^3, g^2, h^2, g * e^2 * h, (e * f^{-1})^2, e^{-1} * g * e * h, \\
& f^{-1} * g * f * h, f^{-1} * e^{-2} * f * g, a^e = a * b * c, a^f = a * d, a^g = a * d, a^h = a * c, \\
& b^e = b * c * d, b^f = b * c * d, b^g = b * c * d, b^h = b * d, c^e = c, c^f = c * d, c^g = c, \\
& c^h = c, d^e = c * d, d^f = c, d^g = d, d^h = d \rangle
\end{aligned}$$

Upon asking Magma if this is isomorphic to M we determine this to be false. This introduces the possibility of a central extension existing between our subgroups. Similar to the code before, we will check each relation in Q to determine how, the elements in K act on Q. We proceed using the following code:

```

> for q,r,s,t in [0..1] do if E^4
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 1 0

```



```

> for q,r,s,t in [0..1] do if F^3
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
1 1 1 1
> for q,r,s,t in [0..1] do if G^2
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 0 0
> for q,r,s,t in [0..1] do if H^2
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 0 0
> for q,r,s,t in [0..1] do if G*E^2*H
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 0 0
> for q,r,s,t in [0..1] do if (E*F^-1)^2
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 1 1 1
> for q,r,s,t in [0..1] do if E^-1*G*E*H
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 1 0
> for q,r,s,t in [0..1] do if F^-1*G*F*H
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 0 0 0
> for q,r,s,t in [0..1] do if F^-1*E^-2*F*G
for|if> eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
for> end for;
0 1 1 1

```

Now, since these elements already exist in the presentation  $Z$ , we change the entry  $e^4$  to  $e^4 = c$ , as Magma tells us, until we achieve the final presentation for  $Z$ , determine it in permutations and ask Magma if our constructed group which is  $2^4 : \bullet S_4$  is in fact isomorphic to  $M$ . The following Magma command shows this thus ending our mixed extension.

```

> Z<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^2,b^2,c^2,d^2,
> (a,b),(a,c),(a,d),(b,c),(b,d),(c,d),e^4=c,f^3=a*b*c*d,g^2,
> h^2,g*e^2*h,(e*f^-1)^2=b*c*d,e^-1*g*e*h=c,f^-1*g*f*h,
> f^-1*e^-2*f*g=b*c*d,a^e=a*b*c,a^f=a*d,a^g=a*d,

```

```

> a^h=a*c,b^e=b*c*d,b^f=b*c*d,b^g=b*c*d,b^h=b*d,
> c^e=c,c^f=c*d,c^g=c,c^h=c,d^e=c*d,d^f=c,d^g=d,
> d^h=d>;
>
> f4,Z4,k4:=CosetAction(Z,sub<Z|Id(Z)>);
> s,t:=IsIsomorphic(Z4,M);
> s;
true

```

### 3.7 Conclusion

Isomorphism types rely on a person's ability to break down groups into smaller pieces and investigate them according to known groups. Magma is very convenient in this process and allows groups to be identified and verified quickly. Identifying the largest abelian subgroup of  $N$ , is typically the first step in the process. Once this has been done we can investigate the remaining pieces by constructing either a semidirect, central, or mixed extension and determining if our group is isomorphic to the group we have constructed.

## Chapter 4

# Double Coset Enumeration

### 4.1 Introduction

Double coset enumeration was a technique invented by Todd in the 1930's. His technique was designed to solve word problems in group theory. Double coset enumeration gives us a faster way to verify single coset enumeration. Curtis, an author related closely to the our field of study has used manual double coset enumeration to generate many of the famous simple sporadic groups. While the process has been around for nearly a century, the process's intricacy never falters.

Double coset enumeration is most practical when used to determine finite groups as we can construct a Cayley diagram to verify the symmetry of our work. The process can be challenging and requires techniques such as substitution and conjugation. While the process, can be completed manually in most instances Magma is very handy when the process requires a maximal subgroup containing our control group  $N$ . This chapter will show examples of manual double coset enumeration performed over the control group  $N$  and Maximal double coset enumeration performed with Magma.

## 4.2 Manual Double Coset Enumeration of

### $PSL_3(4) : 2$ over $M_{10}$

When we begin a double coset enumeration, there are a few checks we can perform to help us know we have in fact completed the process correctly. The first check is to determine the number of single cosets that will appear in our Cayley diagram. To do this, we calculate  $\frac{|G|}{|N|} = \frac{|PSL_3(4):2|}{|M_{10}|} = \frac{40,320}{720} = 56$ . This means the sum of all single cosets should equal exactly 56 in our Cayley diagram. The next helpful piece of information is to ask Magma to determine:

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>#DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
3
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This tells us there are 3 double cosets and gives us a rough estimate of what our Cayley diagram will look like. We now proceed with the manual double coset enumeration of  $PSL_3(4) : 2$  over  $M_{10}$ .

Given the progenitor  $2^{*10} : M_{10}$  factored by the relation  $((y^{-1}x)^3t)^5$  where  $x = (1, 2)(4, 5)(6, 7)(8, 9)$  and  $y = (1, 3, 4, 6)(7, 8, 10, 9)$  we proceed allowing  $t \sim t_1$  and  $\Pi = (y^{-1}x)^3$  we compute the relation as follows:

$$\begin{aligned} ((y^{-1}x)^3t)^5 &= e \\ (\Pi t)^5 &= e \\ \Pi^5 t \Pi^4 t \Pi^3 t \Pi^2 t \Pi t &= e \\ (1, 2, 3, 4, 5, 6, 8, 7)(9, 10)t_5 t_7 t_3 t_6 t_1 &= e \\ (1, 2, 3, 4, 5, 6, 8, 7)(9, 10)t_5 t_7 &= t_1 t_6 t_3 \end{aligned}$$

Let  $G = \frac{2^{*10}:M_{10}}{(1,2,3,4,5,6,8,7)(9,10)t_5 t_7 t_3 t_6 t_1 = e}$  with  $N = \langle (1, 2)(4, 5)(6, 7)(8, 9), (1, 3, 4, 6), (7, 8, 10, 9) \rangle$  and  $t \sim t_1$ . Using our definition of a double coset  $NwN = \{Nw^n | n \in N\}$  we consider our first double coset  $NeN = \{Ne^n | n \in N\}$  which we denote by  $[*]$ . Our control group  $N$  is transitive on  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  so it has a single orbit  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ . We choose a representative from the orbit, 1 in this case, and determine what double coset  $Nt_1$  belongs to. This new double coset will be labeled as  $[1]$  and will contain  $Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9,$

$Nt_{10}$ }. We will now investigate the coset stabilizer  $N^{(1)}$  which is the point stabilizer  $N^1$ .

Considering  $N^{(1)}$  there are 72 elements in the point stabilizer. The 72 elements are generated by the following permutations  $N^{(1)} = \langle (2, 8, 3, 7)(4, 9, 6, 5), (2, 8, 3, 7)(4, 9, 6, 5), (3, 7, 10, 9)(4, 8, 5, 6) \rangle$  and this indicates that there are  $\frac{|N|}{|N^{(1)}|} = \frac{720}{72} = 10$  possible single cosets. Considering the generators of  $N^{(1)}$  we note that the orbits of  $N^{(1)}$  on  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  are  $\{1\}$ ,  $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ . We take a representative from each orbit and determine to which double coset  $Nt_1t_1$  and  $Nt_1t_2$  belong. Since our  $t$ 's are all of order 2,  $Nt_1t_1 \in Ne \in [*]$  so this double coset goes back to the double coset  $[*]$ . Now,  $Nt_1t_2 \in Nt_1t_2N$  is a new double coset and our 9 symmetric generators are carried to our next double coset  $[1\ 2]$ .

We continue the process by considering the coset stabilizer  $N^{(12)}$  which we determine using Magma. Now  $N^{(12)} \geq N^{12} = \langle (3, 8, 10, 6)(4, 9, 5, 7), (3, 7, 10, 9)(4, 8, 5, 6) \rangle$  and the number of single cosets  $\frac{|N|}{|N^{(12)}|} \leq \frac{720}{8} = 90$ . Due to our diligence at the beginning of this process we know that there are a total of 56 single cosets in our Cayley Diagram. We have already used 11 of those single cosets in our double cosets  $[*]$  and  $[1]$ . This means we have 45 single cosets remaining and only the coset  $[1\ 2]$  to put them in. This indicates our cosets have equal names and the relation increases our  $|N^{(12)}|$  from 8 to 16. To understand why this has happened, we look to the relation  $(1, 2, 3, 4, 5, 6, 8, 7)(9, 10)t_5t_7 = t_1t_6t_3$ .

We can use Magma to tell us which cosets are equal but then must solve the relation by hand. The code for this can be found in the appendix and Magma tells us that  $Nt_1t_2 = Nt_2t_1$ . To prove this we must look to our  $N$ . We determine a permutation in  $N$  which fixes  $t_1$  and  $t_3$  but sends  $t_6$  to  $t_2$  using Magma. We proceed as follows:

$$((1, 2, 3, 4, 5, 6, 8, 7)(9, 10)t_5t_7)^{(2,9,10,6)(4,8,7,5)} = (t_1t_6t_3)^{(2,9,10,6)(4,8,7,5)}$$

$$(1, 9, 3, 8, 4, 2, 7, 5)(10, 6)t_4t_5 = t_1t_2t_3$$

$$(1, 9, 3, 8, 4, 2, 7, 5)(10, 6)t_4t_5t_3 = t_1t_2 \text{ (we save this momentarily).}$$

$$((1, 2, 3, 4, 5, 6, 8, 7)(9, 10)t_5t_7)^{(1,4,7)(2,6,5)(8,9,10)} = (t_1t_6t_3)^{(1,4,7)(2,6,5)(8,9,10)}$$

$$(4, 6, 3, 7, 2, 5, 9, 1)(10, 8)t_2t_1 = t_4t_5t_3 \text{ (now substituting to the saved relation above).}$$

$$(1, 9, 3, 8, 4, 2, 7, 5)(10, 6)[(4, 6, 3, 7, 2, 5, 9, 1)(10, 8)t_2t_1] = t_1t_2$$

$$Nt_2t_1 = Nt_1t_2.$$

Which verifies our relation Magma told us as well as the fact that there are 56 single cosets in our Cayley Diagram. Similarly the other 44 cosets that are equal can be proved by hand. Finally, the orbits of  $N^{(12)}$  on the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  are  $\{1, 2\}$  and  $\{3, 4, 5, 6, 7, 8, 9, 10\}$ . Now,  $Nt_1t_2t_2 \in Nt_1 \in [1]$  so  $\{1, 2\}$  go back to  $[1]$ . Considering  $Nt_1t_2t_3 = Nt_4t_5$  we conjugate  $(3, 10)(4, 5)(6, 8)(7, 9)t_2t_1 = t_1t_2$  by  $(1, 4)(2, 5)(3, 10)(6, 7) \in N$  and note  $(1, 2)(3, 10)(6, 9)(7, 8)t_5t_4 = t_4t_5$  or more importantly  $Nt_5t_4 = Nt_4t_5$  since  $Nt_4t_5$  is a single coset in  $[1\ 2]$ . This implies  $Nt_1t_2t_3 = Nt_5t_4 = Nt_4t_5 \in [12]$  so all 8 elements in the same orbit as 3 are sent back to  $[1\ 2]$  thus completing the double coset enumeration of  $PSL_3(4) : 2$  over  $M_{10}$  and producing this Cayley diagram:

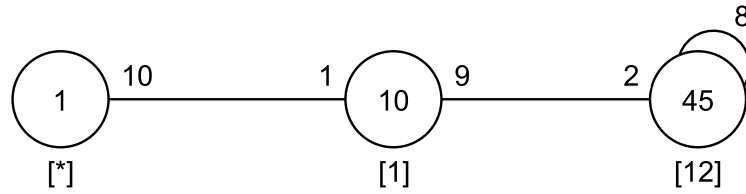


Figure 4.1: Cayley Diagram  $PSL_3(4) : 2$  over  $M_{10}$

### 4.3 Manual Double Coset Enumeration of $PGL_2(11)$ over $D_{12}$

The following finite presentation was original discovered on the progenitor  $2^{*12} : (2^2 \times 3) : \bullet 2^2$ . Later it was discovered that  $N$  had changed from  $(2^2 \times 3) : \bullet 2^2$  to the subgroup  $D_{12}$ . This finite presentation is the result of using the changing  $N$  method and factoring  $3 \bullet PGL_2(11)$  by  $C_3$  presented in Chapter 2. The resulting double coset enumeration that will be conducted is  $PGL_2(11)$  over  $D_{12}$ . Before we begin we will construct a table that lists the relation and all the conjugations of (1), (13), and (19) used in the following lemmas. **We will also denote  $R(t'_i s)$  for right multiplication (where  $t'_i s$  are any  $t$ 's),  $L(t'_i s)$  for left multiplication, and abbreviate conj. for conjugation.**

Table 4.1: Relations for  $PGL_2(11)$  over  $D_{12}$

(1) $t_4 t_2 t_1 = e$	(13) $yt_1 t_3 t_1 t_3 t_1 = e$	(19) $zt_3 t_4 t_3 t_1 t_4 t_6 t_2 t_6 t_1 = e$
(2) $t_6 t_5 t_3 = e$	(14) $yt_3 t_1 t_3 t_1 t_3 = e$	(20) $z^{-1} t_3 t_2 t_3 t_1 t_2 t_5 t_4 t_5 t_1 = e$
(3) $t_2 t_1 t_4 = e$	(15) $yt_2 t_5 t_2 t_5 t_2 = e$	(21) $zt_1 t_6 t_1 t_3 t_6 t_4 t_5 t_4 t_3 = e$
(4) $t_5 t_3 t_6 = e$	(16) $yt_5 t_2 t_5 t_2 t_5 = e$	(22) $z^{-1} t_5 t_4 t_5 t_2 t_4 t_6 t_1 t_6 t_2 = e$
(5) $t_3 t_6 t_5 = e$	(17) $yt_4 t_6 t_4 t_6 t_4 = e$	(23) $z^{-1} t_5 t_4 t_5 t_2 t_4 t_6 t_1 t_6 t_2 = e$
(6) $t_6 t_3 t_5 = e$	(18) $yt_6 t_4 t_6 t_4 t_6 = e$	(24) $z^{-1} t_3 t_2 t_3 t_4 t_5 t_4 t_5 t_1 = e$
(7) $t_1 t_4 t_2 = e$		(25) $z^{-1} t_5 t_4 t_5 t_1 t_6 t_1 t_6 t_2 = e$
(8) $t_4 t_1 t_2 = e$		(26) $zt_2 t_3 t_2 t_6 t_1 t_6 t_1 t_5 = e$
(9) $t_3 t_5 t_6 = e$		
(10) $t_2 t_4 t_1 = e$		
(11) $t_5 t_6 t_3 = e$		
(12) $t_1 t_2 t_4 = e$		

Given  $\frac{2^{*6} : D_{12}}{(y)t_1 t_3 t_1 t_3 t_1, (e)t_4 t_2 t_1, (1,4,2)(3,6,5)t_3 t_4 t_3 t_1 t_4 t_6 t_2 t_6 t_1}$ ,  $N = \langle (1, 2)(3, 5), (1, 3)(2, 5)(4, 6), (1, 4, 2)(3, 6, 5) \rangle$ . We verify the  $\# \text{DoubleCosets}(G, \text{sub} \langle G|x, y, z \rangle, \text{sub} \langle G|x, y, z \rangle) = 15$  using Magma and note  $\frac{|G|}{|N|} = \frac{PGL_2(11)}{D_{12}} = \frac{1,320}{12} = 110$  single cosets. We consider our first double coset  $NeN = \{Ne^n | n \in N\}$  which we will denote  $[*]$ . Since  $N$  is transitive on  $\{1,2,3,4,5,6\}$  we have the single orbit  $\{1,2,3,4,5,6\}$ . We choose a representative from this orbit ( $t_1$ ) which carries us to the double coset  $Nt_1N = \{Nt_1^n | n \in N\}$  denoted  $[1]$ . From here, we consider the coset stabilizer  $N^{(1)}$  which is equal to the point stabilizer  $N^1$ .  $N^1 = \langle e, (2, 4)(5, 6) \rangle$ , to determine the number of single cosets

we know  $|N| \geq | \langle Id(G), (2, 4)(5, 6) \rangle | \geq 2$ . Therefore, the number of single cosets in  $Nt_1N = \frac{|N|}{|N^1|} = \frac{12}{2} = 6$ . The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}, \{3\}, \{2, 4\}$ , and  $\{5,6\}$ . We extract a representative from each orbit and determine to which double coset they belong.

The possible new double cosets we will consider are  $Nt_1t_1, Nt_1t_3, Nt_1t_2$ , and  $Nt_1t_5$ . We know  $t_1^2 = \{e\}$  so  $Nt_1t_1 \in [*]$ .

**Lemma 4.1.**  $t_1t_2 \in [1]$

*Proof.* Given:  $t_1t_2t_4 = e$  by relation (12)

$$t_1t_2t_4 = e \implies t_1t_2 = t_4 \in [1] \qquad R(t_4)$$

□

Since we right multiplied by  $t_4$  that is indicated by  $R(t_4)$  to the right of our work. Considering  $Nt_1t_3$  defined as  $Nt_1t_3N = \{N(t_1t_3)^n | n \in N\}$  We look to the coset stabilizer  $N^{(13)}$ . We have  $N^{(13)} = \langle e, (2, 4)(5, 6) \rangle$  which gives us the inequality:

Computing  $N^{(13)}$  in  $N$  we obtain,

$$\begin{aligned} |N^{(13)}| & \\ & \geq |N^{13}| \\ & \geq | \langle Id(G), (2, 4)(5, 6) \rangle | \geq 2 \end{aligned}$$

This implies the number of single cosets in  $Nt_1t_3N = \frac{|N|}{|N^{(13)}|} = \frac{12}{2} = 6$ . We now consider the orbits on  $\{1,2,3,4,5,6\}$  which are  $\{1\}, \{3\}, \{2,4\}$ , and  $\{5,6\}$ . As before we choose a representative from each orbit and produce the list of possible new double cosets  $Nt_1t_3t_1, Nt_1t_3t_3, Nt_1t_3t_2$ , and  $Nt_1t_3t_5$ .



The result is obvious that  $Nt_1t_3t_3N = Nt_1N \in [1]$ . The relation  $yt_1t_3t_1t_3t_1 = e$  produces  $Nyt_1t_3t_1N = Nt_1t_3N \in [13]$ .

**Lemma 4.2.**  $t_1t_3t_5 \in [15]$

*Proof.* Given:  $t_4t_2t_1 = e$  by relation (1)

$$\begin{aligned}
 e = t_4t_2t_1 &\implies t_1 = t_4t_2 && R(t_1) \\
 t_1t_3t_5 = t_4t_2t_3t_5 &&& R(t_3t_5) \\
 &= (t_4t_2)(t_3t_5) = t_1t_6 && \text{relations : (1), (9)} \\
 &\implies t_1t_3t_5 = t_1t_6 \in [15] && \text{equality}
 \end{aligned}$$

□

$Nt_1t_3t_2$  will be determined to be a new double coset but we will investigate this further later. We now investigate the double coset  $Nt_1t_5$ .

We begin with  $Nt_1t_5$  by determining the number of single cosets as follows:

$$\begin{aligned}
 |N^{(15)}| & \\
 &\geq |N^{15}| \\
 &\geq | \langle Id(G) \rangle | \geq 1
 \end{aligned}$$

The total number of single cosets is  $\frac{|N|}{|N^{(15)}|} = \frac{12}{1} = 12$ . The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$  so we will investigate the possible new double cosets  $Nt_1t_5t_1$ ,  $Nt_1t_5t_2$ ,  $Nt_1t_5t_3$ ,  $Nt_1t_5t_4$ ,  $Nt_1t_5t_5$ , and  $Nt_1t_5t_6$ . Clearly,  $Nt_1t_5t_5 \in [1]$ . The new double cosets we will investigate later are  $Nt_1t_5t_1$ ,  $Nt_1t_5t_2$ , and  $Nt_1t_5t_4$ .

**Lemma 4.3.**  $t_1t_5t_3 \in [13]$

*Proof.* Given:  $t_1t_3t_5 = t_1t_6$  by Lemma(4.2)

$$\begin{aligned}
 t_1t_6^{(2,4)(5,6)} &= t_1t_3t_5^{(2,4)(5,6)} && \text{conj.} \\
 t_1t_5 = t_1t_3t_6 &\implies t_1t_5t_3 = t_1t_3t_6t_3 && R(t_3) \\
 &= t_1t_3(t_6t_3) = t_1t_3t_5 && \text{relation : (6)} \\
 \implies t_1t_5t_3 &= t_1t_3t_5 = t_1t_6 \in [15] && \text{equality}
 \end{aligned}$$

□

**Lemma 4.4.**  $t_1t_5t_6 \in [13]$

*Proof.* Given:  $t_4t_2t_1 = e$  by relation (1).

$$\begin{aligned}
 e = t_4t_2t_1 &\implies t_1 = t_4t_2 && R(t_1) \\
 \implies t_1t_5t_6 &= t_4t_2t_5t_6 && R(t_5t_6) \\
 &= (t_4t_2)(t_5t_6) = t_1t_3 && \text{relations : (1), (11)} \\
 \implies t_1t_5t_6 &= t_1t_3 \in [13]
 \end{aligned}$$

□

We now consider the double coset  $Nt_1t_3t_2$ . When we calculate the point stabilizer we notice there is no element that stabilizes 1, 2 and 3. Even though the  $N^{132} = \langle e \rangle$  Magma indicates to us that any permutation that fixes 1 and 3, and contains (2,4) belongs in the coset stabilizing group. We will now show  $[132] = [134]$  which will put (2,4)(5,6) in the coset stabilizing group.

**Lemma 4.5.**  $[132] = [134]$

*Proof.* Given:  $yt_1t_3t_1t_3t_1 = e$  by relation (13)

$$\begin{aligned}
 yt_1t_3t_1t_3t_1 = e &\implies yt_1t_3t_1 = t_1t_3 && R(t_1t_3) \\
 yt_1t_3t_1t_4 = t_1t_3t_4 &&& R(t_4) \\
 yt_1t_3(t_1t_4) = t_1t_3t_2 &&& \text{relation : (7)} \\
 \implies yt_1t_3t_2 = t_1t_3t_4 &&& 
 \end{aligned}$$

□

So we calculate the number of single cosets as follows:

$$\begin{aligned}
 |N^{(132)}| & \\
 &\geq |N^{132}| \\
 &\geq | \langle Id(G), (2,4)(5,6) \rangle | \geq 2
 \end{aligned}$$

Implying,  $\frac{|N|}{|N^{(132)}|} = \frac{12}{2} = 6$  and the orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}, \{3\}, \{2,4\}, \{5,6\}$ . Our possible new double cosets are  $Nt_1t_3t_2t_1, Nt_1t_3t_2t_3, Nt_1t_3t_2t_2$ , and  $Nt_1t_3t_2t_5$ . As before  $Nt_1t_3t_2t_2 \in [13]$  taking  $\{2,4\}$  with it.  $Nt_1t_3t_2t_3$  and  $Nt_1t_3t_2t_5$  turn out to be new double cosets and we prove  $Nt_1t_3t_2t_1 \in [132]$ .

**Lemma 4.6.**  $t_1t_3t_2t_1 \in [132]$

*Proof.* Given  $yt_1t_3t_2 = t_1t_3t_4$  (Lemma 4.5)

$$\begin{aligned}
 \text{Note : } t_1t_3(t_4) = t_1t_3t_2t_1 &&& \text{relation : (3)} \\
 \implies t_1t_3t_2t_1 = t_1t_3t_4 = yt_1t_3t_2 \in [132] &&& 
 \end{aligned}$$

□

We continue to the double coset  $Nt_1t_5t_1$ .

The number of single cosets for  $Nt_1t_5t_1$  are determined to be:

$$\begin{aligned} |N^{(151)}| & \\ & \geq |N^{151}| \\ & \geq | \langle Id(G) \rangle | \geq 1 \end{aligned}$$

That is,  $\frac{|N|}{|N^{(151)}|} = \frac{12}{1} = 12$  single cosets. The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{5\}$ , and  $\{6\}$ . Taking a representative from each orbit we determine which are new and which are sent to existing double cosets.  $Nt_1t_5t_1t_1 \in [15]$  while  $Nt_1t_5t_1t_3, Nt_1t_5t_1t_5$ , and  $Nt_1t_5t_1t_6$  all turn out to be new double cosets.

**Lemma 4.7.**  $t_1t_5t_1t_2 \in [154]$

*Proof.* Fact:  $t_1t_5t_1t_2 = t_1t_5t_1t_2$

$$t_1t_5t_1t_2 = t_1t_5(t_1t_2) = t_1t_5t_4 \in [154] \quad \text{relation : (12)}$$

□

**Lemma 4.8.**  $t_1t_5t_1t_4 \in [152]$

*Proof.* Fact:  $t_1t_5t_1t_4 = t_1t_5t_1t_4$

$$t_1t_5t_1t_4 = t_1t_5(t_1t_4) = t_1t_5t_2 \in [152] \quad \text{relation : (7)}$$

□

We will now investigate  $Nt_1t_5t_2$ . Once again, the point stabilizer  $N^{152} = \langle e \rangle$ , but we will show  $[152] = [625]$  which will increase the coset stabilizer.

**Lemma 4.9.**  $[152] = [625]$

*Proof.* Given:  $yt_2t_5t_2t_5t_2 = e$  relation: (15)

$$yt_2t_5t_2t_5t_2 = e \implies yt_2t_5 = t_2t_5t_2 \quad R(t_2t_5t_2)$$

$$t_4yt_2t_5 = t_4t_2t_5t_2\star \quad L(t_4)$$

$$\text{Note : } (t_4y)t_2t_5 = yt_6t_2t_5$$

$$\text{Note : } (t_4t_2)t_5t_2 = t_1t_5t_2 \quad \text{relation : (1)}$$

$$\star \implies yt_6t_2t_5 = t_1t_5t_2 \quad \square \quad \text{substitution}$$

□

Since,  $[152] = [625]$  any permutation sending 1 to 6, 5 to 2, and 2 to 5 belongs in the coset stabilizer  $N^{(152)}$ . We calculate the number of single cosets as follows:

$$\begin{aligned} |N^{(152)}| & \\ & \geq |N^{152}| \\ & \geq | \langle Id(G), (1, 6)(2, 5)(3, 4) \rangle | \geq 2 \end{aligned}$$

This implies there exist a total of  $\frac{|N|}{|N^{(152)}|} = \frac{12}{2} = 6$  single cosets. The orbits on  $\{1, 2, 3, 4, 5, 6\}$  are  $\{1, 6\}$ ,  $\{2, 5\}$ , and  $\{3, 4\}$ . Taking a representative from each of the orbits we have  $Nt_1t_5t_2t_1$ ,  $Nt_1t_5t_2t_2$ , and  $Nt_1t_5t_2t_4$ . As before,  $Nt_1t_5t_2t_2 \in [15]$ .

**Lemma 4.10.**  $t_1t_5t_2t_1 \in [154]$

*Proof.* Fact:  $t_1t_5t_2t_1 = t_1t_5t_2t_1$

$$t_1t_5t_2t_1 = t_1t_5(t_2t_1) = t_1t_5t_4 \in [154] \quad \text{relation : (3)}$$

□

**Lemma 4.11.**  $t_1t_5t_2t_4 \in [151]$

*Proof.* Fact:  $t_1t_5t_2t_4 = t_1t_5t_2t_4$

$$t_1t_5t_2t_4 = t_1t_5(t_2t_4) = t_1t_5t_1 \in [151] \quad \text{relation : (10)}$$

□

We will now continue on with the double coset  $Nt_1t_5t_4$ .

Investigating  $N^{(154)}$  we discover the following:

$$\begin{aligned} |N^{(154)}| & \\ & \geq |N^{154}| \\ & \geq | \langle Id(G) \rangle | \geq 1 \end{aligned}$$

The number of single cosets in  $[1 \ 5 \ 4] = \frac{|N|}{|N^{(154)}|} = \frac{12}{1} = 12$ . The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{5\}$ , and  $\{6\}$ . Our possible new double cosets are  $Nt_1t_5t_4t_1$ ,  $Nt_1t_5t_4t_2$ ,  $Nt_1t_5t_4t_3$ ,  $Nt_1t_5t_4t_4$ ,  $Nt_1t_5t_4t_5$ , and  $Nt_1t_5t_4t_6$ .  $Nt_1t_5t_4t_4 \in [15]$  while the only new double coset turns out to be  $Nt_1t_5t_4t_3$ .

**Lemma 4.12.**  $t_1t_5t_4t_1 \in [152]$

*Proof.* Fact:  $t_1t_5t_4t_1 = t_1t_5t_4t_1$

$$t_1t_5t_4t_1 = t_1t_5(t_4t_1) = t_1t_5t_2 \in [152] \quad \text{relation : (8)}$$

□

**Lemma 4.13.**  $t_1t_5t_4t_2 \in [151]$

*Proof.* Fact:  $t_1t_5t_4t_2 = t_1t_5t_4t_2$

$$t_1t_5t_4t_2 = t_1t_5(t_4t_2) = t_1t_5t_1 \quad \text{relation : (1)}$$

□

**Lemma 4.14.**  $t_1t_5t_4t_5 \in [1516]$

*Proof.* Given:  $z^{-1}t_3t_2t_3t_1t_2t_5t_4t_5t_1 = e$  by relation (20)

$$z^{-1}t_3t_2t_3t_1t_2t_5t_4t_5t_1 = e \implies z^{-1}t_3t_2t_3t_1t_2 = t_1t_5t_4t_5\star \quad R(t_1t_5t_4t_5)$$

$$z^{-1}t_3t_2t_3(t_1t_2) = z^{-1}t_3t_2t_3t_4\star \quad \text{relation : (12)}$$

$$t_1t_5t_4t_5 = z^{-1}t_3t_2t_3t_4 \in [1516]. \quad \text{by : } \star$$

□

**Lemma 4.15.**  $t_1t_5t_4t_6 \in [1325]$

*Proof.* Fact:  $t_1t_5t_4t_6 = t_1t_5t_4t_6$

$$t_1t_5t_4t_6 = t_1t_5(t_4t_6) = t_1t_5(yt_4t_6t_4) \quad \text{relation : (17)}$$

$$= (t_1t_5y)t_4t_6t_4 = yt_3t_2t_4t_6t_4$$

$$= yt_3(t_2t_4)t_6t_4 = yt_3t_1t_6t_4 \quad (10)$$

$$\implies t_1t_5t_4t_6 = yt_3t_1t_6t_4 \in [1325] \quad \text{equality}$$

□

We investigate the new double coset  $Nt_1t_3t_2t_3$ .

This next double coset turns out to have equal names. We determine that  $[1323] = [1343]$  as we recall:

**Lemma 4.16.**  $[1323] = [1343]$

*Proof.* Given:  $yt_1t_3t_2 = t_1t_3t_4$  (Lemma 4.6)

$$yt_1t_3t_2 = t_1t_3t_4 \implies yt_1t_3t_2t_3 = t_1t_3t_4t_3 \quad \square \quad R(t_3)$$

□

We calculate the number of single cosets as:

$$\begin{aligned}
|N^{(1323)}| & \\
&\geq |N^{1323}| \\
&\geq | \langle Id(G), (2,4)(5,6) \rangle | \geq 2
\end{aligned}$$

The number of single cosets in  $[1\ 3\ 2\ 3] = \frac{|N|}{|N^{(1323)}|} = \frac{12}{2} = 6$ . The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}$ ,  $\{3\}$ ,  $\{2, 4\}$ ,  $\{5, 6\}$ .  $Nt_1t_3t_2t_3t_1$  turns out to be our final new double coset as we will prove later.  $Nt_1t_3t_2t_3t_3 \in [132]$  as usual. We discover  $Nt_1t_3t_2t_3t_2 \in [1323]$  as indicated by the following:

**Lemma 4.17.**  $t_1t_3t_2t_3t_2 \in [1323]$

*Proof.* Given:  $zt_3t_4t_3t_1t_4t_6t_2t_6t_1 = e$  by (19)

$$\begin{aligned}
zt_3t_4t_3(t_1t_4)t_6t_2t_6t_1 = e &\implies zt_3t_4t_3t_2t_6t_2t_6t_1 = e && \text{relation : (7)} \\
\implies zt_3t_4t_3t_2 = t_1t_6t_2t_6 &&& R(t_1t_6t_2t_6) \\
\implies (t_2zt_3t_4t_3t_2 = t_2t_1t_6t_2t_6) = \star &&& L(t_2) \\
\text{Note : } (t_2z)t_3t_4t_3t_2 = zt_1t_3t_4t_3t_2 &&& \\
\implies z(t_1t_3t_4)t_3t_2 = zyt_1t_3t_2t_3t_2 &&& \text{Lemma(4.6)} \\
\text{Note : } (t_2t_1)t_6t_2t_6 = t_4t_6t_2t_6 &&& \text{relation : (3)} \\
\star = zyt_1t_3t_2t_3t_2 = t_4t_6t_2t_6 &&& \text{Recall} \\
\implies zyt_1t_3t_2t_3t_2 = t_4t_6t_2t_6 \in [1323] &&& \text{substitution} \\
\implies t_1t_3t_2t_3t_2 = z^{-1}yt_4t_6t_2t_6 \in [1323] &&& L(z^{-1}y)
\end{aligned}$$

□



**Lemma 4.18.**  $t_1t_3t_2t_3t_6 \in [1325]$

*Proof.* Given:  $yt_1t_3t_2 = t_1t_3t_4$  Lemma(4.6)

$$\begin{aligned}
 yt_1t_3t_2 = t_1t_3t_4 &\implies (yt_1t_3t_2t_3t_6 = t_1t_3t_4t_3t_6) = \star && R(t_3t_6) \\
 \text{Note : } t_1t_3t_4(t_3t_6) &= t_1t_3t_4t_5 && \text{relation : (5)} \\
 = (t_1t_3t_4)t_5 &= yt_1t_3t_2t_5 && \text{Lemma(4.6)} \\
 \star = yt_1t_3t_2t_3t_6 &= yt_1t_3t_2t_5 \in [1325] \quad \square && \text{Substitution}
 \end{aligned}$$

□

We look into the coset stabilizer  $N^{(1325)}$  and come to the following conclusion:

$$\begin{aligned}
 |N^{(1325)}| & \\
 &\geq |N^{1325}| \\
 &\geq | \langle Id(G) \rangle | \geq 1
 \end{aligned}$$

We calculate the single cosets in  $[1\ 3\ 2\ 5]$  as  $\frac{|N|}{|N^{(1325)}|} = \frac{12}{1} = 12$  single cosets. The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{5\}$ , and  $\{6\}$ . We know  $Nt_1t_3t_2t_5t_6 \in [132]$ . We will prove where each of the remaining possible new double cosets are sent.

**Lemma 4.19.**  $t_1t_3t_2t_5t_1 \in [1516]$

*Proof.* Given:  $zt_3t_4t_3t_2 = t_1t_6t_2t_6$  Lemma 4.17 (line 2)

$$\begin{aligned}
 (zt_3t_4t_3t_2)^y &= (t_1t_6t_2t_6)^y \implies (zt_1t_6t_1t_5 = t_3t_4t_5t_4) = \star && \text{conj.} \\
 \text{Note : } t_3(t_4)t_5t_4 &= t_3t_1t_2t_5t_4 && \text{relation : (12)} \\
 = t_3t_1(t_2t_5)t_4 &= t_3t_1(yt_2t_5t_2)t_4 && \text{relation : (15)} \\
 = (t_3t_1y)t_2t_5t_2t_4 &= yt_1t_3t_2t_5t_1 \\
 \star \implies yt_1t_3t_2t_5t_1 &= zt_1t_6t_1t_5 && \text{Substitution} \\
 \implies t_1t_3t_2t_5t_1 &= yt_1t_6t_1t_5 \in [1516]
 \end{aligned}$$

□

**Lemma 4.20.**  $t_1t_3t_2t_5t_2 \in [154]$

*Proof.* Fact:  $t_1t_3t_2t_5t_2 = t_1t_3t_2t_5t_2$

$$\begin{aligned}
 t_1t_3t_2t_5t_2 &= t_1t_3(t_2t_5t_2) = t_1t_3(yt_2t_5) && \text{relation : (15)} \\
 &= (t_1t_3y)t_2t_5 = yt_3t_1t_2t_5 \\
 &= yt_3(t_1t_2)t_5 = yt_3t_4t_5 && \text{relation : (12)} \\
 \implies t_1t_3t_2t_5t_2 &= yt_3t_4t_5 \in [154] && \text{equality}
 \end{aligned}$$

□

**Lemma 4.21.**  $t_1t_3t_2t_5t_3 \in [1325]$

*Proof.* Given:  $yt_1t_3t_2 = t_1t_3t_4$  (Lemma 4.6)

$$\begin{aligned}
 yt_1t_3t_2 &= t_1t_3t_4 \implies t_1t_3t_2 = yt_1t_3t_4 && L(y) \\
 \implies t_1t_3t_2t_5t_3 &= yt_1t_3t_4(t_5t_3) && R(t_5t_3) \\
 &= yt_1t_3t_4(t_5t_3) = yt_1t_3t_4t_6 && \text{relation : (4)} \\
 \implies t_1t_3t_2t_5t_3 &= yt_1t_3t_4t_6 \in [1325] && \text{equality}
 \end{aligned}$$

□

**Preliminary work for Lemma 4.22:**

The following relations will be used to prove Lemma 4.22 we will call the result  $\alpha$ :

*Proof.* Given :  $t_6t_2t_5 = yt_1t_5t_2$  result from Lemma(4.9)

$$\begin{aligned}
 (t_6t_2t_5)^{(1,4)(3,6)} &= yt_1t_5t_2^{(1,4)(3,6)} \implies t_3t_2t_5 = yt_4t_5t_2 && \text{conj.} \\
 t_3t_2t_5t_4 &= yt_4t_5t_2t_4 && R(t_4) \\
 &= yt_4t_5(t_2t_4) = yt_4t_5t_1 && \text{relation : (10)} \\
 \implies (t_3t_2t_5t_4 = yt_4t_5t_1) &= \alpha && \text{equality}
 \end{aligned}$$

□

Notes about the proof of Lemma 4.22: We will obtain  $zyt_2t_6t_1t_5$  on the right side and then show the left side simplifies to  $t_1t_3t_2t_5t_4$  only using relation substitution that will not effect the right hand side.

**Lemma 4.22.**  $t_1t_3t_2t_5t_4 \in [1543]$

*Proof.* Given:  $z^{-1}t_5t_4t_5t_2t_4t_6t_1t_6t_2 = e$  by relation (23)

$$z^{-1}t_5t_4t_5t_2t_4t_6t_1t_6t_2 = e \implies z^{-1}t_5t_4t_5t_2t_4 = t_2t_6t_1t_6 \quad R(t_2t_6t_1t_6)$$

$$z^{-1}t_5t_4t_5t_2t_4 = t_2t_6t_1(t_6) = t_2t_6t_1t_5t_3 \quad \text{relation : (4)}$$

$$z^{-1}t_5t_4t_5t_2t_4 = t_2t_6t_1t_5t_3 \implies z^{-1}t_5t_4t_5t_2t_4t_3 = t_2t_6t_1t_5 \quad R(t_3)$$

$$z^{-1}t_5(t_4)t_5t_2t_4t_3 = t_2t_6t_1t_5 \implies z^{-1}t_5t_1t_2t_5t_2t_4t_3 = t_2t_6t_1t_5 \quad \text{relation : (12)}$$

$$z^{-1}t_5t_1(t_2t_5t_2)t_4t_3 = t_2t_6t_1t_5 \implies z^{-1}t_5t_1(yt_2t_5)t_4t_3 = t_2t_6t_1t_5 \quad \text{relation : (15)}$$

$$\begin{aligned} z^{-1}(t_5t_1y)t_2t_5t_4t_3 = t_2t_6t_1t_5 &\implies z^{-1}yt_2t_3t_2t_5t_4t_3 = t_2t_6t_1t_5 \\ &\implies t_2t_3t_2t_5t_4t_3 = yzt_2t_6t_1t_5 \star \quad R(yz) \end{aligned}$$

$$\text{Note : } t_2(t_3t_2t_5t_4)t_3 = t_2(yt_4t_5t_1)t_3 \quad \text{relation : } \alpha$$

$$= (t_2y)t_4t_5t_1t_3 = yt_5t_4t_5t_1t_3$$

$$= yt_5t_4t_5(t_1t_3) = yt_5t_4t_5(yt_1t_3t_1) \quad \text{relation : (13)}$$

$$= (yt_5t_4t_5y)t_1t_3t_1 = t_2t_6t_2t_1t_3t_1$$

$$= t_2t_6t_2t_1(t_3)(t_1) = t_2t_6t_2t_1t_6t_5t_2t_4 \quad \text{relations : (2), (10)}$$

$$= t_2t_6t_2t_1t_6(t_5t_2)t_4 = t_2t_6t_2t_1t_6(yt_5t_2t_5)t_4 \quad \text{relation : (15)}$$

$$= (t_2t_6t_2t_1t_6y)t_5t_2t_5t_4 = yt_5t_4t_5t_3t_4t_5t_2t_5t_4$$

$$= yt_5t_4(t_5t_3)t_4(t_5)t_2t_5t_4 = yt_5t_4(t_6)t_4(t_6t_3)t_2t_5t_4 \quad \text{relations : (4), (6)}$$

$$= yt_5(t_4t_6t_4t_6)t_3t_2t_5t_4 = yt_5(yt_4)t_3t_2t_5t_4 \quad \text{relation : (18)}$$

$$= (yt_5y)t_4t_3t_2t_5t_4 = t_2t_4t_3t_2t_5t_4$$

$$= (t_2t_4)t_3t_2t_5t_4 = t_1t_3t_2t_5t_4 \quad \text{relation : (10)}$$

$$\star = t_2t_3t_2t_5t_4t_3 = zyt_2t_6t_1t_5 \quad \text{Recall}$$

$$\implies t_1t_3t_2t_5t_4 = t_2t_3t_2t_5t_4t_3 = zyt_2t_6t_1t_5 \in [1543] \quad \text{equality}$$

□

**Lemma 4.23.**  $t_1t_3t_2t_5t_6 \in [1323]$

*Proof.* Fact:  $t_1t_3t_2t_5t_6 = t_1t_3t_2t_5t_6$

$$t_1t_3t_2t_5t_6 = t_1t_3t_2(t_5t_6) = t_1t_3t_2t_3 \in [1323] \quad \text{relation : (11)}$$

□

With the double coset [1325] fully investigated we move on to the double coset [1513]. We determine there is an equal name for [1513].

**Preliminary work for Lemma 4.24:**

*Proof.* Given:  $t_6t_2t_5 = yt_1t_5t_2$  by Lemma(4.9)

$$(t_6t_2t_5)^{(1,3)(2,6)(4,5)} = (yt_1t_5t_2)^{(1,3)(2,6)(4,5)} \implies (t_2t_6t_4 = yt_3t_4t_6) = \alpha_1 \quad \text{conj.}$$

□

*Proof.* Given:  $zt_3t_4t_3t_2 = t_1t_6t_2t_6$  by Lemma(4.17) line 2

$$(zt_3t_4t_3t_2)^{(1,6)(2,5)(3,4)} = (t_1t_6t_2t_6)^{(1,6)(2,5)(3,4)} \implies (t_4t_3t_4t_5 = zt_6t_1t_5t_1) = \beta_1 \quad \text{conj.}$$

□

**Lemma 4.24.**  $[1513] = [3431]$

*Proof.* Given:  $zt_3t_4t_3t_2 = t_1t_6t_2t_6$  by Lemma(4.17) line 2

$$\begin{aligned}
\text{Note : } &zt_3t_4t_3(t_2) = zt_3t_4t_3t_1t_4 && \text{relation(7)} \\
\implies &zt_3t_4t_3t_1t_4 = t_1t_6t_2t_6 && \text{substitution} \\
\implies &(zt_3t_4t_3t_1 = t_1t_6t_2t_6t_4) = \star && R(t_4) \\
\text{Note : } &t_1t_6(t_2t_6t_4) = t_1t_6(yt_3t_4t_6) && \text{preliminaries : } \alpha_1 \\
&= (t_1t_6y)t_3t_4t_6 = yt_3t_4t_3t_4t_6 \\
&= yt_3t_4t_3t_4(t_6) = yt_3t_4t_3t_4t_5t_3 && \text{relation : (4)} \\
&= yt_3(t_4t_3t_4t_5)t_3 = yt_3(zt_6t_1t_5t_1)t_3 && \text{preliminaries : } \beta_1 \\
&= y(t_3z)t_6t_1t_5t_1t_3 = yzt_6t_6t_1t_5t_1t_3 \\
&= yz(t_6t_6)t_1t_5t_1t_3 = yzt_1t_5t_1t_3 && t_6^2 = e \\
\implies &t_1t_6t_2t_6t_4 = yzt_1t_5t_1t_3 && \text{equality} \\
\star = &zt_3t_4t_3t_1 = t_1t_6t_2t_6t_4 = yzt_1t_5t_1t_3 && \text{substitution} \\
\implies &yt_3t_4t_3t_1 = t_1t_5t_1t_3 && L(z^{-1}y)
\end{aligned}$$

□

This implies our coset stabilizer  $N^{(1513)}$  proceeds as follows

$$\begin{aligned}
|N^{(1513)}| & \\
&\geq |N^{1513}| \\
&\geq | \langle Id(G), (1, 3)(2, 6)(4, 5) \rangle | \geq 2
\end{aligned}$$

This implies our orbits on  $\{1,2,3,4,5,6\}$  are  $\{1, 3\}$ ,  $\{2, 6\}$ , and  $\{4, 5\}$ . The double coset  $Nt_1t_5t_1t_3t_3 \in [151]$ .

**Lemma 4.25.**  $t_1t_5t_1t_3t_2 \in [1515]$

*Proof.* Given:  $t_1t_5t_1t_3 = yt_3t_4t_3t_1$  result from Lemma(4.24)

$$\begin{aligned}
\text{Note : } t_1t_5t_1(t_3) &= t_1t_5t_1t_5t_6 && \text{relation : (11)} \\
t_1t_5t_1t_3 &= t_1t_5t_1t_5t_6 = yt_3t_4t_3t_1 && \text{substitution} \\
t_1t_5t_1t_5t_6 &= yt_3t_4t_3t_1 \implies (t_1t_5t_1t_5 = yt_3t_4t_3t_1t_6) = \star && R(t_6) \\
\text{Note : } y(t_3t_4t_3t_1)t_6 &= y(z^{-1}t_1t_6t_2t_6t_4)t_6 && \text{relation : (19)} \\
&= yz^{-1}t_1t_6t_2(t_6t_4t_6) = yz^{-1}t_1t_6t_2(yt_6t_4) && \text{relation : (18)} \\
&= (yz^{-1}t_1t_6t_2y)t_6t_4 = z^{-1}t_3t_4t_5t_6t_4 \\
&= z^{-1}t_3t_4(t_5t_6)t_4 = z^{-1}t_3t_4t_3t_4 && \text{relation : (11)} \\
&= z^{-1}t_3t_4t_3(t_4) = z^{-1}t_3t_4t_3t_1t_2 && \text{relation : (12)} \\
&= z^{-1}(t_3t_4t_3t_1)t_2 = z^{-1}yt_1t_5t_1t_3t_2 && \text{Line(2)} \\
&\implies yt_3t_4t_3t_1t_6 = z^{-1}yt_1t_5t_1t_3t_2 && \text{equality} \\
\star &= t_1t_5t_1t_5 = yt_3t_4t_3t_1t_6 = z^{-1}yt_1t_5t_1t_3t_2 && \text{substitution} \\
&\implies t_1t_5t_1t_3t_2 = yzt_1t_5t_1t_5 \in [1515] && L(yz)
\end{aligned}$$

□

**Preliminary work for Lemma 4.26**

*Proof.* Given:  $z^{-1}t_3t_2t_3t_4t_5t_4t_5t_1 = e$  relation (24) we declare the result  $\alpha_2$

$$z^{-1}t_3t_2t_3t_4t_5t_4t_5t_1 = e \implies z^{-1}t_3t_2t_3t_4 = t_1t_5t_4t_5 = \alpha_2 \quad R(t_1t_5t_4t_5)$$

□

**Lemma 4.26.**  $t_1t_5t_1t_3t_4 \in [1516]$

*Proof.* Given:  $t_1t_5t_1t_3t_2 = yzt_1t_5t_1t_5$  result from Lemma 4.25

$$\begin{aligned}
\text{Note : } t_1t_5t_1t_3t_2 &= t_1t_5t_1t_3t_4t_1 && \text{relation : (1)} \\
\implies t_1t_5t_1t_3t_4t_1 &= yzt_1t_5t_1t_5 && \text{substitution} \\
\implies t_1t_5t_1t_3t_4 &= yzt_1t_5t_1t_5t_1\star && R(t_1) \\
\text{Note : } yz(t_1t_5t_1t_5)t_1 &= yz(yt_3t_4t_3t_1t_6)t_1 && \text{Lemma(4.24)Line(3)} \\
&= zt_3t_4t_3(t_1)t_6(t_1) = zt_3t_4t_3t_2t_4t_6t_4t_2 && \text{relations : (10), (1)} \\
&= zt_3t_4t_3t_2(t_4t_6t_4)t_2 = zt_3t_4t_3t_2(yt_4t_6)t_2 && \text{relation : (17)} \\
&= z(t_3t_4t_3t_2y)t_4t_6t_2 = zyt_1t_6t_1t_5t_4t_6t_2 \\
&= zyt_1t_6t_1t_5t_4(t_6)t_2 = zyt_1t_6t_1t_5t_4t_5t_3t_2 && \text{relation : (4)} \\
&= zyt_1t_6(t_1t_5t_4t_5)t_3t_2 = zyt_1t_6(z^{-1}t_3t_2t_3t_4)t_3t_2 && \text{preliminary : } \alpha_2 \\
&= zy(t_1t_6z^{-1})t_3t_2t_3t_4t_3t_2 = yt_2t_3t_3t_2t_3t_4t_3t_2 \\
&= y(t_2(t_3t_3)t_2)t_3t_4t_3t_2 = yt_3t_4t_3t_2 && t_3^2 = t_2^2 = e \\
\implies yzt_1t_5t_1t_5t_1 &= yt_3t_4t_3t_2 && \text{equality} \\
\star \implies t_1t_5t_1t_3t_4 &= yzt_1t_5t_1t_5t_1 = yt_3t_4t_3t_2 \in [1516] && \text{substitution}
\end{aligned}$$

□

Moving on to the next double coset, we discover  $[1515]$  has an equal name.

**Lemma 4.27.**  $[1515] = [3434]$

*Proof.* Given:  $yzt_1t_5t_1t_5 = t_1t_5t_1t_3t_2$  by Lemma(4.25)

$$\begin{aligned}
yzt_1t_5t_1t_5 &= t_1t_5(t_1t_3t_2) = t_1t_5(yt_1t_3t_4) && \text{Lemma(4.5)} \\
&= (t_1t_5y)t_1t_3t_4 = yt_3t_2t_1t_3t_4 \\
&= yt_3(t_2t_1)t_3t_4 = yt_3t_4t_3t_4 && \text{relation : (3)} \\
\implies yzt_1t_5t_1t_5 &= yt_3t_4t_3t_4 && \text{equality} \\
\implies t_1t_5t_1t_5 &= z^{-1}t_3t_4t_3t_4 \implies [1515] = [3434] && L(z^{-1}y)
\end{aligned}$$

□

This implies our coset stabilizer  $N^{(1513)}$  proceeds as follows

$$\begin{aligned} |N^{(1515)}| & \\ & \geq |N^{1515}| \\ & \geq | \langle Id(G), (1, 3)(2, 6)(4, 5) \rangle | \geq 2 \end{aligned}$$

resulting in a total number of  $\frac{|N|}{|N^{(1515)}|} = \frac{12}{2} = 6$  single cosets. Once again, the orbits on  $\{1, 2, 3, 4, 5, 6\}$  are  $\{1, 3\}$ ,  $\{2, 6\}$ , and  $\{4, 5\}$  as we begin investigating our possible new double cosets.  $Nt_1t_5t_1t_5t_5 \in [151]$  so this takes the symmetric generators  $\{4\}$  and  $\{5\}$ . We now show the following relations:

**Lemma 4.28.**  $t_1t_5t_1t_5t_1 \in [1516]$

*Proof.*  $yzt_1t_5t_1t_5t_1 = yt_3t_4t_3t_2$  Lemma(4.26) result final line.

$$yzt_1t_5t_1t_5t_1 = yt_3t_4t_3t_2 \implies t_1t_5t_1t_5t_1 = z^{-1}t_3t_4t_3t_2 \in [1516] \quad R(yz^{-1})$$

□

**Lemma 4.29.**  $t_1t_5t_1t_5t_6 \in [1513]$

*Proof.* Fact:  $t_1t_5t_1t_5t_6 = t_1t_5t_1t_5t_6$

$$t_1t_5t_1t_5t_6 = t_1t_5t_1(t_5t_6) = t_1t_5t_1t_3 \in [1513] \quad \text{relation : (11)}$$

□

With this double coset completely investigated we move on to the double coset  $Nt_1t_5t_1t_6$ .



We determine the coset stabilizer

$$\begin{aligned} |N^{(1516)}| & \\ & \geq |N^{1516}| \\ & \geq | \langle Id(G) \rangle | \geq 1 \end{aligned}$$

This indicates there are  $\frac{|N|}{|N^{(1516)}|} = \frac{12}{1} = 12$  single cosets. The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{5\}$ , and  $\{6\}$ . There are no new double cosets left so we determine where each coset belongs. As before  $Nt_1t_5t_1t_6t_6 \in [151]$  and we prove the remaining relations.

### Preliminary work for Lemma 4.30

*Proof.* Given:  $zt_2t_3t_2t_6t_1t_6t_1t_5 = e$  by (26) we declare the result  $\alpha_3$ .

$$zt_2t_3t_2t_6t_1t_6t_1t_5 = e \implies (zt_2t_3t_2t_6 = t_5t_1t_6t_1) = \alpha_3 \quad R(t_5t_1t_6t_1)$$

□

**Lemma 4.30.**  $t_1t_5t_1t_6t_1 \in [1325]$

*Proof.* Fact:  $t_1t_5t_1t_6t_1 = t_1t_5t_1t_6t_1$

$$\begin{aligned} t_1t_5t_1t_6t_1 &= t_1(t_5t_1t_6t_1) = t_1(zt_2t_3t_2t_6) && \text{preliminary : } \alpha_3 \\ &= z(t_4t_2)t_3t_2t_6 = zt_1t_3t_2t_6 && \text{relation : (1)} \\ &= z(t_1t_3t_2)t_6 = z(yt_1t_3t_4)t_6 && \text{Lemma(4.5)} \\ \implies t_1t_5t_1t_6t_1 &= zyt_1t_3t_4t_6 \in [1325] && \text{equality} \end{aligned}$$

□

**Lemma 4.31.**  $t_1t_5t_1t_6t_2 \in [154]$

*Proof.* Given:  $t_1t_5t_1t_6t_1 = zyt_1t_3t_4t_6$  Lemma(4.30) result

$$\begin{aligned}
t_1t_5t_1t_6t_1 = zyt_1t_3t_4t_6 &\implies t_1t_5t_1t_6 = zyt_1t_3t_4t_6t_1 && R(t_1) \\
\implies t_1t_5t_1t_6t_2 = zyt_1t_3t_4t_6t_1t_2 &&& R(t_2) \\
zyt_1t_3t_4t_6(t_1t_2) = zyt_1t_3t_4t_6t_4 &&& \text{relation : (12)} \\
= zyt_1t_3(t_4t_6t_4) = zyt_1t_3yt_4t_6 &&& \text{relation : (17)} \\
= z(yt_1t_3y)t_4t_6 = zt_3t_1t_4t_6 &&& \\
= zt_3(t_1t_4)t_6 = zt_3t_2t_6 &&& \text{relation(7)} \\
\implies zyt_1t_3t_4t_6t_1t_2 = zt_3t_2t_6 &&& \text{equality} \\
\implies t_1t_5t_1t_6t_2 = zt_3t_2t_6 \in [154] &&& \text{substitution}
\end{aligned}$$

□

**Lemma 4.32.**  $t_1t_5t_1t_6t_3 \in [1515]$

*Proof.* Fact:  $t_1t_5t_1t_6t_3 = t_1t_5t_1t_6t_3$

$$t_1t_5t_1t_6t_3 = t_1t_5t_1(t_6t_3) = t_1t_5t_1t_5 \in [1515] \quad \text{relation : (6)}$$

□

**Lemma 4.33.**  $t_1t_5t_1t_6t_4 \in [1543]$

*Proof.* Given:  $t_1t_5t_1t_6t_2 = zt_3t_2t_6$  result from Lemma(4.31)

$$\begin{aligned}
t_1t_5t_1t_6t_2 = zt_3t_2t_6 &\implies t_1t_5t_1t_6t_2t_2t_4 = zt_3t_2t_6t_2t_4 && R(t_2t_4) \\
\implies t_1t_5t_1t_6t_4 = zt_3t_2t_6t_2t_4 &&& t_2^2 = e \\
\text{Note : } zt_3t_2t_6(t_2t_4) = zt_3t_2t_6t_1 &&& \text{relation : (10)} \\
\implies t_1t_5t_1t_6t_4 = zt_3t_2t_6t_1 \in [1543] &&& \text{substitution}
\end{aligned}$$

□

**Lemma 4.34.**  $t_1t_5t_1t_6t_5 \in [1513]$

*Proof.* Fact:  $t_1t_5t_1t_6t_5 = t_1t_5t_1t_6t_5$

$$t_1t_5t_1t_6t_5 = t_1t_5t_1(t_6t_5) = t_1t_5t_1t_3 \in [1513] \quad \text{relation : (2)}$$

□

From here we will investigate the double coset  $Nt_1t_5t_4t_3$  and prove its relations.

We discover that  $[1543] = [6234]$ . We prove this as follows:

**Lemma 4.35.**  $[1543] = [6234]$

*Proof.* Given:  $t_1t_5t_1t_6t_2 = zt_3t_2t_6$  result from Lemma(4.31)

$$\begin{aligned} t_1t_5t_1t_6t_2^{(1,4)(3,6)} &= (zt_3t_2t_6)^{(1,4)(3,6)} \\ \implies t_4t_5t_4t_3t_2 &= z^{-1}t_6t_2t_3 && \text{conj} \\ \implies z^{-1}t_6t_2t_3t_4 &= t_4t_5t_4t_3t_2t_4 && R(t_4) \\ \text{Note : } t_4t_5t_4t_3(t_2t_4) &= t_4t_5t_4t_3t_1 && \text{relation : (10)} \\ &= t_4t_5(t_4)t_3t_1 = t_4t_5t_2t_1t_3t_1 && \text{relation : (3)} \\ &= t_4t_5t_2(t_1t_3t_1) = t_4t_5t_2yt_1t_3 && \text{relation : (13)} \\ &= (t_4t_5t_2y)t_1t_3 = yt_6t_2t_5t_1t_3 \\ &= (yt_6t_2t_5)t_1t_3 = t_1t_5t_2t_1t_3 && \text{Lemma(4.9)} \\ &= t_1t_5(t_2t_1)t_3 = t_1t_5t_4t_3 && \text{relation : (3)} \\ \implies t_4t_5t_4t_3t_2t_4 &= t_1t_5t_4t_3 && \text{equality} \\ \implies z^{-1}t_6t_2t_3t_4 &= t_1t_5t_4t_3 \implies [1543] = [6234] && \text{substitution} \end{aligned}$$

□

This increases our coset stabilizer accordingly:

$$\begin{aligned}
|N^{(1543)}| & \\
& \geq |N^{1543}| \\
& \geq | \langle Id(G), (1, 6)(2, 5)(3, 4) \rangle | \geq 2
\end{aligned}$$

The orbits on  $\{1,2,3,4,5,6\}$  are  $\{1,6\}, \{2,5\}$ , and  $\{3,4\}$ .  $Nt_1t_5t_4t_3t_2 \in [154]$  as many times in the past. We investigate the other two relations we must show.

**Lemma 4.36.**  $t_1t_5t_4t_3t_1 \in [1516]$

*Proof.* Given  $zt_3t_4t_3t_2 = t_1t_6t_2t_6$  from Lemma(4.17) line 2

$$\begin{aligned}
(t_3t_4t_3t_2)^{(1,6,2,3,4,5)} &= (z^{-1}t_1t_6t_2t_6)^{(1,6,2,3,4,5)} = t_4t_5t_4t_3 = z^{-1}t_6t_2t_3t_2 && \text{conj.} \\
\text{Note : } z^{-1}t_6t_2t_3(t_2) &= z^{-1}t_6t_2t_3t_4t_1 && \text{relation : (8)} \\
&= (z^{-1}t_6t_2t_3t_4)t_1 = t_1t_5t_4t_3t_1 && \text{Lemma(4.35)} \\
\implies z^{-1}t_6t_2t_3t_2 &= t_1t_5t_4t_3t_1 && \text{equality} \\
\implies z^{-1}t_6t_2t_3t_2 &= t_1t_5t_4t_3t_1 = t_4t_5t_4t_3 \in [1516] && \text{substitution}
\end{aligned}$$

□

**Lemma 4.37.**  $t_1t_5t_4t_3t_2 \in [1325]$

*Proof.* Given  $t_1t_5t_4t_3 = z^{-1}t_6t_2t_3t_4$  result from Lemma(4.24)

$$\begin{aligned}
t_1t_5t_4t_3 &= z^{-1}t_6t_2t_3t_4 \implies t_1t_5t_4t_3t_2 = z^{-1}t_6t_2t_3t_4t_2 && R(t_2) \\
\text{Note : } z^{-1}t_6t_2t_3(t_4t_2) &= z^{-1}t_6t_2t_3t_1 && \text{relation : (1)} \\
&= z^{-1}t_6t_2(t_3t_1) = z^{-1}t_6t_2(yt_3t_1t_3) && \text{relation : (14)} \\
&= z^{-1}(t_6t_2y)t_3t_1t_3 = z^{-1}yt_4t_5t_3t_1t_3 \\
&= z^{-1}yt_4(t_5t_3)t_1t_3 = z^{-1}yt_4t_6t_1t_3 && \text{relation : (4)} \\
\implies z^{-1}t_6t_2t_3t_4t_2 &= z^{-1}yt_4t_6t_1t_3 && \text{equality} \\
\implies z^{-1}t_6t_2t_3t_4t_2 &= t_1t_5t_4t_3t_2 = z^{-1}yt_4t_6t_1t_3 \in [1325] && \text{substitution}
\end{aligned}$$

□

We move on to our final double coset  $Nt_1t_3t_2t_3t_1$  and show that this is the point where the group terminates.

This double coset has 12 equal names. This means that the coset stabilizing group is  $N$  itself. If we can put the elements  $\langle x, y, z \rangle \in N^{(13231)}$  then we will have effectively proved that the coset stabilizing group is all of  $N$  which will give us one single coset. We will first prove a relation that will help us show what elements belong in the coset stabilizing group. We will start by proving  $[13231] = [25152]$ .

### Preliminary work for Lemma 4.38

*Proof.* Given:  $t_1t_5t_1t_6t_2 = zt_3t_2t_6$  Lemma(4.31) we declare this result  $\alpha_4$ .

$$t_1t_5t_1t_6(t_2) = zt_3t_2t_6 \implies t_1t_5t_1t_6t_4t_1 = zt_3t_2t_6 \quad \text{relation : (8)}$$

$$\implies t_1t_5t_1t_6t_4 = zt_3t_2t_6t_1 \quad R(t_1)$$

$$\implies (z^{-1}t_1t_5t_1t_6t_4 = t_3t_2t_6t_1) = \alpha_4 \quad R(z^{-1})$$

□

**Lemma 4.38.**  $[13231] = [25152]$

*Proof.* Given:  $t_1t_3t_2t_5t_4 = yzt_2t_6t_1t_5$  Lemma(4.38) result

$$\begin{aligned}
t_1t_3t_2t_5t_4 = yzt_2t_6t_1t_5 &\implies t_1t_3t_2(t_5t_6)t_1 = yzt_2t_6t_1t_5t_4t_6t_1 && R(t_4t_6t_1) \\
\implies t_1t_3t_2t_3t_1 = yzt_2t_6t_1t_5t_4t_6t_1 &&& \text{relation : (11)} \\
\text{Note : } yzt_2t_6t_1t_5t_4(t_6)(t_1) = yzt_2t_6t_1t_5t_4t_3t_5t_2t_4 &&& \text{relations : (9), (10)} \\
= yzt_2t_6t_1t_5t_4t_3(t_5t_2)t_4 = yzt_2t_6t_1t_5t_4t_3(yt_5t_2t_5)t_4 &&& \text{relation : (16)} \\
= (yzt_2t_6t_1t_5t_4t_3y)t_5t_2t_5t_4 = zt_5t_4t_3t_2t_6t_1t_5t_2t_5t_4 &&& \\
= zt_5t_4(t_3t_2t_6t_1)t_5t_2t_5t_4 = zt_5t_4z^{-1}t_1t_5t_1t_6t_5t_2t_5t_4 &&& \text{preliminaries : } \alpha_4 \\
= z(t_5t_4z^{-1})t_1t_5t_1t_6t_5t_2t_5t_4 = t_6t_1t_1t_5t_1t_6t_4t_5t_2t_5t_4 &&& \\
= t_6(t_1t_1)t_5t_1t_6t_4t_5t_2t_5t_4 = t_6t_5t_1t_6t_4t_5t_2t_5t_4 &&& t_1^2 = e \\
= (t_6t_5)t_1t_6t_4t_5t_2t_5t_4 = t_3t_1t_6t_4t_5t_2t_5t_4 &&& \text{relation : (2)} \\
= (t_3t_1t_6t_4)t_5t_2t_5t_4 = (yt_1t_5t_4t_6t_5)t_2t_5t_4 &&& \text{Lemma(4.15)} \\
= yt_1t_5t_4(t_6)t_5t_2t_5t_4 = yt_1t_5t_4t_3t_5t_5t_2t_5t_4 &&& \text{relation : (9)} \\
= yt_1t_5t_4t_3(t_5t_5)t_2t_5t_4 = yt_1t_5t_4t_3t_2t_5t_4 &&& t_5^2 = e \\
= y(t_1t_5t_4t_3t_2)t_5t_4 = y(yz^{-1}t_4t_6t_1t_3)t_3t_6t_4 &&& \text{Lemma(4.37)} \\
= yy * z^{-1}t_4t_6t_1t_3t_3t_6t_4 = z^{-1}t_4t_6t_1t_6t_4 &&& t_3^2 = e \\
= z^{-1}(t_4t_6)t_1t_6t_4 = z^{-1}(yt_4t_6t_4)t_1t_6t_4 &&& \text{relation : (17)} \\
= z^{-1}yt_4t_6(t_4t_1)t_6t_4 = z^{-1}yt_4t_6t_2t_6t_4 &&& \text{relation : (8)} \\
\implies t_1t_3t_2t_3t_1 = z^{-1}yt_4t_6t_2t_6t_4 \in [25152] &&& 
\end{aligned}$$

□

Now that we have the relation  $t_1t_3t_2t_3t_1 = yz^{-1}t_4t_6t_2t_6t_4$  we can use this to see N is contained in the coset stabilizing group. We intend to obtain an elements of the same length and order of x,y, and z in the coset stabilizing group as this will generate the entire group N in the coset stabilizer.

Given:  $x$  representative:  $xz^{-1} = (1, 4)(3, 6)$

$$\begin{aligned} (t_1 t_3 t_2 t_3 t_1)^{(1,4)(3,6)} &= (yz^{-1} t_4 t_6 t_2 t_6 t_4)^{(1,4)(3,6)} \\ \implies t_4 t_6 t_2 t_6 t_4 &= yz t_1 t_3 t_2 t_3 t_1 \\ \implies (1, 4)(3, 6) &\in N^{(13231)} \end{aligned}$$

Note:  $Nt_1 t_3 t_2 t_3 t_1 = Nt_4 t_6 t_1 t_4 t_6$  from the proof of  $[13231] = [25152]$ .

Conjugating  $yz^{-1} t_4 t_6 t_2 t_6 t_4 = t_1 t_3 t_2 t_3 t_1$  by  $z = (1, 4, 2)(3, 6, 5)$  gives:

$$\begin{aligned} (yz^{-1} t_4 t_6 t_2 t_6 t_4)^{(1,4,2)(3,6,5)} &= (t_1 t_3 t_2 t_3 t_1)^{(1,4,2)(3,6,5)} \\ \implies yz^{-1} t_2 t_5 t_3 t_5 t_2 &= t_4 t_6 t_1 t_6 t_4 \in [13231] \\ \implies (1, 4, 2)(3, 6, 5) &\in N^{(13231)} \end{aligned}$$

Last we prove  $[13231] = [31513]$  as this will allow us to put  $y$  in the stabilizer.

**Lemma 4.39.**  $[13231] = [31513]$

*Proof.* Fact:  $t_1 t_3 t_2 t_3 t_1 = t_1 t_3 t_2 t_3 t_1$

$$\begin{aligned} t_1 t_3 t_2 t_3 t_1 &= (t_1 t_3 t_2) t_3 t_1 = y t_1 t_3 t_4 t_3 t_1 && \text{Lemma(4.5)} \\ &= y t_1 t_3 t_4 (t_3 t_1) = y t_1 t_3 t_4 (y t_3 t_1 t_3) && \text{relation : (14)} \\ &= y (t_1 t_3 t_4 y) t_3 t_1 t_3 = t_3 t_1 t_6 t_3 t_1 t_3 \\ &= t_3 t_1 (t_6 t_3) t_1 t_3 = t_3 t_1 t_5 t_1 t_3 && \text{relation : (6)} \\ \implies t_1 t_3 t_2 t_3 t_1 &= t_3 t_1 t_5 t_1 t_3 \implies [13231] = [31513] && \text{equality} \end{aligned}$$

□

Conjugating the relation  $t_1t_3t_2t_3t_1 = t_3t_1t_5t_1t_3$  by  $y = (1,3)(2,5)(4,6)$  gives:

$$\begin{aligned} (t_1t_3t_2t_3t_1)^{(1,3)(2,5)(4,6)} &= (t_3t_1t_5t_1t_3)^{(1,3)(2,5)(4,6)} \\ \implies t_3t_1t_5t_1t_3 &= t_1t_3t_2t_3t_1 \in [13231] \\ \implies y &= (1,3)(2,5)(4,6) \in N^{(13231)} \end{aligned}$$

Finally with  $\langle xz^{-1}, y, z \rangle \in N^{(13231)}$  we have  $N \in N^{(13231)}$  which implies

$$\begin{aligned} |N^{(13231)}| &\geq |N^{13231}| \\ &\geq |\langle Id(G), xz^{-1}, y, z \rangle| \geq 12 \end{aligned}$$

This implies all of distinct double cosets  $[13231] = [25152] = [31513] = [46164] = [52325] = [13431] = [64346] = [46264] = [25452] = [31613] = [64546] = [52625]$  are equal. Which shows  $\frac{|N|}{|N^{(13231)}|} = \frac{12}{12} = 1$ . With the orbits on  $\{1,2,3,4,5,6\}$  being  $\{1,2,3,4,5,6\}$ . Allowing  $Nt_1t_3t_2t_3t_1t_1 \in Nt_1t_3t_2t_3$  and all t's are carried back, thus completing the double coset enumeration of  $PGL_2(11)$  over  $D_{12}$ . Which gives us the following Cayley Diagram.



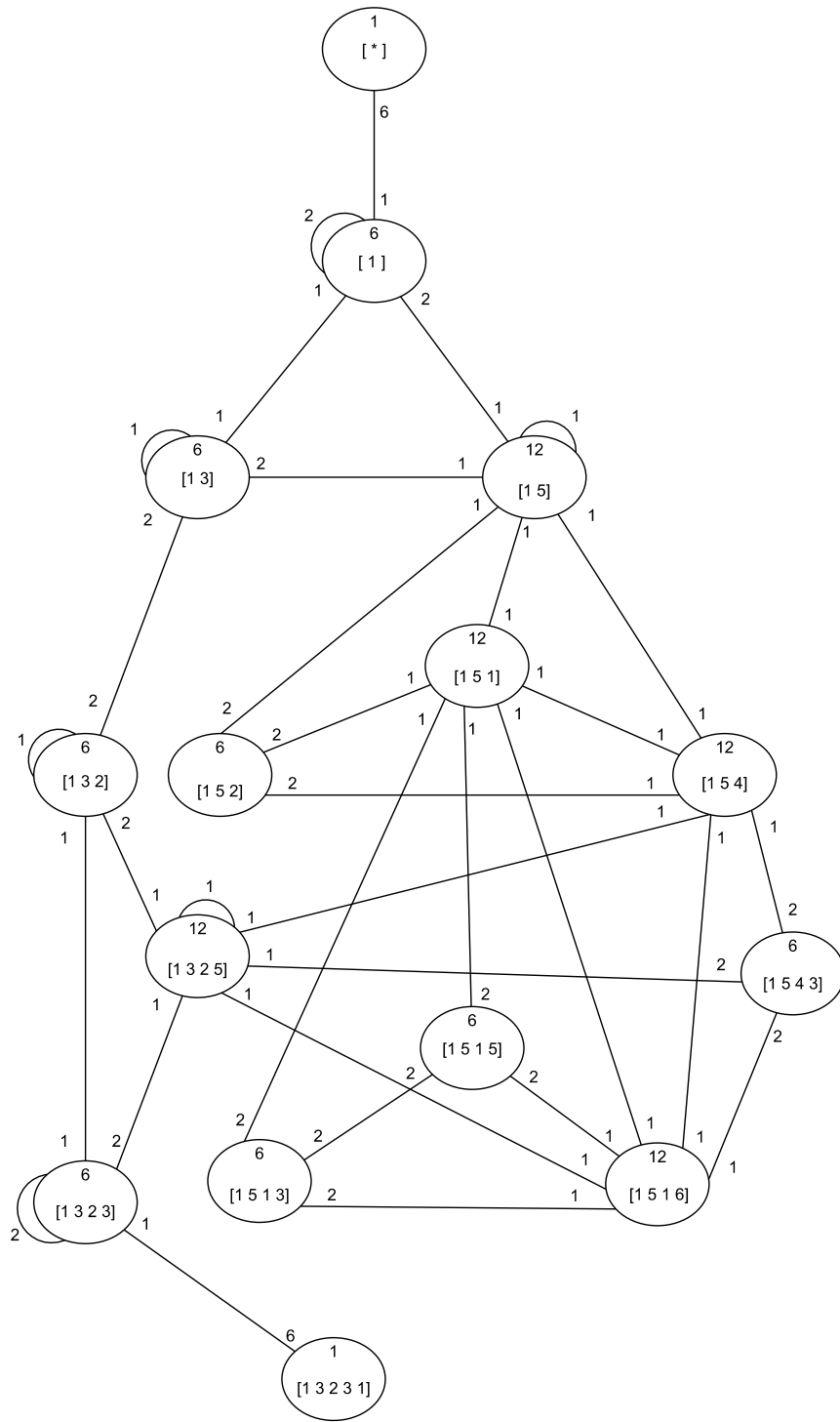


Figure 4.2: Cayley Diagram  $PGL_2(11)$  over  $D_{12}$

## 4.4 Double Coset Enumeration over a Maximal Subgroup

Double coset enumeration is typically performed over the control group  $N$ . At times the process becomes cumbersome when the number of double cosets is a considerably large number. For example, when calculating the double coset enumeration of  $M_{12} : 2$  over  $2^2 : (2^2 \times 3)$ , there are a total of 98 double cosets which we check using Magma. In this case, we will be required to determine how the 12  $t$ 's interact amongst the 98 double cosets. This could result in determining where 1,176 cosets are sent which while possible, is a very challenging task.

### 4.4.1 Is Maximal Double Coset Enumeration Applicable?

Double coset enumeration over a maximal subgroup may not always be possible. For example, consider  $\frac{2^5:5:4}{(2,5,4,3)t_3t_2t_5t_4t_3t_2t_5, (1,3,4,2,5)t_4t_5t_3t_2t_1t_4t_5} \cong Sz(8) \times 2$ . We can factor out the 2 from our  $Sz(8) \times 2$  leaving only the  $Sz(8)$  of order 29,120. If we conduct our checks as we did in section 4.2 we have  $\frac{|G|}{|N|} = \frac{29,120}{20} = 1456$  single cosets and a total of 79 double cosets. While definitely possible to complete by hand it is unnecessary if we can find a suitable maximal subgroup of  $Sz(8)$ . Inputting the following code:

```
G<x, y, z, t>:=Group<x, y, z, t|x^4, y^2, z^5, t^2, x^-2*y, (y*z^-1)^2,
x*z^-1*x^-1*z^-2, (y*t*z^-1)^2, z^-1*t*z*x*t*x^-1, (t*x^-1)^7,
(t*z^-1)^7, x^-1*t*z*t*x^-1*t*x*t*z*t*y*t*y*t*x*t*z^-1*t*y*t>;
```

```
[1] Order 14          Length 2080
    Permutation group acting on a set of cardinality 29120
    Order = 14 = 2 * 7
[2] Order 20          Length 1456
    Permutation group acting on a set of cardinality 29120
    Order = 20 = 2^2 * 5
[3] Order 52          Length 560
    Permutation group acting on a set of cardinality 29120
    Order = 52 = 2^2 * 13
[4] Order 448         Length 65
    Permutation group acting on a set of cardinality 29120
    Order = 448 = 2^6 * 7
```

and using the knowledge that the order of a subgroup must divide the order of the group it is contained in we see that only  $M[2]$  subgroup is suitable to contain an  $N$  of order 20. This of course doesn't help us any as our original  $N$  is of order 20, so unless we decide to perform the double coset enumeration over  $N$ , we can save this discovery for later. While this example did not produce a suitable  $M$ , we will now investigate an example which will.

#### 4.4.2 Double Coset Enumeration of $M_{12} : 2$ over Maximal Subgroup $2^4 : \bullet S_4$

We consider  $M_{12} : 2$  over  $2^2 : (2^2 \times 3)$  again as mentioned previously. As before if we attempt the double coset enumeration over  $N$ , we will have roughly 1,200 calculations and 98 double cosets. Fortunately, this case does have a suitable maximal subgroup that allows us to perform double coset enumeration over  $M$ . To begin, we repeat the process of asking Magma for Maximal subgroups of  $M$  as follows:

```
> M:=MaximalSubgroups(G1);
> M;
Conjugacy classes of subgroups
-----

[ 1]   Order 120           Length 1584
       Permutation group acting on a set of cardinality 3960
       Order = 120 = 2^3 * 3 * 5
[ 2]   Order 144           Length 1320
       Permutation group acting on a set of cardinality 3960
       Order = 144 = 2^4 * 3^2
[ 3]   Order 216           Length 880
       Permutation group acting on a set of cardinality 3960
       Order = 216 = 2^3 * 3^3
[ 4]   Order 1320          Length 144
       Permutation group acting on a set of cardinality 3960
       Order = 1320 = 2^3 * 3 * 5 * 11
[ 5]   Order 1320          Length 144
       Permutation group acting on a set of cardinality 3960
       Order = 1320 = 2^3 * 3 * 5 * 11
[ 6]   Order 480           Length 396
```

- Permutation group acting on a set of cardinality 3960  
Order =  $480 = 2^5 * 3 * 5$
- [ 7] Order 384                      Length 495  
Permutation group acting on a set of cardinality 3960  
Order =  $384 = 2^7 * 3$
- [ 8] Order 384                      Length 495  
Permutation group acting on a set of cardinality 3960  
Order =  $384 = 2^7 * 3$
- [ 9] Order 95040                      Length 1  
Permutation group acting on a set of cardinality 3960  
Order =  $95040 = 2^6 * 3^3 * 5 * 11$

In this case, we have 4 suitable candidates which are all divisible by  $N^{2^2} : (2^2 \times 3)$ . We then ask Magma to compute conjugates and determine that  $M[7]$ 's subgroup of order 384 is the only maximal subgroup containing  $N$ . Now we proceed with our  $G = \frac{2^{*12}:2^2:(2^2 \times 3)}{xt_{12}t_{11}t_{12}, z^2t_5t_{12}t_9t_5t_{12}}$  and  $M = \langle f(x), f(y), f(z), f(w), f(v), f(z * t * v * t * z * v * t * z^{-1} * t * z * t * z^{-1} * t) \rangle$  where  $M \cong (2^4) : \bullet S_4$ .

Consider the first double coset of our Cayley diagram. Since we are performing double coset enumeration over  $M$  our definition becomes  $MwN = \{Mw^n | n \in N\}$ . That is,  $MeN = \{Me^n | n \in N\} = \{Me | e \in N\} = \{M\}$ . We denote this first double coset as  $[*]$  containing just one single coset. Our  $N$  is transitive on  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  so it has a single orbit  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ . Now taking a representative from this orbit, 12 in this case, and right multiplying to the double coset  $Me$  we obtain the new double coset  $Mt_{12}N = \{Mt_{12}^n | n \in N\} = \{Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7, Mt_8, Mt_9, Mt_{10}, Mt_{11}, Mt_{12}\}$ . Now, considering the coset stabilizer  $M^{(12)}$ , which is equal to the point stabilizer  $M^{12}$ .  $M^{(1)} = \langle (1, 5)(2, 3)(4, 7)(6, 9), (1, 2)(3, 5)(4, 9)(6, 7) \rangle = \{e, (1, 5)(2, 3)(4, 7)(6, 9), (1, 2)(3, 5)(4, 9)(6, 7), (1, 3)(2, 5)(4, 6)(7, 9)\}$  so the number of double cosets in  $Mt_{12}N$  denoted  $[12]$  is at most  $\frac{|N|}{|M^{(12)}|} = \frac{48}{4} = 12$ . Now, considering the generators of  $M^{(12)}$  we determine the orbits on  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  are  $\{8\}, \{10\}, \{11\}, \{12\}, \{1, 5, 2, 3\}, \{4, 7, 9, 6\}$ . We proceed by taking a representative from each orbit, in this case,  $\{1\}, \{4\}, \{8\}, \{10\}, \{11\}, \{12\}$ , and determine which double coset they belong to. Since our  $t$ 's are order 2 we know,  $Mt_{12}t_{12} \in M \in [*]$ . From our first relation we have  $Mt_{12}t_{11}N = Mt_{12}N \in [12]$ . Using conjugation, we determine the relations

$$Mt_{12}t_8 = Mt_{12}$$

$$Mt_{12}t_{10} = Mt_{12}$$

Thus these two double cosets also exist in [12] as well. We then determine the double cosets  $Mt_{12}t_1$  and  $Mt_{12}t_4$  are both new double cosets and move forward as [12 1] and [12 4] respectively.

Now we consider the next double coset  $Mt_{12}t_1N = \{M(t_{12}t_1)^n | n \in N\} = \{Mt_{12}t_1, Mt_4t_{10}, Mt_{10}t_2, Mt_3t_4, Mt_7t_{10}, \dots, Mt_3t_6\}$  for a total of 48 distinct, equivalent single cosets. The coset stabilizer  $N^{(121)} \geq N^{121} = \langle e \rangle$ . This implies, the number of single cosets is at most  $\frac{|N|}{|M^{(121)}|} = \frac{48}{1} = 48$ . The orbits on  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  are  $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$  and we will now determine to which double coset  $Mt_{12}t_1t_1N, Mt_{12}t_1t_2N, Mt_{12}t_1t_3N, Mt_{12}t_1t_4N, Mt_{12}t_1t_5N, Mt_{12}t_1t_6N, Mt_{12}t_1t_7N, Mt_{12}t_1t_8N, Mt_{12}t_1t_9N, Mt_{12}t_1t_{10}N, Mt_{12}t_1t_{11}N,$  and  $Mt_{12}t_1t_{12}N$  belong. Obviously,  $Mt_{12}t_1t_1N = Mt_{12} \in [12]$ . We now consider  $Mt_{12}t_1t_2$  and determine which double coset it belongs to. Since we are working in a maximal subgroup M of G rather than N, our relations are no longer equal to the identity as earlier examples and are now equal to some permutation found in M. With the use of Magma, we can determine this relation and verify where cosets belong. We find  $Mt_{12}t_1t_2 \in [12 1]$  and determine using Magma that the relation  $Mt_{12}t_1t_2 = Mx \cdot t_{11}t_1$ . Since  $x \in M$  we have,  $Mt_{12}t_1t_2 = Mt_{11}t_1$ . Now, using our definition of a double coset defined to be  $Mt_{12}t_1N = \{M(t_{12}t_1)^n | n \in N\}$ , we conjugate  $Mt_{12}t_1N$  by the permutation  $(4, 7)(6, 9)(8, 10)(11, 12) \in N$  such that:  
 $M(t_{12}t_1)^{(4,7)(6,9)(8,10)(11,12)} = Mt_{11}t_1$  implying  $Mt_{12}t_1t_2 = Mt_{11}t_1 = Mt_{12}t_1 \in [12 1]$ .

This process can then be repeated on the remaining cosets in similar fashion with different relations to determine where cosets belong. Those relations produce the following list:

$$Mt_{12}t_1t_3 = Mt_{12}t_4 \in [12 4],$$

$$Mt_{12}t_1t_4 \in [12 1 4],$$

$$Mt_{12}t_1t_5 \in [12\ 1\ 5],$$

$$Mt_{12}t_1t_6 = Mt_{12}t_4 \in [12\ 4],$$

$$Mt_{12}t_1t_7 \in [12\ 1\ 7],$$

$$Mt_{12}t_1t_8 \in [12\ 1\ 8],$$

$$Mt_{12}t_1t_9 \in [12\ 1\ 9],$$

$$Mt_{12}t_1t_{10} \in [12\ 1\ 10],$$

$$Mt_{12}t_1t_{11} \in [12\ 1\ 11],$$

$$\text{and } Mt_{12}t_1t_{12} = Mt_{12}t_1 \in [12\ 1].$$

This process was also repeated for the remaining 14 double cosets and established the following Cayley diagram. The Magma code for the remaining double cosets can be found in the appendix of this document.

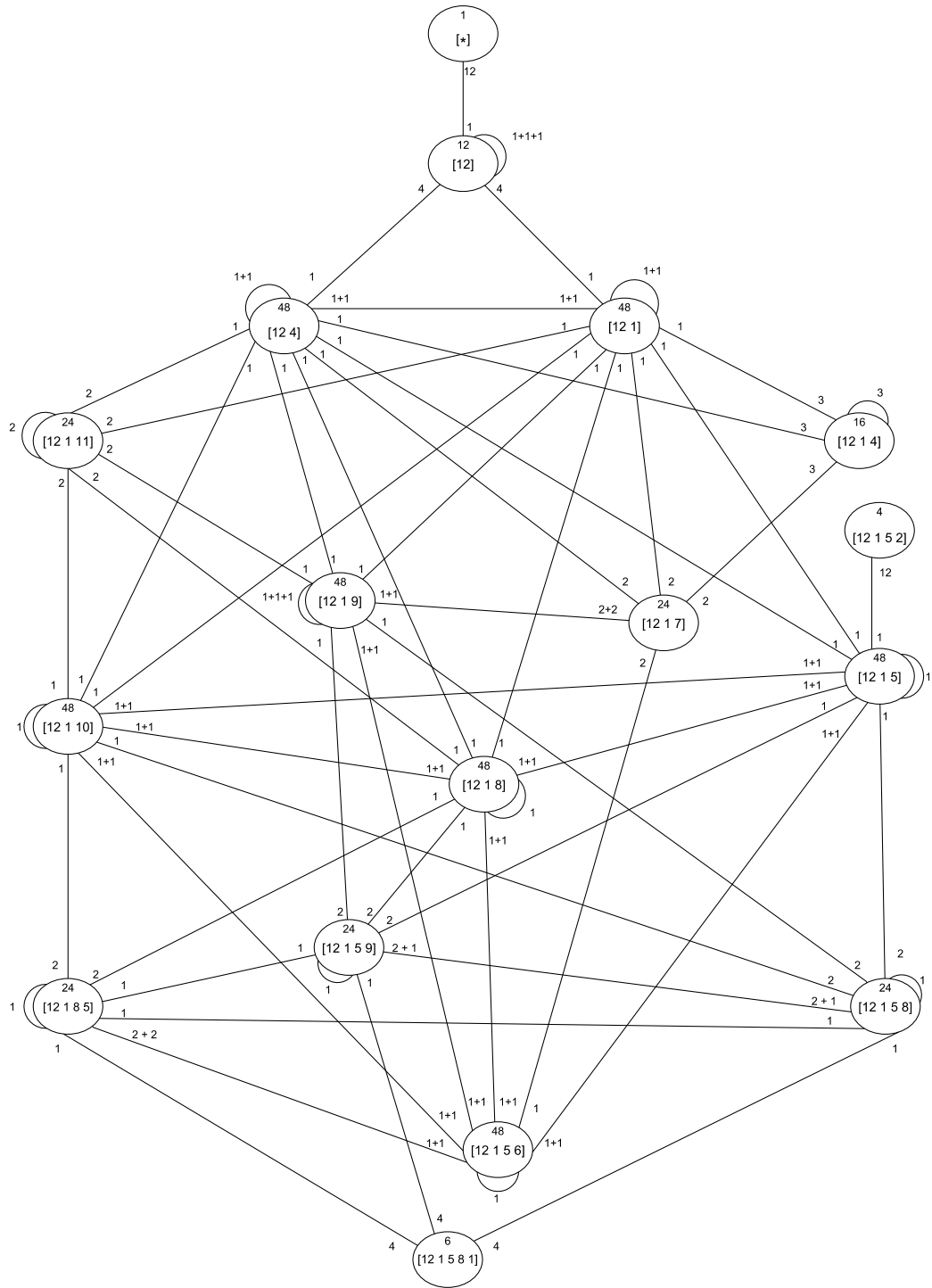


Figure 4.3: Cayley Diagram of  $M_{12}$  over Maximal Subgroup  $M \cong 2^4 : \bullet S_4$

#### 4.4.3 Double Coset Enumeration of $U_4(2) : 2$ over Maximal Subgroup $2^4 : S_5$

Consider the progenerator  $\frac{2^{*12}:D_3 \times D_4}{wvt_7t_{11}t_3t_{10}t_{12},yt_{12}t_9t_{12},t_7t_{12}t_7t_{12},xzyt_{12}t_6t_{11}t_4t_{12}t_6t_{11}t_4t_{12}} \cong U_4(2) : 2$ . Our N in this case is order 48. We wish to find a suitable maximal subgroup that contains N which we can use to perform our maximal double coset enumeration over. To find this M we consider the following:

```
> M:=MaximalSubgroups(G1);
> M;
Conjugacy classes of subgroups
-----
[1]      Order 1296          Length 40
      Permutation group acting on a set of cardinality 1080
      Order = 1296 = 2^4 * 3^4
[2]      Order 1296          Length 40
      Permutation group acting on a set of cardinality 1080
      Order = 1296 = 2^4 * 3^4
[3]      Order 1440          Length 36
      Permutation group acting on a set of cardinality 1080
      Order = 1440 = 2^5 * 3^2 * 5
[4]      Order 1152          Length 45
      Permutation group acting on a set of cardinality 1080
      Order = 1152 = 2^7 * 3^2
[5]      Order 1920          Length 27
      Permutation group acting on a set of cardinality 1080
      Order = 1920 = 2^7 * 3 * 5
[6]      Order 25920         Length 1
      Permutation group acting on a set of cardinality 1080
      Order = 25920 = 2^6 * 3^4 * 5
```

In this case  $\frac{|M|}{|N|}$  for all maximal subgroups of G. Through trial and error using Magma we determine which subgroups contain our N using the following code:

```
> C:=Conjugates(G1,M[5]\`subgroup);
> CC:=Setseq(C);
> for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i]
> and f(z) in CC[i] and f(w) in CC[i] and f(v) in CC[i]
> then i; end if; end for
27
```



Both the subgroups  $M[1]$  and  $M[5]$  contain our  $N$  and since the order of  $M[5]$ 's subgroup is larger we will proceed using this  $M$ . The rest of the code to determine the relation which will produce our  $M$  from our  $G$  presentation will be included in the appendix.

$M$  is determined to be  $(2^4) : S_5$ . We use the following code to verify how many double and single cosets our Cayley diagram should contain:

```
> #DoubleCosets(G,M,sub<G|x,y,z,w,v>);
5
> #G/#M;
27
```

and now we begin the process of double coset enumeration of  $U_4(2) : 2$  over the maximal subgroup  $2^4 : S_5$ .

One major difference from this point forward is the way we define our double cosets. When we perform double coset enumeration over  $N$ , our double coset is defined  $NwN = \{Nw^n | n \in N\}$ . This allows us to set our relations found in  $N$  equal to the  $\text{Id}(N)$  and establish relations manually. Since we have decided to proceed using  $M$  our new definition of a double coset for this set is  $MwN = \{Mw^n | n \in N\}$  and our relations are now equal to an unknown permutation in  $M$ . This makes solving relations by hand almost impossible. To complete this process, we will use Magma to determine the relations. The presentation and symmetric generators will be included in the appendix. With this in mind we begin the process.

If we allow  $G = \frac{2^{*12}:D_3 \times D_4}{wvt_7t_{11}t_3t_{10}t_{12},yt_{12}t_9t_{12},t_7t_{12}t_7t_{12},xzyt_{12}t_6t_{11}t_4t_{12}t_6t_{11}t_4t_{12}}$ ,  
 $M := \text{sub} \langle G | f(x), f(y), f(z), f(w), f(v), f(x * t * x * t * y * z * v * t * y * v * t * y * v * t * v) \rangle$ ,  
and  $t \sim t_{12}$  we can complete the process of double coset enumeration over the maximal subgroup and produce the following Cayley Diagram. The Magma code for this maximal double coset enumeration can also be found in the appendix.

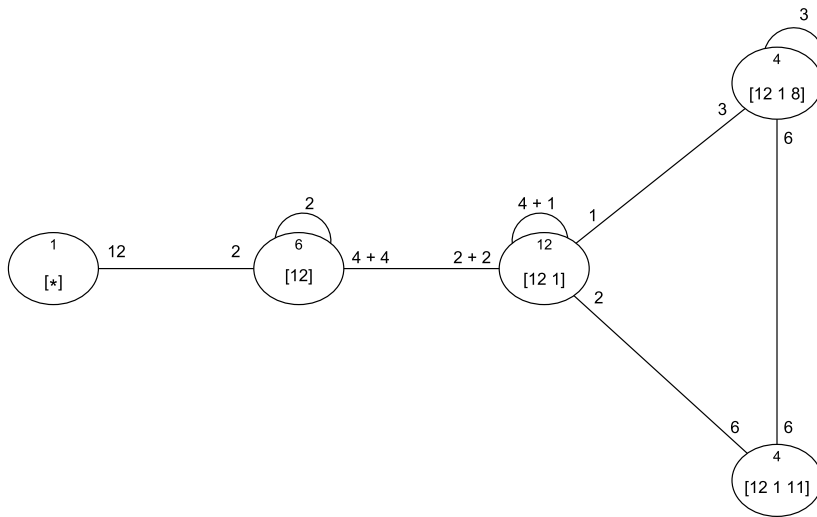


Figure 4.4: Cayley Diagram of  $U_4(2) : 2$  over Maximal Subgroup  $2^4 : S_5$

## 4.5 Conclusion

Double coset enumeration is a challenging process. While the process turns out to be repetitive, proving relations can be a daunting task. Luckily, Magma helps us determine how the relations interact with one another to determine where double cosets belong. Then the user's job becomes solving the puzzles embedded in the group.

## Chapter 5

# Monomial Progenitors

### 5.1 Introduction

There are other ways to construct progenitors than simply stabilizing a single  $t_i$ . One such way is inducing a linear character from the quotient group up to  $G$ . Since this can be performed the question of whether or not induction can be performed on simple groups arises. Of course, simple groups contain only trivial normal subgroups but they do have plenty of maximal subgroups. While this fact cannot guarantee induction will take place it does not prevent it from being attempted. In fact, while it is rare that a monomial progenitor can be constructed in this fashion it can be done. This section will demonstrate the full process of constructing monomial progenitors.

### 5.2 Constructing the Monomial Progenitor

$$17^{*10} :_m PGL_2(9)$$

#### 5.2.1 Choosing an Control Group

Before we start inducing a character up to  $G$  we can choose our control group  $N$ . For this investigation  $N$  was chosen to be  $PGL_2(9)$ :

```
>T1=TransitiveGroups(12);
>for i in [1..#T] do if IsIsomorphic(T[i],PGL(2,9)) then i;
end if; end for;
```

182

```
>G:=TransitiveGroup(12,182);
```

This same process was used for the second example in this chapter but will not be shown. We will now go through the process of creating a monomial progenitor.

## 5.2.2 Inducing and Verifying a Linear Character

We will construct a monomial presentation of  $17^{*10} :_m PGL_2(9)$  by first giving a presentation of  $PGL_2(9)$ . That presentation is:

$$\langle x, y, z | x^3, y^5, z^2, y^{-1} * z * y^{-1} * x * z * y^{-1}, (x^{-1} * y^{-1})^4, (x^{-1}, y^{-1})^2 \rangle.$$

To achieve a monomial presentation we must induce a linear character from a subgroup  $H$  of  $G$ . To insure we get an irreducible character we must choose a subgroup with index matching the degree of an irreducible character of  $G$ . We consider the character table of  $G = PGL_2(9)$  in Table 5.1 and note  $G$  has characters  $\chi_{.1}, \chi_{.2}, \dots, \chi_{.11}$ . We proceed using  $\chi_{.11}$  and look for a subgroup of order 72 so that  $\frac{|G|}{|H|} = 10$ . This produces the index as follows:

$$[G : H] = [PGL_2(9) : 3^2 : 8] = 10$$

Since the index of the two groups is 10. If a matrix representation exists it will be represented by a  $10 \times 10$  matrix.

We produce a character table for  $3^2 : 8$  in table 5.2. We will verify the induction  $\chi_{.5}$  of  $3^2 : 8$  to  $\chi_{.11}$  of  $PGL_2(9)$  by considering the irreducible characters  $\phi$  (of  $H$ ) and  $\phi^G$  (of  $G$ ).  $G = PGL_2(9)$  is generated by  $xx$ ,  $yy$ , and  $zz$  where  $xx = (1, 3, 5)(2, 4, 12)(7, 9, 11)$ ,  $yy = (1, 7, 3, 9, 5)(2, 4, 6, 8, 10)$ , and  $zz = (1, 12)(2, 3)(4, 11)(5, 10)(6, 9)(7, 8)$ . Eliminating the information we will not use from the Character tables of  $G$  and  $H$  we produce Tables 5.3 for  $G$ , and 5.4 for  $H$ . Using our definition of induction we reproduce  $\phi^G$  using  $\phi$  (of  $H$ ).

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \text{ where } n = \frac{|G|}{|H|} = \frac{720}{72} = 10.$$

$$\phi_1^G = \frac{10}{1} \sum_{w \in H \cap C_1} \phi(w)$$

which implies  $\phi_1^G = \frac{10}{1}(\phi(1)) = 10(1) = 10$ .

$$\phi_2^G = \frac{10}{36} \sum_{w \in H \cap C_2} \phi(w)$$

$$\phi_2^G = \frac{10}{36} \sum_{w \in H \cap C_2} \phi((2, 8)(5, 11)(6, 12)(7, 9))$$

which implies  $\phi_2^G = \frac{10}{36}(\phi(0)) = \frac{5}{18}(0) = 0$ .

$$\phi_3^G = \frac{10}{45} \sum_{w \in H \cap C_3} \phi(w)$$

$$\phi_3^G = \frac{10}{45} \sum_{w \in H \cap C_3} \phi((2, 8)(5, 11)(6, 12)(7, 9))$$

which implies  $\phi_3^G = \frac{10}{45}(9(\phi((2, 8)(5, 11)(6, 12)(7, 9)))) = \frac{2}{9}(9(-1)) = -2$ .

$$\phi_4^G = \frac{10}{80} \sum_{w \in H \cap C_4} \phi(w)$$

$$\phi_4^G = \frac{10}{80} \sum_{w \in H \cap C_4} \phi((1, 11, 5)(2, 4, 8)(6, 10, 12))$$

which implies  $\phi_4^G = \frac{10}{80}(8(\phi((1, 11, 5)(2, 4, 8)(6, 10, 12)))) = \frac{1}{8}(8(1)) = 1$ .

$$\phi_5^G = \frac{10}{90} \sum_{w \in H \cap C_5} \phi(w)$$

$$\phi_5^G = \frac{10}{90} \sum_{w \in H \cap C_5} \phi(0)$$

which implies  $\phi_5^G = \frac{1}{9}(\phi(0)) = \frac{1}{9}(0) = 0$ .

$$\phi_6^G = \frac{10}{72} \sum_{w \in H \cap C_6} \phi(w)$$

$$\phi_6^G = \frac{10}{72} \sum_{w \in H \cap C_6} \phi(0)$$

which implies  $\phi_6^G = \frac{10}{72}(\phi(0)) = \frac{5}{36}(0) = 0$ .

$$\phi_7^G = \frac{10}{72} \sum_{w \in H \cap C_7} \phi(w)$$

$$\phi_7^G = \frac{10}{72} \sum_{w \in H \cap C_7} \phi(0)$$

which implies  $\phi_7^G = \frac{10}{72}(\phi(0)) = \frac{5}{36}(0) = 0$ .

$$\phi_8^G = \frac{10}{90} \sum_{w \in H \cap C_8} \phi(w)$$

$$\phi_8^G = \frac{10}{90} \sum_{w \in H \cap C_8} \phi((1, 4, 3, 10)(2, 7, 6, 11, 8, 9, 12, 5))$$

$$\phi_8^G = \frac{10}{90}(9(\phi((1, 4, 3, 10)(2, 7, 6, 11, 8, 9, 12, 5)))) = \frac{1}{9}(9)(-Z_1) = (-Z_1)$$

note:  $-Z_1 = -Z_2$  since they are both primitive eighth roots.

$$\phi_9^G = \frac{10}{90} \sum_{w \in H \cap C_9} \phi(w)$$

$$\phi_9^G = \frac{10}{90} \sum_{w \in H \cap C_9} \phi((1, 4, 3, 10)(2, 9, 6, 5, 8, 7, 12, 11))$$

$$\phi_9^G = \frac{10}{90}(9(\phi(((1, 4, 3, 10)(2, 9, 6, 5, 8, 7, 12, 11)))) = \frac{1}{9}(9)(Z_1) = (Z_1)$$

note:  $Z_1 = Z_2$  similarly.

$$\phi_{10}^G = \frac{10}{72} \sum_{w \in H \cap C_{10}} \phi(w)$$

$$\phi_{10}^G = \frac{10}{72} \sum_{w \in H \cap C_{10}} \phi(0)$$

which implies  $\phi_{10}^G = \frac{10}{72}(\phi(0)) = \frac{5}{36}(0) = 0$ .

$$\text{Finally, } \phi_{11}^G = \frac{10}{72} \sum_{w \in H \cap C_{11}} \phi(w)$$

$$\phi_{11}^G = \frac{10}{72} \sum_{w \in H \cap C_{11}} \phi(0)$$

which implies  $\phi_{11}^G = \frac{10}{72}(\phi(0)) = \frac{5}{36}(0) = 0$ .

so  $\phi \uparrow_H^G = 10, 0, -2, 1, 0, 0, 0, -Z_2, Z_2, 0, 0$  and we have verified that  $\chi_{.5}$  of  $3^2 : 8$  induces  $\chi_{.11}$  of  $PGL_2(9)$ .

Table 5.1: Character Table of G

$\chi$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	$C_{11}$
$\chi_{.1}$	1	1	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	1	1	1	1	-1	-1	-1	-1
$\chi_{.3}$	8	2	0	-1	0	$Z_1$	$Z_1\#2$	0	0	$-Z_1\#2$	$-Z_1$
$\chi_{.4}$	8	-2	0	-1	0	$Z_1\#2$	$Z_1$	0	0	$Z_1$	$Z_1\#2$
$\chi_{.5}$	8	2	0	-1	0	$Z_1\#2$	$Z_1$	0	0	$-Z_1$	$-Z_1\#2$
$\chi_{.6}$	8	-2	0	-1	0	$Z_1$	$Z_1\#2$	0	0	$Z_1\#2$	$Z_1$
$\chi_{.7}$	9	1	1	0	1	-1	-1	-1	-1	1	1
$\chi_{.8}$	9	-1	1	0	1	-1	-1	1	1	-1	-1
$\chi_{.9}$	10	0	2	1	-2	0	0	0	0	0	0
$\chi_{.10}$	10	0	-2	1	0	0	0	$Z_2$	$-Z_2$	0	0
$\chi_{.11}$	10	0	-2	1	0	0	0	$-Z_2$	$Z_2$	0	0

# denotes algebraic conjugation.

$Z_1$  is the primitive fifth root of unity.  $Z_2$  is the primitive eighth root of unity.

Table 5.2: Character Table of H

$\chi$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	1	1	1	1	-1	-1	-1	-1
$\chi_{.3}$	1	1	1	-1	-1	I	-I	I	-I
$\chi_{.4}$	1	1	1	-1	-1	-I	I	-I	I
$\chi_{.5}$	1	-1	1	I	-I	$Z_1$	$Z_1\#3$	$-Z_1$	$-Z_1\#3$
$\chi_{.6}$	1	-1	1	-I	I	$Z_1\#3$	$Z_1$	$-Z_1\#3$	$-Z_1$
$\chi_{.7}$	1	-1	1	-I	I	$-Z_1\#3$	$-Z_1$	$Z_1\#3$	$Z_1$
$\chi_{.8}$	1	-1	1	I	-I	$-Z_1$	$-Z_1\#3$	$Z_1$	$Z_1\#3$
$\chi_{.9}$	8	0	-1	0	0	0	0	0	0

# denotes algebraic conjugation.

I is the primitive fourth root of unity.  $Z_1$  is the primitive eighth root of unity.

Table 5.3:  $\chi_{.11}$  of G

$\phi^G$	Class	Size	Class Representative
10	$C_1$	1	Id(G)
0	$C_2$	36	(1, 12)(2, 3)(4, 11)(5, 10)(6, 9)(7, 8)
-2	$C_3$	45	(2, 10)(3, 11)(4, 6)(7, 9)
1	$C_4$	80	(1, 3, 5)(2, 4, 12)(7, 9, 11)
0	$C_5$	90	(1, 5)(2, 4, 10, 6)(3, 7, 11, 9)(8, 12)
0	$C_6$	72	(1, 7, 3, 9, 5)(2, 4, 6, 8, 10)
0	$C_7$	72	(1, 3, 5, 7, 9)(2, 6, 10, 4, 8)
$-Z_2$	$C_8$	90	(1, 8, 5, 12)(2, 11, 4, 9, 10, 3, 6, 7)
$Z_2$	$C_9$	90	(1, 12, 5, 8)(2, 9, 6, 11, 10, 7, 4, 3)
0	$C_{10}$	72	(1, 6, 7, 10, 3, 12, 11, 2, 9, 4)(5, 8)
0	$C_{11}$	72	(1, 10, 11, 4, 7, 12, 9, 6, 3, 2)(5, 8)

Table 5.4:  $\chi_{.5}$  of H

$\phi$	Class	Size	Class Representative
1	$D_1$	1	Id(H)
-1	$D_2$	9	(2, 8)(5, 11)(6, 12)(7, 9)
1	$D_3$	8	(1, 11, 5)(2, 4, 8)(6, 10, 12)
I	$D_4$	9	(1, 3)(2, 6, 8, 12)(4, 10)(5, 7, 11, 9)
-I	$D_5$	9	(1, 3)(2, 12, 8, 6)(4, 10)(5, 9, 11, 7)
$Z_1$	$D_6$	9	(1, 4, 3, 10)(2, 9, 6, 5, 8, 7, 12, 11)
$Z_1\#3$	$D_7$	9	(1, 10, 3, 4)(2, 5, 12, 9, 8, 11, 6, 7)
$-Z_1$	$D_8$	9	(1, 4, 3, 10)(2, 7, 6, 11, 8, 9, 12, 5)
$-Z_1\#3$	$D_9$	9	(1, 10, 3, 4)(2, 11, 12, 7, 8, 5, 6, 9)

Equivalencies from table to construct matrix:  $1 = 1$ ,  $-1 = 16$ ,  $I = 13$ ,  $-I = 4$ ,  $Z_1 = 8$ ,  $Z_1\#3 = 2$ ,  $-Z_1 = 9$ , and  $-Z_1\#3 = 15$ .



With induction taking place we now verify the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} 0 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A(yy) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$A(zz) = \begin{bmatrix} 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \end{bmatrix}$$

### 5.2.3 Verifying the Monomial Representation

Since we have a linear character  $\phi$  of the subgroup  $H$  of index  $n$  in  $G$  we let  $G = Ht_1 \cup Ht_2 \cup Ht_3 \cup \dots \cup Ht_{10}$  where the  $t_i$ 's are transversals of  $G$  acting on  $H$ . That is  $G = He \cup H(1, 11, 3)(2, 12, 10)(4, 6, 8) \cup H(1, 9, 11)(2, 4, 10)(6, 8, 12) \cup H(1, 8, 9, 10, 7, 6, 3, 2, 11, 12)(4, 5) \cup H(1, 8, 11, 2, 7, 6, 5, 12, 3, 4)(9, 10) \cup H(1, 5, 9, 3, 7)(2, 10, 8, 6, 4) \cup H(1, 5, 9, 3, 7)(2, 10, 8, 6, 4) \cup H(1, 7, 9, 5)(2, 8, 4, 12)(3, 11)(6, 10) \cup H(2, 10, 8, 6, 12)(3, 9, 5, 11, 7) \cup H(1, 3, 11, 9)(2, 10, 8, 6)(4, 12)(5, 7)$ .

Continuing the process in a 10x10 matrix:

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \dots & \phi(t_1xt_{10}^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \phi(t_{10}xt_1^{-1}) & \dots & \dots & \phi(t_{10}xt_{10}^{-1}) \end{bmatrix}$$

We will calculate all 100 elements of the matrix using the following calculation using the  $t_1 = e$ ,  $t_2 = (2, 12)(3, 5)(6, 10)(7, 11)$ , ..., and  $t_{10} = (1, 10, 11, 12, 5, 8, 9, 2)(3, 6, 7, 4)$  with each  $t_i$  corresponding to each transversal respectively from the  $G$  listed before. To conserve space we will calculate the multiplication of  $t_ixt_j^{-1}$  and list the resulting permutation rather than show the entire process. For example:  $\phi(t_1xt_1^{-1}) =$

$\phi(e(1, 3, 5)(2, 4, 12)(7, 9, 11)e^{-1}) = \phi((1, 3, 5)(2, 4, 12)(7, 9, 11))$  We will bypass the middle step and go directly to what  $\phi(t_1xt_1^{-1})$  is equal to. Since  $(1, 3, 5)(2, 4, 12)(7, 9, 11) \notin H$   $\phi((1, 3, 5)(2, 4, 12)(7, 9, 11)) = 0$  We proceed as follows:

Row 1:

$$\phi(t_1xt_1^{-1}) = \phi((1, 3, 5)(2, 4, 12)(7, 9, 11)) = 0$$

$$\phi(t_1xt_2^{-1}) = \phi((1, 11, 7, 9)(2, 8, 6, 4)(3, 5)(10, 12)) = 13$$

$$\phi(t_1xt_3^{-1}) = \phi((1, 3, 5, 11, 7)(4, 8, 6, 12, 10)) = 0$$

$$\phi(t_1xt_4^{-1}) = \phi((1, 6, 7, 8)(2, 5, 12, 3, 4, 11, 10, 9)) = 0$$

$$\phi(t_1xt_5^{-1}) = \phi((1, 12, 11, 2, 3, 6, 7, 10, 9, 8)(4, 5)) = 0$$

$$\phi(t_1xt_6^{-1}) = \phi((1, 9, 11, 3)(2, 6, 8, 10)(4, 12)(5, 7)) = 0$$

$$\phi(t_1xt_7^{-1}) = \phi((1, 5, 7)(2, 6, 12)(3, 11, 9)) = 0$$

$$\phi(t_1xt_8^{-1}) = \phi((1, 11)(2, 8)(3, 9)(6, 10)) = 0$$

$$\phi(t_1xt_9^{-1}) = \phi((1, 7, 3, 9, 5)(2, 4, 6, 8, 10)) = 0$$

$$\phi(t_1xt_{10}^{-1}) = \phi((2, 12, 6, 8, 10)(3, 7, 11, 5, 9)) = 0$$

Row 2:

$$\phi(t_2xt_1^{-1}) = \phi((1, 7, 9, 11, 5)(4, 6, 8, 12, 10)) = 0$$

$$\phi(t_2xt_2^{-1}) = \phi((1, 7, 9)(2, 10, 8)(3, 11, 5)) = 0$$

$$\phi(t_2xt_3^{-1}) = \phi((1, 7)(2, 10)(4, 12)(5, 11)) = 16$$

$$\phi(t_2xt_4^{-1}) = \phi((1, 10, 5, 12, 9, 2, 3, 6)(4, 7, 8, 11)) = 0$$

$$\phi(t_2xt_5^{-1}) = \phi((1, 2, 11, 6)(3, 12, 9, 8, 5, 4, 7, 10)) = 0$$

$$\phi(t_2xt_6^{-1}) = \phi((1, 3, 9, 11)(2, 4, 8, 12)(5, 7)(6, 10)) = 0$$

$$\phi(t_2xt_7^{-1}) = \phi((1, 9, 3, 5, 7)(4, 12, 10, 6, 8)) = 0$$

$$\phi(t_2xt_8^{-1}) = \phi((2, 12, 6)(3, 11, 9)(4, 10, 8)) = 0$$

$$\phi(t_2xt_9^{-1}) = \phi((1, 11, 9, 5)(2, 12)(3, 7)(4, 8, 6, 10)) = 0$$

$$\phi(t_2xt_{10}^{-1}) = \phi((1, 5, 9, 3)(2, 6, 10, 12)(4, 8)(7, 11)) = 0$$

Row 3:

$$\phi(t_3xt_1^{-1}) = \phi((1, 11, 3, 5)(2, 12, 6, 8)(4, 10)(7, 9)) = 13$$

$$\phi(t_3xt_2^{-1}) = \phi((3, 5)(4, 12)(7, 9)(8, 10)) = 0$$

$$\phi(t_3xt_3^{-1}) = \phi((1, 9, 7)(2, 8, 10)(3, 5, 11)) = 0$$

$$\begin{aligned}
\phi(t_3xt_4^{-1}) &= \phi((1, 2, 11, 6)(3, 4, 9, 10, 5, 12, 7, 8)) = 0 \\
\phi(t_3xt_5^{-1}) &= \phi((1, 8, 11, 12, 7, 10, 3, 6)(2, 5, 4, 9)) = 0 \\
\phi(t_3xt_6^{-1}) &= \phi((1, 11, 9, 3)(2, 12, 8, 4)(5, 7)(6, 10)) = 0 \\
\phi(t_3xt_7^{-1}) &= \phi((1, 3, 11, 5, 7)(2, 4, 10, 6, 8)) = 0 \\
\phi(t_3xt_8^{-1}) &= \phi((1, 3, 9)(2, 4, 6)(8, 12, 10)) = 0 \\
\phi(t_3xt_9^{-1}) &= \phi((1, 5)(2, 6, 10, 4)(3, 9, 11, 7)(8, 12)) = 0 \\
\phi(t_3xt_{10}^{-1}) &= \phi((1, 3, 7, 11)(2, 4)(5, 9)(6, 10, 12, 8)) = 0
\end{aligned}$$

Row 4:

$$\begin{aligned}
\phi(t_4xt_1^{-1}) &= \phi((1, 8, 11, 2, 7, 6, 5, 12, 3, 4)(9, 10)) = 0 \\
\phi(t_4xt_2^{-1}) &= \phi((1, 6, 5, 2, 7, 4, 3, 8)(9, 12, 11, 10)) = 0 \\
\phi(t_4xt_3^{-1}) &= \phi((1, 6, 5, 8, 9, 4, 11, 10)(2, 7, 12, 3)) = 0 \\
\phi(t_4xt_4^{-1}) &= \phi((2, 10, 8)(3, 5, 11)(4, 12, 6)) = 0 \\
\phi(t_4xt_5^{-1}) &= \phi(e) = 1 \\
\phi(t_4xt_6^{-1}) &= \phi((1, 10, 5, 12, 9, 2, 3, 6)(4, 7, 8, 11)) = 0 \\
\phi(t_4xt_7^{-1}) &= \phi((1, 8, 3, 6, 11, 2, 9, 10)(4, 7, 12, 5)) = 0 \\
\phi(t_4xt_8^{-1}) &= \phi((1, 2)(3, 8)(4, 5)(6, 9)(7, 10)(11, 12)) = 0 \\
\phi(t_4xt_9^{-1}) &= \phi((1, 10, 3, 4)(2, 11, 12, 7, 8, 5, 6, 9)) = 0 \\
\phi(t_4xt_{10}^{-1}) &= \phi((1, 10, 11, 6, 7, 8, 3, 12)(2, 5, 4, 9)) = 0
\end{aligned}$$

Row 5:

$$\begin{aligned}
\phi(t_5xt_1^{-1}) &= \phi((1, 8, 7, 6)(2, 9, 10, 11, 4, 3, 12, 5)) = 0 \\
\phi(t_5xt_2^{-1}) &= \phi((1, 6, 3, 2, 9, 12, 5, 10)(4, 11, 8, 7)) = 0 \\
\phi(t_5xt_3^{-1}) &= \phi((1, 6, 11, 2)(3, 8, 7, 12, 5, 10, 9, 4)) = 0 \\
\phi(t_5xt_4^{-1}) &= \phi((2, 8, 10)(3, 11, 5)(4, 6, 12)) = 0 \\
\phi(t_5xt_5^{-1}) &= \phi((2, 10, 8)(3, 5, 11)(4, 12, 6)) = 0 \\
\phi(t_5xt_6^{-1}) &= \phi((1, 10, 11, 6, 7, 8, 3, 12)(2, 5, 4, 9)) = 0 \\
\phi(t_5xt_7^{-1}) &= \phi((1, 8, 9, 10, 3, 4, 5, 2)(6, 7, 12, 11)) = 15 \\
\phi(t_5xt_8^{-1}) &= \phi((1, 2, 7, 10, 3, 4, 11, 8)(5, 12, 9, 6)) = 0 \\
\phi(t_5xt_9^{-1}) &= \phi((1, 10, 5, 12, 9, 2, 3, 6)(4, 7, 8, 11)) = 0 \\
\phi(t_5xt_{10}^{-1}) &= \phi((1, 10, 3, 4)(2, 11, 12, 7, 8, 5, 6, 9)) = 0
\end{aligned}$$

Row 6:

$$\phi(t_6xt_1^{-1}) = \phi((2, 10, 8, 6, 12)(3, 9, 5, 11, 7)) = 0$$

$$\phi(t_6xt_2^{-1}) = \phi((1, 3, 9, 5)(2, 12, 10, 6)(4, 8)(7, 11)) = 0$$

$$\phi(t_6xt_3^{-1}) = \phi((1, 11, 7, 3)(2, 4)(5, 9)(6, 8, 12, 10)) = 0$$

$$\phi(t_6xt_4^{-1}) = \phi((1, 12, 3, 8, 7, 6, 11, 10)(2, 9, 4, 5)) = 0$$

$$\phi(t_6xt_5^{-1}) = \phi((1, 4, 3, 10)(2, 9, 6, 5, 8, 7, 12, 11)) = 0$$

$$\phi(t_6xt_6^{-1}) = \phi((1, 7, 9)(3, 5, 11)(4, 6, 12)) = 0$$

$$\phi(t_6xt_7^{-1}) = \phi((1, 7, 5, 3)(2, 10, 8, 12)(4, 6)(9, 11)) = 0$$

$$\phi(t_6xt_8^{-1}) = \phi((1, 5, 3, 7, 11)(2, 6, 4, 8, 10)) = 0$$

$$\phi(t_6xt_9^{-1}) = \phi(e) = 1$$

$$\phi(t_6xt_{10}^{-1}) = \phi((1, 9, 7)(3, 11, 5)(4, 12, 6)) = 0$$

Row 7:

$$\phi(t_7xt_1^{-1}) = \phi((1, 11)(2, 4)(3, 7)(6, 12)) = 0$$

$$\phi(t_7xt_2^{-1}) = \phi((2, 8, 6)(3, 7, 11)(4, 10, 12)) = 0$$

$$\phi(t_7xt_3^{-1}) = \phi((1, 9)(3, 7)(4, 10)(6, 8)) = 0$$

$$\phi(t_7xt_4^{-1}) = \phi((1, 2, 5, 4, 3, 10, 9, 8)(6, 11, 12, 7)) = 8$$

$$\phi(t_7xt_5^{-1}) = \phi((1, 8)(2, 3)(4, 11)(5, 6)(7, 12)(9, 10)) = 0$$

$$\phi(t_7xt_6^{-1}) = \phi((1, 11, 7, 9, 5)(2, 6, 12, 8, 10)) = 0$$

$$\phi(t_7xt_7^{-1}) = \phi((1, 3, 9)(2, 6, 4)(5, 11, 7)) = 0$$

$$\phi(t_7xt_8^{-1}) = \phi((1, 3)(2, 8)(4, 12, 10, 6)(5, 9, 7, 11)) = 0$$

$$\phi(t_7xt_9^{-1}) = \phi((1, 5, 9, 3, 11)(2, 4, 12, 8, 10)) = 0$$

$$\phi(t_7xt_{10}^{-1}) = \phi((1, 3, 5, 7)(2, 12, 8, 10)(4, 6)(9, 11)) = 0$$

Row 8:

$$\phi(t_8xt_1^{-1}) = \phi((1, 9)(2, 8, 12, 4)(3, 7, 11, 5)(6, 10)) = 0$$

$$\phi(t_8xt_2^{-1}) = \phi((1, 9, 3, 7)(2, 6, 12, 8)(4, 10)(5, 11)) = 0$$

$$\phi(t_8xt_3^{-1}) = \phi((2, 6, 4, 10, 12)(3, 7, 9, 11, 5)) = 0$$

$$\phi(t_8xt_4^{-1}) = \phi((1, 8, 11, 4, 3, 10, 7, 2)(5, 6, 9, 12)) = 0$$

$$\phi(t_8xt_5^{-1}) = \phi((1, 10, 7, 8, 5, 12, 3, 2)(4, 11, 6, 9)) = 0$$

$$\phi(t_8xt_6^{-1}) = \phi((1, 5, 9, 7, 11)(2, 10, 8, 12, 6)) = 0$$

$$\phi(t_8xt_7^{-1}) = \phi((2, 8, 4)(3, 9, 7)(6, 10, 12)) = 0$$

$$\phi(t_8xt_8^{-1}) = \phi((1, 7, 3)(4, 12, 8)(5, 11, 9)) = 1$$

$$\phi(t_8xt_9^{-1}) = \phi((1, 3, 11, 9)(2, 10, 8, 6)(4, 12)(5, 7)) = 0$$

$$\phi(t_8xt_{10}^{-1}) = \phi((1, 11, 7, 3, 5)(2, 10, 8, 4, 6)) = 0$$

Row 9:

$$\phi(t_9xt_1^{-1}) = \phi((1, 3, 11, 9)(2, 10, 8, 6)(4, 12)(5, 7)) = 0$$

$$\phi(t_9xt_2^{-1}) = \phi((1, 11, 9, 3)(2, 12, 8, 4)(5, 7)(6, 10)) = 0$$

$$\phi(t_9xt_3^{-1}) = \phi((1, 3, 9, 11)(2, 4, 8, 12)(5, 7)(6, 10)) = 0$$

$$\phi(t_9xt_4^{-1}) = \phi((1, 6, 3, 2, 9, 12, 5, 10)(4, 11, 8, 7)) = 0$$

$$\phi(t_9xt_5^{-1}) = \phi((1, 12, 3, 8, 7, 6, 11, 10)(2, 9, 4, 5)) = 0$$

$$\phi(t_9xt_6^{-1}) = \phi((1, 9, 7)(3, 11, 5)(4, 12, 6)) = 0$$

$$\phi(t_9xt_7^{-1}) = \phi((1, 5, 9, 7, 11)(2, 10, 8, 12, 6)) = 0$$

$$\phi(t_9xt_8^{-1}) = \phi((1, 11, 7, 9, 5)(2, 6, 12, 8, 10)) = 0$$

$$\phi(t_9xt_9^{-1}) = \phi((1, 7, 9)(3, 5, 11)(4, 6, 12)) = 0$$

$$\phi(t_9xt_{10}^{-1}) = \phi(e) = 1$$

Row 10:

$$\phi(t_{10}xt_1^{-1}) = \phi((1, 5, 9, 3, 7)(2, 10, 8, 6, 4)) = 0$$

$$\phi(t_{10}xt_2^{-1}) = \phi((1, 5, 9, 11)(2, 12)(3, 7)(4, 10, 6, 8)) = 0$$

$$\phi(t_{10}xt_3^{-1}) = \phi((1, 5)(2, 4, 10, 6)(3, 7, 11, 9)(8, 12)) = 0$$

$$\phi(t_{10}xt_4^{-1}) = \phi((1, 4, 3, 10)(2, 9, 6, 5, 8, 7, 12, 11)) = 0$$

$$\phi(t_{10}xt_5^{-1}) = \phi((1, 6, 3, 2, 9, 12, 5, 10)(4, 11, 8, 7)) = 0$$

$$\phi(t_{10}xt_6^{-1}) = \phi(e) = 1$$

$$\phi(t_{10}xt_7^{-1}) = \phi((1, 11, 3, 9, 5)(2, 10, 8, 12, 4)) = 0$$

$$\phi(t_{10}xt_8^{-1}) = \phi((1, 9, 11, 3)(2, 6, 8, 10)(4, 12)(5, 7)) = 0$$

$$\phi(t_{10}xt_9^{-1}) = \phi((1, 9, 7)(3, 11, 5)(4, 12, 6)) = 0$$

$$\phi(t_{10}xt_{10}^{-1}) = \phi((1, 7, 9)(3, 5, 11)(4, 6, 12)) = 0.$$

Each  $\phi$  of H corresponded with a conjugacy class of either H or G. If the element is in a conjugacy class from H (seen in table 5.4) we write the value of  $\phi$  for that class. Since our matrix was produced in cyclotomic field 8, we needed to produce an order 8 element in 17. In this case, 8 was chosen as the element of order 8 but other elements of 17 could

have been chosen such as 2 which would have produced a different representation. To complete this process, the matrices for yy and zz should also be verified by repeating the process above but we will proceed without verifying each specific matrix but will instead prove a faithful representation exists.

To verify the faithful representation recall  $|PGL_2(9)| = 720$  and  $PGL_2(9)$  was generated by  $\langle x, y, z | x^3, y^5, z^2, \dots \rangle$ . Since  $|A(x)| = 3$ ,  $|A(y)| = 5$  and  $|A(z)| = 2$  with  $|A(x) \cdot A(y) \cdot A(z)| = 10$ , the order of our index, we know that  $\langle A(x), A(y), A(z) \rangle$  is a faithful representation of  $PGL_2(9)$ . We proceed and produce a permutation representation.

### 5.2.4 Constructing the Permutation Representation

We worked in  $\mathbb{Z}_{17}$  on matrices of degree 10x10 which implies we are producing a  $17^{*10} :_m PGL_2(9)$  progenitor permutation representation based on the monomial representation  $PGL_2(9) = \langle A(x), A(y), A(z) \rangle$ . We consider the matrix entries for  $a_{i,j}$  for  $A(x)$ ,  $A(y)$ , and  $A(z)$ . We know there are 10 t's, one for each column of our matrix and because we have a semi-direct product in our progenitor that the elements of  $PGL_2(9)$  act as automorphisms of  $\langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle * \langle t_4 \rangle * \langle t_5 \rangle * \langle t_6 \rangle * \langle t_7 \rangle * \langle t_8 \rangle * \langle t_9 \rangle * \langle t_{10} \rangle$ . We want to represent x, y, and z as permutations. To do so we observe:

$$a_{i,j} = 1 \text{ if the automorphism takes } t_i \rightarrow t_j$$

$$a_{i,j} = n \text{ if the automorphism takes } t_i \rightarrow t_j^n$$

that  $17^{*10}$  is a free product of ten cyclic groups of order 17. More specifically, we will construct a table with ten t's of order 17 labeled from 1...160 found in table 5.5. Viewing the first entry of  $A(x)$   $a_{1,2} = 13$  this implies  $t_1 \rightarrow t_2^{13}$  and so on for each power of  $t_1$  until we reach the identity. The top number of the chart labels each element, the middle of the chart shows what the automorphism produces and the bottom number shows what that produced element is numbered on the top of the chart. The same process is conducted for generators y and z but tables 5.6 and 5.7 will only show part of the process.

Table 5.5: Automorphisms of  $A(\mathbf{xx})$ 

1	2	3	4	5	6	7	8	9	10	11	12	13	14
$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$	$t_7$	$t_8$	$t_9$	$t_{10}$	$t_1^2$	$t_2^2$	$t_3^2$	$t_4^2$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_2^{13}$	$t_3^{16}$	$t_1^{13}$	$t_5$	$t_7^{15}$	$t_9$	$t_4^8$	$t_8$	$t_{10}$	$t_6$	$t_2^9$	$t_3^{15}$	$t_1^9$	$t_5^2$
122	153	121	5	147	9	74	8	10	6	72	143	81	15
15	16	17	18	19	20	21	22	23	24	25	26	27	28
$t_5^2$	$t_6^2$	$t_7^2$	$t_8^2$	$t_9^2$	$t_{10}^2$	$t_1^3$	$t_2^3$	$t_3^3$	$t_4^3$	$t_5^3$	$t_6^3$	$t_7^3$	$t_8^3$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_7^{13}$	$t_9^2$	$t_4^{16}$	$t_8^2$	$t_{10}^2$	$t_6^2$	$t_2^5$	$t_3^{14}$	$t_1^5$	$t_5^3$	$t_7^{11}$	$t_3^9$	$t_4^7$	$t_8^3$
127	19	154	18	20	16	42	133	41	25	107	29	64	28
29	30	31	32	33	34	35	36	37	38	39	40	41	42
$t_9^3$	$t_{10}^3$	$t_1^4$	$t_2^4$	$t_3^4$	$t_4^4$	$t_5^4$	$t_6^4$	$t_7^4$	$t_8^4$	$t_9^4$	$t_{10}^4$	$t_1^5$	$t_2^5$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_{10}^3$	$t_6^3$	$t_2$	$t_3^{13}$	$t_1$	$t_5^4$	$t_7^9$	$t_9^4$	$t_4^{15}$	$t_8^4$	$t_{10}^4$	$t_6^4$	$t_2^{14}$	$t_3^{12}$
30	26	2	123	1	35	87	39	144	38	40	36	132	113
43	44	45	46	47	48	49	50	51	52	53	54	55	56
$t_3^5$	$t_4^5$	$t_5^5$	$t_6^5$	$t_7^5$	$t_8^5$	$t_9^5$	$t_{10}^5$	$t_1^6$	$t_2^6$	$t_3^6$	$t_4^6$	$t_5^6$	$t_6^6$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_1^{14}$	$t_5^5$	$t_7^7$	$t_9^5$	$t_4^6$	$t_8^5$	$t_{10}^5$	$t_6^5$	$t_2^{10}$	$t_3^{11}$	$t_1^{10}$	$t_5^6$	$t_7^5$	$t_9^6$
131	45	67	49	54	48	50	46	92	103	91	55	47	59
57	58	59	60	61	62	63	64	65	66	67	68	69	70
$t_7^6$	$t_8^6$	$t_9^6$	$t_{10}^6$	$t_1^7$	$t_2^7$	$t_3^7$	$t_4^7$	$t_5^7$	$t_6^7$	$t_7^7$	$t_8^7$	$t_9^7$	$t_{10}^7$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_4^{14}$	$t_8^6$	$t_{10}^6$	$t_6^6$	$t_2^6$	$t_3^{10}$	$t_1^6$	$t_5^7$	$t_7^3$	$t_9^7$	$t_4^5$	$t_8^7$	$t_{10}^7$	$t_6^7$
134	58	60	56	52	93	51	65	27	69	44	68	70	66



71	72	73	74	75	76	77	78	79	80	81	82	83	84
$t_1^8$	$t_2^8$	$t_3^8$	$t_4^8$	$t_5^8$	$t_6^8$	$t_7^8$	$t_8^8$	$t_9^8$	$t_{10}^8$	$t_1^9$	$t_2^9$	$t_3^9$	$t_4^9$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_2^2$	$t_3^9$	$t_1^2$	$t_5^8$	$t_7$	$t_9^8$	$t_4^{13}$	$t_8^8$	$t_{10}^8$	$t_6^8$	$t_2^{15}$	$t_3^8$	$t_1^{15}$	$t_5^9$
12	83	11	75	7	79	124	78	80	76	142	73	141	85

85	86	87	88	89	90	91	92	93	94	95	96	97	98
$t_5^9$	$t_6^9$	$t_7^9$	$t_8^9$	$t_9^9$	$t_{10}^9$	$t_1^{10}$	$t_2^{10}$	$t_3^{10}$	$t_4^{10}$	$t_5^{10}$	$t_6^{10}$	$t_7^{10}$	$t_8^{10}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_7^{16}$	$t_9^9$	$t_4^4$	$t_8^9$	$t_{10}^9$	$t_6^9$	$t_2^{11}$	$t_3^7$	$t_1^{11}$	$t_5^{10}$	$t_7^{14}$	$t_9^{10}$	$t_4^{12}$	$t_8^{10}$
157	89	34	88	90	86	102	63	101	95	137	99	114	98

99	100	101	102	103	104	105	106	107	108	109	110	111	112
$t_9^{10}$	$t_{10}^{10}$	$t_1^{11}$	$t_2^{11}$	$t_3^{11}$	$t_4^{11}$	$t_5^{11}$	$t_6^{11}$	$t_7^{11}$	$t_8^{11}$	$t_9^{11}$	$t_{10}^{11}$	$t_1^{12}$	$t_2^{12}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_{10}^{10}$	$t_6^{10}$	$t_2^7$	$t_3^6$	$t_1^7$	$t_5^{11}$	$t_7^{12}$	$t_9^{11}$	$t_4^3$	$t_8^{11}$	$t_{10}^{11}$	$t_6^{11}$	$t_2^3$	$t_3^5$
100	96	62	53	61	105	117	109	24	108	110	106	22	43

113	114	115	116	117	118	119	120	121	122	123	124	125	126
$t_3^{12}$	$t_4^{12}$	$t_5^{12}$	$t_6^{12}$	$t_7^{12}$	$t_8^{12}$	$t_9^{12}$	$t_{10}^{12}$	$t_1^{13}$	$t_2^{13}$	$t_3^{13}$	$t_4^{13}$	$t_5^{13}$	$t_6^{13}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_1^3$	$t_5^{12}$	$t_7^{10}$	$t_9^{12}$	$t_4^{11}$	$t_8^{12}$	$t_{10}^{12}$	$t_6^{12}$	$t_2^{16}$	$t_3^4$	$t_1^{16}$	$t_5^{13}$	$t_7^8$	$t_9^{13}$
21	115	97	119	104	118	120	116	152	33	151	125	77	129

127	128	129	130	131	132	133	134	135	136	137	138	139	140
$t_7^{13}$	$t_8^{13}$	$t_9^{13}$	$t_{10}^{13}$	$t_1^{14}$	$t_2^{14}$	$t_3^{14}$	$t_4^{14}$	$t_5^{14}$	$t_6^{14}$	$t_7^{14}$	$t_8^{14}$	$t_9^{14}$	$t_{10}^{14}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_4^2$	$t_8^{13}$	$t_{10}^{13}$	$t_6^{13}$	$t_2^{12}$	$t_3^3$	$t_1^{12}$	$t_5^{14}$	$t_7^6$	$t_9^{14}$	$t_4^{10}$	$t_8^{14}$	$t_{10}^{14}$	$t_{10}^{14}$
14	128	130	126	112	23	111	135	57	139	94	138	140	136

141	142	143	144	145	146	147	148	149	150	151	152	153	154
$t_1^{15}$	$t_2^{15}$	$t_3^{15}$	$t_4^{15}$	$t_5^{15}$	$t_6^{15}$	$t_7^{15}$	$t_8^{15}$	$t_9^{15}$	$t_{10}^{15}$	$t_1^{16}$	$t_2^{16}$	$t_3^{16}$	$t_4^{16}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_2^8$	$t_3^2$	$t_1^8$	$t_5^{15}$	$t_7^4$	$t_9^{15}$	$t_4$	$t_8^{15}$	$t_{10}^{15}$	$t_6^{15}$	$t_2^4$	$t_3$	$t_1^4$	$t_5^{16}$
72	13	71	145	37	149	4	148	150	146	32	3	31	155

155	156	157	158	159	160
$t_5^{16}$	$t_6^{16}$	$t_7^{16}$	$t_8^{16}$	$t_9^{16}$	$t_{10}^{16}$
↓	↓	↓	↓	↓	↓
$t_7^2$	$t_9^{16}$	$t_4^9$	$t_8^{16}$	$t_{10}^{16}$	$t_6^{16}$
17	159	84	158	160	156

Table 5.6: Partial Y Table for  $G \cong PGL_2(9)$ 

1	2	3	...	...	...	158	159	160
$t_1$	$t_2$	$t_3$	...	...	...	$t_8^{16}$	$t_9^{16}$	$t_{10}^{16}$
↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_3$	$t_4^8$	$t_2^{13}$	...	...	...	$t_{10}^{16}$	$t_5^8$	$t_7^{16}$
3	74	122	...	...	...	160	75	157

Table 5.7: Partial Z Table for  $G \cong PGL_2(9)$ 

1	2	3	...	...	...	158	159	160
$t_1$	$t_2$	$t_3$	...	...	...	$t_8^{16}$	$t_9^{16}$	$t_{10}^{16}$
↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_2^8$	$t_1^{15}$	$t_4^{16}$	...	...	...	$t_5^{16}$	$t_6^2$	$t_7^8$
72	141	154	...	...	...	155	16	77

Using table 5.5, we can construct permutations using our labels for each automorphism. Consider element #1,  $t_1$  from the table. This produces the permutation (1,122,33) if we follow each element and its corresponding automorphism number labeling we produced. Repeating the process for x, y, and z, we produce the following permutations:

$$\begin{aligned}
X = & (1,122,33)(2,153,31)(3,121,152)(4,5,147)(6,9,10)(7,74,75)(11,82,73)(12,143,71) \\
& (13,81,142)(14,15,127)(16,19,20)(17,154,155)(21,42,113)(22,133,111)(23,41,132) \\
& (24,25,107)(26,29,30)(27,64,65)(32,123,151)(34,35,87)(36,39,40)(37,144,145) \\
& (43,131,112)(44,45,67)(46,49,50)(47,54,55)(51,92,63)(52,103,61)(53,91,102) \\
& (56,59,60)(57,134,135)(62,93,101)(66,69,70)(72,83,141)(76,79,80)(77,124,125) \\
& (84,85,157)(86,89,90)(94,95,137)(96,99,100)(97,114,115)(104,105,117)(106,109,110) \\
& (116,119,120)(126,129,130)(136,139,140)(146,149,150)(156,159,160)
\end{aligned}$$

$$\begin{aligned}
Y = & (1,3,122,14,6)(2,74,36,31,33)(4,86,81,83,142)(5,88,90,87,19)(7,39,15,8,10) \\
& (9,85,128,130,127)(11,13,82,34,16)(12,154,76,71,73)(17,79,35,18,20)(21,23,42,54,26)
\end{aligned}$$

(22,64,116,111,113)(24,96,91,93,102)(25,98,100,97,59)(27,119,55,28,30)(29,95,48,50,47)  
 (32,144,156,151,153)(37,159,75,38,40)(41,43,132,94,46)(44,106,101,103,62)  
 (45,108,110,107,99)(49,105,138,140,137)(51,53,92,114,56)(52,134,66,61,63)  
 (57,69,115,58,60)(65,118,120,117,139)(67,109,135,68,70)(72,124,146,141,143)  
 (77,149,155,78,80)(89,125,148,150,147)(104,136,131,133,112)(84,126,121,123,152)  
 (129,145,158,160,157)

Z=(1,72)(2,141)(3,154)(4,153)(5,8)(6,79)(7,20)(9,146)(10,87)(11,152)(12,121)(13,144)  
 (14,143)(15,18)(19,126)(21,62)(22,101)(23,134)(24,133)(25,28)(26,69)(27,60)(29,106)  
 (30,97)(31,142)(32,81)(33,124)(34,123)(35,38)(36,149)(37,80)(39,86)(17,40)(16,159)  
 (41,52)(42,61)(43,114)(44,113)(45,48)(46,59)(47,100)(49,66)(50,107)(51,132)(53,104)  
 (54,103)(55,58)(56,139)(57,120)(63,94)(64,93)(65,68)(70,117)(71,122)(73,84)(74,83)  
 (75,78)(76,129)(77,160)(82,151)(85,88)(89,156)(90,127)(91,112)(92,131)(95,98)(96,119)  
 (99,136)(102,111)(105,108)(109,116)(110,137)(115,118)(125,128)(130,147)(135,138)  
 (67,140)(145,148)(150,157)(155,158)

and we have completed the process of creating a permutation representation from our matrices.

### 5.2.5 Creating a Presentation of the Progenitor

To construct a presentation for the progenitor we must choose a  $t$  to normalize from our ten choices  $\langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle * \langle t_4 \rangle * \langle t_5 \rangle * \langle t_6 \rangle * \langle t_7 \rangle * \langle t_8 \rangle * \langle t_9 \rangle * \langle t_{10} \rangle$ . Let  $t \sim t_1$  and we must find permutations in  $PGL_2(9)$  which normalize  $\langle t_1 \rangle$ , or fix the following set:  $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, t_1^{13}, t_1^{14}, t_1^{15}, t_1^{16}, t_1^{17}\}$ . This is a defining characteristic of a monomial progenitors. Monomial progenitors fix a set of  $t$ 's while permutation progenitors fix only one specific  $t_i$ . We discover the normalizer of  $\langle t_1 \rangle$  in  $PGL_2(9)$  is generated by 4 elements which we will label A, B, C, and D:

A = (1, 81, 121, 141, 151, 71, 31, 11)(2, 19, 8, 83, 130, 4, 7, 5)(3, 90, 14, 17, 15, 12, 39, 18)(6, 16, 36, 76, 156, 146, 126, 86)(9, 88, 123, 150, 84, 87, 85, 82)(10, 34, 37, 35, 32, 79, 38, 13)(20, 74, 77, 75, 72, 159, 78, 33)(21, 91, 41, 101, 131, 61, 111, 51)(22,

59, 28, 93, 50, 24, 27, 25)(23, 100, 54, 57, 55, 52, 119, 58)(26, 56, 116, 66, 136, 106, 46, 96)(29, 98, 43, 110, 94, 97, 95, 92)(30, 114, 117, 115, 112, 69, 118, 53)(40, 154, 157, 155, 152, 149, 158, 73)(42, 99, 48, 103, 140, 44, 47, 45)(49, 108, 133, 70, 104, 107, 105, 102)(60, 64, 67, 65, 62, 139, 68, 113)(63, 120, 134, 137, 135, 132, 109, 138)(80, 144, 147, 145, 142, 129, 148, 153)(89, 128, 143, 160, 124, 127, 125, 122),

$B = (2, 40, 156)(3, 127, 84)(4, 13, 87)(5, 89, 148)(6, 152, 130)(7, 14, 33)(8, 75, 39)(9, 128, 15)(10, 36, 122)(12, 80, 146)(16, 142, 90)(17, 34, 73)(18, 155, 79)(19, 88, 35)(20, 76, 82)(22, 120, 136)(23, 47, 94)(24, 53, 97)(25, 99, 108)(26, 132, 50)(27, 54, 113)(28, 65, 119)(29, 48, 55)(30, 116, 42)(32, 160, 126)(37, 74, 153)(38, 145, 159)(43, 137, 104)(44, 93, 107)(45, 109, 68)(46, 112, 140)(49, 138, 95)(52, 70, 106)(56, 102, 100)(57, 114, 63)(58, 135, 69)(59, 98, 115)(60, 66, 92)(62, 110, 96)(64, 133, 117)(67, 134, 103)(72, 150, 86)(77, 154, 143)(78, 125, 149)(83, 147, 124)(85, 129, 158)(105, 139, 118)(123, 157, 144),$

$C = (2, 74, 15)(3, 159, 130)(4, 125, 142)(5, 82, 34)(6, 127, 38)(7, 158, 36)(8, 126, 157)(9, 40, 153)(10, 33, 129)(12, 154, 35)(13, 149, 90)(14, 85, 122)(16, 87, 78)(17, 148, 76)(18, 86, 147)(19, 80, 143)(20, 73, 89)(22, 64, 55)(23, 139, 50)(24, 45, 102)(25, 92, 114)(26, 47, 118)(27, 138, 116)(28, 46, 137)(29, 120, 133)(30, 113, 49)(32, 144, 75)(37, 128, 156)(39, 160, 123)(42, 54, 95)(43, 119, 140)(44, 135, 62)(48, 136, 117)(52, 134, 115)(53, 109, 100)(56, 97, 68)(57, 108, 66)(58, 96, 107)(59, 70, 103)(60, 63, 99)(65, 112, 104)(67, 98, 106)(69, 110, 93)(72, 124, 155)(77, 88, 146)(79, 150, 83)(84, 145, 152)(94, 105, 132),$  and

$D = (1, 121, 151, 31)(2, 123, 6, 129)(3, 36, 9, 32)(4, 17, 88, 155)(5, 154, 147, 78)(7, 128, 75, 84)(8, 145, 14, 37)(10, 40, 160, 130)(11, 81, 141, 71)(12, 83, 16, 89)(13, 76, 19, 72)(15, 144, 127, 158)(18, 125, 34, 77)(20, 80, 150, 90)(21, 41, 131, 111)(22, 43, 26, 49)(23, 116, 29, 112)(24, 57, 98, 135)(25, 134, 107, 68)(27, 48, 65, 94)(28, 105, 54, 117)(30, 120, 140, 50)(33, 156, 39, 152)(35, 124, 87, 148)(38, 85, 74, 157)(42, 133, 46, 139)(44, 97, 108, 115)(45, 114, 67, 58)(47, 138, 55, 104)(51, 91, 101, 61)(52, 93, 56, 99)(53, 66, 59, 62)(60, 70, 110, 100)(63, 106, 69, 102)(64, 137, 118, 95)(73, 146, 79, 142)(82, 143, 86, 149)(92, 103, 96, 109)(113, 136, 119, 132)(122, 153,$

126, 159)

For a presentation we must convert these permutations into words which we find to be:

$$\begin{aligned} A &= xy^{-1}x^{-1}zyx, \\ B &= y^2x^{-1}, \\ C &= x^{-1}zy^{-1}z, \text{ and} \\ D &= xzxy^{-1}zy^2 \end{aligned}$$

In the permutations of A, B, C, and D, we find that A sends  $1 \rightarrow 81$  or  $t_1 \rightarrow t_1^9$  and D sends  $1 \rightarrow 121$  or  $t_1 \rightarrow t_1^{13}$  Which implies the following respectively:

$$\begin{aligned} A &\rightarrow t^{xy^{-1}x^{-1}zyx}t^{-9} = \text{Id} \\ D &\rightarrow t^{xzy^{-1}zy^2}t^{-13} = \text{Id} \end{aligned}$$

which is the monomial part of the progenitor.

The other two words for B and C fix  $t_1$  and will commute with t as follows

$$\begin{aligned} B &\rightarrow (t, y^2x^{-1}) \\ C &\rightarrow (t, x^{-1}zy^{-1}z) \end{aligned}$$

and adding these to our presentation for  $PGL_2(9)$  we obtain conclude the process and obtain the monomial progenitor:

$$17^{*10} :_m PGL_2(9) = \langle x, y, z, t|x^3, y^5, z^2, t^{17}, y^{-1} * z * y^{-1} * x * z * y^{-1}, (x^{-1} * y^{-1})^4, (x^{-1}, y^{-1})^2, t^{xy^{-1}x^{-1}zyx}t^{-9}, t^{xzy^{-1}zy^2}t^{-13}, (t, y^2x^{-1}), (t, x^{-1}zy^{-1}z) \rangle .$$

### 5.3 Constructing the Monomial Progenitor $13^{*2} :_m (2^2 \times 3) :$

2

We will construct a monomial presentation of  $13^{*2} :_m (2^2 \times 3) : 2$  by first giving a presentation of  $(2^2 \times 3) : 2$ . That presentation is:

$$\langle x, y, z|x^4, y^2, z^3, (x^{-1} * y)^2, (x, z), y * z^{-1} * y * z \rangle ; .$$

To achieve a monomial presentation we must induce a linear character from a subgroup  $H$  of  $G$ . To insure we get an irreducible character we must choose a subgroup with index matching the degree of an irreducible character of  $G$ . We consider the character table of  $G = (2^2 \times 3) : 2$  and obtain tables 5.8 and 5.9 because we hope to induce the linear character  $\text{CG}[14]$  using  $\text{CH}[6]$ . We note the following as our reasoning of pursuing a linear character:  $[G : H] = [(2^2 \times 3) : 2 : 12] = 2$

Table 5.8:  $\chi_{14}$  of  $G$ 

$\phi^G$	Class	Size	Class Representative
2	$C_1$	1	Id( $G$ )
-2	$C_2$	1	(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)
0	$C_3$	2	(1, 7)(3, 9)(5, 11)
0	$C_4$	2	(1, 10)(2, 5)(3, 12)(4, 7)(6, 9)(8, 11)
$-2 - 2 * J$	$C_5$	1	(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)
$2 * J$	$C_6$	1	(1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8)
0	$C_7$	2	(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)
$-2 * J$	$C_8$	1	(1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)
$2 + 2 * J$	$C_9$	1	(1, 11, 9, 7, 5, 3)(2, 12, 10, 8, 6, 4)
0	$C_{10}$	2	(1, 11, 9, 7, 5, 3)(2, 6, 10)(4, 8, 12)
0	$C_{11}$	2	(1, 3, 5, 7, 9, 11)(2, 10, 6)(4, 12, 8)
0	$C_{12}$	2	(1, 2, 9, 10, 5, 6)(3, 4, 11, 12, 7, 8)
0	$C_{13}$	2	(1, 6, 5, 10, 9, 2)(3, 8, 7, 12, 11, 4)
0	$C_{14}$	2	(1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6)
0	$C_{15}$	2	(1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)

Where  $J$  is the 3rd root of unity

Table 5.9:  $\chi_{.6}$  of H

$\phi^G$	Class	Size	Class Representative
1	$D_1$	1	Id(G)
-1	$D_2$	1	(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)
J	$D_3$	1	(1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8)
-J - 1	$D_4$	1	(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)
$-I$	$D_5$	1	(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)
$I$	$D_6$	1	(1, 10, 7, 4)(2, 11, 8, 5)(3, 12, 9, 6)
1+ J	$D_7$	2	(1, 11, 9, 7, 5, 3)(2, 12, 10, 8, 6, 4)
$-J$	$D_8$	1	(1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)
$Z_1$	$D_9$	1	(1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)
$Z_1\#5$	$D_{10}$	2	(1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6)
$-Z_1$	$D_{11}$	2	(1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8)
$-Z_1\#5$	$D_{12}$	2	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)

Where J is the 3rd root of unity,

I is the 4th root of unity and  $Z_1$  is Cyclotomic Field 12.

Using the tables above we verify the induction similar to the earlier example. We will verify the induction  $\chi_{.6}$  of 12 to  $\chi_{.14}$  of  $(2^2 \times 3) : 2$  by considering the irreducible characters  $\phi$  (of H) and  $\phi^G$  (of G).  $G = (2^2 \times 3) : 2$  is generated by xx, yy, and zz where  $xx = (1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)$ ,  $yy = (1, 7)(3, 9)(5, 11)$ , and  $zz = (1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)$ . Using our definition of induction we reproduce  $\phi^G$  using  $\phi$  (of H).

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \text{ where } n = \frac{|G|}{|H|} = \frac{24}{12} = 2.$$

$$\phi_1^G = \frac{2}{1} \sum_{w \in H \cap C_1} \phi(w)$$

$$\text{which implies } \phi_1^G = \frac{2}{1}(\phi(1)) = 2(1) = 2.$$

$$\phi_2^G = \frac{2}{1} \sum_{w \in H \cap C_2} \phi(w)$$

$$\phi_2^G = \frac{2}{1} \sum_{w \in H \cap C_2} \phi((1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12))$$

which implies  $\phi_2^G = \frac{2}{1}(\phi(-1)) = \frac{2}{1}(-1) = -2$ .

$$\phi_3^G = \frac{2}{2} \sum_{w \in H \cap C_3} \phi(w)$$

$$\phi_3^G = \frac{2}{2} \sum_{w \in H \cap C_3} \phi((1, 7)(3, 9)(5, 11))$$

which implies  $\phi_3^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_4^G = \frac{2}{2} \sum_{w \in H \cap C_4} \phi(w)$$

$$\phi_4^G = \frac{2}{2} \sum_{w \in H \cap C_4} \phi((1, 10)(2, 5)(3, 12)(4, 7)(6, 9)(8, 11))$$

which implies  $\phi_4^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_5^G = \frac{2}{1} \sum_{w \in H \cap C_5} \phi(w)$$

$$\phi_5^G = \frac{2}{1} \sum_{w \in H \cap C_5} \phi((1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12))$$

which implies  $\phi_5^G = \frac{2}{1}(\phi(-J-1)) = \frac{2}{1}(-J-1) = -2 - 2 * J$ .

$$\phi_6^G = \frac{2}{1} \sum_{w \in H \cap C_6} \phi(w)$$

$$\phi_6^G = \frac{2}{1} \sum_{w \in H \cap C_6} \phi((1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8))$$

which implies  $\phi_6^G = \frac{2}{1}(\phi((1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8))) = \frac{2}{1}(J) = 2 * J$ .

$$\phi_7^G = \frac{2}{2} \sum_{w \in H \cap C_7} \phi(w)$$

$$\phi_7^G = \frac{2}{2} \sum_{w \in H \cap C_7} \phi((1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12))$$



which implies  $\phi_7^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_8^G = \frac{2}{1} \sum_{w \in H \cap C_8} \phi(w)$$

$$\phi_8^G = \frac{2}{1} \sum_{w \in H \cap C_8} \phi((1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12))$$

which implies  $\phi_8^G = \frac{2}{1}(\phi(-J)) = \frac{2}{1}(-J) = -2 * J$

$$\phi_9^G = \frac{2}{1} \sum_{w \in H \cap C_9} \phi(w)$$

$$\phi_9^G = \frac{2}{1} \sum_{w \in H \cap C_9} \phi((1, 11, 9, 7, 5, 3)(2, 12, 10, 8, 6, 4))$$

which implies  $\phi_9^G = \frac{2}{1}\phi(1 + J)$  or  $\frac{2}{1}(1 + J) = 2 + 2 * J$

$$\phi_{10}^G = \frac{2}{2} \sum_{w \in H \cap C_{10}} \phi(w)$$

$$\phi_{10}^G = \frac{2}{2} \sum_{w \in H \cap C_{10}} \phi((1, 11, 9, 7, 5, 3)(2, 6, 10)(4, 8, 12))$$

which implies  $\phi_{10}^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_{11}^G = \frac{2}{2} \sum_{w \in H \cap C_{11}} \phi(w)$$

$$\phi_{11}^G = \frac{2}{2} \sum_{w \in H \cap C_{11}} \phi((1, 3, 5, 7, 9, 11)(2, 10, 6)(4, 12, 8))$$

which implies  $\phi_{11}^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_{12}^G = \frac{2}{2} \sum_{w \in H \cap C_{12}} \phi(w)$$

$$\phi_{12}^G = \frac{2}{2} \sum_{w \in H \cap C_{12}} \phi((1, 3, 5, 7, 9, 11)(2, 10, 6)(4, 12, 8))$$

which implies  $\phi_{12}^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_{13}^G = \frac{2}{2} \sum_{w \in H \cap C_{13}} \phi(w)$$

$$\phi_{13}^G = \frac{2}{2} \sum_{w \in H \cap C_{13}} \phi((1, 6, 5, 10, 9, 2)(3, 8, 7, 12, 11, 4))$$

which implies  $\phi_{13}^G = \frac{2}{2}(\phi(0)) = \frac{2}{2}(0) = 0$ .

$$\phi_{14}^G = \frac{2}{2} \sum_{w \in H \cap C_{14}} \phi(w)$$

$$\phi_{14}^G = \frac{2}{2} \sum_{w \in H \cap C_{14}} \phi((1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6))$$

which implies  $\phi_{14}^G = \frac{2}{2}(2\phi(Z_1 \# 5 - Z_1 \# 5)) = \frac{2}{2}(2)(0) = 0$ .

$$\phi_{15}^G = \frac{2}{2} \sum_{w \in H \cap C_{15}} \phi(w)$$

$$\phi_{15}^G = \frac{2}{2} \sum_{w \in H \cap C_{15}} \phi((1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2))$$

which implies  $\phi_{15}^G = \frac{2}{2}(2\phi(Z_1 - Z_1)) = \frac{2}{2}(2)(0) = 0$ .

so  $\phi \uparrow_H^G = 2, -2, 0, 0, -2 - 2 * J, 2 * J, 0, -2 * J, 2 + 2 * J, 0, 0, 0, 0, 0, 0$  and we have verified that  $\chi_{.6}$  of  $C_{12}$  induces  $\chi_{.14}$  of  $(2^2 \times 3) : 2$ . With induction taking place we now verify the monomial representation has the following generators:

$$A(xx) = \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix}, A(yy) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } A(zy) = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}, \text{ and}$$

Since we have a linear character  $\phi$  of the subgroup H of index n in G we let  $G = Ht_1 \cup Ht_2$  where the  $t_1$  and  $t_2$  are transversals of G acting on H. That is  $G = He \cup H(1, 7)(3, 9)(5, 11)$ . Using the same process as earlier in this chapter we will verify the 4 elements of all three matrices using the following calculation  $t_1 = e$ , and  $t_2 = (1, 7)(3, 9)(5, 11)$ .

A[xx]: Row 1

$$\phi(t_1xt_1^{-1}) = \phi((1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)) = 5$$

$$\phi(t_1xt_2^{-1}) = \phi((1, 4)(2, 11)(3, 6)(5, 8)(7, 10)(9, 12)) = 0$$

Row 2:

$$\phi(t_2xt_1^{-1}) = \phi((1, 10)(2, 5)(3, 12)(4, 7)(6, 9)(8, 11)) = 0$$

$$\phi(t_2xt_2^{-1}) = \phi((1, 10, 7, 4)(2, 11, 8, 5)(3, 12, 9, 6)) = 8$$

A[yy]: Row 1

$$\phi(t_1yt_1^{-1}) = \phi((1, 7)(3, 9)(5, 11)) = 0$$

$$\phi(t_1yt_2^{-1}) = \phi(e) = 1$$

Row 2:

$$\phi(t_2yt_1^{-1}) = \phi(e) = 1$$

$$\phi(t_2yt_2^{-1}) = \phi((1, 7)(3, 9)(5, 11)) = 0$$

A[zz]: Row 1

$$\phi(t_1zt_1^{-1}) = \phi((1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)) = 9$$

$$\phi(t_1zt_2^{-1}) = \phi((1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)) = 0$$

Row 2:

$$\phi(t_2zt_1^{-1}) = \phi((1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)) = 0$$

$$\phi(t_2zt_2^{-1}) = \phi((1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)) = 9$$

The reader should note that in the Character Table, I refers to an element of order 4 in cyclotomic field 12. This means we are working in the field 13 and two elements of order four are 5 and 8. They are also inverses of each other which is why one I is positive and one I is negative. Similarly 3 is of order 3 in 13 and the element -J - 1 represents  $J^2$  or  $3^2$  which is 9.

We will construct automorphism tables for xx, yy and zz and then calculate the permutation representation. Each t has order 13 so we will have 12 copies of  $t_1$  and  $t_2$ . Using table 5.10, we can construct permutations using our labels for each automorphism just as in our previous example. We obtain permutations:

xx = (1, 9, 23, 15)(2, 16, 24, 10)(3, 19, 21, 5)(4, 6, 22, 20)(7, 13, 17, 11)(8, 12, 18, 14),  
 yy = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24),  
 and zz = (1, 17, 5)(2, 18, 6)(3, 9, 11)(4, 10, 12)(7, 19, 23)(8, 20, 24)(13, 21, 15)(14, 22, 16).

Which will be our permutation representation for our progenitor.

Table 5.10: Automorphisms of A(xx)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
$t_1$	$t_2$	$t_1^2$	$t_2^2$	$t_1^3$	$t_2^3$	$t_1^4$	$t_2^4$	$t_1^5$	$t_2^5$	$t_1^6$	$t_2^6$	$t_1^7$	$t_2^7$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_1^5$	$t_2^8$	$t_1^{10}$	$t_2^3$	$t_1^2$	$t_2^{11}$	$t_1^7$	$t_2^6$	$t_1^{12}$	$t_2$	$t_1^4$	$t_2^9$	$t_1^9$	$t_2^4$
9	16	19	6	3	22	13	12	23	2	7	18	17	8

15	16	17	18	19	20	21	22	23	24
$t_1^8$	$t_2^8$	$t_1^9$	$t_2^9$	$t_1^{10}$	$t_2^{10}$	$t_1^{11}$	$t_2^{11}$	$t_1^{12}$	$t_2^{12}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_1$	$t_2^{12}$	$t_1^6$	$t_2^7$	$t_1^{11}$	$t_2^2$	$t_1^3$	$t_2^{10}$	$t_1^8$	$t_2^5$
1	24	11	14	21	4	5	20	15	10

Table 5.11: Partial Y Table for  $G \cong (2^2 \times 3) : 2$

1	2	3	...	...	...	22	23	24
$t_1$	$t_2$	$t_1^2$	...	...	...	$t_2^{11}$	$t_1^{12}$	$t_2^{12}$
↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_2$	$t_1$	$t_2^2$	...	...	...	$t_1^{11}$	$t_2^{12}$	$t_1^{12}$
2	1	4	...	...	...	21	24	23

Table 5.12: Partial Z Table  $G \cong (2^2 \times 3) : 2$

1	2	3	...	...	...	22	23	24
$t_1$	$t_2$	$t_1^2$	...	...	...	$t_2^{11}$	$t_1^{12}$	$t_2^{12}$
↓	↓	↓	↓	↓	↓	↓	↓	↓
$t_1^9$	$t_2^9$	$t_1^5$	...	...	...	$t_2$	$t_1^4$	$t_2^4$
17	18	9	...	...	...	16	7	8

Once again we create a presentation for the progenitor from our two choices

for  $t$ .  $\langle t_1 \rangle * \langle t_2 \rangle$  Let  $t \sim t_1$  and we must find permutations in  $(2^2 \times 3) : 2$  that stabilize  $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}, t_1^{11}, t_1^{12}, t_1^{13}\}$ . We discover the normalizer of  $\langle t_1 \rangle$  in  $(2^2 \times 3) : 2$  is generated by 2 elements which we will label A and B:

$$A = (1, 15, 23, 9)(2, 10, 24, 16)(3, 5, 21, 19)(4, 20, 22, 6)(7, 11, 17, 13)(8, 14, 18, 12)$$

$$B = (1, 5, 17)(2, 6, 18)(3, 11, 9)(4, 12, 10)(7, 23, 19)(8, 24, 20)(13, 15, 21)(14, 16, 22)$$

For a presentation we must convert these permutations into words which we find to be:

$$A = x^{-1},$$

$$B = z^{-1},$$

In the permutations of A, we find that A sends  $1 \rightarrow 15$  or  $t_1 \rightarrow t_1^{12}$  and B sends  $1 \rightarrow 5$  or  $t_1 \rightarrow t_1^3$  Which implies the following respectively:

$$A \rightarrow t^{x^{-1}} t^{-12} = \text{Id}$$

$$B \rightarrow t^{z^{-1}} t^{-3} = \text{Id}$$

which is the monomial part of the progenitor and adding these to our presentation for  $(2^2 \times 3) : 2$  we conclude the process and obtain the monomial progenitor:

$$13^{*2} :_m (2^2 \times 3) : 2 = \langle t, x, y, z | t^{13}, x^4, y^2, z^3, (x^{-1} * y)^2, (x, z), y * z^{-1} * y * z, t^{x^{-1}} t^{-12}, t^{z^{-1}} t^{-3} \rangle .$$

## 5.4 Faithful Representations of Monomial Presentations

A faithful representation may not always be possible to construct. Consider the example from chapter 2,  $\mathbb{Z}_2 \wr D_4$ . We constructed this wreath product by hand and it had the following generators:  $(1, 5), (2, 6), (3, 7), (4, 8), (1, 2, 3, 4)(5, 6, 7, 8), (1, 3)(5, 7)$ . We can view the Character table of this group G, and investigate the subgroups H which induce a linear character on G. Using the following code we can produce a list of all inductions that take place among G and H. The user of this code should know, z should be chosen based on the Chi table of G. In this example the first column of the chi table

of  $G$  was 4 for the values 15 through 20. Since  $G$  is order 128,  $H$  should be chosen so that  $\frac{G}{H} = 4$  which implies the order of  $H = 32$  which also should be entered into the code. So  $z$  is in [15..20] while  $\#H$  eq 32 in the code and should be modified as the user requires.

```
Sg:=Subgroups(G);
CG:=CharacterTable(G);
for x in [1..#Sg] do
H:=Sg[x] `subgroup;
CH:=CharacterTable(H);
for y in [1..#CH] do
for z in [15..20] do
if Induction(CH[y],G) eq CG[z] and #H eq 32
then x,y,z;
end if; end for; end for; end for;
```

Items on this list are of the form 159 2 17 for instance. This indicates to the user, that the subgroup  $H$  listed as number 159 has  $\chi.2$  in the character table which induces up to  $G$  for the  $\chi.17$  of  $G$ . This will of course need to be entered and verified using the following code:

```
>H:=Sg[159] `subgroup;
>CH:=CharacterTable(H);
>I:=Induction(CH[2],G);
>I eq CG[17];
true
```

From here we investigate the character table of  $H$  to determine which cyclotomic field we are working in. Since there is no indicator at the bottom of the table, this implies we are working in  $\text{CyclotomicField}(2)$  or  $\mathbb{Z}_3$ . Now that we know which field we are working in, the transversals are used to construct the matrices. Fortunately, we have code that allows us to construct the matrices using Magma. These matrices are representations of our elements of  $G$ . Since matrices can be multiplied, multiplying a matrix by itself until it produces the identity matrix determines the order of the matrix. If the order of this matrix multiplication matches the order of our permutation from  $G$ , we call this a faithful representation of that element. If this is not the case, we call this an unfaithful representation and it will not produce the faithful representation of  $G$  we hope for so we terminate the induction we have attempted and pursue a new one.

Using the following code we will verify that the matrix representation of our generator  $x$  of  $G$  is faithful. We must also verify the other generators of  $G$  this work can be found in the Appendix.

```
>T:=Transversal(G,H);
>C:=CyclotomicField(2);
>GG:=GL(4,C);
>X:=[[C.1,0,0,0]: i in [1..4]];
>for i,j in [1..4] do X[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
>X[i,j]:=CH[2](T[i]*xx*T[j]^-1); end if; end for;
> Order(GG!X);
2
```

Since our generator from  $G = \mathbb{Z}_2 \wr D_4$  is of order 2, we consider this faithful. The rest of our matrices are also faithful to the generators of  $G$ , but this of course may not always be the case. We must also check any products that take place in our presentation to make sure they produce a faithful representation. Our progenitor for  $G$  has the relation  $(e * f)^2 = Id(G)$ . So naturally our matrix multiplication must have the same order and it does.

From here we can use the matrix function in Magma, if we define how Magma should label the entries. Since we are working in cyclotomic field 2, we need an element in  $\mathbb{Z}_3$  of order 2. Since  $2^2 = 1 \pmod{3}$ , 2 is the element we seek. So we label the mat function as follows:

```
mat := function(n,p,D,k)
for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
if CH[n](T[i]*p*T[j]^-1) eq C.1 then D[i,j]:=2; end if;
if CH[n](T[i]*p*T[j]^-1) eq C.1^2 then D[i,j]:=1; end if;
if CH[n](T[i]*p*T[j]^-1) in {1,-1} then
D[i,j]:=CH[n](T[i]*p*T[j]^-1);
end if; end if; end for;
return D;
end function;
```

Now, we take the set of all four by four invertible matrices over the field  $\mathbb{Z}_3$ , and using matrices, attempt to construct a group isomorphic to our  $G$ . Using the following code for each of generators, we determine suitable generators for our subgroup of  $GL(4,3)$ :

```

>GG:=GL(4,3);
>X:=[0,0,0,0]: i in [1..4]];
>mat(2,xx,X,4);
>XX:=GG!mat(2,xx,X,4);
>Order(GG!XX);

> XX;          > YY;          > ZZ;          > WW;          > UU;
[2 0 0 0]      [2 0 0 0]      [2 0 0 0]      [1 0 0 0]      [0 1 0 0]
[0 1 0 0]      [0 2 0 0]      [0 2 0 0]      [0 2 0 0]      [0 0 1 0]
[0 0 2 0]      [0 0 1 0]      [0 0 2 0]      [0 0 2 0]      [0 0 0 1]
[0 0 0 2]      [0 0 0 2]      [0 0 0 1]      [0 0 0 2]      [1 0 0 0]

> VV;
[2 0 0 0]
[0 0 0 2]
[0 0 2 0]
[0 2 0 0]

```

Now that we have our matrices labeled XX, YY, ..., VV, we can create a subgroup of  $GL(4,3)$  using the following command and check if it is isomorphic to our G.

```

>HH:=sub<GG|XX,YY,ZZ,WW,UU,VV>;
>#HH;
128
>IsIsomorphic(HH,G);
true

```

Since this is true, this subgroup of order 32 produced a faithful monomial representation of G. From here we can proceed to introduce a free product to our N and construct the progenitor. The following tables list faithful and unfaithful representations for all proper subgroups of G.



Table 5.13: Faithful Representations for  $G \cong Z_2 \wr D_4$ 

Order of G	Generators of G	Isomorphism of G																														
#G = 128	S:=Sym(8) x:=S!(1, 5), y:=S!(2, 6), z:=S!(3, 7), w:=S!(4, 8), u:=S!(1, 2, 3, 4)(5, 6, 7, 8), v:=S!(1, 3)(5, 7)	$G \cong Z_2 \wr D_4$ (constructed by hand)																														
Order of H	Generators of H	Faithful Representation																														
#H = 32	Subgroup[159] x:=(1, 3)(5, 7), y:=(3, 7)(4, 8), z:=(1, 5)(2, 6)(3, 7)(4, 8), w:=(3, 7), u:=(1, 5)(3, 7)	<table style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>X</th> <th>Y</th> <th>Z</th> <th>W</th> </tr> </thead> <tbody> <tr><td>[2 0 0 0]</td><td>[2 0 0 0]</td><td>[2 0 0 0]</td><td>[1 0 0 0]</td></tr> <tr><td>[0 1 0 0]</td><td>[0 2 0 0]</td><td>[0 2 0 0]</td><td>[0 2 0 0]</td></tr> <tr><td>[0 0 2 0]</td><td>[0 0 1 0]</td><td>[0 0 2 0]</td><td>[0 0 2 0]</td></tr> <tr><td>[0 0 0 2]</td><td>[0 0 0 2]</td><td>[0 0 0 1]</td><td>[0 0 0 1]</td></tr> </tbody> </table> <table style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>U</th> <th>V</th> </tr> </thead> <tbody> <tr><td>[0 1 0 0]</td><td>[2 0 0 0]</td></tr> <tr><td>[0 0 1 0]</td><td>[0 0 0 2]</td></tr> <tr><td>[0 0 0 1]</td><td>[0 0 2 0]</td></tr> <tr><td>[1 0 0 0]</td><td>[0 2 0 0]</td></tr> </tbody> </table>	X	Y	Z	W	[2 0 0 0]	[2 0 0 0]	[2 0 0 0]	[1 0 0 0]	[0 1 0 0]	[0 2 0 0]	[0 2 0 0]	[0 2 0 0]	[0 0 2 0]	[0 0 1 0]	[0 0 2 0]	[0 0 2 0]	[0 0 0 2]	[0 0 0 2]	[0 0 0 1]	[0 0 0 1]	U	V	[0 1 0 0]	[2 0 0 0]	[0 0 1 0]	[0 0 0 2]	[0 0 0 1]	[0 0 2 0]	[1 0 0 0]	[0 2 0 0]
X	Y	Z	W																													
[2 0 0 0]	[2 0 0 0]	[2 0 0 0]	[1 0 0 0]																													
[0 1 0 0]	[0 2 0 0]	[0 2 0 0]	[0 2 0 0]																													
[0 0 2 0]	[0 0 1 0]	[0 0 2 0]	[0 0 2 0]																													
[0 0 0 2]	[0 0 0 2]	[0 0 0 1]	[0 0 0 1]																													
U	V																															
[0 1 0 0]	[2 0 0 0]																															
[0 0 1 0]	[0 0 0 2]																															
[0 0 0 1]	[0 0 2 0]																															
[1 0 0 0]	[0 2 0 0]																															
#H = 32	Subgroup[163] x:=(3, 7), y:=(1, 3)(5, 7), z:=(1, 5)(3, 7), w:=(1, 3, 5, 7)(2, 4, 6, 8), u:=(1, 5)(2, 6)(3, 7)(4, 8)  CH[11] up to CG[16]	<table style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>X</th> <th>Y</th> <th>Z</th> <th>W</th> </tr> </thead> <tbody> <tr><td>[1 0 0 0]</td><td>[0 1 0 0]</td><td>[1 0 0 0]</td><td>[0 -1 0 0]</td></tr> <tr><td>[0 1 0 0]</td><td>[1 0 0 0]</td><td>[0 1 0 0]</td><td>[-1 0 0 0]</td></tr> <tr><td>[0 0 0 2]</td><td>[0 0 1 0]</td><td>[0 0 0 3]</td><td>[0 0 1 0]</td></tr> <tr><td>[0 0 3 0]</td><td>[0 0 0 1]</td><td>[0 0 2 0]</td><td>[0 0 0 1]</td></tr> </tbody> </table> <table style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>U</th> <th>V</th> </tr> </thead> <tbody> <tr><td>[0 0 1 0]</td><td>[-1 0 0 0]</td></tr> <tr><td>[0 0 0 3]</td><td>[0 -1 0 0]</td></tr> <tr><td>[0 2 0 0]</td><td>[0 0 0 -1]</td></tr> <tr><td>[1 0 0 0]</td><td>[0 0 -1 0]</td></tr> </tbody> </table>	X	Y	Z	W	[1 0 0 0]	[0 1 0 0]	[1 0 0 0]	[0 -1 0 0]	[0 1 0 0]	[1 0 0 0]	[0 1 0 0]	[-1 0 0 0]	[0 0 0 2]	[0 0 1 0]	[0 0 0 3]	[0 0 1 0]	[0 0 3 0]	[0 0 0 1]	[0 0 2 0]	[0 0 0 1]	U	V	[0 0 1 0]	[-1 0 0 0]	[0 0 0 3]	[0 -1 0 0]	[0 2 0 0]	[0 0 0 -1]	[1 0 0 0]	[0 0 -1 0]
X	Y	Z	W																													
[1 0 0 0]	[0 1 0 0]	[1 0 0 0]	[0 -1 0 0]																													
[0 1 0 0]	[1 0 0 0]	[0 1 0 0]	[-1 0 0 0]																													
[0 0 0 2]	[0 0 1 0]	[0 0 0 3]	[0 0 1 0]																													
[0 0 3 0]	[0 0 0 1]	[0 0 2 0]	[0 0 0 1]																													
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[0 0 0 3]	[0 -1 0 0]																															
[0 2 0 0]	[0 0 0 -1]																															
[1 0 0 0]	[0 0 -1 0]																															

Order of H	Generators of H	Faithful Representation																														
#H = 32	Subgroup[167] x:=(3, 7), y:=(1, 3)(5, 7), z:=(1, 5)(3, 7), w:=(1, 3)(2, 4)(5, 7)(6, 8), u:=(1, 5)(2, 6)(3, 7)(4, 8)  CH[2] up to CG[17]	<table style="margin-left: auto; margin-right: auto;"> <tr> <td>X</td> <td>Y</td> <td>Z</td> <td>W</td> </tr> <tr> <td>[-1 0 0 0]</td> <td>[0 1 0 0]</td> <td>[-1 0 0 0]</td> <td>[0 -1 0 0]</td> </tr> <tr> <td>[0 -1 0 0]</td> <td>[1 0 0 0]</td> <td>[0 -1 0 0]</td> <td>[-1 0 0 0]</td> </tr> <tr> <td>[0 0 0 -1]</td> <td>[0 0 -1 0]</td> <td>[0 0 0 1]</td> <td>[0 0 -1 0]</td> </tr> <tr> <td>[0 0 -1 0]</td> <td>[0 0 0 -1]</td> <td>[0 0 1 0]</td> <td>[0 0 0 -1]</td> </tr> </table> <table style="margin-left: auto; margin-right: auto;"> <tr> <td>U</td> <td>V</td> </tr> <tr> <td>[0 0 1 0]</td> <td>[-1 0 0 0]</td> </tr> <tr> <td>[0 0 0 1]</td> <td>[0 -1 0 0]</td> </tr> <tr> <td>[-1 0 0 0]</td> <td>[0 0 1 0]</td> </tr> <tr> <td>[0 1 0 0]</td> <td>[0 0 0 -1]</td> </tr> </table>	X	Y	Z	W	[-1 0 0 0]	[0 1 0 0]	[-1 0 0 0]	[0 -1 0 0]	[0 -1 0 0]	[1 0 0 0]	[0 -1 0 0]	[-1 0 0 0]	[0 0 0 -1]	[0 0 -1 0]	[0 0 0 1]	[0 0 -1 0]	[0 0 -1 0]	[0 0 0 -1]	[0 0 1 0]	[0 0 0 -1]	U	V	[0 0 1 0]	[-1 0 0 0]	[0 0 0 1]	[0 -1 0 0]	[-1 0 0 0]	[0 0 1 0]	[0 1 0 0]	[0 0 0 -1]
X	Y	Z	W																													
[-1 0 0 0]	[0 1 0 0]	[-1 0 0 0]	[0 -1 0 0]																													
[0 -1 0 0]	[1 0 0 0]	[0 -1 0 0]	[-1 0 0 0]																													
[0 0 0 -1]	[0 0 -1 0]	[0 0 0 1]	[0 0 -1 0]																													
[0 0 -1 0]	[0 0 0 -1]	[0 0 1 0]	[0 0 0 -1]																													
U	V																															
[0 0 1 0]	[-1 0 0 0]																															
[0 0 0 1]	[0 -1 0 0]																															
[-1 0 0 0]	[0 0 1 0]																															
[0 1 0 0]	[0 0 0 -1]																															
#H = 64	$\emptyset$	No Faithful Representations Exist																														

Table 5.14: Induction of Character up to  $G \cong \mathbb{Z}_2 \wr D_4$ 

Subgroup[159] up to G =	$\chi.2 = 1 -1 1 -1 1 -1 1 -1 -1 1 -1 1 -1 1 1 -1 1 -1$ $\chi.17 = 4 -4 0 -2 0 0 2 -2 2 0 0 0 2 -2 0 0 0 0 0 0$
Subgroup[163] up to G =	$\chi.11 = 1 -1 -1 1 -1 1 1 -1 -I I I -I I -I -I I -1 1 I -I$ I is a 4 <sup>th</sup> root of unity $\chi.16 = 4 -4 0 2 0 0 -2 -2 2 0 0 0 -2 2 0 0 0 0 0 0$
Subgroup[167] up to G =	$\chi.2 = 1 -1 1 -1 1 -1 1 -1 -1 1 -1 1 -1 1 1 -1 1 -1$ $\chi.17 = 4 -4 0 -2 0 0 2 -2 2 0 0 0 2 -2 0 0 0 0 0 0$

#### 5.4.1 Determining Permutation Representation from a Faithful Representation Using Magma $3^{*4} : \mathbb{Z}_2 \wr D_4$

When writing monomial progenitors it is important to obtain the most efficient presentation possible. Unfortunately, the presentation for  $\mathbb{Z}_2 \wr D_4$  when calculated by hand has many unnecessary relations that can be eliminated if we use another version of the wreath product. We know that  $\text{TransitiveGroup}(8,35) \cong \mathbb{Z}_3 \wr D_4$ . Magma also has a command  $W := \text{WreathProduct}(\text{CyclicGroup}(2), \text{DihedralGroup}(4))$ , which is isomorphic to both of these previously mentioned groups. Letting N be the group defined at the beginning of section 5.3, and W be the WreathProduct constructed through Magma the presentation comparison leaves little doubt which presentation is more efficient.

```
> W:=WreathProduct(CyclicGroup(2),DihedralGroup(4));
> FPGroup(W);
```

```

Finitely presented group on 3 generators
Relations
$.1^4 = Id($)
$.2^2 = Id($)
$.3^2 = Id($)
($.1^-1 * $.2)^2 = Id($)
($.1 * $.3 * $.2)^2 = Id($)
($.1 * $.3 * $.1^-1 * $.3)^2 = Id($)
$.1^-1 * $.3 * $.1^-2 * $.3 * $.1^2 * $.3 * $.1^2 * $.3
* $.1^-1 = Id($)
> N:=TransitiveGroup(8,35);
> FPGGroup(N);
Finitely presented group on 6 generators
Relations
$.1^2 = Id($)
$.2^2 = Id($)
$.3^2 = Id($)
$.4^2 = Id($)
$.5^4 = Id($)
$.6^2 = Id($)
($.1 * $.2)^2 = Id($)
($.1 * $.3)^2 = Id($)
($.2 * $.3)^2 = Id($)
($.1 * $.4)^2 = Id($)
($.2 * $.4)^2 = Id($)
($.3 * $.4)^2 = Id($)
$.1 * $.5^-1 * $.4 * $.5 = Id($)
$.2 * $.5^-1 * $.1 * $.5 = Id($)
$.3 * $.5^-1 * $.2 * $.5 = Id($)
$.1 * $.6 * $.3 * $.6 = Id($)
($.2 * $.6)^2 = Id($)
($.4 * $.6)^2 = Id($)
($.5^-1 * $.6)^2 = Id($)

```

Due to the efficiency of the presentation generated by Magma we will produce our progenitor using  $G \cong W$  generated by  $x = (1, 3, 5, 7)(2, 4, 6, 8)$ ,  $y = (1, 7)(2, 8)(3, 5)(4, 6)$ , and  $z = (1, 2)$  with presentation  $\langle x, y, z \mid x^4, y^2, z^2, (x^{-1} * y)^2, (x^{-1} * z * y)^2, (x * z * x^{-1} * z)^2, x^{-1} * z * x^{-2} * z * x^2 * z * x^2 * z * x^{-1} \rangle$ . This  $G$  has a different character table which we must investigate to produce a faithful monomial representation. The same process that was conducted earlier in this section was reproduced here, which identified  $\chi.2$  of  $H$  inducing  $\chi.19$  of  $G$  and producing a faithful representation. Once we have this

faithful representation, we can produce the permutations that will be used.

Using the following code we construct the faithful representation for our new  $G(W)$ , and produce our monomial representation.

```

> S:=Sym(8);
> xx:=S!(1, 3, 5, 7)(2, 4, 6, 8);
> yy:=S!(1, 7)(2, 8)(3, 5)(4, 6);
> zz:=S!(1, 2);
> G:=sub<S|xx,yy,zz>;
> Sg:=Subgroups(G);
> CG:=CharacterTable(G);
> for x in [1..#Sg] do
for> H:=Sg[x] `subgroup;
for> CH:=CharacterTable(H);
for> for y in [1..#CH] do
for|for> for z in [15..20] do
for|for|for> if #H eq 32 and Induction(CH[y],G) eq CG[z]
for|for|for|if> then x,y,z;
for|for|for|if> end if;
for|for|for> end for;
for|for> end for;
for> end for;
> H:=Sg[159] `subgroup;
> S:=Sym(8);
> xh:=S!(1, 5)(2, 6);
> yh:=S!(5, 6)(7, 8);
> zh:=S!(7, 8);
> wh:=S!(1, 2)(3, 4)(5, 6)(7, 8);
> uh:=S!(1, 2)(7, 8);
> H:=sub<S|xh,yh,zh,wh,uh>;
> CH:=CharacterTable(H);
> I:=Induction(CH[2],G);
> I eq CG[19];
true
> T:=Transversal(G,H);
> C:=CyclotomicField(2);
> GG:=GL(4,3);
> X=[[C.1,0,0,0]: i in [1..4]];
> for i,j in [1..4] do X[i,j]:=0; end for;
> Y=[[C.1,0,0,0]: i in [1..4]];
> for i,j in [1..4] do Y[i,j]:=0; end for;
> Z=[[C.1,0,0,0]: i in [1..4]];

```

```

> for i,j in [1..4] do Z[i,j]:=0; end for;
> for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H then
for|if> X[i,j]:=CH[2]( T[i]*xx*T[j]^-1); end if; end for;
> for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H then
for|if> Y[i,j]:=CH[2]( T[i]*yy*T[j]^-1); end if; end for;
> for i,j in [1..4] do if T[i]*zz*T[j]^-1 in H then
for|if> Z[i,j]:=CH[2]( T[i]*zz*T[j]^-1); end if; end for;
> Order(GG!X);
4
> Order(GG!Y);
2
> Order(GG!Z);
2
> GG!X;
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]
> GG!Y;
[0 2 0 0]
[2 0 0 0]
[0 0 0 2]
[0 0 2 0]
> GG!Z;
[2 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 2]
> mat := function(n,p,D,k)
function> for i,j in [1..k] do if T[i]*p*T[j]^-1 in H then
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq C.1
function|for|if> then D[i,j]:=2; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) eq C.1^2
function|for|if> then D[i,j]:=1; end if;
function|for|if> if CH[n](T[i]*p*T[j]^-1) in {1,-1}
function|for|if> then D[i,j]:=CH[n](T[i]*p*\
T[j]^-1); end if;
function|for|if> end if; end for;
function> return D;
function> end function;
> GG:=GL(4,3);
> X:=[[0,0,0,0]: i in [1..4]];
> mat(2,xx,X,4);
[ 0, 1, 0, 0 ],

```

```

      [ 0, 0, 1, 0 ],
      [ 0, 0, 0, 1 ],
      [ 1, 0, 0, 0 ]
> XX:=GG!mat(2,xx,X,4);
> Order(GG!XX);
4
> Y:=[[0,0,0,0]: i in [1..4]];
> mat(2,yy,Y,4);
      [ 0, -1, 0, 0 ],
      [ -1, 0, 0, 0 ],
      [ 0, 0, 0, -1 ],
      [ 0, 0, -1, 0 ]
> YY:=GG!mat(2,yy,Y,4);
> Order(GG!YY);
2
> Z:=[[0,0,0,0]: i in [1..4]];
> mat(2,zz,Z,4);
      [ -1, 0, 0, 0 ],
      [ 0, 1, 0, 0 ],
      [ 0, 0, -1, 0 ],
      [ 0, 0, 0, -1 ]
> ZZ:=GG!mat(2,zz,Z,4);
> Order(GG!ZZ);
2
> HH:=sub<GG|XX,YY,ZZ>;
> #HH;
128
> IsIsomorphic(HH,G);
true

```

Now, that we have established a faith representation, we can use Magma, to determine the permutations that were constructed by hand in the previous two sections of this chapter. Using the following code produces the following permutations:

```

perm := function(n, p, mat)
C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do
s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];

```

```

for i in [1..n] do
j:=0; done:=0;
repeat
if z[1,i]/u^j in Z then
  if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
  done:=1;
end if; end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do
for a in [1..n] do
  w[a+i*n]:= (Z!w[a]+i*n-1) mod (p*n) + 1;
end for; end for;
S:=Sym(n*p);
w:=S!w;
return w;
end function;

> perm(4,2,GG!X);
(1, 4, 3, 2) (5, 8, 7, 6)
> perm(4,2,GG!Y);
(1, 6) (2, 5) (3, 8) (4, 7)
> perm(4,2,GG!Z);
(1, 5) (3, 7) (4, 8)

```

From this point we proceed by stabilizing one of our  $t$ 's. Since we did not construct the permutation by hand we must calculate how Magma labeled  $t_1$  and  $t_1^2$  if we wish to stabilize  $t_1$ . Luckily the function stores them in a rather simple manner. We know our progenitor will be  $3^{*4} : W$  since we are working in cyclotomic field 3 with matrices of rank 4. This indicates we have 4  $t$ 's to choose from of order 3. This is important since Magma labels  $t_1$  as  $\{1\}$  and will label  $t_1^2$  immediately after it labels  $t_2, t_3$ , and  $t_4$  meaning Magma will label  $t_1^2$  as  $\{5\}$ . The function always works in a similar way and counts  $t_1, t_2, \dots, t_n, t_1^2, t_2^2, \dots, t_n^2$  labeling them respectively 1, 2, ..., n, n+1, n+2, ... n+n. This indicates that any permutation sending 1 to 5, is in our stabilizer of  $t_1$ . We notice, the permutation for  $Z$  is the one that does this from above and now we can complete the monomial progenitor.

Using the permutations we obtained from the matrices and the stabilizers of 1 and 5 in the following code we produce the words  $t_1$  commutes with that will produce our infinite progenitor.

```

> S:=Sym(8);
> xx:=S!(1, 4, 3, 2)(5, 8, 7, 6);
> yy:=S!(1, 6)(2, 5)(3, 8)(4, 7);
> zz:=S!(1, 5)(3, 7)(4, 8);
> N:=sub<S|xx,yy,zz>;

> NN<x,y,z>:=Group<x,y,z|x^4,y^2,z^2,(x^-1*y)^2,(x^-1*z*y)^2,
(x*z*x^-1*z)^2,x^-1*z*x^-2*z*x^2*z*x^2*z*x^-1>;

> Stabilizer(N,{1,5});
Permutation group acting on a set of cardinality 8
Order = 32 = 2^5
      (2, 8)(3, 7)(4, 6)
      (3, 7)(4, 8)
      (3, 7)
      (1, 5)(3, 7)(4, 8)

> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..128]];
> for i in [2..128] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;

> for i in [1..128] do if ArrayP[i] eq N!(2, 4)(6, 8) then
for|if> print Sch[i]; end if; end for;
x * z * y * z * x * z * x * z
> for i in [1..128] do if ArrayP[i] eq N!(3, 7)(4, 8) then
for|if> print Sch[i]; end if; end for;
x^2 * z * x^-1 * z * x^-1

```



```

> for i in [1..128] do if ArrayP[i] eq N!(3, 7) then
for|if> print Sch[i]; end if; end for;
x^2 * z * x * z * x * z
> for i in [1..128] do if ArrayP[i] eq N!(1, 5)(3, 7)(4, 8)
for|if> then print Sch[i]; end if; end for;
z

```

Since the only entity in the stabilizer that sends 1 to 5 is  $z$  itself we have,  $t^z = t^2$  as one relation and the remaining three words commuting with  $t_1$  as the following relations:  $(t, x*z*y*z*x*z*x*z)$ ,  $(t, x^2*z*x^{-1}*z*x^{-1})$ , and  $(t, x^2*z*x*z*x*z)$ . Which produces our final infinite monomial progenitor presentation:

$$G \langle x, y, z, t \mid x^4, y^2, z^2, t^3, (x^{-1}*y)^2, (x^{-1}*z*y)^2, (x*z*x^{-1}*z)^2, x^{-1}*z*x^{-2}*z*x^2*z*x^2*z*x^{-1}, (t, x*z*y*z*x*z*x*z), (t, x^2*z*x^{-1}*z*x^{-1}), (t, x^2*z*x*z*x*z), t^z = t^2 \rangle$$

Finally, to insure that this process was done correctly and the progenitor is in fact  $3^{*4} : \mathbb{Z}_2 \wr D_4$  where the order of  $\mathbb{Z}_2 \wr D_4$  is 128, we can use Grindstaff's Lemma to check the order of our progenitor calculated as  $3^4 \times 128 = 10,368$ . This process will be carried out in the next chapter of this document with other examples as well.

## 5.5 Character Table Construction

This section will first demonstrate the ability to produce a character table for the abelian group  $G \cong 3 \times 4$  by hand. This process will be conducted using the guidelines defined in chapter 1 using definitions 1.6 thru 1.65 and theorems 1.85 thru 1.89. We will then move on to show that we can produce a character table for other groups that have been investigated in this study.

### 5.5.1 Constructing a Character Table for $3 \times 4$

To begin the process we consider the generators of  $\mathbb{Z}_3$  and  $\mathbb{Z}_4$ ,  $\mathbb{Z}_3 = \langle a \rangle$  and  $\mathbb{Z}_4 = \langle b \rangle$ . Now, before we begin the chart we must determine the conjugacy classes of our table. The following list is that is generated as follows:

$$\langle a \rangle \times \langle b \rangle = \{e, a, a^2, b, b^2, b^3, ab, a^2b, ab^2, a^2b^2, ab^3, a^2b^3\}$$

Now, Magma lists elements from the conjugacy class in increasing order fashion. This study will use the following order and labels for the character table we will construct:

Table 5.15: Conjugacy Classes  $3 \times 4$  with Representative

# on Table	Conjugacy Class	Permutation
1	$e$	$\text{Id}(G)$
2	$b^2$	$(4,6)(5,7)$
3	$a$	$(1,2,3)$
4	$a^2$	$(1,3,2)$
5	$b$	$(4,5,6,7)$
6	$b^3$	$(4,7,6,5)$
7	$a^2b^2$	$(1,3,2)(4,6)(5,7)$
8	$ab^2$	$(1,2,3)(4,6)(5,7)$
9	$ab$	$(1,2,3)(4,5,6,7)$
10	$a^2b$	$(1,3,2)(4,5,6,7)$
11	$ab^3$	$(1,2,3)(4,7,6,5)$
12	$a^2b^3$	$(1,3,2)(4,7,6,5)$

Now, we will set up the table so it can be filled in. We know the number of conjugacy classes is equal to the number of irreducible characters and will refer back to table 5.15 when we make calculations throughout this process. We construct the following empty character table:

Table 5.16: Initial Character Table for  $3 \times 4$ 

Classes	1	2	3	4	5	6	7	8	9	10	11	12
Size	1	1	1	1	1	1	1	1	1	1	1	1
Order	1	2	3	3	4	4	6	6	12	12	12	12
$\chi.1$												
$\chi.2$												
$\chi.3$												
$\chi.4$												
$\chi.5$												
$\chi.6$												
$\chi.7$												
$\chi.8$												
$\chi.9$												
$\chi.10$												
$\chi.11$												
$\chi.12$												

We will begin systematically filling out the chart using our definitions from chapter 1. The reader should note that a w in the chart refers to a  $3^{rd}$  root of unity, an I refers to a  $4^{th}$  root of unity, and a K refers to a  $12^{th}$  root of unity.

Now, we begin by examining  $\chi.1$  by substituting  $w^0$  and  $I^0$  into each conjugacy class above. Since, both entries  $w^0 = 1$ , and  $I^0 = 1$  we have the following result:

$\chi.1$  evaluated at  $w^0, I^0 = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$

This indicates our first row is all 1's. Now, we match our 5.15 table with the corresponding number on table 5.16. We will evaluate the 2nd row as it is more interesting. We will substitute the relation  $w^0 i^1$  into each conjugacy class. Since a is paired with w, wherever we see "a" in the conjugacy class  $a = w^0 = 1$ . Similarly, wherever we see a "b" in a conjugacy class we will substitute  $b = i^1 = i$ .

The following calculations will represent this idea  $C_1$  represents column 1.  $C_1 \chi.2$  represents column 1 of  $\chi.2$ :

$$\begin{aligned}
C_1 = e &\implies C_{1\chi.2} = e = 1 \\
C_2 = b^2 &\implies C_{2\chi.2} = b^2 = (i)^2 = -1 \\
C_3 = a &\implies C_{3\chi.2} = a = w^0 = 1 \\
C_4 = a^2 &\implies C_{4\chi.2} = a^2 = (w^0)^2 = 1 \\
C_5 = b &\implies C_{5\chi.2} = b = i = I \\
C_6 = b^3 &\implies C_{6\chi.2} = b^3 = (i)^3 = -I \\
C_7 = a^2b^2 &\implies C_{7\chi.2} = a^2b^2 = (w^0)^2(i)^2 = -1 \\
C_8 = ab^2 &\implies C_{8\chi.2} = ab^2 = w^0(i)^2 = -1 \\
C_9 = ab &\implies C_{9\chi.2} = ab = w^0i = I \\
C_{10} = a^2b &\implies C_{10\chi.2} = a^2b = (w^0)^2i = I \\
C_{11} = ab^3 &\implies C_{11\chi.2} = ab^3 = w^0(i)^3 = -I \\
C_{12} = a^2b^3 &\implies C_{12\chi.2} = a^2b^3 = (w^0)^2(i)^3 = -I
\end{aligned}$$

Inserting these entries in their appropriate place from top to bottom, left to right in the table gives us  $\chi_{.1}$  and  $\chi_{.2}$  filled in. Similarly we will repeat this process for all other entries of the table. Now we consider  $a = w^0, b = i^2$  for the  $\chi_{.3}$  entries.

$$\begin{aligned}
C_1 = e &\implies C_{1\chi.3} = e = 1 \\
C_2 = b^2 &\implies C_{2\chi.3} = b^2 = (i^2)^2 = 1 \\
C_3 = a &\implies C_{3\chi.3} = a = w^0 = 1 \\
C_4 = a^2 &\implies C_{4\chi.3} = a^2 = (w^0)^2 = 1 \\
C_5 = b &\implies C_{5\chi.3} = b = i^2 = -1 \\
C_6 = b^3 &\implies C_{6\chi.3} = b^3 = (i^2)^3 = -1
\end{aligned}$$

$$\begin{aligned}
C_7 = a^2b^2 &\implies C_{7\chi.3} = a^2b^2 = (w^0)^2(i^2)^2 = 1 \\
C_8 = ab^2 &\implies C_{8\chi.3} = ab^2 = w^0(i^2)^2 = 1 \\
C_9 = ab &\implies C_{9\chi.3} = ab = w^0i^2 = -1 \\
C_{10} = a^2b &\implies C_{10\chi.3} = a^2b = (w^0)^2i^2 = -1 \\
C_{11} = ab^3 &\implies C_{11\chi.3} = ab^3 = w^0(i^2)^3 = -1 \\
C_{12} = a^2b^3 &\implies C_{12\chi.3} = a^2b^3 = (w^0)^2(i^2)^3 = -1
\end{aligned}$$

If we considered  $a = w^0, b = i^3$  we would produce  $\chi_{.4} = 1, -1, 1, 1, -I, I, -1, -1, -I, -I, I, I$  respectively.

Next, if we consider  $a = w^2, b = i^0$  we obtain  $\chi_{.5}$  as follows:

$$\begin{aligned}
C_1 = e &\implies C_{1\chi.5} = e = 1 \\
C_2 = b^2 &\implies C_{2\chi.5} = b^2 = i^0 = 1 \\
C_3 = a &\implies C_{3\chi.5} = a = w^2 = w^2 \\
C_4 = a^2 &\implies C_{4\chi.5} = a^2 = (w^2)^2 = w^4 = w \\
C_5 = b &\implies C_{5\chi.5} = b = i^0 = 1 \\
C_6 = b^3 &\implies C_{6\chi.5} = b^3 = (i^0)^3 = 1 \\
C_7 = a^2b^2 &\implies C_{7\chi.5} = a^2b^2 = (w^2)^2(i^0)^2 = w^4 = w \\
C_8 = ab^2 &\implies C_{8\chi.5} = ab^2 = w^2(i^0)^2 = w^2 \\
C_9 = ab &\implies C_{9\chi.5} = ab = w^2i^0 = w^2 \\
C_{10} = a^2b &\implies C_{10\chi.5} = a^2b = (w^2)^2i^0 = w^4 = w \\
C_{11} = ab^3 &\implies C_{11\chi.5} = ab^3 = w^2(i^0)^3 = w^2 \\
C_{12} = a^2b^3 &\implies C_{12\chi.5} = a^2b^3 = (w^2)^2(i^0)^3 = w^4 = w
\end{aligned}$$

For  $\chi_6$  we consider  $a = w^2, b = i$ . and determine  $\chi_6$  as follows

$$\begin{aligned}
C_1 = e &\implies C_{1\chi_6} = e = 1 \\
C_2 = b^2 &\implies C_{2\chi_6} = b^2 = (i)^2 = -1 \\
C_3 = a &\implies C_{3\chi_6} = a = w^2 = w^2 \\
C_4 = a^2 &\implies C_{4\chi_6} = a^2 = (w^2)^2 = w^4 = w \\
C_5 = b &\implies C_{5\chi_6} = b = i = I \\
C_6 = b^3 &\implies C_{6\chi_6} = b^3 = (i)^3 = -I \\
C_7 = a^2b^2 &\implies C_{7\chi_6} = a^2b^2 = (w^2)^2(i)^2 = w^4(-1) = -w \\
C_8 = ab^2 &\implies C_{8\chi_6} = ab^2 = w^2(i)^2 = w^2(-1) = -w^2 \\
C_9 = ab &\implies C_{9\chi_6} = ab = w^2i = w^2i \\
C_{10} = a^2b &\implies C_{10\chi_6} = a^2b = (w^2)^2i = w^4 = wi \\
C_{11} = ab^3 &\implies C_{11\chi_6} = ab^3 = w^2(i)^3 = -w^2i \\
C_{12} = a^2b^3 &\implies C_{12\chi_6} = a^2b^3 = (w^2)^2(i)^3 = w^4 = -wi
\end{aligned}$$

Now, for the columns that resulted in a combination of w and i. Magma will label this as  $Z1$  which as indicated at the bottom of the table resides in  $\mathbb{Z}_{12}$ . So rather than force the user to calculate which field is being used Magma has made reading the table more simple. For the remainder of this table construction  $k = w^2i$ , and  $k^2 = wi$ , which implies  $-k = -w^2i$  and  $-k^2 = -wi$ .

We will demonstrate the process one more time for  $\chi_8$  (since  $\chi_7$  will only contain w as a root) as the remainder of the chart is repetitive and then present the final result of our work by hand. Taking  $a = w^2$  and  $b = i^3$  we compute  $\chi_8$  as:

$$\begin{aligned}
C_1 = e &\implies C_{1\chi.8} = e = 1 \\
C_2 = b^2 &\implies C_{2\chi.8} = b^2 = (i^3)^2 = -1 \\
C_3 = a &\implies C_{3\chi.8} = a = w^2 = w^2 \\
C_4 = a^2 &\implies C_{4\chi.8} = a^2 = (w^2)^2 = w^4 = w \\
C_5 = b &\implies C_{5\chi.8} = b = i^3 = -I \\
C_6 = b^3 &\implies C_{6\chi.8} = b^3 = (i^3)^3 = I \\
C_7 = a^2b^2 &\implies C_{7\chi.8} = a^2b^2 = (w^2)^2(i^3)^2 = w^4(-1) = -w \\
C_8 = ab^2 &\implies C_{8\chi.8} = ab^2 = w^2(i)^2 = w^2(-1) = -w^2 \\
C_9 = ab &\implies C_{9\chi.8} = ab = w^2i^3 = -w^2i = -k \\
C_{10} = a^2b &\implies C_{10\chi.8} = a^2b = (w^2)^2i^3 = w^4 = -wi = -k^2 \\
C_{11} = ab^3 &\implies C_{11\chi.8} = ab^3 = w^2(i^3)^3 = w^2i = k \\
C_{12} = a^2b^3 &\implies C_{12\chi.8} = a^2b^3 = (w^2)^2(i^3)^3 = w^4 = wi = k^2
\end{aligned}$$

after completing the rest of the calculations, the following character table is produced which can be compared to the character table of  $\mathbb{Z}_3 \times \mathbb{Z}_4$  in Magma, to see they are copies of one another.

Table 5.17: Final Character Table for  $3 \times 4$ 

w, i	Classes	1	2	3	4	5	6	7	8	9	10	11	12
	Size	1	1	1	1	1	1	1	1	1	1	1	1
	Order	1	2	3	3	4	4	6	6	12	12	12	12
$w^0, i^0$	$\chi.1$	1	1	1	1	1	1	1	1	1	1	1	1
$w^0, i^1$	$\chi.2$	1	-1	1	1	I	-I	-1	-1	I	I	-I	-I
$w^0, i^2$	$\chi.3$	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1
$w^0, i^3$	$\chi.4$	1	-1	1	1	-I	I	-1	-1	-I	-I	I	I
$w^2, i^0$	$\chi.5$	1	1	$w^2$	w	1	1	w	$w^2$	$w^2$	w	$w^2$	w
$w^2, i^1$	$\chi.6$	1	-1	$w^2$	w	I	-I	-w	$-w^2$	k	$k^2$	-k	$-k^2$
$w^2, i^2$	$\chi.7$	1	1	$w^2$	w	-1	-1	w	$w^2$	$-w^2$	-w	$-w^2$	-w
$w^2, i^3$	$\chi.8$	1	-1	$w^2$	w	-I	I	-w	$-w^2$	-k	$-k^2$	k	$k^2$
$w^1, i^0$	$\chi.9$	1	1	w	$w^2$	1	1	$w^2$	w	w	$w^2$	w	$w^2$
$w^1, i^2$	$\chi.10$	1	-1	w	$w^2$	I	-I	$-w^2$	-w	$k^2$	k	$-k^2$	-k
$w^1, i^2$	$\chi.11$	1	1	w	$-w^2$	-1	-1	$w^2$	w	-w	$-w^2$	-w	$-w^2$
$w^1, i^3$	$\chi.12$	1	-1	w	$w^2$	-I	I	$-w^2$	-w	$-k^2$	-k	$k^2$	k

## 5.6 Lifting Character to $H \cong 3^2 : 8$

Let  $G \cong PGL_2(9)$  with subgroup  $H \cong 3^2 : 8$ . We know a faithful irreducible monomial representation of  $G$  exists from the subgroup  $H$  as  $\chi_5^H$  induces up to  $\chi_{11}^G$ . Now, we will investigate  $H$  and find a suitable normal subgroup,  $H'$ , or the derived group  $H$ . Considering the isomorphism type of  $H$ , we know the largest abelian subgroup of  $H$  is  $3^2$ . So we can determine the generators of  $H$  using the following commands:

```
> S:=Sym(12);
> x:=S!(1, 6, 11, 8, 3, 2, 5, 12)(4, 7, 10, 9);
> y:=S!(1, 11, 3, 5)(2, 12, 6, 8)(4, 10)(7, 9);
> z:=S!(1, 3)(2, 6)(5, 11)(8, 12);
> w:=S!(2, 6, 10)(4, 8, 12)(5, 11, 9);
> u:=S!(1, 7, 3)(2, 6, 10)(5, 9, 11);
> H:=sub<S|x,y,z,w,u>;
>
> dH:=DerivedGroup(H);
> dH;
Permutation group dH acting on a set of cardinality 12
Order = 9 = 3^2
(1, 3, 7)(2, 6, 10)(4, 12, 8)
(1, 3, 7)(2, 10, 6)(5, 11, 9)
```

Now,  $H' = \langle (1, 3, 7)(2, 6, 10)(4, 12, 8), (1, 3, 7)(2, 10, 6)(5, 11, 9) \rangle \cong 3^2$  We want to find a character table for  $H/H' \cong 8$ , but we first need to know what  $H/H'$  is composed of. We know  $|\frac{H}{H'}| = \frac{72}{9} = 8$ . We expect 8 transversals which create the set:

$$H/H' = \{H(e), H(1, 6, 11, 8, 3, 2, 5, 12)(4, 7, 10, 9), H(1, 11, 3, 5)(2, 12, 6, 8)(4, 10)(7, 9), H(1, 3)(2, 6)(5, 11)(8, 12), H(1, 8, 5, 6, 3, 12, 11, 2)(4, 9, 10, 7), H(1, 2, 11, 12, 3, 6, 5, 8)(4, 7, 10, 9), H(1, 5, 3, 11)(2, 8, 6, 12)(4, 10)(7, 9), H(1, 12, 5, 2, 3, 8, 11, 6)(4, 9, 10, 7)\}$$

We know  $H/H' \cong \mathbb{Z}_8$ . So we determine the conjugacy classes as follows:



Table 5.18: Conjugacy Classes  $H' \cong 8$ 

# on Table	Conjugacy Class	Permutation
1	e	Id(H)
2	$x^4$	(1, 3)(2, 6)(5, 11)(8, 12)
3	$x^2$	(1, 11, 3, 5)(2, 12, 6, 8)(4, 10)(7, 9)
4	$x^6$	(1, 5, 3, 11)(2, 8, 6, 12)(4, 10)(7, 9)
5	$x$	(1, 6, 11, 8, 3, 2, 5, 12)(4, 7, 10, 9)
6	$x^3$	(1, 8, 5, 6, 3, 12, 11, 2)(4, 9, 10, 7)
7	$x^5$	(1, 2, 11, 12, 3, 6, 5, 8)(4, 7, 10, 9)
8	$x^7$	(1, 12, 5, 2, 3, 8, 11, 6)(4, 9, 10, 7)

Now,  $H/H' = \langle H(1, 6, 11, 8, 3, 2, 5, 12)(4, 7, 10, 9) \rangle$  if we consider Magma's labeling of the character table is, it indicates a fourth root of unity and eighth root of unity. When proceeding from here we will need to substitute a value for  $I$ . The following calculation will help us determine this.

$$\begin{aligned} n &= \cos \frac{2\pi}{8} + i \sin \frac{2\pi}{8} \implies n^2 = (\cos \frac{2\pi}{8} + i \sin \frac{2\pi}{8})^2 = \cos^2(\frac{2\pi}{8}) + 2i \sin \frac{2\pi}{8} \cos \frac{2\pi}{8} + \sin^2(\frac{2\pi}{8}) \\ &= \frac{1}{2} + 2i(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}) - \frac{1}{2} = 2i \frac{1}{2} = i \implies n^2 = i \end{aligned}$$

It follows that anywhere in our chart where we have indicated  $n^2$  shall be replaced with  $I$ . Similarly,  $(n^2)^2 = n^4 = i^2 = -1$ , and  $(n^2)^3 = n^6 = i^3 = -I$ . We compute the entries of our character table for  $H/H'$  as:

Table 5.19: Initial Character Table of  $H/H' \cong \mathbb{Z}_8$ 

n	Classes	1	2	3	4	5	6	7	8
	Size	1	1	1	1	1	1	1	1
	Order	1	2	4	4	8	8	8	8
$n^0$	$\chi_{.1}^\bullet$	$(n^0)^0$	$(n^0)^4$	$(n^0)^2$	$(n^0)^6$	$(n^0)^1$	$(n^0)^3$	$(n^0)^5$	$(n^0)^7$
$n^1$	$\chi_{.2}^\bullet$	$(n^1)^0$	$(n^1)^4$	$(n^1)^2$	$(n^1)^6$	$(n^1)^1$	$(n^1)^3$	$(n^1)^5$	$(n^1)^7$
$n^2$	$\chi_{.3}^\bullet$	$(n^2)^0$	$(n^2)^4$	$(n^2)^2$	$(n^2)^6$	$(n^2)^1$	$(n^2)^3$	$(n^2)^5$	$(n^2)^7$
$n^3$	$\chi_{.4}^\bullet$	$(n^3)^0$	$(n^3)^4$	$(n^3)^2$	$(n^3)^6$	$(n^3)^1$	$(n^3)^3$	$(n^3)^5$	$(n^3)^7$
$n^4$	$\chi_{.5}^\bullet$	$(n^4)^0$	$(n^4)^4$	$(n^4)^2$	$(n^4)^6$	$(n^4)^1$	$(n^4)^3$	$(n^4)^5$	$(n^4)^7$
$n^5$	$\chi_{.6}^\bullet$	$(n^5)^0$	$(n^5)^4$	$(n^5)^2$	$(n^5)^6$	$(n^5)^1$	$(n^5)^3$	$(n^5)^5$	$(n^5)^7$
$n^6$	$\chi_{.7}^\bullet$	$(n^6)^0$	$(n^6)^4$	$(n^6)^2$	$(n^6)^6$	$(n^6)^1$	$(n^6)^3$	$(n^6)^5$	$(n^6)^7$
$n^7$	$\chi_{.8}^\bullet$	$(n^7)^0$	$(n^7)^4$	$(n^7)^2$	$(n^7)^6$	$(n^7)^1$	$(n^7)^3$	$(n^7)^5$	$(n^7)^7$

Which simplifies to:

n	Classes	1	2	3	4	5	6	7	8
	Size	1	1	1	1	1	1	1	1
	Order	1	2	4	4	8	8	8	8
$n^0$	$\chi_{.1}^\bullet$	$n^0$	$n^0$	$n^0$	$n^0$	$n^0$	$n^0$	$n^0$	$n^0$
$n^1$	$\chi_{.2}^\bullet$	$n^0$	$n^4$	$n^2$	$n^6$	$n^1$	$n^3$	$n^5$	$n^7$
$n^2$	$\chi_{.3}^\bullet$	$n^0$	$n^8$	$n^4$	$n^4$	$n^2$	$n^6$	$n^2$	$n^6$
$n^3$	$\chi_{.4}^\bullet$	$n^0$	$n^4$	$n^6$	$n^2$	$n^3$	$n^1$	$n^7$	$n^5$
$n^4$	$\chi_{.5}^\bullet$	$n^0$	$n^8$	$n^8$	$n^8$	$n^4$	$n^4$	$n^4$	$n^4$
$n^5$	$\chi_{.6}^\bullet$	$n^0$	$n^4$	$n^2$	$n^6$	$n^5$	$n^7$	$n^1$	$n^3$
$n^6$	$\chi_{.7}^\bullet$	$n^0$	$n^8$	$n^4$	$n^4$	$n^6$	$n^2$	$n^6$	$n^2$
$n^7$	$\chi_{.8}^\bullet$	$n^0$	$n^4$	$n^6$	$n^2$	$n^7$	$n^5$	$n^3$	$n^1$

And substituting  $n^0 = n^8 = 1$ ,  $n^2 = i$ ,  $n^4 = -1$ , and  $n^6 = -I$  we also note, that  $(-1) * n^1 = n^5$ , and  $(-1) * n^3 = n^7$  so we will replace  $n^5 = -n$  and  $n^7 = -n^3$  and construct our final  $H/H'$  character table as:



Now, using our definition of lifting character, we begin lifting each irreducible character from  $H/H'$  to  $H$ . We note,  $\chi(h) = \chi^\bullet(h)$  for  $h \in H$ .

$\chi_{.1}$  calculations

$$\chi_{.1}(h) = \chi_{.1}^\bullet(h) = 1 \quad \forall h \in H/H'$$

$\chi_{.2}$  calculations

$$\begin{aligned} \chi_{.2}(e) &= \chi_{.2}^\bullet(e) = 1 \\ \chi_{.2}(x^4) &= \chi_{.2}^\bullet(x^4) = -1 \\ \chi_{.2}(w) &= \chi_{.2}^\bullet(H') = 1 \\ \chi_{.2}(x^2) &= \chi_{.2}^\bullet(x^2) = I \\ \chi_{.2}(x^6) &= \chi_{.2}^\bullet(x^6) = -I \\ \chi_{.2}(x^5 * w) &= \chi_{.2}^\bullet(x) = n \\ \chi_{.2}(x^3) &= \chi_{.2}^\bullet(x^3) = n^3 \\ \chi_{.2}(x^5) &= \chi_{.2}^\bullet(x^5) = -n \\ \chi_{.2}(x^7) &= \chi_{.2}^\bullet(x^7) = -n^3 \end{aligned}$$

$\chi_{.3}$  calculations

$$\begin{aligned} \chi_{.3}(e) &= \chi_{.3}^\bullet(e) = 1 \\ \chi_{.3}(x^4) &= \chi_{.3}^\bullet(x^4) = 1 \\ \chi_{.3}(w) &= \chi_{.3}^\bullet(H') = 1 \\ \chi_{.3}(x^2) &= \chi_{.3}^\bullet(x^2) = -1 \\ \chi_{.3}(x^6) &= \chi_{.3}^\bullet(x^6) = -1 \\ \chi_{.3}(x^5 * w) &= \chi_{.3}^\bullet(x) = I \\ \chi_{.3}(x^3) &= \chi_{.3}^\bullet(x^3) = -I \\ \chi_{.3}(x^5) &= \chi_{.3}^\bullet(x^5) = I \\ \chi_{.3}(x^7) &= \chi_{.3}^\bullet(x^7) = -I \end{aligned}$$

$\chi_{.4}$  calculations

$$\begin{aligned}\chi_{.4}(e) &= \chi_{.4}^\bullet(e) = 1 \\ \chi_{.4}(x^4) &= \chi_{.4}^\bullet(x^4) = -1 \\ \chi_{.4}(w) &= \chi_{.4}^\bullet(H') = 1 \\ \chi_{.4}(x^2) &= \chi_{.4}^\bullet(x^2) = -I \\ \chi_{.4}(x^6) &= \chi_{.4}^\bullet(x^6) = I \\ \chi_{.4}(x^5 * w) &= \chi_{.4}^\bullet(x) = n^3 \\ \chi_{.4}(x^3) &= \chi_{.4}^\bullet(x^3) = n \\ \chi_{.4}(x^5) &= \chi_{.4}^\bullet(x^5) = -n^3 \\ \chi_{.4}(x^7) &= \chi_{.4}^\bullet(x^7) = -n\end{aligned}$$

$\chi_{.5}$  calculations

$$\begin{aligned}\chi_{.5}(e) &= \chi_{.5}^\bullet(e) = 1 \\ \chi_{.5}(x^4) &= \chi_{.5}^\bullet(x^4) = 1 \\ \chi_{.5}(w) &= \chi_{.5}^\bullet(H') = 1 \\ \chi_{.5}(x^2) &= \chi_{.5}^\bullet(x^2) = 1 \\ \chi_{.5}(x^6) &= \chi_{.5}^\bullet(x^6) = 1 \\ \chi_{.5}(x^5 * w) &= \chi_{.5}^\bullet(x) = -1 \\ \chi_{.5}(x^3) &= \chi_{.5}^\bullet(x^3) = -1 \\ \chi_{.5}(x^5) &= \chi_{.5}^\bullet(x^5) = -1 \\ \chi_{.5}(x^7) &= \chi_{.5}^\bullet(x^7) = -1\end{aligned}$$

$\chi_{.6}$  calculations

$$\begin{aligned}\chi_{.6}(e) &= \chi_{.6}^\bullet(e) = 1 \\ \chi_{.6}(x^4) &= \chi_{.6}^\bullet(x^4) = -1 \\ \chi_{.6}(w) &= \chi_{.6}^\bullet(H') = 1 \\ \chi_{.6}(x^2) &= \chi_{.6}^\bullet(x^2) = I \\ \chi_{.6}(x^6) &= \chi_{.6}^\bullet(x^6) = -I \\ \chi_{.6}(x^5 * w) &= \chi_{.6}^\bullet(x) = -n\end{aligned}$$

$$\chi_{.6}(x^3) = \chi_{.6}^\bullet(x^3) = -n^3$$

$$\chi_{.6}(x^5) = \chi_{.6}^\bullet(x^5) = n$$

$$\chi_{.6}(x^7) = \chi_{.6}^\bullet(x^7) = n^3$$

$\chi_{.7}$  calculations

$$\chi_{.7}(e) = \chi_{.7}^\bullet(e) = 1$$

$$\chi_{.7}(x^4) = \chi_{.7}^\bullet(x^4) = 1$$

$$\chi_{.7}(w) = \chi_{.7}^\bullet(H') = 1$$

$$\chi_{.7}(x^2) = \chi_{.7}^\bullet(x^2) = -1$$

$$\chi_{.7}(x^6) = \chi_{.7}^\bullet(x^6) = -1$$

$$\chi_{.7}(x^5 * w) = \chi_{.7}^\bullet(x) = -I$$

$$\chi_{.7}(x^3) = \chi_{.7}^\bullet(x^3) = I$$

$$\chi_{.7}(x^5) = \chi_{.7}^\bullet(x^5) = -I$$

$$\chi_{.7}(x^7) = \chi_{.7}^\bullet(x^7) = I$$

$\chi_{.8}$  calculations

$$\chi_{.8}(e) = \chi_{.8}^\bullet(e) = 1$$

$$\chi_{.8}(x^4) = \chi_{.8}^\bullet(x^4) = -1$$

$$\chi_{.8}(w) = \chi_{.8}^\bullet(H') = 1$$

$$\chi_{.8}(x^2) = \chi_{.8}^\bullet(x^2) = -I$$

$$\chi_{.8}(x^6) = \chi_{.8}^\bullet(x^6) = I$$

$$\chi_{.8}(x^5 * w) = \chi_{.8}^\bullet(x) = -n^3$$

$$\chi_{.8}(x^3) = \chi_{.8}^\bullet(x^3) = -n$$

$$\chi_{.8}(x^5) = \chi_{.8}^\bullet(x^5) = n^3$$

$$\chi_{.8}(x^7) = \chi_{.8}^\bullet(x^7) = n$$

This produces the following character table of H.

Table 5.22: Lifted  $H \cong 3^2 : 8$  Character Table

Classes	e	$x^4$	w	$x^2$	$x^6$	x	$x^3$	$x^5$	$x^7$
Size	1	9	8	9	9	9	9	9	9
Order	1	2	3	4	4	8	8	8	8
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	I	-I	n	$n^3$	-n	$-n^3$
$\chi_{.3}$	1	1	1	-1	-1	I	-I	I	-I
$\chi_{.4}$	1	-1	1	-I	I	$n^3$	n	$-n^3$	-n
$\chi_{.5}$	1	1	1	1	1	-1	-1	-1	-1
$\chi_{.6}$	1	-1	1	I	-I	-n	$-n^3$	n	$n^3$
$\chi_{.7}$	1	1	1	-1	-1	-I	I	-I	I
$\chi_{.8}$	1	-1	1	-I	I	$-n^3$	-n	$n^3$	n
$\chi_{.9}$	a	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	$\beta_6$	$\beta_7$	$\beta_8$

Finally we can produce  $\chi_{.9}$  using theorem 1.87 which states that the sum of squares of degrees of the distinct irreducible characters of  $G$  is equal to  $|G|$  at the degree of character  $\chi(1)$ . What this allows us to do is take the first column of our character table and construct the equation:  $(\chi_{.1}(1))^2 + (\chi_{.2}(1))^2 + \dots + (\chi_{.9}(1))^2 = |G|$ . This implies  $1^2 + 1^2 + \dots + 1^2 + a^2 = 72$  Now,  $8 + a^2 = 72 \implies a = 8, -8$  but  $a$  cannot be negative so  $a = 8$ . Now that we have the full first column we can use it to determine the remaining entries in  $\chi_{.9}$  using definition 1.61:

$$\sum_{i=1}^k \chi_{\alpha}^{(i)} \overline{\chi_{\beta}^{(j)}} = 0$$

Now we construct the inner product from  $\chi_{.9}^{(1)}$  to  $\chi_{.9}^{(j)}$ , where  $\chi_{.9}^{(1)}$  is column 1 and  $\chi_{.9}^{(j)}$  is all columns for  $2 \leq j \leq 9$ .

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(2)}} &= 1 * \bar{1} + 1 * \overline{-1} + 1 * \bar{1} + 1 * \overline{-1} + 1 * \bar{1} + 1 * \overline{-1} + 1 * \bar{1} + 1 * \overline{-1} + 1 * \bar{1} + 1 * \overline{-1} + 8 * \overline{\beta_1} = 0 \\ \implies 8\overline{\beta_1} &= 0 \implies \beta_1 = 0 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(3)}} &= 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 1 * \bar{1} + 8 * \overline{\beta_2} = 0 \\ \implies 8 + 8\overline{\beta_2} &= 0 \implies \beta_2 = -1 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(4)}} &= 1 * \bar{1} + 1 * \bar{I} + 1 * \overline{-1} + 1 * \overline{-I} + 1 * \bar{1} + 1 * \bar{I} + 1 * \overline{-1} + 1 * \overline{-I} + 8 * \overline{\beta_3} = 0 \\ \implies 1 - i - 1 + i + 1 - i - 1 + i + 8\overline{\beta_3} &= 0 \implies 8\overline{\beta_3} = 0 \implies \beta_3 = 0 \end{aligned}$$

$$\sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(5)}} = 1 * \bar{1} + 1 * \overline{-I} + 1 * \overline{-1} + 1 * \bar{I} + 1 * \bar{1} + 1 * \overline{-I} + 1 * \overline{-1} + 1 * \bar{I} + 8 * \overline{\beta_4} = 0$$

$$\implies 1 + i - 1 - i + 1 + i - 1 - i + 8\overline{\beta_4} = 0 \implies 8\beta_4 = 0 \implies \beta_4 = 0$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(6)}} &= 1*\overline{1} + 1*\overline{n} + 1*\overline{I} + 1*\overline{n^3} + 1*\overline{-1} + 1*\overline{-n} + 1*\overline{-I} + 1*\overline{-n^3} + 8*\overline{\beta_5} = 0 \\ \implies 1 - n - i - n^3 - 1 + n + i + n^3 + 8\overline{\beta_5} &= 0 \implies 8\beta_5 = 0 \implies \beta_5 = 0 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(7)}} &= 1*\overline{1} + 1*\overline{n^3} + 1*\overline{-I} + 1*\overline{n} + 1*\overline{-1} + 1*\overline{-n^3} + 1*\overline{I} + 1*\overline{-n} + 8*\overline{\beta_6} = 0 \\ \implies 1 - n^3 + i - n - 1 + n^3 - i + n + 8\overline{\beta_6} &= 0 \implies 8\beta_6 = 0 \implies \beta_6 = 0 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(8)}} &= 1*\overline{1} + 1*\overline{-n} + 1*\overline{I} + 1*\overline{-n^3} + 1*\overline{-1} + 1*\overline{n} + 1*\overline{-I} + 1*\overline{n^3} + 8*\overline{\beta_7} = 0 \\ \implies 1 + n - i + n^3 - 1 - n + i - n^3 + 8\overline{\beta_7} &= 0 \implies 8\beta_7 = 0 \implies \beta_7 = 0 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^k \chi_{.9}^{(1)} \overline{\chi_{.9}^{(9)}} &= 1*\overline{1} + 1*\overline{-n^3} + 1*\overline{-I} + 1*\overline{n} + 1*\overline{-1} + 1*\overline{n^3} + 1*\overline{I} + 1*\overline{n} + 8*\overline{\beta_8} = 0 \\ \implies 1 + n^3 + i + n - 1 - n^3 - i - n + 8\overline{\beta_8} &= 0 \implies 8\beta_8 = 0 \implies \beta_8 = 0 \end{aligned}$$

Finally, substituting the values into the Lifted chart from before we produce the complete character table of H:

Table 5.23: Complete Character Table  $H \cong 3^2 : 8$

Classes	e	$x^4$	w	$x^2$	$x^6$	x	$x^3$	$x^5$	$x^7$
Size	1	9	8	9	9	9	9	9	9
Order	1	2	3	4	4	8	8	8	8
$\chi_{.1}$	1	1	1	1	1	1	1	1	1
$\chi_{.2}$	1	-1	1	I	-I	n	$n^3$	-n	$-n^3$
$\chi_{.3}$	1	1	1	-1	-1	I	-I	I	-I
$\chi_{.4}$	1	-1	1	-I	I	$n^3$	n	$-n^3$	-n
$\chi_{.5}$	1	1	1	1	1	-1	-1	-1	-1
$\chi_{.6}$	1	-1	1	I	-I	-n	$-n^3$	n	$n^3$
$\chi_{.7}$	1	1	1	-1	-1	-I	I	-I	I
$\chi_{.8}$	1	-1	1	-I	I	$-n^3$	-n	$n^3$	n
$\chi_{.9}$	8	0	-1	0	0	0	0	0	0

## 5.7 Monomial Progenitor Results

A constant theme of this document has been that we construct an infinite progenitor, factor the progenitor by relations, and produce finite images of known groups.



While any given progenitor is not guaranteed to produce interesting results, we do this with hopes of discovering various simple groups, especially sporadic ones. The following subsections describe the results of such progenitors.

### 5.7.1 $17^{*10} :_m PGL_2(9)$ Results

This progenitor only produced the homomorphic image  $PGL_2(9)$  which can be obtained via the Grindstaff lemma relations. This was the only finding for this progenitor.

### 5.7.2 $13^{*2} :_m ((2^2 \times 3) : 2)$ Results

The progenitor  $13^{*2} :_m ((2^2 \times 3) : 2)$  factored by first order relations only obtained a single finite homomorphic image isomorphic to  $13^2 : ((2 \times 13) : (2 \times 3))$  of order 26,364.

$$\begin{aligned} G \langle x, y, z, t \rangle := & \text{Group} \langle x, y, z, t \mid x^4, y^2, z^3, t^{13}, (x^{-1} * y)^2, \\ & (x, z), y * z^{-1} * y * z, t^x * x^{-1} * t^{-12}, t^z * z^{-1} * t^{-3}, (x^2 * z^{-1} * t)^{12}, \\ & (x * y * z^{-1} * t)^6, (y * z * t)^6 \rangle; \end{aligned}$$

This is appeared to be the most efficient presentation of that group.

### 5.7.3 $3^{*4} :_m (\mathbb{Z}_2 \wr D_4)$ Results

The progenitor  $3^{*4} :_m (\mathbb{Z}_2 \wr D_4)$  factored by first order relations:

$$\begin{aligned} G \langle x, y, z, t \rangle := & \text{Group} \langle x, y, z, t \mid x^4, y^2, z^2, t^3, (x^{-1} * y)^2, (x^{-1} * z * y)^2, \\ & (x * z * x^{-1} * z)^2, x^{-1} * z * x^{-2} * z * x^2 * z * x^2 * z * x^{-1}, (t, x * z * y * z * x * z * x * z), \\ & (t, x^2 * z * x^{-1} * z * x^{-1}), (t, x^2 * z * x * z * x * z), t^z = t^2, (((x * z)^4) * t)^a, (((x * z * x)^2) * t)^b, \\ & ((x^2) * t)^c, ((z^x) * t)^d, ((z * x^{-1} * z * x) * t)^e, ((x * y) * t)^f, ((x^2 * z * x * z * x * z) * t)^g, \\ & ((x^2 * z * x * y * z) * t)^h, ((y) * t)^i, ((x^2 * y * z * x) * t)^j, (((x^{-1} * z)^2) * t)^k, ((x^{-1} * z * y) * t)^l, \\ & ((x^2 * z * x * z * y * z) * t)^m, ((z * x^2) * t)^n, ((z * x^{-1} * z * y) * t)^o, ((z * x^{-1} * z * x * y) * t)^p, \\ & ((x) * t)^q, ((x^2 * y * z) * t)^r, ((x^{-1} * z) * t)^s \rangle; \end{aligned}$$

Note: Only q and s will be shown in the presentation table as they were the only factors which produced finite images.

Table 5.24: Finite Presentations of  $3^{*4} :_m \mathbb{Z}_2 \wr D_4$ 

q	s	Order of G	Name of G
9	0	725,760	$2 \times S_9$
10	0	57,600	$(2 \times A_5^2) : D_4$
0	9	48,522,240	$(2^2)^\bullet PGL_4(3)$

## 5.8 Conclusion

The goals of this chapter were to show that an irreducible character can be induced up to  $G$  from the subgroup  $H$ . If we can produce a faithful monomial representation of our control group  $N$ , we can create a monomial progenitor that will be expressed on a free product that corresponds with the degree and field of the matrix that representation is expressed in. While this process can be completed the results vary just as they would with a progenitor calculated on  $2^{*n}$ .

Unfortunately no sporadic groups were found in these progenitors run in the background. That is the ultimate discovery and reason for developing this process. While some interesting groups were found on these progenitors, there is more to be desired. Progenitors that are too large,  $17^{*10} :_m PGL_2(9)$  for instance, appear to stress test Magma. We are hopeful that future developments in the servers will allow research of even higher order groups.

## Appendix A

# MAGMA Code for DCE of $PGL_2(11)$ over $\mathbf{N} \cong D_{12}$

```

S:=Sym(6);
X:=S!(1, 2)(3, 5);
Y:=S!(1, 3)(2, 5)(4, 6);
Z:=S!(1, 4, 2)(3, 6, 5);
N:=sub<S|X,Y,Z>;

f,N1,k:=CosetAction(N,sub<N|Id(N)>);

G<x,y,z,t>:=Group<x,y,z,t|x^2,y^2,z^3,y^x=y,z^x=z^2,z^y=z,
t^2,(t,x*z),(y*t)^5,(z^-1*t)^3,
x*y*t*x*y*t*x*y*t *y*t*x*t *y*t*y* z^-1*t*y*z*t* y*z*t>;

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
CompositionFactors(G1);

#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);

IN:=sub<G1|f(x),f(y),f(z)>;

ts := [ Id(G1): i in [1 .. 12] ];
ts[1]:=f(t);
ts[2]:=f(t^(x));
ts[3]:=f(t^(y));
ts[4]:=f(t^(z));
ts[5]:=f(t^(y*x));
ts[6]:=f(t^(y*z));

```

```

HH:=sub<G|x,y,z>;

Index(G,HH);
#G/#IN; /* 110 */

prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y,z>)]]
where null is [Integers() | ];
  for i := 1 to 6 do
    cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;

for i in [1..110] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

Orbits(N);

N1:=Stabiliser(N,1);
SSS:={[1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1s:=N1;

T1:=Transversal(N,N1s);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;

```

```
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;
```

```
Orbits(N1s);
```

```
N13:=Stabiliser(N1,3);
SSS:={[1,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N13s:=N13;
```

```
T13:=Transversal(N,N13s);
for i in [1..#T13] do
ss:=[1,3]^T13[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;
```

```
Orbits(N13s);
```

```
N15:=Stabiliser(N1,5);
SSS:={[1,5]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N15s:=N15;
```

```
T15:=Transversal(N,N15s);
for i in [1..#T15] do
ss:=[1,5]^T15[i];
cst[prodim(1, ts, ss)] := ss;
end for;
```

```

m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N15s);

N132:=Stabiliser(N13,2);
SSS:={[1,3,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N132s:=N132;

for g in N do if 1^g eq 1 and 3^g eq 3 and 2^g eq 4
then N132s:=sub<N|N132s,g>; end if; end for;
#N132s;

T132:=Transversal(N,N132s);
for i in [1..#T132] do
ss:=[1,3,2]^T132[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N132s);

N151:=Stabiliser(N15,1);
SSS:={[1,5,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;

N151s:=N151;

```

```

T151:=Transversal(N,N151s);
for i in [1..#T151] do
ss:=[1,5,1]^T151[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N151s);

N152:=Stabiliser(N15,2);
SSS:={[1,5,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N152s:=N152;

for g in N do if 1^g eq 6 and 5^g eq 2 and 2^g eq 5
then N152s:=sub<N|N152s,g>; end if; end for;
#N152s;

T152:=Transversal(N,N152s);
for i in [1..#T152] do
ss:=[1,5,2]^T152[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N152s);

N154:=Stabiliser(N15,4);
SSS:={[1,5,4]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

```

```

then print Rep(Seqq[i]);
end if; end for; end for;
N154s:=N154;

T154:=Transversal(N,N154s);
for i in [1..#T154] do
ss:=[1,5,4]^T154[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N154s);

N1323:=Stabiliser(N132,3);
SSS:={[1,3,2,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[2]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1323s:=N1323;

for g in N do if 1^g eq 1 and 3^g eq 3 and 2^g eq 4
and 3^g eq 3
then N1323s:=sub<N|N1323s,g>; end if; end for;
#N1323s;

T1323:=Transversal(N,N1323s);
for i in [1..#T1323] do
ss:=[1,3,2,3]^T1323[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1323s);

N1325:=Stabiliser(N132,5);
SSS:={[1,3,2,5]}; SSS:=SSS^N;

```



```

#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[2]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1325s:=N1325;

T1325:=Transversal(N,N1325s);
for i in [1..#T1325] do
ss:=[1,3,2,5]^T1325[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1325s);

N1513:=Stabiliser(N151,3);
SSS:={1,5,1,3}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1513s:=N1513;

for g in N do if 1^g eq 3 and 5^g eq 4 and 1^g eq 3
and 3^g eq 1
then N1513s:=sub<N|N1513s,g>; end if; end for;
#N1513s;

T1513:=Transversal(N,N1513s);
for i in [1..#T1513] do
ss:=[1,5,1,3]^T1513[i];
cst[prodim(1, ts, ss)] := ss;
end for;

```

```

m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1513s);

N1515:=Stabiliser(N151,5);
SSS:={[1,5,1,5]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[1]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1515s:=N1515;

for g in N do if 1^g eq 3 and 5^g eq 4 and 1^g eq 3
and 5^g eq 4
then N1515s:=sub<N|N1515s,g>; end if; end for;
#N1515s;

T1515:=Transversal(N,N1515s);
for i in [1..#T1515] do
ss:=[1,5,1,5]^T1515[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1515s);

N1516:=Stabiliser(N151,6);
SSS:={[1,5,1,6]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[1]*ts[6] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);

```

```

end if; end for; end for;
N1516s:=N1516;

T1516:=Transversal(N,N1516s);
for i in [1..#T1516] do
ss:=[1,5,1,6]^T1516[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1516s);

N1543:=Stabiliser(N154,3);
SSS:={1,5,4,3}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5]*ts[4]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1543s:=N1543;

for g in N do if 1^g eq 6 and 5^g eq 2 and 4^g eq 3
and 3^g eq 4
then N1543s:=sub<N|N1543s,g>; end if; end for;
#N1543s;

T1543:=Transversal(N,N1543s);
for i in [1..#T1543] do
ss:=[1,5,4,3]^T1543[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N1543s);

N13231:=Stabiliser(N1323,1);
SSS:={1,3,2,3,1}; SSS:=SSS^N;
#(SSS);

```

```

Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[2]*ts[3]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N13231s:=N13231;

for g in N do if 1^g eq 2 and 3^g eq 5 and 2^g eq 1
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 3 and 3^g eq 1 and 2^g eq 5
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 4 and 3^g eq 6 and 2^g eq 1
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 5 and 3^g eq 2 and 2^g eq 3
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 1 and 3^g eq 3 and 2^g eq 4
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 6 and 3^g eq 4 and 2^g eq 3
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 4 and 3^g eq 6 and 2^g eq 2
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 2 and 3^g eq 5 and 2^g eq 4
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 3 and 3^g eq 1 and 2^g eq 6
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 6 and 3^g eq 4 and 2^g eq 5
then N13231s:=sub<N|N13231s,g>; end if; end for;
for g in N do if 1^g eq 5 and 3^g eq 2 and 2^g eq 6
then N13231s:=sub<N|N13231s,g>; end if; end for;
#N13231s;

T13231:=Transversal(N,N13231s);
for i in [1..#T13231] do
ss:=[1,3,2,3,1]^T13231[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N13231s);

```

## Appendix B

# MAGMA Code Maximal DCE

## $M_{12}$

```

/* =====Determining suitable Maximal Subgroup of $M_{12}$ =====
===== */

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 9)(8, 11)(10, 12);
Z:=S!(1, 4, 8)(2, 6, 10)(3, 7, 11)(5, 9, 12);
W:=S!(1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 11);
V:=S!(1, 5)(2, 3)(4, 6)(7, 9)(8, 11)(10, 12);
N:=sub<S|X,Y,Z,W,V>;
f,N1,k:=CosetAction(N,sub<N|Id(N)>);

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^3,
w^2,v^2,y^x=y,z^x=z,z^y=z,w^x=w,w^y=w,w^z=v,v^x=v,v^y=v,
v^z=w*v,v^w=v,t^2,(t,x*y*w),(t,y*v),(x*t)^3,(z*t)^5>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);
CompositionFactors(G1);

#DoubleCosets(G,sub<G|x,y,z,w,v>,sub<G|x,y,z,w,v>);

M:=MaximalSubgroups(G1);
/* Magma stores this differently each time.
Will change CC below */

```

```

#M;
for i in [1..9] do #M[i]` subgroup; end for;

/* We check 480, 144 or 384 they are divisible by 48 (#N) */

/* C:=Conjugates(G1,M[2]`subgroup);
Didn't produce anything with for i in [1..#CC] below */

/* C:=Conjugates(G1,M[6]`subgroup); Didn't work */
/* C:=Conjugates(G1,M[8]`subgroup); Didn't work */

C:=Conjugates(G1,M[7]`subgroup);
CC:=Setseq(C);
for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i]
and f(z) in CC[i] and f(w) in CC[i] and f(v) in CC[i]
then i; end if; end for;
/* Use the number given here was 332 first time */

CC[332];

for g in G1 do if Order(g) eq 2 and
CC[332] eq sub<G1|f(x),f(y),f(z),f(w),f(v),g>
then A:=g; break; end if; end for;

W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
function(W)
  w7 := W.2 * W.4; w8 := w7 * W.3; w9 := w8 * W.6;
  w10 := w9 * W.2; w11 := w10 * W.6; w12 := w11 * W.3; w13
  := w12 * W.5; w14 := w13 * W.6; w3 := W.3^-1; w15
  := w14 * w3; w16 := w15 * W.6; w17 := w16 * W.3; w18
  := w17 * W.6; w19
  := w18 * w3; w20 := w19 * W.6; return w20;
end function;

AA:=function(w)
  w7 := W.2 * W.4; w8 := w7 * W.3; w9 := w8 * W.6;
  w10 := w9 * W.2; w11 := w10 * W.6; w12 := w11 * W.3; w13
  := w12 * W.5; w14 := w13 * W.6; w3 := W.3^-1; w15
  := w14 * w3; w16 := w15 * W.6; w17 := w16 * W.3; w18
  := w17 * W.6; w19
  := w18 * w3; w20 := w19 * W.6; return w20;
end function;

```

```

AA(G);

NN:=G1;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(NN): i in [1..#G]];
for i in [2..#NN] do
P:=[Id(NN): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=f(z^-1); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(v); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;
end for;
PP:=Id(NN);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N2] do if ArrayP[i] eq f(AA(G)) then Sch[i];
end if; end for;
/* z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t relation which should
be added to our progenitor*/

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^3,
w^2,v^2,y^x=y,z^x=z,z^y=z,w^x=w,w^y=w,w^z=v,v^x=v,
v^y=v,v^z=w*v,v^w=v,t^2,(t,x*y*w),(t,y*v),(x*t)^3,(z*t)^5>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

#sub<G1|f(x),f(y),f(z),f(w),f(v),
f(z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t)>;

J:=M[7] `subgroup;
L:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t)>;

IsIsomorphic(J,L);
/* this work verifies we have found the correct relation */

```

```

M:=sub<G|x,y,z,w,v,z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t>;
#DoubleCosets(G,M,sub<G|x,y,z,w,v>);
/* This is a reasonable group to work with */

/* ===== Coset Counter and DCE =====
===== */
G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^3,
w^2,v^2,y^x=y,z^x=z,z^y=z,w^x=w,w^y=w,w^z=v,v^x=v,
v^y=v,v^z=w*v,v^w=v,t^2,(t,x*y*w),(t,y*v),(x*t)^3,(z*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 9)(8, 11)(10, 12);
Z:=S!(1, 4, 8)(2, 6, 10)(3, 7, 11)(5, 9, 12);
W:=S!(1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 11);
V:=S!(1, 5)(2, 3)(4, 6)(7, 9)(8, 11)(10, 12);

N:=sub<S|X,Y,Z,W,V>;
#N;
Set(N);

HH:=sub<G|x,y,z,w,v,z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t>;
#HH;

IN:=sub<G1|f(x),f(y),f(z),f(w),f(v)>;

M:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t)>;

ts := [ Id(G1): i in [1 .. 12] ];
ts[12]:=f(t);
ts[1]:=f(t^(x*y*z));
ts[2]:=f(t^(z*y));
ts[3]:=f(t^(z*x));
ts[4]:=f(t^(x*v*z^2));
ts[5]:=f(t^(z));
ts[6]:=f(t^(v*z^2));
ts[7]:=f(t^(x*z^2));
ts[8]:=f(t^(x*v));
ts[9]:=f(t^(z^2));
ts[10]:=f(t^(v));
ts[11]:=f(t^(x));

```



```

#DoubleCosets (G, sub<G|x, y, z, w, v, z*t*v*t*z*v*t*z^-1*t*z*t*
z^-1*t>,
sub<G|x, y, z, w, v>);
DoubleCosets (G, sub<G|x, y, z, w, v, z*t*v*t*z*v*t*z^-1*t*z*t*
z^-1*t>,
sub<G|x, y, z, w, v>);
Index(G, HH); /* 495 */
#G/#IN; /* 3960 */

prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
return v;
end function;
cst := [null : i in [1 .. Index(G, sub<G|x, y, z, w, v>)]];
where null is [Integers() | ];
for i := 1 to 12 do
  cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;

for i in [1..3960] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N);

N12:=Stabiliser(N,12);
SSS:={[12]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s:=N12;

T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[12]^T12[i];
cst[prodim(1, ts, ss)] := ss;

```

```

end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;

Orbits(N12s);

/* ***** New Double cosets of [12] ***** */

N121:=Stabiliser(N12,1);
SSS:={[12,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N121s:=N121;

T121:=Transversal(N,N121s);
for i in [1..#T121] do
ss:=[12,1]^T121[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N121s);

N124:=Stabiliser(N12,4);
SSS:={[12,4]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N124s:=N124;

T124:=Transversal(N,N124s);
for i in [1..#T124] do

```

```

ss:=[12,4]^T124[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N124s);

/* *****New double cosets of [12, 1] ***** */

N1214:=Stabiliser(N121,4);
SSS:={[12,1,4]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1214s:=N1214;

for g in N do if 12^g eq 2 and 1^g eq 9 and 4^g eq 10
then N1214s:=sub<N|N1214s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 11 and 4^g eq 5
then N1214s:=sub<N|N1214s,g>; end if; end for;
#N1214s; N1214s;

T1214:=Transversal(N,N1214s);
for i in [1..#T1214] do
ss:=[12,1,4]^T1214[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1214s);

N1215:=Stabiliser(N121,5);
SSS:={[12,1,5]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

```

```

then print Rep(Seqq[i]);
end if; end for; end for;
N1215s:=N1215;

T1215:=Transversal(N,N1215s);
for i in [1..#T1215] do
ss:=[12,1,5]^T1215[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1215s);

N1217:=Stabiliser(N121,7);
SSS:={ [12,1,7] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[7] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1217s:=N1217;

for g in N do if 12^g eq 8 and 1^g eq 2 and 7^g eq 4
then N1217s:=sub<N|N1217s,g>; end if; end for;
#N1217s; N1217s;

T1217:=Transversal(N,N1217s);
for i in [1..#T1217] do
ss:=[12,1,7]^T1217[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1217s);

N1218:=Stabiliser(N121,8);
SSS:={ [12,1,8] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do

```

```

if ts[12]*ts[1]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1218s:=N1218;

```

```

T1218:=Transversal(N,N1218s);
for i in [1..#T1218] do
ss:=[12,1,8]^T1218[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1218s);

```

```

N1219:=Stabiliser(N121,9);
SSS:={[12,1,9]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[9] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1219s:=N1219;

```

```

T1219:=Transversal(N,N1219s);
for i in [1..#T1219] do
ss:=[12,1,9]^T1219[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1219s);

```

```

N12110:=Stabiliser(N121,10);
SSS:={[12,1,10]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[10] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

```

```

then print Rep(Seqq[i]);
end if; end for; end for;
N12110s:=N12110;

T12110:=Transversal(N,N12110s);
for i in [1..#T12110] do
ss:=[12,1,10]^T12110[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12110s);

N12111:=Stabiliser(N121,11);
SSS:={ [12,1,11] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[11] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12111s:=N12111;

for g in N do if 12^g eq 11 and 1^g eq 2 and 11^g eq 12
then N12111s:=sub<N|N12111s,g>; end if; end for;
#N12111s; N12111s;

T12111:=Transversal(N,N12111s);
for i in [1..#T12111] do
ss:=[12,1,11]^T12111[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12111s);

/* *****New Double Cosets of [12, 1, 5] ***** */
N12152:=Stabiliser(N1215,2);
SSS:={ [12,1,5,2] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do

```

```

for n in M do
if ts[12]*ts[1]*ts[5]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12152s:=N12152;

for g in N do if 12^g eq 11 and 1^g eq 2 and 5^g eq 3 and
2^g eq 1 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 10 and 1^g eq 3 and 5^g eq 2 and
2^g eq 5 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 8 and 1^g eq 4 and 5^g eq 1 and
2^g eq 3 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 3 and 1^g eq 7 and 5^g eq 6 and
2^g eq 9 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 5 and 1^g eq 9 and 5^g eq 4 and
2^g eq 7 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 1 and 1^g eq 4 and 5^g eq 9 and
2^g eq 6 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 4 and 1^g eq 10 and 5^g eq 11 and
2^g eq 8 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 2 and 1^g eq 6 and 5^g eq 7 and
2^g eq 4 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 6 and 1^g eq 8 and 5^g eq 12 and
2^g eq 10 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 12 and 5^g eq 8 and
2^g eq 11 then N12152s:=sub<N|N12152s,g>; end if; end for;
for g in N do if 12^g eq 9 and 1^g eq 11 and 5^g eq 10 and
2^g eq 12 then N12152s:=sub<N|N12152s,g>; end if; end for;
#N12152s; N12152s;

T12152:=Transversal(N,N12152s);
for i in [1..#T12152] do
ss:=[12,1,5,2]^T12152[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12152s);

N12156:=Stabiliser(N1215,6);
SSS:={[12,1,5,6]}; SSS:=SSS^N;
#(SSS);

```

```

Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[5]*ts[6] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12156s:=N12156;

T12156:=Transversal(N,N12156s);
for i in [1..#T12156] do
ss:=[12,1,5,6]^T12156[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12156s);

N12158:=Stabiliser(N1215,8);
SSS:={[12,1,5,8]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[5]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12158s:=N12158;

for g in N do if 12^g eq 10 and 1^g eq 1 and 5^g eq 5 and
8^g eq 11 then N12158s:=sub<N|N12158s,g>; end if; end for;
#N12158s;
N12158s;

T12158:=Transversal(N,N12158s);
for i in [1..#T12158] do
ss:=[12,1,5,8]^T12158[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;

```



```

Orbits(N12158s);

N12159:=Stabiliser(N1215,9);
SSS:={ [12,1,5,9] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[5]*ts[9] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12159s:=N12159;

for g in N do if 12^g eq 10 and 1^g eq 1 and 5^g eq 5 and
9^g eq 4 then N12159s:=sub<N|N12159s,g>; end if; end for;
#N12159s; N12159s;

T12159:=Transversal(N,N12159s);
for i in [1..#T12159] do
ss:=[12,1,5,9]^T12159[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12159s);

N12185:=Stabiliser(N1218,5);
SSS:={ [12,1,8,5] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[8]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12185s:=N12185;

for g in N do if 12^g eq 12 and 1^g eq 5 and 5^g eq 1 and
8^g eq 8 then N12185s:=sub<N|N12185s,g>; end if; end for;
#N12185s; N12185s;

```

```

T12185:=Transversal(N,N12185s);
for i in [1..#T12185] do
ss:=[12,1,8,5]^T12185[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12185s);

/* ***** 5 letter double cosets ***** */
N121581:=Stabiliser(N12158,1);
SSS:={[12,1,5,8,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[5]*ts[8]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N121581s:=N121581;

for g in N do if 12^g eq 11 and 1^g eq 2 and 5^g eq 3 and
8^g eq 10 and 1^g eq 2 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 8 and 1^g eq 2 and 5^g eq 3 and
8^g eq 12 and 1^g eq 2 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 10 and 1^g eq 5 and 5^g eq 1 and
8^g eq 11 and 1^g eq 5 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 10 and 1^g eq 1 and 5^g eq 5 and
8^g eq 11 and 1^g eq 1 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 8 and 1^g eq 3 and 5^g eq 2 and
8^g eq 12 and 1^g eq 3 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 11 and 1^g eq 3 and 5^g eq 2 and
8^g eq 10 and 1^g eq 3 then N121581s:=sub<N|N121581s,g>;
end if; end for;
for g in N do if 12^g eq 12 and 1^g eq 5 and 5^g eq 1 and
8^g eq 8 and 1^g eq 5 then N121581s:=sub<N|N121581s,g>;

```

```

end if; end for;
#N121581s; N121581s;

T121581:=Transversal(N,N121581s);
for i in [1..#T121581] do
ss:=[12,1,5,8,1]^T121581[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..3960] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N121581s);

/* ===== Relations to be proved ===== */

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^3,w^2,v^2,y^x=y,
z^x=z,z^y=z,w^x=w,w^y=w,w^z=v,v^x=v,v^y=v,v^z=w*v,v^w=v,
t^2,(t,x*y*w),(t,y*v),(x*t)^3,(z*t)^5>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 9)(8, 11)(10, 12);
Z:=S!(1, 4, 8)(2, 6, 10)(3, 7, 11)(5, 9, 12);
W:=S!(1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 11);
V:=S!(1, 5)(2, 3)(4, 6)(7, 9)(8, 11)(10, 12);
N:=sub<S|X,Y,Z,W,V>;
#N;
Set(N);

HH:=sub<G|x,y,z,w,v,z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t>;
#HH;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(v)>;
M:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(z*t*v*t*z*v*t*z^-1*t*z*t*z^-1*t)>;

ts := [ Id(G1): i in [1 .. 12] ];
ts[12]:=f(t);
ts[1]:=f(t^(x*y*z));
ts[2]:=f(t^(z*y));
ts[3]:=f(t^(z*x));
ts[4]:=f(t^(x*v*z^2));
ts[5]:=f(t^(z));
ts[6]:=f(t^(v*z^2));

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```

ts[7]:=f(t^(x*z^2));
ts[8]:=f(t^(x*v));
ts[9]:=f(t^(z^2));
ts[10]:=f(t^(v));
ts[11]:=f(t^(x));

N:=G1; NN:=G;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do

if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=f(z^-1); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(v); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;

end for;
PP:=Id(G1);
for k in [1..#P] do PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for g in M do for h in IN do if ts[12]*ts[8] eq g*(ts[12])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[8] eq f(w * t * w * t * v * t)*ts[11];

for g in M do for h in IN do if ts[12]*ts[10] eq g*(ts[10])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[10] eq f(y * t * y * t * w * t)*ts[11];

for g in M do for h in IN do if ts[12]*ts[11] eq g*(ts[12])^h

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    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[11] eq f(x)*ts[12];

/* [12, 1] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[2]
eq g*(ts[12]*ts[1])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[2] eq f(x)*ts[11]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[12]
eq g*(ts[12]*ts[1])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[12] eq
f(v * t * z^-1 * t * z * t * z^-1 * t * z * t)*ts[12]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[3]
eq g*(ts[12]*ts[4])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[3] eq
f(y * t * y * t * w * t * z^-1)*ts[4]*ts[2];

for g in M do for h in IN do if ts[12]*ts[1]*ts[6]
eq g*(ts[12]*ts[4])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];

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```

    end if; end for;
ts[12]*ts[1]*ts[6] eq f(v * z^-1)*ts[1]*ts[12];

/* [12 1 4] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[4]*ts[1]
eq g*(ts[12]*ts[1]*ts[7])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[4]*ts[1]
eq f(t*x*y*t*y*z*t*z^-1*t*z*t*z^-1*t)*ts[10]*ts[2]*ts[7];

for g in M do for h in IN do if ts[12]*ts[1]*ts[4]*ts[2]
eq g*(ts[12]*ts[1]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[4]*ts[2]
eq f(x * t * w * t * v * t * z)*ts[10]*ts[1]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[4]*ts[3]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[4]*ts[3]
eq f(y * z * t * y * t * w * t * z)*ts[12]*ts[9];

/* [12 1 5] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[1]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;

```

```

ts[12]*ts[1]*ts[5]*ts[1]
eq f(x * w * v)*ts[12]*ts[1]*ts[5];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[3]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[3]
eq f(t * x * w * t * w * t * z^-1)*ts[6]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[4]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[4]
eq f(z * t * w * t * v * t)*ts[8]*ts[5]*ts[12];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[7]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[7]
eq f(x*z*t*z^-1*t*z*t*z^-1*t*z*t)*ts[9]*ts[11]*ts[4];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[10]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[10]
eq f(t * w * t * v * t * x * z)*ts[7]*ts[11]*ts[4];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[11]

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eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[11]
eq f(x*t*z^-1*t*z*t*z^-1*t*z*v*t*w*t)*ts[1]*ts[6]*ts[3];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[12]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[12]
eq f(z*t*v*t*w*t*x*z)*ts[4]*ts[10]*ts[11]*ts[5];

/* [12 1 7] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[7]*ts[1]
eq g*(ts[12]*ts[1]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[7]*ts[1]
eq f(y*t*z*t*z^-1*t*z*t*v*z^-1*t*y*t)*ts[4]*ts[12]*ts[5];

for g in M do for h in IN do if ts[12]*ts[1]*ts[7]*ts[3]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[7]*ts[3]
eq f(z * t * v * t * w * t * z * v)*ts[2]*ts[9]*ts[4]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[7]*ts[6]
eq g*(ts[12]*ts[1]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;

```



```

for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[7]*ts[6]
eq f(x * z * t * w * t * v * t)*ts[10]*ts[1]*ts[4];

for g in M do for h in IN do if ts[12]*ts[1]*ts[7]*ts[8]
eq g*(ts[12]*ts[1]*ts[9])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[7]*ts[8]
eq f(w * t * v * t * w * t)*ts[8]*ts[2]*ts[6];

for g in M do for h in IN do if ts[12]*ts[1]*ts[7]*ts[10]
eq g*(ts[12]*ts[4])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[7]*ts[10]
eq f(y * z * t * w * t * y * t)*ts[6]*ts[1];

/* [12 1 8] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[1]
eq g*(ts[12]*ts[4])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[1]
eq f(t*x*y*t*y*z^-1*t*z*t*z^-1*t*z*t)*ts[5]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[2]
eq g*(ts[12]*ts[1]*ts[10])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;

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for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[2]
eq f(t * x * w * t * w * t)*ts[12]*ts[1]*ts[10];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[3]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[3]
eq
f(y*t*z*t*z^-1*t*z*t*v*z^-1*t*y*t)*ts[12]*ts[5]*ts[1]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[4]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[4]
eq f(z*t*z^-1*t*z*t*x*w*z^-1*t*z*t*w*t)*ts[12]*ts[1]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[6]
eq g*(ts[12]*ts[1]*ts[11])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[6]
eq f(x*t*z*t*z^-1*t*z*t*z^-1*v*t*w*t)*ts[7]*ts[10]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[7]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[7]

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eq f(t * x * w * t * w * t * z^-1)*ts[10]*ts[3]*ts[2]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[9]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[9]
eq f(y * t * y * t * w * t * z^-1)*ts[8]*ts[5]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[10]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[10] eq f(x)*ts[11]*ts[2]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[11]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[11]
eq f(t*z*t*z^-1*t*z*t*z^-1*t)*ts[2]*ts[6]*ts[7];

for g in M do for h in IN do if ts[12]*ts[1]*ts[8]*ts[12]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[12]
eq f(z * w)*ts[3]*ts[4]*ts[9]*ts[10];

/* [12 1 9] */

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for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[1]
eq g*(ts[12]*ts[1]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[1]
eq f(x*z*w*t*z^-1*t*z*t*z^-1*t*z*t)*ts[12]*ts[1]*ts[9];

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for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[2]
eq g*(ts[12]*ts[1]*ts[11])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[2]
eq f(x * v * z)*ts[9]*ts[11]*ts[7];

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for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[3]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[3]
eq f(t * w * t * y * t * y * z^-1)*ts[9]*ts[8]*ts[12]*ts[4];

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for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[4]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[4]
eq f(t*z*t*z^-1*t*z*t*z^-1*v*t*w*t*x)*ts[4]*ts[8]*ts[12]*ts[1];

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for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[5]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h

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    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[5]
eq f(v*t*z*t*z^-1*t*z*t*z^-1*t)*ts[5]*ts[4]*ts[9]*ts[12];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[6]
eq g*(ts[12]*ts[1]*ts[9])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[6]
eq f(x * z * t * w * t * v * t)*ts[9]*ts[12]*ts[2];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[7]
eq g*(ts[12]*ts[1]*ts[7])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[7] eq f(x)*ts[11]*ts[2]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[8]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[8]
eq f(x*t*z*t*z^-1*t*z*t*z^-1*t*v)*ts[10]*ts[3]*ts[2]*ts[9];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[10]
eq g*(ts[12]*ts[1]*ts[9])^h
    then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;

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ts[12]*ts[1]*ts[9]*ts[10]
eq f(x * t * w * t * v * t * z)*ts[1]*ts[7]*ts[12];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[11]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
ts[12]*ts[1]*ts[9]*ts[11] eq f(x * z^-1 * v)*ts[7]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[9]*ts[12]
eq g*(ts[12]*ts[1]*ts[7])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[9]*ts[12]
eq f(w * t * w * t * v * t)*ts[8]*ts[2]*ts[4];

/* [12 1 10] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[1]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[1]
eq f(y*t*y*t*w*z^-1*t*z*t*z^-1*t*z*t)*ts[3]*ts[10];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[2]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[2]
eq f(t * x * y * t * y * t)*ts[12]*ts[1]*ts[8];

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for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[3]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i]; end if;
end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i]; end if;
end for;
ts[12]*ts[1]*ts[10]*ts[3]
eq f(y*z^-1*t*w*t*y*t*z^-1)*ts[1]*ts[7]*ts[6]*ts[5];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[4]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[4]
eq f(t*z*t*z^-1*t*z*t*x*v*z^-1*t*w*t)*ts[6]*ts[12]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[5]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[5]
eq f(x * z * t * v * t * w * t * z)*ts[2]*ts[9]*ts[4];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[6]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[6]
eq f(x*z*t*z^-1*t*z*t*z^-1*t*z*t*w)*ts[1]*ts[7]*ts[6]*ts[10];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[7]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];

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end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[7]
eq f(z*t*w*t*v*z*t*z^-1*t*z*t*z^-1*t)*ts[12]*ts[1]*ts[10];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[8]
eq g*(ts[12]*ts[1]*ts[8])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[8] eq f(x)*ts[11]*ts[2]*ts[10];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[9]
eq g*(ts[12]*ts[1]*ts[11])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i]; end if;
end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[9]
eq f(t * w * t * y * t * y)*ts[2]*ts[6]*ts[1];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[11]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[11]
eq f(z)*ts[5]*ts[7]*ts[6]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[10]*ts[12]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[10]*ts[12]
eq f(w * t * v * t * w * t * z )*ts[3]*ts[9]*ts[2]*ts[4];

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/* [12 1 11] */

for g in M do for h in IN do if ts[12]*ts[1]*ts[11]*ts[1]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[11]*ts[1]
eq f(v*t*z^-1*t*z*t*z^-1*t*z*t)*ts[2]*ts[11];

for g in M do for h in IN do if ts[12]*ts[1]*ts[11]*ts[3]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[11]*ts[3]
eq f(w * z^-1 * t * w * t * v * t)*ts[9]*ts[11]*ts[6];

for g in M do for h in IN do if ts[12]*ts[1]*ts[11]*ts[4]
eq g*(ts[12]*ts[1]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[11]*ts[4]
eq f(x * v * z^-1)*ts[2]*ts[6]*ts[12];

for g in M do for h in IN do if ts[12]*ts[1]*ts[11]*ts[7]
eq g*(ts[12]*ts[1]*ts[11])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[11]*ts[7]
eq f(x * z^-1 * v)*ts[11]*ts[2]*ts[12];

for g in M do for h in IN do if ts[12]*ts[1]*ts[11]*ts[8]

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eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[11]*ts[8]
eq f(x*t*z^-1*t*z*t*z^-1*t*z*v*t*w*t)*ts[3]*ts[6]*ts[2];

/* [12 4] */

for g in M do for h in IN do if ts[12]*ts[4]*ts[1]
eq g*(ts[12]*ts[1]*ts[7])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[1]
eq f(w * t * w * t * v * t * z)*ts[3]*ts[4]*ts[8];

for g in M do for h in IN do if ts[12]*ts[4]*ts[2]
eq g*(ts[12]*ts[1]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[2]
eq f(x * z * v)*ts[1]*ts[4]*ts[11];

for g in M do for h in IN do if ts[12]*ts[4]*ts[3]
eq g*(ts[12]*ts[1])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[3] eq f(z*w)*ts[4]*ts[12];

for g in M do for h in IN do if ts[12]*ts[4]*ts[5]
eq g*(ts[12]*ts[1]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];

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end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[5] eq f(x * z * w)*ts[6]*ts[11]*ts[3];

for g in M do for h in IN do if ts[12]*ts[4]*ts[6]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[6] eq f(x)*ts[11]*ts[4];

for g in M do for h in IN do if ts[12]*ts[4]*ts[7]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[7]
eq f(x * t * w * t * v * t * z)*ts[3]*ts[4]*ts[9];

for g in M do for h in IN do if ts[12]*ts[4]*ts[8]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[8]
eq f(t*x*y*t*y*z*t*z^-1*t*z*t*z^-1*t)*ts[9]*ts[8]*ts[4];

for g in M do for h in IN do if ts[12]*ts[4]*ts[9]
eq g*(ts[12]*ts[1])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[9]
eq f(y * t * w * t * y * t * z)*ts[5]*ts[6];

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for g in M do for h in IN do if ts[12]*ts[4]*ts[10]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[4]*ts[10]
eq f(y*t*y*t*z*v*t*z^-1*t*z*t*z^-1*t)*ts[7]*ts[10]*ts[4];

```

```

for g in M do for h in IN do if ts[12]*ts[4]*ts[11]
eq g*(ts[12]*ts[1]*ts[11])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
ts[12]*ts[4]*ts[11] eq f(x)*ts[4]*ts[12]*ts[6];

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```

for g in M do for h in IN do if ts[12]*ts[4]*ts[12]
eq g*(ts[12]*ts[4])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
ts[12]*ts[4]*ts[12]
eq f(x*w*t*z*t*z^-1*t*z*t*z^-1*t)*ts[12]*ts[4];

```

[12 1 5 8]

```

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[1]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;

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for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[2]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;

```

```

for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[2]
eq f(z*t*z^-1*t*z*t*z^-1*t*z*t)*ts[12]*ts[1]*ts[5]*ts[8];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[3]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[3]
eq f(y*z*t*z^-1*t*z*t*z^-1*t*z*w*t*y*t)*
ts[11]*ts[2]*ts[3]*ts[7];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[4]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[4]
eq f(y*t*z*t*z^-1*t*z*t*y*z^-1*t*w*t)*
ts[9]*ts[12]*ts[8]*ts[2];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[5]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[5]
eq f(x*t*z*t*z^-1*t*z*t*v*z^-1*t*v*t)*ts[2]*ts[9]*ts[3]*ts[4];

for g in M do for h in IN do if ts[12]*ts[1]*ts[5]*ts[8]*ts[6]
eq g*(ts[12]*ts[1]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;

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```

ts[12]*ts[1]*ts[5]*ts[8]*ts[6]
eq f(z*t*w*t*v*z*t*z^-1*t*z*t*z^-1*t)*ts[5]*ts[9]*ts[10];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[8]*ts[10] eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[10]
eq f(t * w * t * v * t * z)*ts[9]*ts[12]*ts[6];

/* [12 1 5 9] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[1]
eq g*(ts[12]*ts[1]*ts[5]*ts[8]*ts[1])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[1]
eq f(y*t*z*t*z^-1*t*z*t*y*z^-1*t*w*t)*
ts[2]*ts[4]*ts[9]*ts[3]*ts[4];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[2]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[2]
eq f(y*z*t*z^-1*t*z*t*z^-1*t*y*z*t*w*t)*
ts[12]*ts[1]*ts[5]*ts[9];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[3]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];

```

```

end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[3]
eq f(z * w * t * v * t * w * t)*ts[2]*ts[9]*ts[3]*ts[4];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[5]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[5]
eq f(y*z*t*z^-1*t*z*t*z^-1*t*z*w*t*y*t)*
ts[11]*ts[2]*ts[3]*ts[10];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[6]
eq g*(ts[12]*ts[1]*ts[8])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[6]
eq f(v * z^-1)*ts[6]*ts[10]*ts[7];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[8]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[9]*ts[8]
eq f(x*t*z^-1*t*z*t*z^-1*t*z*v*t*w*t)*
ts[1]*ts[7]*ts[6]*ts[5];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[9]*ts[10]
eq g*(ts[12]*ts[1]*ts[9])^h

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```

    then A:=g;B:=h; break; end if; end for; end for;
  for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
  end if; end for;
  for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
  ts[12]*ts[1]*ts[5]*ts[9]*ts[10]
  eq f(z^-1)*ts[4]*ts[11]*ts[5];

/* [12 1 8 5] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[2]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
  for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
  end if; end for;
  for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
  ts[12]*ts[1]*ts[8]*ts[5]*ts[2]
  eq f(x * z * v)*ts[11]*ts[2]*ts[3]*ts[4];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[4]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
  then A:=g;B:=h; break; end if; end for; end for;
  for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
  end if; end for;
  for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
  ts[12]*ts[1]*ts[8]*ts[5]*ts[4]
  eq f(x*t*w*t*v*z^-1*t*z*t*z^-1*t*z*t)*
  ts[12]*ts[5]*ts[1]*ts[9];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[6]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
  for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
  end if; end for;
  for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
  end if; end for;
  ts[12]*ts[1]*ts[8]*ts[5]*ts[6]
  eq f(y*z^-1*t*y*t*w*z^-1*t*z*t*z^-1*t*z*t)*
  ts[6]*ts[10]*ts[9];

```



```

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[8]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[5]*ts[8]
eq f(y * t * y * t * w * t)*ts[7]*ts[11]*ts[10]*ts[1];

```

```

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[10]
eq g*(ts[12]*ts[1]*ts[5]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[5]*ts[10]
eq f(y*t*z^-1*t*z*t*z^-1*t*y*z*t*w*t)*
ts[7]*ts[11]*ts[10]*ts[6];

```

```

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[11]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[5]*ts[11]
eq f(x)*ts[12]*ts[5]*ts[8]*ts[1];

```

```

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[5]*ts[12]
eq g*(ts[12]*ts[1]*ts[5]*ts[8]*ts[1])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[8]*ts[5]*ts[12]

```

```

eq f(z*t*z^-1*t*z*t*x*w*z^-1*t*z*t*w*t)*
ts[1]*ts[4]*ts[9]*ts[5]*ts[4];

/*    [12 1 5 6]    */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[1]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[1]
eq f(x * v * z^-1)*ts[11]*ts[2]*ts[10]*ts[3];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[2]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[2]
eq f(t*x*w*t*z*v*t*z^-1*t*z*t*z^-1*t)*ts[12]*ts[5]*ts[8];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[3]
eq g*(ts[12]*ts[1]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[3]
eq f(x * z * t * w * t * v * t)*ts[3]*ts[7]*ts[6];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[4]
eq g*(ts[12]*ts[1]*ts[8])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;

```

```

for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[4]
eq f(z * t * w * t * v * t * x)*ts[10]*ts[3]*ts[11];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[5]
eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[5]
eq f(x*t*z^-1*t*z*t*z^-1*t*z*t*w)*ts[9]*ts[12]*ts[6];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[7]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[7]
eq f(t* z^-1*t*z*t*x*w*z^-1*t*z*t*w*t)*
ts[12]*ts[5]*ts[8]*ts[1];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[8]
eq g*(ts[12]*ts[1]*ts[7])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[8]
eq f(w * t * w * t * v * t * z)*ts[7]*ts[11]*ts[2];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[9] eq g*(ts[12]*ts[1]*ts[10])^h
  then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;

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```

for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[9]
eq f(z^-1 * v)*ts[7]*ts[8]*ts[4];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[10]
eq g*(ts[12]*ts[1]*ts[5]*ts[6])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[10]
eq f(z^-1*t*x*y*t*y*t*z^-1)*ts[12]*ts[5]*ts[1]*ts[9];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[11] eq g*(ts[12]*ts[1]*ts[9])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[11]
eq f(v*t*z*t*z^-1*t*z*t*z^-1*t)*ts[10]*ts[3]*ts[6];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[6]*ts[12]
eq g*(ts[12]*ts[1]*ts[9])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[6]*ts[12]
eq f(t*z^-1*t*z*t*z^-1*t*z*v*t*w*t)*ts[5]*ts[6]*ts[8];

/* [12 1 5 8 1] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[8]*ts[1]*ts[4]
eq g*(ts[12]*ts[1]*ts[8]*ts[5])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];

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```

end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[1]*ts[4]
eq f(t * v * t * w * t * z)*ts[4]*ts[12]*ts[9]*ts[8];

for g in M do for h in IN do if
ts[12]*ts[1]*ts[5]*ts[8]*ts[1]*ts[8]
eq g*(ts[12]*ts[1]*ts[5]*ts[9])^h
then A:=g;B:=h; break; end if; end for; end for;
for i in [1..#G1] do if ArrayP[i] eq A then Sch[i];
end if; end for;
for i in [1..#G1] do if ArrayP[i] eq B then Sch[i];
end if; end for;
ts[12]*ts[1]*ts[5]*ts[8]*ts[1]*ts[8]
eq f(t*v*t*z*v*t*z^-1*t*z*t*z^-1*t*x)*
ts[6]*ts[8]*ts[12]*ts[1];

```

## Appendix C

# Magma Code Maximal DCE

$U_4(2) : 2$

```

/* ===Determining suitable Maximal Subgroup of U_4(2) : 2$ ===
===== */
/* Same method as M_12 */

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 8)(6, 9)(7, 11)(10, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 10)(8, 11)(9, 12);
Z:=S!(3, 5)(7, 10)(11, 12);
W:=S!(1, 2)(3, 5)(4, 6)(7, 10)(8, 9)(11, 12);
V:=S!(1, 4, 9)(2, 6, 8)(3, 7, 12)(5, 10, 11);
N:=sub<S|X,Y,Z,W,V>;

f,N1,k:=CosetAction(N,sub<N|Id(N)>);

NL:=NormalLattice(N);

/* N = 12 : 2^2 */
for i in [1..25] do if IsAbelian(NL[i]) then i;
end if; end for;
IsIsomorphic(NL[15],CyclicGroup(12));

q,ff:=quo<N|NL[15]>;
NLq:=NormalLattice(q);
E:=DirectProduct(CyclicGroup(2),CyclicGroup(2));
IsIsomorphic(q,E);

```

```
G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^2,w^2,v^3,y^x=y,
z^x=z,z^y=z*w,w^x=w,w^y=w,w^z=w,v^x=v^2,v^y=v,v^z=v,
v^w=v,t^2,(t,z*w),(t,x*w*v),(w*v^-1*t)^5,(x*t)^6,(y*t)^3,
(x*w*t)^4,(x*z*y*t)^9>;
```

```
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);
#DoubleCosets(G,sub<G|x,y,z,w,v>,sub<G|x,y,z,w,v>);
M:=MaximalSubgroups(G1);
```

```
for i in [1..6] do #M[i]` subgroup; end for;
C:=Conjugates(G1,M[1]`subgroup);/* works 1296 */
C:=Conjugates(G1,M[5]`subgroup); /* works 1920 */
```

```
C:=Conjugates(G1,M[5]`subgroup);
CC:=Setseq(C);
```

```
for i in [1..#CC] do if f(x) in CC[i] and f(y) in CC[i]
and f(z) in CC[i] and f(w) in CC[i] and f(v) in CC[i] then i;
end if; end for;
```

```
for g in G1 do if Order(g) eq 2 and
CC[27] eq sub<G1|f(x),f(y),f(z),f(w),f(v),g>
then A:=g; break; end if; end for;
```

```
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
```

```
function(W)
  w7 := W.5 * W.4; w8 := W.2 * w7; w9 := w7 * W.1; w10
  := w8 * w9; w11 := W.6 * W.2; w12 := w10 * w11; w13
  := w9 * W.4; w14 := w12 * w13; w15 := w13 * w11; w16
  := w14 * w15; w17 := w15 * W.3; w18 := w16 * w17; w19
  := W.6 * W.1; w20 := w18 * w19; w21 := w9 * w17; w22
  := w20 * w21; w23 := w9 * w19; w24 := w22 * w23; w25
  := w23 * w11; w26 := w24 * w25; w27 := w11 * w21; w28
  := w26 * w27; w29 := w15 * w21; w30 := w28 * w29; w31
  := w29 * w25; w32 := w30 * w31; w33 := w31 * w15; w34
  := w32 * w33; w35 := w19 * w29; w36 := w34 * w35; w37
  := w27 * w19; w38 := w36 * w37; w39 := w31 * W.6; w40
  := w38 * w39; w41 := w39 * w37; w42 := w40 * w41; w43
```

```

:= w35 * W.6; w44 := w42 * w43; w45 := w39 * w41; w46
:= w44 * w45; w47 := w39 * w45; w48 := w46 * w47; w49
:= w47 * W.6; w50 := w48 * w49; w51 := w33 * w39; w52
:= w50 * w51; w53 := w39 * w51; w54 := w52 * w53; w55
:= w51 * w49; w56 := w54 * w55; w57 := w53 * w19; w58
:= w56 * w57; w59 := w57 * w49; w60 := w58 * w59; w61
:= w53 * W.6; w62 := w60 * w61; w64 := w62 * W.1; w63
:= w62^-1; w65 := w64 * w63; return w65;
end function;

```

```

AA:=function(W);
w7 := W.5 * W.4; w8 := W.2 * w7; w9 := w7 * W.1; w10
:= w8 * w9; w11 := W.6 * W.2; w12 := w10 * w11; w13
:= w9 * W.4; w14 := w12 * w13; w15 := w13 * w11; w16
:= w14 * w15; w17 := w15 * W.3; w18 := w16 * w17; w19
:= W.6 * W.1; w20 := w18 * w19; w21 := w9 * w17; w22
:= w20 * w21; w23 := w9 * w19; w24 := w22 * w23; w25
:= w23 * w11; w26 := w24 * w25; w27 := w11 * w21; w28
:= w26 * w27; w29 := w15 * w21; w30 := w28 * w29; w31
:= w29 * w25; w32 := w30 * w31; w33 := w31 * w15; w34
:= w32 * w33; w35 := w19 * w29; w36 := w34 * w35; w37
:= w27 * w19; w38 := w36 * w37; w39 := w31 * W.6; w40
:= w38 * w39; w41 := w39 * w37; w42 := w40 * w41; w43
:= w35 * W.6; w44 := w42 * w43; w45 := w39 * w41; w46
:= w44 * w45; w47 := w39 * w45; w48 := w46 * w47; w49
:= w47 * W.6; w50 := w48 * w49; w51 := w33 * w39; w52
:= w50 * w51; w53 := w39 * w51; w54 := w52 * w53; w55
:= w51 * w49; w56 := w54 * w55; w57 := w53 * w19; w58
:= w56 * w57; w59 := w57 * w49; w60 := w58 * w59; w61
:= w53 * W.6; w62 := w60 * w61; w64 := w62 * W.1; w63
:= w62^-1; w65 := w64 * w63; return w65;
end function;
AA(G);

```

```

NN:=G1;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(NN): i in [1..#G]];
for i in [2..#NN] do
P:=[Id(NN): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;

```



```

if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(v); end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=f(v^-1); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;

end for;
PP:=Id(NN);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq f(AA(G)) then Sch[i];
end if; end for;
/* t * x * y * w * t * v^-1 * t * y * v^-1 * t */

#sub<G1|f(x),f(y),f(z),f(w),f(v),
f(x*t*x*t*y*z*v*t*y*v*t*y*v*t*v)>;

J:=M[5]`subgroup;
L:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(x*t*x*t*y*z*v*t*y*v*t*y*v*t*v)>;

IsIsomorphic(J,L);

M:=sub<G|x,y,z,w,v,x*t*x*t*y*z*v*t*y*v*t*y*v*t*v>;
#DoubleCosets(G,M,sub<G|x,y,z,w,v>);

/* ===== Coset Counter and DCE =====
===== */

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^2,w^2,v^3,y^x=y,
z^x=z,z^y=z*w,w^x=w,w^y=w,w^z=w,v^x=v^2,v^y=v,v^z=v,
v^w=v,t^2,(t,z*w),(t,x*w*v),(w*v^-1*t)^5,(x*t)^6,(y*t)^3,
(x*w*t)^4,(x*z*y*t)^9>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 8)(6, 9)(7, 11)(10, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 10)(8, 11)(9, 12);
Z:=S!(3, 5)(7, 10)(11, 12);
W:=S!(1, 2)(3, 5)(4, 6)(7, 10)(8, 9)(11, 12);

```

```

V:=S!(1, 4, 9)(2, 6, 8)(3, 7, 12)(5, 10, 11);
N:=sub<S|X,Y,Z,W,V>;

HH:=sub<G|x,y,z,w,v,x*t*x*t*y*z*v*t*y*v*t*y*v*t*v>;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(v)>;
M:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(x,y,z,w,v,x*t*x*t*y*z*v*t*y*v*t*y*v*t*v)>;

ts := [ Id(G1): i in [1 .. 12] ];
ts[12]:=f(t);
ts[1]:=f(t^(v*y));
ts[2]:=f(t^(v*y*x));
ts[3]:=f(t^(v));
ts[4]:=f(t^(y*v^2));
ts[5]:=f(t^(w*v));
ts[6]:=f(t^(y*x));
ts[7]:=f(t^(v^2));
ts[8]:=f(t^(w*y));
ts[9]:=f(t^(y));
ts[10]:=f(t^(x));
ts[11]:=f(t^(w));

#DoubleCosets(G, sub<G|x,y,z,w,v,x*t*x*t*y*z*v*t*y*v*t*y*v*t*v>,
sub<G|x,y,z,w,v>);
DoubleCosets(G, sub<G|x,y,z,w,v,x*t*x*t*y*z*v*t*y*v*t*y*v*t*v>,
sub<G|x,y,z,w,v>);

prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
return v;
end function;
cst := [null : i in [1 .. Index(G, sub<G|x,y,z,w,v>)]];
where null is [Integers() | ];
  for i := 1 to 12 do
    cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;

for i in [1..1080] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N);

```

```

/* [12] */

N12:=Stabiliser(N,12);
SSS:={ [12] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12] eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s:=N12;

for g in N do if 12^g eq 11 then
N12s:=sub<N|N12s,g>; end if; end for;
#N12s; N12s;

T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[12]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..1080] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12s);

/* [12 1] */

N121:=Stabiliser(N12,1);
SSS:={ [12,1] }; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N121s:=N121;

for g in N do if 12^g eq 11 and 1^g eq 1 then
N121s:=sub<N|N121s,g>; end if; end for;

```

```

for g in N do if 12^g eq 12 and 1^g eq 4 then
N121s:=sub<N|N121s,g>; end if; end for;
for g in N do if 12^g eq 11 and 1^g eq 4 then
N121s:=sub<N|N121s,g>; end if; end for;
#N121s; N121s;

T121:=Transversal(N,N121s);
for i in [1..#T121] do
ss:=[12,1]^T121[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..1080] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N121s);

/* [12 1 8] */

N1218:=Stabiliser(N121,8);
SSS:={[12,1,8]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[8] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1218s:=N1218;

for g in N do if 12^g eq 11 and 1^g eq 1 and 8^g eq 8 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 3 and 1^g eq 4 and 8^g eq 2 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 1 and 8^g eq 6 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 5 and 1^g eq 4 and 8^g eq 2 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 9 and 8^g eq 6 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 10 and 1^g eq 1 and 8^g eq 6 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 12 and 1^g eq 4 and 8^g eq 6 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 10 and 1^g eq 9 and 8^g eq 8 then

```

```

N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 3 and 1^g eq 9 and 8^g eq 2 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 11 and 1^g eq 4 and 8^g eq 8 then
N1218s:=sub<N|N1218s,g>; end if; end for;
for g in N do if 12^g eq 5 and 1^g eq 9 and 8^g eq 2 then
N1218s:=sub<N|N1218s,g>; end if; end for;
#N1218s; N1218s;

T1218:=Transversal(N,N1218s);
for i in [1..#T1218] do
ss:=[12,1,8]^T1218[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..1080] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1218s);

/* [12,1,11] */

N12111:=Stabiliser(N121,11);
SSS:={[12,1,11]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[12]*ts[1]*ts[11] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12111s:=N12111;

for g in N do if 12^g eq 3 and 1^g eq 4 and 11^g eq 5 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 6 and 1^g eq 5 and 11^g eq 4 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 1 and 11^g eq 10 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 8 and 1^g eq 5 and 11^g eq 9 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 7 and 1^g eq 9 and 11^g eq 10 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 8 and 1^g eq 10 and 11^g eq 9 then
N12111s:=sub<N|N12111s,g>; end if; end for;

```

```

for g in N do if 12^g eq 12 and 1^g eq 4 and 11^g eq 11 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 2 and 1^g eq 10 and 11^g eq 1 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 2 and 1^g eq 11 and 11^g eq 1 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 3 and 1^g eq 9 and 11^g eq 5 then
N12111s:=sub<N|N12111s,g>; end if; end for;
for g in N do if 12^g eq 6 and 1^g eq 11 and 11^g eq 4 then
N12111s:=sub<N|N12111s,g>; end if; end for;
#N12111s; N12111s;

```

```

T12111:=Transversal(N,N12111s);
for i in [1..#T12111] do
ss:=[12,1,11]^T12111[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..1080] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12111s);

```

/\* ===== Relations to be proved by hand ===== \*/

```

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^2,w^2,
v^3,y^x=y,z^x=z,z^y=z*w,w^x=w,w^y=w,w^z=w,v^x=v^2,
v^y=v,v^z=v,v^w=v,t^2,(t,z*w),(t,x*w*v),(w*v^-1*t)^5,(x*t)^6,
(y*t)^3,(x*w*t)^4,(x*z*y*t)^9>;

```

```

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

```

```

S:=Sym(12);
X:=S!(1, 2)(3, 5)(4, 8)(6, 9)(7, 11)(10, 12);
Y:=S!(1, 3)(2, 5)(4, 7)(6, 10)(8, 11)(9, 12);
Z:=S!(3, 5)(7, 10)(11, 12);
W:=S!(1, 2)(3, 5)(4, 6)(7, 10)(8, 9)(11, 12);
V:=S!(1, 4, 9)(2, 6, 8)(3, 7, 12)(5, 10, 11);
N:=sub<S|X,Y,Z,W,V>;
IN:=sub<G1|f(x),f(y),f(z),f(w),f(v)>;
M:=sub<G1|f(x),f(y),f(z),f(w),f(v),
f(t*x*y*w*t*v^-1*t*y*v^-1*t)>;

```

```

ts := [ Id(G1): i in [1 .. 12] ];
ts[12]:=f(t);

```

```

ts[1]:=f(t^(v*y));
ts[2]:=f(t^(v*y*x));
ts[3]:=f(t^(v));
ts[4]:=f(t^(y*v^2));
ts[5]:=f(t^(w*v));
ts[6]:=f(t^(y*x));
ts[7]:=f(t^(v^2));
ts[8]:=f(t^(w*y));
ts[9]:=f(t^(y));
ts[10]:=f(t^(x));
ts[11]:=f(t^(w));

NN:=G1;

Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=Id(NN): i in [1..#G]];
for i in [2..#NN] do
P:=Id(NN): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(z); end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=f(w); end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=f(v); end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=f(v^-1); end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=f(t); end if;

end for;
PP:=Id(NN);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

/* [12] */

for g in M do for h in IN do if
ts[12]*ts[1] eq g*(ts[12]*ts[1])^h
then "true"; break; end if; end for; end for;

for g in M do for h in IN do if
ts[12]*ts[3] eq g*(ts[12]*ts[1])^h
then "true"; break; end if; end for; end for;

```

```

/* [12 1] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[2] eq g*(ts[12])^h
then true; break; end if; end for; end for;

for g in M do for h in IN do if
ts[12]*ts[1]*ts[3] eq g*(ts[12]*ts[1])^h
then true; break; end if; end for; end for;

for g in M do for h in IN do if
ts[12]*ts[1]*ts[9] eq g*(ts[12]*ts[1])^h
then true; break; end if; end for; end for;

for g in M do for h in IN do if
ts[12]*ts[1]*ts[11] eq g*(ts[12]*ts[1]*ts[8])^h
then true; break; end if; end for; end for;

/* [12 1 8] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[1] eq g*(ts[12]*ts[1]*ts[8])^h
then true; break; end if; end for; end for;

for g in M do for h in IN do if
ts[12]*ts[1]*ts[8]*ts[3] eq g*(ts[12]*ts[1]*ts[11])^h
then true; break; end if; end for; end for;

/* [12 1 11] */

for g in M do for h in IN do if
ts[12]*ts[1]*ts[11]*ts[2] eq g*(ts[12]*ts[1]*ts[8])^h
then true; break; end if; end for; end for;

```



## Appendix D

# Magma Code for Extension Types

### D.0.1 Direct Product $G \cong S_3 \times PGL_2(7) \times S_5$

```
G<x, y, z, w, u, t>:=Group<x, y, z, w, u, t | x^2, y^2, z^2, w^2, u^3, y^x=y,
z^x=z, z^y=z*w, w^x=w, w^y=w, w^z=w, u^x=u^2, u^y=u, u^z=u,
u^w=u, t^2, (t, z*w), (t, x*w*u), (x*w*t)^6, (z*y*u*t)^6, (x*t)^3>;
```

```
f, G1, k:=CosetAction(G, sub<G | x, y, z, w, u>);
CompositionFactors(G1);
```

```
NL:=NormalLattice(G1);
IsIsomorphic(Sym(3), NL[3]);
q, ff:=quo<G1 | NL[3]>;
```

```
NL1:=NormalLattice(q);
IsIsomorphic(NL1[5], PGL(2, 7));
```

```
E:=DirectProduct(PGL(2, 7), Sym(3));
IsIsomorphic(E, NL[17]);
```

```
q1, ff:=quo<G1 | NL[17]>;
#q1;
IsIsomorphic(q1, Sym(5));
```

```
E1:=DirectProduct(E, Sym(5));
if IsIsomorphic(E1, G1) then true; end if;
```

### D.0.2 Central Extension $G \cong S_3 \times PGL_2(7) \times S_5$

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^2,z^3,w^2,y^x=y*w,
z^x=z,z^y=z,w^x=w,w^y=w,w^z=w,t^2,(t,y*w),
(y*z*t^x)^3,((w*z^-1)*t)^4,(x*t)^6,(x*y*z*t)^6>;
```

```
f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
CompositionFactors(G1);
H<a>:=Group<a|a^2>;
```

```
NL:=NormalLattice(G1);
NL;
```

```
q,ff:=quo<G1|NL[2]>;
NLq:=NormalLattice(q);
```

```
H:=PGU(4,2);
```

```
K<b,c>:=Group<b,c|b^3,c^6,
(b^-1*c*b^-1*c^-1)^2,
(b*c^2*b^-1*c^-2)^2,
c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
(c^-1*b^-1)^9,
(b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2>;
f2,K2,k2:=CosetAction(K,sub<K|Id(K)>);
s,t:=IsIsomorphic(K2,NLq[2]);
```

```
T:=Transversal(G1,NL[2]);
B:=t(f2(b));
C:=t(f2(c));
```

```
for i in [1..#T] do if ff(T[i]) eq B then i; break;
end if; end for;
for i in [1..#T] do if ff(T[i]) eq C then i; break;
end if; end for;
```

```
for d in NLq[3] do if Order(d) eq 2 and d
notin NLq[2] and NLq[3]eq sub<q|NLq[2],d>
then D:=d; break; end if; end for;
```

```
B:=ff(B);
```

```

C:=ff(C);
N:=sub<q|B,C>;

NN<b,c>:=Group<b,c|b^3,c^6,(b^-1*c*b^-1*c^-1)^2,
(b*c^2*b^-1*c^-2)^2,c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
(c^-1*b^-1)^9,(b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2>;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=C^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq B^D then print Sch[i];
end if; end for;
c * b * c * b^-1 * c * b * c^-2 * b^-1 * c^-1
for i in [1..#N] do if ArrayP[i] eq C^D then print Sch[i];
end if; end for;
b^-1 * c * b * c * b^-1 * c^2 * b * c^-1 * b

/* (U(4,2) : 2)
Z<b,c,d>:=Group<b,c,d|b^3,c^6,(b^-1 * c * b^-1 * c^-1)^2,
(b * c^2 * b^-1 * c^-2)^2,
c^-3 * b * c^-3 * b^-1 * c^3 * b * c^3 * b^-1,
(c^-1 * b^-1)^9,
(b * c^-2 * b^-1 * c^-1 * b * c^-1 * b^-1 * c^-1)^2,
d^2,
b^d= c * b * c * b^-1 * c * b * c^-2 * b^-1 * c^-1,
c^d= b^-1*c*b*c*b^-1*c^2*b*c^-1*b>;
f4,Z1,k4:=CosetAction(Z,sub<Z|Id(Z)>);

```

```

s,t:=IsIsomorphic(Z1,q);

/* Now, we form a Central extension among these elements. */
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^2,y^2,z^3,w^2,y^x=y*w,
z^x=z,z^y=z,w^x=w,w^y=w,w^z=w,t^2,(t,y*w),
(y*z*t^x)^3,((w*z^-1)*t)^4,(x*t)^6,(x*y*z*t)^6>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w>);
CompositionFactors(G1);

H<a>:=Group<a|a^2>;

NL:=NormalLattice(G1);
T:=Transversal(G1,NL[2]);
q,ff:=quo<G1|NL[2]>;

Z<b,c,d>:=Group<b,c,d|b^3,c^6,(b^-1*c*b^-1*c^-1)^2,
(b*c^2*b^-1*c^-2)^2,
c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
(c^-1*b^-1)^9,
(b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2,
d^2,b^d=c*b*c*b^-1*c*b*c^-2*b^-1*c^-1,
c^d=b^-1*c*b*c*b^-1*c^2*b*c^-1*b>;
f4,Z1,k4:=CosetAction(Z,sub<Z|Id(Z)>);
s,t:=IsIsomorphic(Z1,q);

B:=t(f4(b));
C:=t(f4(c));
D:=t(f4(d));

for i in [1..#T] do if ff(T[i]) eq B then i; break;
end if; end for;
for i in [1..#T] do if ff(T[i]) eq C then i; break;
end if; end for;
for i in [1..#T] do if ff(T[i]) eq D then i; break;
end if; end for;

B:=T[2424];
C:=T[20394];
D:=T[7920];

for i in [0..1] do if (C^-1*B^-1)^9 eq A^i then i;
break; end if; end for;

```

```

H<a,b,c,d>:=Group<a,b,c,d|b^3,c^6,(b^-1*c*b^-1*c^-1)^2,
(b*c^2*b^-1*c^-2)^2,
c^-3*b*c^-3*b^-1*c^3*b*c^3*b^-1,
(c^-1*b^-1)^9=a,
(b*c^-2*b^-1*c^-1*b*c^-1*b^-1*c^-1)^2,
d^2,b^d=c*b*c*b^-1*c*b*c^-2*b^-1*c^-1,
c^d=b^-1*c*b*c*b^-1*c^2*b*c^-1*b,a^2>;
f4,H1,k4:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H1,G1);

```

### D.0.3 Semi-Direct Product $2^2 : (2^2 \times 3)$

```

S:=Sym(12);
X:=S!(1,2)(3,5)(4,6)(7,9)(8,10)(11,12);
Y:=S!(1,3)(2,5)(4,7)(6,9)(8,11)(10,12);
Z:=S!(1,4,8)(2,6,10)(3,7,11)(5,9,12);
W:=S!(1,2)(3,5)(4,7)(6,9)(8,12)(10,11);
U:=S!(1,5)(2,3)(4,6)(7,9)(8,11)(10,12);
N:=sub<S|X,Y,Z,W,U>;

f,N1,k:=CosetAction(N,sub<N|Id(N)>);

NL:=NormalLattice(N1);

Center(N1) eq NL[6];
X:=[2,2];
IsIsomorphic(NL[6],AbelianGroup(GrpPerm,X));

H<a,b>:=Group<a,b|a^2,b^2,(a*b)^2>;
f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);

q, ff:=quo<N1|NL[6]>;

IsIsomorphic(Alt(4),q);

Q<c,d,e>:=Group<c,d,e|c^3,d^2,e^2,c^-1*d*c*e,
(d*e)^2,c*d*c^-1*d*e>;
f3,Q3,k3:=CosetAction(Q,sub<Q|Id(Q)>);

T:=Transversal(N1,NL[6]);

s,t:=IsIsomorphic(Q3,q);

for i in [1..#T] do if ff(T[i]) eq q.1 then i;

```

```

    end if; end for;
/* store as C */

for i in [1..#T] do if ff(T[i]) eq q.4 then i;
end if; end for;

for i in [1..#T] do if ff(T[i]) eq q.5 then i;
end if; end for;

Generators(NL[6]);

A:=N1!(1, 9) (2, 10) (3, 11) (4, 12) (5, 20) (6, 19) (7, 18) (8, 17)
(13, 33) (14, 34) (15, 35) (16, 36) (21, 37) (22, 38) (23, 39)
(24, 40) (25, 44) (26, 43) (27, 42) (28, 41) (29, 48) (30, 47)
(31, 46) (32, 45);
B:=N1!(1, 5) (2, 6) (3, 7) (4, 8) (9, 20) (10, 19) (11, 18) (12, 17)
(13, 25) (14, 26) (15, 27) (16, 28) (21, 29) (22, 30) (23, 31)
(24, 32) (33, 44) (34, 43) (35, 42) (36, 41) (37, 48) (38, 47)
(39, 46) (40, 45);

C:=N1!(1, 13, 21) (2, 14, 22) (3, 15, 23) (4, 16, 24) (5, 25, 29)
(6, 26, 30) (7, 27, 31) (8, 28, 32) (9, 33, 37) (10, 34, 38)
(11, 35, 39) (12, 36, 40) (17, 41, 45) (18, 42, 46)
(19, 43, 47) (20, 44, 48);
D:=N1!(1, 6) (2, 5) (3, 8) (4, 7) (9, 19) (10, 20) (11, 17) (12, 18)
(13, 35) (14, 36) (15, 33) (16, 34) (21, 45) (22, 46) (23, 47)
(24, 48) (25, 42) (26, 41) (27, 44) (28, 43) (29, 40) (30, 39)
(31, 38) (32, 37);
E:=N1!(1, 17) (2, 18) (3, 19) (4, 20) (5, 12) (6, 11) (7, 10) (8, 9)
(13, 26) (14, 25) (15, 28) (16, 27) (21, 39) (22, 40) (23, 37)
(24, 38) (29, 46) (30, 45) (31, 48) (32, 47) (33, 43) (34, 44)
(35, 41) (36, 42);

for i,j in [1..2] do if A^C eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if A^D eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if A^E eq A^i*B^j then i,j;
break; end if; end for;

for i,j in [1..2] do if B^C eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if B^D eq A^i*B^j then i,j;
break; end if; end for;

```

```

for i,j in [1..2] do if B^E eq A^i*B^j then i,j;
break; end if; end for;

for i,j in [1..2] do if C^3 eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if D^2 eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if E^2 eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if C^-1*D*C*E eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if (D*E)^2 eq A^i*B^j then i,j;
break; end if; end for;
for i,j in [1..2] do if C*D*C^-1*D*E eq A^i*B^j then i,j;
break; end if; end for;

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,(a*b)^2,
c^3,d^2,e^2,c^-1*d*c*e,(d*e)^2,c*d*c^-1*d*e,
a^c=a,a^d=a,a^e=a,b^c=b,b^d=b,b^e=b>;
f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H2,N1);
true

```

#### D.0.4 Semi-Direct Product $2^8 : A_5$

```

G<x,y,z,w,u,t>:=Group<x,y,z,w,u,t|x^2,y^2,z^3,w^2,u^2,y^x=y,
z^x=z,z^y=z,w^x=w,w^y=w,w^z=u,u^x=u,u^y=u,u^z=w*u,u^w=u,
t^2,(t,x*y*w),(t,y*u),(x*z*t)^5,(y*z^-1*t)^5,
(x*y*u*z^-1*t)^5>;

f,G1,k:=CosetAction(G,sub<G|x,y,z,w,u>);
CompositionFactors(G1);

/* isomorphism type */
NL:=NormalLattice(G1);

for i in [1..#NL] do if IsAbelian(NL[i]) then i;
end if; end for;

q, ff:=quo<G1|NL[5]>;

FPGroup(q);

```

```

X:=[2,2,2,2,2,2,2,2];
E:=DirectProduct(Alt(5),AbelianGroup(GrpPerm,X));
IsIsomorphic(E,G1);

/* 2^8 */
H<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^2,b^2,c^2,d^2,
e^2,f^2,g^2,h^2,(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),(a,h),
(b,c),(b,d),(b,e),(b,f),(b,g),(b,h),(c,d),(c,e),(c,f),(c,g),
(c,h),(d,e),(d,f),(d,g),(d,h),(e,f),(e,g),(e,h),(f,g),(f,h),
(g,h)>;

/* q = A_5 */
Z<i,j>:=Group<i,j|i^3,j^3,(i*j^-1*i^-1*j^-1)^2,
(i^-1*j*i^-1*j^-1)^2>;

T:=Transversal(G1,NL[5]);

ff(T[3]) eq q.1;
ff(T[14]) eq q.2;

for i in [1..#T] do if ff(T[i]) eq q.2 then i; end if; end for;

A:=G1!(2, 4)(3, 6)(7, 17)(10, 13)(15, 56)(18, 50)(19, 59)
(22, 38)(23, 29)(26, 41)(27, 32)(28, 31)(30, 68)(33, 156)
(36, 53)(37, 40)(39, 64)(42, 152)(43, 45)(44, 57)(46, 141)
(47, 117)(48, 86)(49, 52)(51, 89)(54, 171)(55, 58)(60, 113)
(61, 146)(62, 116)(63, 65)(66, 102)(67, 69)(70, 80)(71, 77)
(73, 83)(74, 142)(76, 109)(78, 154)(81, 136)(84, 138)(85,197)
(87, 147)(88, 90)(91, 130)(92, 98)(93, 167)(94, 105)(95,131)
(96, 104)(97, 134)(99, 149)(100, 128)(101, 170)(103, 123)
(106, 125)(107, 110)(111, 139)(114, 144)(115, 204)(118, 148)
(119, 120)(121, 126)(122, 133)(124, 132)(127, 168)(129, 151)
(145, 160)(150, 169)(157, 182)(158, 159)(161, 181)(162, 164)
(163, 248)(165, 184)(166, 185)(172, 190)(173, 191)(174, 178)
(175, 192)(176, 193)(177, 258)(179, 195)(180, 229)(183, 218)
(194, 239)(196, 241)(198, 209)(199, 206)(202, 232)(203, 257)
(205, 213)(207, 211)(208, 244)(210, 214)(212, 246)(215, 217)
(216, 222)(219, 283)(220, 250)(221, 247)(223, 251)(224, 254)
(225, 235)(226, 234)(227, 255)(228, 237)(230, 236)(231, 240)
(233, 253)(238, 259)(242, 277)(243, 245)(249, 264)(252, 256)
(260, 282)(261, 268)(262, 290)(263, 271)(265, 269)(266, 270)
(267, 285)(272, 291)(273, 280)(274, 281)(275, 288)(276, 278)

```



(279, 289) (284, 286) (287, 296) (292, 299) (293, 297) (294, 305)  
 (295, 298) (300, 304) (301, 303) (302, 307) (306, 312) (308, 315)  
 (309, 313) (310, 320) (311, 314) (316, 319) (317, 318);

B:=G1! (5, 11) (7, 10) (8, 14) (13, 17) (15, 23) (18, 22) (19, 27)  
 (26, 36) (28, 55) (29, 56) (30, 142) (31, 58) (32, 59) (33, 109)  
 (37, 49) (38, 50) (39, 131) (40, 52) (41, 53) (42, 134) (43, 67)  
 (44, 136) (45, 69) (46, 73) (51, 103) (54, 106) (57, 81) (60, 84)  
 (63, 88) (64, 95) (65, 90) (66, 122) (68, 74) (70, 114) (71, 111)  
 (72, 135) (75, 137) (76, 156) (77, 139) (78, 107) (79, 140)  
 (80, 144) (82, 143) (83, 141) (85, 181) (89, 123) (91, 94)  
 (92, 168) (93, 118) (96, 124) (97, 152) (98, 127) (99, 126)  
 (100, 193) (101, 129) (102, 133) (104, 132) (105, 130)  
 (108, 153) (110, 154) (112, 155) (113, 138) (115, 158) (121, 149)  
 (125, 171) (128, 176) (145, 162) (148, 167) (150, 173) (151, 170)  
 (157, 165) (159, 204) (160, 164) (161, 197) (163, 205) (166, 210)  
 (169, 191) (172, 179) (174, 192) (175, 178) (177, 226) (180, 225)  
 (182, 184) (183, 198) (185, 214) (190, 195) (194, 252) (196, 203)  
 (199, 250) (202, 230) (206, 220) (207, 215) (208, 251) (209, 218)  
 (211, 217) (212, 222) (213, 248) (216, 246) (219, 263) (221, 243)  
 (223, 244) (224, 237) (227, 238) (228, 254) (229, 235) (231, 253)  
 (232, 236) (233, 240) (234, 258) (239, 256) (241, 257) (242, 273)  
 (245, 247) (249, 261) (255, 259) (260, 262) (264, 268) (265, 270)  
 (266, 269) (267, 286) (271, 283) (272, 304) (274, 279) (275, 278)  
 (276, 288) (277, 280) (281, 289) (282, 290) (284, 285) (287, 293)  
 (291, 300) (292, 298) (294, 301) (295, 299) (296, 297) (302, 319)  
 (303, 305) (306, 309) (307, 316) (308, 314) (310, 317) (311, 315)  
 (312, 313) (318, 320);

C:=G1! (1, 187) (2, 62) (3, 117) (4, 116) (5, 137) (6, 47) (7, 148)  
 (8, 143) (9, 25) (10, 167) (11, 75) (12, 20) (13, 93) (14, 82)  
 (15, 69) (16, 186) (17, 118) (18, 26) (19, 58) (21, 24) (22, 36)  
 (23, 45) (27, 31) (28, 32) (29, 43) (30, 154) (33, 156) (34, 200)  
 (35, 188) (37, 63) (38, 53) (39, 64) (40, 65) (41, 50) (42, 66)  
 (44, 139) (46, 141) (48, 87) (49, 88) (51, 89) (52, 90) (54, 91)  
 (55, 59) (56, 67) (57, 111) (60, 113) (61, 119) (68, 78) (70, 80)  
 (71, 81) (72, 79) (73, 83) (74, 107) (76, 109) (77, 136) (84, 138)  
 (85, 164) (86, 147) (92, 98) (94, 106) (95, 131) (96, 170)  
 (97, 133) (99, 149) (100, 193) (101, 104) (102, 152) (103, 123)  
 (105, 125) (108, 112) (110, 142) (114, 144) (115, 184)  
 (120, 146) (121, 126) (122, 134) (124, 151) (127, 168) (128, 176)  
 (129, 132) (130, 171) (135, 140) (145, 161) (150, 173) (153, 155)  
 (157, 159) (158, 182) (160, 181) (162, 197) (163, 218) (165, 204)  
 (166, 220) (169, 191) (172, 179) (174, 192) (175, 178) (177, 256)

(180, 257) (183, 248) (185, 250) (189, 201) (190, 195) (194, 234)  
 (196, 235) (198, 213) (199, 214) (202, 228) (203, 229) (205, 209)  
 (206, 210) (207, 245) (208, 246) (211, 243) (212, 244) (215, 247)  
 (216, 251) (217, 221) (219, 261) (222, 223) (224, 236) (225, 241)  
 (226, 239) (227, 240) (230, 254) (231, 255) (232, 237) (233, 238)  
 (242, 276) (249, 263) (252, 258) (253, 259) (260, 279) (262, 274)  
 (264, 271) (265, 286) (266, 285) (267, 270) (268, 283) (269, 284)  
 (272, 292) (273, 288) (275, 280) (277, 278) (281, 290) (282, 289)  
 (287, 294) (291, 299) (293, 301) (295, 300) (296, 305) (297, 303)  
 (298, 304) (302, 314) (306, 317) (307, 311) (308, 319) (309, 310)  
 (312, 318) (313, 320) (315, 316);

D:=G1!(1, 25) (2, 117) (3, 62) (4, 47) (5, 155) (6, 116) (7, 13)  
 (8, 140) (9, 187) (10, 17) (11, 112) (12, 186) (14, 79) (15, 56)  
 (16, 20) (18, 63) (19, 59) (21, 35) (22, 88) (23, 29) (24, 188)  
 (26, 37) (27, 32) (28, 31) (30, 78) (33, 144) (34, 201) (36, 49)  
 (38, 90) (39, 131) (40, 41) (42, 125) (43, 45) (44, 111) (46, 138)  
 (48, 61) (50, 65) (51, 103) (52, 53) (54, 97) (55, 58) (57, 139)  
 (60, 83) (64, 95) (66, 105) (67, 69) (68, 154) (70, 76) (71, 136)  
 (72, 82) (73, 113) (74, 110) (75, 108) (77, 81) (80, 109) (84, 141)  
 (85, 204) (86, 146) (87, 119) (89, 123) (91, 133) (92, 168) (93, 148)  
 (94, 102) (96, 132) (98, 127) (99, 126) (100, 150) (101, 151)  
 (104, 124) (106, 152) (107, 142) (114, 156) (115, 197) (118, 167)  
 (120, 147) (121, 149) (122, 130) (128, 169) (129, 170) (134, 171)  
 (135, 143) (137, 153) (145, 182) (157, 160) (158, 161) (159, 181)  
 (162, 184) (163, 198) (164, 165) (166, 212) (172, 174) (173, 193)  
 (175, 195) (176, 191) (177, 237) (178, 190) (179, 192) (180, 231)  
 (183, 205) (185, 246) (189, 200) (194, 230) (196, 238) (199, 251)  
 (202, 252) (203, 227) (206, 223) (207, 221) (208, 250) (209, 248)  
 (210, 222) (211, 247) (213, 218) (214, 216) (215, 243) (217, 245)  
 (219, 270) (220, 244) (224, 226) (225, 253) (228, 258) (229, 240)  
 (232, 256) (233, 235) (234, 254) (236, 239) (241, 259) (242, 289)  
 (249, 286) (255, 257) (260, 278) (261, 267) (262, 275) (263, 265)  
 (264, 284) (266, 283) (268, 285) (269, 271) (272, 298) (273, 281)  
 (274, 280) (276, 282) (277, 279) (287, 301) (288, 290) (291, 295)  
 (292, 304) (293, 294) (296, 303) (297, 305) (299, 300) (302, 308)  
 (306, 310) (307, 315) (309, 317) (311, 316) (312, 320) (313, 318)  
 (314, 319);

E:=G1!(2, 3) (4, 6) (7, 13) (10, 17) (15, 43) (18, 88) (19, 28)  
 (22, 63) (23, 67) (26, 49) (27, 55) (29, 69) (30, 57) (31, 59)  
 (32, 58) (33, 144) (36, 37) (38, 65) (39, 51) (40, 53) (41, 52)  
 (42, 133) (44, 68) (45, 56) (46, 138) (47, 116) (48, 119) (50, 90)  
 (54, 105) (60, 83) (61, 87) (62, 117) (64, 89) (66, 97) (70, 76)

(71, 110) (73, 113) (74, 136) (77, 107) (78, 139) (80, 109)  
 (81, 142) (84, 141) (85, 159) (86, 120) (91, 125) (92, 121)  
 (93, 148) (94, 171) (95, 123) (96, 132) (98, 126) (99, 127)  
 (100, 169) (101, 151) (102, 134) (103, 131) (104, 124)  
 (106, 130) (111, 154) (114, 156) (115, 161) (118, 167)  
 (122, 152) (128, 150) (129, 170) (145, 184) (146, 147)  
 (149, 168) (157, 164) (158, 197) (160, 165) (162, 182)  
 (163, 243) (166, 206) (172, 178) (173, 176) (174, 190)  
 (175, 179) (177, 256) (180, 227) (181, 204) (183, 207)  
 (185, 199) (191, 193) (192, 195) (194, 234) (196, 253)  
 (198, 215) (202, 228) (203, 231) (205, 221) (208, 216)  
 (209, 217) (210, 220) (211, 218) (212, 223) (213, 247)  
 (214, 250) (219, 284) (222, 244) (224, 236) (225, 238)  
 (226, 239) (229, 255) (230, 254) (232, 237) (233, 241)  
 (235, 259) (240, 257) (242, 278) (245, 248) (246, 251)  
 (249, 266) (252, 258) (260, 289) (261, 269) (262, 281)  
 (263, 285) (264, 270) (265, 268) (267, 271) (272, 297)  
 (273, 275) (274, 290) (276, 277) (279, 282) (280, 288)  
 (283, 286) (287, 300) (291, 293) (292, 303) (294, 295)  
 (296, 304) (298, 305) (299, 301) (302, 313) (306, 316)  
 (307, 309) (308, 318) (310, 311) (312, 319) (314, 320)  
 (315, 317);

F:=G1!(1, 201) (2, 61) (3, 87) (4, 146) (5, 79) (6, 147) (7, 132)  
 (8, 112) (9, 200) (10, 104) (11, 140) (12, 35) (13, 96) (14, 155)  
 (15, 23) (16, 24) (17, 124) (18, 88) (19, 27) (20, 188) (21, 186)  
 (22, 63) (25, 34) (26, 49) (28, 55) (29, 56) (30, 57) (31, 58)  
 (32, 59) (33, 60) (36, 37) (38, 65) (39, 126) (40, 53) (41, 52)  
 (42, 134) (43, 67) (44, 68) (45, 69) (46, 70) (47, 86) (48, 117)  
 (50, 90) (51, 98) (54, 106) (62, 119) (64, 121) (66, 122)  
 (71, 110) (72, 137) (73, 114) (74, 136) (75, 135) (76, 138)  
 (77, 107) (78, 139) (80, 141) (81, 142) (82, 153) (83, 144)  
 (84, 109) (85, 184) (89, 92) (91, 94) (93, 170) (95, 149)  
 (97, 152) (99, 131) (100, 191) (101, 167) (102, 133) (103, 127)  
 (105, 130) (108, 143) (111, 154) (113, 156) (115, 164)  
 (116, 120) (118, 151) (123, 168) (125, 171) (128, 173)  
 (129, 148) (145, 159) (150, 176) (157, 161) (158, 160)  
 (162, 204) (163, 248) (165, 197) (166, 250) (169, 193)  
 (172, 175) (174, 195) (177, 236) (178, 179) (180, 241)  
 (181, 182) (183, 218) (185, 220) (187, 189) (190, 192)  
 (194, 228) (196, 229) (198, 209) (199, 210) (202, 234)  
 (203, 235) (205, 213) (206, 214) (207, 211) (208, 212)  
 (215, 217) (216, 223) (219, 266) (221, 247) (222, 251)  
 (224, 256) (225, 257) (226, 232) (227, 233) (230, 258)

(231, 259) (237, 239) (238, 240) (242, 262) (243, 245)  
 (244, 246) (249, 284) (252, 254) (253, 255) (260, 273)  
 (261, 285) (263, 269) (264, 286) (265, 271) (267, 268)  
 (270, 283) (272, 293) (274, 276) (275, 289) (277, 290)  
 (278, 281) (279, 288) (280, 282) (287, 304) (291, 297)  
 (292, 301) (294, 298) (295, 305) (296, 300) (299, 303)  
 (302, 315) (306, 320) (307, 308) (309, 318) (310, 312)  
 (311, 319) (313, 317) (314, 316);

G:=G1!(5, 8) (7, 17) (10, 13) (11, 14) (15, 19) (18, 36)  
 (22, 26) (23, 27) (28, 43) (29, 32) (30, 110) (31, 45)  
 (33, 113) (37, 88) (38, 41) (39, 121) (40, 90) (42, 171)  
 (44, 77) (46, 80) (49, 63) (50, 53) (51, 92) (52, 65)  
 (54, 152) (55, 67) (56, 59) (57, 71) (58, 69) (60, 156)  
 (64, 126) (66, 130) (68, 107) (70, 141) (72, 108)  
 (73, 144) (74, 78) (75, 82) (76, 84) (79, 112) (81, 111)  
 (83, 114) (85, 182) (89, 98) (91, 102) (93, 167)  
 (94, 133) (95, 99) (96, 104) (97, 106) (100, 190)  
 (101, 170) (103, 168) (105, 122) (109, 138) (115, 160)  
 (118, 148) (123, 127) (124, 132) (125, 134) (128, 172)  
 (129, 151) (131, 149) (135, 153) (136, 139) (137, 143)  
 (140, 155) (142, 154) (145, 204) (150, 178) (157, 197)  
 (158, 164) (159, 162) (161, 165) (163, 198) (166, 208)  
 (169, 174) (173, 175) (176, 179) (177, 254) (180, 203)  
 (181, 184) (183, 205) (185, 244) (191, 192) (193, 195)  
 (194, 232) (196, 225) (199, 222) (202, 239) (206, 216)  
 (207, 221) (209, 248) (210, 251) (211, 247) (212, 250)  
 (213, 218) (214, 223) (215, 243) (217, 245) (219, 261)  
 (220, 246) (224, 258) (226, 228) (227, 231) (229, 257)  
 (230, 256) (233, 259) (234, 237) (235, 241) (236, 252)  
 (238, 253) (240, 255) (242, 262) (249, 263) (260, 273)  
 (264, 271) (265, 286) (266, 285) (267, 270) (268, 283)  
 (269, 284) (272, 305) (274, 276) (275, 289) (277, 290)  
 (278, 281) (279, 288) (280, 282) (287, 299) (291, 294)  
 (292, 296) (293, 295) (297, 298) (300, 301) (302, 320)  
 (303, 304) (306, 315) (307, 310) (308, 312) (309, 311)  
 (313, 314) (316, 317) (318, 319);

H:=G1!(1, 12) (2, 4) (3, 6) (5, 143) (7, 124) (8, 137)  
 (9, 16) (10, 96) (11, 82) (13, 104) (14, 75) (15, 59)  
 (17, 132) (18, 50) (19, 56) (20, 187) (21, 34) (22, 38)  
 (23, 32) (24, 200) (25, 186) (26, 41) (27, 29) (28, 45)  
 (30, 136) (31, 43) (33, 80) (35, 201) (36, 53) (37, 40)  
 (39, 98) (42, 133) (44, 142) (46, 113) (47, 117) (48, 86)

(49, 52) (51, 126) (54, 105) (55, 69) (57, 74) (58, 67)  
 (60, 141) (61, 146) (62, 116) (63, 65) (64, 92) (66, 97)  
 (68, 81) (70, 156) (71, 78) (72, 112) (73, 138) (76, 114)  
 (77, 154) (79, 108) (83, 84) (85, 160) (87, 147) (88, 90)  
 (89, 121) (91, 125) (93, 101) (94, 171) (95, 168)  
 (99, 103) (100, 174) (102, 134) (106, 130) (107, 111)  
 (109, 144) (110, 139) (115, 182) (118, 129) (119, 120)  
 (122, 152) (123, 149) (127, 131) (128, 178) (135, 155)  
 (140, 153) (145, 197) (148, 151) (150, 172) (157, 204)  
 (158, 184) (159, 165) (161, 162) (163, 217) (164, 181)  
 (166, 199) (167, 170) (169, 190) (173, 179) (175, 176)  
 (177, 202) (180, 238) (183, 247) (185, 206) (188, 189)  
 (191, 195) (192, 193) (194, 224) (196, 231) (198, 245)  
 (203, 253) (205, 211) (207, 213) (208, 222) (209, 243)  
 (210, 250) (212, 251) (214, 220) (215, 248) (216, 244)  
 (218, 221) (219, 269) (223, 246) (225, 227) (226, 230)  
 (228, 256) (229, 259) (232, 258) (233, 257) (234, 236)  
 (235, 255) (237, 252) (239, 254) (240, 241) (242, 279)  
 (249, 285) (260, 276) (261, 284) (262, 288) (263, 266)  
 (264, 267) (265, 283) (268, 286) (270, 271) (272, 305)  
 (273, 274) (275, 290) (277, 289) (278, 282) (280, 281)  
 (287, 299) (291, 294) (292, 296) (293, 295) (297, 298)  
 (300, 301) (302, 320) (303, 304) (306, 315) (307, 310)  
 (308, 312) (309, 311) (313, 314) (316, 317) (318, 319);

T[3];

I:=G1!(1, 2) (3, 9) (4, 12) (5, 15) (6, 16) (7, 18) (8, 19)  
 (10, 22) (11, 23) (13, 26) (14, 27) (17, 36) (20, 47)  
 (21, 48) (24, 61) (25, 62) (28, 72) (29, 75) (30, 42)  
 (31, 79) (32, 82) (33, 85) (34, 86) (35, 87) (37, 93)  
 (38, 96) (39, 100) (40, 101) (41, 104) (43, 108) (44, 102)  
 (45, 112) (46, 115) (49, 118) (50, 124) (51, 128) (52, 129)  
 (53, 132) (54, 81) (55, 135) (56, 137) (57, 106) (58, 140)  
 (59, 143) (60, 145) (63, 148) (64, 150) (65, 151) (66, 78)  
 (67, 153) (68, 105) (69, 155) (70, 157) (71, 97) (73, 158)  
 (74, 130) (76, 159) (77, 91) (80, 160) (83, 161) (84, 162)  
 (88, 167) (89, 169) (90, 170) (92, 172) (94, 139) (95, 173)  
 (98, 174) (99, 175) (103, 176) (107, 122) (109, 181)  
 (110, 171) (111, 152) (113, 182) (114, 165) (116, 186)  
 (117, 187) (119, 188) (120, 189) (121, 190) (123, 191)  
 (125, 154) (126, 178) (127, 192) (131, 193) (133, 136)  
 (134, 142) (138, 184) (141, 197) (144, 164) (146, 200)  
 (147, 201) (149, 195) (156, 204) (163, 219) (166, 177)  
 (168, 179) (180, 242) (183, 249) (185, 194) (196, 260)

(198, 261) (199, 202) (203, 262) (205, 263) (206, 224)  
 (207, 264) (208, 254) (209, 265) (210, 226) (211, 266)  
 (212, 256) (213, 267) (214, 252) (215, 268) (216, 258)  
 (217, 269) (218, 270) (220, 237) (221, 271) (222, 239)  
 (223, 236) (225, 273) (227, 274) (228, 251) (229, 275)  
 (230, 250) (231, 276) (232, 244) (233, 277) (234, 246)  
 (235, 278) (238, 279) (240, 280) (241, 281) (243, 283)  
 (245, 284) (247, 285) (248, 286) (253, 288) (255, 282)  
 (257, 289) (259, 290) (272, 302) (287, 306) (291, 307)  
 (292, 308) (293, 309) (294, 310) (295, 311) (296, 312)  
 (297, 313) (298, 314) (299, 315) (300, 316) (301, 317)  
 (303, 318) (304, 319) (305, 320);

T[14];

J:=G1!(1, 18, 15) (3, 48, 47) (4, 62, 61) (5, 85, 42)  
 (6, 87, 86) (7, 30, 100) (8, 115, 102) (9, 22, 19)  
 (10, 81, 128) (11, 145, 106) (12, 26, 23) (13, 78, 150)  
 (14, 157, 105) (16, 36, 27) (17, 77, 169) (20, 49, 32)  
 (21, 40, 43) (24, 53, 31) (25, 63, 56) (28, 188, 52)  
 (29, 186, 37) (33, 177, 219) (34, 65, 67) (35, 41, 45)  
 (38, 58, 200) (39, 242, 166) (44, 175, 101)  
 (46, 194, 249) (50, 69, 201) (51, 260, 185)  
 (54, 82, 161) (55, 189, 90) (57, 174, 148) (59, 187, 88)  
 (60, 202, 261) (64, 262, 199) (66, 79, 160) (68, 176, 104)  
 (70, 224, 263) (71, 173, 96) (72, 158, 97) (73, 254, 264)  
 (74, 192, 129) (75, 159, 130) (76, 226, 265) (80, 256, 266)  
 (83, 252, 267) (84, 258, 268) (89, 273, 206) (91, 153, 162)  
 (92, 274, 210) (93, 139, 172) (94, 140, 197) (95, 275, 251)  
 (98, 276, 250) (99, 277, 244) (103, 278, 246) (107, 190, 118)  
 (108, 181, 122) (109, 232, 283) (110, 179, 170) (111, 195, 151)  
 (112, 182, 152) (113, 234, 284) (114, 239, 271) (116, 119, 146)  
 (121, 288, 214) (123, 281, 223) (124, 154, 191) (125, 155, 204)  
 (126, 279, 220) (127, 282, 208) (131, 289, 212) (132, 136, 193)  
 (133, 137, 184) (134, 143, 164) (135, 165, 171) (138, 230, 286)  
 (141, 228, 285) (142, 178, 167) (144, 237, 270) (149, 290, 216)  
 (156, 236, 269) (163, 302, 180) (168, 280, 222) (183, 306, 233)  
 (196, 213, 307) (198, 308, 231) (203, 211, 309) (205, 310, 235)  
 (207, 311, 229) (209, 312, 255) (215, 313, 225) (217, 314, 241)  
 (218, 315, 238) (221, 316, 240) (227, 247, 317) (243, 318, 253)  
 (245, 319, 259) (248, 320, 257) (287, 291, 303) (292, 293, 305)  
 (294, 297, 304) (295, 296, 301);

for q,r,s,t,u,v,w,x in [1..2] do if A^I eq  
 A^q\*B^r\*C^s\*D^t\*E^u\*F^v\*G^w\*H^x then q,r,s,t,u,v,w,x;

```

end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if A^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if B^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if B^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if C^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if C^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if D^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if D^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if E^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if E^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if F^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if F^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if G^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if G^J eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if H^I eq
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
for q,r,s,t,u,v,w,x in [1..2] do if H^J eq

```

```
A^q*B^r*C^s*D^t*E^u*F^v*G^w*H^x then q,r,s,t,u,v,w,x;
end if;end for;
```

```
Z<a,b,c,d,e,f,g,h,i,j>:=Group<a,b,c,d,e,f,g,h,i,j|i^2,j^3,
a^2,b^2,c^2,d^2,e^2,f^2,g^2,h^2,(a,b),(a,c),(a,d),(a,e),
(a,f),(a,g),(a,h),(b,c),(b,d),(b,e),(b,f),(b,g),(b,h),(c,d),
(c,e),(c,f),(c,g),(c,h),(d,e),(d,f),(d,g),(d,h),(e,f),(e,g),
(e,h),(f,g),(f,h),(g,h),(i*j^-1)^5,a^i=a*g*h,a^j=b*d*e,b^i=b,
b^j=a*h,c^i=d,c^j=f,d^i=c,d^j=d,e^i=b*c*d*e,e^j=a*b*d*f*h,
f^i=c*d*f*h,f^j=h,g^i=g,g^j=c*d*e*g,h^i=h,h^j=c>;
f,Z1,k:=CosetAction(Z,sub<Z|Id(Z)>);
IsIsomorphic(Z1,G1);
```

#### D.0.5 Mixed Extension $2^4 : \bullet S_4$

```
S:=Sym(12);
X:=S!(1,2)(3,5)(4,6)(7,9)(8,10)(11,12);
Y:=S!(1,3)(2,5)(4,7)(6,9)(8,11)(10,12);
Z:=S!(1,4,8)(2,6,10)(3,7,11)(5,9,12);
W:=S!(1,2)(3,5)(4,7)(6,9)(8,12)(10,11);
V:=S!(1,5)(2,3)(4,6)(7,9)(8,11)(10,12);
N:=sub<S|X,Y,Z,W,V>;

G<x,y,z,w,v,t>:=Group<x,y,z,w,v,t|x^2,y^2,z^3,
w^2,v^2,y^x=y,z^x=z,z^y=z,w^x=w,w^y=w,w^z=v,v^x=v,
v^y=v,v^z=w*v,v^w=v,t^2,(t,x*y*w),(t,y*v),(x*t)^3,(z*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y,z,w,v>);

Ms:=MaximalSubgroups(G1);
M:=Ms[7]`subgroup;
NL:=NormalLattice(M);

for i in [1..#NL] do if IsAbelian(NL[i]) then i;
end if; end for;

X:=[2,2,2,2];
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,X));

q, ff:=quo<M|NL[4]>;

K<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,d^2,(a,b),(a,c),(a,d),
(b,c),(b,d),(c,d)>;
```



```

Q<e, f, g, h>:=Group<e, f, g, h|e^4, f^3, g^2, h^2, g*e^2*h,
(e*f^-1)^2, e^-1*g*e*h, f^-1*g*f*h, f^-1*e^-2*f*g>;
f3, Q3, k3:=CosetAction(Q, sub<Q|Id(Q)>);

```

```

T:=Transversal(M, NL[4]);
for i in [1..#T] do if ff(T[i]) eq q.1 then i;
end if; end for;
T[4];
for i in [1..#T] do if ff(T[i]) eq q.2 then i;
end if; end for;
T[2];
for i in [1..#T] do if ff(T[i]) eq q.3 then i;
end if; end for;
T[7];
for i in [1..#T] do if ff(T[i]) eq q.4 then i;
end if; end for;
T[10];

```

```

A:=NL[4].1;
B:=NL[4].2;
C:=NL[4].3;
D:=NL[4].4;
E:=T[4];
F:=T[2];
G:=T[7];
H:=T[10];

```

```

for q, r, s, t in [0..1] do if A^E
eq A^q*B^r*C^s*D^t then q, r, s, t; end if;
end for;
for q, r, s, t in [0..1] do if A^F
eq A^q*B^r*C^s*D^t then q, r, s, t; end if;
end for;
for q, r, s, t in [0..1] do if A^G
eq A^q*B^r*C^s*D^t then q, r, s, t; end if;
end for;
for q, r, s, t in [0..1] do if A^H
eq A^q*B^r*C^s*D^t then q, r, s, t; end if;
end for;
for q, r, s, t in [0..1] do if B^E
eq A^q*B^r*C^s*D^t then q, r, s, t; end if;
end for;
for q, r, s, t in [0..1] do if B^F

```

```

eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if B^G
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if B^H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if C^E
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if C^F
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if C^G
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if C^H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if D^E
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if D^F
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if D^G
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if D^H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;

for q,r,s,t in [0..1] do if E^4
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if F^3
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if G^2
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if H^2

```

```

eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if G*E^2*H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if (E*F^-1)^2
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if E^-1*G*E*H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if F^-1*G*F*H
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;
for q,r,s,t in [0..1] do if F^-1*E^-2*F*G
eq A^q*B^r*C^s*D^t then q,r,s,t; end if;
end for;

Z<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^2,b^2,
c^2,d^2,(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),e^4=c,
f^3=a*b*c*d,g^2,h^2,g*e^2*h,(e*f^-1)^2=b*c*d,
e^-1*g*e*h=c,f^-1*g*f*h,f^-1*e^-2*f*g=b*c*d,
a^e=a*b*c,a^f=a*d,a^g=a*d,a^h=a*c,
b^e=b*c*d,b^f=b*c*d,b^g=b*c*d,b^h=b*d,
c^e=c,c^f=c*d,c^g=c,c^h=c,
d^e=c*d,d^f=c,d^g=d,d^h=d
>;
f4,Z4,k4:=CosetAction(Z,sub<Z|Id(Z)>);

IsIsomorphic(Z4,M);

```

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