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An Introduction to Lie Algebra

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An Introduction to Lie Algebra

A Thesis

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in

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by

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Abstract

An (associative) algebra is a vector space over a field equipped with an associative, bilinear multiplication. By use of a new bilinear operation, any associative algebra morphs into a nonassociative abstract Lie algebra, where the new product in terms of the given associative product, is the commutator. The crux of this paper is to investigate the commutator as it pertains to the general linear group and its subalgebras. This forces us to examine properties of ring theory under the lens of linear algebra, as we determine subalgebras, ideals, and solvability as decomposed into an extension of abelian ideals, and nilpotency, as decomposed into the lower central series and eventual zero subspace. The map sending the Lie algebra L to a derivation of L is called the adjoint representation, where a given Lie algebra is nilpotent if and only if the adjoint is nilpotent. Our goal is to prove Engel's Theorem, which states that if all elements of L are ad-nilpotent, then L is nilpotent.

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Chapter 1

Introduction

Abstract Lie algebras are algebraic structures used in the study of Lie groups. They are vector space endomorphisms of linear transformations that have a new operation that is neither commutative nor associative, but referred to as the bracket operation, or commutator. Recall that an associative algebra preserves bilinear multiplication between elements in a vector space V over a field F, where $V \times V \to V$ is defined by $(x, y) \to xy$ for $x, y \in V$. The bracket operation yields a new bilinear map, over the same vector space, that turns any associative algebra into a Lie Algebra, where $V \times V \to V$ is now defined by $[x, y] \to (xy - yx)$. Central to the study of Lie algebras are the classical algebras, which will be explored throughout this paper to derive isomorphisms, identify simple algebras, and determine solvability. After examining some natural Lie algebra examples, we will delve into the derived algebra, which is analogous to the commutator subgroup of a group, and make use of it to define a sequence of ideals called the derived series, which is solvable. We will then use a different sequence of ideals called the descending central series to classify nilpotent algebras. All of this will arm us with the necessary tools to prove Engel's Theorem, which states that if all elements of L are ad-nilpotent, then L is nilpotent. We must first recall some basic group axioms, ring axioms, and vector space properties as a precursor to the study of Lie algebra.

The study of Lie Algebra requires a thorough understanding of linear algebra, group, and ring theory. The following provides a cursory review of these subjects as they will appear within the scope of this paper. Unless otherwise stated, our refresher for group theory was derived from Aigli Papantonopoulou's text **Algebra Pure and Applied** [Aig02].

Definition 1.1. *An associative algebra is an algebra A whose associative rule is associative: $x(yz) = (xy)z$ for all $x, y, z \in A$*

Definition 1.2. *A nonempty set G equipped with an operation $*$ on it is said to form a **group** under that operation if the operation obeys the following laws, called the **group axioms**:*

- ***Closure**: For any $a, b \in G$, we have $a * b \in G$.*
- ***Associativity**: For any $a, b, c \in G$, we have $a * (b * c) = (a * b) * c$.*
- ***Identity**: There exists an element $e \in G$ such that for all $a \in G$ we have $a * e = e * a = a$. Such an element $e \in G$ is called an **identity** in G.*
- ***Inverse**: For each $a \in G$ there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. Such an element $a^{-1} \in G$ is called an **inverse** of a in G.*

Proposition 1.3. ***(Basic group properties)*** *For any group G*

- *The identity element of G is unique.*
- *For each $a \in G$, the inverse a^{-1} is unique.*
- *For any $a \in G$, $(a^{-1})^{-1} = a$*
- *For any $a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$*
- *For any $a, b \in G$, the equations $ax = b$ and $ya = b$ have unique solutions or, in other words, the left and right cancellation laws hold.*

Example 1.0.1. *The **general linear group**, denoted $gl(n)$, consists of the set of invertible $n \times n$ matrices. By the above, given that multiplication of invertible $n \times n$ matrices is associative, and each invertible matrix has an inverse and an identity, $gl(n)$ forms a group under multiplication.*

Definition 1.4. *nonempty subset H of a group G is a* ***subgroup*** *of G if H is a group under the same operation as G. We use the notation $H \subset G$ to mean that H is a subset of G, and $H \leq G$ to mean that H is a subgroup of G. For a group G with identity element e, $\{e\}$ is a subgroup of G called the* ***trivial subgroup****. For any group G, G itself is a subgroup of G, called the* ***improper subgroup****. Any other subgroup of G besides the two above is called a* ***nontrivial proper subgroup*** *of G.*

Definition 1.5. *The number of elements in a group G is called the* ***order*** *of G and is denoted $|G|$. G is a* ***finite*** *group if $|G|$ is finite. If $a \in G$, the* ***order*** *$|a|$ of a in G is the least positive integer n such that $a^n = e$. If there exists no such integer n, then $|a|$ is infinite.*

Definition 1.6. *A map $\varphi : G \to G^{'}$ from a group G to a group G' is called a* ***homomorphism*** *if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$ (in $\varphi(ab)$ the product is being taken in G, while in $\varphi(a)\varphi(b)$ the product is being taken in G').*

Proposition 1.7. *(**Basic group homomorphism properties**) Let $\varphi : G \to G'$ be a homomorphism. Then*

- *$\varphi(e) = e'$, where e is the identity of G and e' the identity of G'.*
- *$\varphi(a^{-1}) = (\varphi(a))^{-1}$ for any $a \in G$*
- *$\varphi(a^n) = \varphi(a)^n$ for any $n \in Z$*
- *If $|a|$ is finite, then $|\varphi(a)|$ divides $|a|$.*
- *If H is a subgroup of G, then $\varphi(H) = \{\varphi(x)|x \in H\}$ is a subgroup of G'*
- *If K is a subgroup of G', then $\varphi^{-1}(K) = \{x \in G|\varphi(x) \in K\}$ is a subgroup of G.*

Example 1.0.2. *For any group G, the* ***identity*** *map is always a homomorphism, since if $\varphi : G \to G'$ is the identity $\varphi(x) = x$, then $\varphi(xy) = xy = \varphi(x)\varphi(y)$*

Definition 1.8. *For the homomorhpism φ, the* ***kernel*** *of φ is the set $\{x \in G|\varphi(x) = e'\}$, denoted $Ker\ \varphi$.*

Definition 1.9. *Let G be a group, H a subgroup of G, and $a \in G$. Then the set $aH = \{aH|h \in H\}$ is called a* ***left coset*** *of H in G, and the set $Ha = \{ha|h \in H\}$ is called a* ***right coset*** *of H in G.*

Definition 1.10. *The group consisting of the cosets of H in G under the operation $(aH)(bH) = (ab)H$ is called the **quotient group** of G by H, written G/H.*

Definition 1.11. *If G is any group, then the **center** of G, denoted $Z(G)$, consists of the elements of G that commute with every element of G. In other words,*

$$Z(G) = \{x \in G | xy = yx \quad \forall y \in G\} \tag{1.1}$$

*Note that $ey = y = ye$ for all $y \in G$, so $e \in Z(G)$, and the center is a nonempty subset of G. Let $a \in G$. Then the **centralizer** of a in G, denoted $C_G(a)$, is the set of all elements of G that commute with a. In other words*

$$C_G(a) = \{y \in G | ay = ya\} \tag{1.2}$$

Note that for any $a \in G$ we have $Z(G) \subseteq C_G(a)$. In other words, the center is contained in the centralizer of any element [Aig02].

Definition 1.12. *Let G be a group and H a subgroup of G. If for all $g \in G$ we have $gH = Hg$, then we say H is a **normal** subgroup of G and write $H \triangleleft G$. (Recall the normal subgroup test: A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G) .*

The following material for group and ring theory was derived from Joseph A. Gallian's **Contemporary Abstract Algebra** [Gal04].

Definition 1.13. *An isomorphism from a group G onto itself is called an **automorphism** of G.*

Definition 1.14. *Let G be a group, and let $a \in G$. The function φ_a defined by $\varphi_a(x) = axa^{-1}$ for all $x \in G$ is called the **inner automorphism** of G induced by a.*

Definition 1.15. *A set R equipped with two operations, written as addition and multiplication, is said to be a **ring** if the following four **ring axioms** are satisfied, for any elements, a, b, and c in R:*

- *R is an Abelian group under addition.*
- ***Closure**: $ab \in R$*

- ***Associativity***: $a(bc) = (ab)c$.
- ***Distributivity***: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Definition 1.16. *A subring A of a ring R is called a (two-sided)* ***ideal*** *of R if for every $r \in R$ and every $a \in A$ both ra and ar are in A.*

Theorem 1.17. ***Ideals are Kernels*** *Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal A is the kernel of the mapping $r \to r + A$ from R to R/A.*

Definition 1.18. *A* ***unity*** *(or identity) in a ring is a nonzero element that is an identity under multiplication. A nonzero element of a commutative ring with unity that has a multiplicative inverse is called a* ***unit*** *of the ring.*

Theorem 1.19. ***First Isomorphism Theorem for Rings*** *Let φ be a ring homomorphism from $R \to S$. Then the mapping from $R/ker\varphi$ to $\varphi(r)$, given by $r + ker\varphi \to \varphi(r)$ is an isomorphism. In symbols, $R/ker\varphi \cong \varphi(r)$*

Theorem 1.20. ***Second Isomorphism Theorem for Rings*** *If A is a subring of R and B an ideal of R, then $A \cap B$ is an ideal of A and $A/A \cap B$ is isomorphic to $(A+B)/B$. (Recall that $\{A + B = a + b | a \in A, b \in B\}$)*

Theorem 1.21. ***Third Isomorphism Theorem for Rings*** *Let A and B be ideals of a ring R with $B \subseteq A$. A/B is an ideal of R/B and $(R/B)(A/B)$ is isomorphic to R/A.*

Theorem 1.22. ***Correspondence Theorem*** *Let I be an ideal of a ring R. There exists a bijection between the set of all ideals J of R such that $I \subset J$ and the set of all ideals of R/I:*

$$\{J | I \quad \text{is an ideal of} \quad R, I \subset J\} \to \{K | K \quad \text{is an ideal of} \quad r/I\}$$
$$J \to J/I$$

Definition 1.23. *In a ring R the* ***characteristic*** *of R, denoted char R, is the least positive integer n such that $n \cdot a = 0$ for all $a \in R$. If no such n exists, we say char $R = 0$.*

Definition 1.24. *A* ***field*** *is a commutative ring with unity in which every nonzero element is a unit.*

Unless otherwise stated, the material for our vector space review was derived from Stephen H. Friedberg et al's, **Linear Alebra** [FIS03].

Definition 1.25. *Let F be a field. A set V equipped with two operations, written as addition and multiplication, is said to be a **vector space** over F if*

- *V is an Abelian group under addition. For all $a, b \in F$ and all $u, v \in V$,*
- *The product $av \in V$ is defined*
- $a(v + w) = av + aw$
- $a(bv) = (ab)v$
- $1v = v$

*An element of $v \in V$ is called a **vector**. The **identity** element of V under addition is called the **zero vector** and written **0**. An element of $a \in F$ is called a **scalar**, and the operation of forming av is called **scalar multiplication** [Aig02].*

Definition 1.26. *A subset U of a vector space V over a field F is called a **subspace** of V if U is a vector space over F with the operations of addition and scalar multiplication defined on V*

Example 1.0.3. *If $T : V \to W$ is a linear map, the **kernel** of T and the **image (range)** of T defined by*

$$ker(T) = \{x \in W : T(x) = 0\}$$
$$im(T) = \{w \in W : T(x), x \in W\}$$

are subspaces of V.

Definition 1.27. *Let V be a vector space over a field F. A vector space homomorphism that maps V to itself is called an **endomorphism** of V*

Definition 1.28. *Let V and W be vector spaces. We say that V is **isomorphic** to W if there exists a linear transformation $T : V \to W$ that is invertible. Such a transformation is called an **isomorphism** from V to W.*

Definition 1.29. *Let V be a vector space over C. An* ***inner product*** *on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in C, denoted $\langle x, y\rangle$, such that for all x, y, z in V and all c in C, the following hold:*

1. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

2. $\langle cx, y\rangle = c\,\langle x, y\rangle$

3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$

4. $\langle x, x\rangle > 0$ *if* $x \neq 0$

Definition 1.30. *Let V be a vector space over a field F. A function f from the set $V \times V$ of ordered pairs of vectors to F is called a* ***bilinear form*** *on V if f is linear in each variable when the other variable is held fixed; that is, f is a bilinear form on V if*

1. $f(\alpha x_1 + x_2, y) = \alpha f(x_1, y) + f(x_2, y)$

2. $f(x, \alpha y_1 + y_2) = \alpha f(x, y_1) + f(x, y_2)$

Any nondegenerate bilinear form on V consists of all operators x on V under which the form f is **infinitesimally invariant**, i.e., that satisfies $f(x(v), w) + f(v, x(w)) = 0$. [FIS03]

The kernel of a symmetric bilinear form is given as: $ker f = \{v : f(v, w) = 0 \quad \text{for all} \quad w \in V\}$. That is, for all $v \in W$, there exists a $w \in V$ such that $f(v, w) = 0$.

Definition 1.31. *An* ***alternating space*** *is defined as the set of all vector spaces V and alternate bilinear forms such that $f : V \times V \to F$ where:*

1. $f(x, y+z) = f(x, y) + f(x, z)$

2. $f(x+y, z) = f(x, z) + f(y, z)$

3. $f(ax, y) = af(x, y) = f(x, ay)$

4. $f(x, x) = 0$ *which implies that f(x, y)=-f(y,x)*

Definition 1.32. *In matrix terms, an n-linear function $\gamma : M_{n\times n}(F) \to F$ is called* ***alternating*** *if, for each $A \in M_{n\times n}(F)$, we have $\gamma(A) = 0$ whenever two adjacent rows of A are equal. [FIS03]*

Theorem 1.33. *Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices and D and E be $q \times m$ matrices. Then*

1. $A(B+C) = AB + AC$ *and* $(D+E)A = DA + EA$

2. $\alpha(AB) = (\alpha A)B = (A\alpha B)$

3. $I_m A = A = AI_m$

4. *If V is an n-dimensional vector space with an ordered basis B, then* $[I_V]_B = I_n$

Definition 1.34. *Let A, B and C be matrices where $C = c_{ij}$, $A = a_{jk}$, and $B = b_{kj}$. Then by* ***matrix multiplication***

$$C = AB \quad \textit{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The formula for the ***determinant*** *of A, denoted $det(A)$ or $|A|$, where A is a $n \times n$ matrix can be expressed as cofactor expansion along the 1st row of A*

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot det(\tilde{A}_{ij})$$

Where the scalar $c_{ij} = (-1)^{i+j} det(\tilde{A}_{ij})$ is called the ***cofactor*** *of the entry of A if row i, column j.*

Example 1.0.4. *The Laplace expansion of A where A is a 2×2 matrix is*

$$det(A) = A_{11}(-1)^{1+1} det\tilde{A}_{11} + A_{12}(-1)^{1+2} det\tilde{A}_{12}$$

Definition 1.35. *(Adjoint of a Matrix) Let A be an $n \times n$ matrix. The matrix $B = [b_{ij}]$ with $b_{ij} = c_{ji}$ (for c_{ij} as defined in Definition 1.33), for $1 \leq i, j \leq n$ is called the* ***Adjoint*** *of A, denoted $Adj(A)$.*

Definition 1.36. *The* ***trace*** *of a matrix is the sum of its diagonal entries.*

Definition 1.37. *Given an $n \times n$ matrix $A = \{a_{ij}\}$, the* ***transpose*** *of A is the matrix $A^T = \{b_{ij}\}$, where $b_{ij} = a_{ji}$.*

Definition 1.38. *If the transpose of a matrix is equal to the negative of itself, the matrix is said to be* ***skew symmetric****. This means that for a matrix to be skew symmetric, $A^t = -A$.*

Definition 1.39. *Let F be a field and V, W vector spaces over the field F. A function $T : V \to W$ is said to be a* ***linear transformation*** *from V to W if for all $c, d \in F$, and all $x, y \in V$ we have*

$$T(cx + yd) = cT(x) + dT(y)$$

The following are simple properties of linear transformations:

- $T(0) = 0$
- $T(cx + y) = cT(x) + T(y)$
- $T(x - y) = T(x) - T(y)$
- For $x_1, x_2, ..., x_n \in V$ and $a_1, a_2, ..., a_n \in F$, we have
$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$$

Example 1.0.5. *Consider the map $T : M_{n\times n}(F) \to M_{n\times m}(F)$ defined by $T(A) = A^t$. We can show that T is a linear transformation. Let $A = A_{ij}$ and $B = B_{ij}$*

$$(A + B)^t = (A_{ij} + B_{ij})^t = (A_{ij})^t + (B_{ij})^t = A^t + B^t$$
$$T(cA_{ij}) = (cA_{ij})^t = c(A_{ij})^t = cA^t$$

Definition 1.40. *A* ***basis****, β, for a vector space V is a linearly independent subset of V that generates V. If there exists a finite set that forms a basis for V over a field F, then the number n of vectors in such a basis $\{v_1 ... v_n\}$ is called the* ***dimension*** *of V over F, written dim F_V. If there exists no finite basis for V over F, then V is said to be* ***infinite dimensional*** *over F.*

Definition 1.41. *Let $T = T_A : V \to V$ be a linear transformation, where V is a finite-dimensional vector space over a field F, and let $0 \neq v \in V$ be such that $T_A(v) = \lambda v$ for some $\lambda \in F$. Then v is called an* ***eigenvector*** *of T (and of A), and λ the corresponding* ***eigenvalue*** *of T (and of A).*

Definition 1.42. *A is* ***invertible*** *if there exists an $n \times n$ matrix B such that $AB = BA = I$.*

Definition 1.43. *a matrix $M \in M_{n\times n}(F)$ is called* ***orthogonal*** *if $MM^t = I$.*

Theorem 1.44. *If A is a square matrix of order $n > 1$, then $A(adjA) = (adjA)A = |A|I_n$. The (i, j)th element of $A(adjA)$ is:*

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{ij}A_{jk} = \begin{cases} |A| \Leftrightarrow i = j \\ 0 \Leftrightarrow i \neq j \end{cases}$$

Therefore $A(adjA)$ is a scalar matrix with diagonal elements all equal to $|A|$.

$$\Rightarrow A(AdjA) = I_{(n)}$$
$$\Rightarrow (AdjA)A = |A|I_{(n)}$$

Where $|A|$ represents the determinant of A.[Eve66]

Proposition 1.45. *The following equation denotes the inverse of A:*

$$A^{-1} = \frac{AdjA}{|A|}$$

Definition 1.46. *If A is a square matrix of order n, then the λ matrix $[A - \lambda I_{(n)}]$ is called the* ***characteristic matrix*** *of A. The determinant $|A - \lambda I_{(n)}|$ is called the* ***characteristic determinant*** *of A, and the expansion of it is a polynomial of degree n:*

$$f(\lambda) = (-1)^n[\lambda^n - p_1\lambda^{n-1} + p_2\lambda^{n-2} + ..._(- 1)^n p_n]$$

[Eve66]

Theorem 1.47. *In the characteristic function $f(\lambda)$ of matrix A, $p_n = |A|$ and $p_1 = trA$. [Eve66]*

Chapter 2

Definitions and First Examples

2.1 The Notion of a Lie Algebra and Linear Lie Algebras

Let R be a fixed commutative ring (or a field). An associative R-algebra is a group A that is abelian under addition, and has the structure of both a ring and an R-module such that the scalar multiplication satisfies $r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$ for all $r \in R$ and $x, y \in A$. Additionally, A contains a unique element 1 such that $1 \cdot x = x = x \cdot 1$ for all $x \in A$. A is therefore an R-module together with (1) an R-bilinear map $A \times A \to A$, called the product, and (2) the multiplicative identity, such that multiplication is associative: $x(yz) = (xy)z$, for all x, y, and z in A. If one negates the requirement for associativity, then one obtains a non-associative algebra. If A itself is commutative, it is called a commutative R-algebra. The commutator of two elements x and y of a ring or an associative algebra is defined by $xy-yx$. (The anticommutator of two elements x and y of a ring or an associative algebra is defined by $\{x, y\} = xy + yx$). Any algebra A over F, where F is a vector space with associative multiplication can be made into a Lie algebra L via the commutator, yielding a structure similar to a ring modulo L. The commutator is also referred to as the the bracket operation. In order to prove Engel's Theorem, we shall restrict ourselves to the following definition of abstract Lie algebra, in contrast to the concrete Lie algebra definition featured in universal enveloping algebras and the Poincare Birkhoff Witt Theorem. Additionally, we will assume all base fields have characteristic $\neq 2$. Unless otherwise stated, the material for this chapter is derived from [Hum72].

Definition 2.1. *A vector space L over a field F, is a Lie algebra if there exists a bilinear multiplication $L \times L \to L$, with an operation, denoted $(x, y) \mapsto [xy]$, such that:*

1. *It is **skew symmetric** where $[x, x] = 0$ for all x in L (this is equivalent to $[x, y] = -[y, x]$ since character $F \neq 2$).*

2. *It satisfies the **Jacobi identity** $[x[yz]] + [y[zx]] + [z[xy]] = 0$ $(x, y, z \in L)$.*

Example 2.1.1. *Given an n dimensional vector space End (V), the set of all all linear maps $V \mapsto V$ with associative multiplication $(x, y) \mapsto xy$ for all x, y, where xy denotes functional composition, observe, End (V) is an associative algebra over F. Let us define a new operation on End (V) by $(x, y) \mapsto xy - yx$. If we denote $xy - yx$ by $[x, y]$, then End (V) together with the map $[\cdot, \cdot]$ satisfies Definition 2.1, and is thus a Lie algebra.*

Proof. The first two bracket axioms are satisfied immediately. The only thing left to prove is the Jacobi identity. Given $x, y, z \in$ End (V), we have by use of the bracket operation:

$$
\begin{aligned}
&[x[yz]] + [y[zx]] + [z[xy]] = 0 \\
&= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y + z(xy - yx) - (xy - yx)z \\
&= xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy + zxy - zyx - xyz + yxz \\
&= (xyz - xyz) + (xzy - xzy) + (yzx - yzx) + (zyx - zyx) + (yxz - yxz) = 0
\end{aligned}
$$

□

Definition 2.2. *The Lie algebra End (V) with bracket $[x, y] = xy - yx$, is denoted as $\mathfrak{gl}(V)$, the **general linear algebra**.*

Example 2.1.2. *We can show that real vector space R^3 is a Lie algebra. Recall the following cross product properties when a, b and c represent arbitrary vectors and α, β and γ represent arbitrary scalars:*

1. $a \times b = -(b \times a)$,

2. $\alpha \times (\beta b + \gamma c) = \beta(a \times b) + \gamma(a \times c)$ *and* $(\alpha a + \beta b) \times c = \alpha(a \times c) + \beta(b \times c)$,

([MT03])
Note, $a \times a = -(a \times a)$, by property (1), letting $b = a$, therefore, $a \times a = 0$. By the above properties, the cross product is both skew symmetric (property 1) and bilinear (property 2). By vector triple product expansion: $x \times (y \times z) = y(x \cdot z) - z(x \cdot y)$. To show that the cross product satisfies the Jacobi identity, we have:

$$\begin{aligned}
[x[y,z]] + [y[z,x]] + [z[x,y]] &= x \times (y \times z) + y \times (z \times x) + z \times (x \times y) \\
&= [y(x \cdot z) - z(x \cdot y)] + [z(y \cdot x) - x(y \cdot z)] + [x(z \cdot y) - y(z \cdot x)] \\
&= 0 \quad \{\textit{Since the dot product is commutative}\}
\end{aligned}$$

Therefore, by definition 2.1, the real vector space R^3 is a Lie algebra

Definition 2.3. *A subspace K of L is called a (Lie)* ***subalgebra*** *if $[xy] \in K$ whenever $x, y \in K$.*

2.2 Classical Lie Algebras

Classical algebras are finite-dimensional Lie algebras. Each classical algebra A_l, B_l, C_l, and D_l has an associated algebra, represented by symmetric, skew symmetric, and orthogonal matrices. Let s be an $n \times n$ matrix. We shall prove that each classical algebra representation, equipped with basis and dimension satisfying $xs + sx^t = 0$ (x^t =transpose of x), is a subalgebra of the linear Lie algebra $\mathfrak{gl}(V)$. Note: a proper subalgebra maintains dimension less than that of $\mathfrak{gl}(n)$.

($\mathbf{A_l}$) The set of all endomorphisms of V having trace zero is denoted by $\mathfrak{sl}(V)$, the **special linear algebra**. Letting $x, y \in \mathfrak{sl}(V)$, we can show that $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$.

Proof.

$$tr[x,y] = tr(xy) - tr(yx) = 0 \quad \{\text{Since the trace of a matrix preserves bilinearity}\}$$

$$tr([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) = tr[x,0] + tr[y,0] + tr[z,0] = 0$$

Therefore, both Lie algebra axioms are satisfied, hence by Definition 2.3, $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$.

□

The dimension of $\mathfrak{sl}(V)$ is found by determining the number of linearly independent matrices of $\mathfrak{sl}(V)$ that yield trace 0. By counting the matrices $e_{ij}(i \neq j)$, and adding them to the matrices $h_i = e_{ij} - e_{i+1,i+1}$, we find dim $\mathfrak{sl}(V) = l + (l+1)^2 - (l+1)$

($\mathbf{C_l}$) The set of endomorphisms of V having dim $V = 2l$, that satisfies $f(v, w) = -f(w, v)$, is denoted as $\mathfrak{sp}(V)$, the **symplectic algebra**. Let s be a nondegenerate, skew symmetric matrix where $s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. Recall that a matrix $A \in M_{n\times n}(F)$ is called **skew-symmetric** if $A^t = -A$. We can show $\mathfrak{sp}(2l, F)$ is a subalgebra of $\mathfrak{gl}(V)$ if we partition x as $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ where $m, n, p, q \in \mathfrak{gl}(v)$.

Proof. Given $x, y \in \mathfrak{sp}(V)$,

$$\begin{aligned}
&f([xy](v), w) + f(v, [xy](w)) \\
=&f((xy - yx)v, w) + f(v, (xy - yx)w) \quad \{\text{By definition of the bracket operation}\} \\
=&f(xy(v), w) - f(yx(v), w) + f(v, xy(w)) - f(v, yx(w)) \quad \{\text{By definition of bilinearity}\} \\
=&f(x(y(v)), w) - f(y(x(v)), w) + f(v, x(y(w)) - f(v, (y(x(w)) \\
=&-f(y(v), x(w)) + f(x(v), y(w)) - f(x(v), y(w)) + f(y(v), x(w)) \quad \{\text{By skew symmetry}\} \\
=&0
\end{aligned}$$

Therefore $[x, y] \in \mathfrak{sp}(V)$, hence $\mathfrak{sp}(V)$ is a subalgebra of $\mathfrak{gl}(V)$ when $\mathfrak{sp}(v)$ is of even dimension. We show that $\mathfrak{sp}(V)$ requires even dimensionality by considering the determinant of x when n is odd. From $sx = -x^t s$,

$$\begin{aligned}
det(sx) = det(-x^t s) \iff det(s)det(x) &= det(-x^t)det(s) \\
\iff det(x) &= det(-x^t) \\
&= (-1)^n det(x^t) \\
&= (-1)^{2k-1} det(x^t) \\
&= -det(x^t)
\end{aligned}$$

but then $det(x) = -det(x^t)$, therefore, $det(x) = 0$ $\square$

($\mathbf{B_l}$) The set of all endomorphisms of V with dimension $2l^2 + l$ satisfying $f(x(v), w) = -f(v, x(w))$, such that f is a non-degenerate bilinear form on V with matrix $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$, corresponds to the **orthogonal algebra**, denoted as $\mathfrak{o}(V)$, or $\mathfrak{o}(2l+1, F)$. Recall a matrix $A \in M_{n\times n}(F)$ is called **orthogonal** if $AA^t = I$. We can partition x as $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$. The orthogonal algebra is a subalgebra of $\mathfrak{gl}(V)$, where $f(x(v), w) = -f(v, x(w))$ (the same conditions as for ($\mathbf{C_l}$).

Example 2.2.1. *When $V = F^n$, take for $f(x(v), y)$ the form $\sum X_i Y = x^t y$ where $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$. The corresponding Lie algebra is $\mathfrak{o}(n, F)$*

($\mathbf{D_l}$) Let D_l be an an orthogonal algebra with $s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. Let $m = 2l$, then D_l consists of all the matrices with dimension $2l^2 - l$, satisfying $f(x(v), w) = -f(v, x(w))$.

Definition 2.4. *The following is a list of several other subalgebras consisting of the upper, lower, and triangular matrices of $\mathfrak{gl}(n, F)$:*

- *Let $\mathfrak{n}(n, F)$ be the **strictly upper triangular matrices** $(a_{ij}) = 0$ if $i \geq j$*
- *Let $\mathfrak{d}(n, F)$ be the set of all **diagonal matrices** (a_{ij}), $a_{ij} = 0$ if $i \neq j$.*
- *Let $\mathfrak{t}(n, F)$ be the set of **upper triangular matrices** (a_{ij}), $a_{ij} = 0$ if $i > j$.*

Example 2.2.2. *Given algebras $\mathfrak{t}(n, F), \mathfrak{o}(n, F), \mathfrak{n}(n, F)$, we can compute the dimension of each algebra, by exhibiting a basis.*

- *For $\mathfrak{n}(n, F) = \begin{pmatrix} 0 & a_{12} & ... & a_{1n} \\ 0 & 0 & ... & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$*

  Using the formula for the area of a triangle: $A = \frac{bh}{2}$, the dimension for $\mathfrak{n}(n, F)$ is therefore $\frac{n(n-1)}{2}$

- *For* $\mathfrak{o}(n, F) = \begin{pmatrix} a_{11} & 0 & ... & 0 \\ 0 & a_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$
  The dimension of $\mathfrak{o}(n, F)$ *is clearly* n.
- *For* $\mathfrak{t}(n, F) = \begin{pmatrix} a_{11}+ & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & ... & a_{nn} \end{pmatrix}$
  Again using the formula for the area of a triangle, the dimension of $\mathfrak{t}(n, F)$ *is* $\frac{n(n+1)}{2}$

Definition 2.5. *The (external)* ***direct sum*** *of two Lie algebras* L, L'*, written* $L \oplus L'$ *is the vector space direct sum, with* $[\cdot, \cdot]$ *defined "componentwise":* $[(x_1, y_1), (x_2, y_2)]$ *where* $\{(x, 0)\}, \{(y, 0)\}$ *are ideals and thus "nullify" each other. That is,* $[L, L'] = 0$. *[Sam90]*

Example 2.2.3. *We can prove that* $\mathfrak{t}(n, F)$ *is the direct sum of* $\mathfrak{o}(n, F)$ *and* $\mathfrak{n}(n, F)$*. By the above definition, we must show* $[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = 0$ *where*
$\mathfrak{t}(n, F) = \mathfrak{o}(n, F) + \mathfrak{n}(n, F)$.

Proof. Let $A \in \mathfrak{n}(n, F)$ and $B \in \mathfrak{o}(n, F)$, and $C \in \mathfrak{t}(n, F)$:

$$\begin{pmatrix} 0 & a_{12} & ... & a_{1n} \\ 0 & 0 & ... & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} b_{11} & 0 & ... & 0 \\ 0 & b_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & a_{12} & ... & a_{1n} \\ 0 & b_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & ... & b_{nn} \end{pmatrix}$$

where $A + B$ is an upper diagonal matrix. Therefore $\mathfrak{t}(n, F) = \mathfrak{o}(n, F) + \mathfrak{n}(n, F)$. To show $A \cap B = \{0\}$, we have

$$\begin{aligned} (AB)_{ij} &= \sum_{k=1}^{i} A_{ik}B_{kj} + \sum_{k=i+1}^{n} A_{ik}B_{kj} \\ &= \sum_{k=1}^{i} 0 \cdot B_{kj} + \sum_{k=i+1}^{n} A_{ik} \cdot 0 \end{aligned}$$

Where $A_{ik} = 0$ in the first sum since $i \geq k$ for A strictly upper triangular. For the second sum $k \geq i > j - 1$, thus $B_{kj} = 0$. Therefore $A \cap B = \{0\}$, meaning

$AB - BA = [AB] = 0$. Therefore $[\mathfrak{o}(n,F), \mathfrak{n}(n,F)] \subseteq \mathfrak{n}(n,F)$. We can construct a basis e_{ij} for $\mathfrak{n}(n,F)$ such that e_{ij} is 1 for the (ij) element, and zero elsewhere. By definition of the commutator:

$$[e_{ii}, e_{ij}] = e_{ii}e_{ij} - e_{ij}e_{ii} = e_{ij}$$

By the above, when $i \geq j$, $\mathfrak{n}(n,F) \subseteq [\mathfrak{o}(n,F), \mathfrak{n}(n,F)]$, and thus
$[\mathfrak{o}(n,F), \mathfrak{n}(n,F)] = \mathfrak{n}(n,F)$. Since $\mathfrak{o}(n,F) + \mathfrak{n}(n,F) = \mathfrak{t}(n,F)$:

$$\begin{aligned}[\mathfrak{o}(n,F) + \mathfrak{n}(n,F), \mathfrak{o}(n,F) + \mathfrak{n}(n,F)] &= [\mathfrak{o}(n,F), \mathfrak{o}(n,F)] + [\mathfrak{n}(n,F), \mathfrak{n}(n,F)] \\ &\quad [\mathfrak{n}(n,F), \mathfrak{o}(n,F)] + [\mathfrak{o}(n,F), \mathfrak{n}(n,F)] \\ &= 0 + 0 + [\mathfrak{o}(n,F), \mathfrak{n}(n,F)] \subseteq \mathfrak{n}(n,F)\end{aligned}$$

Conversely:

$$\mathfrak{n}(n,F) = [\mathfrak{o}(n,F), \mathfrak{n}(n,F)] \subseteq [\mathfrak{t}(n,F), \mathfrak{t}(n,F)]$$

Therefore $[\mathfrak{t}(n,F), \mathfrak{t}(n,F)] = \mathfrak{n}(n,F)$ and thus $\mathfrak{t}(n,F) = \mathfrak{o}(n,F) + \mathfrak{n}(n,F)$ □

2.3 Lie Algebras of Derivations

The derivative between two functions f and g is a linear operator that satisfies the Leibniz rule:

1. $(fg)' = f'g + fg'$

2. $(\alpha f)' = \alpha f'$ where α is a scalar.

Given an algebra A over a field F, the **derivation** of A is the linear map δ such that $\delta(f,g) = f\delta(g) + \delta(f)g$ for all $f, g \in A$. The set of all derivations of A is denoted by Der(A). Given $\delta \in Der(A)$, $f, g \in A$, and $\alpha \in F$. By property 2.,

$$(\alpha\delta)(fg) = \alpha\delta(fg) = \alpha(f\delta(g) + \delta(f)g) = \alpha f\delta(g) + \alpha\delta(f)g$$

where the Leibniz rule is satisfied if and only if $af = fa$ where F is a field.

Example 2.3.1. *Let $x, y \in End(V)$, and $\delta, \delta' \in Der(V)$. By definition of the commutator, $[\delta, \delta'] = (\delta\delta' - \delta'\delta)$:*

- *Der*(V) *is a vector subspace of End*(V)*:*

$$\begin{aligned}\delta[x,y] &= \delta(xy-yx) = \delta(xy)-\delta(yx)\\ &= x\delta(y)+\delta(x)y-(y\delta(x)+\delta(y)x)\\ &= \delta(y)x-y\delta(x)+x\delta(y)-\delta(y)x\\ &= [\delta(x),y]+[x,\delta(y)]\end{aligned}$$

- *The commutator* $[\delta,\delta']$ *of two derivations* $\delta,\delta'\in Der(V)$ *is again a derivation.*

$$\begin{aligned}([\delta,\delta'](x))y+x([\delta,\delta'](y)) &= ((\delta\delta'-\delta'\delta)(x))y+x(\delta\delta'-\delta'\delta)(y)\\ &= (\delta\delta'(x)-\delta'\delta(x))y+x(\delta\delta'(y)-\delta'\delta(y))\\ &= \delta\delta'(x)y-\delta'\delta(x)y+x\delta\delta'(y)-x\delta'\delta(y)\\ &= \delta(\delta'(x)y+x\delta'(y))-\delta'(\delta(x)y+x\delta(y))\\ &= \delta(\delta'(x)y)+\delta'(x)\delta(y)+\delta(x)\delta'(y)+x\delta(\delta'(y))\\ &\quad -\delta'(\delta(x)y)-\delta(x)\delta'(y)-\delta'(x)\delta(y)-x\delta'\delta(y)\\ &= \delta(\delta'(x)y+x\delta'(y))-\delta'(\delta(x)y+x\delta(y))\\ &= \delta(\delta'(xy))-\delta'(\delta(xy))\\ &= [\delta,\delta'](xy)\end{aligned}$$

Definition 2.6. *Given* $x\in L$*, the map* $y\to[x,y]$*, is an endomorphism of* L*, denoted* $\boldsymbol{ad}_x$*, where* $\boldsymbol{ad}_x$ *is an inner derivation. Derivations of the form* $[x[yz]]=[[xy]z]+[y[xz]]$ *are* ***inner****. All others are* ***outer****.*

Example 2.3.2. *By example 2.3.1, the collection of derivations, Der*(V)*, satisfies skew symmetry. By definition 2.6, Der*(V) *satisfies the Jacobi identity. Therefore Der*(V) *defines a Lie algebra.*

Definition 2.7. *The map* $L\to DerL$ *sending* x *to* ad_x *is called the* ***adjoint representation*** *of* L*.*

This is akin to taking the ad homomorphism of $\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$. To show ad is a homomorphism:

$$ad_{([x,y])}=[ad_{(x)},ad_{(y)}]=ad_{(x)}ad_{(y)}-ad_{(y)}ad_{(x)}$$

if and only if:

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$
$$0 = [x,[y,z]] + [y,[z,x]] + [z,[x,y]]$$

That is, if and only if the Jacobi identity is satisfied, where $[[x,y],z] = -[z,[x,y]]$ and $-[y,[x,z]] = [y,[z,x]]$ by skew symmetry.

Definition 2.8. *A representation is **faithful** if its kernal is zero, i.e., if $\phi(x) = 0 \Leftrightarrow x = 0$, where the adjoint operator defines the trivial representation. [Sam90]*

2.4 Abstract Lie Algebras

Let L be a finite dimensional vector space over F. Any vector space can be made into a Lie algebra simply by setting $[x,y] = 0$ for all vectors x and y in L. The resulting Lie algebra is called **abelian**. Recall that a group G under multiplication is said to be **Abelian** if G obeys the **commutative** law or, that is, if for every $a, b \in G$ we have $ab = ba$ [Aig02]. If an associative algebra is abelian and therefore commutative, then $ab = ba \Leftrightarrow [a,b] = 0$, where $[a,b]$ is the associated commutator.

Example 2.4.1. *Given L as a Lie algebra, with $[xy] = 0$ for all $x, y \in L$,*

$$[xy] = (xy - yx) = 0 \Leftrightarrow xy = yx$$

Therefore L under trivial Lie multiplication, is abelian. Similarly, $[yx] = 0$.

Definition 2.9. *If L is any Lie algebra with basis $x_1, ..., x_n$, then the multiplication table for L is determined by the **structural constants**, the set of scalars $\{b_{ijk}\}$ such that $[x_i, x_j] = \sum_{k=1}^{n} b_{ijk} x_k$.*

Example 2.4.2. *By use of the bracket axioms, the following set of structural constants define an abstract Lie algebra:*

1. $b_{iik} = 0 = b_{ijk} + b_{jik}$

2. $\sum (b_{ijk} b_{klm} + b_{jlk} b_{klm} + b_{lik} b_{kjm}) = 0$

Proof.

1. Recall $[xx] = 0$ for all $x \in L$.

$$[x_i, x_i] = 0 = \sum_{k=1}^{n} b_{iik} x_k$$
$$= b_{ii1}x_1 + b_{ii2}x_2 + ... + b_{iin}x_n = 0.$$

Then, since x_i is a basis, $b_{iik} = 0$ for all k.

2. Given $[x_i, x_j] = \sum_{k=1}^{n} b_{ijk} x_k$, notice $b_{ijk} + b_{jik} = 0$ if and only if

$$\sum_k (b_{ijk} + b_{jik}) x_k = 0 :$$
$$\Leftrightarrow \sum_k b_{ijk} x_k + \sum_k b_{jik} x_k = 0 \Leftrightarrow [x_i x_j] + [x_j x_i] = 0$$

But:

$$[x_i x_j] = x_i x_j - x_j x_i$$
$$[x_j x_i] = x_j x_i - x_i x_j$$
$$\Leftrightarrow [x_i x_j] + [x_j x_i] = x_i x_j - x_j x_i + x_j x_i - x_i x_j = 0$$

□

Definition 2.10. *The assorted systems of structural constants of a given Lie algebra relative to its basis form an* ***orbit*** *(the set of all transformations of one element) under this action [Sam90].*

Example 2.4.3. *The orbit of the system $c_{ijk} = 0$ for all i, j, k forms a natural isomorphism to the abelian algebra $[X, Y] = 0$ of dimension n.*

Example 2.4.4. *Consider a Lie algebra L with basis $x_1, ...x_n$. The structural constants are enumerated as followed: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. By use of the Lie bracket axioms, We have:*

$$[e_1, e_2] = e_3$$
$$[e_1, e_3] = -e_2$$
$$[e_2, e_3] = e_1$$

Therefore, by definition, the structural constants are:

$$b_{121} = 0, b_{122} = 0, b_{123} = 1,$$
$$b_{131} = 0, b_{132} = -1, b_{133} = 0,$$
$$b_{231} = 1, b_{232} = 0, b_{233} = 0$$

Example 2.4.5. *We can use structural constants to show that a three dimensional vector space with basis (x, y, z) such that $[xy] = z, [xz] = y, [yz] = 0$ defines a Lie algebra.*

Proof. the structural constants are:

$$b_{121} = b_{122} = 0, b_{123} = 1,$$
$$b_{131} = b_{133} = 0, b_{132} = 1,$$
$$b_{231} = b_{232} = b_{233} = 0$$

Where $b_{ijk} = b_{ijk} - b_{jik} = 0$ for all structural constants above (by skew-symmetry). By the Jacobi identity, we have:

$$[x[yz]] + [z[xy]] + [y[zx]] = [x, 0] + [z, z] + [y. - y] = 0 + 0 + 0 = 0$$

Therefore the above basis defines a Lie algebra structure over F. □

Definition 2.11. *For any field F, in the vector space*

$$F^n = \{(b_1, b_2, b_3..., b_n)|b_i \in F \quad for \quad 1 \leq i \leq n\}$$

a basis is formed by the vectors

$$e_1 = (1, 0, ..., 0)$$
$$e_2 = (0, 1, ..., 0)$$
$$\vdots$$
$$e_n = (0, 0, ..., 1)$$

Where $\{e_1, e_2, ..., e_n\}$ is called the ***standard basis*** *for F^n over F.*

Theorem 2.12. *Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if they are of equal dimension.*

Corollary 2.13. *Let V be a vector space over F. Then V is isomorphic to F^n if and only if dim(V) $= n$ [FIS03]*

Example 2.4.6. *By Corollary 2.13, there exists an isomorphism between $\mathfrak{gl}(\mathfrak{n})$ and F^n.*

Chapter 3

Ideals and Homomorphisms

3.1 Ideals

Lie algebra ideals are intrinsic to much that follow. In group theory, ideals are used to define the kernel of a group, normal subgroups, and the structure of nilpotent and solvable groups. In a ring, every kernal is an ideal of a homomorphism, and every subgroup is normal. By definition of the normal subgroup, if H is a subgroup of G, for all $g \in G$ we have $gH = Hg \Leftrightarrow [g, H] = 0$. Unless otherwise stated, the material for this chapter is derived from [Hum72].

Definition 3.1. *A subspace I of a Lie algebra L is called an* ***ideal*** *of L if $x \in I, y \in L$ together imply $[x, y] \in I$. The construction of ideals in Lie algebra is analogous to the construction of normal subgroups in group theory.*

By skew-symmetry, all Lie algebra ideals are automatically two sided. That is, if $[x, y] \in I$, then $[y, x] \in I$. The kernel of a Lie algebra L and L itself are trivial ideals contained in every Lie algebra.

Example 3.1.1. *The set of all inner derivations ad_x, $x \in L$, is an ideal of $Der(L)$. Let $\delta \in Der(L)$. By definition of inner derivations, for all $y \in L$:*

$$\begin{aligned}[\delta, ad_x](y) &= (\delta(ad_x) - (ad_x)\delta)(y) = \delta[x, y] - ad_x(\delta(y)) \\ &= [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] \qquad \{Since\ \delta[x, y] = [\delta(x), y] + [x, \delta(y)]\} \\ &= ad(\delta(x)y)\end{aligned}$$

Therefore, by Definition 3.1, ad_x is an ideal of $Der(L)$.

Definition 3.2. *The **center** of a Lie algebra L is the set*

$$Z(L) = \{z \in L | [x, z] = 0 \textit{ for all } x \in L\}$$

The center respresents the collection of elements in L for which the adjoint action $\mathrm{ad}_{(z)}$ yeilds the zero derivation. It can therefore be thought of as the kernel of the adjoint map from L to $\mathrm{Der}(L)$. Clearly, the center is an ideal, therefore $Z(L)$ satisfies skew symmetry. Given $y \in L$, $[z, [x, y]] = -[x, [y, z]] - [y, [z, x]]$, both $[y, z]$ and $[z, x]$ belong to $Z(L)$ by definition, therefore, $Z(L)$ satisfies the Jacobi identity. If $Z(L) = L$, then L is said to be albelian.

Definition 3.3. *The **centralizer** of a subset X of L is defined to be*

$$C_L(X) = \{x \in L | [x, X] = 0\}$$

By the Jacobi, $C_L(L)$ is a subalgebra of L where $C_L(L) = Z_L$.

Definition 3.4. *Consider the Lie algebra L, having two ideals I and J, the following are ideals of L:*

$$I + J = \{x + y | x \in I, y \in J\}$$
$$[I, J] = \left\{\sum x_i y_i | x_i \in I, y_i \in J\right\}$$

Proposition 3.5. *Let L be a Lie algebra and let I, J be subsets of L. The subspace spanned by elements of the form $x + y$ $(x \in I, y \in J)$, is denoted $I + J$, and the subspace spanned by elements of the form $[x, y]$, $(x \in I, y \in J)$, is denoted $[I, J]$. If I, I_1, I_2, J, K are subspaces of L, then*

1. $[I_1 + I_2, J] \subseteq [I_1, J] + [I_2, J]$
2. $[I, J] = [J, I]$
3. $[I, [J, K]] \subseteq [J, [K, I]] + [K, [I, J]]$

[Wan75]

If I is an ideal of L, then the quotient space L/I (whose elements are the linear cosets $L + I$) induces the well defined bracket operation, where $[x + I, y + I] = [x, y] + I$ for $x, y \in L$ [Sam90].

Definition 3.6. *By use of the bracket operation $[\cdot,\cdot]$, the quotient space L/I becomes a* ***quotient Lie algebra*** *[Sam90].*

Definition 3.7. *If the Lie algebra L has no ideals except itself and 0, and if moreover $[LL] \neq 0$, we call L* ***simple****.*

Example 3.1.2. *Let L be a Lie algebra, and $L = \mathfrak{sl}(\mathfrak{n}, F)$ where char $F \neq 2$. It can be shown that L is simple if we take as a standard basis for L the three matrices*

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We will begin by computing the matrices of ad_x, ad_h, and ad_y.

- *ad_x:*

$$ad_x(x) = [x,x] = 0 = 0 \cdot x + 0 \cdot h + 0 \cdot y$$

$$ad_x(h) = [x,h] = xh - hx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2x = -2 \cdot x + 0 \cdot y + 0 \cdot h$$

$$ad_x(y) = [x,y] = xy - yx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h = 0 \cdot x + 0 \cdot y + h$$

$$\textit{Therefore } ad_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- *ady:*

$$ad_y(x) = [y,x] = yx - xy = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h$$

$$ad_y(y) = [y,y] = 0 = 0 \cdot x + 0 \cdot h + 0 \cdot y$$

$$ad_y(h) = [y,h] = yh - hy = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 2y$$

$$\textit{Therefore } ad_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- *ad h:*

$$ad_h(x) = [h,x] = hx - xh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2x = 2 \cdot x + 0 \cdot h + 0 \cdot y$$

$$ad_h(h) = [h,h] = 0 = 0 \cdot x + 0 \cdot h + 0 \cdot y$$

$$ad_h(y) = [h,y] = hy - yh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2y = 0 \cdot x + (-2) \cdot y + 0 \cdot h$$

$$\textit{Therefore } ad_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Let $I \neq 0$ be an ideal of L where the eignevectors of x, y, h correspond to the distinct eigenvalues $2, -2$ and 0 respectively (since char $F \neq 2$). Let $ax + by + ch \in I$ be *arbitrary. If ad_x is applied twice:*

$$ad_x(ax + by + ch) = a[x, x] + b[x, y] + c[x, h] = 0 + bh - 2cx$$
$$ad_x(0 + bh - 2cx) = b[x, h] - 2c[x, x] = -2bx$$

then $-2bx \in I$. If ad y is applied twice:

$$ad_y(ax + by + ch) = a[y, x] + b[y, y] + c[y, h] = -ah + 0 + 2cy$$
$$ad_y(-ah + 0 + 2cy) = -a[y, h] + 2c[y, y] = -2ay$$

then $-2ay \in I$. So if a or b is nonzero, than either y or x is contained in I, thus $I = L$. If a and b are both zero, then $ch \in I$ where $ch \neq 0$. Thus, again $I = L$. Therefore L is simple. [Hum72]

Example 3.1.3. *We can show that $\mathfrak{sl}(3, F)$ is simple, unless char $F = 3$, using the standard basis h_1, h_2, $e_{ij}(i \neq j)$. If $I \neq 0$ is an ideal, then I is the direct sum of eigenspaces for ad h_1 or ad h_2. We will determine the eigenvalues of by afixing h_1 and h_2 to e_{ij}, i.e, take the adjoint using diagonal basis matrices. By definition of basis matrices, for $\mathfrak{sl}(3, F)$, $h_1 = e_{11} - e_{22}$ and $h_2 = e_{22} - e_{33}$.*

$$\begin{aligned} ad_{h_1}(e_{12}) &= [h_1, e_{12}] = h_1 e_{12} - e_{12} h_1 \\ &= (e_{11} - e_{22})e_{12} - e_{12}(e_{11} - e_{22}) = e_{12} - (-e_{12}) = 2e_{12} \end{aligned}$$

Continuing this way, we construct the table below to denote the eigenvalues ad h_1, ad h_2 acting on e_{ij}

$[h_1, e_{ij}]$	$[h_2, e_{ij}]$
$[h_1, e_{12}] = 2e_{12}$	$[h_2, (e_{12})] = -1e_{12}$
$[h_1, e_{21}] = -2e_{21}$	$[h_2, (e_{21})] = 1e_{21}$
$[h_1, e_{13}] = 1e_{13}$	$[h_2, (e_{13})] = 1e_{13}$
$[h_1, (e_{31})] = -1e_{31}$	$[h_2(e_{31})] = -1e_{31}$
$[h_1, (e_{23})] = -1e_{23}$	$[h_2(e_{23})] = 2e_{23}$
$[h_1, (e_{32})] = 1e_{32}$	$[h_2(e_{32})] = -2e_{32}$

Figure 3.1: Table of eigenvalues

Therefore the eigenvalues are $2, 1, -1, -2$ and 0, and since char $F \neq 3$, the eigenvalues are distinct. If I is an ideal, then by definition,

$$adh_k(I) = [adh_k, I] \subset I$$

To show that $\mathfrak{sl}(3,F)$ is simple, recall the dimension of $\mathfrak{sl}(l+1,F)$ is $l+(l+1)^2-(l+1) = 2+(2+1)^2-(2+1) = 8$. When $dim I = 1$, $I = Fx$, but I is not an ideal. When $dim I = 2$, I is spanned by x and y, but I is not an ideal. Continuing this way, by the linearity of the adjoint operator, it is evident that $I \neq 0$ is an ideal if and only if $dim I = 8$. Therefore $\mathfrak{sl}(3,F)$ is simple.

Definition 3.8. *The **normalizer** of a subalgebra K of L, is denoted by $N_L(K)$, where $N_L(K) = \{x \in L | [x,k] \subset K\}$ is a subalgebra of L. If $K = N_L(K)$, we call K **self-normalizing**.*

In the normalizer, K is the largest ideal of L that absorbs L. If K is an ideal of L, then $N_L(K) = L$.

Example 3.1.4. *We can show that $\mathfrak{t}(n,F)$ and $\mathfrak{o}(n,F)$ are self-normalizing subalgebras of $\mathfrak{gl}(n,F)$, whereas $\mathfrak{n}(n,F)$ has a normalizer $\mathfrak{t}(n,F)$. Note: for all upper triangular matrices, the diagonal entries of the matrix are precisely it's eigenvalues.*

- *By definition of the normalizer, we must show $N_L(\mathfrak{t}(n,F)) = \{b \in \mathfrak{gl}(n,f) | [b, \mathfrak{t}(n,f)] \in \mathfrak{t}(n,f)\}$ where $N_L(\mathfrak{t}(n,F)) = \mathfrak{t}(n,f)$ is a subalgebra of $\mathfrak{gl}(n,f)$. By the definition of structural constants, $b = \sum_{k=1}^{n} b_{ijk}x_k$.*

$$\begin{aligned}
[b, e_{kk}] &= \sum_{k=1}^{n} b_{ijk}x_k e_{kk} - e_{kk}\sum_{k=1}^{n} b_{ijk}x_k \\
&= \sum_{i,j=1}^{n} b_{ij}e_{ij}e_{kk} - \sum_{i,j=1}^{n} b_{ij}e_{kk}e_{ij} \\
&= \sum_{i,j=1}^{n} b_{ij}\delta_{jk}e_{ik} - \sum_{i,j=1}^{n} b_{ij}\delta_{ki}e_{kj} \qquad \because e_{ij}e_{kl} = \delta_{jk}e_{il} \\
&= \sum_{i=1}^{n} b_{ik}e_{ik} - \sum_{j=1}^{n} b_{kj}e_{kj} \in \mathfrak{t}(n,F)
\end{aligned}$$

  Where $\sum_{i=1}^{n} b_{ik}e_{ik}$ implies that $b_{ik} = 0$, for $i < k$ and $\sum_{j=1}^{n} b_{kj}e_{kj}$ implies that $b_{jk} = 0$ for $j < k$. Therefore $b_{kl} = 0$ for $k < l$ as required, and thus $b \in \mathfrak{t}(n,f)$.

- *To show that $\mathfrak{o}(n,F)$ is self normalizing, we must show:*

$N_L(\mathfrak{o}(n,F)) = \{b \in \mathfrak{gl}(n,F) | [b, \mathfrak{o}(n,F)] \in \mathfrak{o}(n,F)\}$ *where* $N_L(\mathfrak{o}(n,F)) = \mathfrak{o}(n,f)$.

$$\begin{aligned}
[b, e_{kk}] &= \sum_{k=1}^{n} b_{ijk} x_k e_{kk} - e_{kk} \sum_{k=1}^{n} b_{ijk} x_k \\
&= \sum_{i,j=1}^{n} b_{ij} e_{ij} e_{kk} - \sum_{i,j=1}^{n} b_{ij} e_{kk} e_{ij} \\
&= \sum_{i,j=1}^{n} b_{ij} \delta_{jk} e_{ik} - \sum_{i,j=1}^{n} b_{ij} \delta_{ki} e_{kj} \quad \because e_{ij} e_{kl} = \delta_{jk} e_{il} \\
&= \sum_{i=1}^{n} b_{ik} e_{ik} - \sum_{j=1}^{n} b_{kj} e_{kj} \in \mathfrak{o}(n,F)
\end{aligned}$$

Where $\sum_{i=1}^{n} b_{ik} e_{ik}$ *implies that* $b_{ik} = 0$ *for* $i \neq j$ *and* $\sum_{j=1}^{n} b_{kj} e_{kj}$ *implies that* $b_{kj} = 0$ *for* $j \neq k$. *Therefore* $b_{kl} = 0$ *for* $k \neq l$ *as required and thus* $b \in \mathfrak{o}(n,f)$ *which makes* $o(n,f)$ *self normalizing.*

- *To show that* $\mathfrak{n}(n,F)$ *has a normalizer in* $\mathfrak{t}(n,F)$, *let* $x \in \mathfrak{gl}(n,F)$, $x \notin \mathfrak{o}(n,F)$, *then:*

$$[x, \mathfrak{n}(n,F)] \not\subseteq \mathfrak{n}(n,F) \quad \textit{where}$$
$$\mathfrak{n}(n,F)\mathfrak{t}(n,F) \subset \mathfrak{n}(n,F) \quad \textit{and} \quad \mathfrak{t}(n,F)\mathfrak{n}(n,F) \subset \mathfrak{n}(n,F)$$

Therefore $[\mathfrak{n}(n,F), \mathfrak{t}(n,F)] \subset \mathfrak{n}(n,F)$. *Let* $b \notin \mathfrak{t}(n,F)$, *so* $b_{ij} \neq 0$ *when* $i > j$.

$$\begin{aligned}
[e_{ji}, b] &= \sum_{k=1}^{n} b_{ijk} e_{ji} x_k - \sum_{k=1}^{n} b_{ijk} x_k e_{ji} \\
&= \sum_{i,j=1}^{n} b_{ij} e_{ji} e_{ij} - \sum_{i,j=1}^{n} b_{ij} e_{ij} e_{ji} \\
&= \sum_{i,j=1}^{n} b_{ij} \delta_{ii} e_{jj} - \sum_{i,j=1}^{n} b_{ij} \delta_{jj} e_{ii}
\end{aligned}$$

$= \sum_{j=1}^{n} b_{ij} e_{jj} - \sum_{i=1}^{n} b_{ij} e_{ii} \neq 0$ *Since its* (j,j) *entry is* b_{ij}. *Therefore* b *is not in the normalizer of* $\mathfrak{t}(n,F)$.

3.2 Homomorphisms and Representations

Homomorphisms as they appear in ring theory articulate nicely in Lie algebra. Vector space homomorphisms are linear maps where the group action is abelian under addition and preserves scalar multiplication. Given a Lie algebra L, the next few definitions allow us to visualize Lie algebra homomorphisms, and generalize the well known isomorphism theorems to vector spaces modulo I.

Definition 3.9. *A linear transformation $\phi : L \to L'$ is called a* ***homomorphism*** *if $\phi([x, y]) = [\phi(x), \phi(y)]$, for all $x, y \in L$. ϕ is called a* ***monomorphism*** *if its kernal is zero, an* ***epimorhpism*** *if its image equals L', and an* ***isomorphism*** *if ϕ is both a monomorphism and epimorphism, that is, if ϕ is bijective.*

Example 3.2.1. *Let φ be a homomorphism $\varphi : L \to L'$, such that L, L' are Lie algebras over F. We can show that Ker φ is an ideal of L. Let $x \in L$ and $s \in L$. Now*

$$\varphi[s, x] = [\varphi(s), \varphi(x)] = [\varphi(s), 0] = 0$$

The adjoint representation of a Lie algebra L is a homomorphism using the map ad: $L \to \mathfrak{gl}(V)$. By definition, ker ad$=Z(L)$. If L is a simple Lie algebra, then $Z(L) = 0$, thus by Definition 3.9, $\text{ad}_L \to \mathfrak{gl}(L)$ is a monomorphism, and therefore an invertible homomorphism, which proves that any simple Lie algebra is isomorphic to a linear Lie algebra.

Definition 3.10. *The* ***special linear group SL(n, F)*** *denotes the kernel of the homomorphism*

$$det : GL(n, F) \to F^x = \{x \in F | x \neq 0\}$$

where F is a field.

We will now forge a proof of Lie algebra isomorphism theorems analogous to the ring theory isomorphisms described in the introduction.

Proposition 3.11. *Let L and L' be Lie algebras*

1. *If φ: $L \to L'$ is a homomorphism of Lie algebras, then $L/Ker\varphi \cong Im\varphi$. If L has an ideal I, included in $Ker\varphi$, then the map $\psi : L/I \to L'$ defines a unique homomorphism that makes the following diagram commute (π = canonical map):*

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & L' \\ & L/I \arrow[u, "\psi"'] \end{tikzcd}$$

2. Let I and J be ideals of L such that $I \subset J$, then J/I is an ideal of L/I and $(L/I)/(J/I)$ is naturally isomorphic to L/J.

3. Let I, J be ideals of L, then there exists a natural isomorphism between $(I+J)/J$ and $I/(I \cap J)$.

Proof.

1. Let $\varphi : L \to L'$ be a homomorphism of Lie algebras. Our first goal is to show given I is any ideal of L included in $Ker\varphi$, there exists a unique homomorphism $\psi : L/I \to L'$. Let $Ik, Il \in L/I$. To show ψ is a group isomorphism, note that $j = j'l$. If $j \in Il$ then $\varphi(j'l) = \varphi(j')\varphi(l) = \varphi(l)$, since φ is a homomorphism. Therefore ψ is well defined. For all $j, k \in I$:

$$\psi(IkIl) = \psi(Ikl) = \varphi(kl) = \varphi(k)\varphi(l) = \psi(Il)\psi(Ik)$$

Therefore ψ is a homomorphism. Let $ker\varphi = I$,

$$\varphi(Il) = 1 \iff \varphi(l) = 1 \iff l \in ker\varphi \iff l \in I \iff Il = I$$

2. Fist note: if $i \in I \cap J$, then by definition of the intersection, $[i, x] \in I$ and $[i, x] \in J$, so $[i, x] \in I \cap J$. Therefore, $I \cap J$ satisfies the definition of an ideal. Given $x + I \in L/I$ and $j + I \in J/I$, we have:

$$[x + I, j + I] = [x, j] + [x, I] + [I, j] + [I, I] = [x, j] + I \in J/I$$

Where $[x, j] \in L$. Therefore, J/I is an ideal of L/I. Additionally, given $x + I, y + I \in L/I$, $x + I = y + I$ mod J/I implies that $(x - y) + I \in J/I$ and therefore $x - y = j \in J$. Thus x and y are equivalent mod J in L.

3. Let $i_1 + j_1, i_2 + j_2 \in I + J$. If $i_1 + j_1 = i_2 + j_2$ mod J, then $(i_1 - i_2) = j_2 - j_1 = j \in J$, but $i_1 - i_2 \in I$. Therefore $i_1 - i_2 \in I \cap J$

□

Example 3.2.2. *Since $\mathfrak{n}(n, F)$ is in the kernal of $\mathfrak{t}(n, F)$, by Proposition 3.11, we can show $\mathfrak{t}(n, F)/\mathfrak{n}(n, F) \cong \mathfrak{o}(n, F)$. Given $x, z \in \mathfrak{t}(n, F), y \in \mathfrak{n}(n, F)$, we have*

$$\begin{aligned} [(x + y), (z + y)] &= [x, z] + [x, y] + [y, z] + [y, y] \\ &= [x, z] + y \quad \{\textit{Since } y \in \mathfrak{n}(n, F)\} \end{aligned}$$

Where $[x,z] \in \mathfrak{t}(n,F)$. Any upper triangular matrix can be diagonalized, therefore $[x,z] \in \mathfrak{o}(n,F)$ and thus $[x,z]+y \in \mathfrak{o}(n,F)$.

$$(x+y)+(z+y) = (x+z)+y$$

Since the sum of any two upper triangular matrices is also upper triangular, $(x+z) \in \mathfrak{t}(n,F)$, and therefore $(x+z)+y \in \mathfrak{o}(n,F)$.

Example 3.2.3. *For small values of l, isomorphisms occur among certain classical algebras. We can show that A_1, B_1, C_1 are all isomorphic. Additionally, we will show that B_2 is isomorphic to C_2, and D_3 is isomorphic to A_3. We can form our desired isomorphisms given the following root system decomposition for classical algebras $A_n - D_n$:*

A_l:

$$\begin{aligned} &e_{ij} \quad (i \neq j) \\ &h_i = e_{ii} - e_{i+1,i+1} \quad (1 \leq i \leq l) \end{aligned}$$

B_l:

$$\begin{aligned} &e_{ii} - e_{l+i,l+i} \quad (2 \leq i \leq l+1) \\ &e_{1,l+i+1} - e_{i+1,1} \quad (1 \leq i \leq l) \\ &e_{1,i+1} - e_{l+i+1,1} \quad (1 \leq i \leq l) \\ &e_{i+1,j+1} - e_{l+j+1,l+i+1} \quad (1 \leq i \neq j \leq l) \\ &e_{i+1,l+j+1} - e_{j+1,l+i+1} \quad (1 \leq i < j \leq l) \\ &e_{i+l+1,j+1} - e_{j+l+1,i+1} \quad (1 \neq j < i \leq l) \end{aligned}$$

C_l:

$$\begin{aligned} &e_{ii} - e_{l+i,l+i} \quad (1 \leq i \leq l) \\ &e_{ij} - e_{l+j,l+i} \quad (1 \leq i \neq j \leq l) \\ &e_{i,l+i} \quad (1 \leq i \leq l) \\ &e_{i,l+j} + e_{j,l+i} \quad (1 \leq i < j \leq l) \end{aligned}$$

D_l

$$\begin{aligned} &e_{i,j} - e_{l+j,l+i} \quad (1 \le i, j \le l) \\ &e_{i,l+j} - e_{j,l+i} \quad (1 \le i < j \le l) \\ &e_{l+i,j} - e_{l+j,i} \quad (1 \le i < j \le l) \end{aligned}$$

[Hum72], [Wan75]

- *To prove* $A_1 \cong B_1$*, we must show* $\mathfrak{sl}(2, F) \cong \mathfrak{o}(3, F)$*. The diagonal basis matrix for* A_1 *is* $e_{11} - e_{22}$*:*

$$\begin{aligned} [e_{11} - e_{22}, e_{12}] &= (e_{11} - e_{22})e_{12} - e_{12}(e_{11} - e_{22}) = e_{12} + e_{12} = 2e_{12} \\ [e_{11} - e_{22}, e_{21}] &= (e_{11} - e_{22})e_{21} - e_{12}(e_{11} - e_{21}) - 2e_{21} \end{aligned}$$

The following map enumerates the root system from A_1 *to* B_1 *where* $\mathfrak{sl}(2, F)$ *is a* 2×2 *matrix and* $\mathfrak{o}(3, F)$ *is a* 3×3 *matrix:*

$$\begin{aligned} e_{11} - e_{22} &\to 2(e_{11} - e_{22}) \\ e_{12} &\to 2(e_{13} - e_{21}) \\ e_{21} &\to 2(e_{12} - e_{31}) \end{aligned}$$

Similarly, the following map signifies the root system from B_1 *to* C_1 *where* $\mathfrak{sp}(2, F)$ *is a* 2×2 *matrix:*

$$\begin{aligned} 2(e_{11} - e_{22}) &\to e_{11} - e_{22} \\ 2(e_{13} - e_{21}) &\to e_{12} \\ 2(e_{12} - e_{31}) &\to e_{21} \end{aligned}$$

Notice that the basis matrices for A_1 *and* C_1 *are identical, therefore by the transitive property, the isomorphism between* A_1 *and* C_1 *is immediate. and* $A_1 \cong B_1 \cong C_1$

- *For* $B_2 \cong C_2$*, we must show* $\mathfrak{o}(5, F) \cong \mathfrak{sp}(4, F)$ *by adjoining the diagonal matrices from* B_2, C_2 *to corresponding eigenvectors to determine the eigenvalues involved in their respective basis matrices. The diagonal matrices for* B_2 *are as followed:*

$$\begin{aligned} h_1 &= e_{22} - e_{44} \\ h_2 &= e_{33} - e_{55} \end{aligned}$$

Denote $\lambda_1 = [h_1, x]$ *and* $\lambda_2 = [h_2, x]$*, where* x *is an eigenvector, then eigenvalue* $\lambda = (\lambda_1, \lambda_2)$*. Recall that* $[h_i, x] = -[h_i, x^t]$

$$\begin{aligned}
[h_1, e_{21} - e_{14}] &= h_1(e_{21} - e_{14}) - (e_{21} - e_{14})h_1 \\
&= (e_{22} - e_{44})(e_{21} - e_{14}) - (e_{21} - e_{14})(e_{22} - e_{44}) = e_{21} - e_{14} = 1 \\
[h_2, e_{21} - e_{14}] &= (e_{33} - e_{55})(e_{21} - e_{14}) - (e_{21} - e_{14})(e_{33} - e_{55}) = 0 \\
[h_1, e_{12} - e_{41}] &= e_{41} - e_{12} = -1 \\
[h_2, e_{21} - e_{14}] &= 0
\end{aligned}$$

Therefore, the eigenvalue for eigenvector $e_{21} - e_{14}$ *is* $(1, 0) = \alpha$*, and the eigenvalue for eigenvector* $e_{12} - e_{41}$ *is* $(-1, 0) = -\alpha$*. The remaining eigenvalues are derived similarly:*

$\lambda_1 = [h_1, x]$	$\lambda_2 = [h_2, x]$	$\lambda = (\lambda_1, \lambda_2)$
$[h_1, e_{14} - e_{21}] = 1$	$[h_2, e_{14} - e_{21}] = 0$	$\alpha = (1, 0)$
$[h_1, e_{12} - e_{41}] = -1$	$[h_2, e_{12} - e_{41}] = 0$	$-\alpha = (-1, 0)$
$[h_1, e_{32} - e_{45}] = -1$	$[h_2, e_{32} - e_{45}] = 1$	$\beta = (-1, 1)$
$[h_1, e_{23} - e_{54}] = 1$	$[h_2, e_{23} - e_{54}] = -1$	$-\beta = (1, -1)$
$[h_1, e_{15} - e_{31}] = 0$	$[h_2, e_{15} - e_{31}] = 1$	$\alpha + \beta = (1, 0) + (-1, 1) = (0, 1)$
$[h_1, e_{51} - e_{13}] = 0$	$[h_2, e_{51} - e_{13}] = -1$	$-(\alpha + \beta) = (-1, 0) + (1, -1) = (0, -1)$
$[h_1, e_{25} - e_{34}] = 1$	$[h_2, e_{25} - e_{34}] = 1$	$2\alpha + \beta = 2(1, 0) + (-1, 1) = (1, 1)$
$[h_1, e_{52} - e_{43}] = -1$	$[h_2, e_{52} - e_{43}] = -1$	$-(2\alpha + \beta) = -[2(1, 0) + (-1, 1)] = (-1, -1)$

Figure 3.2: Table of eigenvalues for B_2

The diagonal matrices for C_2 *are as followed, where* $l = 2$*:*

$$h'_1 = e_{11} - e_{33}, \quad h'_2 = e_{22} - e_{44}$$

Denote λ'_1 *as the eigenvalue of* $[h'_1, x]$ *and* λ'_2 *as the eigenvalue of* $[h'_2, x]$ *where* x *is an eigenvector. Then eigenvalue* $\lambda' = (\lambda'_1, \lambda'_2)$*.*

λ_1'	λ_2'	$\lambda' = (\lambda_1', \lambda_2')$
$[h_1', e_{21} - e_{34}] = -1$	$[h_2', e_{21} - e_{34}] = 1$	$\alpha' = (-1, 1)$
$[h_1', e_{12} - e_{43}] = 1$	$[h_2', e_{12} - e_{43}] = -1$	$-\alpha' = (1, -1)$
$[h_1', e_{13}] = 2$	$[h_2', e_{13}] = 0$	$\beta' = (2, 0)$
$[h_1', e_{31}] = -2$	$[h_2', e_{31}] = 0$	$-\beta' = (-2, 0)$
$[h_1', e_{14} + e_{23}] = 1$	$[h_2', e_{14} + e_{23}] = 1$	$\alpha' + \beta' = (-1, 1) + (2, 0) = (1, 1)$
$[h_1', e_{41} + e_{32}] = -1$	$[h_2', e_{41} + e_{32}] = -1$	$-(\alpha' + \beta') = -[(-1, 1) + (2, 0)] = (-1, -1)$
$[h_1', e_{51} - e_{13}] = 0$	$[h_2', e_{51} - e_{13}] = -1$	$-(\alpha' + \beta') = (-1, 0) + (1, -1) = (0, -1)$
$[h_1', e_{24}] = 0$	$[h_2', e_{24}] = 2$	$2\alpha' + \beta' = 2(-1, 1) + (2, 0) = (0, 2)$
$[h_1', e_{42}] = 0$	$[h_2', e_{42}] = -2$	$-(2\alpha' + \beta') = -[2(-1, 1) + (2, 0)] = (0, -2)$

Figure 3.3: Table of eigenvalues for C_2

We can construct a linear transformation, where

$$H_1' = -\frac{1}{2}h_1' + \frac{1}{2}h_2' \quad H_2' = \frac{1}{2}h_1 + \frac{1}{2}h_2$$

Let the isomorphism between B_2 and C_2 be defined by the following map:

$$\alpha(h_i) = \alpha' H_i' \quad 1 \le i \le 2 \quad \beta(h_i) = \beta' H_i' \quad 1 \le i \le 2$$

Therefore, $B_2 \cong C_2$, where:

$$\begin{aligned}
e_{22} - e_{44} &\mapsto -\tfrac{1}{2}(e_{11} - e_{33}) + \tfrac{1}{2}(e_{22} - e_{44}) \\
e_{33} - e_{55} &\mapsto \tfrac{1}{2}(e_{11} - e_{33}) + \tfrac{1}{2}(e_{22} - e_{44}) \\
e_{21} - e_{14} &\mapsto \tfrac{\sqrt{2}}{2}(e_{21} - e_{34}) \\
e_{12} - e_{41} &\mapsto \tfrac{\sqrt{2}}{2}(e_{12} - e_{43}) \\
e_{32} - e_{45} &\mapsto e_{13} \\
e_{23} - e_{54} &\mapsto e_{31} \\
e_{15} - e_{31} &\mapsto \tfrac{\sqrt{2}}{2}(e_{14} + e_{23}) \\
e_{31} - e_{41} &\mapsto \tfrac{\sqrt{2}}{2}(e_{41} + e_{32}) \\
e_{25} - e_{34} &\mapsto e_{24} \\
e_{43} - e_{52} &\mapsto e_{42}
\end{aligned}$$

- *For $A_3 \cong D_3$, we must show $\mathfrak{sl}(4, F) \cong \mathfrak{o}(6, F)$. The diagonal matrices for A_3 are:*

$$h_1 = e_{11} - e_{22}, \quad h_2 = e_{22} - e_{33}, \quad h_3 = e_{33} - e_{44}$$

Denote $\lambda_1 = [h_1, x], \lambda_2 = [h_2, x]$, *and* $\lambda_3 = [h_3, x]$. *To find the eigenvalue that corresponds to eigenvector* e_{13}, *we have:*

$$[h_1, e_{13}] = (e_{11} - e_{22})e_{13} - e_{13}(e_{11} - e_{22}) = e_{13}$$

$$[h_2, e_{13}] = (e_{22} - e_{33})e_{13} - e_{13}(e_{22} - e_{33}) = -e_{13}$$

$$[h_3, e_{13}] = (e_{33} - e_{44})e_{13} - e_{13}(e_{33} - e_{44}) = -e_{13}$$

Therefore eigenvalue $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *corresponding to* e_{13} *can be written as* $\alpha = (1, -1, -1)$.

The diagonal matrices for D_3 *are:*

$$h_1' = e_{11} - e_{44}, \quad h_2' = e_{22} - e_{55}, \quad h_3' = e_{33} - e_{66}$$

For D_3, *denote* $\lambda_1' = [h_1', x'], \lambda_2' = [h_2', x'], \lambda_3' = [h_3', x']$ *where* x' *is an eigenvector. Let* $\lambda' = (\lambda_1', \lambda_2', \lambda_3')$. *The eigenvalues for* A_3 *and* D_3 *are listed on the following table:*

x	$\lambda = (\lambda_1, \lambda_2, \lambda_3)$	x'	$\lambda' = (\lambda_1', \lambda_2', \lambda_3')$
e_{13}	$\alpha(1, 1, -1)$	$e_{26} - e_{25}$	$\alpha' = (0, 1, 1)$
e_{31}	$\alpha = (-1, 1, 1)$	$e_{62} - e_{53}$	$\alpha = (0, -1, -1)$
e_{24}	$\beta(-1, 1, 1)$	$e_{23} - e_{65}$	$\beta'(0, 1, -1)$
e_{42}	$-\beta' = (1, -1, -1)$	$e_{32} - e_{56}$	$-\beta'(0, -1, 1)$
e_{41}	$\gamma = (-1, 0, -1)$	$e_{12} - e_{54}$	$\gamma' = (1, -1, 0)$
e_{14}	$\gamma = (1, 0, 1)$	$e_{21} - e_{45}$	$-\gamma'(-1, 1, 0)$
e_{43}	$(\alpha + \gamma) = (0, 1, -2)$	$e_{16} - e_{34}$	$(\alpha' + \gamma') = (1, 0, 1)$
e_{34}	$-(\alpha + \gamma) = (0, -1, 2)$	$e_{61} - e_{43}$	$-(\alpha' + \gamma') = (1, 0, 1)$
e_{21}	$\beta + \gamma = (-2, 1, 0)$	$e_{13} - e_{64}$	$\beta' + \gamma' = (1, 0, -1)$
e_{12}	$-(\beta + \gamma) = (2, -1, 0)$	$e_{31} - e_{46}$	$-(\beta' + \gamma') = (-1, 0, 1)$
e_{23}	$\alpha + \beta + \gamma = (-1, 2, -1)$	$e_{15} - e_{24}$	$\alpha' + \beta' + \gamma' = (1, 1, 0)$
e_{32}	$-(\alpha + \beta + \gamma) = (1, -2, 1)$	$e_{51} - e_{42}$	$\alpha' + \beta' + \gamma' = (1, 1, 0)$

Figure 3.4: Table of eigenvalues for A_3 and D_3

We can construct a linear transformation:

$$H_1' = -h_1' + h_3', \quad H_2' = h_1' + h_2', \quad H_3' = -h_1' - h_3'$$

The insomorphism between A_3 *and* D_3 *is thus obtained by defining a map between diagonal matrices;*

$$\alpha(h_i) = \alpha'(H_i'), \quad \beta(h_i) = \alpha'(H_i'), \quad \gamma(h_i) = \gamma(H_i')$$

Therefore $A_3 \cong D_3$, where:

$$\begin{aligned}
e_{11} - e_{22} &\mapsto -(e_{11} - e_{44}) + (e_{33} - e_{66}) \\
e_{22} - e_{33} &\mapsto (e_{11} - e_{44}) + (e_{22} - e_{55}) \\
e_{33} - e_{44} &\mapsto -(e_{11} - e_{44}) - (e_{33} - e_{66}) \\
e_{13} &\mapsto e_{26} - e_{25} \\
e_{31} &\mapsto e_{62} - e_{53} \\
e_{24} &\mapsto e_{23} - e_{65} \\
e_{42} &\mapsto e_{32} - e_{56} \\
e_{41} &\mapsto e_{12} - e_{54} \\
e_{14} &\mapsto e_{21} - e_{45} \\
e_{43} &\mapsto e_{16} - e_{34} \\
e_{34} &\mapsto e_{61} - e_{43} \\
e_{21} &\mapsto e_{13} - e_{64} \\
e_{12} &\mapsto e_{31} - e_{46} \\
e_{23} &\mapsto e_{15} - e_{24} \\
e_{32} &\mapsto e_{51} - e_{42}
\end{aligned}$$

3.3 Automorphisms

The set of **inner automorphisms** of a ring, or associative algebra A, is given by the conjugation element, using right conjugation, such that:

$$\varphi_a : A \to A$$
$$\varphi_a(x) = a^{-1}xa$$

Given $x, y \in A$:

$$\varphi_a(xy) = a^{-1}(xy)a = a^{-1}xaa^{-1}ya = (a^{-1}xa)(a^{-1}ya) = \varphi_a(x)\varphi_a(y)$$

Where, φ_a is an (invertible) homomorphism that contains φ_a^{-1}. Therefore φ_a constitutes an isomorphism onto itself. Since the composition of conjugation is associative, $a^{-1}xa$ is often denoted as x^a.

Definition 3.12. *An* ***automorphism*** *of L is an isomorphism of L onto itself. Aut L denotes the group of all such.*

Example 3.3.1. *Consider $L \subseteq \mathfrak{gl}(V)$, and $g \in \mathfrak{gl}(V)$ where g is an invertible endomorphism. Define a mapping $\phi : L \to L'$ where $xLx^{-1} = L$, then $x \to gxg^{-1}$, where ϕ is an automophism of L. The same can be said for $\mathfrak{sl}(V)$ since the trace of a matrix is invariant under a change of basis.*

Example 3.3.2. *Let L be a Lie algebra such that $L = \mathfrak{sl}(n, F), g \in GL(n, F)$. The map $\varphi : L \to L$ defined by $x \to -gx^tg^{-1}(x^t =$ the transpose of $x)$ belongs to Aut L. When $n = 2$, $g =$ the identity matrix, we can prove that Aut L is inner.*

$$\begin{aligned} tr(-gx^tg^{-1}) = tr(-gg^{-1}x^t) = tr(-x^t) = -tr(x) \quad & \{Since\ g = the\ identity\ matrix\} \\ \Rightarrow tr(x) = 0 \Leftrightarrow tr(-gx^tg^{-1}) = 0 & \end{aligned}$$

Therefore, the map is a linear automorphism of $\mathfrak{sl}(n, F)$. If we apply the transpose to the commutator, for $x, y \in L$, we have:

$$\begin{aligned} [x, y]^t &= (xy - yx)^t = (xy)^t - (yx)^t \\ &= y^tx^t - x^ty^t = [y^tx^t] \end{aligned}$$

Therefore:

$$\begin{aligned} \varphi[x, y] &= -g[x, y]^tg^{-1} = -g[y^t, x^t]g^{-1} \qquad \{By\ properties\ of\ the\ transpose\} \\ &= g(y^tx^t - x^ty^t)g^{-1} = gy^tx^tg^{-1} - gx^ty^tg^{-1} \\ &= [gy^tg^{-1}, gx^tg^{-1}] = gy^tg^{-1}gx^tg^{-1} - gx^tg^{-1}gy^tg^{-1} \\ &= [\varphi(x), \varphi(y)] \end{aligned}$$

Therefore, φ is a homomorphism. Thus Aut L is inner.

Example 3.3.3. *An automorphism of the form $exp(ad_x)$, with ad_x **nilpotent**, i.e., $(ad_x)^k = 0$ for some $k > 1$, is called **inner**.*

To derive the power series expansion for $exp(ad_x)$, we begin by constructing the Taylor series expansion using the differential operator for matrix A. Given an invertible matrix A, the solution to $x' = A\vec{x}$ might be $x'(t) = \lambda ve^{\lambda t}$. Multiplying by A, we have:

$$Ax(t) = Ave^{\lambda t}$$

$$\text{where} \quad Ax(t) = x'(t) \Leftrightarrow \lambda v = Av$$

where λ is an eignenvalue of A with corresponding eigenvector v. Then the power series expansion to define $e^{\lambda t}$ when λ is complex becomes:

$$e^{\lambda t} = 1 + (\lambda t) + \frac{1}{2!}(\lambda t)^2 + ... + \frac{1}{n!}(\lambda t)^n$$

[Eve66] Let char$F = 0$, and $x \in L$ such that ad$_x$ is nilpotent. That is, $(ad_x)^k = 0$ for $k > 0$. Then, since ad$_x$ is finite dimensional, the above series expansion generalizes to:

$$\begin{aligned} exp(ad_x) &= \sum_{k=0}^{\infty} \frac{ad^k_{(x)}}{k!} \\ &= 1 + ad_x + \frac{(ad_x)^2}{2!} + \frac{(ad_x)^3}{3!} + ... + \frac{(ad_x)^{k-1}}{(k-1)!} \end{aligned}$$

where exp(ad$_x$) $\in$ Aut$L$. To show this, recall that Leibniz's rule for the product of derivations is:

$$\frac{\delta^n}{n!}(xy) = \sum_{i=0}^{n} \frac{1}{i!}\delta^i(x)\frac{1}{(n-1)!}\delta^{n-i}(y)$$

Where δ is an arbitrary nilpotent derivation of L. Given $x, y \in L$:

$$\begin{aligned} exp(\delta(x))exp(\delta(y)) &= \left(\sum_{i=0}^{n-1} \frac{\delta^i(x)}{i!}\right)\left(\sum_{j=0}^{n-1} \frac{\delta^j(y)}{j!}\right) \\ &= \sum_{n=0}^{2k-2} \frac{\delta^n(xy)}{n!} \\ &= \sum_{n=0}^{k-1} \frac{\delta^n(xy)}{n!} \end{aligned}$$

Where the first equality holds because $exp(ad_x) = \sum_{k=0}^{\infty} \frac{ad^k_{(x)}}{k!}$, and the second equality satisfies Leibniz's product rule. Since δ is a nilpotent derivation of L, the last equality holds because $\delta^k = 0$. Therefore, the composition of two inner automorphisms is again an inner automorphism, thus the derivation of an inner automorphisms $exp(ad(x))$ is a homomorphism:

$$[exp(\delta(x)), exp(\delta(y))] = exp([\delta(x), \delta(y)])$$

Where exp δ is invertible by exhibiting the inverse: $1 - n + n^2 - n^3 + ... \pm n^{k-1}$, exp $\delta = n + 1$. Thus, $exp(ad_x) \in AutL$. The subgroup of all such inner automorphisms $exp(ad_x)$ is called $Int(L)$ and defines a normal subgroup of $Aut(L)$.

Definition 3.13. *When ad_x is nilpotent, the inner automorphism constructed is called Int L. For $\phi \in$ Aut $l, x \in L$,*

$$\phi(ad_x)\phi^{-1} = ad_{\phi(x)} \quad when \quad \phi exp(ad_x)\phi^{-1} = exp(ad_{\phi(x)})$$

Example 3.3.4. *Let σ be the automorphism of $\mathfrak{sl}(2,F)$ give by the following: let $L \in \mathfrak{sl}(2,F)$, with standard basis (x,y,h). Define $\sigma =$exp ad $x\cdot$ exp ad $(-y)\cdot$ exp ad x. We can show that $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$. From example 3.1.2, we have* $[x,y] = h, [h,y] = -2y, [h,x] = -2x$

$$\begin{aligned}
\sigma(x) &= exp\ adx \cdot exp\ ad(-y) \cdot exp\ adx(x) \\
&= exp\ adx \cdot (1 + ad(-y) + \frac{ad(-y)^2}{2!})x \\
&= exp\ adx \cdot (x + (-ady)(x) + \frac{1}{2!}(ad(-y))^2(x)) \\
&= exp\ adx \cdot (x - [y,x] + \frac{1}{2!}(ad(-y))(h)) \\
&= exp\ adx \cdot (x + h + [-y,h]) \\
&= (1 + adx + \frac{ad(x)^2}{2!})(x + h - y) \\
&= (x + h - y) + ([x,x] + [x,h] - [x,y]) + (\frac{[x[x,x]]}{2!} + \frac{[x,-2x]}{2!} - \frac{[x,h]}{2!}) \\
&= (x + h - y) + (-2x - h) + (x) \\
&= -y \\
\sigma(y) &= exp\ adx \cdot exp\ ad(-y) \cdot exp\ adx(y) \\
&= exp\ adx \cdot (1 + adx + \frac{ad(x)^2}{2!})(y) \\
&= exp\ adx \cdot (y + [x,y] + \frac{[x,[x,y]]}{2}) \\
&= exp\ adx \cdot (1 + ad(-y) + \frac{ad(-y)^2}{2!})(y + h - x) \\
&= exp\ adx \cdot [(y + h - x) + ([-y,y] + [-y,h] - [-y,x]) \\
&+ (\frac{[-y,[-y,y]]}{2!} + \frac{[-y,[-y,h]]}{2!} - \frac{[-y,[-y,x]]}{2!})] \\
&= exp\ adx \cdot (y + h - 2y - x - h + y) \\
&= (1 + adx + \frac{ad(x)^2}{2!})(-x) \\
&= -x
\end{aligned}$$

$$
\begin{aligned}
\sigma(h) &= exp\ adx \cdot exp\ ad(-y) \cdot exp\ adx(h) \\
&= exp\ adx \cdot exp\ ad(-y) \cdot (1 + adx + \frac{ad(x)^2}{2!})(h) \\
&= exp\ adx \cdot exp\ ad(-y) \cdot (h + [x,h] + \frac{[[x,h],h]}{2!}) \\
&= exp\ adx \cdot (1 + ad(-y) + \frac{ad(-y)^2}{2!})(h - 2x) \\
&= exp\ adx \cdot [(h - 2x) + ([-y,h] - 2[-y,x]) + (\frac{[-y,[-y,h]]}{2!}) - \frac{2[-y,[-y,x]]}{2!})] \\
&= exp\ adx \cdot [(h - 2x) + (-2y - 2h) + (0 + 2y)] \\
&= exp\ adx(-h - 2x) \\
&= (1 + adx + \frac{ad(x)^2}{2!})(-h - 2x) \\
&= (-h - 2x) + (-[x,h] - 2[x,x]) + (-\frac{[x,[x,h]]}{2} - 2\frac{[x,[x,x]]}{2}) \\
&= (-h - 2x) + (2x - 0) \\
&= -h.
\end{aligned}
$$

Chapter 4

Solvable and Nilpotent Lie Algebras

4.1 Solvability

In this section, we will decompose L into a collection of subalgebras. Given L as the direct sum of ideals, the derivation of L^i for $i = 1, 2...n$, consists of a mapping such that the commutator $[L^i, L^i]$ is zero.

Definition 4.1. *The derived series of a Lie algebra L is a sequence of ideals of L where*

$$L^0 = L, L^1 = [LL], L^2 = [L^1L^1], ..., L^i = [L^{i-1}L^{i-1}].$$

Given a Lie algebra L, and an ideal I, we have:

$$[L, I^1] = [L, [I, I]] \subseteq [I, [I, L]] + [I, [L, I]] \subseteq [I, I] + [I, I] = I^1 \quad \text{[Wan75]}$$

Therefore, if I is an ideal of L, then so is I^1. Unless otherwise stated, the material for this chapter is derived from [Hum72].

Example 4.1.1. *Given I is an ideal of L, we can show I^{n+1} is an ideal of L using induction.*

Proof. Since I is an ideal of L, $I^0 = I$ is also an ideal of L, as is I^1 by the above. Assume I^n is an ideal of L. Let $x \in L, y, z \in I^n$.

$$
\begin{aligned}
[x[yz]] &= -[y[zx]] - [z[xy]] && \{\text{By the Jacobi identity}\}\\
&\in [y, I^n] + [z, I^n] && \{\text{Since } [z,x],[x,y] \in I^n\}\\
&\in [I^n, I^n] + [I^n, I^n] && \{\text{Since } y, z \in I^n\}\\
&\in I^{n+1} + I^{n+1}\\
&\in I^{n+1}
\end{aligned}
$$

Therefore I^{n+1} is an ideal, making each member of the derived series of I an ideal of L. □

Definition 4.2. *A group G is said to be **solvable** if it has a subnormal series*

$$G = G_0 \geq G_1 \geq G_2 \geq ... \geq G_n = e \tag{4.1}$$

in which the factors G_i/G_{i+1} are all Abelian for all $i, 0 \leq i \leq n-1$.

By subnormal, we mean that for each i, $0 \leq i \leq n-1$, G_i is normal in G_{i+1}. Given a Lie algebra L, and series length $n > 0$ we say that L is solvable if $L^{(n)} = 0$, that is, a Lie algebra is solvable if its derived series terminates in the zero subalgebra. The derived subalgebra of a finite dimensional solvable Lie algebra over a field of characteristic 0 is nilpotent, thus making abelian algebras solvable and conversely simple algebras nonsolvable.

Lemma 4.3. *Let $0 \to J \to L \to I \to 0$ be an exact sequence of Lie algebras. Then L is solvable if and only if both J and I are solvable. [Sam90]*

Proposition 4.4.

1. *If L is a solvable Lie algebra, then so are all of the subalgebras and homomorphic images of L.*

2. *If I is a solvable ideal of L such that the quotient L/I is solvable, then L itself is solvable.*

3. *If I, J are solvable ideals of L, where L is a solvable Lie algebrra, then $I + J$ is solvable.*

Proof.

1. Let K be a subalgebra of L, then by definition, $K^{(i)} \subset L^{(i)}$, therefore $K^{(i)}$ is solvable for all i. If we define a map $\varphi : L \rightarrow M$ where φ is an epimorphism, then by induction, we can show $\varphi(L^{(i)}) = M^{(i)}$.

2. Let $(L/I)^{(n)} = 0$. By construction of the canonical homomorphism $\pi : L \rightarrow \frac{L}{I}$, we have that $\pi(L^{(n)}) = 0$ by part 1. or $L^{(n)} \subset I = ker\pi$. If $I^{(m)} = 0$, then $(L^{(i)})^{(i)} = L^{(i+j)} \Rightarrow L^{(n+m)} = 0$ (since $L^{(n)} \subset I$).

3. By Proposition 3.11 (part 3.): if I, J are ideals of L, then from the sequence $0 \rightarrow I \rightarrow I + J \rightarrow \frac{I+J}{J} \rightarrow 0$, there exists a natural isomorphism between the third term $(I + J)/J$ and $I/(I \cap J)$. Now $(I + J)/J$ is solvable by the Lemma 4.3, since the homomorphic image of I is solvable. Therefore $I/(I \cap J)$ is solvable by part 2.

□

As a consequence of Proposition 4.4, if L is solvable, then L possesses a unique maximal solvable ideal, called the **radical of** L, or **Rad** L. When L is **semisimple**, Rad $L = 0$. Recall, L is simple if it contains no non trivial ideals (i.e., only contains the trivial ideals, 0 and L itself), that is, if Rad $L = 0$. Therefore, every simple algebra is semisimple.

Example 4.1.2. *L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset ... \supset L_k = 0$ such that:*

- *L_{i+1} is an ideal of L_i.*
- *Each quotient $\frac{L_i}{L_{i+1}}$ is abelian.*

Proof.

$\Rightarrow$

- Since L is solvable, there exists a sequence of ideals of L such that $L \supseteq L^{(1)} \supseteq ... \supseteq L^{(i)} = 0$ for some i. By definition, each ideal forms a subalgebra of L. By Proposition 4.4, $L^{(i+1)} \subseteq L^{(i)}$ where $L^{(i+1)}$ and $L^{(i)}$ are ideals of L.

- Since $L^{(i+1)}$ is an ideal of $L^{(i)} \subseteq L$, $L^{(i+1)}$ is an ideal of L. Since L is solvable, then by definition of the subnormal series, we can show each quotient $\frac{L^{(i)}}{L^{(i+1)}}$ is abelian. By definition of the derived series, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$. Let $[x, y] \in L^{(i+1)}$, with $x, y \in L^{(i)}$:

$$\begin{aligned}[x + L^{(i+1)}, y + L^{(i+1)}] &= [x, y] + [x, L^{(i+1)}] + [L^{(i+1)}, y] + [L^{(i+1)}, L^{(i+1)}] \\ &= [x, y] + L^{(i+1)}\end{aligned}$$

Therefore $\frac{L^{(i)}}{L^{(i+1)}}$ is abelian.

$\Leftarrow$ Given $\frac{L^{(k-1)}}{L^{(k)}}$ is abelian and $L = L_0 \supset L_1 \supset ... \supset L_k = 0$, then the derived series for L terminates in the zero subalgebra. Therefore, by defintion, L_k is solvable, making $\frac{L^{(k-1)}}{L^{(k)}}$ solvable, and thus L_{k-1} is solvable by Proposition 4.4. Similarly, $\frac{L_{k-2}}{L_{k-1}}$ is solvable, and thus L_{k-2} is solvable. Continuing this way, $\frac{L_0}{L_1}$ is abelian and thus L_1 is solvable. Therefore $L_0 = L$ is solvable. □

Example 4.1.3. *We can show that L is solvable if and only if ad_L is solvable.*

Proof.
$\Rightarrow$ Let $L^{(n)} = 0$. By induction, when $n = 0$, $L^{(0)} = L$, and $ad_{L^{(0)}} = ad_L = (ad_L)^{(0)}$. Therefore ad_L is solvable for $n = 0$. Assume $ad_{L^{(n)}} = (ad_L)^{(n)}$,

$$\begin{aligned} ad_{L^{(n+1)}} &= ad[L^{(n)}, L^{(n)}] && \{\text{By definition of the derived series}\} \\ &= [(ad_L)^{(n)}, (ad_L)^{(n)}] && \big\{\text{Since } ad_{[xy]} = [ad_x, ad_y]\big\} \\ &= (ad_L)^{(n+1)} && \big\{\text{Since } L^{n+1} = [L^n, L^n]\big\} \end{aligned}$$

Therefore ad_L is solvable.
$\Leftarrow$ Since ad_L is solvable, $(ad_L)^{(n)} = ad_L^{(n)} = 0$, by the induction above. Therefore $L^{(n)} \subseteq Z(L)$. Therefore $\frac{L}{Z(L)}$ is solvable and thus L is solvable.

□

4.2 Nilpotency

In this section, we shall explore the descending central series, defined as a recursive sequence of subalgebras that ends in the zero subspace. We shall rewrite this series using the quotient space to determine the effect of having a nilpotent algebra as it relates to it's subalgebras and their homomorphic images.

Definition 4.5. *The* ***descending central series*** *is a sequence of ideals of* L *defined as* $L^0 = L, L^1 = [LL], L^2 = [LL^1], ..., L^i = [LL^{i-1}]$*. We call* L ***nilpotent*** *if for some* $n > 0$*,* $L^n = 0$*.*

Thus L^i is spanned by *long* brackets $[X_1[X_2[...X_{I+1}]...]$ (which we abbreviate to $[X_1X_2...X_{i+1}]$) [Sam90]. Notice that $L^n = 0$ makes any abelian algebra nilpotent. Therefore nilpotent implies solvable since $L^{(i)} \subset L^i$ for all i. However solvable does not imply nilpotent.

Example 4.2.1. *The algebra* $\mathfrak{n}(n, F)$ *of strictly upper triangular matrices is nilpotent. A matrix is strictly upper triangular if and only if* $i > j - k$*. So Given* $B^k_{ij} = 0$*, implies* $B^k = 0$ *for* $k \geq m$ *if* B *is an* $m \times m$ *matrix. Using induction, we have* $B^1 = 0$ *for* $i.j - 1$*. Assume* $B^k_{ij} = 0$ *for* $i, j - (k-1)$*:*

$$\begin{aligned} B^k_{ij} = BB^{k-1}_j &= \sum_{r=1}^{i} B_{ir}B^{k-1}_{rj} + \sum_{r=i+1}^{m} B_{ir}B^{k-1}_{rj} \\ &= \sum_{r=1}^{i} 0 \cdot B^{k-1}_{rj} + \sum_{r=i+1}^{m} B_{ir} \cdot 0 = 0 \end{aligned}$$

Note that the first term goes to zero, since $r \leq i$ *where* B_{ir} *is strictly triangular. The second term goes to zero since* $r \geq i + 1 > (j - k) + 1 = j - (k - 1)$*. Therefore, by the induction step,* $B^k_{ij} = 0$*.*
We can show a similar result for lower triangular matrices using the following indices:

$$\begin{aligned} m \geq j.i \geq 1 \quad C_{ij} = D_{ij} = 0 \\ m \geq i \geq 1 \quad C_{ii} = D_{ii} = 0 \end{aligned}$$

Which would prove that $(CD)_{ij} = 0$ *[Eve66]*

Proposition 4.6.

1. *If* L *is a nilpotent Lie algebra, then all the subalgebras and homomorphic images of* L *are nilpotent.*

2. *If* L *is a Lie algebra such that* $L/Z(L)$ *is nilpotent, then* L *is nilpotent.*

3. *If* $L \neq 0$ *is a nilpotent Lie algebra, then* $Z(L) \neq 0$

Proof.

1. Let K be a subalgebra, then by definition $K^i \subset L^i$. Similarly, if $\varphi : L \to M$ is an epimorphism, we can show, by induction on i, that $\varphi(L^i) = M^i$

2. Let $L^n \subset Z(L)$, then

$$\begin{aligned} L^{n+1} &= [L, L^n] && \{\text{By definition of the descending central series}\} \\ &= [L, Z(L)] = 0 && \{\text{By definition of the center}\} \end{aligned}$$

3. If L is nilpotent, $L^n = \{0\}$, and $L^{n-1} \neq \{0\}$. By definition of the descending central series, $L^{n-1} = [L, L^n]$, which implies that $L^{n-1} \subseteq Z(L)$. Therefore $Z(L) \neq 0$.

□

Example 4.2.2. *Let I be an ideal of L. Then each member of the descending central series of I is also an ideal of L.*

Proof. Since I is an ideal of L, $I^0 = I$ is also an ideal of L. Assume $I^n = [I, I^{n-1}]$ is an ideal of L. Let $x \in L, y \in I, z \in I^n$.

$$\begin{aligned} [x[yz]] &= -[y[zx]] - [z[xy]] && \{\text{By the Jacobi identity}\} \\ &\in [y, I^n] + [z, I] && \{\text{Since } [z, x] \in I^n, [x, y] \in I\} \\ &\in [I, I^n] + [I^n, I] && \{\text{Since } y \in I, z \in I^n\} \\ &= I^{n+1} + I^{n+1} \\ &= I^{n+1} \end{aligned}$$

Therefore I^{n+1} is an ideal, then each member of the descending central series of I is an ideal of L. □

Example 4.2.3. *We can show that L is nilpotent if and only if ad_L is nilpotent.*

Proof. Using induction, let $n = 0$, then $L^0 = L$. Therefore $\mathrm{ad}_{L^0} = (ad_L)^0 = ad_L$. Let $ad_{L^n} = (ad_L)^n$. Then

$$\begin{aligned} ad_{L^{n+1}} &= [ad_L, ad_{L^n}] && \{\text{By definition of the descending central series}\} \\ &= [ad_L, (ad_L)^n] && \{\text{Since } ad_{L^n} = (ad_L)^n\} \\ &= (ad_L)^{n+1} \end{aligned}$$

$\Rightarrow$ Let $L^n = 0$ for some n. By the above induction, $\text{ad}_{L^n} = (ad_L)^n = 0$ and thus ad_L is nilpotent.
$\Leftarrow$ Given ad_L is nilpotent, $(ad_L)^n = 0$. Since $(ad_L)^n = ad_{L^n} = 0$, $L^n = 0$, therefore L is nilpotent. $\square$

Example 4.2.4. *The sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal.*

Proof. Let I, J be nilpotent ideals of L. By definition, $I^m, J^n = 0$ for some m, n. Let $n \geq m$. Let $x \in I, y \in J$, then:

$$\begin{aligned} [x+y, x+y] &= [x,x] + [x,y] + [y,x] + [y,y] \\ &\subseteq [I,I] + [I,J] + [J,I] + [J,J] \\ &= [I,I] + [J,J] + [I,J] \\ \subseteq [I,I] + [J,J] + I \cap J &= [I+J, I+J] \quad \text{[Sam90]} \end{aligned}$$

Assume by induction, $(I+J)^k \subseteq I^n + J^m + I \cap J$. Our aim is to show the there exists a k such that $(I+J)^k = 0$ where $(I+J)^k = [I+J, (I+J)^{k-1}]$, by definition of the descending central series. Let $k = n+m$, by the use of binomial expansion:

$$\begin{aligned} (I+J)^k = (I+J)^{n+m} &= I^{n+m} + I^{n+m-1}J + ... + IJ^{n+m-1} + J^{n+m} \\ &= 0 + I^{n+m-1}J + ... + IJ^{n+m-1} + 0 \quad \{\text{Since } I^n, J^m = 0\}. \\ &\subseteq I^k \cap J + J \cap I^k \end{aligned}$$

where $(I+J)^k = (I+J)^{n+m} = 0$. Thus $I+J$ is a nilpotent ideal. $\square$

Example 4.2.5. *Let charF* $= 2$. *We can prove that* $L = \mathfrak{sl}(2, F)$ *is nilpotent. Recall the standard basis for L:*

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$[x,z] = [(e_{11} - e_{22}), e_{21}] = (e_{11} - e_{22})(e_{21}) - (e_{21})(e_{11} - e_{22}) = -e_{21} - e_{21} = -2e_{21} = -2z$

$[x,y] = [(e_{11} - e_{22}), e_{12}] = (e_{11} - e_{22})(e_{12}) - (e_{12})(e_{11} - e_{22}) = e_{12} - (-e_{12}) = 2e_{12} = 2y$

$[y,z] = [e_{12}, e_{21}] = e_{12}e_{21} - e_{21}e_{12} = e_{11} - e_{22} = x$

Therefore, by definition of the lower central series,

$$L^1 = [\mathfrak{sl}(2,F), \mathfrak{sl}(2,F)] = \begin{pmatrix} 0 & -2y & 2z \\ 2y & 0 & -x \\ -2z & x & 0 \end{pmatrix}$$

Recall $[x,x] = 0$. *Since* $charF = 2$, *then* $[x,z] = 0 = [x,y]$, *and*

$$[\mathfrak{sl}(2,F), \mathfrak{sl}(2,F)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{pmatrix} = Fx.$$ *Therefore by the Jacobi relation,*

$$L^2 = [\mathfrak{sl}(2,F), [\mathfrak{sl}(2,F), \mathfrak{sl}(2,F)]] = [\mathfrak{sl}(2,F), Fx] = 0.$$

4.3 Proof of Engel's Theorem

The purpose of Engel's Theorem is to connect the property of Lie algebra nilpotence to operators on a vector space. A Lie algebra is nilpotent if for a sufficiently long sequence $\{x_i\}$ of elements of L, the nested adjoint $ad(x_n)[...ad(x_2)[ad(x_1)[y]]]$ is zero for all $y \in L$. That is, applying the adjoint sufficiently many times will kill any element belonging to L, rending each $x \in L$ ad-nilpotent. Engel's theorem states the converse as true. In preparation for Engel's theorem, we must first observe the following lemma, then prove Theorem 4.8.

Lemma 4.7. *Let A be a nilpotent operator on a vector space V, then*

1. *There exists a non-zero $v \in V$ such that $Av = 0$.*

2. *ad_A is a nilpotent operator on $\mathfrak{gl}(v)$.*

Theorem 4.8. *Let L be a subalgebra of $\mathfrak{gl}(v)$, with V finite dimensional. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists a nonzero $v \in V$ for which $L.v = 0$.*

Proof. Using induction on dim$L$. If dim $L = 0$, then clearly $L.v = 0$ and the theorem is true. If dim $L = 1$, then recall that a single nilpotent linear transformation contains at least one eigenvector corresponding to eigenvalue 0, thus the theorem is true and $L.v = 0$.

Given L is a nilpotent Lie algebra, $L_{(n)} = 0$ for all $x_1, ..., x_n, x \in L$. Let $ad_x \in \mathfrak{gl}(L)$ with $x \in L$ ad-nilpotent, then:

$$\begin{aligned} [x_1, [x_{2,}[...[x_{n,x}]...] &= 0 \\ \Leftrightarrow (ad_{x_1})(ad_{x_2})...(ad_{x_n})(x) &= 0 \\ \Leftrightarrow (ad_L)^n &= 0 \end{aligned}$$

[GG78]

Suppose $K \neq L$ is any nontrivial subalgebra of L. Consider the vector spaces of K and L. Since K is a subalgebra of a nilpotent endomorphism, $ad_k(l + K) = ad_k(l) + K$ is a well defined map where, for $k' \in K$:

$$\begin{aligned} ad_k(l + k' + K) &= ad_k(l) + ad_k(k') + K \\ &= ad_k(l) + K \end{aligned}$$

Thus ad_k induces a nilpotent linear transformation on the quotient subspace L/K. Since K is a subalgebra of L, $\dim K < \dim L$. From the induction assumption and Lemma 4.7, it follows that there exists some non-zero vector $x + k \neq k$ in L/K such that $x + k$ is killed by the image in $\mathfrak{gl}(L/K)$. By definition, $N_L(K) = \{x \in L | [x, k] \subset K\}$, therefore K is properly in the normalizer of K in L since for $x \in L, y \in K$, $[x, y] \subset K$ where $x \notin K$. Let K be a maximal proper subalgebra of L, then every $x \in L$ has $[x, k] \in K$ for all $k \in K$, which makes $N_L(K)$ an ideal of L. Consider the dimension of L/K. If $\dim L/K > 1$, then there exists a 1 dimensional subalgebra $Fv/K \subseteq L/K$. By the correspondance theorem, $K \subseteq Fv \subseteq L$, but L is not one dimensional. Therefore, K has codimension 1, which makes L/K one dimensional. Since L/K is 1-dimensional, $\mathrm{Span}\{l \in L\} = F(z + L)K$ for some nilpotent endomorphism $z \in L$. For each $l \in L$, $l + K = \alpha z + K$, $k \in K$. Therefore $\alpha z \in K$, and thus $l = \alpha z + k$ for $z \in l - k$.

By induction, $W = \{v \in V | k.v = 0\} \neq 0$. By the above, K is an ideal of L. If $x \in L, y \in K, w \in V$, then since $yx - xy = -[x, y]$ in End (V), we have:

$$
\begin{aligned}
(yx) \cdot w - (xy) \cdot w = -[x, y] \cdot w & \\
(yx) \cdot w &= (xy) \cdot w - [x, y] \cdot w \\
&= x(y(w)) - 0 \cdot w && \{\text{since } [x, y] \text{ is an ideal of } K\} \\
&= x \cdot 0 && \{\text{since } (y(w)) = 0, y \in K\} \\
&= 0
\end{aligned}
$$

Since z is a nilpotent endomorphism in L, L stabilizes W, therefore, $z : W \to W$ is nilpotent. For $z^n = 0$ ($z^{n-1} \neq 0$), there exists a $u \in V$ such that $z^{n-1}u \neq 0$. Let $v_0 = z^{n-1}u \in W$. Then $zv_0 = z^n = 0$, Thus, there exists an eigenvector $v_0 \in W(v \neq 0)$ for which $K(v_o) = 0$. Therefore, $L.v = 0$ since $L = K + Fz$. □

Theorem 4.9. ***Engel I*** *Let V be a vector space; let L be a sub Lie algebra of the general linear Lie algebra $\mathfrak{gl}(V)$, consisting entirely of nilpotent operators. Then L is a nilpotent Lie algebra.*

Proof. By Theorem 4.8, if we apply the dual representation (inverse transpose) of L on V^t; the operators are nilpotent. Let $\lambda \neq 0$ be a function on V that is anulled by L, then the space $(L \cdot v)$ spanned by all Xv with $X \in L$, is a proper subspace of V, and in fact, is in the kernal of λ where $\lambda(Xv) = X^t\lambda(v) = 0$. Recall $(L \cdot v)$ is invariant under L, so we can iterate the argument. Let $m = dimV$, then the abbreviated iteration $X_1 \cdot X_2 ... X_m$, vanishes, since for every X_i, the dimension of V is decreased by at least 1 where any long bracket $[X_1 X_2 ... X_k]$ expands by the bracket operation into a sum of products of kX's. [Sam90] □

Theorem 4.10. ***Engel II*** *If all elements of L are ad-nilpotent, then L is nilpotent.*

Proof. Given a Lie algebra of L having only ad-nilpotent elements, since the adoint is a linear transformation, $\mathrm{ad}_L \subset \mathfrak{gl}(L)$, which satisfies the hypothesis of Theorem 4.8 when $L \neq 0$. Thus there exists an $x \in L$ such that $[L, x] = 0$, which implies that the center of L is nonzero. Therefore $L/Z(L)$ consists of ad-nilpotent elements, where dim $ad_{L/Z(L)} <$ dim$L$. Using an induction argument on the dimension of L, similar to Theorem 4.8, it follows that $L/Z(L)$ is nilpotent. Therefore, by Proposition 4.6 (part 2.), if $L/Z(L)$ is nilpotent, then so is L. □

Corollary 4.11. *Let L be nilpotent, and K be an ideal of L. Then if $k \neq 0, k \cap Z(L) \neq 0$*

Proof. Since K is an ideal of L, L induces a linear transfomation on K via the adjoint representation, therefore there exists $x \in K(k \neq 0)$. Therefore $[L, x] = 0$ by definition of nilpotency, and thus $x \in K \cap Z(L)$ as desired. $\square$

Chapter 5

Conclusion

Now that we have covered the basics, we can further explore the root system of the classical Lie algebras. The full set of roots of a Lie algebra L can be generated from the set of simple roots and their associated Weyl reflections. If we further explore nilpotent subalgebras of a Lie algebra that are self-normalising, we invariably must investigate cartan subalgebras and the killing form.

Differentiating a Lie group action yeids a Lie algebra action. A Lie algebra is a linearization of a Lie group action, and allows one a doorway to study Lie groups. The foundation of Lie theory is the exponential map relating Lie algebras to Lie groups, known as the Lie group-Lie algebra correspondence. We briefly touched on this map earlier in section 3.3, dealing with Lie algebra automorphisms.

The question arises: what real world applications do Lie algebras possess? Classical Lie algebra representations have applications in physics. They can be used to study fluid dynamics. Symmetries in physics take on the structure of mathematical groups, where continuous groups represent Lie groups. Collisions of vector bosons in quantum field theory is tied to Lie algebra, specifically gauge theory, and as such, becomes a mechanism to interpolate physical interactions. The algebra can be promoted to a group and interpreted as a symmetry (gauge symmetry). Lie algebras are thus integral to describing theories of nature.

Bibliography

[Aig02] Papantonopoulou Aigli. *Algebra Pure and Applied.* Prentice Hall, New Jersey, 2002.

[Eve66] Howard Eves. *Elementary Matrix Theory.* Dover Publications, Inc, New York, 1966.

[FIS03] Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence. *Linear Algebra.* Pearson Education, Inc, New Jersey, 2003.

[Gal04] Joseph A. Gallian. *Contemporary Abstract Algebra.* Cengage Learning, New York, 2004.

[GG78] Morikuni Goto and Frank D. Grosshans. *Semisimple Lie Algebras.* Marcel Dekker, Inc, New York, NY, 1978.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York, 1972.

[MT03] Jerrold E. Marsden and Anthony J. Tromba. *Vector Calculus.* Wiley, New York, NY, 2003.

[Sam90] Hans Samelson. *Notes on Lie Algebras.* Springer-Verlag, New York, 1990.

[Wan75] Zhe-Xian Wan. *Lie Algebra.* Pergamon Press, Oxford, 1975.