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PLANAR GRAPHS, BIPLANAR GRAPHS AND GRAPH THICKNESS

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A graph is planar if it can be drawn on a piece of paper such that no two edges cross. The smallest complete and complete bipartite graphs that are not planar are $K_5$ and $K_{3,3}$. A biplanar graph is a graph whose edges can be colored using red and blue such that the red edges induce a planar subgraph and the blue edges induce a planar subgraph. In this thesis, we determine the smallest complete and complete bipartite graphs that are not biplanar.
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Chapter 1

Introduction

Within the branch of combinatorics exists a field of mathematics called graph theory. Graphs can be used to model many everyday circumstances, such as transportation networks and electrical circuits. Consider a map of a given region of the world. There exist man made cities with transportation pathways such as roads, freeways, or railways between the cities. Through graph theory, such a map can be modeled in a very simple way. A city could be represented by a circle, called a vertex, with lines, called edges, drawn between the circles to represent transportation pathways between different cities. Consider Figure 1.1.

![Figure 1.1: An example of a transportation network.](image)

Note that not every city is directly connected by a road to every other city, such as Cities 1 and 2. Thus to travel from one city to another, a person may need to take a path through multiple cities to get to their destination. In order to reach the destination
of City 2 from City 1, a person would need to travel to City 3 first then to City 2, or travel from City 1 to City 3 to City 4 to City 2. Another interesting observation is if transportation through City 3 was blocked, then City 1 would be, in a sense, isolated from Cities 2 and 4. In this way, graphs can be used to encode important information from a transportation network, giving us a relatively simplified perspective of an otherwise complicated system.

Graphs can also be used to model electrical circuits in a computer or even the electrical circuit within a single computer chip. In these cases we would model separate electrical components as circles with lines drawn between them to represent electrical pathways. This is a very similar model to that of the model of transportation networks. Consider Figure 1.2.

![Figure 1.2: Two models of the same circuit.](image)

Although both models are equally valid and encode the same information about components and electrical pathways, there exists a significant difference in structure. In the model on the left in Figure 1.2, the electrical pathway connecting Components 1 and 4 crosses the electrical pathway connecting Components 2 and 3. In contrast, the model on the right in Figure 1.2 does not contain any pathways that cross. This is a key structural difference that was not important within the transportation network. On a circuit board, if electrical pathways cross the communication between the components will not function properly. With this in mind, the graph on the right in Figure 1.2
would provide a better model for an electrical circuit. A graph that can be drawn on a piece of paper without having any two lines cross is called a planar graph. Modeling electrical circuits as planar graphs provides our motivation for studying planar graphs. In particular, we are interested in being able to determine precisely when a given graph is planar. For, if it is, such a graph could potentially serve as a model for an electrical circuit embedded on a circuit board.

Intuitively, an example may exist in which the number of electrical pathways is too great in comparison to the number of electrical components preventing us from placing the circuit on a single circuit board. We might view such a circuit as one that requires the use of the front and back sides of a circuit board to place the electrical pathways in a way that no two pathways cross. Modeling these more complex circuits would require graphs that can be decomposed in some way to two planar graphs. We call a graph biplanar when such a decomposition is possible. Hence, biplanar graphs model electric circuits which require the use of at most two sides of a circuit board. Figure 1 provides an example of a biplanar model of the planar model in Figure 1.2.

![Biplanar model of the electrical circuit in Figure 1.2.](image)

As electrical circuits become more complex there would become a point at which the graph model of the circuit would require even more than two sides of a circuit board. An electrical circuit of significant complexity may require more than two layers in order to connect all components in a prescribed way. The graph modeling such a circuit would then require a decomposition consisting of more than two planar graphs. The number of planar graphs required to model such a circuit is a parameter we will investigate in this thesis and is called the thickness of the graph that models the circuit.
Given a graph that serves as a model for an electrical circuit, determining the thickness of a graph will tell us the minimum number of layers needed in a computer chip in order to successfully build the circuit. Assuming more layers implies an increased cost in production, we are in some sense minimizing the cost of microchip production. This application is one factor that motivates the study of biplanar graphs.
Chapter 2

Graphs and Planar Graphs

2.1 Graphs

The concept of a graph grew from a problem Leonhard Euler had developed concerning the city of Königsberg.

The city of Königsberg, at the time when Euler contemplated it, was located in Prussia. The city was divided into four sections with seven bridges as seen in Figure 2.1. Euler’s original problem asked the inhabitants of Königsberg whether or not it was possible to walk around the city and cross each bridge exactly once. Euler was not concerned with the starting and ending point to be the same. Although the problem was quite simple, Euler needed to derive a new concept in order to generate a proof that would be rigorous enough to withstand mathematical scrutiny.

Thus Euler developed the graph as a tool to solve his problem. A graph $G$ is a
finite nonempty set $V$ of objects, called *vertices* together with a possibly empty set $E$ of 2-element subsets of $V$ called *edges*. Graphs are commonly viewed as drawings where the vertices are points or circles and the edges as lines occurring between two vertices. Two distinct vertices $u$ and $v$ are *adjacent* if the edge $\{u,v\}$ is contained in the edge set $E$ of $G$. Rather than denoting the edges as a two element subset of the vertex set, we will shorten the notation to the edge $uv$ for any two connected vertices $u$ and $v$.

**Example 2.1.** Let $G$ have the vertex set $V = \{v_1, v_2, v_3, v_4\}$ and the edge set $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_2, v_4\}\}$. The graph of $G$ can be drawn in the following way:

![Graph](image)

Euler constructed the graph in Figure 2.2 to model the bridges of Königsberg. The vertices represent the land masses separated by the river Pregel, and the edges represent the bridges. By developing some of the first theorems about graphs, Euler was able to give a rigorous proof that no such walk around Königsberg was possible. In particular, Euler proved that there was no such way to traverse the corresponding graph model unless there were either an even number of edges at all vertices, or there were exactly two vertices at which there were an odd number of edges.

Notation should be clarified at this point. Let $G$ be a graph with edge set $E$ and vertex set $V$. We will use $|E| = m$ to denote the number of elements in the edge set $E$. Additionally, $|V| = n$ will denote the number of elements in the vertex set $V$. Finally we will denote the number of 2-element sets of $E$ containing a vertex $v$ an an element as $\text{deg}(v)$. A graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ if $V'$ is a subset of $V$.
and $E'$ is a subset of $E$. Additionally for any vertex $v$ in $V$ but not in $V'$, no 2-element set of $E'$ can contain $v$. A graph $G' = (V', E')$ is a proper subgraph of a graph $G = (V, E)$ if $V'$ is a proper subset of $V$ and $E'$ is a proper subset of $E$ such that $E'$. Throughout this thesis we will denote the subgraph of $G = (V, E)$ obtained by removing a single edge $e$ as $G - e$. Additionally, a proper subgraph of $G$ may be obtained by removing a vertex $v$ from $V$ and removing all elements that include $v$ from $E$. This is denoted by $G - v$.

A graph is simple if its edge set has no repeated elements and every element of the edge set consists of two distinct vertices of the set $V$. Although there are other types of graphs, it is important to note that for the remainder of the thesis.

Since the graphs we study here are simple there exists a maximum number of edges a graph could have. A graph $G$ is a complete graph if for any two distinct vertices $v, u \in V$ then the edge $\{v, u\}$ is contained in the edge set $E$. A complete graph with $n$ distinct vertices is denoted $K_n$. Additionally, for a complete graph $K_n$, it is important to note there are exactly $n(n - 1)/2$ edges and $\text{deg}(v) = n - 1$ for ever $v \in V$.

A graph $G$ is complete bipartite if the vertex set $V$ can be partitioned into two parts $V_1$ and $V_2$ such that the edge $\{u, w\} \in E$ if and only if $u$ and $w$ are in different parts. Complete bipartite graphs are denoted by $K_{a,b}$ where $|V_1| = a$ and $|V_2| = b$. Additionally, for a complete bipartite graph $K_{a,b}$ has exactly $a \cdot b$ edges.

It is important to properly define complete and complete bipartite graphs because they play a vital role throughout graph theory and especially within this thesis.

### 2.2 Planar Graphs

As mentioned in Chapter 1, it may be possible to draw a graph on a piece of paper in such a way that the edges do not cross, although this is not always possible. A graph $G$ is planar if there exists a way to draw the graph without edges crossing. Such a drawing is called a planar embedding of $G$.

A path in a graph $G$ is a sequence of vertices $(v_1, v_2, ..., v_n)$ such that $v_i$ and $v_{i+1}$ are adjacent for $1 \leq i \leq n$ and each vertex may only be only appear once in the sequence. Such a path starting at a vertex $v_i$ and terminating at a vertex $v_j$ is called a $v_i, v_j$-path. A cycle in a graph $G$ is a path that begins and ends at the same vertex.

A graph $G$ is connected if between any two vertices $u, v \in V$, there exists a $u, v$-path. A graph $G$ has components if $G$ is disconnected. That is, $G$ is disconnected if
there exist at least two proper subgraphs \( H_1 \) and \( H_2 \) of \( G \) such that for any \( v \in H_1 \) and \( u \in H_2 \) then there does not exist a \( v, u \)-path.

Let \( G = (V, E) \) be a graph with \( \kappa \) components. The graph \( G \) contains a cut vertex if there exists a vertex \( v \) such that the number of components in the proper subgraph \( G - v \) increases. A graph \( G \) is \( n \)-connected if \( n \) vertices must be removed in order to increase the number of components of \( G \).

In order to better understand the introduced definitions, consider the following example:

![Figure 2.3: A simple graph.](image)

Between any two vertices \( i \) and \( j \) in Figure 2.3 there exists an \( i, j \)-path. For example there exists a 1,7-path by traveling along the edges \{1, 3\},\{3, 4\},\{4, 7\}. Thus, this graph is connected and has exactly one component. Additionally Vertex 4 is a cut vertex in \( G \) since \( G - 4 \) results in components \( G_1 \) with the vertex set \( V_1 = \{1, 2, 3\} \) and \( G_2 \) with the vertex set \( V_2 = \{5, 6, 7\} \). Since \( G \) contains a cut vertex, \( G \) is only 1-connected.

The simplest form of a planar graph is a tree or a forest. A graph \( G \) is a tree if it does not contain any cycles. A graph \( G \) is a forest if each component of \( G \) is a tree. A region of a graph is a section of a graph that is enclosed by a cycle. Additionally there exists an unbounded region on the exterior of the graph. The vertices and edges that create the cycle enclosing the region are said to be incident with the region.

**Theorem 2.2** (Euler’s Polyhedral Formula). Let \( G \) be a connected graph with \( n \) vertices, \( m \) edges, and \( r \) regions. The graph \( G \) obeys the equality \( n - m + r = 2 \).
Proof. We proceed by induction on $m$. Let $m = 0$. Since $G$ is a connected graph, $n = 1$ and since the only region of $G$ that exists is the exterior region, then $r = 1$. We obtain the equality $1 - 0 + 1 = 2$ and so the formula holds for $m = 0$.

Let $m = 1$. The only edge must be between two vertices. Thus $n = 2$. Since $G$ can not contain any cycles, then the only region is the exterior region. Thus $r = 1$. We obtain the equality $2 - 1 + 1 = 2$ and so the formula holds for $m = 1$.

Now consider $m \geq 2$. The graph $G$ is either a tree or contains a cycle. If $G$ is a tree, then $m = n - 1$ and since the only region is the exterior region then $r = 1$. The formula holds true since $n - (n - 1) + 1 = 2$. If $G$ contains a cycle, then there exists an edge $e$ such that $e$ is incident with two regions. Consider the subgraph $G - e$ having $n'$ vertices, $m'$ edges and $r'$ regions. The new graph of $G - e$, $n' = n$, $m' = m - 1$ and $r' = r - 1$. By the induction hypothesis, $m' \leq m$ so the equality $n' - m' + r' = 2$ holds true. We obtain $n - (m - 1) + (r - 1) = 2$ implies $n - m + r = 2$ is true. That is, Euler’s polyhedral formula holds for $G$, and so holds in general.

Euler’s Polyhedral Formula is the first important tool in which to study planar graphs. Through this identity, an upper bound on the number of edges in a planar graph $G$ with $n$ vertices can be found.

**Theorem 2.3.** If $G$ is a planar graph with $n \geq 3$ vertices and $m$ edges then $m \leq 3n - 6$.

Proof. Let $G$ be a planar graph. Since a simple graph with three vertices cannot have more than three edges the result holds when $n = 3$. Suppose $n \geq 4$. Since $G$ is a planar graph than it obeys Euler’s Identity, having $n$ vertices, $m$ edges and $f$ faces, $n - m + f = 2$. Since every region of $G$ is composed of at least 3 edges, and every edge is incident with exactly two regions we see that inequality $3r \leq 2m$ holds true. By multiplying both sides of Euler’s Polyhedral Identity we see that $6 = 3n - 3m + 3r$. Since we know that $3r \leq 2m$, then $3n - 3m + 3r \leq 3n - m$. By combining the equations we obtain $6 \leq 3n - m$ or $m \leq 3n - 6$.

A graph $G$ is maximal planar if $G$ is planar and the addition of an edge $e$ to the edge set $E$ of $G$ produces a nonplanar graph.
Corollary 2.4. If $G$ is a maximal planar graph with $n \geq 3$ vertices and $m$ edges, then $m = 3n - 6$

We will now establish some lemmas for maximal planar graphs that we will utilize later on in chapter 3.

Lemma 2.5. Let $G$ be a maximal planar graph. Every region of $G$ is of length three.

Proof. Let $G$ be a maximal planar graph. Suppose the $G$ had a region $R$ of length greater than three. Since $R$ is bounded by a cycle, then there exists a path $P$ of length three. Let $v_1, v_2$ and $v_3$ be the vertices of $P$. At least two among $v_1, v_2$ and $v_3$ are not adjacent or else $P$ is part of a cycle of length three. Without loss of generality let $v_1$ and $v_3$ not be adjacent. Adding the edge $\{v_1, v_3\}$ to the edge set of $G$, $G$ would still be planar. But this is a contradiction to the definition of $G$ being maximal planar.

\[\square\]

Lemma 2.6. Let $G$ be a maximal planar graph of order $n \geq 4$

Proof. Let $G = (V, E)$ be a maximal planar graph. Suppose there exists a vertex $v$ such that $\deg(v) \leq 2$. Let $\deg(v) = 0$ than clearly the edge $\{v, u\}$ can be added to $G$ for any vertex $u$ distinct from $v$ and the resulting graph will still be planar. Hence $G$ is not maximal planar which is a contradiction. Suppose $\deg(v) = 1$. Thus $v$ is adjacent to exactly one vertex, say $u \in V$. Then $u$ must be adjacent to some other vertex say $w$. Then the edge $\{v, w\}$ can be added to $G$, since $u$ and $w$ are adjacent, adding the edge $\{v, w\}$ would create a $u, w$-path through $v$ which is homeomorphic to the edge $\{u, w\}$ [GM12]. Therefore $G$ is not maximal planar which is a contradiction.

Suppose $\deg(v) = 2$. Let $v_i$ and $v_j$ be the vertices adjacent to $v$. At most $v_i$ and $v_j$ are adjacent. There must exists a face containing $v, v_i$ and $v_j$ plus at least another vertex. Then there exists some vertex $v_k$ adjacent to $v_i$ or $v_j$. Then the edge $v_i, v_k$ can be added to $E$. Thus $G$ is not maximal planar. Thus for a maximal planar graph, every vertex must be of at least degree three.

\[\square\]

Lemma 2.7. Let $G$ be a maximal planar graph with a vertex $v$ of degree three. Let $v_1, v_2$ and $v_3$ be the three vertices adjacent to $v$. Then $v_1, v_2$ and $v_3$ are pairwise adjacent.
Proof. Let $G$ be a graph with a vertex $v$ of degree three. Suppose at least two of the three vertices are not adjacent say $v_1$ and $v_2$. Then the vertices $v, v_1$ and $v_2$ are all part of the same region. Since $G$ is maximal planar, by Lemma 2.5 every region is a cycle of length three. Then $v, v_1$, and $v_2$ are a cycle of length three. Thus $v_1$ and $v_2$ are adjacent which is a contradiction. Therefore all three vertices adjacent to $v$ are pairwise adjacent.

\[\square\]

**Theorem 2.8.** $K_5$ is not planar

**Proof.** Suppose that $K_5$ is planar. Then $K_5$ has exactly five vertices and ten edges. Then $K_5$ satisfies the inequality $m \leq 3n - 6$. But $10 \leq 9$, which is false. Thus $K_5$ is not planar. \[\square\]

We will now prove that $K_{3,3}$ is not planar. However, $K_{3,3}$ has six vertices and nine edges which does not contradict Theorem 2.3. Hence we will need to employ a different technique.

**Theorem 2.9.** $K_{3,3}$ is not planar.

**Proof.** Suppose $K_{3,3}$ is planar. We obtain the equality, by Euler’s Polyhedral Formula, $n - m + r = 2$ where $n = 6$ and $m = 9$. It follows that $r = 5$ regions. Additionally, since $K_{3,3}$ contains no cycles of length three, then each region must be incident with at least four edges. By summing the edges over each face, we count each edge twice and each face at least four times. Hence, $4r \leq 2m$. Since $r = 5$ and $m = 9$, we obtain $20 \leq 18$, a contradiction. Thus $K_{3,3}$ is cannot be planar. \[\square\]

Since neither $K_5$ nor $K_{3,3}$ are planar, it is clear that any graph $G$ that contains either $K_5$ or $K_{3,3}$ as a subgraph, cannot be planar. Thus, we have a necessary but not sufficient condition for a graph to be planar.

A graph $H$ is a subdivision of a graph $G$ if $H$ is obtained from $G$ by replacing any of the edges of $G$ by arbitrarily long paths. Two graphs $H$ and $H'$ are homeomorphic if there is some graph $G$ from which $H$ and $H'$ are each obtained from a series of subdivisions.

**Theorem 2.10.** Let $G$ be a plane graph. Then for any region $R$ having boundary $B$, there exists an embedding such that $B$ is on the boundary of the exterior region.
Proof. Let $G$ be a given plane graph with regions $R_1, R_2, \ldots, R_n$. Then the boundary of a region $R_i$ is the collection of edges incident with the region. Let $B_1, B_2, \ldots, B_n$ be the boundaries of $R_1, R_2, \ldots, R_n$, respectively. We construct a planar embedding of $G$ having $B_1$ on the boundary of the exterior region. We may assume $R_1$ is not already the exterior region otherwise we are done. If $G$ is not connected, we construct the desired embedding working only in the component of $G$ containing $R_1$. Thus we lose no generality in assuming $G$ is connected. We begin by embedding the boundary $B_1$ in the plane. Next we identify a region $R_i$ with boundary $B_i$ that is incident with at least one vertex of $B_1$. We then embed the boundary $B_i$ on the interior of boundary $B_1$. Next we identify a region $R_j$ with boundary $B_j$ which is incident to at least one vertex of either $R_1$ or $R_i$ and embed the boundary $B_j$ on the interior of $B_1$.

If at some point in this process we are unable to embed any additional boundaries on the interior of $B_1$, then either no remaining boundaries have a vertex in common with any previously embedded boundaries or no matter how we try to embed another boundary $B_k$ on the interior of $B_1$, planarity is violated. In the first case, we have a collection of boundaries that are disjoint from embedded boundaries. Then $G$ must be disconnected, which is a contradiction to our assumption that $G$ is connected. In the second case, since every edge of $G$ that is not a bridge is incident with exactly two regions, all remaining boundaries can only share edges with boundaries that are not already in common to two embedded boundaries. Thus the set of edges that are contained in exactly one embedded boundary bound a region $R$ in the current partial embedding of $G$. It follows that all remaining boundaries must be on the interior of $R$. Hence, any of the remaining boundaries may be embedded without violating planarity.

From Theorem 2.11, it is easy to see that for any vertex $v$ or edge $e$ of a planar graph $G$, there exists a planar embedding of $G$ such that $v$ or $e$, respectively, is may also be positioned incident to the exterior region. This fact allows us to construct new planar graphs from existing planar graphs. Consider two planar graphs $G_1$ and $G_2$ with given planar embeddings. We construct a planar gluing by identifying either a vertex or edge in each of $G_1$ and $G_2$, making the edge or vertex incident with the exterior region in each graph and taking the union of the planar embeddings of $G_1$ and $G_2$ by overlapping the planar embeddings precisely at the vertex or edge. Through planar gluing we are able to
construct larger planar graphs by gluing smaller planar graphs together.

**Theorem 2.11** (Kuratowski’s Theorem). A graph $G$ is planar if and only if $G$ contains no subgraph that is homeomorphic to $K_5$ or $K_{3,3}$

**Proof.** By Theorem 2.5 and Theorem 2.6, if $G$ is a planar graph, then $G$ cannot contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$. Thus we will verify that if a graph contains no subgraph homeomorphic to $K_5$ or $K_{3,3}$, then $G$ is planar.

To the contrary, suppose there exists a nonplanar graph that contains no subgraph homeomorphic to $K_5$ or $K_{3,3}$. We may assume that $G$ is an example of such a graph that is minimal with respect to the cardinality of its edge set. We lose no generality in assuming $G$ is connected, for if $G$ is disconnected, our argument applies to each connected component of $G$. If $G$ is not 2-connected, then there exists a vertex $v$ in $G$ whose deletion results in a disconnected graph. Let $C_1$ be a component of $G - v$ and let $C_2$ be the union of the remaining components of $G - v$. By the minimality of $G$, we see that the subgraph of $G$ obtained by deleting all the vertices of $C_1$ is planar. Similarly, the subgraph of $G$ obtained by deleting all the vertices of $C_2$ is planar. That is, the induced subgraph $G_1$ on $V(C_1) \cup \{v\}$ is planar, as is the induced subgraph $G_2$ on $V(C_2) \cup \{v\}$.

Thus, we can construct a planar gluing of $G_1$ and $G_2$ at the vertex $v$ to produce $G$. As $G$ is nonplanar, this contradiction implies $G$ must be 2-connected. We now show $G$ is 3-connected.

Suppose $G$ is not 3-connected. Then there exist two vertices $v_1$ and $v_2$ such that $G - \{v_1, v_2\}$ is a disconnected graph. Consider the partition $\{H_1, H_2\}$ of the edge set of $G$ such that $H_1$ is one of the components of $G - \{v_1, v_2\}$ and $H_2$ the union of the remaining components of $G - \{v_1, v_2\}$.

Suppose that $v_1$ and $v_2$ are adjacent. Consider the partitions $F_1 = H_1 \cup \{v_1, v_2\}$ and $F_2 = H_2 \cup \{v_1, v_2\}$. By the minimality of $G$, each of $F_1$ and $F_2$ induce planar subgraphs of $G$. Thus we can construct a planar gluing of $F_1$ and $F_2$ at $v_1v_2$ producing the graph $G$. Thus $G$ is planar, a contradiction.

Now suppose $v_1$ and $v_2$ are not adjacent. Since $G$ is nonplanar, the graph $G' = G \cup v_1v_2$ is also nonplanar. Moreover, since $G - \{v_1, v_2\}$ is disconnected, so is $G' - \{v_1, v_2\}$. Let $H_1$ be a connected component of $G - \{v_1, v_2\}$ and let $H_2$ be the collection of the remaining components of $G - \{v_1, v_2\}$. Let $F_1 = (G - H_2) + v_1v_2$ and $F_2 = (G - H_1) + v_1v_2$ be subgraphs of $G'$. Since $G'$ is not planar, either $F_1$ or $F_2$ is not...
planar. Indeed, suppose both $F_1$ and $F_2$ are planar, then we may construct an embedding of $F_1$ with $v_1v_2$ on the exterior region by Theorem 2.11 and also construct an embedding of $F_2$ with $v_1v_2$ as an exterior region. Hence we may assume $F_1$ is not planar.

Since $F_1 - v_1v_2$ is a proper subgraph of $G$ then, by minimality, $F_1 - v_1v_2$ is planar. Since $G$ is 2-connected, then there exists some $v_1, v_2$-path contained $F_2$ other than the edge $v_1v_2$. Thus, in $G$ there exists a subgraph homeomorphic to $F_1$. Therefore $G$ has a subgraph that is not planar, contradicting the minimality of $G$. We conclude that $G$ is 3-connected.

The minimality of $G$ guarantees that there exists an edge $e = uv$ such that the subgraph $H = G - e$ is planar and 2-connected. For the remainder of the proof, we assume $H$ is embedded in the plane. Since $H$ is 2-connected, there exists a cycle in $H$ containing $u$ and $v$. Among all such cycles, let $C$ be one of maximum length. We let $C = v_0v_1v_2...v_kv_0$, and we may assume $u = v_0$ and $v = v_l$ for some $l$ with $0 \leq l \leq k$.

We will call an $x,y$-path in $G$ that only has vertices $x$ and $y$ in common with $C$ an $x,y$-chordal path of $C$. If for all $s$ and $t$ with $0 < s < l$ and $l < t \leq k$, there does not exist a $v_s, v_l$-chordal path of $C$ in $H$, then $e$ can be added to $H$ while preserving planarity, a contradiction. Therefore, there must exist such a $v_s, v_l$-path in $H$. We illustrate this structure in Figure 2.2.

Figure 2.4: A picture of the subgraph $H$. 
We will now develop a case analysis with the cases being determined by where cordial paths terminate on \( C \).

Case 1: Let the \( v_a, v_b \)-chordal path of \( C \) exist such that neither \( v_a \) nor \( v_b \) are any of \( v_0, v_s, v_l \) or \( v_t \). Suppose \( v_a \) and \( v_b \) were between \( v_0 \) and \( v_l \). Then the \( v_a, v_b \)-chordal path could be embedded parallel to both the \( v_0, v_l \) path and the edge \( e \). Thus such a path would not cross the edge \( e \) and so \( e \) could be added to \( G \) preserving planarity. Similarly, if \( v_a \) and \( v_b \) are between the vertices from \( v_l \) for \( l < i \leq k \), then we again are able to generate a planar embedding of \( G \).

Suppose \( v_a \) and \( v_b \) are between the vertices from \( v_i \) for \( s < i < t \). Such a \( v_a, v_b \)-chordal path would be able to be drawn on the exterior of our cycle in such a way that it was parallel to the cycle. Thus such an embedding of the chordal path would not affect the planarity of \( G \). Similarly, if \( v_a \) and \( v_b \) are between the vertices from \( v_i \) for \( t < i \leq k \) or \( 0 < i < s \) then the \( v_a, v_b \)-chordal path could be placed parallel to the cycle and the \( v_s, v_l \)-path in such a way to not affect the planarity of \( G \) when \( e \) is embedded.

Thus finally, without lost in generality consider if \( v_a \) is one of the vertices \( v_i \) such that \( 0 < i < s \) and \( v_b \) is one of the vertices \( v_j \) for \( l < j < t \). Such a \( v_a, v_b \)-chordal path must intersect the embedding of the edge \( e \) as seen in Figure 2.5. Thus if such a chordal path existed \( G \) is not planar. But such a graph \( G \) contains a subgraph homeomorphic to \( K_{3,3} \). This is a contradiction to our assumption \( G \) has no subgraph homeomorphic to neither \( K_{3,3} \) nor \( K_5 \). Thus such a chordal path can not exist. A similar argument holds for if \( v_a \) was one of the vertices \( v_i \) such that \( s < i < l \) and \( v_b \) was one of the vertices \( v_j \) for \( t < j < k \).

Case 2: Let the \( v_a, v_b \)-chordal path of \( C \) exist such that one of \( v_a \) or \( v_b \) is either \( v_0, v_s, v_l \) or \( v_t \). Without lost of generality let \( v_a \) be \( v_0 \), and let \( v_b \) be one one the cycle vertices but not \( v_s, v_l \), or \( v_t \). Regardless of our choice of \( v_b \), we are able to embed the \( v_a v_b \) chordal path in such a way that it is parallel to \( e \) and thus does not intersect it. Such a chordal path results in \( G \) being planar which is a contradiction to our assumption that \( G \) is not planar. So consider if \( v_b \) was one of the vertices \( v_i \) such that \( l < i < t \) and there existed a vertex \( v_c \) such that \( s < c < l \). Then we know from Case 1 a \( v_b, v_c \)-chordal path is planar. Then consider if the \( v_a, v_b \)-chordal path and the \( v_b v_c \) chordal path shared at least a vertex \( w \). We are then no longer able to embed the chordal paths in such a way that \( G \) is planar as seen in Figure 2.6. But such chordal paths result in \( G \) having a
subgraph that is homeomorphic to $K_{3,3}$ which is a contradiction to our assumption $G$ has no such subgraph. Regardless of our choices for $v_a$ and $v_b$, we are always able to identify a $v_c$ such that this argument holds.

Case 3: Let the $v_a,v_b$-chordal path of $C$ exist such that $v_a$ is either $v_0, v_s, v_l$ or $v_t$ and $v_b$ is either $v_0, v_s, v_l$ or $v_t$ but not the same as $v_a$. Without lost of generality let $v_a$ be $v_s$ and $v_b$ be $v_t$. If this was the only $v_a,v_b$-chordal path in $G$ then we would be able to embed the chordal path parallel to the $v_s,v_t$-path in such a way that the chordal path does not intersect the edge $e$ and thus $G$ would be planar, which is a contradiction to our assumption that $G$ is not planar. A similar argument holds true for if $v_a$ is $v_0$ and $v_b$ is $v_l$. Thus consider if two such chordal paths existed the first being a $v_0,v_l$-chordal path and the other being a $v_s,v_t$-chordal path. Since we know if the chordal paths do not intersect we are able to embed them in such a way that $G$ is planar, let the two chordal paths intersect.

Suppose the $v_0,v_l$-chordal path and the $v_s,v_t$-chordal path intersect and share at least two common vertices. As demonstrated in Figure 2.7, such a structure must in fact intersect some part of $G$, mainly it must intersect with some part of the edge $e$. Thus $G$ is not planar. But, $G$ would have a subgraph that is homeomorphic to $K_{3,3}$ which is a contradiction to our assumption that $G$ has no such subgraph. Thus suppose that $v_0,v_l$-chordal path and the $v_s,v_t$-chordal path intersect at exactly one vertex, namely $w$. The structure would again intersect $e$ and thus $G$ is not planar as seen in Figure 2.8. But $G$ would have a subgraph that is homeomorphic to $K_5$ which is a contradiction to our assumption that $G$ has no such subgraph.

From the case analysis, it becomes clear that with any such chordal path, either we are able to embed the chordal path in such a way that $G$ is planar, or anytime $G$ is not planar, $G$ has a subgraph that is homeomorphic to either $K_{3,3}$ or $K_5$. Thus since no such chordal path exists then there is no such graph $G$. 

\[\square\]
Figure 2.5: Structure of Case 1 of Kuratowski’s Theorem.

Figure 2.6: Structure of Case 2 of Kuratowski’s Theorem.
Figure 2.7: Structure of Case 3 of Kuratowski’s Theorem.

Figure 2.8: Structure of Case 4 of Kuratowski’s Theorem.
Chapter 3

Biplanar Graphs

Our study of biplanar graphs is inspired by the model of an electrical circuit that was given in Chapter 1, in which it was necessary to utilize two sides of a chip. A graph \( G = (V, E) \) is biplanar if there exists a partition of \( E \) into two parts \( E_1, E_2 \) such that subgraphs \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) are planar graphs. If a graph \( G \) is biplanar then we will state that \( G \) has a biplanar decomposition into the subgraphs \( G_1 \) and \( G_2 \). The ultimate goal in our study of biplanar graphs is to provide a characterization of forbidden subgraph analogous to Kuratowski’s Theorem. Unfortunately, there exist examples of non-biplanar graphs that are homeomorphic to biplanar graphs (See Corollary 3.10). Hence, a completely analogous characterization is impossible. With this in mind, we dedicate this chapter to an investigation of the smallest complete and complete bipartite graphs that are not biplanar.

The goal of this chapter is to prove the following Theorems.

**Theorem 3.1.** The graph \( K_9 \) is the smallest complete graph that is not biplanar.

**Theorem 3.2.** The graphs \( K_{7,7} \), \( K_{6,9} \) and \( K_{5,12} \) are the smallest complete bipartite graphs that are not biplanar.

### 3.1 Complete Biplanar Graphs

We will first show that the complete graph \( K_9 \) is the smallest complete graph that is not biplanar.

**Lemma 3.3.** The complete graph \( K_8 \) is biplanar.
Proof. As seen in Figure 3.1 $K_8$ has a biplanar decomposition.

![Figure 3.1: A biplanar decomposition of $K_8$.](image)

In order to show that $K_9$ is not biplanar, then we will use Lemmas, 2.5, 2.6 and 2.7 from chapter two. The original inspiration for the proof can be found here [BHK62].

Lemma 3.4. The complete graph $K_9$ is not biplanar.

Proof. Suppose that $K_9$ is biplanar. Then $K_9$ has a biplanar decomposition into $G = (V, E_1)$ and $\overline{G} = (V, E_2)$. We first show that both $G$ and $\overline{G}$ must be connected. Suppose, to the contrary, that either $G$ or $\overline{G}$ is not connected. With no loss in generality, we may assume $\overline{G}$ is not connected. Consider the following cases determined by the number of components in $\overline{G}$:

Case 1: Suppose $\overline{G}$ has at least four components. Then there exists a partition of the components into two parts $P_1$ and $P_2$ such that each part has at least three vertices. Then, in $G$, every vertex in $P_1$ is adjacent to every vertex in $P_2$. Since both parts are of size at least three, $G$ has a $K_{3,3}$ subgraph. By Kuratowski’s Theorem, $G$ is not planar, which is a contradiction to our assumption that $K_9$ is biplanar.

Case 2: Suppose $\overline{G}$ has exactly three components $C_1$, $C_2$ and $C_3$. If each component has more than one vertex, then we can partition the components into two parts $P_1$ and $P_2$ such that each part has at least three vertices. Since each of $P_1$ and $P_2$ have at least three vertices, $G$ has a $K_{3,3}$ subgraph. By Kuratowski’s Theorem, $G$ is not planar, which is a contradiction to our assumption that $K_9$ is biplanar.
Now suppose that exactly one component, say \( C_1 \), of \( \overline{G} \) contains exactly one vertex. Further assume that \( C_3 \) has at least as many vertices as \( C_2 \). Since there are exactly nine vertices, it follows that \( C_3 \) must have at least three vertices. Consider a partition of vertices of \( \overline{G} \) into two parts \( P_1 \) and \( P_2 \), such that \( P_1 \) contains the vertex of \( C_1 \) and vertices of \( C_2 \), and \( P_2 \) contains the vertices of \( C_3 \). Since both \( P_1 \) and \( P_2 \) contain at least three vertices, \( G \) has a \( K_{3,3} \) subgraph. By Kuratowski’s Theorem, \( G \) is not planar which again is a contradiction to our assumption that \( K_9 \) is biplanar.

Finally, suppose that two components, say \( C_1 \) and \( C_2 \), have exactly one isolated vertex. Let \( C_3 \) contain the remaining vertices. This is actually a proper subgraph of Case 3 in which the vertices in \( C_1 \) and \( C_2 \) would be adjacent. So instead consider Case 3.

Case 3: Suppose \( \overline{G} \) has exactly two components. Let \( C_1 \) and \( C_2 \) be the components of \( \overline{G} \) and assume \( C_2 \) has at least as many vertices as \( C_1 \). Suppose \( C_1 \) has at least three vertices in it, then it follows \( C_2 \) must also have at least three vertices. Then, \( G \) contains a \( K_{3,3} \). By Kuratowski’s Theorem, \( G \) is not planar, which is a contradiction to our assumption that \( K_9 \) is biplanar.

Now suppose \( C_1 \) has exactly two vertices, say \( v_1 \) and \( v_2 \). Then, in \( G \), \( v_1 \) and \( v_2 \) are not adjacent but are both adjacent to all the vertices \( v_3, ..., v_9 \). In order to minimize the edges in \( \overline{G} \), we will let \( G \) be maximal. It follows that \( v_3, ..., v_9 \) are degree at most four. If \( v_i, 3 \leq i \leq 9 \) is degree at least 5 then \( G \) contains a \( K_{3,3} \) as a subgraph making \( G \) not planar by Kuratowski’s Theorem which is a contradiction to our assumption \( K_9 \) is biplanar. Thus the vertices \( v_3, ..., v_9 \) form a cycle. Then \( \overline{G} \) contains the graph as in Figure 3.2. Thus \( \overline{G} \) contains a subdivision of \( K_5 \) and is not planar by Kuratowski’s Theorem, which is a contradiction to our assumption that \( K_9 \) is biplanar.

Suppose \( C_1 \) has exactly one vertex, say \( v_1 \). In order to reduce the number of edges in \( \overline{G} \), suppose that \( G \) is maximal planar. Then, by Corollary 2.4, \( G \) has exactly \( 3n - 6 = 21 \) edges. Since we know eight of the edges are incident with \( v_1 \) there exist thirteen edges between \( v_2, ..., v_9 \). In order for \( \overline{G} \) to be planar, by Theorem 2.3, it must have \( m \leq 3n - 6 \) edges. Since we know \( v_1 \) is an isolated vertex, all edges must be between \( v_2, ..., v_9 \) except for the thirteen edges in \( G \). Thus \( \overline{G} \) must have less than \( m \leq 3(8) - 6 = 18 \) edges. But \( \overline{G} \) must have all the remaining edges except the thirteen in \( G \) and thus has \( 8^2/2 - 13 = 19 \) edges. Thus \( \overline{G} \) is not planar by Theorem 2.3 which is a contradiction to our assumption that \( K_9 \) is biplanar.
Figure 3.2: Structure of G in Case 3.1.

From Cases 1, 2, and 3, it is clear that G and $\overline{G}$ must both be connected if $K_9$ biplanar. Let $d_1, d_2, ..., d_9$ be the degrees of the vertices of $G$ such that $d_1 \leq d_2 \leq ... \leq d_9$. With no loss in generality, we may assume $G$ is maximal planar so that the cardinality of $E_2$ is minimized.

If any vertex has degree eight in $G$, then that vertex must be an isolated vertex in $G$. Since $G$ is connected, this is impossible. Additionally since $G$ is maximal planar, by Lemma 2.6, every vertex has degree at least three in $G$. Thus $3 \leq d_i \leq 7$ for $i = 1, 2, ..., 9$.

In the case analysis that follows, we narrow down the possibilities of the degree sequence for $G$. In Case 1, we will show no two vertices may have degree seven. In case two we show that there must exist at most one vertex of degree three. In case three we show there must exist at most one vertex of degree four. Then finally, we show the only degree sequence possible does not have a corresponding planar graph.

Case 1: Suppose that $G$ has a degree sequence such that $d_1 = d_2 = 7$. Let the two vertices of degree seven be labeled $v_1$ and $v_2$. Suppose $v_1$ and $v_2$ are not adjacent in $G$. Then $v_1$ and $v_2$ are adjacent in $\overline{G}$. Then $v_1$ and $v_2$ together with the edge $\{v_1, v_2\}$ are a component in $\overline{G}$. Thus $\overline{G}$ has two components which is a contradiction to $\overline{G}$ being connected. Thus $v_1$ and $v_2$ are adjacent in $G$.

Case 1.1: Suppose there exists a vertex $v_9$ in $G$ that is not adjacent to $v_1$ and is not adjacent to $v_2$. Then since $v_1$ and $v_2$ are adjacent in $G$ and of degree seven in $G$, then $v_1$ and $v_2$ are adjacent to vertices $v_3, v_4, ..., v_8$. Suppose that $v_9$ was adjacent to three
or more of $v_3, v_4, ..., v_8$. Then the graph $G$ contains a $K_{3,3}$ subgraph since $v_1, v_2$ and $v_9$ are adjacent to at least three common vertices. Therefore, $v_9$ must have degree less than three. But this contradicts Lemma 2.6 and thus there every vertex must be adjacent to either $v_1, v_2$, or both.

Hence there must exist two vertices $v_3$ and $v_4$ such that $v_3$ is adjacent to $v_1$ but not $v_2$ in $G$ and $v_4$ is adjacent to $v_2$ but not $v_1$ in $G$. Suppose that either $v_3$ or $v_4$ has degree greater than four. We may assume $v_3$ is this vertex. Then $v_3$ is adjacent to $v_2$ and at least four other vertices. Thus $v_3$ is adjacent to at least three vertices among $v_5, v_6, ..., v_9$. We may assume that $v_3$ is adjacent to $v_5, v_6$ and $v_7$. It follows $G$ contains a $K_{3,3}$ subgraph having a partition sets $\{v_1, v_2, v_4\}$ and $\{v_5, v_6, v_7\}$. We may conclude that $v_3$ and $v_4$ have degree at most four.

Suppose $v_4$ has degree four. Since $v_4$ is adjacent to $v_2$ and not $v_1$, then $v_4$ is adjacent to three other vertices among $v_3, v_5, v_6, ..., v_9$. If $v_4$ is adjacent to three vertices among $v_5, v_6, ..., v_9$, then $G$ once again contains a $K_{3,3}$ subgraph. This implies that $G$ would not be planar by Kuratowski’s Theorem which is a contradiction to our assumption that $K_9$ is biplanar. We may conclude that $v_3$ and $v_4$ are both adjacent to $v_5$ and $v_6$.

Suppose that $v_3$ is either not adjacent to $v_5$ or not adjacent to $v_6$. We may assume $v_3$ is not adjacent to $v_5$. Then $v_3, v_4, v_5$ are incident with some region $R$. But since $v_3$ and $v_5$ are not adjacent then there exists some vertex $v_i$ also a incident with $R$ for $i \in 7, 8, 9$. Thus $R$ is a region of length at least four in the maximal planar graph $G$. But by Lemma 2.5, every region of a maximal planar graph is of length three. With this contradiction, we conclude that $v_3$ and $v_4$ are both adjacent to $v_5$ and $v_6$. Figure 3.1 illustrates the structure present in $G$ within our subcase.

Suppose that none of $v_7, v_8$ and $v_9$ are adjacent to $v_5$. In $G$, none of $v_7, v_8$ and $v_9$ are not adjacent to $v_3$ or $v_4$ since each of $v_3$ and $v_4$ are degree four. Then in $\overline{G}$, the vertices $v_3, v_4$, and $v_5$ are adjacent to all the vertices $v_7, v_8$, and $v_9$ creating a $K_{3,3}$ subgraph in $\overline{G}$. Thus, by Kuratowski’s Theorem, $\overline{G}$ is not planar which is a contradiction to our assumption that $K_9$ is biplanar. A similar argument holds when $v_7, v_8$ and $v_9$ are all not adjacent to $v_6$.

The other possible case is if there exists a path $P$ of length three consisting of two vertices among $v_7, v_8$ and $v_9$ with one of the end points being either $v_5$ or $v_6$. Without
loss of generality, let $P = v_9v_8v_5$. Then $v_7$ must be adjacent to $v_6$. Then we obtain the graph in Figure 3.4.

In $\overline{G}$ the vertices $v_3$ and $v_4$ are adjacent to $v_7, v_8$ and $v_9$. Additionally, $v_6$ is adjacent to $v_8$ and $v_9$. But $v_5$ must be adjacent to $v_6$ and $v_7$. Then through a subdivided edge, $v_6$ is adjacent to $v_7$. Thus $\overline{G}$ contains a subgraph homeomorphic to $K_{3,3}$ making $\overline{G}$ and thus, by Kuratowski’s Theorem, $G$ nonplanar which is a contradiction to our assumption that $K_9$ is biplanar. Note that regardless of choices for the vertices for $P$, the same argument can be applied. The degree of vertices $v_3$ and $v_4$ must be equal to three.
Since both $v_3$ and $v_4$ are of degree three, then by Lemma 2.7, $v_3$ and $v_4$ may have exactly zero, one or two neighbors in common. These three possibilities are the following cases.

Case 1.4: Let $v_3$ and $v_4$ be of degree three and be adjacent to exactly zero common vertices. Then both $v_3$ and $v_4$ are adjacent to four distinct vertices, say $v_5, v_6, v_7$ and $v_8$. Let $v_3$ be adjacent to $v_5$ and $v_6$, and let $v_4$ be adjacent to $v_7$ and $v_8$. Then $v_9$ is not adjacent to $v_3$ or $v_4$. Additionally, $v_9$ is adjacent to either one or two of $v_5, v_6, v_7$, or $v_8$.

Suppose that $v_9$ is adjacent to one of $v_5, v_6, v_7$, or $v_8$, say $v_5$. Then $v_9, v_5, v_3$ are not adjacent to $v_4, v_7$ or $v_8$ in $G$. Thus in $\overline{G}$, $v_9, v_5, v_3$ are adjacent to $v_4, v_7$ and $v_8$ creating a $K_{3,3}$ subgraph in $\overline{G}$. Then $\overline{G}$ is not planar which is a contradiction. Then $v_9$ must be adjacent to two of $v_5, v_6, v_7$, or $v_8$.

Suppose that $v_9$ is adjacent to both $v_5$ and $v_6$. Then $v_1, v_3$ and $v_9$ are all adjacent to $v_2, v_5$ and $v_6$ in $G$. Thus $G$ has a $K_{3,3}$ subgraph and, by Kuratowski’s Theorem, $G$ is not planar which is a contradiction to our assumption that $K_9$ is biplanar. A similar argument holds if $v_9$ is adjacent to $v_7$ and $v_8$.

Thus $v_9$ must be adjacent to one of $v_5$ or $v_6$ and one of $v_7$ or $v_8$. Let $v_9$ be adjacent to $v_5$ and $v_7$. Then $v_5$ and $v_7$ are adjacent to exactly two vertices of $v_6, v_8, ... v_9$. If either $v_5$ or $v_7$ were adjacent to any additional vertex of the set $v_6, v_8, v_9$, then $G$ would have a $K_{3,3}$ and, by Kuratowski’s Theorem, $G$ is not planar which is a contradiction to our assumption that $K_9$ is biplanar. So $v_5$ is not adjacent to $v_4, v_7$ nor $v_8$ and $v_7$ is not adjacent to $v_3, v_5$ or $v_6$. Then, in $\overline{G}$, $v_3$ and $v_5$ are adjacent to $v_4, v_7$ and $v_8$. Also $v_6$ is adjacent to $v_4$ and $v_7$ in $\overline{G}$. Additionally, $v_9$ is adjacent to $v_6$ and $v_8$ in $\overline{G}$. Thus $\overline{G}$ has a subgraph homeomorphic to $K_{3,3}$ and, by Kuratowski’s Theorem, is not planar which is a contradiction to our assumption that $K_9$ is biplanar.

Case 1.5: Let $v_3$ and $v_4$ be of degree three and be adjacent to exactly one common vertex. Let $v_5$ be that common vertex, let $v_6$ be adjacent to $v_3$ and $v_7$ be adjacent to $v_4$. Suppose that there existed a path $P$ of length three consisting of $v_8$ and $v_9$ with either $v_6$ or $v_7$. Let $P = v_6, v_8, v_9$. Then $G$ has the graph seen in Figure 3.5.

Then in $\overline{G}$, vertices $v_3$ and $v_6$ are adjacent to vertices $v_4, v_7$ and $v_9$. Additionally $v_8$ is adjacent to $v_4$ and $v_7$. Finally, since $v_5$ is adjacent to $v_8$ and $v_9$, then $\overline{G}$ has a subgraph homeomorphic to $K_{3,3}$. Then $\overline{G}$ is not planar which is a contradiction.
Since there does not exist a path $P$ of length three, then vertices $v_8$ and $v_9$ must not be adjacent. Then $v_8$ must be adjacent to $v_6$ or $v_7$, and $v_9$ must be adjacent to the opposite vertex. Without loss of generality, let $v_8$ be adjacent to $v_7$ and $v_9$ be adjacent to $v_6$. Then vertices $v_3, v_6$, and $v_9$ are not adjacent to $v_4, v_7$ nor $v_8$. Then in $\overline{G}$, vertices $v_3, v_6$, and $v_9$ are adjacent to $v_4, v_7$ and $v_8$. Then $\overline{G}$ has a $K_{3,3}$ subgraph and, by Kuratowski’s Theorem, $\overline{G}$ is nonplanar which is a contradiction to our assumption that $K_9$ is biplanar.

Case 1.6: Let $v_3$ and $v_4$ be of degree three and be adjacent to exactly two common vertices. This case follows a similar argument from that in case 1.2 in which there either exists a path $P$ of length three resulting in $\overline{G}$ has a subgraph homeomorphic to $K_{3,3}$ and, by Kuratowski’s Theorem, $\overline{G}$ is nonplanar which is a contradiction to our assumption that $K_9$ is biplanar.

From Cases 1.1 through 1.6 we conclude that, there does not exist a maximal planar graph $G$ with an edge set $E_1$ with two vertices are of degree seven such that $\overline{G}$ is planar.

Case 2: Let $G$ have a degree sequence such that $d_8 = d_9 = 3$. Since $G$ has two vertices of degree three let them be labeled $v_8$ and $v_9$. By Lemma 2.7, $v_8$ and $v_9$ may have either zero, one, or two common neighbors. We will consider these possibilities in the following subcases.

Subcase 2.1: Suppose $v_8$ and $v_9$ have exactly two common vertices. Let the two common vertices be $v_6$ and $v_7$. Let $v_5$ be the other vertex adjacent $v_8$ and let $v_4$ to be the vertex adjacent to $v_9$. Since every vertex adjacent to $v_8$ is adjacent and every
vertex adjacent to $v_9$ is adjacent, then $v_6$ and $v_7$ are of at least degree five. Since we are considering the decomposition of $K_9$, then there exists three other vertices $v_1, v_2$ and $v_3$. A graph of Subcase 2.1 can be seen in Figure 3.6.

![Figure 3.6: Structure of Subcase 2.1.](image)

By Case 1, it is clear that no two vertices may be of degree seven. In particular, $v_7$ and $v_8$ may not both be of degree seven. Additionally, since no vertex may be of degree eight, $v_1, v_2$ and $v_3$ may not all be adjacent to either $v_6$ nor $v_7$.

Suppose that $v_1, v_2$ and $v_3$ were not adjacent to neither $v_6$ nor $v_7$. Then in $\bar{G}$, $v_6, v_8$ and $v_9$ would all be adjacent to $v_1, v_2$ and $v_3$. Thus $k_{3,3}$ would be a subgraph of $\bar{G}$ making $\bar{G}$ not planar. But this is a contradiction, so at least one of $v_1, v_2$ or $v_3$ is adjacent to $v_7$. Without lost of generality, let the vertices $v_1$ and $v_6$ be adjacent and let the vertices $v_3$ and $v_7$ be adjacent. We will now consider two subcases for if the edge $v_4, v_5$ is in the edge set $E_1$ or if $v_4, v_5$ is in the edge set $E_2$.

Subcase 2.1.1: Suppose the edge $v_4, v_5$ was in the edge set $E_1$. Then without lost of generality, we may consider the edge $v_4, v_5$ to be part of the region with vertices $v_7, v_4$ and $v_5$. Since $v_7$ must be adjacent to at least one of $v_1, v_2$ or $v_3$ then without lost of generality let $v_3$ be adjacent to $v_7$. Suppose $v_3$ was the only vertex of $v_1, v_2$ or $v_3$ to be adjacent to $v_7$. Then a graph of subcase 2.1.1 can be seen in Figure 3.7.

The vertices $v_8$ and $v_9$ are not adjacent to $v_1, v_2$ or $v_3$, and vertices $v_3$ and $v_7$ are not adjacent to vertices $v_1$ nor $v_2$. Then all of these vertices must be adjacent in $\bar{G}$. Then $\bar{G}$ contains a subgraph homeomorphic to $K_5$ and, by Kuratowski’s Theorem, is not planar. This is a contradiction to our assumption that $K_9$ is biplanar.

This is the only case to consider with the edge $v_4v_5$ being in the edge set $E_1$. 
If two vertices of $v_1, v_2$ or $v_3$ were adjacent to $v_7$ and the last vertex was adjacent to $v_6$, then the edge going around $v_7$ may instead be drawn around $v_6$ creating an isomorphic graph.

**Subcase 2.1.2:** Suppose the edge $v_4, v_5$ was not in the edge set $E_1$. Suppose that $v_2$ was not adjacent to $v_6$ nor $v_7$. Then in a very similar argument to Case 2.1.1, $\overline{G}$ has a subgraph isomorphic to $K_{3,3}$ making $\overline{G}$ not planar which is a contradiction.

**Subcase 2.2:** Suppose $v_8$ and $v_9$ have one common vertex. Let $v_7$ be that common vertex. Let $v_5$ and $v_6$ be the other two vertices adjacent to $v_8$ and let $v_3$ and $v_4$ be the vertices adjacent to $v_9$. Let $v_1$ and $v_2$ be vertices of at least degree three. Then Case 2.2 has a graph like below.

Since $v_7$ is already of degree six, then $v_7$ can at most be adjacent to either $v_1$ or $v_2$. Then consider the following cases

**Subcase 2.2.1** Suppose neither $v_1$ nor $v_2$ are adjacent to $v_7$. Suppose further
that the edge $v_5, v_4$ is not in the edge set $E_1$. Then in the graph $G$, $v_5, v_4$ and $v_7$ are part of some face. But since $v_5, v_4$ is not in $E_1$ then there must exist some other vertex $v_i$ part of the same face. Then $G$ has a face of size greater then three which is a contradiction since $G$ is maximal. Then the edge $v_5, v_4$ is in $E_1$. Similarly, $v_3, v_6$ is in $E_1$.

Figure 3.9: Structure of Case 2.2.1.

Subcase 2.2.2 Suppose either $v_1$ or $v_2$ is adjacent to $v_7$. Without lost of generality, let $v_1$ be adjacent to $v_7$. Similar to in Case 2.2.1, then either the edge $v_6, v_3$ or the edge $v_4, v_5$ is in $E_1$. Without lost of generality, let the edge $v_4, v_5$ be in $E_1$.

Figure 3.10: Structure of Case 2.2.2.

Suppose that $v_1$ was not adjacent to $v_3$. Then $v_1, v_7$ and $v_3$ are part of some
face with at least one more vertex. Then either $v_1$ is adjacent to $v_2$, or one of $v_4$ or $v_5$. If $v_1$ was adjacent to $v_2$ then $v_2$ is adjacent to some $v_i$. In order to make $G$ maximal planar, every face must be of size three. Thus there exists an edge that can be added to $E_1$. But since the edge $v_2, v_7$ makes $v_7$ a degree eight vertex, then $v_1$ must be adjacent to $v_3$. A similar argument holds for $v_1$ and $v_6$ to be adjacent.

Subcase 2.3 Suppose $v_8$ and $v_9$ have zero common vertices. Then $v_8$ is adjacent to three vertices that are adjacent. Let these three vertices be $v_5, v_6, v_7$. Then $v_9$ is adjacent to three vertices that are adjacent. Let these three vertices be $v_2, v_3, v_4$. Then $v_1$ is the remaining vertex. Not from Cases 2.1 and 2.2, it is not possible for $v_1$ to be of degree three or else it would fall into one of the previous cases of degree three adjacency. Thus $v_1$ must be of degree greater than three. Additionally, $v_1$ is not adjacent to $v_9$ nor $v_9$ and thus could only be of max degree six.

Case 3: Let $G$ have a degree sequence such that $d_i = d_j = 4$. By Case 1 and Case 2, then $G$ must have at most one degree seven vertex, and one degree three vertex. Let $v_8$ and $v_9$ be the vertices of degree four. For each construction of the maximal planar graph $G$ with having two vertices of degree four, every case ends up with $G$ having at least two vertices of degree three. Thus such a case is not possible.

From the Case 2 and Case 3 it follows that $G$ must have exactly one vertex of degree three and exactly one vertex of degree four. Then the degree sequence of $G$ must take on the form $5, 5, 5, 5, 5, 5, 3, 4$. It is left to show that this degree sequence does not have a planar graph representation. Suppose that the degree sequence did have a planar graph representation. Then there would exist a degree five vertex that was adjacent to five other degree five vertices. Each of these vertices would then be at least adjacent to two other degree vertices in the set of five degree five vertices. This would form a cycle like construct seen in Figure 3.11.

Since there is a degree three vertex, it must be adjacent to exactly two adjacent of the outer degree five vertices. If the degree three vertex was adjacent to three adjacent then the center of the three degree three vertices would be a degree four. A similar argument holds true for the degree four vertex. Then there exist exactly three cases, if the degree three and degree four vertex are adjacent to exactly zero one or two of the same vertices.

Case 1: The degree three and degree four vertex are adjacent to exactly zero of
the same vertices. Then the graph $G$ has the following structure:

Figure 3.12: Structure of degree sequence Case 1.

The degree four vertex would have to be adjacent to at least one of the other vertices in the structure. If the degree four vertex was adjacent to one of the other vertices, then it would form a cycle of length three. One of the vertex would then be locked into an interior region and would not be able to have any additional edges adjacent to it forcing the degree to be less than five, a contradiction.

Case 2: The degree three and degree four vertex are adjacent to exactly one of the same vertices. Then the graph $G$ has the following structure:

Since the vertex with all edges complete is degree five, and since every region must be of length three, then the degree three and degree four vertices are adjacent. In a similar fashion, the degree four and the degree five vertex that is also adjacent to the
degree three vertex are adjacent creating the structure seen in Figure 3.14.

From this it is clear that the last vertex could only be of maximum degree three. Then the graph $G$ is not maximum planar which is a contradiction.

Case 3: The degree three and degree four vertex are adjacent to exactly two of the same vertices. Then the graph $G$ has the structure seen in Figure 3.15.

The vertices in this graph labeled with $a$ and $b$ are of degree five. Similar to in Case 2, the two vertices adjacent to the degree four are already degree five, so in order
to keep each region a cycle of length three, then the degree four must be adjacent to $a$ and $b$ making it a degree five, a contradiction.

From these cases it becomes clear that it is not possible to construct a maximal degree sequence of 5, 5, 5, 5, 5, 5, 5, 4, 3. So there does not exist a degree sequence correlating to a maximal planar graph $G$. Thus $K_9$ is not planar.

Lemma 3.5. The graph $K_9$ is minimal non-biplanar.

Proof. As seen in Figure 3.16, it is possible to remove a single edge $e = \{2, 4\}$ and generate a biplanar decomposition. Thus the graph $K_9 - e$ is biplanar.

\[ \Box \]

3.2 Complete Bipartite Biplanar Graphs

Although the proof that $K_9$ is not biplanar is very technical, it follows directly from expanding Euler’s Polyhedral Identity to show the complete bipartite graphs $K_{7,7}$ and $K_{6,9}$ are not biplanar. First we will expand Euler’s Polyhedral Identity to biplanar bipartite graphs.
Theorem 3.6. A the size and order of a biplanar bipartite graph must satisfy the inequality \( m \leq 4n - 8 \).

Proof. Let \( G \) be a planar bipartite graph. By Euler’s Polyhedral Identity, in order for \( G \) to be planar it must satisfy the equation \( n - m + r = 2 \) or \( r = 2 + m - n \). Since \( G \) is bipartite, \( G \) contains no cycles of length three. Then every region of \( G \) must be incident with at least four edges and since every edge is incident with exactly two regions we obtain the following inequality \( 2m \geq 4r \). By substitution, we obtain the inequality \( 2 + m - n \leq m/2 \), which simplifies to \( m \leq 2n - 4 \). Since a biplanar decomposition of a bipartite graph results in two planar graphs, the number of edges and vertices in a biplanar bipartite graph must satisfy the inequality \( m \leq 4n - 8 \). \( \Box \)

Lemma 3.7. The complete bipartite graph \( K_{7,7} \) is not biplanar.

Proof. Suppose the complete bipartite graph \( K_{7,7} \) is biplanar. By Theorem 3.2, the size and order of \( K_{7,7} \) must satisfy the inequality \( m \leq 4n - 8 \). By substituting 49 for \( m \) and 14 for \( n \), we obtain \( 49 \leq 4(14) - 8 \), or \( 49 \leq 48 \), which is a contradiction. Thus \( K_{7,7} \) is not biplanar. \( \Box \)

Lemma 3.8. The complete bipartite graph \( K_{6,9} \) is not biplanar.

Proof. Suppose that the complete bipartite graph \( K_{6,9} \) is biplanar. By Theorem 3.2, the size and order of \( K_{6,9} \) must satisfy the inequality \( m \leq 4n - 8 \). By substituting 54 for \( m \)
and 15 for \( n \) we obtain \( 6(9) \leq 4(15) - 8 \) or \( 54 \leq 52 \) which is a contradiction. Thus \( K_{6,9} \) is not biplanar.

**Lemma 3.9.** The complete bipartite graph \( K_{5,13} \) is not biplanar.

*Proof.* Suppose that the complete bipartite graph \( K_{5,13} \) is biplanar. By Theorem 3.2, the size and order of \( K_{5,13} \) must satisfy the inequality \( m \leq 4n - 8 \). By substituting 60 for \( m \) and 18 for \( n \) we obtain \( 60 \leq 4(18) - 8 \) or \( 65 \leq 64 \) which is a contradiction. Thus \( K_{5,13} \) is not biplanar.

**Lemma 3.10.** The graphs \( K_{7,7}, K_{6,9} \) and \( K_{5,13} \) are the smallest complete bipartite graphs that are not biplanar.

*Proof.* As seen in Figure 3.2, the complete bipartite graph \( K_{6,8} \) is biplanar. Since \( K_{6,7} \) is a proper subgraph of \( K_{6,8} \), it follows that \( K_{6,7} \) is biplanar as seen in Figure 3.17. Additionally, \( K_{5,12} \) has a biplanar decomposition as seen in Figure 3.18.

**Corollary 3.11.** Biplanar graphs are not closed under subdivision of an edge.

*Proof.* Let \( G \) be the graph \( K_9 - \{v_2,v_4\} \) as in Lemma 3.5. Let the edge \( e = \{v_2, v_4\} \) be subdivided with a vertex \( v \) such that the resulting edges are \( \{2,v\} \) and \( \{v,4\} \). By placing the edge \( \{2,v\} \) into one edge set and \( \{v,4\} \) into the other edge set, each decomposition of \( G \) is still is still planar. However, since \( K_9 \) is not biplanar, biplanarity is not closed under subdivisions of edges.

Note that Corollary 3.10 shows biplanar graphs are not closed under subdivision of an edge, then biplanar graphs are not closed under graph homeomorphism.

**Theorem 3.12.** The graphs \( K_9, K_{7,7}, K_{6,9} \) and \( K_{5,13} \) are the smallest complete and complete bipartite graphs that are not biplanar.

Theorem 3.12 follows directly from Lemmas 3.5, 3.5, and 3.12. As with Kuratowski’s Theorem, we have identified the smallest complete and complete bipartite graphs that are not biplanar. Thus for any graph \( G \) to be biplanar, \( G \) must not have one of \( K_9, K_{7,7}, K_{6,9} \) nor \( K_{5,13} \) as a subgraph.
Figure 3.17: A biplanar decomposition of $K_{6,8}$. 
Chapter 4

Graph Thickness

As with planar graphs, biplanar graphs can only have a certain number of edges in comparison to the number of vertices before the decomposition becomes impossible. This was shown through the cases of $K_9, K_{7,7}$ and $K_{6,9}$. The thickness of a graph $G$ is smallest number partitions of the edge set $E$ of $G$ such that each subgraph $G_1 = (V, E_1), G_2 = (V, E_2) ... G_n = (V, E_n)$ is a planar graph. The thickness of a graph will be denoted by $\theta(G)$.

From the previous section, it is clear that $\theta(K_8) = 2$ since $K_8$ is biplanar.

**Theorem 4.1.** For a graph $G$, $\theta(G) \geq \lceil m/(3n - 6) \rceil$

*Proof.* Let $G$ be a given graph. A maximal planar graph has $3n - 6$ edges. By $m/(3n - 6)$ would give a minimum number of planes in which it would take to embed $G$. Since $m/(3n - 6)$ may be rational and not an integer, it is sufficient to consider $\lceil m/(3n - 6) \rceil$ as the minimum number of planes needed to planar embed $G$. Thus $\theta(G) \geq \lceil m/(3n - 6) \rceil$.

It directly follows from Theorem 4.2 that a complete graph has a minimum thickness.

**Theorem 4.2.** For a complete graph $K_n$, then $\theta(K_n) \geq \lfloor (n + 1)/6 \rfloor + 1$

*Proof.* Let $G = K_n$ for some $n$. By the Theorem 4.2, $\theta(G) \geq \lceil m/(3n - 6) \rceil$. Since $G$ is complete, then it has exactly $n(n - 1)/2$ edges. Then $\theta(G) \geq \lceil n(n - 1)/2(3n - 6) \rceil$.
or $\theta(G) \geq \lceil n(n - 1)/6(n - 2) \rceil$. Then by dividing with remainder, $\theta(G) \geq \lceil (n + 1)/6 + 2/6(n - 2) \rceil \geq \lceil (n + 1)/6 \rceil + 1$. Thus $\theta(K_n) \geq \lceil (n + 1)/6 \rceil + 1$.

In a very similar manor of expanding maximal planar, we are able to expand the result of Euler's Polyhedral Identity for biplanar graphs to give a minimum for graph thickness.

**Theorem 4.3.** *For all simple bipartite graphs $G$, $\theta(G) \geq \lceil m/(2n - 4) \rceil$*

*Proof.* Let $G$ be a bipartite graph. As seen before, $m \leq 2n - 4$. Then for any given partition of the edges of a bipartite graph, there may exist at most $2n - 4$. Then by considering $\lceil m/(2n - 4) \rceil$ would give a minimum number of parts needed for the partition of the edges of $G$. Thus for all bipartite graphs $G$, $\theta(G) \geq \lceil m/(2n - 4) \rceil$

Although these are theorems only develop the foundations for studying graph thickness, they create a very powerful tool set. From these theorems, exact formulas for graph thickness have been determined for all complete graphs and most complete bipartite graphs. More information that we did not have time to investigate in this thesis is available in [Bei97].
Chapter 5

Conclusion

Through this thesis, we were able to develop the required tools from introducing a graph to major theorems that play a key role in determining which graphs are planar and biplanar along with graph thickness. By building up to the proof that $K_9$ is not biplanar, we introduced many structural theorems that create an idea of what maximal planar graphs can look like. Although this thesis is self contained, there is still a lot of work to be done in the topic of biplanar graphs.

Within future research, we would further investigate more classifications of biplanar graphs including multipartite. Additionally, we would add edges to smaller biplanar bipartite graphs to determine if there exist any other cases of graphs that are not complete and are not biplanar. After these cases are studied, we would be able to generate all forbidden subgraphs for biplanar graphs.

In an attempt to still expand Kuratowski's Theorem, we would also further investigate additional structures within biplanar graphs. Through these structures we would hope to find some function on graphs that would assist in determining if a given graph is biplanar or not.
Bibliography


