Solving Absolute Value Equations and Inequalities on a Number Line

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SOLVING ABSOLUTE VALUE EQUATIONS AND
INEQUALITIES ON A NUMBER LINE

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Teaching: Mathematics

by
Melinda Antoinette Curtis
September 2016
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ABSTRACT

Absolute value has often been taught procedurally. Many students struggle with solving absolute value equations and inequalities because they do not have an understanding of the underlying concepts. This study was designed to determine to what extent solving absolute value equations and inequalities by using the concept of distance on a number line is an effective method. The claim is that if students use the distance concept on a number line, they will develop the necessary conceptual understanding in addition to just a procedural knowledge that will lead to the success with and flexibility in the use of strategies for more challenging problems. The following questions were addressed in this study: How and to what extent can using a number line develop a conceptual understanding of absolute value equations and inequalities? What solution strategies do students tend to use to solve absolute value equations and inequalities? Does the strategy depend on the complexity of the problem? To what extent do they exhibit flexibility in their use of strategies? What extensions are students able to make? What misconceptions do they have? In this study, lessons and assessments were implemented based on the “best practice” of using multiple representations with a focus on conceptual understanding of absolute value. The lessons were consistent with current content standards. Students completed a pre and post assessment, and some students were selected to participate in a 15 minute interview based on their responses from their assessments. The results were analyzed qualitatively and show that
students struggled with remembering the procedure for solving absolute value equations and inequalities. The results also show that students were more successful when using the distance concept on a number line.
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CHAPTER ONE

INTRODUCTION

Background

Recently, there has been a new change in mathematics education. Math teachers are learning about new standards called the Common Core. The California Common Core Standards have replaced the California State Mathematics Content Standards for California Public Schools adopted in December 1997. Some people believe the Common Core Standards came out of nowhere. However, those people involved in writing the standards devoted time collaborating and developing these standards so the achievement levels of students in not only California, but all over the country, would increase. This process started in 2008. Before the Common Core Standards, each state had their own set of standards. When students would move from state to state, they could be faced with different expectations. The Common Core standards were developed so most states would have common standards and expectations. In California, the new standards are in the beginning stages of implementation (2015).

Along with these content standards, there are eight Standards for Mathematical Practice (SMPs): Make sense of problems and persevere in solving them (SMP1), Reason abstractly and quantitatively (SMP2), Construct viable arguments and critique the reasoning of others (SMP3), Model with
mathematics (SMP4), Use appropriate tools strategically (SMP5), Attend to precision (SMP6), Look for and make use of structure (SMP7), and Look for and express regularity in repeated reasoning (SMP8). These standards are described as habits of mind for doing mathematics. Teachers are required to learn how to engage their students in using the SMPs in their daily lessons. This is an adjustment because in the past there was a focus on procedural and rote learning. In the Common Core Standards, students need to develop a conceptual understanding of mathematical topics. Common Core encourages students to comprehend the process it takes to get the answer, rather than focusing on just the answer. Students will see different approaches to the same problem. In the past, students have been able to use a variety of different math techniques and most likely not understand why they work. The state tests were multiple choice and problems were largely focused on skills rather than concepts, so students could provide just the answer with no justification.

New Courses and Curriculum

The new changes of the curriculum allow for two different course pathways for high school. It was a district’s responsibility to make the decision to stay with a traditional path (Algebra I, Geometry and Algebra II) or an integrated path (Integrated Math I, II, III). My district, Beaumont Unified School District, decided to take an integrated math approach. This was a new approach for most teachers because they would need to find ways to show how different topics in math are related. For instance, the same problem could have an algebraic and
geometric solution. The Common Core content standards and SMPs require teachers to implement several different strategies, and one challenge to this is that the selection of textbooks that support these standards are limited. This concerns teachers because they do not know how they are going to teach these new standards and what resources they are going to use. With the implementation of the Common Core, teachers need to decide what curriculum they will use.

Curriculum Design Teams

Half way through the 2013 - 2014 school year, my district decided to put together Curriculum Design Teams (CDTs) for the three new integrated courses (Math I, II and III). The CDTs were made up of teachers, and the goal was to design units of study. These units were going to be used by mathematics teachers within the district. There were several days devoted to these units of study. The units replaced a new mathematics textbook adoption. Again, there were not many textbooks to support the Common Core curriculum, so the district believed this would be a better option. When there is a new textbook adoption, teachers have a textbook and resources for many years. On the other hand, the newly designed units of study are intended to be more flexible. Lessons and resources can be developed, modified or taken out as needed. While designing units of study, teachers developed some of the material, some was taken from other districts that have already designed units, and some was taken from websites that have Common Core lessons. To ensure that the units continuously
improve, the district created venting teams. The teachers on the venting teams make necessary changes to the units based on the feedback given by teachers that taught the units of study. The goal is to continue perfecting these units of study so teachers will have sufficient resources for the new courses. Carrying out all of these new courses at once can be overwhelming. So the decision was made to first implement Math I by replacing Algebra I. The next year will be Math II replacing Geometry and the year after will be Math III replacing Algebra II.

**Implementing the New Curriculum**

In the 2014-2015 school year, I taught three integrated Math I courses. It was a bit difficult using the units of study because this was the first year and most of the resources were practice worksheets. There were only a few resources that were discovery lessons. Instead of giving students several worksheets focusing on procedures, I decided to collaborate with two other teachers and develop lessons that would provide students with a style of learning that would involve developing the SMPs. Students struggled at first with this type of curriculum, but slowly they began to use their critical thinking skills, justify their reasoning, and develop a conceptual understanding of various topics as evidenced by their group and class discussions and how they were able to explain their reasoning for solving problems.

One of the reasons why the Math I units of study resources were more practice based was because Common Core was new to teachers. For several years the normal routine for math classes was a lecture by the teacher and
students would take notes. Then students would complete a worksheet or a textbook assignment to practice the material the teacher presented. It was difficult for teachers to try and design lessons that were a different style. Most teachers have not been exposed to this type of teaching. The Master of Arts in Teaching (MAT) Mathematics program at California State University, San Bernardino (CSUSB), has helped me to understand how the Common Core curriculum should look in the classroom. Before the program, I believe I was like most teachers. Growing up, my education experience in mathematics was full of direct instruction and practice. The MAT program allowed me to understand topics. My experiences in the MAT classes were different than those in any other mathematics class I have ever taken. There was more of an emphasis on process and comprehending why a procedure works.

Professional Development on Absolute Value

Along with my district providing CDTs, the district had the funds to allow mathematics teachers to attend Professional Development (PD). One of the PDs offered through the Riverside County Office of Education (RCOE) was called Strengthening Mathematics Instruction (SMI). During the PD, we had to complete an exercise describing the meaning of some given absolute value equations, for example, $|x + 3| = 8$ and $|x - 1| = |2x + 3|$. We were asked to solve these equations using two different approaches. Most of us started by using a procedure to solve the equations: rewriting the two expressions in the absolute value with no absolute value. One equation would be the expression in
the absolute value equal to the positive number from the original absolute value equation. The other equation would have the same expression, but the number it is equal to will change to negative. For example, $|x + 3| = 8$ can be rewritten into the following two equations: $x + 3 = 8$ and $|x + 3| = -8$. Solving the equations independently gives two solutions to the absolute value equality. But, what does this process actually mean? Why are there two solutions? Then the conversation started to take a turn. It was not just about a procedure, but understanding what absolute value means and using the definition of absolute value as distance between two numbers to solve these equations. One person suggested using a number line. This immediately sparked my interest because this was not a method I used. The person explained the steps taken to use a number line to solve one of the absolute value equations. This method did not have a long process and there were fewer computations. This type of solution allowed us to use our knowledge about distance, which is important not only in absolute value equations and inequalities, but in several mathematical topics. The leader of the PD provided another example of an absolute value equation, but this time the numbers were not as easy to compute and there were slight changes in the format of the equation. However, we were still able to solve this equation using the concept of distance on a number line. After using the number line, we were able to observe and reflect on both procedural and conceptual methods. The leader did not talk explicitly about the connections
between the two methods, but we were able to look at the two methods and develop our own connections.

**Designing a Unit on Absolute Value Equations and Inequalities**

Often, solving absolute value equations has been taught to students only as a procedure. When I ask my students to describe the meaning of absolute value, they tell me that the solution will always be positive number. This response indicated that they might be lacking a conceptual understanding of absolute value. A lack of understanding of this key concept may lead to difficulties in understanding how to solve multi-step equations and inequalities with absolute value. Before the PD, I had never seen the technique of using a number line to solve absolute value equations. In the past, I would teach absolute value equations and inequalities by a procedure only. For example, in the equation \(|x - 2| = 3\), the two values that could be equal to the expression in the absolute value are -3 and 3. This is because \(|-3| = 3\) and \(|3| = 3\). Students can understand that \(|-3| = 3\) and \(|3| = 3\) are true statements using the concept of distance because -3 and 3 are the same distance from the origin. Students would need to find the two values of \(x\) that would make the expression in the absolute value equal to -3 and 3. For inequalities, I would introduce compound inequalities and then have students solve absolute value inequalities. The process was similar to that for equations except now students would have “and” and “or” statements. For the most part, the solution involved procedural rather than conceptual thinking. The textbooks I have used in the past would approach
the solving the problem in the same procedural way. Like me, the two other
teachers I was working with had not seen the number line approach before. So
for the next unit in Math I that involved absolute value equations and inequalities,
we felt it would be beneficial to design lessons where students are solving
equations and inequalities using the concept of distance on a number line. The
first lesson that we designed had no equations or inequalities. It focused on the
distance definition of absolute value. We felt this would set the foundation for
more complex problems. After that, we wanted to have students evaluate simple
absolute value expressions like |3| or |-6|. For each problem, students were
asked to: (a) write out the meaning of each absolute value expression using a
sentence, and (b) draw a number line representation of each absolute value
expression and use it to simplify the expression. This helped students to solve
elementary equations with absolute value such as |x| = 6 and |x − 3| = 2, which
were introduced in the next lesson. For each equation students were given
directions analogous to those in the previous task: (a) write out the meaning of
each absolute value equation using a sentence and then (b) draw a number line
representation of each absolute value equation and use that to find all possible
values for x. Students were able to explore more complex equations in the
subsequent lessons. Then inequalities with absolute value were
introduced. This stage of the lessons focused on showing students that instead
of finding two values that have the same distance from a number, the solution
may consist of a range of values depending on the inequality. In the equation
\(|x - 3| = 4\), the distance between \(x\) and 3 is 4. The two values that would have a distance of 4 from 3 are -1 and 7. If we take the same problem and consider the corresponding inequality, \(|x - 3| \leq 4\), then a solution would require us to find all values \(x\) such that the distance between \(x\) and 3 is less than or equal to 4. The solution set includes not only -1 and 7, but it could also include numbers such as 2.4 and 3.6. Both values 2.4 and 3.6 have a distance 0.6 from the center number 3. In fact, there are infinitely many values that have a distance from 3 that is less than or equal to 4.

**Other Connections to Absolute Value Equations and Inequalities**

All of these lessons were engaging and involved opportunities for students to think conceptually about absolute value equations and inequalities, but I still believe there is room for improvement. It would be good to connect absolute value equations and inequalities to topics introduced in earlier units. For instance, students learned about geometric transformations in a previous unit, like translations, rotations and reflections. There were also lessons on composition of transformations. The lessons allowed students to observe ordered pairs and determine the rule(s) for the transformation. For example, students were given ordered pairs of a preimage \(A(1, 1), B(2, 3), C(3, 1)\) and ordered pairs of the image \(A'(-1, 2), B'(0, 4), C'(1, 2)\). Students were then asked to plot the preimage and image and determine the type of transformation that maps the preimage to the image, which in this case is a translation. The rule would need to be developed, so the transformation is a translation two units to the left.
and one unit up in relation to the original plane. By that description, the mapping would be \((x, y) \rightarrow (x - 2, y + 1)\). With the prior experience of working with geometric transformations, students could be exposed to solving absolute value equations and inequalities using the distance concept and transformation of the number line. This allows students to see another approach (geometric) and connect to topics that they learned before. Also, there can be lessons that extend to graphing absolute value equations and inequalities with two variables in the Cartesian plane. For instance, the absolute value inequality \(|x - 2| < 3\) can be solved by graphing the functions \(y_1 = |x - 2|\) and \(y_2 = 3\) and determining the values for \(x\) that make the graph of \(y_1\) below the graph of the line \(y_2 = 3\). Also, the absolute value inequality with two variables \(y < |x - 3|\) can be graphed in the plane. These topics are introduced in Math III (or Algebra II).

However, using a number line as a tool develops a conceptual understanding of absolute value equations and inequalities. This approach can help students make connections to other concepts.

Goal and Research Questions

The goal of this study was to determine to what extent solving absolute value equations and inequalities by using the concept of distance on a number line is an effective method. The study should find that if students use the number line, they will develop conceptual understanding in addition to a procedural
knowledge, which in turn will lead to success with and flexibility in the use of strategies for more challenging problems.

I addressed the following questions in this study:

- How and to what extent can using a number line develop a conceptual understanding of absolute value equations and inequalities?
- What solution strategies do students tend to use to solve absolute value equations and inequalities?
- Does the strategy depend on the complexity of the problem?
- To what extent do students exhibit flexibility in their use of strategies?
- What extensions are students able to make?
- What misconceptions do students have?

Significance

Beaumont Unified School District encourages teachers to continue learning about effective teaching strategies and methods. Since the district is allowing teachers to design the curriculum, the lessons designed during this study could be a resource for other teachers. This study will encourage teachers to think about absolute value equations and inequalities conceptually. The results of this study may lead to the lessons being adapted into the units of study. If teachers use a number line to solve absolute value equations and inequalities, then they could share the method with other teachers. A
professional development workshop could be developed to show teachers this method. In the long run, students could benefit from learning these topics conceptually. They can apply their understanding to other problems. Also, thinking through these types of problems can lead to students wanting to think more conceptually about other mathematical topics and the culture in high school mathematics classes could start to change.

I have taught an intermediate algebra course at California State University, San Bernardino (CSUSB). Absolute value equations and inequalities are taught in this class. This is one of the areas that students have challenges. I find that students understand the procedure when it is first presented to them. However, later in the course they forget how to solve the problem correctly. This could be a good opportunity to show other professors/instructors this method, so they can help their students be more successful. Students may better retain solutions strategies for absolute value equations and inequalities because the lessons will help them conceptually understand the topics. This could start to change the culture at the higher educational level as well.
CHAPTER TWO
LITERATURE REVIEW

Introduction

To improve the lessons that I have already designed, I needed to research
the methods that have been used to teach absolute value. Has the approach
using a number line been implemented and researched elsewhere? If so, in what
grade levels? To what extent? I started researching the definition of absolute
value and the procedures taken to solve absolute value equations and
inequalities.

Misconceptions about Absolute Value

Teachers and textbooks often define absolute value as the “distance from
zero on a number line”. This is a geometric definition of absolute value (Wade,
2012). When absolute value is introduced, mathematics teachers often hear
students say, "The answer is always positive." The teacher may have not said
this statement, but I believe students come to this conclusion based on solutions
to problems that involve simplification. Students are able to understand $|4| = 4$
and $|-4| = 4$. Then students are faced with the problem $|0|$, i.e. absolute value of
zero is zero. The solution is not positive because zero is neither a negative nor a
positive number. Students then change their statement to, "The answer is
always positive or zero." This statement is then carried to solving absolute value
equations. Students might change a problem like $|x - 3| = 5$ into $x + 3 = 5$, or
rewrite \( |x - 3| = -5 \) as \( x + 3 = 5 \). These misconceptions appear to stem from their initial conception that “absolute value makes everything positive.” Several students believe that negatives do not make sense when working with absolute value. In my experience, students become discouraged and confused when the equation \( |x + 3| = 5 \) is solved for \( x \) correctly using a method such as first rewriting the equation as \( x + 3 = 5 \) or \( x + 3 = -5 \), and then solving each equation separately to get two answers \( x = 2 \) or \( x = -8 \). Students often say, “I thought the answer is always positive. Why is one of the answers negative?” (Ponce, 2008)

\[
|x| = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases}
\]

A formal analytic definition of absolute value is:

This definition often initially confuses students because they think the expression \( -x \) is negative which goes against their understanding that the absolute value of a quantity is positive. Students with this misconception do not understand that \( x < 0 \) is a critical part to this definition (Ponce, 2008). It is a critical part because if \( x \) is less than 0, then \( x \) is to the left of 0 on the number line. Hence, the additive opposite of \( x \) (i.e. \( -x \)) is to the right of 0 on the number line and must be positive. For example if \( x \) is \( -2 \), then \( -(−2) \) is equal to 2.

How Absolute Value is Often Taught

In Integrated Math 1 (Algebra 1), students are introduced to solving equations with absolute value such as \( 3|x - 10| = 12 \). Before the Common Core curriculum, the mathematics textbook that I used would show how to solve this
problem by using an algorithm such as the following: the first step is to treat the absolute value expression as a variable. To isolate the variable (absolute value expression) divide both sides of the equation by 3. The result of the first step would get a new equation: $|x - 10| = 4$. The next step would be to set up two equations as $x - 10 = 4$ or $x - 10 = -4$. Lastly, solve each equation. There will be two solutions $x = 14$ or $x = 6$. Most students would be satisfied with these solutions because they are both positive. However, they may not understand why there are two different solutions. They may wonder why we need to isolate the absolute value. This process just consists of procedural steps and is not based on conceptual understanding of the problem. “When absolute value problems become more complex, students often do not have sufficient conceptual understanding to make any sense of what is happening mathematically” (Ellis and Bryson 2011).

**International Research**

Equations and inequalities are woven into several mathematical subjects, like algebra, trigonometry and calculus. “Some research has found that there are epistemological obstacles in the teaching of the concept absolute value” (Almog and Ilany, 2012). There are different definitions used to describe absolute value, but each description comes with more underlying definitions. In most countries, absolute value is taught in a pure algorithmic manner, resulting in a sequence of routine procedures (Almog and Ilany, 2012). In a study conducted in Israel by Almog and Ilany (2012), students were given tasks related to absolute value
inequalities and some students were selected to participate in an interview. The tasks were given to students after they have been taught absolute value equations and inequalities possibly with procedural methods only. One of the issues that the researchers wanted to focus on was: Do students use a number line? If so do they use it correctly? The researchers found students using various techniques for solving the inequalities such as representing their answers through immediate solution without algebraic manipulations, using a rule, using a number line with test values, or partitioning the solution via cases. In their first task, \(|x| > 3\), about a third of the students just gave the immediate solution \(x > 3\) or \(x < -3\). One of these students explained in writing, “Only these numbers can be the solution. A number greater than three remains greater than three, and a number smaller than minus three in absolute value is greater than three” (Almog and Ilany, 2012). In a follow-up interview, when this student was asked how he arrived at this, he explained that he thought about the numbers whose distance from zero is greater than three. This shows that he is thinking of absolute value as distance from zero even though it is not communicated in his written response. Without the interview, the researchers may not have known his true understanding of the absolute value as distance. With regard to the students who used the following rule: when \(|x| > a\) (for \(a \geq 0\)) the solution is \(x > a\) or \(x < -a\), the interviews of some of these students showed they did not have a deep understanding. The results of this study showed that students did not often use the number line. The students that used a number line divided the number
line into intervals according to thinking the task as an equation. Then they found the intervals on the number line that would make the inequality true using test values. It is not clear if the students who used the number line did so by using the concept of distance.

A Different Approach to Teaching

There is often a strong urge by teachers to give students a step-by-step procedure to solve any type of equation or inequality. However, “it is more important to demonstrate the logic of math than to memorize rules” (Wakefield, 2001). Teachers need to focus on reasoning and developing meaning, rather than telling how to get the right answers. In the article, Teaching Young Children to Think About Math (Wakerfield, 2001), there are three principles stated to guide teachers: (1) Encourage Children to Think, (2) Encourage Children to Think about Thinking, and (3) Encourage Representations of Thinking. These principles have some similarities to the SMPs. The first strategy allows students to think about their answers and determine if it make sense. In the past, everyone was expected to do the problem the same way and depend on the teacher to determine if the answer makes sense. This may be fine for those students who are already motivated, but the students who are not motivated to memorize rules lose confidence in their ability to learn math. These students often start to dislike math and view math as a set of random tricks. If, on the other hand, the teacher asks if anyone took different approaches to solve the
same problem, students can challenge each other's thinking. The second principle encourages students to conceptualize their answer and make sense of their reasoning process. This skill is not developed instantly. “Young children who are challenged to think about their thinking will become skilled at doing it” (Wakefield, 2001). Students develop a solution that makes sense to them. When they share their solution with a peer and get different answers, they can defend their logic. The third principle encourages young children to use multiple representations. They can represent their thinking by using words, pictures, or symbols. “When teachers encourage children to reflect on thinking and use representation of thinking in their classrooms, it evitably leads to more thinking, and more thinking about thinking” (Wakefield, 2001). This process develops a more complex and sophisticated mental activity. These three strategies are great for students, so why are there not more teachers using these principles? This approach can be more challenging for teachers. The textbook often allows teachers to teach math, but with no logic or reasoning. Teachers need to grow in their logic, so they can help their students think this way too. “Before teachers can prepare classroom environments that encourage students to think, they must first learn to be thinkers themselves. It takes a lot of thinking and a lot of thinking about thinking…to grow and develop” (Wakefield, 2001).
Procedural and Conceptual Knowledge

The balance of procedural knowledge and conceptual knowledge is often debated in math education. The Common Core standards have required teachers to transition into using lessons that have more conceptual understanding built in. However, should teachers forget about procedural knowledge? Is procedural knowledge just as important as conceptual knowledge? In the article, *Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process*, Rittle-Johnson, Siegler, and Alibali (2001) defined procedural knowledge as “the ability to execute action sequences to solve problems.” Conceptual knowledge was defined as “implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain.” In some cases, it can be more efficient to use procedural knowledge for standard arithmetic problems. Students can use their conceptual knowledge in novel types of tasks. Most students believe that mathematics only consists of procedural knowledge. “They see mathematics as a collection of rules, procedures, and facts that must be remembered” (Richland, Stigler, and Holyoak 2012). When a topic is first introduced to students they should use conceptual knowledge. Yet, as they develop an understanding of the topic they can make connections between the procedural and conceptual knowledge to apply their skills to similar problems. If they do not start with understanding a topic conceptually, this may affect their understanding when they apply procedures to more challenging problems. The
Common Core Mathematics Standards emphasize three components to learning mathematics: computational and procedural skills, conceptual understanding, and problem solving. These three strands of mathematics should be intertwined and mutually reinforced through instruction. Common Core instruction is not solely on conceptual understanding. However, teaching conceptually is important and was not stressed heavily in the past. The other instructional methods are important as well (“California Common Core State Standards: Mathematics”, 2013).

Absolute Value Content Standards

The next part of my research has been looking at the content standards for Mathematics adopted in 1997 and the Common Core Mathematics Standards to compare where absolute value is introduced. In the Mathematics Content Standards for CA Public School (adopted in 1997), absolute value was first introduced in Grade 7. In these standards, according to Standard 2.5, students need to “understand the meaning of absolute value of a number, interpret the absolute value as distance of the number from zero on a number line, and determine the absolute values of real numbers.” This was the basics, like \(|3| = 3\) or \(|-10| = 10\). In the past, students would take Algebra 1 after 7th grade math. In the Algebra 1 standards, Standard 3.0 states, “students solve equations and inequalities involving absolute value” (Mathematics Content Standards for California Public Schools: K-12, 2009). Note that this standard does not
specifically mention any particular reasoning process that could or should be emphasized while solving the equations or inequalities. However, the Common Core standards introduce this concept in grade 6. The standard that involves the definition of absolute value is CCSS.MATH.CONTENT.6.NS.7.C, “understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation” (California Common Core State Standards: Mathematics, 2013). Some of the previous standards involved students understanding the position of integers on a number line and the ordering of different numbers, and also interpreting inequality statements and relating them to their position on the number line. Students understand the position of numbers on the number line, which leads them into the absolute value. In the Common Core standards for Algebra I, under Reasoning with Equations and Inequalities A-REI, students should “understand solving equations as a process of reasoning and explain the reasoning.” Standard 3.1 states “solve one-variable equations and inequalities involving absolute value, graphing the solutions, and interpreting them in context.”

Absolute Value as a Concept of Distance

As previously stated, absolute value is first introduced in the content standards as the distance from zero. In the article A Conceptual Approach to Absolute Value Equations and Inequalities, the authors state that, “the
transformational approach extends the initial conceptual understanding to realizing that the distance between two values is the absolute value of their difference - that the distance from $x$ to $b$ is $|x - b|$ (Ellis and Bryson, 2011). To find the distance between two numbers on the number line involves finding the difference between them, which is why subtraction is used. Students need to first understand this concept before moving on. Then students can start connecting the symbolic to the verbal phrase. For example, for the symbolic statement $|x - a| = b$ the verbal translation is “$x$ is $b$ units from $a$ in both directions.” A contextual example might be my friend lives at 12 Fairhill Drive, and I live 5 houses away (house numbers change by units of 1). Where would my house be located? One would realize that you could move 5 units from 12 in the positive or negative direction. Students can make a visual representation of this on a number line. A teacher can then have students write a mathematical equation that fits this scenario. The correct equation would be $|x - 12| = 5$. The initial point changes from zero to 12. The equation is solved visually first by using a number line. If needed, the solution is recorded in algebraic notation. Next, is the case of $|x - a| = c$. How do we approach this type of problem by using a number line? What does this equation conceptually mean? For example, for $|x + 6| = 3$, students can look at the expression $x + 6$ as $x - (-6)$. So the initial point will be $-6$. If students are unable to develop this concept, then they can look at the equation as $|x + 6| = 0$. Verbally, this equation can be translated as “the distance between $x$ and negative six is
zero.” What value does $x$ need to be to make this equation equal to zero? This value would be $-6$. The original equation can be rewritten as $|x - (-6)| = 3$, and so $x$ is 3 units from $-6$ in either direction. This is expressed algebraically by $x = -6 - 3$ and $x = -6 + 3$, so the solutions are $x = -9$ and $x = -3$ (Ellis and Bryson, 2011).

Using the concept of distance on a number line eliminates the challenges with the expressions $|x - a|$ and $|a - x|$. “Students whose understanding is restricted to the procedural algorithm must mentally think though the distributive property and the property of opposites and apply the definition of absolute value before they can grasp or retain the equivalence” (Ellis and Bryson, 2011). Using distance can help with showing that these two expressions are equal. We know that the distance from home to your friend’s house is the same distance from your friend’s house back home (Ellis and Bryson, 2011). In other words, the distance between $x$ and $a$ is the same as the distance between $a$ and $x$. Hence, solving an equation like $|3 - x| = 5$ really is not any more complicated than solving $|x - 3| = 5$. Without this understanding that these are equivalent, it would seem that students would find solving $|3 - x| = 5$ more difficult than $|x - 3| = 5$ since it is not in a “standard” form.

This method of using distance on a number line can be carried into absolute value inequalities. Students can verbally express the meaning of the inequality using the distance concept and then use a number line to help find the solution. For example, $|x - 2| > 5$ can be expressed as “the distance from $x$ to $2$
is greater than 5.” On a number line, the initial point is 2 (just like in absolute value equations). The value \(-3\) is 5 units from 2 on the left side of 2 on the number line and the value 7 is 5 units from 2 on the right side of 2 on the number line. However, the inequality allows there to be several numbers whose distance from 2 is greater than 5 (outside of \(-3\) and 7). The solution is \(x < -3\) or \(x > 7\). “When the inequality symbol is the is-less-than symbol, the problem becomes finding points whose distance is less than a given value from the [initial] point ...” (Ellis and Bryson, 2011). So students could use their prior knowledge of absolute value equations to help with absolute value inequalities. This is an extension to their learning of absolute value equations. “Students can check their solution by substituting within each of the graph’s intervals” (Ellis and Bryson, 2011).

The distance concept on a number line can also be used for absolute value equations/inequalities that involve expressions with nonzero coefficients besides 1. According to Ellis and Bryson, students would need to treat this case as if the coefficient is 1. “This approach can also be thought of as substituting a single variable with coefficient 1 for something more complex, a strategy students will use later in trigonometry and calculus” (Ellis and Bryson, 2011). Recognizing the common structure between the two types of inequalities, with and without leading coefficient 1, is SMP 7 Look for and Make Use of Structure. Also, this idea is similar to \(u\)-substitution, which is a common structure idea used in more advanced topics in Algebra, i.e. \(|2x - 9| = 4\) has the same structure as
\[ |u - 9| = 4 \] where \( u = 2x \). After this initial idea, the absolute value can be solved using the methods previously described. The last step would be to substitute the original expression \((2x)\) and divide by the coefficient, which can also be considered the scale factor. Let's look at \(|2x - 9| = 4\). Remember \(2x\) behaves like a single unit. The distance from \(2x\) and 9 is 4. The initial value is 9, so 5 and 13 are 4 units from 9 to the left and right of 9 on the number line. Since the coefficient is 2, divide both values by 2. The solution set is \(x = \frac{5}{2}\) and \(x = \frac{13}{2}\). “We have found that students have more success doing the division after setting up the visual representation with the coefficient intact. Errors from incorrectly adding or subtracting fractional values are visually eliminated” (Ellis and Bryson, 2011).

The ability to identify and represent the same mathematical idea in different ways allows students to understand the idea conceptually. Moving from one representation to another also deepens one’s understanding, strengthens one’s ability to solve problems and allows one to be more flexible in using strategies. “Connectedness between different representations develops insights into understandings of the essence as well as the many factors of a concept” (Even, 1998). Often, solving absolute value equations and inequalities is taught procedurally, with algebraic manipulation. Showing students how to solve these problems using the concept of distance on a number line will provide another representation (geometric) that is more visual in nature. This representation can allow students to use verbal representations, make connections between the
geometric representation and the symbolic representations of a procedure, and as a result, students can deepen their understanding of absolute value. The way that mathematics topics are represented creates a foundation to understanding mathematical ideas. “When students gain access and when they can create representations to capture mathematical concepts or relationships, they acquire a set of tools that significantly expand their capacity to model and interpret physical, social, and mathematical phenomena” (NCTM Principles and Standards for School Mathematics, 2000).

Missing in the Current Research/Literature

There have been articles written about using a number line to solve absolute value equations and inequalities. However, in my review of the literature, there has not been any study with a specific emphasis on solving absolute value equations and inequalities using the concept of distance and transformations on the number line. Also, the studies were conducted either at the college level or internationally. In these studies, students did not use the number line to find distance. Rather, the number line was used as a tool to test numbers to find the interval that makes an inequality true. Also, none of the research studies or practitioner articles addressed absolute value equations and inequalities involving two absolute value expressions (for example, $|x - 3| = |x + 6|$).
Connection Between this Study and Existing Research/Literature

The topic addressed in this study fits into the existing research because equations and inequalities are topics used throughout higher-level mathematical topics. Solving absolute value equations and inequalities this way allows students to be exposed to techniques that they will use in higher-level mathematics classes. Research has shown that multiple representations are important in helping students deepen their understanding. The concept of distance on a number line is another representation that students can connect to other representations and to procedural methods. There have been teacher practitioner articles that have described this method used in middle and high school. These articles describe that this method allows students to conceptually understand absolute value equations and inequalities, allows them to apply their prior knowledge of distance on a number line to help find solutions, and helps them with more complex absolute value equations and inequalities. However, these articles give anecdotal evidence rather than specific evidence or student data.

I predict students will have more success with solving absolute value equations and inequalities using a number line. Before finding the solutions, students will need to verbalize in words the meaning of the absolute value equation or inequality. This first step helps students to start thinking about how they would use a number line. The number line provides a visual representation. By using this method, the stress of applying several
computations decreases. When students have to solve these types of equations and inequalities or ones similar, they will conceptually be able to work their through the problem to find the solution.
CHAPTER THREE
METHODOLOGY

Lessons

I implemented conceptual lessons on solving absolute value equations and inequalities in two Honors Algebra II classes at Beaumont High School in the Beaumont Unified School District. These lessons were based on the ideas presented in the NCTM article, *A Conceptual Approach to Absolute Value Equations and Inequalities*, by Ellis and Bryson (2011). Absolute value equations is a concept covered in Algebra I, so students should have some prior knowledge of this topic. In Algebra II, solving absolute value equations is reviewed. Absolute value inequalities is a new concept to most students. The following lessons took 6 days to implement (see Appendix C for handouts).

- Activity 1, Cabazon, was designed for students to start thinking about distance on a number line. The idea was for students to determine where they could place a new Cabazon on a number line. The coordinate may be negative, but the distance from the original Cabazon to Beaumont High School would be the same as the distance from Beaumont High School to the new Cabazon.

- Activity 2, Mini Lesson on Distance, asked to find the distance between two values, i.e. what is the distance from 2 to 3? Students had to explain their answer, represent their answer as an equation and on a number
line. The goal was for students to understand that distance is associated with difference or subtraction. This concept is important because it leads to the initial values in absolute value equations and inequalities. For example, \(|x - 3| = 5\), the distance from 3 to a number \(x\) is 5. Since distance is associated with subtraction, 3 would not be negative.

- Activity 3, Understanding Absolute Value Equations, was implemented on the same day as activity 2. There was time given for students to work individually on this task. Students compared their answers with their group members. Time was given for students to share with the class. As a class, we were able to develop the academic vocabulary to define absolute value, describing the meaning of absolute value equations and representing absolute value in terms of distance on a number line.

- Activity 4, Multiple Representations, continues building on concepts from previous activities. Students were given several absolute value equations to represent and solve in three different ways: verbally (translate the problem using distance), graphically on a number line, and algebraically.

- Activity 5, Isolating the Absolute Value, extended the previous activity so that students had to determine what to do when there is a number on the outside of the absolute value and it is equal to a number. For example, 
  \[3|\frac{x}{2}| = 18\]  and  \[|-2x| + 4 = 10\]. Students explored the meaning of these equations.
• Activity 6, Absolute Value Inequalities, and Activity 7, Pairs Check: Absolute Value Inequalities, both extended the concepts that students learned about in absolute value equations and applied them to absolute value inequalities. A pairs check is an activity done in a group of 2 - 3 students. Each group is given one paper. The students take turns alternating solving problems. Before each student begins the next problem, they must check the previous problem done by their partner. If they agree with the answer, then the student writes their initials by the problem.

• Activity 8 was a domino card activity. This activity was a resource from the Strengthening Mathematics Instruction (SMI) professional development. There were sets of 24 cards. One card has the word START on the left side and a problem, absolute value equation or inequality, on the right side. This is the card that starts the domino. The next card will have the answer, the solution represented on the number line, to the problem on the start card on the right and a new problem on the right side. The goal is the match each absolute value equation or inequality, on the right side of each card, with the correct number line representation, on the left side of another card. All the 24 cards must be matched correctly. The last card has a number line on the left and the word END on the right. If you get to this card and have not used all the
other cards, then there is a mistake. Also, there was no pencils/pen or paper used during this activity.

- The first class that I implemented this activity, I gave students only the first 12 cards to match in their groups. When circulating the room, there were several great conversations. I noticed that students would first say out loud the meaning of the absolute value equation or inequality and then they would determine which number must be in the middle of the number line. Students applied the concept of distance on the number line. It was like they were visualizing the number line in their heads. I was surprised to see that students finished quicker than I had anticipated. Each group was paired with another group to check their cards. Most groups matched the cards correctly. The groups that did not match the cards correctly had an opportunity to discuss their mistakes and misconceptions in their groups and as a class. I decided to change how I would implement the domino activity in my other Honors Algebra 2 class. Instead of giving them half of the cards, each group was given all 24 cards. They still could not use pencils/pens or paper. The outcome was somewhat different. Students still had great conversations about the cards and they followed the same process like the other period, stating the meaning of the equation or inequality and going from there to find the correct number
line. However, students did not finish matching the cards as quickly. None of the groups finished during that one class period. Most groups finish by the end of the period the next day.

Pre- and Post-Assessments

Before the lessons were implemented, a pre-assessment was given to determine students’ prior knowledge of solving absolute value equations and inequalities and the approaches they take to solve these types of problems. The assessment consisted of 10 questions, and students were given two days, five questions each day to complete the assessment. Most questions on the assessment focused on determining student’s level of conceptual understanding of absolute value equations and inequalities since students were asked to explain their thinking. For example, one problem stated, “the solution to an absolute value equation is shown on the given number line. Write this equation and explain your reasoning.” Other problems could have been done procedurally depending on the approach the student decided to take on the problem. Written responses on the pre-assessment were analyzed to identify specific conceptual understandings and misconceptions that students had with regard to absolute value. Their responses were also checked for procedural fluency and quality of explanations including connections between concepts and procedures. The results will be discussed in a subsequent chapter.
The outcome of this assessment gave me an insight into students’ prior knowledge. It also gave me a better understanding of how the lessons should be implemented and questions I could ask students during the lessons to help them develop conceptual understanding of absolute value. I used the lessons that I had already created as a foundation and revised them based on research that was found on this topic and previous experience with teaching these concepts. I continued to develop student thinking on the concept of absolute value through the lessons and also connected the conceptual approach to procedural approach. The lessons took 6 days (one hour each) to implement and consisted of some direct instruction, small group discovery activities, and mathematical discussions, both small group and whole group. The tasks given to students included higher-level demand tasks. Facilitation of activities and discussions ensured that students were connecting procedural approaches to the concept of distance on a number line. The number line was used as a tool to help students deepen their understanding while solving the problems and not just as a way to represent an answer.

The next part of the study was the post-assessment. The questions were similar to the pre-assessment. There were also some extension problems such as solving $|x - 5| = |x + 2|$ and explaining the reasoning process. The goal was to see if students have deepened their conceptual understanding of absolute value equations and inequalities, and if they were flexible with their use of approaches as the problems increased in complexity.
In the article, *Mathematical Tasks as a Framework for Reflection: from Research to Practice*, Stein and Smith (1998) described ways in which tasks can be approached with varying levels of cognitive demand. The lower level demands are tasks that ask students to memorize procedures in a routine manner. Tasks that have students conduct a procedure without connections are also lower-level demand task. The higher-level demand tasks are described as procedures with connections and doing mathematics. “Tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking” (Stein and Smith, 1998). The pre and post assessments for my study were designed to see if students are engaging with the tasks at a low or high level by asking them to not only solve the problems but explain their reasoning. This indicated whether they were using procedures with or without connections to the concept of distance.

Since all students took the assessments as part of a regular classroom lesson routine, assessments were collected using standard school practice. The data from the assessments that was used for this research only came from those students who agreed (and who had parent consent) to have their data used for this study.

**Interviews**

The last part of the study was student interviews. Some of the students who agreed (and who had parent consent) to participate in interviews were
selected for an interview (see Appendix D or interview questions) to help me understand/clarify their thinking and any misconceptions they had as indicated by their responses on the post-assessment. I took notes during the interview process. Students were selected for an interview using the following criteria:

- written explanation is left blank or is incomplete
- written explanation is a list of procedural steps
- written explanation doesn’t match the algebraic and/or pictorial representation
- nonstandard answers, either correct or incorrect
- answer only, student did not elaborate upon their process of solving the task

In order to study and describe student thinking with regard to absolute value and the impact of the lessons on their thinking, data was analyzed qualitatively. In particular, student responses from the assessments were described and categorized in terms of the types of responses they gave, such as procedural versus conceptual, the cognitive level of those responses (procedures without connections to concepts or procedures with connections to concepts), the types of solution strategies and representations used as the complexity of the problems change, as well as error patterns and misconceptions. The pre and post assessments were compared to identify changes in the development of their conceptual understanding and quality of written explanations as a result of the lessons. Interviews helped in understanding student thinking that were not visible
in the written responses. For example, if a student has a correct algebraic solution but has an incomplete or missing written explanation, the interview helped to see if the student can explain the concept verbally. The student may actually have had conceptual understanding but lacked the skills to communicate this in writing. Interviews also helped to identify whether incorrect responses are indeed misconceptions or rather just mistakes due to errors in calculations or lack of attention to detail in written work. The results from the data helped improve student learning and instruction.
CHAPTER FOUR

RESULTS

Pre-Assessment Results

The data from the first part of this study came from the pre-assessment which was used to determine student’s prior knowledge of absolute value equations and inequalities and their level of conceptual understanding of absolute value. The results are discussed here in order by question on the assessment.

Table 1. Pre-Assessment Part 1 Question 1

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Responses:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.) List AT LEAST 3 facts about</td>
<td>“When you solve it, it is always positive” (Participate 27, 2016)</td>
</tr>
<tr>
<td></td>
<td>“Distance from zero” (Participate 5, 2016)</td>
</tr>
<tr>
<td></td>
<td>“If the value inside is an expression the subtraction signs become addition” (Participate 11, 2016)</td>
</tr>
<tr>
<td></td>
<td>“The absolute value of x is x” (Participate 30, 2016)</td>
</tr>
</tbody>
</table>

Most students on question 1 (Table 1) did not mention distance in their responses. They mostly implied that the absolute value of a number will always...
be positive. There there only 8 students out of 39 that stated distance as one of their facts.

Table 2. Pre-Assessment Part 1 Question 2

<table>
<thead>
<tr>
<th>Question:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.) What is the value of</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Responses:</th>
</tr>
</thead>
<tbody>
<tr>
<td>“The value of</td>
</tr>
<tr>
<td>“The value of</td>
</tr>
</tbody>
</table>

All 39 students stated that the absolute value of 4 is 4. However, their explanations indicated whether they conceptually understand this answer. There were 32 students that responded like the first response in Table 2. Again, these students know that the absolute value of a negative number is positive, but their explanations indicate a lack of what this means. There were only 7 students to respond with an explanation that relates to distances.
Table 3. Pre-Assessment Part 1 Question 3

<table>
<thead>
<tr>
<th>Question:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3) The solution to an absolute value equation is shown on the number line below. Write this equation. Explain your reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Responses:</th>
</tr>
</thead>
<tbody>
<tr>
<td>“</td>
</tr>
<tr>
<td>“x =</td>
</tr>
<tr>
<td>“</td>
</tr>
<tr>
<td>“x &gt; 2 &lt; 8 The two dots indicate that x is in the middle of 2 and 8 as represented in the equation. x is greater than 2 but less than 8.” (Participate 33, 2016)</td>
</tr>
<tr>
<td>“2 ≤ x ≤ 8 The possible number can [be equal to] or is greater than 2. The possible number is [equal to] or less than 8.” (Participate 7, 2016)</td>
</tr>
</tbody>
</table>

None of the students in this study were able to get question 3 correct. In fact, there were 10 students that left this question blank. This question was a higher cognitive level because it requires working backwards. Instead of given an absolute value equation and use the number line to find the answers, students were given a number line to help find the absolute value equation. There were some students that had answers with absolute value. However, their answers were based on the spaces between x = 2 and x = 8. The other types of answers that students gave were intervals, such as 2 < x < 8.
Question 4 (Figure 1) is similar to question 3 (Table 3). Students would have to work backwards to find the absolute value inequality that matches the given number line. None of the students were able to find the correct inequality. The responses to this question was similar to question 3. However, there were more students to write intervals such as $x < -8$ or $x > 0$ and $-8 \leq x \leq 0$.

On question 5 (Figure 2), there were 2 students that left this question blank, 10 students to respond yes and 27 students to respond no. The answer is no, which is the response that the major of the students answered. Their explanations showed that they did not conceptually understand why the answer is no. A common response was, “no because negatives in an absolute value becomes positive.” These students could possibly be thinking that the absolute
value equations $|x - 4| = 3$ and $|4 - x| = 3$, both change to $x + 4 = 3$ and $4 + x = 3$. Most students believe that “anything in the absolute value turns positive,” including expressions. This was a common misconception.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Write out what the problem means</th>
<th>Solve for $x$, show all work</th>
<th>Explain how you solved each problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.) $</td>
<td>x</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>(B.) $</td>
<td>x + 4</td>
<td>= 3$</td>
<td></td>
</tr>
<tr>
<td>(C.) $</td>
<td>x - 6</td>
<td>= -2$</td>
<td></td>
</tr>
<tr>
<td>(D.) $</td>
<td>2x - 7</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>(E.) $</td>
<td>x</td>
<td>&lt; 3$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. Pre-Assessment Part 2

The second part of the pre-assessment is shown in Figure 3. Students were given absolute value equations and inequalities. For each problem, students needed to write out what the problem means (before solving the problem), solve for $x$ (showing all work) and explain how they solved each problem.

- Question A: Approximately 18% of students correctly solved this equation (7 students). Most students were able to find $x = 5$ as an answer, but they were missing $x = -5$. The student explanations were procedural. They did not use the concept of distance. For example, one student stated, “The bars show $x$ is positive no matter what the only answer there could be is 5 because it is equal to itself” (Participate 3,
Another student responded, “set $x$ by its negative or positive number” (Participate 29, 2016).

- **Question B:** Approximately 8% of students correctly answered this problem (3 students). The three students that got this problem correct used guess and check. They substituted values that would make the absolute value equation true. Most of the students set up the equation $x + 4 = 3$ and solved for $x$ to correctly get $x = -1$. However, this was only one answer. Students did not use the concept of distance. Their work and explanations were procedural. For example, one student stated, “I put $x + 4$ out of the absolute value bars and brought down the three and subtracted $-4$ and got $x = -1$” (Participate 20, 2016).

- **Question C:** Approximately 3% of students correctly answered this problem (1 student). The one student that responded correctly stated, 

"$|x - 6| = -2$ no solution, the result cannot be negative. It has to be positive" (Participate 10, 2016). This student did not use the concept of distance, but they understood that absolute value has been positive. There was another student that did not state the answer was no solution but instead wrote, “I’m not sure about this one because I was taught that positive numbers are always the outcome with an absolute value sign” (Participate 22, 2016). The majority of the other students used a procedure by setting up equations. For example, some students changed the sign for -6 in the absolute value and solved $x + 6 =$
−2. Some students kept the -6 and changed the −2 to positive and solved \( x - 6 = 2 \). Another equation that students were writing was \( x - 6 = -2 \). They were following a procedure like question B. However, if they would have checked the solution to this equation, \( x = 4 \), they would notice that answer does not result in a true statement. There were some students to describe this problem as the absolute value of \( x - 6 \) is \( -2 \). They did not use distance in their meaning of this problem.

- Question D: Approximately 5% correctly answered this problem (2 students). The two students who got the correct answer used the guess and check method. They substituted values in for “\( x \)” that would make the equation true. Some students described the meaning of this problem as, absolute value of 2 times \( x - 7 \) is equal to 5. Some students wrote the equation \( 2x - 7 = 5 \) and solved for \( x \) to get \( x = 6 \). This is one correct solution to the equation, but they are missing the other solution. Another equation that some students incorrectly created was \( 2x + 7 = 5 \). The students are changing the −7 in the absolute value to positive.

- Question E: There was 0% to get this problem correct. Some students stated the answer as \( x < 3 \) and graphed the inequality on a number line. This does not take into account what happens when \( x \) is less than or equal to −3. There were also students to list some values that would make the inequality true if the values were substituted in for \( x \), such as 0, 1, and 2.
Post - Assessment Results

The data from the post-assessment was to determine if their level of conceptual understand of absolute value equations improved. Also, to determine the flexibility of their strategy depending on the complexity of the problem. The results are discussed here in order by question on the assessment.

Table 4. Post - Assessment Part 1 Question 1

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.) What is the value of</td>
<td>-7</td>
</tr>
<tr>
<td></td>
<td>“7 because the distance of 0 to</td>
</tr>
<tr>
<td></td>
<td>“7 because absolute value deals with distance and there is no such thing as negative distance.” (Participate 16, 2016).</td>
</tr>
<tr>
<td></td>
<td>“7 because anything in the absolute value sign becomes positive” (Participate 5, 2016).</td>
</tr>
</tbody>
</table>

Question 1 (Table 4) on the post - assessment part 1 is similar to question 2 on the pre-assessment part 1. On this question, there were 21 students to use distance in their explanations compared to only 7 students in the pre-assessment.
Table 5. Post - Assessment Part 1 Question 2

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.) The solution to an absolute value equation is shown on the number line below. Write this equation. Explain your reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>“</td>
</tr>
<tr>
<td>“</td>
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<tr>
<td>“</td>
</tr>
</tbody>
</table>

Question 2 (Table 5) on the post - assessment part 1 is similar to question 3 on the pre-assessment part 1. There were 37 students that answered this problem correctly and had a response similar to those in Table 5. The 2 students that did not get this problem correct wrote the inequality |x – 5| < 3 instead of an equation.
Table 6. Post - Assessment Part 1 Question 3

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3.) List AT LEAST 3 facts about $</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>“$</td>
</tr>
<tr>
<td></td>
<td>“The absolute value of $y$ is $y$” (Participate 37, 2016).</td>
</tr>
</tbody>
</table>

Question 3 (Table 6) on the post - assessment part 1 is similar to question 1 on the pre-assessment part 1. There were 27 students that used distance in their explanation. However, there were still several students to state the absolute value of $y$ is $y$. These students failed to recognize that the variable can be positive or negative. If the variable is negative then the absolute value would be the opposite of the variable (i.e. positive)

Table 7. Post - Assessment Part 1 Question 4

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4.) Is there a difference between $</td>
<td>x - 5</td>
</tr>
<tr>
<td></td>
<td>“There is no difference because 5 will still be a positive number because the equation is still subtracting” (Participate 31, 2016).</td>
</tr>
</tbody>
</table>
Question 4 (Table 7) on the post-assessment part 1 is similar to question 5 on the pre-assessment part 1. There were 30 students that stated no to the two absolute value equations being different. These students had responses similar to those in Table 7.

Table 8. Post-Assessment Part 1 Question 5

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5.) The solution to an absolute value equation is shown on the number</td>
</tr>
<tr>
<td>line below. Write this equation. Explain your reasoning.</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>![Number Line]</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>way the distance between -7 and x is 2&quot; (Participate 10, 2016).</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>distance and add or subtract since it's the distance from -7&quot; (Participate 23, 2016).</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>2016).</td>
</tr>
</tbody>
</table>

Question 5 (Table 8) on the post-assessment part 1 is similar to question 3 on the pre-assessment part 1 and question 2 on the post-assessment part 1. The two solutions are negative values on the number line, so students would need to consider this when creating their absolute value equations. There were three students that answered the problem incorrect and 36 students that answered correctly. The mistake that these students made was similar to the
mistake made on question 2 on the post-assessment part 1 (Table 5). They wrote an absolute value inequality instead of an absolute value equation.

Table 9. Post-Assessment Part 1 Question 6

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6.) The solution to an absolute value inequality is shown on the number line below. Write this inequality. Explain your reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
</tr>
<tr>
<td>“$</td>
</tr>
<tr>
<td>“$</td>
</tr>
</tbody>
</table>

Question 6 (Table 9) on the post-assessment part 1 is similar to question 4 on the pre-assessment part 1. There were 34 students that answered this problem correctly. The common mistake for those that answered the problem incorrectly was the inequality symbol. The correct answer is $|x + 4| > 4$, but some students wrote the inequality as $|x + 4| < 4$. Another mistake that students
made was miscounting. For example, some students wrote $|x + 4| > 3$, most likely counting the distance from $-4$ to the two points $0$ and $-8$ as $3$ instead of $4$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Write out what the problem means</th>
<th>Solve for $x$. Show all work</th>
<th>Explain how you solved each problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.) $</td>
<td>x</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>(B.) $</td>
<td>x + 4</td>
<td>= 3$</td>
<td></td>
</tr>
<tr>
<td>(C.) $</td>
<td>x - 6</td>
<td>= -2$</td>
<td></td>
</tr>
<tr>
<td>(D.) $</td>
<td>2x - 7</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>(E.) $</td>
<td>x - 4</td>
<td>&lt; 3$</td>
<td></td>
</tr>
<tr>
<td>(F.) $</td>
<td>x - 5</td>
<td>=</td>
<td>x + 3</td>
</tr>
</tbody>
</table>

Figure 4. Post-Assessment Part 2

The second part of the post-assessment is shown in Figure 4. Students were given absolute value equations and inequalities. As on the pre-assessment, for each problem, students needed to write out what the problem means (before solving the problem), solve for $x$ (showing all work) and explain how they solved each problem.

- **Question A:** Approximately 95% of students correctly answered this problem (37 students). All students used a number line to find their answer. One incorrect answer was 5. This was a common mistake on the pre-assessment. The student is missing the other value that would be a
distance of 5 units from 0. Another mistake was using 5 as the initial value and stating that “the distance from \( x \) to a number is 5. \( x = 0, 10 \).”

- Question B: Approximately 95% of students correctly answered this problem (37 students). Most of these students explain their reasoning using the concept of distance. The response of one student who answered incorrectly was, “The distance from \(-4\) to \( x \) is 3” (Participate 11, 2016). Then this student wrote two equations, \( x + 4 = 3 \) and \( x + 4 = -3 \), and made a calculation error to get the solutions \( x = -1 \) and \( x = -4 \). Making sense of the problem and the answer, they should have realized that \(-4\) cannot be an answer because it is not three units from \(-4\). In any case, this student showed conceptual understanding of the equation. Another student stated the meaning of the problem as “the distance from \( x \) to 4 is 3” (Participate 34, 2016). This student used a number line to solve this problem, 4 was in the middle of the number line and 1 and 7 was on each side of the number line. There was a misunderstanding of using 4 instead of -4. Since absolute value relates to distance and distance can be represented by using subtraction, the problem can be rewritten as \(|x - (-4)| = 3\).

- Question C: Approximately 85% of students correctly answered this problem (33 students). More students were able to recognize that distance cannot be negative. All students described the meaning of this problem as “the distance from \( x \) to 6 is -2.” However, the students that got
this problem incorrect used a number line and placed 6 in the middle of the number line, 4 on the left side and 8 on the right side. Students that answered this problem correctly may have used a number line as well. These students could not complete the number line, so they would write “no solution”. In their explanations, they talked about distance cannot be represented as a negative number.

- **Question D:** Approximately 97% of students correctly answered this problem (38 students). Most of these students explain their reasoning using the concept of distance. The one student that got this problem incorrect, stated the meaning of the problem as “the distance between 7 and \(x\) is 5 divided by 2” (Participate 9, 2016). This student used a number line to solve this problem. The number line had 7 in the middle, 5 on the left side and 9 on the right side. Below the number line, the student wrote \(\frac{5}{2} = 2.5\) and \(\frac{9}{2} = 4.5\).

- **Question E:** Approximately 92% of students correctly answered this problem (36 students). Most of these students explain their reasoning using the concept of distance. This problem was an extension to the pre-assessment. The problem that was similar on the pre-assessment part 2 was \(|x| < 3\). The post-assessment problem had a binomial expression in the absolute value. Students that got this problem incorrect solved the problem algebraically. They created the following inequalities: \(x - 4 < 3\) and \(x - 4 < -3\). The second inequality is incorrect. The students did not
change the inequality symbol. This is an example for mistake that can be made when following a procedure and not conceptually understanding the problem.

- Question F: Approximately 49% of students correctly answered this problem (19 students). This problem was an extension to the pre-assessment part 2. None of the problems from the pre-assessment were similar to this problem. The percentage of students that got this problem correct is low compared to the results of the assessment. However, the percentage is high if we consider it as an extension problem. Most students were able to describe the meaning of this problem as “the distance from $x$ to 5 is the same as the distance from $x$ to -3.” Some students struggled with finding the correct answer, but the students that solved this problem correctly used a number line with -3 and 5 marked, and found the value that is equidistant from -3 and 5 to be 1.

Interviews Results

There were 22 students that agreed to participate in the interview part of this study. However, there were 12 students that were selected for an interview based on their responses to the post-assessment part 2. Below is the criteria used to select students for an interview and the results of the interviews.
Written explanation is left blank or is incomplete:

- Participate 11: There was one student that wrote the meaning of each problem. The student used a number line to help solve the problems. When the student was suppose to explain how he/she solved the problem, he/she used that space provided to solve the same problem algebraically. During the interview I asked the student, what was he/she thinking about problem A? The student stated, “I found the numbers that were 5 spaces from 0. Those numbers are -5 and 5.” I ask the student the same question regarding problem B. The student said, “The middle number shifts to -4 and the two numbers that would be three spaces from -4 is -7 and -1.” Then I ask about problem F. The student replied, “I started with the meaning of the problem. The distance from -3 and 5 is the same as the distance from $x$ and -3. Then I put 5 and -3 on the number line and found the distance between. The answer is 1.” Although the student did not write an explanation, the student was able to explain the problems verbally using appropriate academic vocabulary.

Written explanation is a list of procedural steps:

- Participate 32: There was a student to explain the mean of Question B on the post-assessment as “the distance from $x$ to -4 is 3.” The student then solved the problem using a number line correctly. The student explained how he/she solved the problem as “I solve the problem by plugging both
distance.” The explanation of their solution process was a description of procedural steps, and it was unclear if this student understands the concepts of distance. I asked the student how he/she figured out problem B. The student stated, “I wrote out the problem, the distance from \( x \) to -4 is 3. It made it easier. Then I put -4 in the middle and add 3 to each direction to get -7 and -1." I asked the student why he/she would add 3 to each direction. The student replied, “it the distance.” Then the student said, “I plugged -7 and -1 into the absolute value equation.” I asked the student why he/she plugged those values into the equation. The student took a long pulse. It seemed like he/she did not know why he/she did that step, but he/she replied, “To check the answers.” The interview results show that this student did conceptually understand this problem.

- Participate 21: On Question E, there was a student to describe the meaning of the problem as, “The distance from \( x \) to 4 is less than 3.” This student attempted to solved the problem algebraically by writing to inequalities, \( x - 4 < 3 \) and \( x - 4 < -3 \). Then the student solved the two inequalities to get \( x < 7 \) and \( x < 1 \). The student wrote their explanation as, “change answer to 1 negative and 1 positive then solve from \( x \)’s.” This student did not use the procedure correctly and their explanation of their solution process was a list of procedures, so it was difficult to understand if this student understood this problem conceptually. I asked the student can he/she solve it a different way. The
student immediately told me, “using a number line.” I asked the student to show me. He/she started reading the meaning that he/she wrote. Then the student drew a number line and placed 4 in the middle of the number line. The student placed -7 on the right side of 4 and 7 on the left side of 4. I asked the student why he/she wrote 7 and 7. The student quickly erased -7 and put 1 and stated, “the distance is 3 on both sides.” The student started writing open circles above 1 and 7 and then said, “I have problems with understanding the direction, inwards or outwards.” I asked the student to read the meaning of the problem again. After reading it, the student represented the solutions going inwards. I asked the student why, he/she said, “the distance has to be less than 3. Those are the numbers in the middle.” This student used the algebraic method on all the problems and like this one; he/she made computational errors. The student could have solved the problem using a number line.

**Written explanation doesn’t match the algebraic and/or pictorial representation:**

- Participate 29: For Question C on the post-assessment, one student described the meaning of the problem as, “the distance from $x$ to 6 is -2.” Then the student had a number line with the middle number as -2 and closed dots on -8 and 4. The two dots were connected with a line segment, which represents an inequality instead of an equation. Above
the line segment, the student wrote “No Solution.” Their number line representation did not match the answer. I asked the student what he/she was thinking when solving this problem. The student stated, “The answer cannot be negative.” I asked him/her why? He/she said, “The equation is equal to -2 and the distance cannot be negative.” I realized that the student did not use the correct vocabulary in the initial explanation. However, he/she did understand why it was no solution. I was still curious about the number line representation. I asked about the number line. The student replied, “I thought that the number line did represent no solution.” This student had a misconception on how a number line that represents no solution would look.

• Participate 11: There was another student whose work did not match the answer on Question C on the post-assessment. They were able to explain the meaning of the problem as “the distance from \( x \) to 6 is -2.” Under the work section, he/she drew a number line with 6 in the middle and one 4 on the left side of 6 and another 4 on the right side of 6. In the explanation section, he/she wrote “no solution?” The number line did not match the answer. This student used number lines for all of the questions and only Question C and F were done incorrectly. The student understands how to solve the problems. However, I was wondering if she conceptually understand the steps to these problems. I asked how he/she figured this problem out. The student stated, “I put 6 in the middle and 4 is
2 spaces from 6." I asked the student to read the meaning of the problem. After the student read the meaning, I asked the student to represent -2. The student was silent and could not provide me with an example. Then the student wrote, “The distance from -2 to -4 is -2.” I asked the student to represent this on a number line. He/She drew another number line with -2 in the middle this time and on the left side of -2 the student wrote -4. I asked the student to count the spaces between the two values. The student counted and said, “There are 2 spaces.” I asked the student to compare the previous statement, “the distance from -2 to -4 is -2,” and the number line with the distance of 2. The student said, “The distance would be 2 not -2.” Then he/she wrote under the explanation section, “There is no such thing as a distance of -2.” I asked the student what was his/her answer, the student replied, “no solution.” A follow-up question that I should have asked the student was how you can represent no solution on a number line. I did not think about this until after the study. However, I do feel the student has a better understanding about this problem.

**Nonstandard answers, either correct or incorrect:**

- Participate 33: For question F, there was one student that I interviewed that wrote the meaning of the absolute value equation as, “The distance from \( x \) to 5 is the same as \( x \) to -3.” This student then used a number to
help solve the problem. The number line had 5 in the middle and 2 on the left side of 5 and 8 on the right side of 5. Both 2 and 8 had a dot above the number. Below the number was $x = 2, 8$. The student explained this as, “I put 5 in the middle and found a distance of 3 to each side.” The explanation described the student’s thinking and he/she was not using their description of the meaning of the problem to help him/her. So I ask the student to read their meaning of the problem again. Then I asked him/her if the meaning matches the number line that they created. The student was quick to say, “No.” I asked why they are not connected, the student stated, “Because -3 is not on the number line.” The student drew another number line, but he/she did not know what numbers to place on the number line. I told the student to look back at the meaning of the absolute value equation and write the values that are given on the number line. He/She wrote -3 and 5 on the number line. I asked the student what he/she thinks would be next. The student pulsed and then told me, “I do not know”. Then I talk about the $x$, which is on both sides of the absolute value equation. I asked the student whether $x$ represents the same value on both sides of the absolute value. The student paused and then said, “yes”. So I asked the student which value would be the same distance from -3 and 5. The student started writing the numbers in between -3 and 5 on the number line. Soon the student wrote next to the number line $x = 1$. I noticed that this student understood the meaning of the absolute
value equations, but needed help showing the meaning on the number line.

- There was one student to solve all the problems on the post-assessment algebraically (by writing and solving equations/inequalities) and using a number line. The student solved all the problems correctly except for question F. The student wrote the meaning of the equation as, “The distance from \( x \) to 5 is equal to the distance of \( x \) and -3.” Then the student wrote the equations \(|x - 1| = 4\) and \(|x - 1| = -4\). He/She solved both equations and got \( x = 5 \) and \( x = 3 \). Below the equations, the student drew a number line with -3 on the left side, 5 on the right side and 1 in the middle. Below the number was written \( x = -3 \) and \( x = 5 \). His/Her explanation stated, “I know that 5 and -3 have to have the same distance so I wrote the numbers between them and eliminate to get 1. To verify I checked the equation \(|x - 1| = 4\) and I got -3 and 5.” This student understood the meaning of the absolute value equation and drew the number line correctly. However, I noticed that he/she was having trouble with the understanding the meaning of the answer. I asked the student what was his/her answer to question F. The student stated, “\( x = -3 \) and \( x = 5 \).” I asked the student how they knew they were right and the student stated, “-3 and 5 have to have the same distance. The number will be 1.” I asked was the answer 1 or 5 and -3. The student said “wait” and looked at the problem more. Then the student said “-3 and 5.” I told
the student to look at the meaning of the absolute value equation that was written by the student. Then I asked what are -3 and 5 on the number line. The student said, “the two values on the sides.” All of a sudden, the student wrote under the number line $x = 1$. I asked why the student changed their answer. The student explained, “-3 and 1 have equal distance from 1 and 5.” The correct answer was there on their number line, but the student did not understand why until the interview.

**Answer only, student did not elaborate upon their process of solving the task:**

- There were students that only had an answer and did not elaborate on their process. However, these students did not agree to be interviewed.
CHAPTER FIVE
CONCLUSION

The Common Core State Standards and Standards for Mathematical Practice allow teachers to explore different approaches to teaching so that students will have a conceptual understanding of mathematics. When students are learning a topic conceptually, they can take their prior knowledge of a mathematical topic and apply it to a new topic. For example, in this study, students used their prior knowledge of the concept of distance and a number line to help make sense and solve absolute value equations and inequalities. This conceptual understanding helped them overcome some of their misconceptions, work with more complex problems, and also helped with retention of the material. After the last activity during the lesson phase of the study, the students went on a spring break that was one week long. When they returned from break, they took the post-assessment. The results of the post-assessment show that students were able to retain the information with no review before the assessment were given.
Conclusion Relevant to Research Questions

**How and to what extent can using a number line develop a conceptual understanding of absolute value equations and inequalities?**

In the beginning of this study the results show that students did not have a conceptual understanding of absolute value equations and inequalities. The students were trying procedural methods to solve the problems, and their explanations did not provide a clear understanding. The results of the post-assessment show a significant improvement in their conceptual understanding. Students were able to use the concept of distance on a number line to help solve absolute value equations and inequalities. Students were also able to write an absolute value equation or inequality given a solution set on a number line. That type of inverse question requires a higher level of thinking. Additionally, about half of the students were able to solve an equation with two absolute value expressions by using the concept of distance a number line. Overall, the student explanations improved. Students were able to explain their understanding more clearly.

**What solution strategies do students tend to use to solve absolute value equations and inequalities?**

The strategies that students tended to use on the pre-assessment and post-assessments to solve absolute value equations and inequalities were different. On the pre-assessment, students tended to try to solve the problems algebraically. For example, most students would set up equations to find
solutions. Often students only found one solution. Some students made computational errors and/or set up the wrong equation(s). Another strategy that students used on the pre-assessment was guess and check. They would substitute values into the absolute value equation or inequality to find values that would make the equation or inequality true. On the post-assessment, there was a drastic change in the methods students use to solve the problems. Most students successfully used a number line to represent the concept of distance. They determined the initial value and found the other values that would have the same distance from the initial value. There were two students that solved using a number line and algebraically and one student that solved the problems algebraically only. The students, who solved the problem(s) algebraically, set up two equations or inequalities to find the solutions. Most of the students (76% of the students) that used the algebraic method were able to find all the answers. However, there were some that still made computational errors.

*Does the strategy depend on the complexity of the problem?*

The strategy did depend on the complexity of the problem for the pre-assessment. Students used the algebraic and guess and check methods. However, there were not that many students that used the guess and check method, and this strategy was used on more complex problems. There were only a few students that used both methods, and a majority of the students tried to use the algebraic method. On the post-assessment, the strategy did not
depend on the complexity of the problem. Students tended to use the same method throughout, either an algebraic method or the distance on a number line method.

**To what extent do students exhibit flexibility in their use of strategies?**

There were some students on the pre-assessment that used two different strategies for solving absolute value equations and inequalities. On the post-assessment the student’s strategy did not vary. They would solve the simple problems the same as the complex problems. Students did not exhibit flexibility in their use of strategies.

**What extensions are students able to make?**

There were two problems that were different on the post-assessment than the pre-assessment. There were approximately 92% of students who were able to solve one of these problems. This problem was an absolute value inequality. On the pre-assessment there was an absolute value inequality involving only "x" in the absolute value. On the post-assessment, there was a binomial (two terms) in the absolute value. The inequality symbol was the same on both assessments. Students are able to find the solution to this problem on the post-assessment by using a number line to represent distance. This problem was similar to an absolute value equation. However, students needed to consider that there are more than two solutions, and they would need to express the solutions as an interval on the number line. The other problem was not like
any problem on the pre-assessment. This problem was an equation with absolute value expressions involving binomials on both sides of the equality, i.e. an equation of the form \( |x - a| = |x - b| \). In previous problems involving absolute value equations of the type \( |x - b| = c \), students would find the initial value and then find two values that are the same given distance from the initial value. When there is an absolute value on both sides and both expressions in the absolute value have a leading coefficient of one, then the students need to find a value that is the same distance from two values, although this distance is not explicitly given. However, this was a good extension to see if students can use their understanding of the concept of distance on a number line to find the correct answer in a slightly new situation. The results of the post-assessment showed that approximately a little less than half of the students in the study correctly answered this problem using the concept of distance. However, the students that missed this problem were at least able to describe the meaning of the equation, and this shows significant growth in their conceptual understanding of absolute value.

**What misconceptions do students have?**

One misconception that most students had on the pre-assessment that they overcame on the post-assessment using the concept of distance was the absolute value of a number is always positive. The post-assessment shows that most students started to use distance to describe their explanations, instead of just “positive.” However, there are still some students that still state in their
explanations on the post-assessment that the absolute value makes the answer positive. Students were able to explain the meaning of absolute value equations and inequalities using distance in their explanations. As a result of the post-assessment, student still have some misconceptions about the problem with the absolute value expressions on both sides. Students used the two values given as two separate initial values and created two number lines, instead of locating these two points on the number line first and then finding the midpoint between them to find the point that is equidistant.

Future Recommendations and Research

Before this study, I noticed that students struggled with understanding why the initial value changes sign from the way it appears in the equation/inequality. For example, the initial value for \(|x + 1| = 4\) would be -1. Students would often ask, “Why is it -1 and not 1?” At the time, I did not understand why. I know this was important for students to understand. In this study, there was a lesson that had students associate distance with subtraction. Students then wrote statements that represent distance using absolute value. The goal was to clear up the understanding of the initial value. During the interview process, it was evident that some students still struggle with why the sign changes. When they explained their process in solving a problem, I would often hear, “the sign switches.” Or they would say, “You told us to switch the sign.” Most students were unable to explain why the
signs change during the interview. My recommendation is to take more time to explain this to students. It is important for students to conceptually understand why the signs change. This process will be just another procedure and they may have trouble conceptually understanding more complex problems that involve absolute value. Another recommendation is to not only consider students written work with regard to their understanding, but also to talk with students to get their true understanding they may not be visible. I was able to learn more about what the student was think during the interview process.

It is also important for students to see the connections between mathematical topics. For example, students can make a connection to the distance formula for the distance between two points in the coordinate plane, which also uses the subtraction of coordinates. One reason why is that if you draw the right triangle determined by the two given ordered pairs and use the Pythagorean theorem to derive the distance formula, then one can see that the lengths of the sides of the triangle are found by subtracting the corresponding coordinates. Also, the distance formula is the two-dimensional analogue of the distance on a number line. In fact, if in the following points are substituted into the distance formula \((a, 0)\) and \((b, 0)\). Then the two points are on the number line created by the x-axis. And, the distance between these two points given by the distance formula is \(d = \sqrt{(a - b)^2 + (0 - 0)^2}\) is equal to \(\sqrt{(a - b)^2}\) and \(\sqrt{(a - b)^2} = |a - b|\). The distance between \(a\) and \(b\) on the number line is \(|a - b|\).
Another related concept that comes up later for those students that take calculus is the epsilon and delta definition of a limit. For example, the limit of $f(x)$ as $x$ approaches $c$ is $L$ if for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$, whenever $|x - c| < \delta$. Students who have an understanding of absolute value as distance have a much stronger foundation to understand this calculus concept.

For future research topics, Students can explore when studying the graphs of functions is absolute value equations and inequalities with two variables. An equation with absolute value expressions on both sides of the equal sign can change to two absolute value equations equal to the same variable. For example, the absolute value equation $|x + 3| = |x - 5|$, can be written as $y = |x + 3|$ and $y = |x - 5|$. If the two functions defined by these two equations are graphed in the coordinate plane, then where the two equations intersect gives the answer to the original absolute value equation. Students can see the connection to systems of equations.
APPENDIX A

INFORMED CONSENT
February 05, 2016

Ms. Melinda Curtis
C/o: Prof. Laura Wallace
Department of Mathematics
California State University, San Bernardino
5500 University Parkway
San Bernardino, California 92407

Dear Ms. Curtis:

Your application to use human subjects, titled “Solving Absolute Equations and Inequalities on a Number Line” has been reviewed and approved by the Institutional Review Board (IRB). The attached informed consent document has been stamped and signed by the IRB chairperson. All subsequent copies used must be this officially approved version. A change in your informed consent (no matter how minor the change) requires resubmission of your protocol as amended. Your application is approved for one year from February 05, 2016 through February 04, 2017. One month prior to the approval end date you need to file for a renewal if you have not completed your research. See additional requirements (Items 1 – 4) of your approval below.

Your responsibilities as the researcher/investigator reporting to the IRB Committee include the following 4 requirements as mandated by the Code of Federal Regulations 45 CFR 46 listed below. Please note that the protocol change form and renewal form are located on the IRB website under the forms menu. Failure to notify the IRB of the above may result in disciplinary action. You are required to keep copies of the informed consent forms and data for at least three years. Please notify the IRB Research Compliance Officer for any of the following:

1) Submit a protocol change form if any changes (no matter how minor) are proposed in your research protocol for review and approval of the IRB before implemented in your research,
2) If any unanticipated/adverse events are experienced by subjects during your research,
3) To apply for renewal and continuing review of your protocol one month prior to the protocols end date,
4) When your project has ended by emailing the IRB Research Compliance Officer.

The CSUSB IRB has not evaluated your proposal for scientific merit, except to weigh the risk to the human participants and the aspects of the proposal related to potential risk and benefit. This approval notice does not replace any departmental or additional approvals which may be required.

If you have any questions regarding the IRB decision, please contact Michael Gillespie, the IRB Compliance Officer. Mr. Michael Gillespie can be reached by phone at (909) 537-7588, by fax at (909) 537-7028, or by email at mgillesp@csusb.edu. Please include your application approval identification number (listed at the top) in all correspondence.

Best of luck with your research.

Sincerely,

Judy Sylva

Judy Sylva, Ph.D., Chair
Institutional Review Board
JS/MG

909.537.7588 • fax: 909.537.7028 • http://irb.csusb.edu/
5500 UNIVERSITY PARKWAY, SAN BERNARDINO, CA 92407-2393
Christina Pierce, Principal
Beaumont High School

Dear Mrs. Pierce,

As you know, I am currently a graduate student at Cal State University, San Bernardo. One of my responsibilities is to implement a research study for my masters' project in Mathematics Education. I am seeking your permission to conduct a qualitative study about expanding student’s conceptual understanding of absolute value equations and inequalities. The goal of this research is to determine how and to what extent the use of the distance concept on a number line deepens conceptual understanding of absolute value as well as develops flexibility in the use of strategies for solving absolute value equations and inequalities. In my Honors Algebra 2 and regular Algebra 2 classes, I will be implementing lessons and assessments based on the “best practice” of using multiple representations with a focus on conceptual understanding of absolute value. The lessons are consistent with current content standards. In order to study the effectiveness of these lessons, my research will involve analysis of student data from these assessments as well as 15 minute interviews with some students. Questions asked in the interview will help with understanding student thinking as well as help identify/clarify misconceptions and improve student learning and instruction. I will be asking for all students in these classes to participate in the study. Before implementing this study, I will make sure students and parents understand the following:

- All students will be required to participate in the lessons and assessments as they normally would in my regular classroom routine.
- Allowing the use of data from the assessments and participation in interviews is voluntary.
- There will be no penalty or loss of benefit should the student decide not to participate in interviews or allow their data from the assessments to be used for the research.
- Participant’s information will remain anonymous (i.e., name and as it relates to assessments and interview responses).
- Participant’s parents or guardians will be required to sign a permission slip.

Thank you for your consideration.

Sincerely,

Melinda Curtis

I hereby grant permission for Melinda Curtis to conduct the aforementioned study.

Principal Name (Please Print)  Principal Signature

909.537.5361

5500 UNIVERSITY PARKWAY, SAN BERNARDINO, CA 92407-2393

The California State University • Bakersfield • Channel Islands • Chico • Dominguez Hills • East Bay • Fresno • Fullerton • Humboldt • Long Beach • Los Angeles
Maritime Academy • Monterey Bay • Northridge • Pomona • Sacramento • San Bernardino • San Diego • San Francisco • San Jose • San Luis Obispo • San Marcos • Sonoma • Stanislaus
Hello Algebra II students,

As you may know, I am a college student at Cal State University, San Bernardino. I am currently conducting research for my masters’ project in Math Education. I am researching student understanding of absolute value and how it affects their performance in algebra. During the next couple of weeks, we will be learning about absolute value equations and inequalities to help improve your understanding. You will take an assessment at the beginning and at the end of this unit. Each assessment will take approximately 30 minutes to complete. The lessons and assessments will be a part of your grade as they normally would in our regular classroom routine. For my research, I plan to use the data from the assessments to learn about your thinking about absolute value. I will also interview some students about this. Each interview will take about 15 minutes and will be recorded with a digital recorder. Your participation in this research is completely voluntary. Your grade for the class activities and the assessments will not be affected in any way should you decide not to participate in interviews or allow your data from the assessments to be used for the research. Please indicate below whether or not you agree for your data to be used this way, and also whether you would agree to be interviewed. Thank you for your help.

Sincerely,

Ms. Curtis

☐ I agree for my data to be used.
☐ If selected, I agree to be interviewed.

☐ I do not agree for my data to be used.
☐ I do not agree to be interviewed.

Print Name: ________________________________
Dear Parent(s)/Guardian(s):

My name is Melinda Curtis and I am currently teaching Integrated Math 2, Algebra 2 and Honors Algebra 2 at Beaumont High School. I am also a graduate student at Cal State San Bernardino (CSUSB). As part of my graduate studies I am doing research for my master’s degree project in Math Education under the supervision of Dr. Laura Wallace, Professor of Mathematics at CSUSB.

**Purpose:**
The goal of this research is to determine how and to what extent using the concept of distance on a number line can help to develop a conceptual understanding of absolute value equations and inequalities and flexibility in the use of strategies for solving problems.

**Description:**
I will be asking for all students to participate in my Honors Algebra II and regular Algebra II classes. In these classes, I am designing lessons and assessments based on the “best practice” of using multiple representations with a focus on conceptual understanding. These lessons will be consistent with current content standards. In order to study the effectiveness of these lessons, my research will involve analysis of student data from these assessments as well as 15 minute interviews with some students. Questions asked in the interview will help with understanding participant’s thinking as well as help identify/clarify misconceptions and improve student learning and instruction. Student interviews will be recorded with a digital recorder.

**Participation:**
While all students will be required to participate in the lessons and assessments as they normally would in my regular classroom routine, allowing the use of student data from the assessments and participation in interviews is completely voluntary and a student may withdraw at any time. The lessons and assessments will be a part of the student's grade as usual. However, there will be no penalty and the student's grade will not be affected in any way should they decide not to participate in interviews or allow their data from the assessments to be used for the research.

**Confidentiality:**
Every effort will be made to preserve the confidentiality of the students. No student names or personal information will be mentioned in the study. I will keep all study records locked in a secure location.
The data collected will be destroyed within one year of collection. At the conclusion of this study, I may publish my findings. Information will be presented in summary format and students will not be identified in any publications or presentations.

**Benefits:**
The potential benefits of participation in this study include developing a deeper, conceptual understanding of absolute value. Strategies implemented in this study may provide constructive and timely feedback for students to improve their learning. I hope that student participation in the study will also benefit other learners through dissemination of this study in the future.

**Contact:**
For any questions or concerns, please contact Dr. Laura Wallace at 909-537-7113 or wallace@csusb.edu, or contact Christina Pierce (principal) at (951)845 - 3171 or cpierce@beaumontusd.k12.ca.us.

Please indicate below whether you give permission for your student’s participation in this study and return the signed form to me.

Thank you in advance,

Melinda Curtis

☐ I grant permission for my child’s data to be used in the aforementioned study.

☐ I grant permission for my child to be interviewed, if selected.

☐ I do not grant permission for my child’s data to be used in the aforementioned study.

☐ I do not grant permission for my child to be interviewed.

---

Student Name (Please print)  Parent/Guardian Signature

[Approval stamp]
Queridos
Estimado(s) padre(s)/tutor(es):

Mi nombre es Melinda Curtis y actualmente estoy enseñando matemáticas integrado 2, Álgebra 2 y álgebra 2 Honores en Beaumont High School. Yo también soy un estudiante de posgrado en Cal State San Bernardino (CSUSB). Como parte de mis estudios de doctorado que estoy haciendo la investigación para mi maestría en educación matemática del proyecto bajo la supervisión de la Dra. Laura Wallace, profesor de matemáticas en CSUSB.

Objetivo:
El objetivo de esta investigación es determinar cómo y en qué medida utilizando el concepto de distancia en un número de línea puede ayudar a desarrollar una comprensión conceptual de valor absoluto ecuaciones e inequaciones y flexibilidad en el uso de estrategias para resolver problemas.

Descripción:
Voy a pedir a todos los alumnos para participar en mis honores Álgebra II y álgebra II clases regulares. En estas clases, estoy diseñando lecciones y evaluaciones basadas en las “mejores prácticas” de utilizar múltiples representaciones con un enfoque en la comprensión conceptual. Estas lecciones serán consistentes con las actuales normas de contenido. A fin de estudiar la efectividad de estas lecciones, mi investigación incluirá un análisis de los datos de los alumnos de estas evaluaciones, así como 15 minutos de entrevistas con algunos estudiantes. Preguntas en la entrevista ayudarán a entender el pensamiento del participante, así como ayudar a identificar y aclarar los conceptos erróneos y mejorar la capacidad de aprendizaje de los alumnos y la instrucción. Entrevistas de los estudiantes serán grabados con una grabadora digital.

Participación:
Mientras que todos los estudiantes estarán obligados a participar en las lecciones y las evaluaciones como lo haría normalmente en mi rutina de aula regular, permitiendo el uso de los datos de los alumnos a partir de las evaluaciones y la participación en las entrevistas es completamente voluntario y un estudiante puede retirarse en cualquier momento. Las evaluaciones serán utilizadas para fines formativos, y no habrá ninguna sanción o pérdida de beneficio en caso que el estudiante decida no participar en entrevistas o permitir que los datos de las evaluaciones que se utilizan para la investigación.

Confidencialidad:
se harán todos los esfuerzos posibles para preservar la confidencialidad de los estudiantes. Ningún estudiante nombres o datos personales serán mencionadas en el estudio. Voy a seguir todos los registros del estudio bloqueado en una ubicación segura. Los datos recogidos serán destruidos en el plazo de un año de recolección. En la conclusión de este estudio, puedo publicar mis conclusiones. La información se presentará en formato de resumen y los estudiantes no serán identificados en las publicaciones o presentaciones.
Beneficios:
Los beneficios potenciales de la participación en este estudio incluyen el desarrollo de un más profundo, la comprensión conceptual del valor absoluto. Las estrategias implementadas en este estudio puede proporcionar retroalimentación oportuna y constructiva para los estudiantes para mejorar su aprendizaje. Espero que la participación de los estudiantes en el estudio beneficiará también a otros estudiantes a través de la difusión de este estudio en el futuro.

Contacto:
Para cualquier duda o inquietud, por favor comuníquese con la Dra Laura Wallace en 909-537-7113 o wallace@csusb.edu, o póngase en contacto con Christina Pierce (principal) al (951)845 - 3171 o Cpierce@beaumontusd.k12.ca.us.

Indique a continuación si desea dar permiso para que el estudiante pueda participar en este estudio y devolver el formulario firmado a mí.

Gracias,

Melinda Curtis

☐ Autorizo a mi hijo/a con datos que se utilizarán en el estudio antes mencionado.

☐ Autorizo a mi hijo/a ser entrevistado, si está seleccionada.

☐ Yo no otorgar permiso para que mi hijo/a con datos que se utilizarán en el estudio antes mencionado.

☐ Yo no conceder el permiso para que mi hijo/a ser entrevistados.

__________________________  ______________________________
Nombre del alumno (favor de imprimir)  Padre/madre/tutor firma
Part 1: Pre-Assessment

Answer the following questions using complete sentences. SHOW ALL WORK.

(1.) List AT LEAST 3 facts about |x|.

____________________________________________________________________________
____________________________________________________________________________
____________________________________________________________________________

(2.) What is the value of |-4|? How do you know? Explain your reasoning.

____________________________________________________________________________
____________________________________________________________________________
____________________________________________________________________________

(3.) The solution to an absolute value equation is shown on the number line below. Write this equation. Explain your reasoning.

Absolute value equation: ________________________
Explanation:
____________________________________________________________________________
____________________________________________________________________________
____________________________________________________________________________

(4.) The solution to an absolute value inequality is shown on the number line below. Write this inequality. Explain your reasoning.

Absolute value inequality: ________________________
Explanation:
____________________________________________________________________________
____________________________________________________________________________
____________________________________________________________________________

(5.) Is there a difference between |x - 4| = 3 and |4 - x| = 3? Why or why not?

____________________________________________________________________________
____________________________________________________________________________
____________________________________________________________________________
Part 2: Pre-Assessment

Solve for $x$ in each of the problems. Show all work. Explain your reasoning.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Write out what the problem means</th>
<th>Solve for $x$. Show all work</th>
<th>Explain how you solved each problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.) $</td>
<td>x</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>(B.) $</td>
<td>x + 4</td>
<td>= 3$</td>
<td></td>
</tr>
<tr>
<td>(C.) $</td>
<td>x - 6</td>
<td>= -2$</td>
<td></td>
</tr>
</tbody>
</table>
(D.)
\[|2x - 7| = 5\]

(E.)
\[|x| < 3\]
Post-Assessment Part 1

Answer the following questions using complete sentences. SHOW ALL WORK.

(1.) What is the value of \(|-7|\)? How do you know? Explain your reasoning.
_____________________________________________________________________________________________
_____________________________________________________________________________________________

(2.) The solution to an absolute value equation is shown on the number line below. Write this equation. Explain your reasoning.

Absolute value equation: ________________________
Explanation:
_____________________________________________________________________________________________
_____________________________________________________________________________________________

(3.) List AT LEAST 3 facts about \(|y|\).
_____________________________________________________________________________________________
_____________________________________________________________________________________________
_____________________________________________________________________________________________

(4.) Is there a difference between \(|x - 5| = 3\) and \(|5 - x| = 3\)? Why or why not?
_____________________________________________________________________________________________
_____________________________________________________________________________________________
_____________________________________________________________________________________________

(5.) The solution to an absolute value equation is shown on the number line below. Write this equation. Explain your reasoning.

Absolute value equation: ________________________
Explanation:
_____________________________________________________________________________________________
_____________________________________________________________________________________________

(6.) The solution to an absolute value inequality is shown on the number line below. Write this inequality. Explain your reasoning.

Absolute value inequality: ________________________
Explanation:
_____________________________________________________________________________________________
Solve for \( x \) in each of the problems. Show all work. Explain your reasoning.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Write out what the problem means</th>
<th>Solve for ( x ). Show all work</th>
<th>Explain how you solved each problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.) (</td>
<td>x</td>
<td>= 5)</td>
<td></td>
</tr>
<tr>
<td>(B.) (</td>
<td>x + 4</td>
<td>= 3)</td>
<td></td>
</tr>
<tr>
<td>(C.) (</td>
<td>x - 6</td>
<td>= -2)</td>
<td></td>
</tr>
</tbody>
</table>
(D.) \[|2x - 7| = 5\]

(E.) \[|x - 4| < 3\]

(F.) \[|x - 5| = |x + 3|\]
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Suggested Lessons Outline

Day 1: Pre-assessment part 1

Day 2: Pre-assessment part 2 & Act 1: Cabazon - Introduce absolute value and relating it to distance.

Day 3: Act 2: Mini Lesson on Distance (students need to understand that distance is finding the difference of two values) & Act 3: Understanding Absolute Value Equations

Day 4: Act 4: Multiple Representations of Absolute Value Equations

Day 5: Act 5: Isolating the absolute value

Day 6: Warm-up (review absolute value equations), Act 6: Absolute Value Inequalities Part 1 & Exit Ticket: Absolute Value inequalities

Day 7: Act 7: Pairs Check - Absolute Value Inequalities

Day 8: Warm-up (absolute value inequalities 1) & Domino Activity

Day 9: Post assessment - Part 1

Day 10: Post assessment - Part 2

Day 11: Interviews

Day 12: Interviews
Algebra 2
Act 1: Cabazon

The owners of the discount mall in Cabazon realize that people have to travel a long distance from towns west of Beaumont to shop at this location. To accommodate the residents living in towns west of Beaumont, the owners decide to build another Cabazon. They looked at the distance from Cabazon to Beaumont and decide that the new Cabazon (Cabazon 2) should be the same distance but on the west side of Beaumont.

(1.) Find the distance from Beaumont to Cabazon (east off of the 10 freeway)? (Use Chromebooks)

(2.) What would be the distance from Beaumont to Cabazon 2 (west off of the 10 freeway)?

(3.) If Beaumont is located on a number line with coordinate zero, what would be the coordinate for Cabazon 1 (old) and Cabazon 2 (new)? Explain your reasoning.

(4.) What do you notice about the distance from Beaumont to Cabazon 1 compared to the coordinate of Cabazon 1?

(5.) What do you notice about the distance from Beaumont to Cabazon 2 compared to the coordinate of Cabazon 2?
(7.) What is the definition of ABSOLUTE VALUE related to distance?


(8.) How does the definition of ABSOLUTE VALUE relate to this problem?


(9.) Since Beaumont actually is at the mile marker 3 miles. How could you find the coordinates for Cabazon 1 and Cabazon 2? Explain your reasoning.


(10.) What do you notice about the distance from Beaumont to Cabazon 1 compared to the coordinate of Cabazon 1?


(11.) What do you notice about the distance from Beaumont to Cabazon 2 compared to the coordinate of Cabazon 2?
Algebra 2

Act 2: Distance

(1.) a.) What is distance from 2 to 3? _____

b.) Explain your answer.

c.) Represent this distance as an equation.

d.) Represent this distance on a number line.

(2.) a.) What is the distance from -2 to 4? _____

b.) Explain your answer.

c.) Represent this distance as an equation.

d.) Represent this distance on a number line.

3.) a.) What is the distance from -5 to -2? _____

b.) Explain your answer.

c.) Represent this distance as an equation.

d.) Represent this distance on a number line.
4.) What operation is associated with finding the distance? Explain.

5.) (a.) What do you notice about the distance from -2 to 3 and the distance from 4 and 9? Explain.

(d.) Represent your answer on a number line.
1.) Define Absolute value:

____________________________________________________________________________

(a) Write out the meaning of each absolute value using a sentence
(b) Draw a number line representing each absolute value and use it to simplify the absolute value

2.)

\[ |5| \]

____________________________________________________________________________

3.)

\[ |-5| \]

____________________________________________________________________________

(a) Write out the meaning of each absolute value using a sentence
(b) Draw a number line representing each absolute value and use that to find all possible values for \( x \)

4.)

\[ |x| = 8 \]

____________________________________________________________________________

5.) Compare problems 2 and 3 to problem 4.

____________________________________________________________________________

____________________________________________________________________________

____________________________________________________________________________

____________________________________________________________________________

____________________________________________________________________________

____________________________________________________________________________
(a) Write out the meaning of each absolute value as distance using a sentence
(b) Draw a number line representing each absolute value and use that to find all possible values for x

6.) \[ |x - 7| = 8 \]

7.) \[ |x + 10| = 16 \]

8.) Compare problem 6 to 7.

6.) Write an algebraic problem for:

\textit{The distance from \(x\) to 4 is the same as the distance from 4 to \(x\).}
Act 4: Multiple Representations

Find the missing representation:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Words (translate the problem using distance)</th>
<th>Using a Number line</th>
<th>Algebraically</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) $</td>
<td>x</td>
<td>= 3$</td>
<td></td>
</tr>
<tr>
<td>2.) $</td>
<td>x + 4</td>
<td>= 5$</td>
<td></td>
</tr>
<tr>
<td>3.) $</td>
<td>x - 10</td>
<td>= 3$</td>
<td></td>
</tr>
<tr>
<td>4.) $2</td>
<td>x - 3</td>
<td>= 20$</td>
<td></td>
</tr>
<tr>
<td>5.) $</td>
<td>2x - 10</td>
<td>= 32$</td>
<td></td>
</tr>
<tr>
<td>6.) $</td>
<td>x - 8</td>
<td>= -4$</td>
<td></td>
</tr>
</tbody>
</table>
Reflection:

________________________________________

________________________________________

________________________________________

________________________________________

________________________________________

________________________________________

________________________________________

95
(a.) Write out the meaning of each absolute value using a sentence.
(b.) Draw a number line representing each absolute value and use that to find all possible values for x.
(c.) Explain your reasoning.

1.) |3x| = 11
   (a.) ____________________________
   ____________________________
   ____________________________
   (b.) ____________________________
   (c.) ____________________________

2.) |x| = 5
   (a.) ____________________________
   ____________________________
   ____________________________
   (b.) ____________________________
   (c.) ____________________________

3.) |-x + 2| = 5
   (a.) ____________________________
   ____________________________
   ____________________________
   (b.) ____________________________
   (c.) ____________________________

4.) |3x| = 18
   (a.) ____________________________
   ____________________________
   ____________________________
   (b.) ____________________________
   (c.) ____________________________

5.) |-2x| + 4 = 10
   (a.) ____________________________
   ____________________________
   ____________________________
   (b.) ____________________________
   (c.) ____________________________
Warm-up: Absolute Value equations

Find the missing representation:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Words (translate the problem using distance)</th>
<th>Using a Number line</th>
<th>Explain how you were able to find your answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.)</td>
<td></td>
<td><img src="image" alt="Number line" /></td>
<td>x = [Blank]</td>
</tr>
<tr>
<td>2.)</td>
<td></td>
<td>x +6</td>
<td>= 5</td>
</tr>
<tr>
<td>3.)</td>
<td>The distance from 11 to a number x is 2.</td>
<td><img src="image" alt="Number line" /></td>
<td></td>
</tr>
<tr>
<td>4.)</td>
<td></td>
<td><img src="image" alt="Number line" /></td>
<td>x = [Blank]</td>
</tr>
<tr>
<td>5.)</td>
<td></td>
<td>2x - 1</td>
<td>= 5</td>
</tr>
</tbody>
</table>
(1.) Name the following inequality symbols:

\[
\begin{array}{cccc}
< & \leq & > & \geq \\
\end{array}
\]

(2.) Write out the meaning of each absolute value using a sentence.

| (a.) |x - 4| = 3 | (b.) |x - 4| < 3 |
|------|-------|------|-------|
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |

| (c.) |x + 5| = 2 | (d.) |x + 5| > 2 |
|------|-------|------|-------|
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |

| (e.) |x - 10| = 5 | (f.) |x - 10| ≥ 5 |
|------|-------|------|-------|
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |

(3.) Draw a number line representing each absolute value and use that to find all possible values for \(x\).

| (a.) |x - 4| = 3 | (b.) |x - 4| < 3 |
|------|-------|------|-------|
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |

| (c.) |x - 10| = 5 | (d.) |x - 10| ≥ 5 |
|------|-------|------|-------|
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
| \text{_______________________________________________} | \text{_______________________________________________} |
(4.) How are the absolute value equations and inequalities similar and different? (i.e. number line, answer, meaning, ect.) *Please explain each similarity and difference.*

<table>
<thead>
<tr>
<th>Similar</th>
<th>Different</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(a) Write out the meaning of each absolute value using a sentence
(b) Draw a number line representing each absolute value and use it to simplify the absolute value

| (1.) |x - 3| < 2                              | (2.) |x + 3| ≥ 2                              |
|---------------------------------|---------------------------------|-------|---------------------------------|
|                                  |                                  |       |                                  |
|                                  |                                  |       |                                  |
|                                  |                                  |       |                                  |
|                                  |                                  |       |                                  |
### Algebra 2

**Act 7: Pairs Check - Absolute Value Inequalities**

<table>
<thead>
<tr>
<th></th>
<th>Student A: __________________</th>
<th>Student B:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a.)</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>Meaning:</td>
<td></td>
<td>Meaning:</td>
</tr>
<tr>
<td>Number Line:</td>
<td></td>
<td>Number Line:</td>
</tr>
<tr>
<td>Answer:</td>
<td></td>
<td>Answer:</td>
</tr>
</tbody>
</table>

|   | 2a.) | \(|x - 7| \leq 2\) | 2b.) | \(|x + 5| > 4\) |
| Meaning: | | Meaning: | |
| Number Line: | | Number Line: | |
| Answer: | | Answer: | |

|   | 3a.) | \(|3 - x| < 1\) | 3b.) | \(|x| < 3\) |
| Meaning: | | Meaning: | |
| Number Line: | | Number Line: | |
| Answer: | | Answer: | |

|   | 4a.) | \(|2x - 1| \geq 5\) | 4b.) | \(|7 - x| \leq 2\) |
| Meaning: | | Meaning: | |
| Number Line: | | Number Line: | |
| Answer: | | Answer: | |
5a.) \(|x - 8| < 4\)  
Meaning: 

Number Line: 

Answer: 

5b.) \(|3x - 4| \geq 5\)  
Meaning: 

Number Line: 

Answer: 

Create your own problems. Solve each problem using a number line. DO NOT use any problems from above.

<table>
<thead>
<tr>
<th>a.) Greater than or equal to:</th>
<th>b.) Greater than:</th>
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<tr>
<th>c.) Less than:</th>
<th>d.) Less than or equal to:</th>
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Warm-up: Absolute Value Inequalities 1

(a) Write out the meaning of each absolute value using a sentence
(b) Draw a number line representing each absolute value and use it to simplify the absolute value

| (1.) |2x - 3| < 5 | (2.) 3 ≥ |x - 1|
|------|------|------|
| Answer: ____________________ | Answer: ____________________ |

| (3.) |x - 3| < - 2 | (4.) |x - 3| > - 2 |
|------|------|------|
| Answer: ____________________ | Answer: ____________________ |

<p>| (1.) |2x - 3| &lt; 5 |
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<p>| (2.) 3 ≥ |x - 1|</p>
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<p>| (3.) |x - 3| &lt; - 2 |
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<p>| (4.) |x - 3| &gt; - 2 |
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<td>Answer: ____________________</td>
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ACT 8: DOMINO ACTIVITY

SMI RESOURCE
$2 < |x+2|$  
$0 < |x| < 2$

$1 \leq x$  
$-1 > |x+1|$

$|x| \geq 2$  
$|x+1| < 2$

$|x-2| < 2$  
$0 < |x| \leq 1$
$1 \neq |x|$

$|x-2| = 1$

$|x-2| \geq 1$

$|x-1| < 2$

$|x-1| > 1$

$0 < |x+2|$

$|x-1| = 2$

END
APPENDIX D

INTERVIEW QUESTIONS
Interview Questions

Students that agree to be apart of this study may be selected to be apart of a 15 minute interview. The interviews will be conducted after the post assessment. Questions asked in the interview will help with understanding student thinking as well as help identify/clarify misconceptions and improve student learning and instruction. Below are the Interview Questions:

1. Can you solve it a different way?
2. What are you thinking?
3. How did you figure it out?
4. Why did you _________? [write that, draw that, etc.]
5. You wrote __________. How did that help you?
6. I noticed that you stopped what you were doing just now. What were you thinking?
7. Why did you change your mind (answer)?
8. I do not know what you mean by that. Will you show me?
9. Will you draw a picture of that?
10. Will you show me with ________? (a number line)
11. You started with this (point) and then went to this (point). Tell me about your thinking.
12. Are you right? How do you know?
13. What do you notice?
14. Is there another way to show me? What is it?
15. What strategy did you use?
16. What would happen if…?

Resource:
https://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf
REFERENCES


