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Upset Paths and 2-Majority Tournaments

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A Thesis

Presented to the

Faculty of

California State University,

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In Partial Fulfillment

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Master of Arts

in

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Abstract

In 2005, Alon, et al. proved that tournaments arising from majority voting scenarios have minimum dominating sets that are bounded by a constant that depends only on the notion of what is meant by a majority. Moreover, they proved that when a majority means that Candidate A beats Candidate B when Candidate A is ranked above Candidate B by at least two out of three voters, the tournament used to model this voting scenario has a minimum dominating set of size at most three. This result gives 2-majority tournaments some significance among all tournaments and motivates us to investigate when a given tournament can be considered a 2-majority tournament. In this thesis, we prove, among other things, that the presence of an upset path in a tournament allows us to conclude the tournament is realizable as a 2-majority tournament.

Acknowledgements

I am grateful for the help of many people. First and foremost, to my mother and my father: your love, prayers and encouragement, without which I wouldn't have been able to follow my passion for mathematics. All success I've had is rooted in the effort they invested in my upbringing. Secondly, special thanks go to my husband Hatim for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. To my lovely daughters, Jana and Deem, your unconditional love gives me the power of making a balance in my life. I would also like to thank my brothers and my sisters for always being next to me when my life gets challenging.

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Chapter 1

Introduction

Imagine the following voting scenario: There is an election in which the candidates are ranked by each voter and Candidate A beats Candidate B precisely when Candidate A is ranked above Candidate B in a majority of the voter rankings. We will call this *k-majority voting.* In this thesis, we represent every candidate as a point. If Candidate A beats Candidate B, we draw an arrow from the point representing Candidate A to the point representing Candidate B. The resulting diagram, which we will call a tournament, can be used as a mathematical tool to decide the winner in a contest. We will refer to the points in the tournament as the vertices and the arrows between the vertices as arcs.

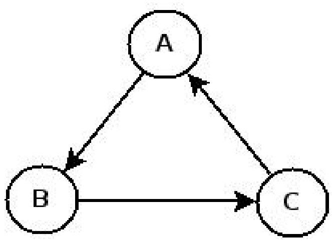


Figure 1.1: A case when we have difficulty picking a winner.

Continuing with our scenario, suppose there are n candidates in an election. You might have imagined that determining a winner is simple, fair, and consistent. This is certainly true when there are only two candidates. It might also be easy when, for example, there are n candidates and there is an arrow directed from Candidate A to every other candidate in the resulting tournament. That means Candidate A beats everyone in the election. However, the task of picking a winner, or winners, is often much more complicated. Consider the following circumstance: Candidate A beats Candidate B, Candidate B beats Candidate C, and Candidate C beats Candidate A as in Figure 1.1.

In this scenario, it seems picking a winner is impossible. In fact, it is complicated whenever this case happens in the resulting tournament. Often, there is not a single best way to determine the winner. In voting theory, one studies the advantages and disadvantages of different methods of selecting a winner in a contest. One method is to find a small collection of vertices, which we call a dominating set, such that for any vertex x in the election, not in the dominating set, there exists a vertex in the dominating set that beats x.

In general, tournaments can have minimum dominating sets that are arbitrarily large [Erd63]. In some voting scenarios, the upper bound for the size of a minimum dominating set is small, no matter how many candidates there are, which makes the task of picking a winner or winners easier. However, in other voting scenarios, the upper bound for the size of a minimum dominating set might be big. The focus of our study is to find what structures in a tournament T that gives T the possibility of having a small upper bound on the size of a minimum dominating set. Specifically, we will search for a structure that, when present in a tournament, will imply that the tournament is a model for a voting scenario in which there is a minimum dominating set of size at most three.

We divide this thesis into six chapters. In this chapter, we introduced the voting problem that inspired us. Chapter two introduces all important definitions and notation that the reader will need to understand our study. Chapter three illustrates previous related work that inspired us to choose this topic. In Chapter four and Chapter five, we develop techniques, identify specific structures, and prove that when these structures are present, we can say that a tournament can have a minimum dominating set of size no greater than three. In Chapter six, we conclude by summarizing and reviewing our findings, and we suggest possible further investigations.

Chapter 2

Preliminaries, Notation and Terminology

In this chapter, we define some relevant terminology and develop some useful notation that we can use to formalize the mathematical model for voting we described in the introduction. We also state some basic results from graph theory. Most of the material that is introduced here can be found in a basic graph theory book such as Graphs and Digraphs, sixth edition, by Gary Chartrand, Linda Lesniak and Ping Zhang [CLZ10].

A *graph* G is a finite nonempty set V of objects called *vertices* together with a possibly empty set E of two element subsets of V called *edges*. A graph is called *complete* when E consists of all two element subsets of V. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(T)$ and $E(H) \subseteq E(G)$. We notate this by $H \subseteq G$.

If $\{u, v\}$ is an edge in a graph G, we write this edge as uv or vu. When the order of vertices in an edge is important, we say that the edge is a *directed edge* or an *arc*. We write uv or $u \to v$ to designate that there is an arc directed from u to v. If all the edges of a graph are directed, then it is called a *digraph*.

Let $D = (V, A)$ be a digraph with a set V of vertices and a set A of arcs, and let $x \in V$. The number of arcs that are directed from x is called the *out degree* of x, written $od(x)$. The number of arcs that are directed to x is called the *in degree* of x, written $id(x)$.

A digraph is called a *tournament* if between every pair of vertices x and y there exists a directed edge, either from x to y or from y to x, but not both. In other words, a tournament T is an orientation of a complete graph. Figure 1.1 is a tournament with the set of vertices $V = \{A, B, C\}$ and the set of arcs $\{AB, BC, CA\}$. In a tournament on n vertices, we say that a vertex x is a *source* if $id(x) = 0$ and $od(x) = n - 1$. We say that a vertex y is a *sink* if $od(y) = 0$ and $id(y) = n - 1$.

In a graph G, for any two vertices $x, y \in G$, we say that x *dominates* y or y *dominates* x if there is an edge between x and y. Also, we say that a set X of vertices *dominates* a set Y of vertices, $X \to Y$, if for every $y \in Y$ there exists $x \in X$ such that $x \to y$. A *dominating set* X in G is a set of vertices such that $X \to V(G)$. A *minimum dominating set* X in G is a smallest set of vertices X such that $X \to V(G)$. The *domination number* of G is the size of a minimum dominating set in G. We notate this $\gamma(G)$. Domination number is a well-studied graph parameter. Let $D = (V, A)$ be a digraph with a set V of vertices and a set A of arcs. If $xy \in A$, we write $x \to y$ and say x *dominates* y or x *beats* y. A *dominating set* X in D is a set of vertices such that $X \to V(D)$. A subset X of the vertex set of a digraph is called a *dominating set* if for any vertex v not in X there exists a vertex $x \in X$ such that x dominates v. There are far fewer papers concerning dominating sets in digraphs and tournaments than in graphs. According to Reid et. al., 90% of the papers on domination consider domination in undirected graphs [RMHH04].

In voting theory, digraphs and tournaments can be used to model a voting scenario and possibly determine a winner. The winner is called the *tournament solution*. When a tournament has a source, a tournament solution is easily found; namely, it is the source. Otherwise, the structure of the dominating set can be used to narrow down a tournament solution.

A path P in a digraph D is a finite sequence of distinct vertices $v_1 v_2 ... v_k$ such that $v_i v_{i+1}$ is an arc in D for $i = 1, 2, ..., k - 1$. We called a path $v_1 v_2 ... v_k$ in a digraph D together with the arc $v_k v_1$ a *cycle*. In other words, a cycle in D is a closed path. In the voting scenario we mentioned in Chapter one, we assumed that there are three candidates: A, B, and C. The relationship between the three is depicted in the tournament shown in Figure 1.1. The path $ABCA$ forms a cycle. This cycle is one structure that makes choosing a tournament solution difficult.

When no cycle exists in a tournament, we call the tournament *acyclic*. A tournament T is *transitive* if for any vertices a, b and c whenever ab and bc are arcs of T, it must be that $ac \in A(T)$. In the next lemma, we will prove that transitive tournaments are acyclic.

Lemma 2.1. *A tournament T is transitive if and only if T is acyclic.*

Proof. Let a tournament $T = (V, A)$ be transitive, we prove that T is acyclic. Suppose not. Then T has at least one cycle. Let $v_1v_2...v_kv_1$, $k \geq 3$ be a cycle in T. Since v_1v_2 and v_2v_3 are arcs in T, and T is transitive, it follows that v_1v_3 is an arc in T. Similarly, $v_1v_4, v_1v_5, ..., v_1v_k \in A(T)$; a contradiction since $v_k \to v_1$. Therefore, T is acyclic.

Conversely, let T be acyclic, we prove that T is transitive. Suppose not. Then there exist vertices v_1, v_2 and v_3 such that v_1v_2, v_2v_3 and v_3v_1 are arcs in T. This makes a cycle and contradicts that T is acyclic. Thus, T is transitive. □

The next lemma gives a well-known and important property of acyclic tournaments.

Lemma 2.2. *If a tournament T is acyclic, then T has exactly one source and one sink.*

Proof. If a tournament T is acyclic, we prove that T has a source. Suppose not. Let a tournament T be acyclic, then there is no vertex v such that $id(v) = 0$. Let $P : u_1u_2...u_k$ be a maximal length path in T. Since $id(u_1) > 0$, there exists a vertex w such that $w \to u_1$. If $w \notin P$, then $w \cup P$ is a longer path; a contradiction. If $w \in P$, then $u_1u_2...wu_1$ is a cycle in T; a contradiction. Therefore, there exists a sink v such that $id(v) = 0$.

If a tournament T is acyclic, we prove that T has a sink. Suppose not. Let a tournament T be acyclic, then there is no vertex u_k such that $od(u_k) = 0$. Let P : $u_1u_2...u_k$ be a maximal length path in T. Since $od(u_k) > 0$, there exist a vertex w such that $u_k \to w$. If $w \notin P$, then $w \cup P$ is a longer path; a contradiction. If $w \in P$, then $u_kw...u_{k-2}u_{k-1}u_k$ is a cycle in T; a contradiction. Therefore, there exists a sink u_k such that $od(u_k) = 0$.

To prove uniqueness of the source, let u_1 and u_2 be two sources in T. Then there is a directed edge between every pair of vertices. If $u_1 \to u_2$, then u_1 is the source but not u_2. If $u_2 \to u_1$, then u_2 is the source, not u_1. Similarly, to prove uniqueness of the sink, let v_1 and v_2 be two sinks in T. Then there is an arc between every pair of vertices. If $v_1 \to v_2$, then v_2 is the sink but not v_1. If $v_2 \to v_1$, then v_1 is the sink, not v_2. □

Note that for any acyclic tournament T, the source x itself is a dominating set in the tournament. In fact, x is the unique minimum dominating set of size one. In this case, the tournament solution can easily be found because x uniquely beats all the vertices. Therefore, x is the winner in the election.

A path in a digraph D that contains all the vertices of D is called a *Hamiltonian path*. A *Hamiltonian cycle* is a closed Hamiltonian path. A *Hamiltonian digraph* is a digraph D that has a Hamiltonian cycle. We can clearly see that not all digraphs have a Hamiltonian path. However, all tournaments must have a Hamiltonian path as we prove in the following lemma.

Lemma 2.3. *Every tournament T has a Hamiltonian path.*

Proof. Suppose not. Let $P : v_1v_2....v_k$ be a longest path in T. Since P is not Hamiltonian, there exists a vertex $x \notin V(P)$. By the maximality of P, $x \to v_k$ and consequently, $x \to v_{k-1}$. If $x \to v_i$ for all $i \in \{1, 2, ..., k\}$, then $x \to v_1$ and the path xP is longer than P. Let r be the largest index in $\{1, 2, ..., k\}$ such that $v_r \to x$. Then $x \to v_{r+1}$ and so $v_1v_2...v_rxv_{r+1}...v_r$ is a path in T longer than P. $\square$

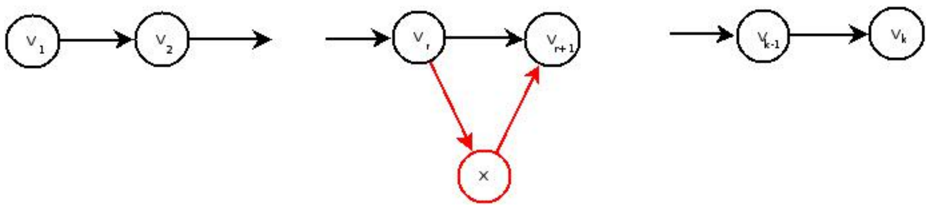


Figure 2.1: Every tournament T has a Hamiltonian path.

A path $P : v_1v_2...v_k$ in a tournament T is called a *forward path* if $v_i \to v_j$ for all i and j such that $1 \leq i < j \leq k$. The next lemma proves that a Hamiltonian forward path exists in any acyclic tournament.

Lemma 2.4. *If a tournament T is acyclic, then T has a Hamiltonian forward path.*

Proof. By Lemma 2.3, $P : v_1v_2....v_\ell$ is a Hamiltonian path in T. If P is not a forward path, then $v_j \to v_i$ for some values of i and j such that $1 \leq i < j \leq k$. Since T is acyclic, by definition, T cannot have any cycles. But $v_jv_{j+1}...v_\ell v_j$ is a cycle, which contradicts our assumption that T is acyclic. Therefore, T has a Hamiltonian forward path.

□

Chapter 3

Voting and k-Majority Tournaments

In this chapter, we will discuss some important results discovered by N. Alon, G. Brightwell, H. A. Kierstead, A. V. Kostochka and P. Winkler [ABK$^+$05]. The topic of this thesis was inspired by these results.

3.1 k-Majority Voting

We will now formally develop the idea of k-majority voting. Each voter orders all the candidates by his preference. The resulting orderings are called *linear orderings*. We use Π to denote the collection of the voters' linear orderings of the candidates. A voter's linear ordering define when one candidate is preferred over another for each individual voter. In other words, linear orderings form a binary relationship between the candidates. Suppose we have linear orderings $\Pi : P_1, P_2, ..., P_r$ on a finite set of candidates. When Candidate x is ranked higher than Candidate y in the linear ordering P_i, we write $x >_i y$. If $x >_i y$ for more than the half values of i in the set $\{1, 2, ..., r\}$, then x beats y in the election. In this case, we say x *dominates* y and write $x \rightarrow y$. Notice that for n candidates, $\{x_1, x_2, ..., x_n\}$, if we have $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow ... \rightarrow x_n \rightarrow x_1$, it is difficult to select a winner or a group of winners in such an election. One way to decide a winner or a group of winners is to find a minimum dominating set on $\{x_1, x_2, ..., x_n\}$. To do so, we need to find a minimum group of candidates, let us denote it X, so that for every candidate x in the election, there is a candidate $y \in X$ such that $y \to x$.

3.2 k-Majority Tournaments

We model a k-majority voting scenario as described in the previous section using a digraph where the vertices represent the candidates. Let $\Pi : P_1, P_2, ..., P_k$ be linear orderings on a finite set of vertices V. When r is odd, the resulting digraph is a tournament since, between any two candidates, one will always be ranked above the other in a majority of the linear orderings [MSW11]. We say that a tournament T is a *k-majority tournament* if T can be realized by some set of $2k-1$ linear orderings on $V(T)$ [ABK$^+$05]. For instance, a 2-majority tournament that has a vertex set V is realized by three linear orderings on V, while a 3-majority tournament is realized by five linear orderings on V.

Alon, et.al. studied upper bounds of the cardinality of a minimum dominating set in a k-majority tournament T [ABK$^+$05]. Let $F(k)$ be the maximum over all k-majority tournaments of the cardinality of a minimum dominating set. They proved that the cardinality of a minimum dominating set in a k-majority tournament is bounded by a value that does not depend on the number of vertices in the tournament, so that $F(k)$ is finite and exists for all $k > 0$. Specifically, they proved that $F(k) \leq (80 + \mathcal{O}(1))\, k \log k$, where the $\mathcal{O}(1)$ term tends to zero as k tends to infinity [ABK$^+$05]. This result is in contrast to the following result by Paul Erös [Erd63]. Our statement of this result follows the statement given by Reid, et al [RMHH04].

Lemma 3.1. *For every $\epsilon > 0$, there is an integer K such that for every $k \geq K$, there exists a tournament T_k with no more than $k^2 2^2(\log 2 + \epsilon)$.*

This result shows that the cardinality of a minimum dominating set in general tournaments can be arbitrarily large.

3.3 2-Majority Tournaments

We are especially interested in 2-majority tournaments. Our interest stems from the following result [ABK$^+$05]. We give the proof, with illustration, that is found in [ABK$^+$05].

Theorem 3.2. *The size of a dominating set is at most three for all 2-majority tournaments ($F(2) = 3$). Moreover, if T does not have a dominating set of size one, then T has a dominating set of size three that induces a directed cycle.*

Proof. Consider a 2-majority tournament $T = (V, A)$ with a set V of vertices and a set A of arcs defined by three linear orderings P_1, P_2 and P_3 of V. Consider the linear order P_3. We look for a vertex c such that every vertex strictly greater than c in P_3 is dominated by some vertex d with $c \geq_3 d$. If no such c exists, then the top vertex in P_3 is a dominating set in T. We may, therefore, assume such a c exists and we choose c to be the least such vertex in P_3. If another vertex dominates c then this contradicts our choice of c.

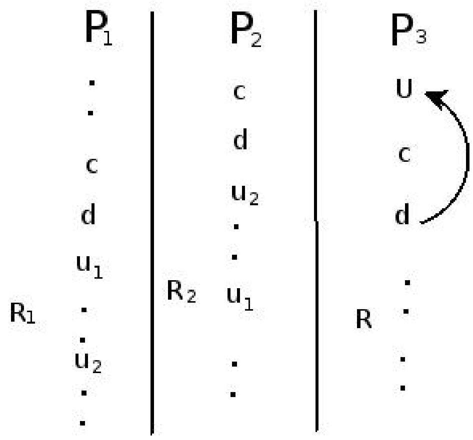


Figure 3.1: The linear orderings on $V(T)$ in a 2-majority tournament.

Let U be the set of vertices above c in P_3. Then $U = \{x \in V : x >_3 c\}$. If U is empty, then the top element in P_3 is c. Therefore, $c \to \{c\}$, and $c \to V$. This implies T again contains a dominating set of size one. Thus, we may assume that U is nonempty. We let D be the nonempty set of vertices not in U that dominates U in T. We can write $D = \{x \in V \backslash U : x \to U\}$. Therefore, for all $d \in D$, $d >_i U$ for $i = 1, 2$ but not for $i = 3$. Let R be the remaining collection of vertices. That is, $R = V \backslash (U \cup D \cup \{c\})$.

Let u_1 be the maximum element of U in P_1 and u_2 be the maximum element of U in P_2. For all $d \in D$, either $c >_1 d$ or $c >_2 d$. Therefore, $c \to D$ since $c >_3 D$.

Any element $x \in V \backslash U$ satisfies $x <_3 \{u_1, u_2\}$. So, if x dominates both u_1 and u_2, then it satisfies $u_1, u_2 <_i x$ for $i = 1, 2$. Therefore, if $d \in D$, then d dominates all of U. It follows that for any vertex $d \in D$, $d >_1 u_1$ and $d >_2 u_2$. Thus $d \to \{u_1, u_2\}$. Since $\{u_1, u_2\}$ is above R in P_3, $u_1 >_1 R_1$ and $u_2 >_2 R_2$, we see that $\{u_1, u_2\}$ dominates all vertices in R. This is illustrated in Figure 3.1. Thus, if we fix a $d \in D$, the set $\{c, d, u_1, u_2\}$ is a dominating set of $V(T)$ of size four, as shown in Figure 3.2.

However, we can do better. Let $R = R_1 \cup R_2$ such that $R_i = \{x \in R : x <_i u_i\}$, and $u_i \to R_i$ for all i in $\{1, 2\}$. Since $u_1, u_2 <_i d$ for both i in $\{1, 2\}$ and $c \to d$, there exists i in $\{1, 2\}$ such that $u_i <_i c$. Let $u_2 <_2 c$. Since c above R in P_3, then $c \to R_2$. It follows that $\{c, d, u_1\} \to V$. If $c \in D$ then $c >_1 R_1$. Then $c \to V$ and we have a dominating set of size one. If this is not the case, then we will have a cycle cdu_1c and the set $\{c, d, u_1\}$ of vertices dominates all of V, as shown in Figure 3.2.

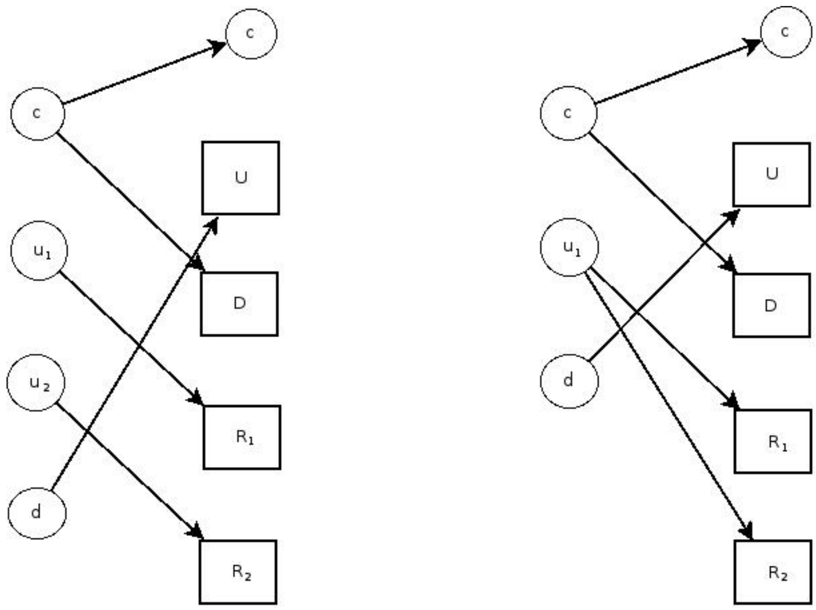


Figure 3.2: Improve a minimum dominating set on a 2-majority tournament.

□

We conclude from Alon, et.al.'s study that $F(2) = 3$, so that there is a small upper bound for all minimum dominating sets in 2-majority tournaments. However, the upper bound for a minimum dominating set in a k-majority tournament when $k \geq 3$ is not tight, and it is relatively large when k is large. In the next section, we consider a method using what is known about 2-majority tournaments to attempt to improve the upper bound on $F(3)$.

We can easily draw a 2-majority tournament given three linear orderings P_1, P_2 and P_3. For example, Let T be a 2-majority tournament on vertex set $V = \{a, b, c, d, e, f\}$ realized by the linear orderings $P_1 : b > a > c > d > e > f$, $P_2 : c > d > a > f > e > b$ and $P_3 : e > f > a > b > c > d$. The resulting tournament can be drawn as follows:

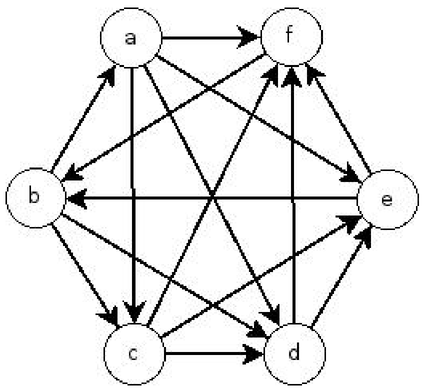


Figure 3.3: The resulting 2-majority tournament from the given linear orderings P_1, P_2 and P_3.

3.4 Inherited 2-Majority Tournaments

Alon, et.al. also investigated the maximum overall 3-majority tournaments of a minimum dominating set [ABK+05]. By way of an example, they proved that $F(3) \geq 4$. However, the upper bound they gave for $F(3)$ was less than or equal to $80(3 \log 3)$, which is a relatively big bound.

One way to improve this upper bound might be to investigate the structural differences that arise between 3-majority tournaments and 2-majority tournaments. A 3-majority tournament is realized by five linear orderings. In a 3-majority tournament T, if we restrict our attention to any three of the linear orderings, these three linear orderings will realize a 2-majority tournament on $V(T)$. We call such a 2-majority tournament T an *inherited* 2*-majority tournament of a* 3*-majority tournament*. Thus, a given 3-majority tournament has $\binom{5}{2} = 10$ different inherited 2-majority tournaments. Let T_i be the inherited 2-majority tournaments on a 3-majority tournament T for $i \in \{1, 2, ..., 10\}$. Let D_i be a minimum dominating set for T_i. Then $|D_i| = 1$ or $|D_i| = 3$ for all i, by Theorem 3.2. If $|D_i| = 3$, then D_i is a 3-cycle.

Proposition 1. *If T_1 and T_2 are inherited* 2*-majority tournaments from a* 3*-majority tournament T such that T_1 and T_2 share exactly one linear ordering and $x \to V$ in both T_1 and T_2, then $x \to V$ in T.*

Proof. We may assume that T_1 has linear orderings P_1, P_2, P_3, and T_2 has linear orderings P_3, P_4, P_5. Let $y \in V$ and suppose $y \to x$ in T_1 and in T_2. Then $x >_i y$ for at least two values of $i \in \{1, 2, 3\}$ and for at least two values of $i \in \{3, 4, 5\}$. This implies $x{>_i}y$ for at least three values of $i \in \{1, 2, 3, 4, 5\}$. Therefore, $x \to y$ in T. Since y was arbitrarily chosen, $x \to V$ in T. □

Proposition 2. *If x is a dominating set shared by at least four of the inherited* 2*-majority tournaments from a* 3*-majority tournament T, then $x \to V$ in T.*

Proof. Let T_1 have linear orderings P_1, P_2, P_3. If T_2 shares exactly one linear ordering with T_1, then, by Proposition 1, $x \to V$ in T. Assume that T_2 shares two linear orderings with T_1. Let T_2 have linear orderings P_1, P_2, P_4. If T_3 shares exactly one linear ordering with T_1 or T_2, then, by Proposition 1, $x \to V$ in T. Assume that T_3 has linear orderings P_1, P_2, P_5. Notice that no two of T_1, T_2 and T_3 share exactly one linear ordering. It follows that T_4 must have linear orderings that share exactly one linear ordering with T_i for some values of $i \in \{1, 2, 3\}$. Thus, by Proposition 1, $x \to V$ in T. □

Chapter 4

Upset Paths and 2-Majority Tournaments

Let $P : v_1 v_2 .... v_k$ be a path in a tournament T. We call P an *upset path* if when the arcs of the path are reversed, the resulting tournament is acyclic with source v_k and sink v_1. The notion of an upset path was first defined by Melcher et al. [MR10]. Let T' be the tournament obtained from T by reversing the arcs in the upset path P. Tournament T' is acyclic with exactly one source v_k and one sink v_1. Obviously, all cycles in the tournament T must contain at least one arc of the upset path.

Figure 4.1: Upset path in the tournament T and the tournament T'.

In Figure 4.1, the tournament T has an upset path $P : abc$. We obtain T' by reversing the arcs in P. Note that T' is acyclic with source c and sink a. In this example, it is easy to check that all cycles in T contain at least one arc of the upset path. It is not too difficult to verify that the last two vertices in an upset path in a tournament T also form a dominating set in T. Vertex c dominates all of $V - \{b\}$ and $b \to c$. Thus, $\{b, c\}$ forms a minimum dominating set in T.

In this chapter, we will derive the relationship between upset paths and 2-majority tournaments. Our method is to investigate tournaments that have an upset path. We will see that the presence of an upset path will imply that other structural properties exist in the tournament. For any tournament T, we say that T is *realizable as a 2-majority tournament* if T can be constructed as a 2-majority tournament using exactly three linear orderings of $V(T)$. Our conjecture is that if we have a tournament T with an upset path, then T is realizable as a 2-majority tournament. To prove this conjecture, the challenge will be to construct three linear orderings of the vertices that will realize T as a 2-majority tournament. Thus, it seems reasonable to investigate structures that exist in tournaments that, when present, will make this process easier.

The structure of a Hamiltonian forward path allows us to construct linear orderings to realize a tournament as a 2-majority tournament.

Lemma 4.1. *If a tournament T is acyclic, then T is realizable as a 2-majority tournament.*

Proof. Since T is acyclic, by Lemma 2.4, there is exists a Hamiltonian forward path. Let $v_1 v_2 v_3 ... v_k$ be a Hamiltonian forward path in T. Therefore, we can write the vertices in T in three linear orderings P_1, P_2 and P_3 as the following: $P_1 : v_1 > v_2 > v_3 > ... > v_k$, $P_2 : v_1 > v_2 > v_3 > ... > v_k$ and P_3 can be anything. Thus, T is realizable as a 2-majority tournament.

□

Lemma 4.2. *Let $|V(T)| \geq 3$. If xy is an upset path in tournament T of length one, then T is not acyclic.*

Proof. Suppose not. Then T is acyclic. Let T' be the tournament obtained from T after reversing the arc xy. In T', y is a source and x is a sink. Therefore, since $|V(T)| \geq 3$, there exists a vertex t such that $t \to x$ and $y \to t$ in T. Then $xytx$ is a cycle in T; a contradiction.

Figure 4.2: The tournament T is not acyclic.

□

Lemma 4.3. *If a tournament T has an upset path, then T is not acyclic.*

Proof. Suppose not. Then T is acyclic and contains an upset path $P : v_1 v_2 ... v_k$. Let T' be the tournament obtained from T by reversing all the arcs in P. By definition of the upset path, v_k is a source in T' and v_1 is a sink in T'. Since T is acyclic, it contains a source s and a sink t. It must be that s and t are distinct from v_k and v_1. Thus, $s \to v_k$ in T' and $v_1 \to t$ in T'; a contradiction.

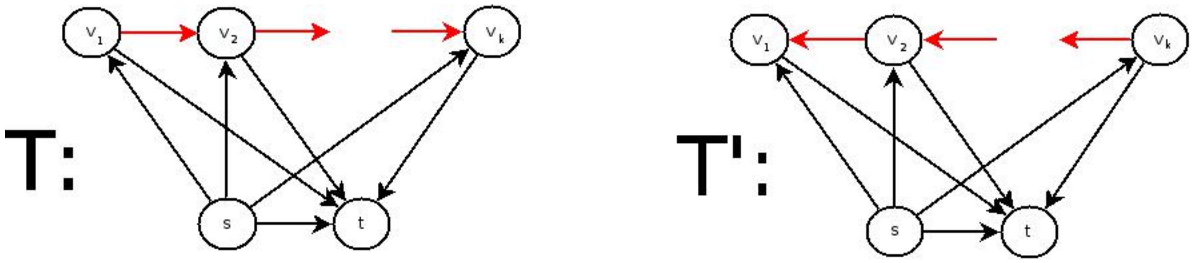


Figure 4.3: The path P is not an upset path.

□

Lemma 4.4. *If $P : av_1 v_2 ... v_k b$ is a shortest upset path in a tournament T, then there must exist a cycle in T containing the arc av_1 and no other arc in P, and there must* *exist a cycle in T containing the arc $v_k b$ and no other arc in P.*

Proof. If no such a cycle containing $v_k b$ exists, then $P : av_1v_2...v_{k-1}$ is a shorter upset path in T; a contradiction. A similar argument applies when no such cycle containing av_1 exists. □

Remark (Number of cycles in T)**.** *Let T be a tournament. If T has an upset path $P : v_1v_2$ of length one, then the number of cycles in T must be greater than or equal one. For all T that has an upset path P of length ≥ 2, then the number of cycles in T must be greater than or equal two.*

Lemma 4.5. *Let $P : v_1v_2...v_k$ be a shortest upset path in a tournament T, then the tournament on $V(T) - V(P)$ must be acyclic.*

Proof. Suppose not, then there exists at least one cycle in $V(T) - V(P)$. By reversing all arcs in P, the resulting tournament is not acyclic. Therefore, P is not an upset path in T. □

Lemma 4.6. *Let T be a tournament on $n \geq 3$ vertices that has an upset path of length one, then T is realizable as a 2-majority tournament.*

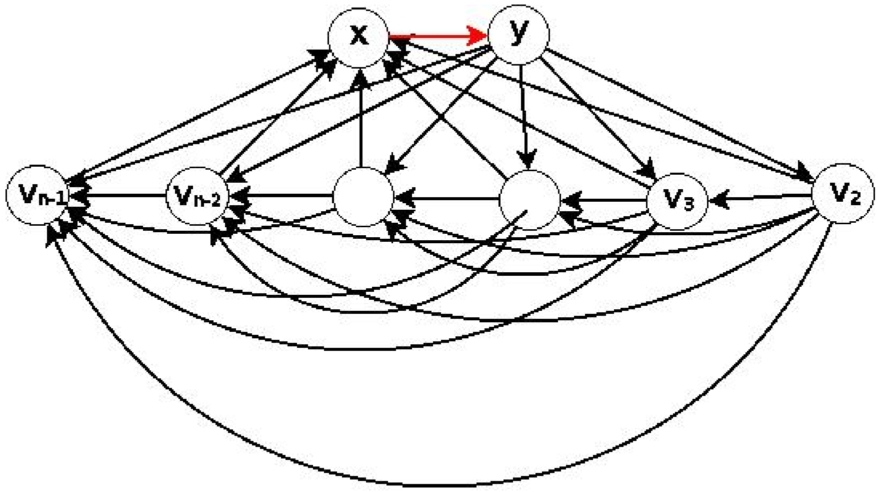
Proof. Let T be a tournament that has an upset path xy of length one. Let T' be the tournament obtained from T by reversing the arc xy. Then T' is acyclic with source y and sink x, and it has a dominating set of size one; $D = \{y\}$. By Lemma 4.1, we can realize T' using three linear orderings P_1, P_2 and P_3. Since y is the source, then y is greater than all vertices in V in at least two linear orders. Moreover, since x is the sink, x is less than all other vertices in at least two linear orderings. Also, because there is no cycle that does not contain the arc xy, we can order the vertices so that $y = v_1, v_2, ..., v_n = x$ forms a Hamiltonian forward path by Lemma 2.4. We use this Hamiltonian path to define the linear orderings on V as following. Let $P_1 : y > v_2 > v_3 > ... > v_{n-1} > x$, $P_2 = P_1$ and P_3: can be anything.

Now we can write T as a 2-majority tournament as shown in Figure 4.4. To do so, we note that:

(i) $y \rightarrow (V(T) - x)$.

(ii) $(V(T) - y) \rightarrow x$.

(iii) By Lemma 2.4 there is a Hamiltonian forward path $v_1, v_2, ..., v_n$ on $V(T) - \{x, y\}$.



$P_1 : y > v_2 > v_3 > ... > v_{n-1} > x$
$P_2 : x > y > v_2 > v_3 > ... > v_{n-1}$
$P_3 : v_2 > v_3 > ... > v_{n-1} > x > y$

Figure 4.4: The tournament T and its linear orderings.

Therefore, T is realizable as a 2-majority tournament as shown in Figure 4.4. $\square$

Lemma 4.7. *If $P : v_1v_2...v_k$ is a shortest upset path on a tournament T of length at least two, then $|V(T) - V(P)| \geq 2$.*

Proof. If $|V(T) - V(P)| = 0$, then $V(T) = V(P)$ and T', a tournament obtained from T by reversing the arcs in P, is transitive. By Lemma 2.4, $v_1v_2....v_k$ is a Hamiltonian forward path in T'. This implies that every consecutive collection of vertices in P is a cycle in T. By reversing the arcs in the subpath $P' : v_1v_2...v_{k-2}v_{k-1}$, we obtain an acyclic tournament. This contradicts the minimality of P.

If $|V(T) - V(P)| = 1$, then let $V(T) - V(P) = \{u\}$. Then $v_k \to u$ and $u \to v_1$. If $u \to v_2$, then there is no cycle in T containing the arc v_1v_2 and no other arc in P.

Hence, the path $v_2v_3...v_k$ is a shorter upset path in T. Thus, we may assume $v_2 \to u$. If $v_i \to u$, for some $i \in \{2, 3, ..., k\}$, then there is no cycle in T containing the arc $v_{k-1}v_k$ and no other arc in P. Thus, the path $v_1v_2...v_{k-1}$ is a shorter upset path in T. Therefore, it must be that for some $i \in \{2, 3, ..., k\}$, $u \to v_i$. Let j be the smallest such i. Then $v_{j-1} \to u$ and $u \to v_j$. Thus, $v_{j-1}uv_jv_{j-1}$ is a 3-cycle in T'; a contradiction. It follows that $|V(T) - V(P)| \geq 2$. □

Lemma 4.8. *If $P : v_1v_2...v_k$ is an upset path of length at least two, then there exist vertices u and w in $V(T) - V(P)$ such that $u \to v_2$ and $v_2 \to w$.*

Proof. By Lemma 4.7, $|V(T)-V(P)| \geq 2$. Let $V(T)-V(P) = \{u_1, u_2, ..., u_\ell\}$. If $u_i \to v_2$ for all $i \in \{1, ..., \ell\}$, then there is no cycle in T that contains the arc v_1v_2 and no other arc in P. Therefore, $v_2v_3...v_k$ is a shorter upset path; a contradiction. Thus, there exists a vertex $u \in V(T) - V(P)$ such that $v_2 \to u$. Symmetrically, there exists a vertex $w \in V(T) - V(P)$ such that $w \to v_{k-1}$. If $v_2 \to w$, then $v_2wv_{k-1}v_{k-2}...v_2$ is a cycle in T'; a contradiction. Therefore, $w \to v_2$. □

Theorem 4.9. *If a tournament T has an upset path of length two or less, then T is realizable as a 2-majority tournament.*

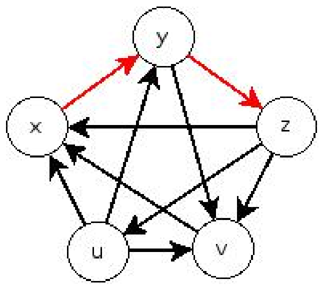
Proof. By Lemma 4.6, we may assume that a shortest upset path has length two. Let xyz be a shortest upset path in T. If $V - \{x, y, z\} = \emptyset$, then $x \to y$, $y \to z$ and $z \to x$. Therefore, xyz is not a shortest upset path.

First, suppose $|V - \{x, y, z\}| = 1$. Let $V - \{x, y, z\} = \{v\}$. We have two cases: If $v \to y$, then xyz is not a minimal upset path in T since yz is an upset path in T; a contradiction. If $y \to v$, then xyz is not a minimal upset path in T since xy is an upset path in T; a contradiction too. Thus, there must be at least two vertices in $V - \{x, y, z\}$

Now let $v, u \in V - \{x, y, z\}$. Suppose $|V - \{x, y, z\}| = 2$, and let T' be the acyclic tournament obtained from T after reversing the arcs in xyz. Then T' has source z and sink x. Note that since z is a source and x is a sink in T', we have $z \to \{x, u, v\}$ and x is dominated by u and v. We divided our argument for the remaining three arcs into cases depending on the direction of the arc between u and v.

If $u \to v$, then, by Lemma 4.4, it must be that either $v \to y$ and $y \to u$ or $y \to v$ and $u \to y$. If $v \to y$ and $y \to u$, then $uvyu$ is a cycle avoiding P; a contradiction. If

$y \to v$ and $u \to y$, then T can be realized as a 2-majority tournament as shown in Figure 4.5.



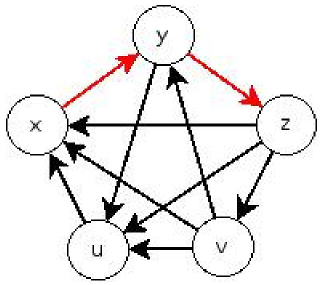
$P_1 : x > y > z > u > v$
$P_2 : u > y > z > v > x$
$P_3 : z > u > v > x > y$

Figure 4.5: The tournament T when $u \to v$ along with linear orderings that make T realizable as a 2-majority tournament.

If $v \to u$, then, by Lemma 4.4, it must be that either $y \to v$ and $u \to y$ or $v \to y$ and $y \to u$. If $y \to v$ and $u \to y$, then $vuyv$ is a cycle avoiding P; a contradiction. If $v \to y$ and $y \to u$, then T realizable as a 2-majority tournament as shown in Figure 4.6.

We need to prove that if $|V - \{x, y, z\}| \geq 3$, then T is realizable as a 2-majority tournament. Let A and B be two sets of vertices such that $A = \{v \in V - \{x, y, z\} : v \to y\}$ and $B = \{v \in V - \{x, y, z\} : y \to v\}$. By Lemma 4.8, there exists at least one vertex in A and at least one vertex in B. It follows that A and B are nonempty sets.

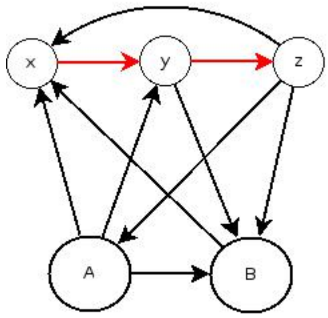
Then for all $a \in A$, $a \to x$ and $z \to a$ and for all $b \in B$, $b \to x$ and $z \to b$. Moreover, if $a \in A$ and $b \in B$ such that $b \to a$, then there exists a cycle $ayba$ that avoids P; a contradiction. Therefore, $a \to b$ for all $a \in A$ and $b \in B$. Since T' is transitive, each of the subtournaments I_A and I_B induced on A and B, respectively, must be transitive. Therefore, by Lemma 2.4, A contains a Hamiltonian forward path $H_A : a_1, a_2.....a_s$ and B contains a Hamiltonian forward path $H_B : b_1, b_2.....b_t$. Thus, T can be realized as a 2-majority tournament determined by the three linear orderings P_1, P_2 and P_3 as shown



$P_1 : x > y > z > v > u$
$P_2 : v > y > z > u > x$
$P_3 : z > v > u > x > y$

Figure 4.6: The tournament T when $v \to u$ along with linear orderings that make T realizable as a 2-majority tournament.

in Figure 4.7.



$P_1 : x > y > z > a_1 > a_2 > ...... > a_s > b_1 > b_2 > ... > b_t$
$P_2 : a_1 > a_2 > ... > a_s > y > z > b_1 > b_2 > ... > b_t > x$
$P_3 : z > b_1 > b_2 > ... > b_t > a_1 > a_2 > ... > a_s > x > y$

Figure 4.7: The tournament T when $a \to b$ along with linear orderings that make T realizable as a 2-majority tournament.

$\square$

Theorem 4.10. *If a tournament T has an upset path, then T is realizable as a 2-majority tournament.*

Proof. Let T be a tournament that has an upset path $P : v_1v_2v_3...v_k$ of length $k-1$. By the definition of an upset path, if we reverse the arcs in P, we will get an acyclic tournament T' which, by Lemma 4.1, can be realized as a 2-majority tournament with a dominating set of size one. For $j = 1, 2, ..., k-1$, let $U_j = \{v \in V(T) - V(P) : v_{j+1} \to v$ and $v \to v_i, \forall i \leq j\}$. Note that no vertex in $V(T) - V(P)$ is dominated by v_1, and no vertex in $V(T) - V(P)$ dominates v_k. For example, $U_1 = \{v \in V(T) - V(P) : v_2 \to v\}$ and $U_2 = \{v \in V(T) - V(P) : v_3 \to v$ and $v \to v_i, \forall i \leq 2\}$.

If the only cycle in T that contain v_1v_2 but not $v_{k-1}v_k$ also avoid $U_1, U_2, ..., U_{k-1}$, then $P - v_1v_2$ is a shorter upset path; a contradiction. Therefore, there must be at least one cycle in T containing v_1v_2 and vertices in $\bigcup_{i=1}^{k-1} U_i$. Similarly, there must be at least one cycle in T containing $v_{k-1}v_k$ and vertices in $\bigcup_{i=1}^{k-1} U_i$. By Lemma 4.6 there exists at least one vertex in each of U_1 and U_{k-1}.

For $1 \leq a < b \leq k-1$, it must be that $x \to y$ for all $x \in U_b$ and $y \in U_a$. By Lemma 4.5, given a set of vertices of the form $X = \{x_1, x_2, ..., x_{k-1}\}$ where $x_i \in U_i$, the tournament on X is acyclic with source x_{k-1} and sink x_1. For each j, by definition, every vertex $x \in U_j$ has the properties: $v_{j+1} \to x$ and $x \to v_i$, for $i = 1, 2, ..., j$. Moreover, we have $v_\ell \to x$, for $\ell = j+2, j+3, ..., k$. If this were not the case, then for some $\ell \in \{j+2, ..., k\}$ we would have $x \to v_\ell$. Upon reversing the arcs in P, the cycle $xv_\ell v_{\ell-1}...v_jx$ will be created, contradicting the assumption that P is an upset path.

Each subtournament on the vertices in U_j is acyclic, which gives us a natural ordering of the vertices in U_j: $\pi_j : x_{j,1} > x_{j,2} > x_{j,3} > ... > x_{j,s_j}$, where $x_{j,1}, x_{j,2}, x_{j,3}, ..., x_{j,s_j}$ is a Hamiltonian forward path in the subtournament on U_j, the vertex $x_{j,1}$ is the source in U_j and the vertex x_{j,s_j} is the sink in U_j. We draw the tournament T in Figure 4.9. The table in Figure 4.8 describes the subtournament on $V(P)$.

In order for three linear orderings P_1, P_2, P_3 on $V(T)$ to realize T as a 2-majority tournament, the subtournament on the vertices in the upset path P must also be realized by the linear orderings P_1, P_2, P_3. Based on the table in Figure 4.8, it is clear that meeting the following four conditions will suffice:

(i) For $v_i, (2 \leq i \leq k)$ we have $v_{i-1} >_1 v_i$ and either $v_{i-1} >_2 v_i$ or $v_{i-1} >_3 v_i$.

vertex in P	what dominates the vertex
v_k	v_{k-1}
v_{k-1}	v_{k-2}
v_{k-2}	v_{k-3}, v_k
v_{k-3}	v_k, v_{k-1}
$\vdots$	$\vdots$
v_i	v_{i-1} and $\{v_k, v_{k-1}, ..., v_{i+2}\}$

Figure 4.8: This table describes the relationships between the vertices in T.

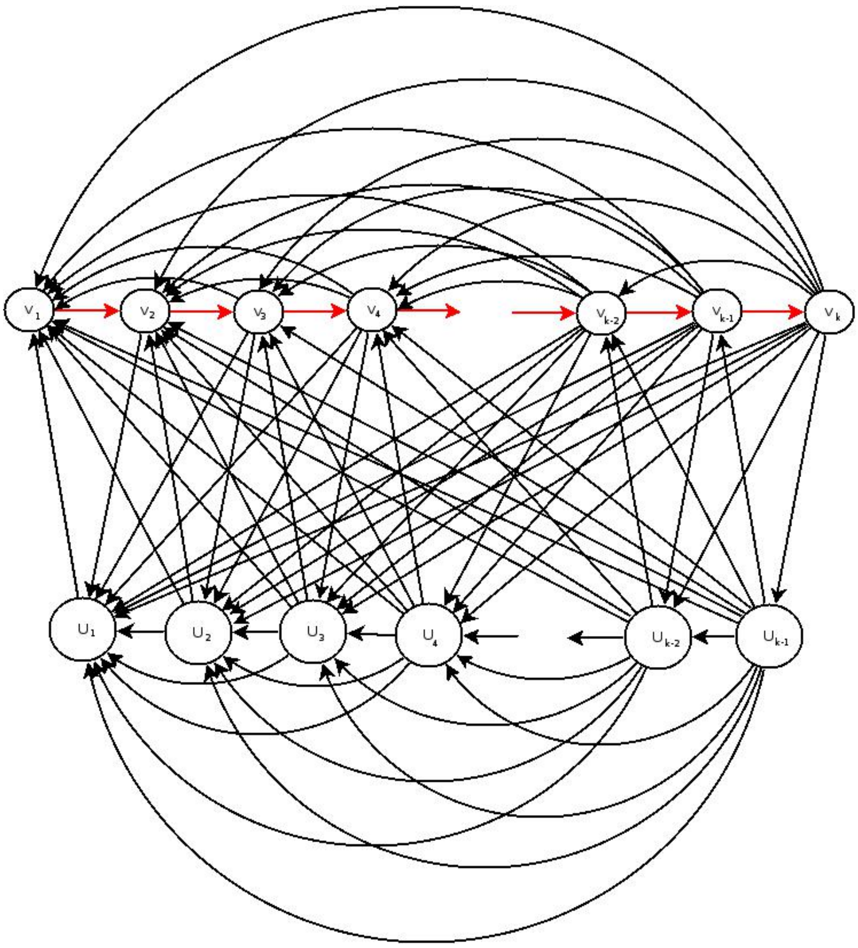


Figure 4.9: The tournament T with an upset path of length k.

(ii) Every vertex in $\{v_k, v_{k-1}, ..., v_{i+2}\}$ is above v_i in P_2 and P_3, when $(1 \leq i \leq k-2)$.

(iii) For $v_i, (1 \leq i \leq k-1)$ we have $v_i >_1 v_{i+1}$ and either $v_i >_2 v_{i+1}$ or $v_i >_3 v_{i+1}$.

(iv) For all $l \in \{1, ..., i-2\}$, $v_i >_2 v_l$ and $v_i >_3 v_l$.

Based on the following three facts, we can determine where each subordering π_j on $V(U_j)$ can be placed:

(i) $\{v_{j+1}, ..., v_k\} \rightarrow \pi_j$.

(ii) $\pi_j \rightarrow \{v_j, v_{j-1}, ..., v_1\}$.

(iii) $\pi_j \rightarrow \{\pi_1, \pi_2, ..., \pi_{j-1}\}$.

There are two cases to represent $V(T)$ whether $|V(P)|$ is odd or even. Please see Figure 4.10 when $|V(P)|$ is even, and Figure 4.11 when $|V(P)|$ is odd.

$$P_1 : v_1 > v_2 > \pi_1 > v_3 > \pi_2 > v_4 > \pi_3 > ... > v_{k-2} > \pi_{k-3} > v_{k-1} > \pi_{k-2} > v_k > \pi_{k-1}$$
$$P_2 : \pi_{k-1} > v_{k-1} > v_k > \pi_{k-2} > \pi_{k-3} > ... > \pi_4 > \pi_3 > v_3 > v_4 > \pi_2 > \pi_1 > v_1 > v_2$$
$$P_3 : v_k > \pi_{k-1} > \pi_{k-2} > v_{k-2} > v_{k-1} > ... > \pi_4 > v_4 > v_5 > \pi_3 > \pi_2\, v_2 > v_3 > \pi_1 > v_1$$

Figure 4.10: Linear orderings when $|V(P)| \equiv 0 \pmod 2$.

$$P_1 : v_1 > v_2 > \pi_1 > v_3 > \pi_2 > v_4 > \pi_3 > ... > v_{k-2} > \pi_{k-3} > v_{k-1} > \pi_{k-2} > v_k > \pi_{k-1}$$
$$P_2 : v_k > \pi_{k-1} > \pi_{k-2} > v_{k-2} > v_{k-1} > ... > \pi_4 > \pi_3 > v_3 > v_4 > \pi_2 > \pi_1 > v_1 > v_2$$
$$P_3 : \pi_{k-1} > v_{k-1} > v_k > \pi_{k-2} > \pi_{k-3} > ... > \pi_4 > v_4 > v_5 > \pi_3 > \pi_2 > v_2 > v_3 > \pi_1 > v_1$$

Figure 4.11: Linear orderings when $|V(P)| \equiv 1 \pmod 2$.

Therefore, T is realizable as 2-majority tournament. $\square$

Remark (Minimum dominating sets in T)**.** *Now we can locate all minimum dominating sets in T. $D = \{v_{k-1}, v_k\}$ is not the only minimum dominating set in T, but $\{v_k, x\}$ is also a minimum dominating set in T, for all $x \in U_{k-1}$.*

Chapter 5

UP-Sets and 2-Majority Tournaments

In this chapter, we assume T is a tournament that contains at least one cycle. Let $\{P_1, P_2, ..., P_h\}$ be a collection of vertex-disjoint paths in a tournament T. We call this collection a *UP-set of size* h if upon reversing the arcs in each set of the paths in this collection, we obtain a new tournament T' that is acyclic and for some not necessarily distinct i and j in $\{1, 2, ..., h\}$, the source in T' is the terminal vertex in P_i and the sink in T' is the initial vertex in P_j. Note that a UP-set of size one is an upset path. Thus, UP-sets generalize the notion of upset paths. Given a UP-set $\mathcal{S} = \{P_1, P_2, ..., P_h\}$, let $\mathcal{A} = \bigcup_{i=1}^{h} A(P_i)$ be the union of all arcs in the paths in $\mathcal{S}$. We call $\mathcal{A}$ the *arc set* of the UP-set.

A *minimal UP-set* is a UP-set whose arc set does not properly contain the arc set of another UP-set. Intuitively, any UP-set should contain a minimal UP-set. We call a tournament T an *h-critical* tournament if it has a minimal UP-set of size h, but no minimal UP-set of size less than h.

Lemma 5.1. *Let* $\mathcal{S} = \{P_1, P_2, ..., P_h\}$ *be a UP-set in a tournament* T*, then* $\mathcal{S}$ *contains a minimal UP-set.*

Proof. Suppose not. Then $\mathcal{S}$, itself, is not a minimal UP-set. Therefore, the arc set $\mathcal{A}$ of $\mathcal{S}$ properly contains the arc set $\mathcal{A}_1$ of another UP-set $\mathcal{S}_1$. Similarly, $\mathcal{S}_1$ is not a minimal UP-set, so $\mathcal{A}_1$ properly contains in $\mathcal{A}_2$, the arc set of another UP-set $\mathcal{S}_2$. Continuing in this fashion, we obtain the following properly nested sequence of arc sets of UP-sets:

$$\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset ....$$

Since $\mathcal{A}$ is a finite set, this nested sequence must terminate, implying that the final set is the arc set of a minimal UP-set contained in $\mathcal{S}$; a contradiction. $\square$

Lemma 5.2. *Let $\mathcal{S} = \{P_1, P_2, ..., P_h\}$ be a minimal UP-set in a tournament T. Let $\mathcal{A} = \bigcup_{i=1}^{h} A(P_i)$ be the union of all arcs in the paths in $\mathcal{S}$. Then, for all $a \in \mathcal{A}$, there exists a cycle C such that $C \cap (\mathcal{A} - \{a\}) = \phi$.*

Proof. Suppose not. Then there exists $a \in \mathcal{A}$ such that for any cycle C in T, either $C \cap \{a\} = \phi$ or C contains a and $|C \cap \mathcal{A} - \{a\}| \geq 1$. If every cycle in T avoids a, then $\mathcal{A} - \{a\}$ is the arc set of a UP-set properly contained in $\mathcal{S}$, which contradicts the minimality of $\mathcal{S}$.

Thus, every cycle C containing a must meet $\mathcal{A} - \{a\}$ in at least one element. All such cycles will be eliminated upon reversing the arcs in $\mathcal{A} - \{a\}$. If arc a does not contain the source nor the sink in T', the tournament obtained after reversing all arcs in $\mathcal{A}$, then $\mathcal{A} - \{a\}$ is the arc set of a UP-set properly contained in $\mathcal{S}$; a contradiction.

Let $a = uv$. If arc a contains both the source v and the sink u in T', then if x is any vertex other than u or v, there is a cycle $uvxu$ in T intersecting $\mathcal{A}$ only in the arc arc a; a contradiction.

Suppose arc $a = uv$ contains exactly one of the source or the sink in T'. Using symmetry, it suffices to assume that u is the sink in T'. If there is no vertex y in T such that vy is an arc in T that is not in $\mathcal{A}$. Then in T', v is a sink; a contradiction. Therefore, such an arc y exists. Since u is a sink in T', $y \to u$ in T and $uvyu$ is a cycle in T disjoint from $\mathcal{A} - \{a\}$; a contradiction.

The result now follows.

$\square$

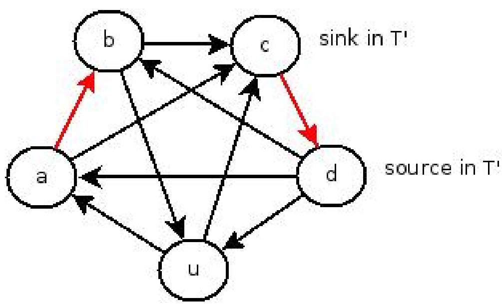
Theorem 5.3. *Let T be a 2-critical tournament with at least one cycle and let $\mathcal{S} = \{P_1, P_2\}$ be a minimal UP-set where P_1 and P_2 are both paths of length one. Then T is realizable as a 2-majority tournament.*

Proof. Let $P_1 : ab$ and $P_2 : cd$. Let T' be the tournament obtained from the tournament T by reversing P_1 and P_2. We divide our argument into cases based on the cardinality of $V - \{a, b, c, d\}$.

First, suppose $|V - \{a, b, c, d\}| = 0$. If, in T', d is the source and c is the sink, then we have $b \to c$, $a \to c$, $d \to b$ and $d \to a$. Thus, in T, there is no cycle that contains ab and not cd. This implies cd is a UP-set of size one; contradicting the minimality of $\mathcal{S}$. Similarly, it cannot be that b is the source in T' and a is the sink in T'.

We deduce that, in T', either d is the source and a is the sink, or b is the source and c is the sink. By symmetry, we may assume that d is the source and a is the sink in T'. If $b \to c$, then $\{abcd\}$ is a UP-set of size one; a contradiction. Therefore, $c \to b$. Then T is acyclic, which is also a contradiction. From this, we conclude that $|V - \{a, b, c, d\}| \geq 1$.

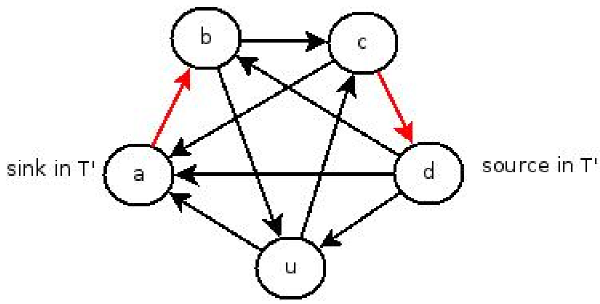
Suppose $|V - \{a, b, c, d\}| = 1$ and let $V - \{a, b, c, d\} = \{u\}$. Up to symmetry, we must consider two scenarios: In T', d is the source and c is the sink, and in T', d is the source and a is the sink. If d is the source and c is the sink, then $d \to u$, $d \to a$, $d \to b$, $u \to c$, $a \to c$ and $b \to c$. By Lemma 5.2, there must be a cycle in T containing the arc ab. It follows that $u \to a$ and $b \to u$. Then T is realizable as a 2-majority tournament with the linear orderings P_1, P_2 and P_3 as shown in Figure 5.1.



$$\begin{aligned} P_1 &: d > u > a > b > c \\ P_2 &: b > u > a > c > d \\ P_3 &: c > d > a > b > u \end{aligned}$$

Figure 5.1: The tournament T when, in T', d is the source and c is the sink along with linear orderings that make T realizable as a 2-majority tournament.

We may now assume that, in T', d is the source and a is the sink, then $d \to u$, $d \to a$, $d \to b$, $u \to a$ and $c \to a$. By Lemma 5.2, there must be a cycle in T containing the arc ab and not cd, and there must be a cycle in T containing the arc cd and not ab. It follows that $b \to u$ and $u \to c$. Therefore, it must be that $b \to c$. Otherwise, $bcub$ is a cycle which contradicts that $\mathcal{S}$ is a minimal UP-set in T. Then T is realizable as a 2-majority tournament with the linear orderings P_1, P_2 and P_3 as shown in Figure 5.2.



$$P_1 : a > b > u > c > d$$
$$P_2 : c > d > u > a > b$$
$$P_3 : d > b > u > c > a$$

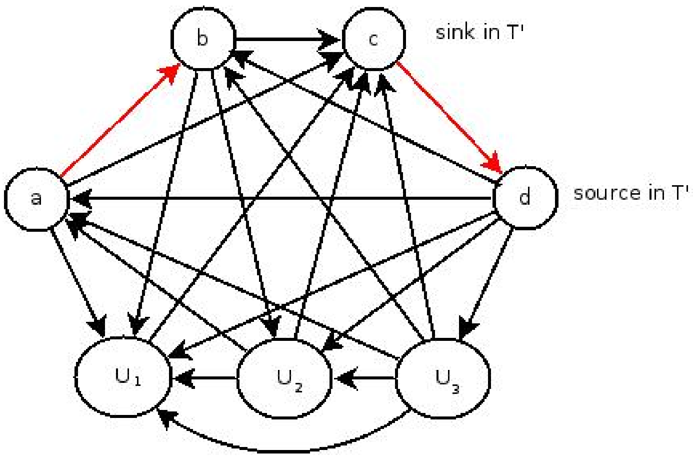
Figure 5.2: The tournament T when, in T', d is the source and a is the sink along with linear orderings that make T realizable as a 2-majority tournament.

Now suppose $|V - \{a, b, c, d\}| = k$. Up to symmetry, we must consider two scenarios: In T', d is the source and c is the sink, and in T', d is the source and a is the sink.

If d is the source and c is the sink, then $d \to a$, $d \to b$, $a \to c$ and $b \to c$. We partition $V - \{a, b, c, d\}$ into four sets of vertices. Let $U_1 = \{u \in V - \{a, b, c, d\} : a \to u \quad \text{and} \quad b \to u\}$. Let $U_2 = \{u \in V - \{a, b, c, d\} : u \to a \quad \text{and} \quad b \to u\}$. Let $U_3 = \{u \in V - \{a, b, c, d\} : u \to a \quad \text{and} \quad u \to b\}$. Since, in T', d is the source and c is the sink, $d \to U_1$, $d \to U_2$, $d \to U_3$, $U_1 \to c$, $U_2 \to c$ and $U_3 \to c$.

Each subtournament on the vertices in U_j is acyclic, which gives us a natural ordering of the vertices in U_j: $\pi_j : x_{j,1} > x_{j,2} > x_{j,3} > ... > x_{j,s_j}$, where, by Lemma 2.4, $x_{j,1}, x_{j,2}, x_{j,3}, ..., x_{j,s_j}$ is a Hamiltonian forward path in the subtournament on U_j. The vertex $x_{j,1}$ is the source in U_j and the vertex x_{j,s_j} is the sink in U_j. We draw the tournament T in Figure 5.3.

By Lemma 5.2, there must be a cycle in T containing the arc ab. It follows that there must be at least one vertex in U_2. Then T is realizable as a 2-majority tournament with the linear orderings P_1, P_2 and P_3 as shown in Figure 5.3.



$$P_1 : d > \pi_3 > \pi_2 > a > b > \pi_1 > c$$
$$P_2 : \pi_3 > b > \pi_2 > a > \pi_1 > c > d$$
$$P_3 : c > d > \pi_3 > a > b > \pi_2 > \pi_1$$

Figure 5.3: The tournament T when, in T', d is the source and c is the sink along with linear orderings that make T realizable as a 2-majority tournament.

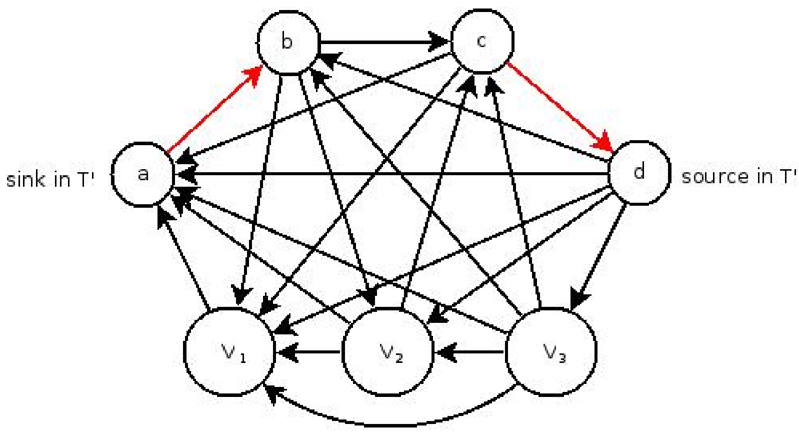
We may now assume that, in T', d is the source and a is the sink, then $d \to a$, $d \to b$ and $c \to a$. There will be two cases: $b \to c$ and $c \to b$.

If $b \to c$, then we partition $V - \{a, b, c, d\}$ into three sets of vertices. Let $V_1 = \{v \in V - \{a, b, c, d\} : b \to v \quad \text{and} \quad c \to v\}$. Let $V_2 = \{v \in V - \{a, b, c, d\} : b \to v \quad \text{and} \quad v \to c\}$. Let $V_3 = \{v \in V - \{a, b, c, d\} : v \to b \quad \text{and} \quad v \to c\}$. Since, in T', d is the source and a is the sink, $d \to V_1$, $d \to V_2$, $d \to V_3$, $V_1 \to a$, $V_2 \to a$ and $V_3 \to a$.

Each subtournament on the vertices in V_j is acyclic, which gives us a natural ordering of the vertices in V_j: $\pi_j : x_{j,1} > x_{j,2} > x_{j,3} > ... > x_{j,s_j}$, where, by Lemma 2.4, $x_{j,1}, x_{j,2}, x_{j,3}, ..., x_{j,s_j}$ is a Hamiltonian forward path in the subtournament on V_j.

The vertex $x_{j,1}$ is the source in V_j and the vertex x_{j,s_j} is the sink in V_j. We draw the tournament T in Figure 5.4.

By Lemma 5.2, there must be a cycle in T containing the arc ab and not cd, and there must be a cycle in T containing the arc cd and not ab. It follows that either $V_2 \neq \phi$ or $V_1 \neq \phi$ and $V_3 \neq \phi$. Then T is realizable as a 2-majority tournament with the linear orderings P_1, P_2 and P_3 as shown in Figure 5.4.



$$
\begin{aligned}
P_1 &: \pi_3 > a > b > \pi_2 > \pi_1 > c > d \\
P_2 &: c > d > \pi_3 > \pi_2 > \pi_1 > a > b \\
P_3 &: d > \pi_3 > b > \pi_2 > c > \pi_1 > a
\end{aligned}
$$

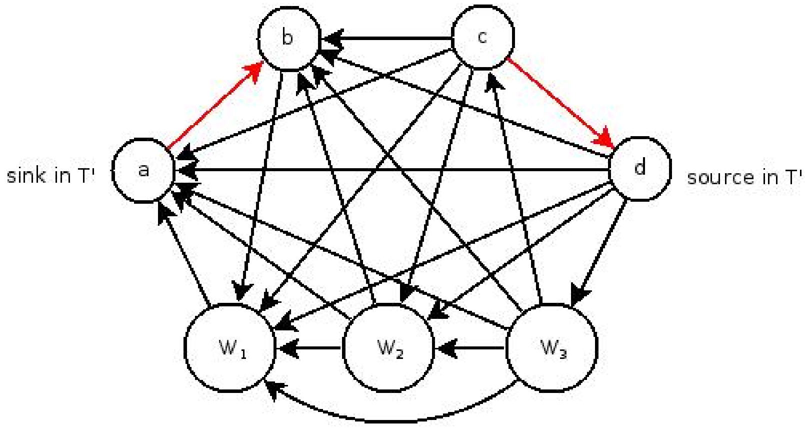
Figure 5.4: The tournament T when, in T', d is the source and a is the sink along with linear orderings that make T realizable as a 2-majority tournament.

If $c \rightarrow b$, then we partition $V - \{a,b,c,d\}$ into three sets of vertices. Let $W_1 = \{w \in V - \{a,b,c,d\} : b \rightarrow w \quad \text{and} \quad c \rightarrow w\}$. Let $W_2 = \{w \in V - \{a,b,c,d\} : w \rightarrow b \quad \text{and} \quad c \rightarrow w\}$. Let $W_3 = \{w \in V - \{a,b,c,d\} : w \rightarrow b \quad \text{and} \quad w \rightarrow c\}$. Since, in T', d is the source and a is the sink, $d \rightarrow W_1$, $d \rightarrow W_2$, $d \rightarrow W_3$, $W_1 \rightarrow a$, $W_2 \rightarrow a$ and $W_3 \rightarrow a$.

Each subtournament on the vertices in W_j is acyclic, which gives us a natural ordering of the vertices in W_j: $\pi_j : x_{j,1} > x_{j,2} > x_{j,3} > ... > x_{j,s_j}$, where, by Lemma

2.4, $x_{j,1}, x_{j,2}, x_{j,3}, ..., x_{j,s_j}$ is a Hamiltonian forward path in the subtournament on W_j. The vertex $x_{j,1}$ is the source in W_j and the vertex x_{j,s_j} is the sink in W_j. We draw the tournament T in Figure 5.5.

By Lemma 5.2, there must be a cycle in T containing the arc ab and not cd, and there must be a cycle in T containing the arc cd and not ab. It follows that $W_1 \neq \phi$ and either $W_2 \neq \phi$ or $W_3 \neq \phi$. Then T is realizable as a 2-majority tournament with the linear orderings P_1, P_2 and P_3 as shown in Figure 5.5.



$$P_1 : a > \pi_3 > b > c > \pi_2 > d > \pi_1$$
$$P_2 : c > d > \pi_3 > \pi_2 > \pi_1 > a > b$$
$$P_3 : d > \pi_3 > c > \pi_2 > b > \pi_1 > a$$

Figure 5.5: The tournament T when, in T', d is the source and a is the sink along with linear orderings that make T realizable as a 2-majority tournament.

□

Chapter 6

Conclusion

The property that 2-majority tournaments have a dominating set of size at most three motivates us to investigate what structures exist in a tournament T that makes T realizable as a 2-majority tournament. Let $T = (V, A)$ be a tournament with a set of vertices V and a set of arcs A. Let $P : v_1 v_2 ... v_k$ be a shortest upset path in T of length $k - 1$. Let T' be the tournament obtained from the tournament T after reversing all arcs of P in T.

Acyclic tournaments have one source and one sink. This property leads to some important results. First, if a tournament T is acyclic, then T has a Hamiltonian forward path. If a tournament T is acyclic, then T is realizable as a 2-majority tournament, and we have derived structures to build the three linear orderings P_1, P_2 and P_3 on $V(T)$. Let $|V(T)| \geq 3$. If T has an upset path of length one, then T is not acyclic. Let $P : v_1 v_2 ... v_k$ be a shortest upset path in a tournament T, then the tournament on $V(T) - V(P)$ must be acyclic.

Moreover, our investigation of tournaments with upset paths has given us some structures that we can find in such tournaments. First, the arc $v_1 v_2$ must be contained in a cycle that contains no other arcs in P. Also, the last arc in P, which is $v_{k-1} v_k$, must be contained in a cycle that contains no other arcs in P. Moreover, if P is an upset path of length at least two in a tournament T, then there must exist at least two vertices in $V(T) - V(P)$. Moreover, there must exist distinct vertices u and w in $V(T) - V(P)$ such that $u \to v_2$ and $v_2 \to w$.

Our observation leads us to the following statements: First, for any tournament

T, such that T has an upset path of length two or less, then T is realizable as a 2-majority tournament, and we have derived structures to build the three linear orderings P_1, P_2 and P_3 on $V(T)$. Second, for any tournament T, such that T has an upset path, then T is realizable as a 2-majority tournament, and we have derived structures to build the three linear orderings P_1, P_2 and P_3 on $V(T)$. The linear orderings P_1, P_2 and P_3 depend on the parity of the number of vertices in P.

We also generalize the idea of the upset path in a tournament T. Let $\mathcal{S} = \{P_1, P_2, ..., P_h\}$ be a UP-set in a tournament T, and let $\mathcal{A} = \bigcup_{i=1}^{h} A(P_i)$ be the union of all arcs in the paths in $\mathcal{S}$. We have shown that every UP-set in T contains a minimal UP-set. Moreover, if $\mathcal{S}$ is a minimal UP-set in a tournament T, then for all $a \in \mathcal{A}$, there exists a cycle C such that $C \cap (\mathcal{A} - \{a\}) = \phi$.

Let T be a 2-critical tournament with at least one cycle and $\mathcal{S} = \{P_1, P_2\}$ be a UP-set where P_1 and P_2 are both paths of length one. Then T is realizable as a 2-majority tournament, and we have derived structures enabling us to build the three linear orderings P_1, P_2 and P_3 on $V(T)$.

Further studies must be done to find other structures in a tournament T that, when present, imply T can be realized as a 2-majority tournament. Something we would like to prove is that if a tournament T is a 2-critical tournament with at least one cycle and $\mathcal{S} = \{P_1, P_2\}$ is a UP-set, where P_1 is a path of length m_1, and P_2 is a path of length m_2, then T is realizable as a 2-majority tournament.

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