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Ádám's Conjecture and Arc Reversal Problems

Claudio D. Salas
California State University - San Bernardino

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A Thesis

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Master of Arts

in

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by

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Approved by:

Jeremy Aikin, Committee Chair

Date

Paul Vicknair, Committee Member

Cory Johnson, Committee Member

Charles Stanton, Chair,
Department of Mathematics

Corey Dunn
Graduate Coordinator,
Department of Mathematics

Abstract

A. Ádám conjectured that for any non-acyclic digraph D, there exists an arc whose reversal reduces the total number of cycles in D. In this thesis we characterize and identify structure common to all digraphs for which Ádám's conjecture holds. We investigate quasi-acyclic digraphs and verify that Ádám's conjecture holds for such digraphs. We develop the notions of arc-cycle transversals and reversal sets to classify and quantify this structure. It is known that Ádám's conjecture does not hold for certain infinite families of digraphs. We provide constructions for such counterexamples to Ádám's conjecture. Finally, we address a conjecture of Reid [Rei84] that Ádám's conjecture is true for tournaments that are 3-arc-connected but not 4-arc-connected.

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Introduction

Suppose a problem is posed to you in which the answer is generally accepted, yet no formal argument for the answer has been found. You are able to simplify the problem and use your simplified model to prove your answer is correct. Then you notice your answer to the problem has motivated and created a new branch of mathematics. This situation occurred with a popular puzzle now called the *Königsberg Bridge Problem*, which was solved by mathematician Leonhard Euler who was living in Köngisberg at the time. Euler's solution to the Königsberg bridge problem is considered the origin of graph theory. In the early 18*th* century, the Prussian city of Königsberg occupied both banks of the river Pregel and the island of Kneiphof, positioned at a point where the river branches into two parts [CLZ15]. There were seven bridges located in various parts of the river. The problem asked whether a route that crossed each of the seven bridges exactly once existed. Several had already thought such a route to be impossible but Euler offered the first mathematical verification as to why such a route was indeed impossible. On August 26, 1735, Euler presented a solution to the problem at the Petersburg Academy. Euler even generalized the problem to any number of land masses and bridges. Although the solution to the problem seemed trivial, Euler was curious as to why neither algebra nor traditional geometry was sufficient to solve the problem. Euler mentioned Leibniz as being the first to adopt a math called geometry of position or positional geometry [CLZ15]. This geometry does not concern measurements nor calculations making use of them. Euler described this geometry as interested only in determination of position and properties of position.

A *graph* G is a finite non-empty set V of objects called *vertices* (we call the singular a *vertex*) together with a possibly empty set E of 2-element subsets of V called *edges*. We now use vertices to represent our position and in such geometry we disregard traditional measurement of distance. For example, in the Königsberg bridge puzzle, we assign a vertex to each piece of land as shown in Figure 1.1. We denote the vertex set of a graph G as $V(G) = \{a, b, c, d\}$. Every bridge in Figure 1.1 can be viewed as an edge between pairs of vertices of G. The bridge between land mass a and land mass b is the edge consisting of vertices a and b, denoted ab. Hence the edge set of the Königsberg graph is $E(G) = \{ab, ba, bc, cb, bd, da, dc\}$. Simplifying the problem in this manner gave Euler a better approach to proving his solution. Although many branches of mathematics do not have an exact date of birth, graph theory does have an exact birth date, and is still a young discipline with many accessible parts to explore and applications to discover.

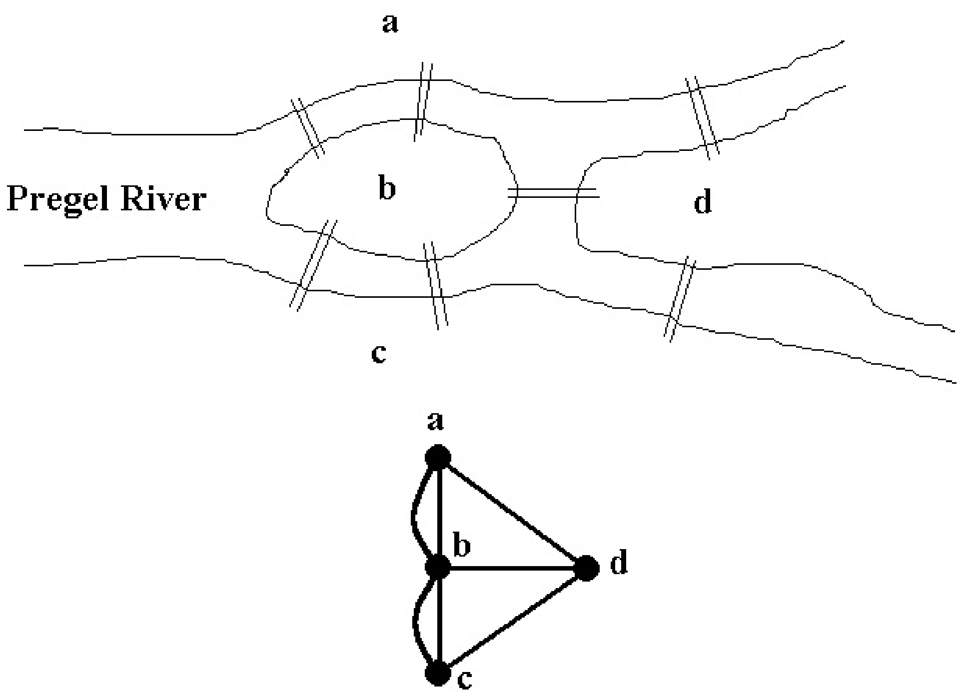

Figure 1.1: The Bridges of Königsberg with graph

Graph theory can be applied to many of our modern day problems. Social media networks such as Facebook can be modeled using a graph. Each person is represented by a vertex and an edge exists between a pair of friends. For two, not necessarily distinct, vertices x and y in a graph G, an $x-y$ *walk* W in G is a sequence of vertices in G, starting at x and ending at y such that consecutive vertices in W are connected by an edge. A walk in which no vertices are repeated is called a *path*. Suppose two Facebook users are chosen at random. What is the shortest path from one user to the other? The answer to this question gives a measure of how inter-connected two users are in a social media network. For example, if a path of length four exists from User a to User b passing through vertices x, y, and z in that order, we denote it as $P = axyzb$. There are different types of graphs with disparate properties. However many of the properties and definitions are shared among diverse types of graphs. For example, the type of graph we study in this thesis is slightly different than what we have defined thus far.

In this thesis we study digraphs. Digraphs can also be applied to modern day problems. Consider networks in which information is given and received. Information flows through networks in certain directions. We can assign a direction to an edge in a graph to model this. Let the edge ab be in a graph. An edge from a directed into b will be denoted as $\langle a, b\rangle$. On the other hand, if the edge is directed from b to a, we denote it as $\langle b, a\rangle$. Directed edges will be referred to as *arcs*. A graph in which all edges are arcs is called a *digraph*. We use $A(D)$ to denote the arc set of a digraph D. Formally, a digraph D is a finite non-empty set V of objects called vertices together with a possibly empty set A of ordered pairs of elments of V called *arcs*.

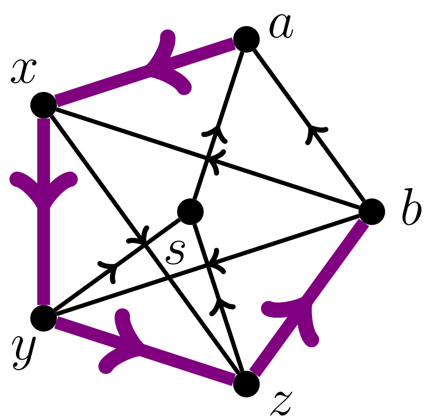

Figure 1.2: A path $P = axyzb$ from Vertex a to Vertex b

Graph theory can also be applied to a favorite past time in Italy. The Italian Serie A soccer league consists of twenty teams. Each team must compete against every team in the league twice, once at their home venue and once away at the opponents home venue. A digraph D can be used to describe all the matches. If Team y beats Team s in a majority of their two match ups, we denote this by an arc from a Vertex y to Vertex s, $\langle y, s\rangle$. In the Italian Serie A, a competition is allowed to be recorded as a draw. If no arc exist between Team v and Team x, we can assume Team v and Team x tied. If no competition ends in a draw then there exists exactly one arc between each distinct pair of vertices. The resulting digraph is called a tournament. Formally a *tournament* is a digraph in which there is exactly one arc between every pair of vertices. At the end of Serie A competition a winner is declared. Clearly, if a team wins every single match, such a team will clearly be declared the ultimate winner of Serie A. Such a winner would be described as a vertex in the corresponding digraph in which all arcs are directed out, also known as a *source*. Figure 1.3 shows Vertex u as a source. The end of competition also brings bad news to the team in last place. The last place team is relegated to a lower level of competition. If a team losses all their matches, they correspond to a sink in the digraph. A *sink* is a vertex in which all arcs are directed in. Figure 1.3 displays Vertex v as a sink. If at the end of competition, several teams win the most matches and win the same amount of matches, it can be difficult to declare a winner. A possible situation can occur as follows. Team a defeats Team b, Team b defeats Team c, and Team c defeats Team a. Which team would be considered the champion? This situation describes a cycle. We define a *cycle* as a digraph with n vertices labeled $v_1, v_2, v_3, ..., v_n$ and arcs v_1v_n and v_iv_{i+1} for $i = 1, 2, 3, ..., n-1$. If the resulting digraph at the end of competition contains cycles, it can be difficult to declare a winner. Figure 1.2 shows a cycle $C = axyzba$.

An *acyclic* digraph is a digraph that contains no cycles. In this thesis we look at digraphs that require one arc reversal to become acyclic and try to characterize them. We will show results concerning digraphs while others will be specified to tournaments. We will prove that an acyclic digraph contains at least one sink and one source while acyclic tournaments contain exactly one sink and one source.

In 1963, A. Ádám published a paper in *Theory of Graphs and its Appplications* in which he made a conjecture. The research in this thesis is motivated by Ádám's

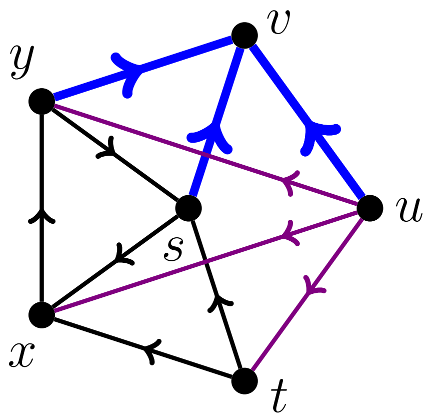

Figure 1.3: A digraph D with a sink v and a source u

conjecture, which states that in any finite digraph D containing at least one cycle, there is at least one arc whose reversal results in a digraph D' in which the total number of cycles is strictly less than the total number of cycles in D [Á64]. Very little was known about Ádám and his conjecture until recently. In 1985 C. Thomassen [Tho85] came forth with counterexamples to Ádám's conjecture. If more than one arc in the same direction is permitted to be shared among two vertices in a digraph, these arcs are called *parallel arcs*. A digraph containing parallel arcs is called a *multidigraph*. The classes of multidigraphs Thomassen used will be investigated this thesis in Chapter 3. In 1988 E.J. Grinberg also published a paper (in Russian) [Gri88], showing more counterexamples of Ádám's conjecture. We will discuss some of the counterexamples to try and better understand where Ádám's conjecture fails and where his conjecture holds. An arc joining a vertex to itself is called a *loop*. In fact, the counterexamples presented so far have led to Ádám's conjecture remaining open for simple digraphs: that do not contain parallel arcs and do not contain loops.

We will also explore digraphs requiring one, two, or three arc reversals for the digraph to become acyclic. The property of a digraph being almost acyclic has a great deal of importance to this research. If a digraph almost satisfies the requirement for an acyclic digraph, say requires no more than three arc reversal to become acyclic, then perhaps we can show that reversing only one of the at most three arcs will reduce the number of cycles in the digraph. In 1992 Josef Jirásek proved that Ádám's conjecture holds for digraphs that become acyclic after up to three arc reversals [Jir92]. We call these digraphs *almost acyclic*. Jirásek's work gave us more structure and methods to explore other types of digraphs for which Ádám's conjecture holds. Jirásek showed that almost acyclic digraphs satisfy Ádám's conjecture. We will, in addition, consider characteristics of digraphs that need more than three arc reversals to be acyclic.

K.B. Reid, in 1984, also made an advancement concerning Ádám's conjecture. Reid's main result in [Rei84] includes an elegant proof with a different method for showing Ádám's conjecture is true for certain digraphs. A digraph D is *strongly connected* if for every $x, y \in D$ there exist both an $x - y$ path and a $y - x$ path. In Reid's result, he compares two digraphs, D and D', where D' is the resulting digraph after one arc reversal in D. Instead of analyzing how close digraph D is to becoming ayclic, Reid is concerned with how strongly connected D and D' are. We analyze Reid's main result in this thesis and then use his method to expand his result.

This project will help build more structure and characterize the types of digraphs for which Ádám's conjecture holds. We will also construct multidigraphs that are counterexamples to Ádám's conjecture. We will conclude with a result proving that Ádám's conjecture holds for a class of tournaments, and in the process, extend Reid's main result.

Chapter 2

Ádám's Conjecture

2.1 Preliminaries

In this thesis we are interested in digraphs, and, as previously stated, we will use $\langle x, y\rangle$ to denote an arc from Vertex x directed to Vertex y. A digraph D_1 is a *subdigraph* of a digraph D if $A(D_1) \subseteq A(D)$ and $V(D_1) \subseteq V(D)$. In this case, we write $D_1 \subseteq D$

For simplicity we will assume the digraphs in this chapter do not contain loops or parallel arcs. We now explain the importance of paths concerning arc reversal problems in the context of Ádám's conjecture. We will use (x, y) to denote the number of directed paths from a vertex x to a vertex y. Knowing the number of paths from one vertex to another will help simplify the manner in which we count the number of cycles that contain an arc. Let $\langle x, y\rangle$ be some arc in a digraph D. If we wish to investigate the number of cycles containing $\langle x, y\rangle$, then we simply look at the number of paths from y to x, which is (y, x). Notice $\langle x, y\rangle$ together with any $y - x$ path creates a cycle. Hence, there are (y, x) cycles containing $\langle x, y\rangle$. Similarly, we can expect to have (x, y) cycles containing the reversal of $\langle x, y\rangle$ (the reversal is $\langle y, x\rangle$). Once establishing how many cycles contain $\langle x, y\rangle$ and how many cycles contain the reversed arc, we can determine the effect reversing the arc $\langle x, y\rangle$ will have on the number of cycles. This perspective will enable us to simplify our arguments in the proofs of many results. We can now formulate Ádám's conjecture by means of directed paths as stated by Jirásek [Jir92].

Conjecture 2.1. *For every digraph D, with at least one cycle, there exists an arc $\langle x, y\rangle$ such that $(x, y) \leq (y, x)$.*

Next we describe some properties of vertices. An arc $\langle x, y\rangle$, starting at Vertex x and directed to a Vertex y, is said to be *incident* from x and incident to y. This will help describe vertices and their relationship to arcs. Another way to describe the relationship between arcs and vertices, specifically to digraphs, is through the degree of a vertex. The *out-degree* of a vertex x is the number of arcs directed away from the vertex, and is denoted $od(x)$. The *in-degree* of a vertex x is the number of arcs directed to the vertex, and is denoted $id(x)$. For example, in Figure 1.2, the arcs containing the Vertex s are $\langle s, a\rangle$, $\langle z, s\rangle$, and $\langle y, s\rangle$. Thus $id(s) = 2$ and $od(s) = 1$.

A digraph is *connected* if there is a path between every pair of its vertices. A digraph that is not connected is called a *disconnected digraph.* In a digraph D, we call a set of arcs A an *arc-cut* if $D - A$ is a disconnected digraph. In a connected digraph D, we refer to a set of vertices X as a *cut* or *cut-vertex* if $D - X$ is a disconnected digraph.

2.2 Jirásek's contributions

In this section, we give results that can be found in Jirásek's paper [Jir92]. Many of the results in this paper are stated without proof, and for these results, we also provide a proof. For those results in [Jir92] that are accompanied by a proof, we provide a proof that closely follows that found in [Jir92].

Ádám's conjecture states:

Conjecture 2.2. *For every digraph $D = (V, A)$ containing a directed cycle there is an arc $\langle x, y\rangle \in A$ whose reversal decreases the total number of cycles in D.*

This conjecture fails for multidigraphs, but remains open for tournaments and simple digraphs. Recall, (x, y) denotes the number of paths from Vertex x to Vertex y. Consider some $y - x$ path in D. The $y - x$ path together with arc $\langle x, y\rangle$ creates a cycle. We denote this cycle as $xP_y y$ where P_y is the $y - x$ path. Thus, we say that any number of n distinct $y - x$ paths together with $\langle x, y\rangle$ will create n distinct cycles. Therefore, there are (y, x) cycles containing $\langle x, y\rangle$. If we reverse $\langle x, y\rangle$ then we will destroy all cycles containing $\langle x, y\rangle$, namely (y, x) cycles. We may however create new cycles containing the reversal of $\langle x, y\rangle$, which is $\langle y, x\rangle$. The new cycles created will use $\langle y, x\rangle$ and some $x - y$ path. Hence there will be (x, y) cycles created after the reversal of $\langle x, y\rangle$. If $(y, x) \geq (x, y)$ then there are more $y - x$ paths than $x - y$ paths which results in more cycles containing

$\langle x, y\rangle$ than the number of cycles containing $\langle y, x\rangle$. Hence, we arrive at Conjecture 2.1, the version of Ádám's conjecture as formulated by Jirásek [Jir92]. Such formulation of Ádám's conjecture provides assistance in proving the following propositions. *Symmetric arcs* are pairs of arcs that are between the same two vertices, but are directed opposite of each other. A digraph D containing symmetric arcs implies a cycle of length two exists in D.

Proposition 2.1. *Ádám's conjecture holds for every digraph containing a symmetric pair of arcs.*

Proof. If $\langle x, y\rangle$ and $\langle y, x\rangle$ are symmetric arcs in a digraph D, then either $(x, y) \leq (y, x)$ or $(y, x) \leq (x, y)$ is true. Without loss of generality, assume $(x, y) \leq (y, x)$. Then there are more $y - x$ paths than $x - y$ paths which implies there are more cycles containing $\langle x, y\rangle$ than there are cycles containing $\langle y, x\rangle$. The reversal of arc $\langle x, y\rangle$ will eliminate (y, x) cycles and create (x, y) cycles. Therefore, the reversal of $\langle x, y\rangle$ will decrease the number of cycles in D since $(x, y) \leq (y, x)$. This results in (x, y) cycles containing the arc $\langle y, x\rangle$. Since $(x, y) \leq (y, x)$, reversal of one of the symmetric arcs decreases the total number of cycles in D. $\square$

Proposition 2.2. *If, in a cut of a connected digraph D, there exists exactly one arc $\langle x, y\rangle$ having the opposite direction than the other arcs in the cut and there is a cycle containing the arc, then reversal arc $\langle x, y\rangle$ decreases the total number of cycles of the digraph.*

Proof. Let digraph D have a cut (V_1, V_2) containing the arc $\langle x, y\rangle$ oriented in the opposite direction of all the other arcs in this cut, where $x \in V_1$ and $y \in V_2$. Then any path from a vertex in V_1 to a vertex in V_2 must contain the arc $\langle x, y\rangle$. Therefore $(x, y) = 1$. Since there is a cycle containing arc $\langle x, y\rangle$ and all arcs in (V_1, V_2) are directed into V_1 except $\langle x, y\rangle$, there exists at least one $y - x$ path. That is, $(y, x) \geq 1$. Therefore $(x, y) \leq (y, x)$. So the reversal of $\langle x, y\rangle$, which is the only arc from V_1 to V_2, will eliminate all $x - y$ paths. Then reversal of $\langle x, y\rangle$ will create (x, y) cycles. After reversal of $\langle x, y\rangle$, there will be no arcs from V_1 to V_2 and thus $(x, y) = 0$. Therefore $(x, y) \leq (y, x)$ and the reversal of $\langle x, y\rangle$ decreases the total number of cycles in D. $\square$

Proposition 2.3. *If a digraph D contains a directed cycle C of length 3 such that the reversal of all the arcs in C decreases the total number of cycles D, then the reversal of any one of the three arcs in C also decreases the total number of cycles in D.*

Proof. Let cycle C be in digraph D where C consists of arcs $\langle x, y\rangle$, $\langle y, z\rangle$, and $\langle z, x\rangle$. It is important to note that the reversal of any one of the arcs in C may produce new cycles, however none of the new cycles will contain either of the two other arcs in C. For example, consider the arc $\langle x, y\rangle$ in C. The reversal of $\langle x, y\rangle$ may produce new cycles, in fact it will produce $(x, y) - 1$ new cycles (where -1 accounts for not being able to use $\langle x, y\rangle$ as an $x - y$ path after reversal). All new cycles produced by the reversal of $\langle x, y\rangle$ will be required to use an $x-y$ path other than $\langle x, y\rangle$. The new cycles will consist of an $x-y$ path together with the arc $\langle y, x\rangle$. There are no new cycles that contain $\langle y, x\rangle$, and $\langle z, x\rangle$ or $\langle y, z\rangle$. So reversing all arcs in C will result in at least $(x, y)-1+(y, z)-1+(z, x)-1$ new cycles. There will also be a new cycle of length three consisting of all the reversed arcs in C. Simultaneously, each cycle in D consisting of at least one of the arcs in C will be eliminated. The number of cycles eliminated is not greater than $(y, x)+(z, y)+(x, z)+1$. Note there is exactly one cycle containing all of the reversed arcs.

Assume to the contrary, that the reversal of any single arc in C does not decrease the total number of cycles in D. Then $(x, y)-1 \geq (y, x)$, $(y, z)-1 \geq (z, y)$, and $(z, x)-1 \geq (x, z)$ hold. By adding these inequalities, we obtain $(x, y) - 1 + (y, z) - 1 + (z, x) - 1 \geq (y, x) + (z, x) + (x, z)$. Then the reversal of all three arcs does not decrease the number of cycles in D, which is a contradiction. $\square$

The previous propositions will enable us to prove two theorems crucial to our findings in arc reversal problems in which Ádám's conjecture holds. Before proving the following, we must define some terms. Recall, a digraph D is called *strongly connected* if for all $x, y \in V(D)$ there exists both an $x - y$ path and a $y - x$ path in D. We now generalize this notion and define what it means for a digraph to be more highly connected.

Definition 2.1. *A digraph D is called strongly 2-connected if for every $x, y \in V(D)$ there exists at least two distinct $x - y$ paths and two distinct $y - x$ paths in D.*

Lemma 2.2. *A strongly connected digraph D is strongly 2-connected if and only if in each of its cuts, there exists at least two arcs in one direction and at least two arcs in the opposite direction.*

Proof. Let D be a strongly 2-connected digraph. Consider a cut (X, Y) in D and consider vertices $x \in X$ and $y \in Y$. Now assume, to the contrary, that D is strongly 2-connected and there exist strictly less than two arcs in (X, Y) directed from one side of the cut to the other. With no loss in equality, we may assume there are fewer than two arcs directed from X to Y. If there are no arcs from X to Y, then there are no $x - y$ paths. Hence D is not strongly connected, which is a contradiction. Similarly, if there exists one arc directed from X to Y, then there exists exactly one $x - y$ path. But this contradicts Definition 2.1 for a strongly 2-connected digraph.

We now prove the reverse direction. Let D be a digraph such that in each of its cuts, there exist at least two arcs in one direction and at least two arcs in the opposite direction. Let (X, Y) be a cut in D and let $x \in X$, $y \in Y$ and arcs $\langle a, b\rangle$, $\langle c, d\rangle$ be arcs from Y to X. Since D is a strongly connected digraph, there exists a $b - x$ path, $d - x$ path, $y - a$ path, and $y - c$ path. Similarly there exist two arcs $\langle e, f\rangle$ and $\langle g, h\rangle$ from X to Y. D being a strongly connected digraph implies there exist an $f - y$ path, $h - y$ path, $x - e$ path, and $x - g$ path. By Definition 2.1, D is strongly 2-connected.

Assume both x and y are in Y. Since D is strongly connected, there exists two paths from x ,through $\langle a, b\rangle$ and $\langle c, d\rangle$, to X. Similarly, since D is strongly connected, there exists two paths from b and two paths from d, through $\langle e, f\rangle$ and $\langle g, h\rangle$, to Y. Then there exists both an $f - y$ path and an $h - y$ path in Y. Therefore, there are two $x - y$ paths in D, mainly $P_{x_1} = xabefy$ and $P_{x_2} = xcdghy$. We use a similar argument to show there are two $y - x$ paths. By Definition 2.1, D is strongly 2-connected □

The characterization for strongly 2-connected digraphs given in Lemma 2.2 [Jir92], is a generalization of the following characterization of strongly connected digraphs.

Lemma 2.3. *If a digraph D is strongly connected, then in each of its cuts there exists at least one arc in one direction and at least one arc in the opposite direction.*

Proof. Let D be strongly connected digraph. Take a cut (X, Y) in D and consider vertices $x \in X$ and $y \in Y$. Suppose D is strongly connected, yet there exists strictly less than one arc in one of the directions in (X, Y). Without loss of generality, assume there are no arcs from X to Y. Then there are no $x - y$ paths. Hence D cannot be strongly connected, which is a contradiction.

□

Lemma 2.4. *If D is a strongly connected digraph, then each of its arcs is contained in a cycle.*

Proof. Suppose, to the contrary, there exists an arc in D not contained in a cycle. Let $x, y \in D$ and $x \neq y$ such that $\langle x, y\rangle$ is not contained in any cycle. Since D is strongly connected, there exists a $y - x$ path. This is a contradiction since $\langle x, y\rangle$ together with the $y - x$ path creates a cycle. $\square$

We define a *component* D_1, of a strongly connected subdigraph of a digraph D, if D_1 is not a proper subdigraph of any connected subdigraph of D [CLZ15]. Now we can describe a class of simple digraphs for which Ádám's conjecture holds.

Theorem 2.5. *Ádám's conjecture holds for every simple digraph containing a nontrivial strongly connected component that is not strongly 2-connnected.*

Proof. Consider a simple digraph D containing a nontrivial strongly connected component which is not strongly 2-connected. Call this component digraph D_1. Since D_1 is strongly connected, each of its arcs is contained in a cycle and every cut of D_1 contains at least one arc in each direction. Since D_1 is not strongly 2-connected, there must exist one cut containing only one arc in one direction and all others in the opposite direction. Otherwise, all cuts would contain at least two arcs in each direction and D_1 would be strongly 2-connected. It is now established that there exist a cut in D_1 with exactly one arc in the opposite direction of all the other arcs. By Proposition 2.2, Ádám's conjecture holds.

$\square$

According to Theorem 2.5, Ádáms conjecture holds for every digraph containing a strongly connected component that is not strongly 2-connected. Before arriving at at Jirásek's main result, we first prove two useful lemma's.

Lemma 2.6. *If a digraph D has all vertices with in-degree at least one and out-degree at least one, then there exists a cycle in D.*

Proof. Let all vertices in a digraph D have in-degree at least one and out-degree at least one. Take a maximum length path P, beginning at vertex u and ending at v. Since $od(u) \geq 1$, there exists $\langle u, u_1\rangle$ for some vertex $u_1 \neq u$. Since $od(u_1) \geq 1$, there exists $\langle u_1, u_2\rangle$. Thus, path $P = uu_1u_2u_3...v$. However $od(v) \geq 1$, and if v has an arc that is directed to a vertex not on path P, we contradict the maximality of path P. So vv must have an arc directed to a vertex already on path P, which creates a cycle.

□

Lemma 2.7. *If a simple digraph D is acyclic, then D has a vertex s with $id(s) = 0$ and a vertex t with $od(t) = 0$.*

Proof. To the contrary, suppose D is a simple acyclic digraph with no vertex s with $id(s) = 0$. Recall, a vertex t is a sink if $od(t) = 0$ and a vertex s is a source if $id(s) = 0$. Consider a maximal length path $P = s, x_1, x_2, ..., x_n, t$ for $s, t, x_i \in V(D)$. Then there exists some arc directed to s. If there exists an arc $\langle k, s \rangle$, for some k not on P, we contradict the maximality of P. Hence there exists an arc $\langle x_i, s \rangle \in A(D)$ for $1 \leq i \leq n$. But $\langle x_i, s \rangle$ together with the $s - x_i$ path is a cycle which contradicts D being acyclic.

Similarly, assume D is a simple acyclic digraph with no vertex t with $od(t) = 0$. Consider a maximal length path $P = s, x_1, x_2, ..., x_n, t$ for $s, t, x_i \in V(D)$. Then there exists some arc directed from t. If there exists an arc $\langle t, k' \rangle$, for some k' not on P, we contradict the maximality of P. Hence there exists an arc $\langle t, x_i \rangle \in A(D)$ for $1 \leq i \leq n$. But $\langle t, x_i \rangle$ together with the $x_i - t$ path is a cycle which contradicts D being acyclic.

□

Now we arrive at Jirásek's main result. Jirásek showed that Ádám's conjecture holds for simple digraphs that become acyclic after the reversal of no more than three arcs [Jir92].

Theorem 2.8. *If after the reversal of at most three arcs of a non-acyclic simple digraph D, all cycles are eliminated, then there exists an arc in D whose reversal decreases the total number of cycles in D.*

Proof. Let D' denote the acyclic digraph obtained after arc reversals. Since D' is acyclic, by Lemma 2.7, there exists at least one vertex s with $id(s) = 0$ and at least one vertex t with $id(t) = 0$ such that $t \neq s$. To arrive at digraph D', all arcs ending at $s \in D$ and all arcs beginning at $t \in D$ must be reversed. For example, if there exists arcs of the form $\langle x, s \rangle$ and $\langle t, y \rangle$ in D for all $x, y \in V(D)$, then since s is a source and t is a sink in D', all such arcs must be reversed in D to arrive at D'.

If D contains a strongly connected component, D_3, that is not strongly 2-connected, then, by Theorem 2.5, Ádám's conjecture holds. If D contains two strongly connected components D_1 and D_2 that are strongly 2-connected, then by Definition 2.1, there exists at least two $x-y$ paths and two $a-b$ paths for $x, y \in V(D_1)$ and $a, b \in V(D_2)$ respectively. Hence, each vertex in D_1 and D_2 have indegree at least two and outdegree at least two. In order to create a sink in D_1 we would need to reverse at least two arcs and the same would apply to D_2. We also may require more arc reversals to create a source. Thus making D acyclic would require at least four arc reversals, which does not satisfy the requirement of at most three arc reversals to arrive at D'. Therefore we cannot have more than one component that is strongly 2-connected.

We now verify that if D contains at most one nontrivial strongly connected component that is strongly 2-connected, then D may be made acyclic with at most three arc reversals. Without loss of generality, assume D is strongly 2-connected. By Definition 2.1, D contains at least two distinct $x-y$ paths and two distinct $y-x$ paths for all $x, y \in V(D)$. Hence, $id(x) \geq 2$ and $od(x) \geq 2$ for every $x \in V(D)$. Then the reversal of two arcs might be required to create a vertex $s \in V(D')$, such that $id(s) = 0$. We now verify that a third arc reversal can result in a sink t and thus arriving at D'. If arcs $\langle y, s\rangle$, $\langle t, s\rangle$, and $\langle t, x\rangle$ for $s, t, x, y \in V(D)$, where $id(s) = 2$ and $od(t) = 2$, are reversed then s a becomes source and t becomes a sink and the resulting digraph is the desired acyclic digraph D'. Figure 2.2 shows s and t before performing the three arc reversals for some digraph D^*. Since $x, y \in V(D)$, we have $id(x) \geq 2$, $id(y) \geq 2$ and $od(x) \geq 2$, $od(y) \geq 2$. Reversing one arc incident from y and one arc incident to x will not result in a sink y nor a source x. Therefore, D' contains exactly one source s and one sink t.

We now verify that reversal of $\langle t, s\rangle$ decreases the total number of cycles. Since D is strongly 2-connected, by Lemma 2.2, there exists a cut with two arcs in one direction and at least two arcs in the opposite direction. Let (V_1, V_2) be a cut in D where $s, x_1, x_2, ..., x_n \in V_1$ and $t, y_m, y_{m-1}, .., y_1, y, x \in V_2$. We must remember that $id(s) = 2$ and $od(t) = 2$. Then (V_1, V_2) contains more than two arcs to V_2 and only $\langle t, s\rangle$ and $\langle y, s\rangle$ directed into V_1. Any arc directed away from s goes to some vertex x_i, for $i = 1, 2, ..., n$, and thus arc $\langle s, x_i\rangle$ remains in V_1. There does not exist an arc $\langle x_i, s\rangle$ in D since $id(s) = 2$ and there already exists $\langle y, s\rangle$ and $\langle t, s\rangle$. The number of cycles in D containing $\langle t, s\rangle$ is (s, t). The $s-t$ paths in D are composed of the $s - x_i$ paths, $x_i - x$ paths, $x - y$ paths, and $y - y_i$ paths. Hence the number of cycles in D containing $\langle t, s\rangle$ is strictly larger than (x, y). After the reversal of $\langle t, s\rangle$ there will be (t, s) new cycles created. With the exception of $\langle t, s\rangle$, notice that every $t-s$ path in D begins with $\langle t, x\rangle$ and ends with

$\langle y, s\rangle$. So the number of $t - s$ paths is (x, y). Therefore $(s, t) \geq (t, s)$. Reversal of $\langle t, s\rangle$ will eliminate more cycles than those created and Ádám's conjecture is satisfied.

□

Figure 2.1 displays the strongly 2-connected digraph D from Theorem 2.8. The dotted arcs represent paths and not necessarily arcs. The thicker arcs represent the three arcs being reversed in D to arrive at D'. From the figure, it is much easier to notice there are more $s - t$ paths than $t - s$ paths.

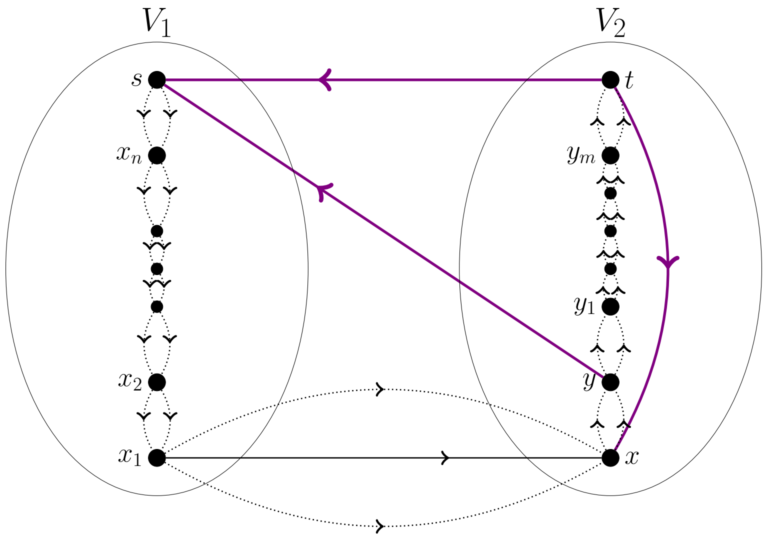

Figure 2.1: Strongly 2-connected digraph D requiring three arc reversals to create an acyclic digraph D'.

We now have structure for the types of simple digraphs that satisfy Ádám's conjecture. The conjecture holds for all simple digraphs containing a nontrivial strongly connected component that is not strongly 2-connected. Then by Lemma 2.3, there exists a cut in a component of a digraph, containing exactly one arc in the opposite direction of all the other arcs in the cut. Furthermore we also know the conjecture holds for simple digraphs that may be strongly 2-connected if the digraph becomes acyclic after at most three arc reversals.

We will continue our efforts with more specific digraphs. In the next section we use directed tournaments which allow for more structure when trying to characterize digraphs that satisfy Ádám's conjecture.

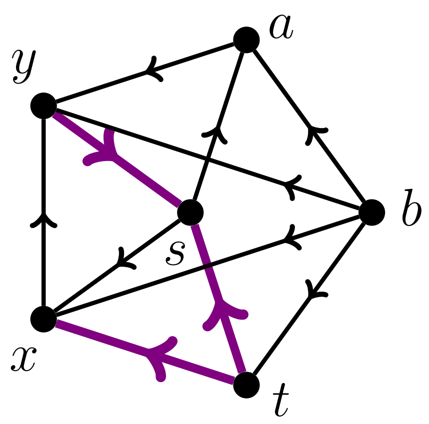

Figure 2.2: A digraph D^* becomes acyclic after at most three arc reversals.

2.3 Tournaments and Reid's result.

In this section we investigate Ádám's conjecture for a specific class of digraphs: tournaments. We say a vertex x *dominates* a vertex y in a digraph D, if the arc $\langle x, y \rangle$ exists. Ádám's conjecture can be thought of in a different manner and applied to tournaments as Reid's result will demonstrate.

Lemma 2.9. *Let T be a strongly connected tournament. If T contains an arc whose reversal produces a tournament T' that is not strongly connected, then T' has strictly fewer cycles than T.*

Proof. Let T be a strongly connected tournament. By Lemma 2.4, each arc in T is contained in a cycle. Then, by Lemma 2.3, each cut of D contains at least one arc in one direction and at least one arc in the opposite direction. Let (X, Y) be a cut in D with $x \in X$ and $y \in Y$. T contains an arc $\langle x, y \rangle$, whose reversal results in a non-strong tournament T'. Hence $\langle x, y \rangle$ is directed opposite to all the other arcs in (X, Y) and after reversal, T' will be a non-strong digraph since no $x - y$ exists in T'. By Proposition 2.2, reversal of $\langle x, y \rangle$ will decrease the total number of cycles in T. $\square$

We will denote a set of a cycles in a tournament T by $C(T)$. The following result by Reid provides a sufficient condition for Ádám's conjecture to hold for strongly connected tournaments [Rei84].

Theorem 2.10. *Suppose that T is a strongly connected tournament such that the reversal of any single arc results in a strongly connected tournament, but that the reversal of some pair of arcs results in a tournament that is not strongly connected. Then Ádám's conjectures holds for T.*

Proof. Let T be a strongly connected tournament such that reversal of any single arc results in a strongly connected tournament, but the reversal of some pair of arcs results in a tournament that is not strongly connected. Since T is a strongly connected tournament, for any distinct pair of vertices x and y in T there exists an $x-y$ path and a $y-x$ path in T. Moreover, this property also holds in the digraph T' that results after the reversal of any single arc. However the reversal of some pair of arcs will result in a tournament T'' that does not have an $x-y$ path for some $x, y \in V(T)$. In order for such a tournament T to exist, there must be a non-empty subset S of vertices from $V(T)$ such that the number of arcs from S to $V(T) - S$ is equal to two. Keep in mind that since T is a tournament, there exists an arc between every distinct pair of vertices. Hence, there will be an arc joining every vertex in $V(T) - S$ with every vertex in S such that these arcs are directed to S with the exception of the two arc from S to $V(T) - S$.

Assume there does not exist exactly two arcs from S to $V(T) - S$. Then there are one or strictly more than two arcs from S to $V(T) - S$. In the first case we assume that only one arc exists from S to $V(T)-S$. Let $\langle x, y\rangle$ be the only arc from S to $V(T)-S$. Hence $x \in S$ and $y \in V(T) - S$. Then reversal of $\langle x, y\rangle$ will result in a tournament that is not strongly connected since there does not exist any path from S to $V(T) - S$ in T'. This contradicts the property that T' remains a strongly connected tournament after the reversal of any single arc. Then there must be more than one arc from S to $V(T)-S$. In the second case, we assume there are more than two arcs from S to $V(T)-S$. Then there exists at least three arcs $\langle x, y\rangle$, $\langle a, b\rangle$, and $\langle w, z\rangle$ from S to $V(T)-S$. Reversal of any one of arc will result in a strongly connected tournament T'. But reversal of a pair of arcs will continue to be a strongly connected tournament, which contradicts the property that the reversal of some pair of arcs results in a tournament that is not strongly connected. Hence there exists exactly two arcs from S to $V(T) - S$. Choose S to be maximal such set. The two arcs from S to $V(T) - S$ occur in one of the following cases:

(1) $\langle x, z\rangle$ and $\langle x, w\rangle$, such that $x \in S$, and $z, w \in V(T) - S$, or

(2) $\langle x, z\rangle$ and $\langle y, z\rangle$, such that $x, y \in S$, and $z \in V(T) - S$, or

(3) $\langle x, w\rangle$ and $\langle y, z\rangle$, such that $x, y \in S$, and $z, w \in V(T) - S$.

Each case will be proven separately.

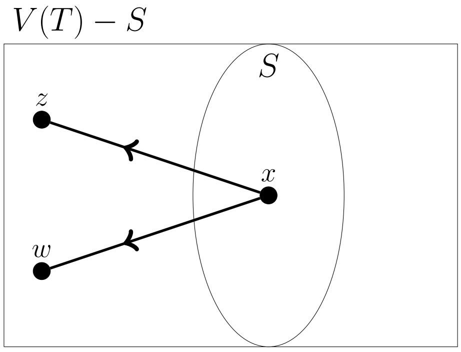

Figure 2.3: Theorem 2.10 Case 1

Case 1: Without loss of generality, assume w dominates z in T and S is maximal. Assume $V(T) - S$ contains only two vertices, w and z. Then $S \cup \{w\}$ implies S can contain w and still have two arcs, $\langle x, z\rangle$ and $\langle w, z\rangle$, to $V(T) - S$. However this contradicts the maximality of S. So $V(T) - S$ must contain more than two vertices, mainly a vertex that is dominated by w which we will name later. Now let T' denote the tournament obtained after reversing the arc $\langle x, w\rangle$. Let vertex w_1 be a vertex in $V(T) - S$, such that $w_1 \neq w$ and w_1 is any vertex that dominates w. Note that every cycle in T' that uses arc $\langle w, x\rangle$ must also use arcs $\langle x, z\rangle$ and $\langle w_1, w\rangle$. Any cycle that does not use arc $\langle w, x\rangle$, is contained in $V(T) - S$ or in S. Consider a function $f(C) : C(T') \to C(T)$ as follows for each cycle in T':

$$f(C) = \begin{cases} C, & \text{if } \langle w, x\rangle \text{ is not an arc of C.} \\ x, w, z, C[z, w_1], w_1, x, & \text{if C is given by w,x,z,...,}w_1\text{,w.} \end{cases} \tag{2.1}$$

Since T and T' are tournaments, there must exist an arc joining w_1 and x. Since the only two arcs from S to $V(T) - S$ are $\langle x, w\rangle$ and $\langle x, z\rangle$, w_1 dominates x in both T and T'. Then every cycle in T' that does contain $\langle w, x\rangle$, will map to itself in T. Every cycle in T' containing $\langle w, x\rangle$, z, and w_1, will map to cycles in T containing $\langle x, w\rangle$, z, and w_1. Therefore $f(C)$ is one-to-one. Next we show $f(C)$ is not onto. If z is the only vertex in $V(T) - S$ dominated by w, then as previously shown, $S \cup \{w\}$ contradicts the maximality of S. There must exist a vertex $v \in V(T) - S$ such that w dominates v and $v \neq z$. The cycle $C_x = x, w, v, x$ in T, does not map from any cycle in T', because any cycle in T' that does not use vertex z must avoid the arc $\langle w, x\rangle$. Since $f(C)$ is not onto, there exists more cycles in T than in T'. Hence Ádám's conjecture holds for such tournaments.

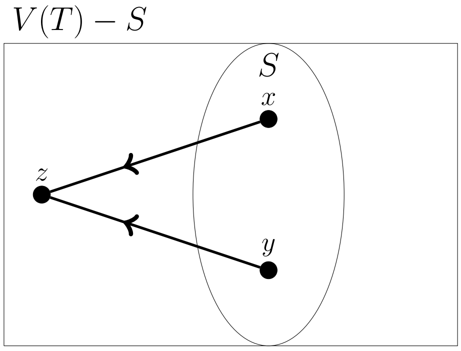

Figure 2.4: Theorem 2.10 Case 2

Case 2: The treatment of this case is similar to Case 1. Without loss of generality, let x dominate y. Since T is a strongly connected tournament, there exists a vertex x_1 in S that dominates x. T is strongly 2-connected, since T requires a pair of arcs to be reversed for the resulting tournament to be a tournament that is not strongly connected. Then there exists at least two paths from y to x_1. One such path travels through $\langle y, z\rangle$. Another such path will use an arc $\langle y, y_1\rangle$ where $y_1 \in S$. If x_1 is in $V(T) - S$, then the $y_1 - x_1$ path will contain an arc from S to $V(T) - S$ that is not $\langle y, z\rangle$ or $\langle x, z\rangle$. This contradicts the existence of only two arcs from S to $V(T) - S$. Hence x_1 must be in S. Consider a function $f(C) : C(T') \to C(T)$ as follows for each cycle in T':

$$f(C) = \begin{cases} C, & \text{if } \langle z, y\rangle \text{ is not an arc of C.} \\ y, z, y_1, C[y_1, x_1], x_1, x, y, & \text{if C is given by z,y,}y_1\text{,...,}x_1\text{,x,z.} \end{cases} \tag{2.2}$$

The reversal of $\langle y, z\rangle$ results in T'. Similarly to Case 1, notice that all cycles in T' not containing the arc $\langle z, y\rangle$ will map to themselves in T. Also cycles in T' containing $\langle z, y\rangle$, x, and x_1, will map to cycles in T containing $\langle y, z\rangle$, x, and x_1. Therefore, $f(C)$ is one-to-one. Since T is strongly 2-connected, y is dominated by some vertex v such that $v \neq x$. Then there exist cycle $C_y = y, z, v, y$ in T. But $f(C)$ does not may any cycle from T' to T, since all cycles in T' that contain the arc $\langle z, y\rangle$ must also contain x and x_1. Hence f(C) is not onto. Then there exists more cycles in T than in T' and Ádám's conjecture holds.

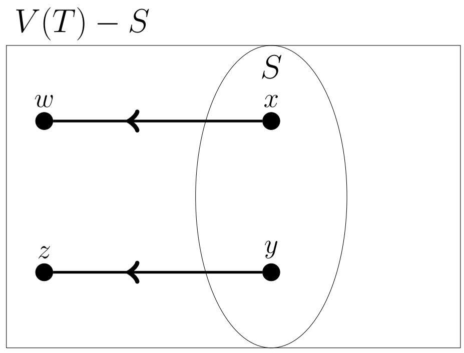

Figure 2.5: Theorem 2.10 Case 3

Case 3: Without loss of generality, assume vertex x dominates y in S. Then either z dominates w or w dominates z in $V(T) - S$. In either case let T' be obtained from T by reversing the arc $\langle y, z\rangle$. Note that any cycle in T' that uses arc $\langle z, y\rangle$, must use arcs $\langle y, y_1\rangle$, $\langle x, w\rangle$, and $\langle z_1, z\rangle$, where $y_1 \in S$ and $z_1 \in V(T) - S$. In the case where z dominates w, $z \neq w$. Define $f(C) : f(T') \to f(T)$ as follows: if C is in $C(T')$ and z dominates w, then:

$$f(C) = \begin{cases} C, & \text{if } \langle z, y\rangle \text{ is not an arc of C.} \\ z, w, C[w, z_1], z_1, y_1, C[y_1, x], x, y, z, & \text{if C is given by z,y,}y_1\text{,...,x,w,...,}z_1\text{,z.} \end{cases} \tag{2.3}$$

If w dominates z, then:

$$f(C) = \begin{cases} C, & \text{if } \langle z, y\rangle \text{ is not an arc of C.} \\ z, y_1, C[y_1, x], x, w, z, & \text{if C is given by z,y,}y_1\text{,...,x,w,z.} \\ z, y_1, C[y_1, x], x, w, C[w, z_1], z_1, y, z & \text{if C is given by z,y,}y_1\text{,...,x,w,...,}z_1\text{,z,} \end{cases} \tag{2.4}$$

In either case there is a mapping for every cycle in T' to T, proving $f(C)$ is one-to-one, because cycles that do not contain the reversed arc $\langle y, z\rangle$, remain unaffected in T and T'. Now we show $f(C)$ is not onto. If z dominates w (respectively if w dominates z), then a cycle of length three in T is given by $C_{y_1} = y, z, w, y$ (respectively, by $C_{y_2} = y, z, w, y$). Notice, $f(C)$ does not map any cycle from T' to C_{y_1} (respectively C_{y_2}) in T. Thus $f(C)$ is not onto and Ádám's conjecture holds. Since in all three cases satisfy Ádám's conjecture, the proof is complete.

□

As a direct consequence of Reid's elegant proof, a new definition is required. A strongly connnected tournament T is *k-arc-connected* if the reversal of fewer than k arcs of T results in a strongly connected tournament. Then Thorem 2.10 can be restated as follows [Rei84].

Theorem 2.11. *Ádám's conjecture is true for strongly connected tournaments that are 2-arc-connected but not 3-arc-connected.*

Reid's proof of Theorem 2.10 may help verify Ádám's conjecture for k-arc-connected tournaments which are not $(k + 1)$-arc-connected tournaments. We will use Reid's proof method to prove a sub-case for $k = 3$ in the final Chapter of this thesis.

Chapter 3

Counterexamples to Ádám's conjecture

So far we have seen results that satisfy Ádám's conjecture and have given structure to such digraphs. We know Ádám's conjecture holds for digraphs containing a nontrivial strongly connected component that is not strongly 2-connected, as shown in Theorem 2.5. Ádám's conjecture holds for digraphs that are strongly 2-connected if the digraph becomes acyclic after the reversal of at most three arcs, as shown in Theorem 2.8. By Theorem 2.10, strongly connected tournaments that are 2-arc-connected but not 3-arc-connected satisfy Ádám's conjecture. We have also verified Ádám's conjecture, in Proposition 2.1, for digraphs containing symmetric arcs. There are, however. counterexamples to Ádám's conjecture. In this chapter we construct a counterexample to Ádám's conjecture.

If in a digraph, there are multiple arcs directed from a Vertex x to a Vertex y, then these arcs are parallel and the digraph is a *multidigraph*. Consider a digraph containing parallel arcs, none of which is contained in a cycle. Reversing one of the parallel arcs will produce cycles of length two and will not eliminate any existing cycles. Thus, multidigraphs provide our first counterexamples to Ádám's conjecture. The first counterexamples using multidigraphs were described by Thomassen [Tho85] and Grinberg [Gri88]. Jirásek then gave further results on counterexamples to Ádám's conjecture [Jir01]. We denote G^P, the multidigraph resulting from a digraph G ,without loops or cycles of length two, by replacing each arc with p parallel arcs. Digraphs of the form

G^P, with at most five vertices, satisfy Ádám's conjecture [Jir01]. However, there exists counterexamples to Ádám's conjecture, of the form G^P, with at least twelve vertices [Jir01]. The question remains open for counterexamples of the form G^P with at most twelve vertices.

We now begin constructing a counterexample, of the form G^P, to Ádám's conjecture. For any natural number n, we will denote the additive cyclic group of integers modulo n as $\mathbb{Z}_n$. For any set A of integers, let $\text{Cay}(\mathbb{Z}_n; A)$ be the *Cayley digraph* whose vertex set is labeled by elements of $\mathbb{Z}_n$ such that there is an arc from i to $i+a \pmod{n}$ for every $i \in \mathbb{Z}_n$ and every $a \in A$. A digraph isomorphic to $\text{Cay}(\mathbb{Z}_n; A)$ for some n and A is called a *circulant digraph*. For the following example we avoid digraphs with symmetric arcs. A digraph D is *Hamiltonian* if D contains a cycle containing every vertex of D. A cycle containing every vertex in the digraph is called a *Hamiltoninan cycle*. Jirásek proved the following theorem [Jir01].

Theorem 3.1. *For $t \geq 1$, the circulant digraph Cay($\mathbb{Z}_{8t+4}$; $2t+1, 2, 4t+4$) has no Hamiltonian cycle and the reversal of any of its arcs results in a Hamiltonian digraph.*

We will not prove this theorem, instead we will simply apply it and contruct a counterexample to Ádám's conjecture. We now build Cayley digraph as described in Theorem 3.1 for $t = 1$. Therefore, our Cayley digraph is $\text{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$. The twelve vertices are labeled: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Each arc will be from Vertex i to Vertex $i + a \pmod{n}$. Note, every vertex will be incident from exactly three arcs. It is easily seen that $4 + 3 \pmod{12}$ is 7, $4 + 2 \pmod{12}$ is 6, and $4 + 8 \pmod{12}$ is 0. Hence, $\langle 4, 7\rangle$, $\langle 4, 6\rangle$, and $\langle 4, 0\rangle$ are arcs in $\text{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$. The remaining arcs are found in a similar manner. $\text{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ is the digraph shown in Figure 3.1, with twelve vertices containing no Hamiltoninan cycles. However, the reversal of any arcs in $\text{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ will result in a Hamiltonian digraph.

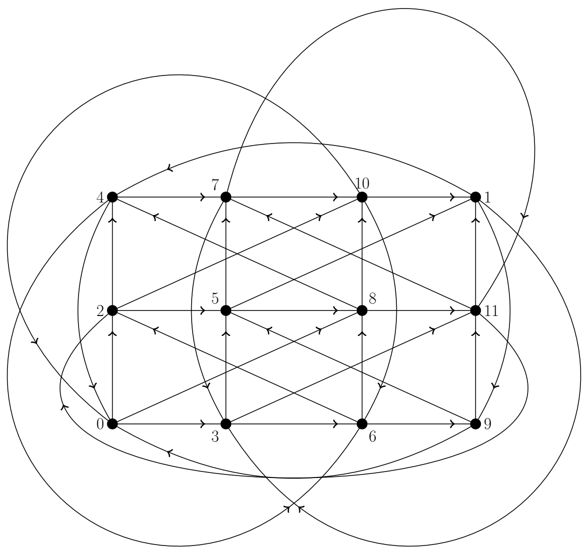

Figure 3.1: Cayley digraph $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$

We previously noted that there exists counterexamples to Ádám's conjecture of the form G^P, with twelve or more vertices [Jir01]. The following proposition formalizes this [Tho85].

Proposition 3.1. *Let G be a digraph such that reversal of any arc increases the length of a longest directed cycle. Then there exists a natural number t, such that for $p \geq t$, G^P is a counterexample to Ádám's conjecture.*

We now check if $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ satisfies Proposition 3.1. There exists a natural number t ($t = 1$ for our Cayley digraph) such that $p \geq t$. By Theorem 3.1, the reversal of any arc in $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ will result in a digraph containing a Hamiltonian cycle. Hence, $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)^P$ is a counterexample to Ádám's conjecture. In fact, $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)^4$ is the smallest known Cayley digraph counterexample to Ádám's conjecture, consisting of

12 vertices and 144 arcs [Jir01]. $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)^4$ is constructed by replacing every arc in $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ with 4 parallel arcs.

We now have a procedure for building counterexamples to Ádám's conjecture. So far, we know there exists counterexamples with twelve or more vertices. However the problem remains open for finding counterexamples with less than 12 vertices.

Chapter 4

Extending results

4.1 Reversal sets and arc-cycle transversals

We continue to investigate digraphs that satisfy Ádám's conjecture. We begin this chapter by characterizing digraphs that become acyclic after one arc reversal. We call such digraph, *quasi-acyclic*. Notice that any quasi-acyclic digraph satisfies Ádám's conjecture and guarantees no new cycles are created.

In an effort to give structure to digraphs that contain one arc whose reversal results in an acyclic digraph, the following procedure was used. Consider a non-acyclic digraph D. Assume that the reversal of an arc $\langle x, y\rangle$ does not create new cycles. First locate one cycle in D, say cycle C_1. Color every arc in C_1 with the same color. If this is the only cycle in D, then the reversal of any of the colored arcs will result in an acyclic digraph. If there are more cycles in D, check where those cycles intersect with the colored arcs. Any colored arc that did not have an intersection with the rest of the cycles in the digraph must be un-colored. After this process is completed for all cycles in D, any remaining colored arcs, are arcs that intersect all cycles in D. We will prove that the reversal of the remaining colored arcs will result in an acyclic digraph. On the other hand, if there exists a cycle that does not intersect any other cycle in D, then there exists a pair of cycles that have no arcs in common. A pair of *disjoint cycles* are two cycles that do not have any arcs in common. This procedure motivates the following lemma.

Lemma 4.1. *If a non-acyclic digraph D has at least one pair of disjoint cycles, then there does not exist a single arc whose reversal results in an acyclic digraph.*

Proof. Let D contain a pair of disjoint cycles C_1 and C_2. Reversal of any one of the arcs in C_1, say $\langle x, y\rangle$ for $x, y \in V(D)$, will eliminate cycle C_1 and any cycle containing $\langle x, y\rangle$. Since C_1 and C_2 are disjoint, reversal of any arc in C_1 will have no affect on C_2. Then C_2 will remain a cycle and the resulting digraph will not be acyclic. $\square$

We will show that digraphs in which an arc is contained in all of the cycles will become acyclic after the reversal of one arc. Furthermore, if the intersection of all the cycles in a digraph is empty, then there is no single arc whose reversal makes the digraph acyclic. This scenario motivates us to define an *arc-cycle transversal* $\mathcal{T}$ which is a minimal acyclic sub-digraph of a digraph D such that any cycle in D has an arc in common with $\mathcal{T}$. We will prove, in Theorem 4.11, that if D has $\mathcal{T}$ with cardinality one ($|\mathcal{T}| = 1$), then D will be acyclic after one arc reversal.

Another definition motivated by Lemma 4.1 is that of a reversal set. We call a subset X of $A(D)$, a *reversal set*, if the reversal of all arcs in X makes D acyclic and X is minimal with respect to this property. We will make use of reversal sets and arc-cycle transversals later in Section 4.2.

Lemma 4.2. *If there exist an arc in $\langle x, y\rangle$ whose reversal makes digraph D acyclic, then there does not exist an $x - y$ path other than the arc $\langle x, y\rangle$.*

Proof. Let $\langle x, y\rangle$ be contained of every cycle. Assume there exists an $x - y$ path, $P_y : x = v_0v_1v_2v_3...v_k = y$. Then there is a vertex v_j in P_y of maximal index such that $v_j \in V(C)$, for some cycle C. Thus $v_jP_yCv_j$ is a cycle in D avoiding $\langle x, y\rangle$. This contradicts $\langle x, y\rangle$ being contained in every cycle.

$\square$

Lemma 4.3. *Let D be a digraph containing at least one cycle. There exists an arc $\langle x, y\rangle$ whose reversal makes D acyclic if and only if $\langle x, y\rangle$ is contained in every cycle in D.*

Proof. Let D be a digraph with at least one cycle. Assume to the contrary: there exists an arc $\langle x, y\rangle$ whose reversal makes D acyclic, but $\langle x, y\rangle$ is not contained in every cycle. Then there exists some cycle C_1 that does not contain $\langle x, y\rangle$. Reversal of $\langle x, y\rangle$ will have no affect on C_1. Hence D will have a cycle after the reversal of $\langle x, y\rangle$, which is a contradiction.

Let $\langle x, y\rangle$ be contained in every cycle. Then every cycle in D will be eliminated after the reversal of $\langle x, y\rangle$. By Lemma 4.2, no new cycles will be created after the reversal of $\langle x, y\rangle$. Hence, the reversal of $\langle x, y\rangle$ will result in an acyclic digraph.

□

As defined earlier a *source* is a vertex s with $id(s) = 0$ and a *sink* t is a vertex with $od(t) = 0$. All acyclic digraphs must contain both a sink and a source.

Theorem 4.4. *If D is an acyclic digraph, then D has a source and a sink*

Proof. Without loss of generality, assume, an acyclic digraph D has no source. Then all vertices $v \in V(D)$ have in-degree at least one. Let path $P : v = v_0v_1v_2...v_k = u$ be of maximal length. Since all vertices have in-degree at least one, there exists and arc directed to v. Then by the maximality of P, there must be an arc from a vertex in P directed to v. This will produce a cycle in D, which is a contradiction. Therefore D must contain both a source and a sink. A symmetric argument can be applied if D has no sink

□

The converse of Theorem 4.4 is false. Refer to Figure 4.1 for a counterexample.

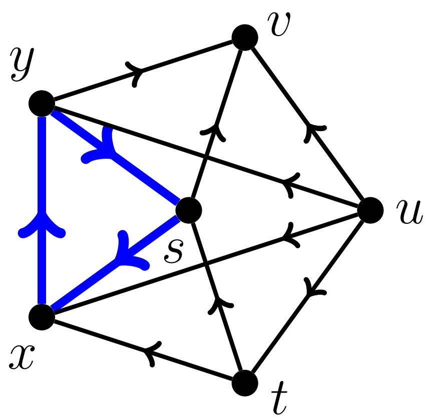

Figure 4.1: A digraph D with a sink v and source u that is not acyclic.

We now have a deeper understanding of quasi-acyclic digraphs. We have also found that any acyclic digraph must contain a source and a sink. We now investigate similar ideas in the context of tournaments.

4.2 Results in tournaments

We now take a look at non-acyclic tournaments. Since tournaments are digraphs, we will be able to apply results from Section 4.1 to tournaments.

Lemma 4.5. *If there exists an arc in tournament T whose reversal makes T acyclic, then there is a vertex v with $id(v) \leq 1$ and a vertex u with $od(u) \leq 1$.*

Proof. Without loss of generality, assume there is no vertex v, with $id(v) \leq 1$, in a non-acyclic tournament T. Then $id(v) \geq 2$ for all v $\in V(T)$. By Theorem 4.4, any acyclic digraph contains both a source and a sink. If all vertices in T have in-degree at least 2 then we require at least two arc reversals for the resulting digraph to contain a vertex with indegree zero. Hence it is impossible to obtain a source with a single arc-reversal. Therefore T, must contain a vertex v with $id(v) \leq 1$ to create a source with a single arc reversal. Similarly, it is impossible to produce a sink by reversing a single arc if every vertex $u \in V(T)$ has $od(u) \leq 2$.

□

Recall, an *arc-cut* (X, Y) of a tournament T is a subset of $A(T)$ such that $T - (X, Y)$ is a disconnected tournament. Every arc in (X, Y) will join one vertex in X and one vertex in Y. For any subset X of $V(T)$, X *induces* a subtournament H where H has vertex set X and arc set consisting of all arcs in T between vertices in X.

Theorem 4.6. *T is a non-acyclic tournament with the property that the reversal of some arc makes T acyclic if and only if there exists a cut (X, Y) such that X and Y induce acyclic sub-tournaments and there is exactly one arc in the cut directed from X to Y.*

Proof. Let T be a non-acyclic tournament such that the reversal of exactly one arc, $\langle x, y\rangle$, makes T acyclic. By Lemma 4.3, all cycles in T must contain $\langle x, y\rangle$. Then there exists a cut (X, Y) in T containing $\langle x, y\rangle$, such that $x \in V(X)$ and $y \in V(Y)$. Refer to Figure 4.2. Any $x - y$ path other than $\langle x, y\rangle$, requires an arc from X to Y. By Lemma 4.2, there does not exist an $x - y$ path other than the arc $\langle x, y\rangle$. Hence, all arcs in (X, Y) are directed from Y to X except $\langle x, y \rangle$. Since all cycles of T contain $\langle x, y \rangle$, $T - (X, Y)$ will result in two acyclic sub-tournaments.

If there exists a cut (X, Y) in tournament T such that X and Y induce acyclic sub-tournaments and exactly one arc in the cut is directed from X to Y, then all cycles in T must contain an arc from (X, Y). There exists only one arc directed from X to Y. Hence, all other arcs in (X, Y) are directed from Y to X. Then every cycle in T must contain $\langle x, y \rangle$. Therefore, by Lemma 4.3, reversal of $\langle x, y \rangle$ will eliminate all cycles in T.

$\square$

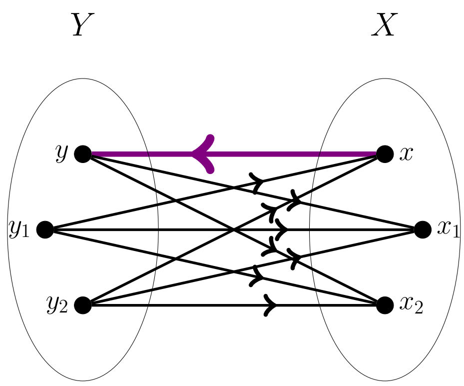

Figure 4.2: Example of Cut (X,Y) $\in T$

We now know that if there exists one arc whose reversal makes T acyclic, then there exists a cut in which one arc will be directed in the opposite direction of all the other arcs in the cut. Furthermore we can reformulate Theorem 4.3 for tournaments as follows.

Theorem 4.7. *There exists an arc $\langle x, y \rangle$ whose reversal makes T acyclic if and only if $\langle x, y \rangle$ is contained in every cycle in T.*

Corollary 4.8. *If the intersection of all the cycles in a tournament T is non empty, then the reversal of any arc in the intersection makes T acyclic.*

Non-acyclic tournaments in which one arc reversal results in an acyclic tournament, are simply tournaments in which all cycles have an arc in common. We will refer to such tournaments as $quasi - acyclic$. By Lemma 4.5, quasi-acyclic tournaments contain a vertex v with $id(v) \leq 1$ and a vertex u with $od(u) \leq 1$ such that u and v share an arc.

Ádám's conjecture so far has been described using tournaments and simple digraphs that become acyclic after one arc reversal. It is obvious that eliminating all cycles in a non-acyclic digraph will satisfy Ádám's conjecture. We now generalize to digraphs that become acyclic after more than one arc reversal. In fact, by Theorem 2.8, any digraph that results in an acyclic digraph after at most three arc reversals, satisfies Ádám's conjeture. By Lemma 4.1, we can say that any tournament with three or fewer mutually disjoint cycles will satisfy Ádám's conjecture. Therefore the cardinality of the arc-cycle transversal $\mathcal{T}$ must be three or less. This motivates the following.

Lemma 4.9. *If a non-acyclic tournament T has an arc-cycle transversal, such that $|\mathcal{T}| \leq 3$, then reversal of $\mathcal{T}$ results in an acyclic tournament T'.*

Lemma 4.9 will help identify the tournaments and simple digraphs for which Ádám's conjecture holds. We must consider the reversal set of a digraph. We now aim to prove Lemma 4.9 is true by showing the cardinality of the reversal set is equal to the cardinality of the arc-cycle transversal.

Lemma 4.10. *Let D be a digraph such that D' is the resulting digraph after reversal of the arc-cycle transversal $\mathcal{T}$. If C is a cycle in D' but not in D and $\mathcal{T} \cap C \neq \emptyset$, then there exists a cycle in D that avoids $\mathcal{T}$.*

Proof. Let $e = \langle x, y\rangle \in \mathcal{T} \cap C$ where $\langle x, y\rangle \in A(D)$ and C is not a cycle in D. Since C is not a cycle in D, C can only share an arc $\langle x, y\rangle$ with $\mathcal{T}$ if $\langle x, y\rangle$ is contained in a cycle in D. Let C_1 be a cycle in D containing $\langle x, y\rangle$. Since C_1 is a cycle in D, then there exist a $y - x$ path, P_y. Similarly, there exists an $x - y$ path P_x, since C is a cycle in D'. Hence, there exists a cycle $C_e = xP_xyP_yx$ in D such that $C_e \cap \mathcal{T} = \emptyset$. □

We will make use of Lemma 4.10 to demonstrate that every cycle must share an arc with the arc-cycle transversal. The final result of this section will be generalized for non-acyclic digraphs and the same proof can be applied to non-acyclic tournaments.

Theorem 4.11. *$\mathcal{T}$ is an arc-cycle transversal in a digraph D if and only if $\mathcal{T}$ is a reversal set in D.*

Proof. We must first show the reversal of the arcs in $\mathcal{T}$ makes D acyclic. Assume, to the contrary, that the reversal of the arcs in $\mathcal{T}$ does not make D acyclic. Let D' be the digraph obtained by reversing the arcs in $\mathcal{T}$ and let C be a cycle in D' whose intersection with $\mathcal{T}$ has minimum cardinality. We first consider the cases when either $\mathcal{T} \cap C = \emptyset$ or $C \subseteq \mathcal{T}$. It is impossible for $\mathcal{T} \cap C = \emptyset$ since $\mathcal{T}$ is an arc-cycle transversal. If $C \subseteq \mathcal{T}$ we contradict the definition of $\mathcal{T}$ being acyclic. Then C is not a cycle in D. Therefore $C \cap \mathcal{T} \neq \emptyset$ and $C \nsubseteq \mathcal{T}$. Then by Lemma 4.10, there exist a cycle C_e that avoids $\mathcal{T}$ by avoiding some arc e. Let $e \in C \cap \mathcal{T}$ where $e = \langle u, v \rangle$. But $(C_e \cup C) - e$ meets $\mathcal{T}$ in fewer elements. This contradicts C having minimal intersection with $\mathcal{T}$.

We now show there is no smaller set of arcs than the set $\mathcal{T} \in D$ whose reversal makes D acyclic. Assume to the contrary, that there exists a set of arcs $\mathcal{T}^\circ$, such that $|\mathcal{T}^\circ| < |\mathcal{T}|$ and the reversal of $\mathcal{T}^\circ$ makes D acyclic. Then there exists an arc $e = \langle u, v \rangle \notin \mathcal{T}^\circ$ and $e \in \mathcal{T}$ where $u, v \in V(D)$. Since $\mathcal{T}$ is an arc-cycle transversal, there exists some cycle C_1 whose only intersection with $\mathcal{T}$ is e and $e \notin C_1 \cap \mathcal{T}^\circ$. This contradicts the reversal of $\mathcal{T}^\circ$ making D acyclic, since reversal of $\mathcal{T}^\circ$ will not reverse any arc in cycle C_1. Cycle C_1 will remain a cycle in D° where D° is the resulting digraph after the reversal of $\mathcal{T}^\circ$. Hence $|\mathcal{T}^\circ| < |\mathcal{T}|$ is false, which results in $|\mathcal{T}^\circ| \geq |\mathcal{T}|$. Thus $\mathcal{T}$ is the smallest set of arcs whose reversal makes D acyclic.

In the reverse direction of the proof, we first prove the following. If X is a reversal set, then X meets every cycle in D. Suppose X does not meet every cycle in D. Then there exists a cycle C such that $C \cap X = \emptyset$. After reversing all arcs in X, all the arcs in C will remain unaffected. Therefore C is still a cycle in D° which contradicts X being a reversal set.

Next we show that X is acyclic. Suppose X is not acyclic. Then there exist some arcs in X that form a cycle C^*. After reversal of all arcs in X, D will become an acyclic digraph D°. All arcs in C^* will have opposite orientation. Thus, after reversal of X, C^* will continue to be a cycle, but oriented in the opposite direction, which contradicts X being a reversal set.

Finally, we prove that there is no reversal set of smaller cardinality. We know the reversal set X meets every cycle in digraph D. Now suppose there exists a reversal set X^* such that $|X^*| < |X|$. Then there exist at least one arc $e \in X$ such that $e \notin X^*$. Since X is a, minimal, reversal set, reversal of arc e is required to eliminate some cycle C such that $C \cap X = e$. Note, cycle C has no other arc in common with X. Reversal of X^* will make D acyclic. Recall, $e \notin X^*$, which implies $C \cap X^* = \emptyset$. Therefore cycle C

is unaffected by the reversal of X^*. Thus cycle C remains a cycle, which contradicts X^* being a reversal set. Then X is the smallest set of arcs whose reversal makes D acyclic. Since reversal of $\mathcal{T}$ results in an acyclic digraph and $\mathcal{T}$ is minimal, $|\mathcal{T}| = |X|$.

□

Theorem 4.11 allows us to use arc-cycle transversals and their cardinality to determine how many arc reversals are required for any simple non-acyclic digraph or tournament to become acyclic. Every non-acyclic digraph will require $|\mathcal{T}| = |X|$ arc reversals to result in an acyclic digraph.

4.3 Applications. Digraphs in which the reversal of at most three arcs makes the digraph acyclic

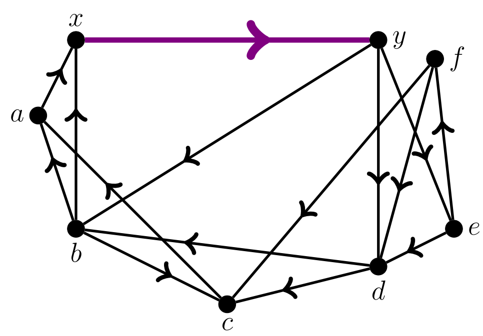

Figure 4.3: Digraph D with $|\mathcal{T}| = |X| = 1$

We now analyze several digraphs and apply results from the previous sections in this thesis. Consider the simple digraph D from Figure 4.3. We can structurize this digraph and organize some of the chaos. We first notice that D is strongly connected. For any vertices $x_i, y_i \in V(D)$ there exist an $x_i - y_i$ path and a $y_i - x_i$ path. Due to the modest size of D, we can count and describe all of the cycles in D. Consider the following cycles in D.

- $C_1 = x, y, b, x$
- $C_2 = x, y, b, a, x$
- $C_3 = x, y, d, c, a, x$
- $C_4 = x, y, d, b, x$
- $C_5 = x, y, d, b, a, x$
- $C_3 = x, y, e, d, c, a, x$
- $C_4 = x, y, e, d, b, x$
- $C_5 = x, y, e, d, b, a, x$
- $C_6 = x, y, e, f, c, a, x$
- $C_7 = x, y, e, f, d, b, a, x$
- $C_8 = x, y, e, f, d, b, x$

Notice, all cycles contain the arc $\langle x, y \rangle$. In fact, $|\mathcal{T}| = 1$, since $\langle x, y \rangle$ is the only arc in the intersection of all the cycles in D. By Lemma 4.3, reversal of $\langle x, y \rangle$ results in an acyclic digraph D'. This indicates, the reversal set X of D only contains $\langle x, y \rangle$. Hence, by Theorem 4.11, $|X| = |\mathcal{T}| = 1$. We can confirm visually that there are no $x - y$ paths other than the arc $\langle x, y \rangle$. Since the only $x - y$ path in D is $\langle x, y \rangle$, by Corollary 4.2, no new cycles are being created with the reversal of $\langle x, y \rangle$. With certainty, we can say that the reversal of $\langle x, y \rangle$ will result in an acyclic digraph D'. Hence, by Theorem 4.4, D' contains a sink and a source. Let $x, y \in V(D)$, such that x is a source and y a sink of D'. Hence, $od(x) = 0$ and $id(y) = 0$.

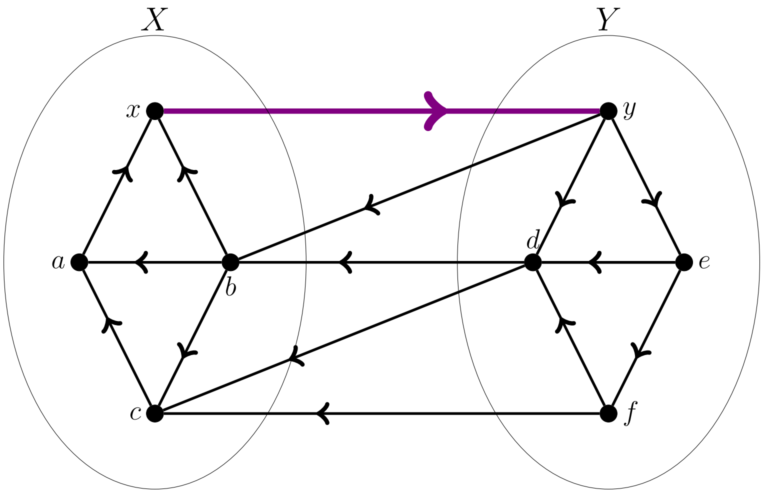

Figure 4.4: Cut (X, Y) in D

We now turn our attention to Figure 4.4, which shows the same digraph D from Figure 4.3. We have restructured Figure 4.3 to show the cut (X, Y) of D where $V(D)$ has been partitioned into two sets, X and Y. The following arcs are in the cut (X, Y).

- $\langle x, y \rangle$
- $\langle y, b \rangle$
- $\langle d, b \rangle$
- $\langle d, c \rangle$
- $\langle f, c \rangle$

All of the arcs in (X, Y) are directed to X except $\langle x, y \rangle$. By Proposition 2.2, reversal of $\langle x, y \rangle$ decreases the total number of cycles in D. We conclude that D, from Figure 4.3, satisfies Ádám's conjecture. Figure 4.5 displays the resulting digraph D' after reversal of $\langle x, y \rangle \in D$.

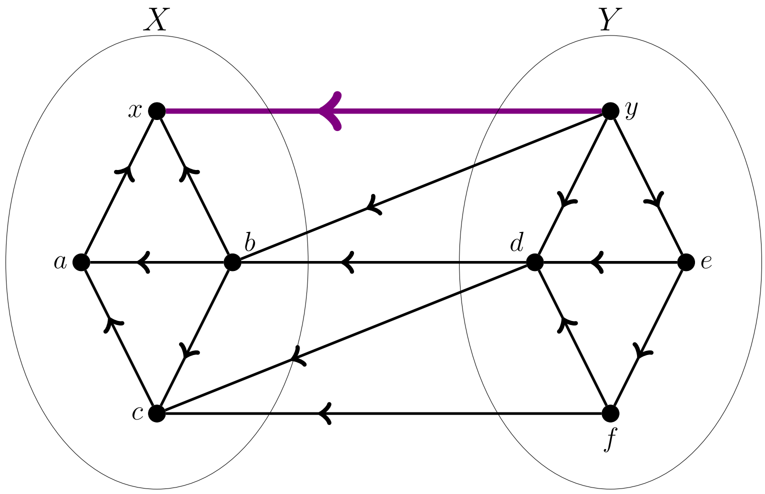

Figure 4.5: Acyclic D' after reversal of $\langle x, y \rangle$

Theorem 2.8 allows us to verify Ádám's conjecture on digraph that are not quasi-acyclic. Due to Theorem 2.8, we can say that any non-acyclic digraph that becomes acyclic after at most three arcs, will satisfy Ádám's conjecture. Digraph D, shown in Figure 4.6, is a non-acyclic digraph. To verify Ádám's conjecture, we must confirm D becomes acyclic after no more than three arc reversals. Digraph D is strongly connected since for all $u, v \in D$, there exists a $u - v$ path and a $v - u$ path. We take a slightly different approach to verify Ádám's conjecture. We will not describe every single cycle in D, as in the previous example. Often, digraphs may be too large, and counting all the cycles can be a long cumbersome task. Instead, we search for disjoint cycles. For example, cycles $C_1 = x, y, c, x$ and $C_2 = x_1, y_1, d, e, b, x_1$ have no arcs in common. We seek the smallest number of disjoint cycles possible. If all other cycles in D have an arc in common with either C_1 or C_2, then we could identify our arc-cycle transversal. Let us claim, the arc-cycle transversal of D has cardinality two. To verify $|\mathcal{T}| = 2$, we must be certain that exactly two arcs, are in intersection of all the cycles. Figure 4.7 shows a cut (X, Y) of D. The cut (X, Y) partitions $V(D)$ into X and Y. All arcs in (X, Y) are directed to X except $\langle x, y \rangle$ and $\langle x_1, y_1 \rangle$. We use arguments of Theorem 2.10. The subdigraph on X does not contain any cycles and neither does the subdigraph on Y. Therefore all cycles in D contain arcs from (X, Y). Note there are three arcs directed to X and only two arcs directed to Y. Hence, all cycles contain either $\langle x, y \rangle$ or $\langle x_1, y_1 \rangle$. At the very least, C_1 and C_2 contain $\langle x, y\rangle$ and $\langle x_1, y_1\rangle$, respectively. Then $|\mathcal{T}| = 2$. Furthermore, reversing $\langle x, y\rangle$ and $\langle x_1, y_1\rangle$ will result in an acyclic digraph D'. The resulting acyclic digraph D' will have a source y and a sink x. Since the reversal of two arcs in D resulted in an acyclic digraph D', the total number of cycles in D is decreased. Hence, D satisfies Ádám's conjecture.

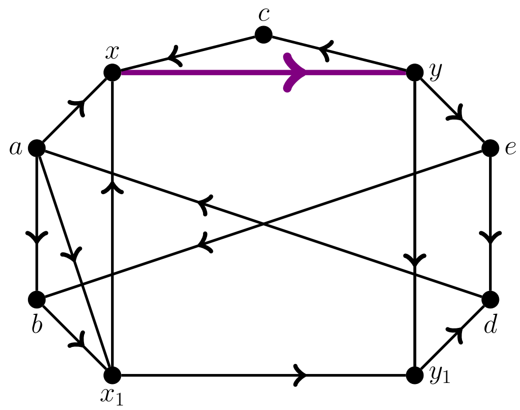

Figure 4.6: Digraph D with $|\mathcal{T}| = |X| = 2$

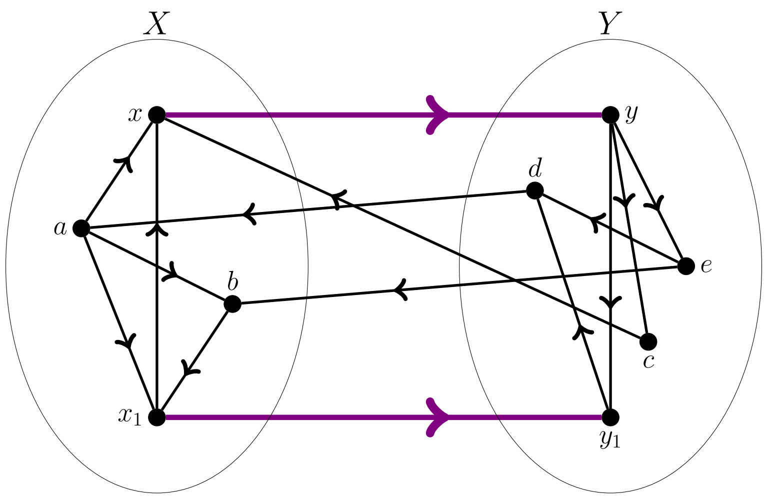

Figure 4.7: Cut (X,Y) of D

We are now able to verify Ádám's conjecture for a class of digraphs and tournaments using several of the results in this thesis along with Jirasék's and Reid's results. We can also confirm that Cayley multidigraphs are counterexamples to Ádám's conjecture. We have arrived at our final result, which extends Reid's main result for tournaments.

4.4 3-arc connected but not 4-arc connected

The final result concerns tournaments. We can reformulate Theorem 2.10 [Rei84] as: Ádám's conjecture is true for strongly connected tournaments that are 2-arc-connected but not 3-arc-connected. The technique Reid used in this proof of Theorem 2.10 is clever and innovative. The goal is to establish a function from T', where T' is the resulting tournament after an arc reversal, to a non-acyclic tournament T. It is then shown that the function, mapping of cycles in T' to cycles in T, is one-to-one. However, if the mapping is not onto, then there exists more cycle in T than in T'. Reid conjectured that perhaps this method of proof can be applied to verify Ádám's conjecture for a k-arc-connected tournaments that are not $(k+1)$-arc-connected. One can interpret the following result as a case for a tournament that is 3-arc connected but not 4-arc connected.

Theorem 4.12. *Let T be a strongly connected tournament containing a vertex x with $od(x) = 3$. If $\langle x, z\rangle$, $\langle x, w\rangle$, and $\langle x, v\rangle$, are arcs directed from x and if $\{z, w, v\}$ induces an acyclic subtournament and the reversal of these three arcs results in a tournament that is not strongly connncted, then T contains an arc whose reversal decreases the number of cycles in T.*

Proof. Let S be a set of vertices in T that contains x and is maximal with respect to the property that $z, w, v \in V(T) - S$ and for all other vertices $y \in V(T) - S$, there is no arc $\langle u, y\rangle$ for any $u \in S$. Without loss of generality, assume w dominates z and v. Also, z dominates v. Let T' be the resulting tournament after the reversal of $\langle x, w\rangle$. If $V(T) - S$ only contains z, w, and v, then $S \cup \{w\}$ contradicts the maximality of S. Thus, $V(T) - S$ contains more than three vertices. There exists $w_1 \in V(T) - S$ such that w_1 dominates w. Notice, any cycle in T' that uses arc $\langle w, x\rangle$ must also use arc $\langle w_1, w\rangle$. We define a function from T' to T, $f(C) : C(T') \to C(T)$ as follows: for each cycle in C in T',

$$f(C) = \begin{cases} C, & \text{if } \langle w, x\rangle \text{ is not an arc of C.} \\ x, w, z, C[z, w_1], w_1, x & \text{if C is given by w,x,z,...,}w_1\text{,w.} \\ x, w, v, C[v, w_1], w_1, x & \text{if C is given by w,x,v,...,}w_1\text{,w.} \\ x, w, z, v, C[v, w_1], w_1, x & \text{if C is given by w,x,z,v,...,}w_1\text{,w.} \end{cases} \tag{4.1}$$

Since T and T' are tournaments, there exists an arc joining w_1 and x. Since there are only three arcs from S to $V(T)-S$, w_1 dominates x. Then every cycle in T' that not does contain $\langle w, x\rangle$ will map to itself in T. Also every cycle in T' containing $\langle w, x\rangle$, z (respectively v), and w_1, will map to cycles in T containing $\langle x, w\rangle$, z (respectively v), and w_1. Hence, every cycle in T' is mapped to a cycle in T as defined by $f(C)$. Therefore, $f(C)$ is one-to-one. Now we show $f(C)$ is not onto. If w only dominates v and z, then $S \cup \{w, v\}$ contradicts the maximality of S. Then there exists some vertex p such that w dominates p and $p \neq z, v$. Since T' and T are tournaments and there are only three arcs from S to $V(T)-S$, there exists an arc $\langle p, x\rangle$. The cycle $C_1 = xwpx$ does not have a pre-image in T', because any cycle in T' that does not use z or v, must avoid $\langle w, x\rangle$. Hence, any cycle in T' that does not contain z, or v, has all vertices in S or all vertices in $V(T)-S$. Since $f(C)$ is not onto, there exist more cycles in T than in T'. Hence Ádám's conjecture holds for such tournaments.

$\square$

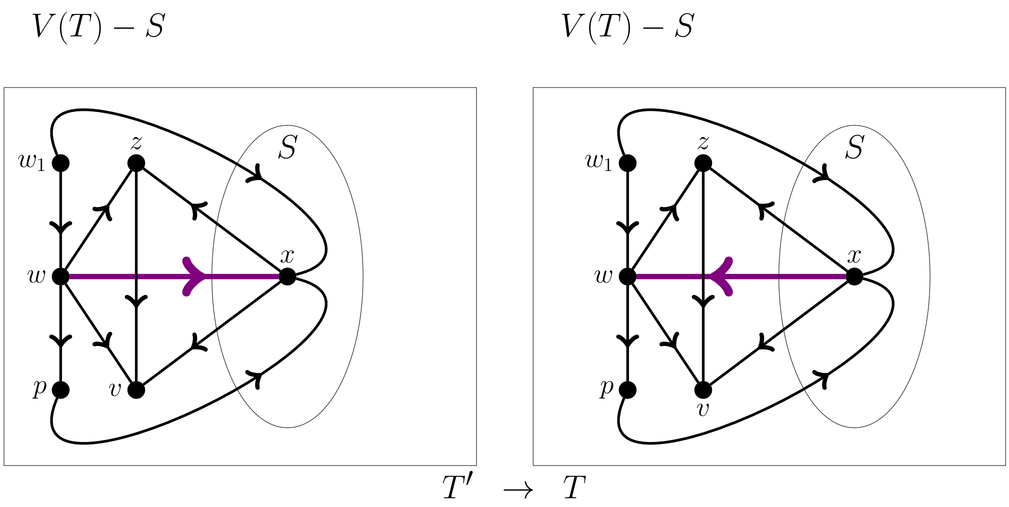

Figure 4.8: Case where $z, w, v \in V(T)$ do not form a cycle

Chapter 5

Conclusion

In this thesis we have investigated structural properties of digraphs that satisfy Ádám's conjecture. Jirásek proved that Ádám's conjecture holds for every digraph containing a pair of symmetric arcs [Jir92]. We noted that all non-acyclic digraphs in which one arc reversal results in an acyclic digraph satisfy Ádám's conjecture. It is obvious that if reversing one arc eliminates all the cycles in the digraph, then the number of cycles is decreased. The results from Section 4.1 yielded several characterizations for quasi-acyclic digraphs. From Lemma 4.1 we are able to determine when a digraph is quasi-acyclic. A pair of disjoint cycles would require two or more arc reversals for a digraph to become acyclic. The notion of disjoint cycles motivated the definitions of an arc-cycle transversal and a reversal set. These definitions were required for a major result in Section 4.2. As a result of Lemma 4.3, we are able to show that if a single arc is in the intersection of all of the cycles of a digraph D, then D is quasi-acyclic. Once a digraph is identified as quasi-acyclic, the digraph satisfies Ádám's conjecture. Lemma 4.2 proved that quasi-acyclic digraphs have $(x, y) = 1$ where the only $x - y$ path is the arc $\langle x, y \rangle$.

Throughout this thesis, we proved and mentioned results concerning acyclic digraphs. Acyclic digraphs can be thought of as digraphs in which the flow of every path is interrupted and a cycle is never acquired. For such interruptions to occur, there are two major requirements. By Theorem 4.4, a sink and a source must be present in an acyclic digraph. However, the converse of Theorem 4.4 is false. In essence, when identifying quasi-acyclic digraphs, we search for an arc reversal that results in a sink and a source. This idea is also applied to non-acyclic tournaments.

While investigating digraphs that sarisfy Ádám's conjecture, we must also understand counterexamples to Ádám's conjecture. The first counterexamples were described by Thomassen [Tho85] and Grinsberg [Gri88] and both classes of counterxamples make use of multidigraphs. In fact, the class of multidigraphs described can be constructed by taking a simple digraph G and replacing each of its arcs by p parallel arcs. These multidigraphs are denoted G^p, and some are counterexamples to Ádám's conjecture, as explained in Chapter 3. In Chapter 3, we also constructed a Cayley digraph counterexample to Ádám's conjecture. We know that there exists Cayley digrpahs with twelve or more vertices that are counterexamples to Ádám's conjecture. There are still no known Cayley digraph counterexamples for Ádám's conjecture with less than twelve vertices.

Ádám's conjecture remains open for simple digraphs. The results of Reid [Rei84], Jirásek [Jir92], Thomassen [Tho85], and other graph theorists, have given a deeper understanding of the classes of digraphs satistying Ádám's conjecture. In this thesis we proved quasi-acyclic digraphs satisfy Ádám's conjecture. Reid's method of proof gave a new approach in the investigation. Reid's results showed that Ádám's conjecture is true for strongly connected tournaments that are 2-arc-connected but not 3-arc-connected. The proof of Theorem 4.12 can be interpreted as case for an extension of Reid's result to tournaments that are 3-arc-connected but not 4-arc-connected. Although we made the transition from digraphs to tournaments, the results for digraphs still hold for tournaments. The added structure of tournaments helped prove certain results that are not obvious and cannot be generalized to all digraphs yet. Perhaps further research will yield a more unified result for all digraphs satisfying Ádám's conjecture.

Theorem 4.12 extends Reid's result. However, in the process of investigating tournaments that are 3-arc connected but not 4-arc connected that satisfy Ádám's conjecture, many cases and subcases were revealed. In continuing the investigation for such tournaments, one must begin from the result of Theorem 4.12. Will Reid's technique yield results for the case where vertices $z, w, v \in V(T) - S$ create a cycle? The question also remains open for tournaments that are 3-arc connected but not 4-arc connected that satisfy Ádám's conjecture. Then perhaps Reid's proof method can be generalized to verify Ádám's conjecture for tournaments that are k-arc connected but not $(k+1)$-arc connected.

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