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THINKING POKER THROUGH GAME THEORY

Damian Palafox

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THINKING POKER THROUGH GAME THEORY

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Damian Palafox

June 2016

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ABSTRACT

Poker is a complex game to analyze. In this project we will use the mathematics of game theory to solve some simplified variations of the game. Probability is the building block behind game theory. We must understand a few concepts from probability such as distributions, expected value, variance, and enumeration methods to aid us in studying game theory. We will solve and analyze games through game theory by using different decision methods, decision trees, and the process of domination and simplification. Poker models, with and without cards, will be provided to illustrate optimal strategies. Extensions to those models will be presented, and we will show that optimal strategies still exist. Finally, we will close this paper with an original work to an extension that can be used as a medium to creating more extensions and, or, different games to explore.

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Chapter 1

Introduction

There are many categories of poker games, and within each game category, there are many variations, and even regional variants of the same game. Casinos could have their own set of rules, while a home game could run the same game completely differently. In general, poker games share some characteristics, there will be betting involved, and for the most part a deck of 52 cards is used in play. In this sense, we can examine the game, which may seem that involves randomness and luck, through a mathematical eye. This will demonstrate that indeed, the game is solvable if we analyze it using game theory. What are the tools needed for us to tackle this project?

In this expository paper we will present known solutions to simplified poker models. Additionally, we will present original work extending some of these models. Before we can understand how the poker models work, some background work in probability is required. Poker caters to some specific probability problems, those that fall into the combinatorial spectrum. Our work in probability will lead us to understand one of the most powerful concepts available to statisticians, Bayes' Theorem. We will also need, as the title of this project suggest, a grasp on game theory. Game theory has many applications in a wide array of fields such as economics, politics, psychology, science, computer science, and biology to name a few. Although, we could spend an entire project studying game theory alone, we will concentrate on decision making and solving 2-person games. We can then think of Chapters 2 through 4 as an introductory stepping stone into the world of poker. We will revise some old concepts while learning new ideas in the process.

The second half of this project will deal exclusively with poker problems. Fer-

guson & Ferguson, father and son, will provide an extensive look into the work of game theorists Émile Borel and John von Neumann, about solving poker models. The poker models provided will be reduced instances of an actual game with a well defined set of rules. These poker models will also serve as a reminder that if a game is played optimally, then a game can be profitable.

In the final chapter, we will attempt to apply game theory to a poker game. Chen and Ankenman provide exhaustive work in their book, about how poker can be solved step by step via mathematics. From their work we will take the game theoretical approach to solving poker. We will start by extending the poker models of Chapter 5 to an actual game of poker. Then, we will demonstrate how to reduce the whole game of poker to a simple 3-card game that we can solve, and thus, we can play optimally. The work done in Chapter 6 can then be applied to any other game, or specific circumstance that we may encounter in real life.

In short, this project will not teach anyone how to play poker, but instead, we will attempt to teach everyone how to think about poker from a mathematical viewpoint.

Chapter 2

Probability

2.1 Probability

Game theory is the mathematical study of games. Game theory can be divided into many branches, but it is usually defined as combinatorial or classical. The combinatorial game theory is the branch that studies enumerations, combinations, and permutations of sets of elements, and the mathematical relations that characterize their properties. Understanding combinatorics is key in solving probability problems as they arise in a game of poker. **Probability** is then defined as a real-valued set function P that assigns, to each event A in the sample space S , a number $P(A)$, called the probability of the event A , such that the following properties are satisfied:

(a) $P(A) \geq 0$.

(b) $P(S) = 1$.

(c) If A_1, A_2, A_3, \dots are events and $A_i \cap A_j = \emptyset, i \neq j$, then $P(A_1 \cup A_2 \cup \dots \cup A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$ for each positive integer k , and $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$ for an infinite, but countable, number of events [HT10].

Theorem 2.1. For each event A , $P(A) = 1 - P(A')$.

Proof. We have

$$S = A \cup A' \quad \text{and} \quad A \cap A' = \emptyset.$$

Thus, from properties (b) and (c), it follows that

$$1 = P(A) + P(A').$$

Hence,

$$P(A) = 1 - P(A').$$

□

[HT10].

We may also define probability as a number. If some number of n trials of an experiment produces n_0 successes, how many occurrences of an event x will we have? As n becomes larger and grows without bound, this number will converge to a specific ratio, thus the probability p of x occurring $p(x)$ times is the ratio

$$p(x) = \lim_{n \rightarrow \infty} \frac{n_0}{n}.$$

For example, what is the probability that we can draw a specific card from a standard deck of cards? We first define a standard deck of cards as follows:

1. There are 52 cards.
2. There are 4 suits, ♠ spades, ♣ clubs, ♥ hearts, and ♦ diamonds.
3. There are 13 ranks in each suit, A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K (The A, “Ace” is both a high and low card in poker games).
4. Each card is unique, ie. there is only one 6 of clubs (6♣), and only one 8 of diamonds (8♦), etc.

It should be evident that the probability of drawing a specific card is then $1/52 \approx 1.92\%$ since each card is unique. If we wanted to draw any one card of any suit, say any club, we know that there are 13 clubs out of 52 possible cards, so $13/52 = 1/4 = 25\%$. It should be clear that since the deck is divided into 4 equal suits, the probability of drawing one card, of any one particular suit, is 25%. Another question we could ask is, what is the probability of drawing any ace or any heart? We know that there are 4 aces in the deck, and that there are 13 hearts in the deck, however, we must be careful and not double count the $A♥$, this is the only card that can both be an ace and a heart at the same time. We can add the probabilities of a card being an ace, or a card being a heart, but we must also subtract the probability that the card is specifically the $A♥$. Then,

there are 13 hearts and 4 aces and only one ace of hearts, so the probability of drawing any ace or any heart is $(\frac{13}{52}) + (\frac{4}{52}) - (\frac{1}{52}) = \frac{16}{52} \approx 30.77\%$.

As demonstrated above, there could be many factors affecting probability and its calculations. For some events, the occurrence of one event may not change the probability of the occurrence of the other. Events of this type are said to be *independent* events. The probability that both A and B occur is called the *joint probability* of A and B . However, when one event has a clear impact on the other, that is, if event A happening affects the probability of event B occurring, then it is said that the events are *dependent*. We may also need to consider the *conditional probability* of A given B . This is the probability that if event B happens, event A will also occur. In summary, we have the following:

- Events A and B are *independent* if and only if $P(A \cap B) = P(A)P(B)$. Otherwise, A and B are called *dependent* events.
- The *conditional probability* of an event A , given that event B has occurred is defined by $P(A|B) = P(A \cap B)/P(B)$ provided that $P(B) > 0$.

Throughout, we will use the following notation:

$P(A \cup B)$ = Probability of A or B occurring.

$P(A \cap B)$ = Probability of A and B occurring.

$P(A|B)$ = Conditional probability of A occurring given B has already occurred.

For mutually exclusive events we have $P(A \cup B) = P(A) + P(B)$.

For all events $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

For independent events we have $P(A \cap B) = P(A)P(B)$.

And for all events $P(A \cap B) = P(A)P(B|A)$.

Example 2.1.1 What is the probability that in rolling two fair dice, we can obtain a sum of 12?

We should note that we have two independent events, as the roll of one die will not have any influence on the other. There is also only one way of obtaining a sum of 12 from two dice, when both dice land on 6, which carries a $1/6$ probability on each die. Let $P(A)$ be the probability that the first die is a 6, and let $P(B)$ be the probability that the second die is a 6. Then

$$P(A \cap B) = P(A)P(B) = (\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}.$$

Example 2.1.2 Three football players will attempt a field goal. Let A_i denote the event that the field goal is made by player i , where, $i = 1, 2, 3$ and let $P(A_1) = 0.5$, $P(A_2) = 0.7$, and $P(A_3) = 0.6$. What is the probability that exactly one player is successful? [HT10].

In this case we have three independent events, since one player, succeeding has no affect on the others' success or failure rate. We want to compute the probability of one success and two failures, but we do not know which player will score and which two will not. Therefore we must calculate the probability that if player 1 scores, both player 2 and player 3 will miss. Also, since any player can score, we will do the same for the other two cases, when player 2 scores the other two miss, and when player 3 scores, both player 1 and player 2 miss. Now, we have

$$P(A_1) + P(A'_1) \rightarrow 0.5 + 0.5 = 1,$$

$$P(A_2) + P(A'_2) \rightarrow 0.7 + 0.3 = 1,$$

$$P(A_3) + P(A'_3) \rightarrow 0.6 + 0.4 = 1.$$

We proceed to calculate,

$$\begin{aligned} &P(A_1 \cap A'_2 \cap A'_3) + P(A'_1 \cap A_2 \cap A'_3) + P(A'_1 \cap A'_2 \cap A_3) \\ &= P(A_1)P(A'_2)P(A'_3) + P(A'_1)P(A_2)P(A'_3) + P(A'_1)P(A'_2)P(A_3) \\ &= (0.5)(0.3)(0.4) + (0.5)(0.7)(0.4) + (0.5)(0.3)(0.6) \\ &= 0.06 + 0.14 + 0.09 \\ &= 0.29. \end{aligned}$$

Example 2.1.3 Given a standard deck of cards, draw two cards without replacement, what is the probability that both cards will be aces?

We now have a pair of events that are dependent, since removing a card from the deck will affect the probability of drawing the second card. Denote event A as the first card is an ace, and event B as the second card is also an ace. We know that the probability of drawing one ace is $4/52$ because there are 4 aces in a 52 card deck. However, the probability of drawing a second ace is now $3/51$, we have removed one card from the deck, namely, an ace. Thus, for these dependent events

$$P(A \cap B) = P(A)P(B|A) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}.$$

We will draw a pair of aces $1/221 \approx 0.45\%$ of the time. Now drawing two aces, or any other pair carries the same percentage. If this is the case, then what is the probability that we can draw any pair? Since there are 13 ranks, we can have 13 types of pairs. Then

$$13\left(\frac{1}{221}\right) = \frac{1}{17} \approx 5.88\%.$$

With these tools at our disposal, we can calculate any combination of cards imaginable.

Example 2.1.4 Given a standard deck of cards, draw three cards without replacement, what is the probability that all cards will be of the same rank, that is three of a kind?

As in the previous example, we have a set of dependent events. Let A denote the event that the first card drawn is any card of any rank. Now B is the event where second card is of the same rank as A , and C the event where the third card is of the same rank as the previous two cards from A and B . The probability of the events are $52/52$, $3/51$, and $2/50$ respectively. Then, for dependent events

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B|A)P(C|B \cap A) \\ &= \left(\frac{52}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) \\ &= \frac{1}{425} \approx 0.2\%. \end{aligned}$$

2.2 Distributions and Expected Value

More often than not, it is necessary to take into account many different probabilities simultaneously. Possible outcomes and their respective probabilities can be characterized by *probability distributions*. Probability distributions are created by taking each outcome and pairing it with its probability.

- The probability distribution D of event A with probability $P(A)$, and event B with probability $P(B)$, ... , and event Z with probability $P(Z)$ is

$$D = \{(A, P(A)), (B, P(B)), \dots, (Z, P(Z))\},$$

where $P(A) + P(B) + \dots + P(Z) = 1$, and $A \cup B \cup \dots \cup Z$ is the sample space of all possible outcomes.

For example, rolling a standard die yields the outcomes 1, 2, 3, 4, 5, or 6. Each result happens $1/6$ of the time. Thus the probability distribution of the roll is $D = \{(1, \frac{1}{6}), (2, \frac{1}{6}), (3, \frac{1}{6}), (4, \frac{1}{6}), (5, \frac{1}{6}), (6, \frac{1}{6})\}$. When a probability distribution is represented by numerical values per every single outcome, we can find the *expected value*, $\langle EV \rangle$, of that distribution. EV is defined as the value of each outcome multiplied by its probability, all summed together. In order to do well at any game, we must play in a way that maximizes expected value.

Example 2.2.1 Roll a fair 6-sided die. We place a bet of one-unit on any particular number. A winning bet pays five-units when the die lands on our chosen number, otherwise, it is considered a losing bet, and we lose our bet of one-unit. What is the expected value of this game? Is this a profitable game that we should play? It does not matter which number we pick. We could pick always the number “2” or we could change our number after every roll. Every single number has the same probability of appearing. It is erroneous to think that a “4” is coming just because it has not been observed in many rolls, or that “1” is the lucky number because that number seems to repeat often. Therefore, let’s assume that we pick any number x that has a $1/6$ probability of showing. When we roll an x we will be paid 5 units, however, we will lose 1 unit when any other of the five numbers show, which will happen $5/6$ of the time. The probability distribution is $D = \{(5, \frac{1}{6}), (-1, \frac{5}{6})\}$. The expected value of the game is $\langle EV \rangle = (\frac{1}{6})(5) + (\frac{5}{6})(-1) = 0$.

Since the expected value of the game is 0, this is not a profitable game to play. If we played this game long enough, we will never show a profit. Suppose we start by losing some games, but winning just once will get us back to where we started. Or perhaps, we could start by winning a few games, but eventually we will lose all of our profits and will be back at zero. Note that if the payouts were changed to six-units per win, we will show a profit of exactly $1/6$ units or ≈ 0.16 units per unit played. If that were the case, we should be happy to play this game.

Example 2.2.2 Given a standard deck of cards, randomly draw a card. We place a bet of one-unit on any **face card** (any K, Q, J). If a face card is drawn we will win three-units, otherwise, we lose our initial bet. What is the probability distribution and the expected value of this game? Should we play this game?

The probability distribution is $D = \{(3, \frac{12}{52}), (-1, \frac{40}{52})\}$. The expected value of the game is $\langle EV \rangle = (\frac{12}{52})(3) + (\frac{40}{52})(-1) = -0.07$. This is a game that we should avoid since the expected value is negative.

The example above shows a characteristic of all casino games which are designed to have negative expected values for the consumers. It should be noted that there are 12 face cards, any of which will win for us, and 40 non-face cards, which we lose to. There is more than triple of the bad cards (3.33) than the good cards. We are only given 3 to 1 to our money, but we will lose more than that per game. We could in fact create our own games, just like above, to give the consumer the illusion of being able to win big, when in fact, they will lose overall. We may now formally define *expected value* of a random variable X with probability distribution P as

$$\langle X \rangle = \sum_{i=1}^n p_i x_i.$$

2.3 Enumeration

When working with probabilities, there may be a multitude of outcomes. So far we have limited our experiments to six-sided dice, or a standard deck of 52 cards. Before we can take on more complex or varied problems, we need to define some enumeration techniques. The first and easiest is the *multiplication principle*. Suppose that a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ... , and r_m different outcomes in the m th stage. If the number of outcomes at each stage is independent of the choices in previous stages and if the composite outcomes are all distinct, then the total procedure has $(r_1)(r_2) \cdots (r_m)$ different composite outcomes [Tuc02].

Example 2.3.1 In California, standard license plates for passenger vehicles are made up of one digit, followed by three letters, and ending with a three digit number. Assuming that leading zeros are permissible, and that repetition is allowed, how many different combinations of standard license plates are there?

Since there are 10 digits, 0-9, and 26 letters, a-z, available, the multiplication principle gives

$$(10)(26)(26)(26)(10)(10)(10) = 175,760,000.$$

Example 2.3.2 In California, standard license plates for passenger vehicles are made up of one digit, followed by three letters, and ending with a three digit number. Suppose leading zeros are not permissible, and repetition is not allowed on either letters or numbers, now, how many different combinations of license plates are there giving these restrictions?

There are 10 digits, 0-9, and 26 letters, a-z, available, the multiplication principle gives

$$(9)(26)(25)(24)(9)(8)(7) = 70,761,600.$$

By having those restrictions, the total possible number of combinations was reduced by 105 million possibilities. Let's take a moment to discuss how the restrictions affected the total number. The first factor is one of nine digits, 1-9, since 0 is not an option. The second factor can be any of the 26 letters, whereas the third factor can be any of the remaining 25 letters, because there is no repetition allowed. Similarly, the fourth factor can consist of any of the remaining 24 letters. Now for the remaining factors, the fifth factor can be any of the remaining nine digits, including zero but excluding the leading digit of the license plate. Thus, the sixth and seventh factors have eight and seven possible choices.

Another counting method is a *permutation*. Suppose that there are n positions which can be filled with n different objects. We can choose the first object in any of n different ways, $n - 1$ ways of choosing the second object, $n - 2$ for the third, ... , and only one remaining way for the last object to be chosen. Thus by the multiplication principle there are $n(n - 1)(n - 2) \cdots (2)(1) = n!$ possible arrangements. Here $n!$ is read as “ n factorial” where $0! = 1$ [HT10].

Example 2.3.3 A soccer coach can field 11 players. Assuming that every player can play in any position, how many different line-ups are there available to the coach? The coach can position the first player in any of the open 11 positions. The second player in any of the remaining 10 positions, ... , the last player will fill the last open position. Then there are

$$11! = (11)(10) \cdots (2)(1) = 39,916,800$$

different line-ups.

Now, what if we had n objects available and only need to fill r positions where $r < n$? Then the number of possible arrangements is

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

Here there are n ways to fill the first position, $n-1$ ways to fill the second position, until only $[n - (r-1)] = (n-r+1)$ ways of filling the r th position. Or

$${}_nP_r = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots(3)(2)(1)}{(n-r)\cdots(3)(2)(1)} = \frac{n!}{(n-r)!}.$$

where each of the ${}_nP_r$ arrangements is called a *permutation of n objects taken r at a time* [HT10].

Example 2.3.4 How many ordered samples of five cards can be drawn without replacement from a standard deck of cards?

$$(52)(51)(50)(49)(48) = \frac{52!}{(52-5)!} = \frac{52!}{47!} = 311,875,200.$$

In poker however, the order of selection is not important. If we have a pair of aces, it does not matter in which manner we receive each card. We could get an ace as the first card, and the last, fifth card could be the second ace. That hand will be equal in value as if we had received both aces as our first two cards.

When we can disregard the order of selection of objects, since subsets are of equal worth, we say that each of the ${}_nC_r$ unordered subsets is called a *combination of n objects taken r at a time*, where

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Here ${}_nC_r$ is read as “ n choose r ” [HT10].

Example 2.3.5 What is the number of five-card hands of poker that can be made from a standard deck if we disregard order?

$${}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!47!} = 2,598,960.$$

We can also calculate ${}_{52}C_5$ as $\binom{52}{5} = \frac{(52)(51)(50)(49)(48)}{(5)(4)(3)(2)(1)} = 2,598,960.$

Example 2.3.6 A poker hand is defined as drawing 5 cards at random without replacement from a standard deck of cards. What is the probability that the hand will consist of one pair, that is two cards of the same rank and three different cards of different ranks?

We first must choose one rank to make a pair. Since there are four cards of the same rank, we can choose any two cards from that set. We then must be careful and choose three cards, one from each of the remaining 12 ranks, where each can be chosen from any of the four available cards. Otherwise we run the risk of making either a second pair, or perhaps a three of a kind.

$$\begin{aligned} & \binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} \\ &= \frac{13}{1} \cdot \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} \cdot \frac{4}{1} \cdot \frac{4}{1} \cdot \frac{4}{1} \\ &= 1,098,240. \end{aligned}$$

Now, since there are 2,598,960 total five-card hands and 1,098,240 are one-pair hands we have

$$\frac{1098240}{2598960} \approx .4227.$$

Thus, there are about 42% of hands that will contain at least one pair.

The figure below displays all possible five-card poker hand combinations.

Hand	Frequency	Probability
Royal Flush	4	0.00015%
Straight Flush	36	0.00138%
Four of a Kind	624	0.02401%
Full House	3,744	0.14405%
Flush	5,108	0.19654%
Straight	10,200	0.39246%
Three of a Kind	54,912	2.11285%
2 Pair	123,552	4.75390%
Pair	1,098,240	42.25690%
High Card	1,302,540	50.11774%

Figure 2.1: 5-Card Poker Hands with Respective Probabilities.

2.4 Variance

Probability distributions have two characteristics which if taken together describe most of the behavior of the distribution for repeated experiments. As previously discussed, the first one is the expected value, which is also called the *mean* of the distribution. The second characteristic is *variance*, a measure of the dispersion of the outcomes from the expectation.

- For a probability distribution P , where each of the n outcomes has a value x_i and a probability p_i , the variance of P , V_p is

$$V_p = \sum_{i=1}^n p_i (x_i - \langle P \rangle)^2.$$

We say that variance is the weighted mean of the squares of the outcomes. The positive square root of the variance is called the *standard deviation*. Thus, we denote variance as σ^2 and standard deviation simply as σ [HT10].

Example 2.4.1 Consider the following probability distribution $D = \{(1, \frac{3}{6}), (2, \frac{2}{6}), (3, \frac{1}{6})\}$. Calculate the expected value, the variance and the standard deviation.

$$\begin{aligned} \langle EV \rangle &= (1)(\frac{3}{6}) + (2)(\frac{2}{6}) + (3)(\frac{1}{6}) = \frac{10}{6} = \frac{5}{3}. \\ \sigma^2 &= \frac{3}{6}(1 - \frac{5}{3})^2 + \frac{2}{6}(2 - \frac{5}{3})^2 + \frac{1}{6}(3 - \frac{5}{3})^2 = \frac{15}{27} = \frac{5}{9}. \\ \sigma &= \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}. \end{aligned}$$

Thus, the expected value is $5/3 \approx 1.66$, the variance is $5/9 \approx 0.56$, and the standard deviation is $\frac{\sqrt{5}}{3} \approx 0.75$.

2.5 Bayes' Theorem

The last item needed for our background work in probability is *Bayes' Theorem*. Before we are ready to proceed, recall the definition of conditional probability from Section 2.1 which was the probability of two events A and B happening. Here, $P(A \cap B)$ is the probability of A , $P(A)$, times the probability of B given that A has occurred, $P(B|A)$. In other words, we have

$$P(A \cap B) = P(A)P(B|A).$$

On the other hand, the probability of A and B is also equal to the probability of B times the probability of A given B ,

$$P(A \cap B) = P(B)P(A|B).$$

Equating the two equations yields:

$$P(A)P(B|A) = P(B)P(A|B),$$

and thus

$$P(A|B) = P(A) \frac{P(B|A)}{P(B)}.$$

The equation above is the basic idea behind Bayes' Theorem [CA06].

From above, we know that one of the basic probability equations given for all dependent events is $P(A \cap B) = P(A)P(B|A)$. Sometimes, it may be preferable to calculate the conditional probability of B given that A has already occurred. For example, in poker, I know the cards in my hand, and now I can use that information to calculate the probability that my opponent has a certain type of hand.

We now define a **partition** of S as a collection of sets $\{B_1, B_2, \dots, B_k\}$, for some positive integer k , where B_1, B_2, \dots, B_k are sets such that:

1. $S = B_1 \cup B_2 \cup \dots \cup B_k$.
2. $B_i \cap B_j = \emptyset$, for $i \neq j$.

If A is any subset of S and $\{B_1, B_2, \dots, B_k\}$ is a partition of S , then A can be decomposed as:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k).$$

Theorem 2.2 (Bayes' Theorem). *Assume that $P(A) > 0$ and that $\{B_1, B_2, \dots, B_k\}$ is a partition of the sample space S such that $P(B_i) > 0$ for $i = 1, 2, \dots, k$. Then*

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^k P(B_i)P(A|B_i)}.$$

Proof. We have, $A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k)$. Then,

$$\begin{aligned} P(A) &= \sum_{i=1}^k P(B_i \cap A) \\ &= \sum_{i=1}^k P(B_i)P(A|B_i). \end{aligned}$$

Since

$$\begin{aligned} P(B_j|A) &= \frac{P(A \cap B_j)}{P(A)} \\ &= \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^k P(B_i)P(A|B_i)}. \end{aligned}$$

□

[WMS08].

Example 2.5.1 Suppose a new player sits at our poker table. We know from previous observations that he is 10% likely to be an aggressive player who will raise any hand 80% of the time. Also, from experience, we can deduce that he is 90% likely to be a passive player who will only raise the top 10% of his hands. On the first hand that he plays, he raises, what is the probability that this player is aggressive? [CA06]. Denote A as the probability that the player plays his first dealt hand. Denote B as the probability that the player is aggressive. Then,

- a) $P(A|B) = 0.8$ is the probability that given that the player is aggressive, he will raise 80% of the time.
- b) $P(A|\bar{B}) = 0.1$ is the probability that given that the player is not aggressive, therefore passive, he will only raise 10% of the time.
- c) $P(B) = 0.1$ is the probability that the player is aggressive, 10%.
- d) $P(\bar{B}) = 0.9$ is the probability that the player is passive, 90%

By Bayes' Theorem we have,

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})} \\ P(B|A) &= \frac{(0.8)(0.1)}{(0.8)(0.1) + (0.1)(0.9)} \\ P(B|A) &= 0.4705 \approx 47.1\% \end{aligned}$$

Just by observing this player raise his first hand, we can adjust the probability that this player is aggressive from 10% to 47%. Now, what will happen if this player raised his second hand in a row? We can redefine events A and B as follows:

- a) $P(A|B) = 0.8$ is the probability that given that the player is aggressive, he will raise 80% of the time.
- b) $P(A|\bar{B}) = 0.1$ is the probability that given that the player is not aggressive, therefore passive, he will only raise 10% of the time.
- c) $P(B) = 0.47$ is the probability that the player is aggressive, 47%.
- d) $P(\bar{B}) = 0.53$ is the probability that the player is passive, 53%

By Bayes' Theorem again we have,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$
$$P(B|A) = \frac{(0.8)(0.47)}{(0.8)(0.47) + (0.1)(0.53)}$$
$$P(B|A) = 0.8764 \approx 87.6\%$$

Thus, by observing this player raise his first two hands in a row, we can deduce that this player is 87% likely to be aggressive.

Chapter 3

Decision Analysis

3.1 Decision Making

Decision making is an everyday process, and decisions we make, big or small, can have a great impact on life. There are many factors that can influence the decision making process, and normally a combination of these factors will play a role. Some of these factors may include conflict, time, competition, attitude, risk, policies, etc. Thus, a great decision-maker is a person who rises above these factors and is able to bring out its best through the process of *rationality of decisions*, making and improving upon decisions made. Hence, deep analysis of previous decisions and that of complexities of the given environment help the process of rational and effective decision making. *Decision theory* comes into play when, more often than not, parameters, inputs, or consequences are not fully known and thus uncertainty exists which complicates the decision making process. Uncertain situations are dealt with by assuming parameters following various probability distribution functions. However, uncertainty must be approached with a determined amount of calculated risk, since the variables involved could change due to the environment and even by our own decisions. Similarly, conditions may change based upon equivalent circumstances. In order to calculate risk we turn to a *decision table*, or *pay-off table* [Sha09].

The decision making process can be broken down into four steps:

1. Identify all possible states of nature, events available, or factors affecting the decisions.
2. List out various courses of action open to the decision maker.

3. Identify the pay-offs for various strategic solutions under all known events or states of nature.
4. Decide to choose from amongst these alternatives under given conditions with identified pay-offs.

Tactical decisions are day-to-day decisions having impact on the immediate business environment and the resultant outcome. These types of decisions are but only one of many other types of decision categories. Tactical decisions are affected by decision making environments. One such environment is **uncertainty**. Decision making under uncertainty is the absence of past data, where it may not be possible to estimate the probabilities of occurrence of different states of nature. There exist some useful criteria when there is no prior data that the decision maker can rely upon, and there is no method of computing the expected pay-off for any strategy. Sharma, in his book *Operations Research*, provides the following methods to compute expected pay-offs:

1. Maximin criterion: The decision maker adopts a pessimistic approach and tries to maximize his security, for a worst case scenario we must make the best of the situation and choose the highest pay-off.
2. Minimax criterion: Of the best possible scenario, we will try to secure the minimum of the maximums available.
3. Maximax: We start getting greedy and will go for maximum pay-off, totally optimistic, we choose the highest reward of the best case scenario.
4. Laplace criterion: There is no definite information about the probability of occurrences, so we assume that each one is just as likely. This is the sum of all probabilities for a specific strategy.
5. Hurwicz Alpha criterion: A combination of maximin and maximax. The decision maker utilizes α as a degree of optimism where $0 < \alpha < 1$, where 0 is total pessimism and 1 is total optimism. The decision D_i is defined by

$$D_i = \alpha M_i + m_i(1 - \alpha),$$

where M_i is the maximum pay-off from any of the outcomes from the i th strategy and m_i is the minimum pay-off from any of the outcomes from the i th strategy.

6. Regret criterion: We consider the dissatisfaction associated with not having obtained the best return in investment. Regret is computed as the difference in pay-off of the outcome and the largest pay-off which could have been obtained under the corresponding state of nature. Also called the *opportunity loss table*.

Example 3.1.1 A bakery usually sells the following three products, lemon cake, coconut cookies, or glazed donuts. Next year, the expected sales are highly uncertain, and the owner decides to scale back to selling just one product. It is estimated that profits will reflect the table below [Sha09].

Table 3.1: Products and Profits

Product	Estimated profit in thousands for the indicated quantities		
	5,000	10,000	20,000
Lemon Cake	15	25	45
Coconut Cookies	20	55	65
Glazed Donuts	25	40	70

Which product should the bakery sell under the different criterion?

- (a) Under the maximin criterion, we want to pick the best outcome from all of the minimums. That is, if we only make five thousand of any product, the best result will be the donuts. Thus, the bakery should make the donuts and expect a \$25,000 profit.
- (b) The maximax is the best of the best, thus donuts will yield \$70,000 profit if the bakery makes 20,000 of them.
- (c) If we use the Laplace method, where we assume that each outcome is just as likely, we have for lemon cakes,

$$\frac{1}{3}(15) + \frac{1}{3}(25) + \frac{1}{3}(45) = 28.33.$$

For coconut cookies,

$$\frac{1}{3}(20) + \frac{1}{3}(55) + \frac{1}{3}(65) = 46.67.$$

And for glazed donuts,

$$\frac{1}{3}(25) + \frac{1}{3}(40) + \frac{1}{3}(70) = 45.$$

Thus, under this criteria, the bakery should sell the cookies.

- (d) Under the Hurwicz alpha criteria, assume $\alpha = 0.6$, then for lemon cakes we have,

$$0.6(45) + 15(1 - 0.6) = 33.$$

For coconut cookies,

$$0.6(65) + 20(1 - 0.6) = 47.$$

And for glazed donuts,

$$0.6(70) + 25(1 - 0.6) = 52.$$

Thus, the bakery should sell the donuts.

- (e) For the regret criteria, we must subtract the highest possible pay-off under each state of nature from the outcomes associated with each of the possible events. From the opportunity loss table, we want to minimize future regret, and as such, the bakery should choose to sell the coconut cookies.

Table 3.2: Opportunity Loss Table

Product	5,000	10,000	20,000	Max Regret
Lemon Cake	$(25-15)=10$	$(55-25)=30$	$(70-45)=25$	30
Coconut Cookies	$(25-20)=5$	$(55-55)=0$	$(70-65)=5$	5
Glazed Donuts	$(25-25)=0$	$(55-40)=15$	$(70-70)=0$	15

It is important to note that a person cannot predict the outcome of any event, by selecting an action we can expect some conclusions to occur which may be beneficial, insignificant or even harmful. Thus, there is risk involved and in order to reach the optimal decision we can use probability distributions out of the probable outcomes based upon previously collected data. This is another type of decision making category which we will call decision making under *risk*. We can make our decisions according to a model that will yield the largest expected profit value which is called *expected money value* or EMV for short. EMV is used in order to calculate risk and facilitate the decision making process. EMV is calculated with the following equation.

$$EMV(C_i) = \sum_{i=1}^n P_i O_i.$$

Here,

C_i = Course of action i ,

P_i = Probability of occurrence of outcome O ,

n = Number of possible outcomes,

O_i = The pay-off expected or outcome of action i .

The equation above for EMV is another way of illustrating the expected value of a probability distribution.

Example 3.1.2 A trading company is considering expansion. We need to determine whether to operate from an existing office and cover the area by traveling, or to open a new locale closer to the new market. We come up with the following data: [Sha09].

Table 3.3: Current vs. Expansion

Alternatives	States of Nature	Probability	Pay-off
A. Operate from current office	i) Increase in demand by 30%	60%	50
	ii) No change	40%	5
B. Open new office	i) Increase in demand by 30%	70%	40
	ii) No change	30%	-10

The expected pay-off for A is $0.6(50) + 0.4(5) = 32$, and the expected pay-off for B is $0.7(40) + 0.3(-10) = 25$. As a result, we decide to stay at the current location and operate by trading locally and by traveling to the new market.

Decision making under *conflict* arises when there is more than one option open for optimal gain, mainly due to the action of others in the field. This is where game theory comes in handy. Some concepts that can be analyzed by game theory is *expected value of perfect information* or EVPI. This is the average outcome of the states of nature of the level of information. That is to say, the information is available and the probabilities are known and hence we can calculate the best option that results on the highest profit possible. So $EVPI = (\text{best outcome of first state of nature})(\text{probability of first state of nature}) + (\text{best outcome of second state of nature})(\text{probability of second state of nature}) + \dots + (\text{best outcome of last state of nature})(\text{probability of last state$

of nature). There is also the concept of *expected profit with perfect information* or EPPI. EPPI is the maximum obtainable monetary gain given that all possible parameters are known and accounted for. We can then define this relationship as

$$EVPI = EPPI - EMV,$$

where, EPPI is the expected profit with perfect information, and EMV is the expected monetary value.

Example 3.1.3 As a manufacturer we want to increase our business. There are two choices available to choose from, first, expansion of the existing capacity at a cost of 8 units, or second, modernization of current capacity at a cost of 5 units. We estimate a 35% probability of having a high demand versus a 65% probability of having no change on demand. Additionally, when demand is high, we will be earning 12 units if we decide to expand, against 6 units if we decide to modernize. If there is no change in demand, we will earn 7 units on expansion versus 5 units for modernization. Let S_1 be the state of nature pertaining to high demand with probability P_1 at 35%. Let S_2 be the state of nature corresponding to no change in demand with probability P_2 at 65%. Also, list the courses of action as A_1 for expansion, and A_2 for modernization. The conditional profits will be given by the difference of the new revenue and the cost of expansion or modernization [Sha09].

Table 3.4: Conditional Profits

States of Nature	A_1	A_2
S_1	$12 - 8 = 4$	$6 - 5 = 1$
S_2	$7 - 8 = -1$	$5 - 5 = 0$

Then, the expected monetary values are:

$$EMV(A_1) = 4(0.35) + (-1)(0.65) = 0.75,$$

and

$$EMV(A_2) = 1(0.35) + 0(0.65) = 0.35.$$

Thus we choose to expand since this course of action will maximize EMV. Now, we need to calculate the EPPI by choosing the optimal course of action for each state of nature, and then multiplying by the corresponding probability.

Table 3.5: Expected Profit with Perfect Information

States of Nature	Probability	Optimal Course of Action	Expected Profit
S_1	0.35	A_1	$4(0.35) = 1.4$
S_2	0.65	A_2	$0(0.65) = 0$

Therefore EPPI is $1.4 + 0 = 1.4$.

Hence,

$$EPPI - EMV = EVPI$$

$$1.4 - 0.75 = 0.65.$$

3.2 Decision Trees

Decision making may involve multiple stages and at each stage, each decision will open up different outcomes with their respective pay-offs. A **decision tree** facilitates this process. A decision tree is made up of nodes, branches, probabilities, and the resultant pay-offs of each outcome. Decision trees are graphical tools that allows us to see how decisions affect the outcomes.

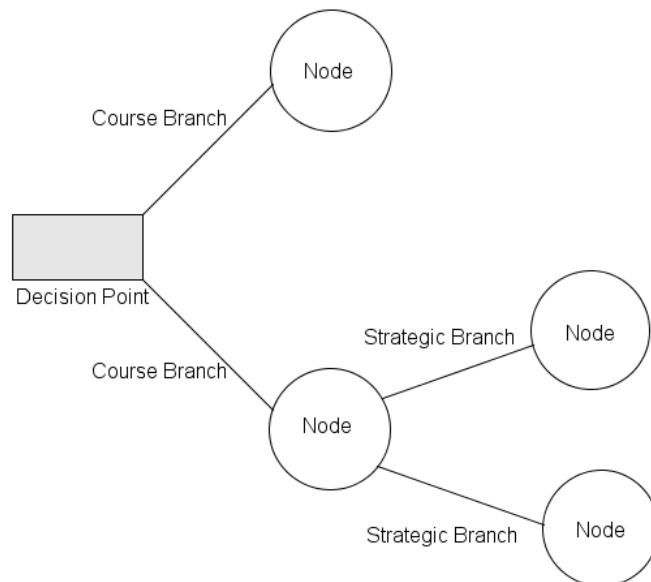


Figure 3.1: Decision Tree Diagram.

Possibly the most important tool available to a decision maker is *posterior probabilities and analysis*. By evaluating old decisions, new and improved models can be derived. With new information, new alternatives can be considered. This may drastically change future expected pay-offs. Bayes' Theorem, is used to calculate the effects and relationships of prior probability, the initial probability statement with pay-offs, posterior probability, and allows us to revise probability statements due to post data analysis.

Example 3.2.1 Suppose we are offered two investment opportunities, call them A and B . The probability of success on A is 70% and on investment B is 40%. Both investments require a capital of \$2,000 to get started. Investment A returns \$3,000 while investment B returns \$5,000. If either investment fails, we lose our initial capital of \$2,000. We can only partake in one investment at a time. Thus, our options are, to take investment A and stop, or if successful we may continue onto investment B . Or we may start with investment B and stop, or if successful continue with investment A . Which is the best course of action? [Sha09].

We have the following courses of action:

- i. Take A and stop.
- ii. Take B and stop.
- iii. Take A and if successful, take B .
- iv. Take B and if successful, take A .

We begin by filling in the decision tree, shown on the next page, with the actual investments values, and from there we can calculate the expected values of each decision. For example, taking on investment A and succeeding gives a payout of \$3000, but if we fail, we will lose the capital investment of \$2000. If we are successful at A , we reach decision point 2. From here we can choose to stop, at which point we will collect \$3000, or we may continue with investment B . If B is successful, we collect the new pay-out of \$5000 plus we still have the amount from A for a total of \$8000. However, if investment B fails, we lose \$2000, but we still have \$1000 from the initial gain from A . The same logic follows if we decide to take investment B first. Note that if both investments succeed, we will receive \$8000, but depending

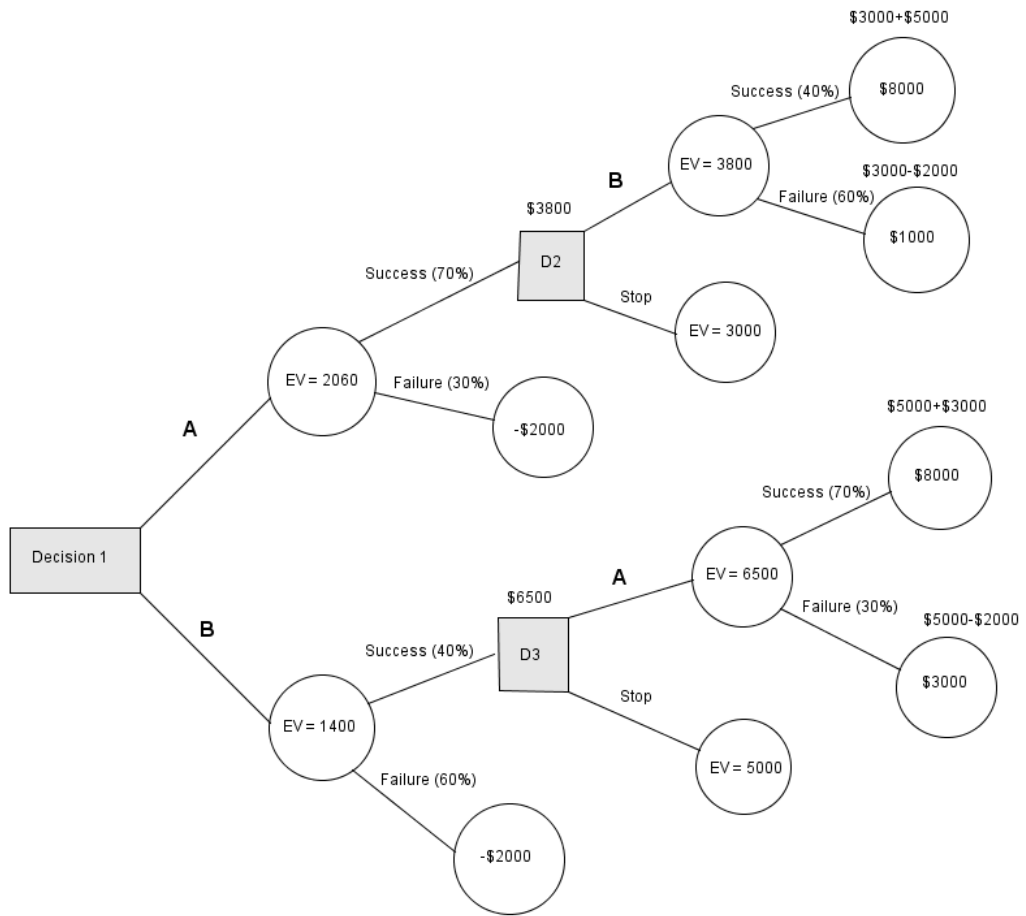


Figure 3.2: Investment Opportunities.

on which investment we take first, the pay-out for failing on the second investment will be different. Also, failing at any investment first carries the same negative expectancy of $-\$2000$. Thus, if we want maximum pay-out, we must follow the path that carries the greatest expected value for our choices.

The expected value of A succeeding and then B succeeding is given by $(8000)(0.6) + (1000)(0.4) = 3800$. Thus, $\$3800$ is the expected value at decision point 2. The expected value of taking only A is $(3800)(0.7) + (-2000)(0.3) = 2060$. Thus $\$2060$ is the expected value of investment A succeeding alone.

If the investments are reversed, the expected value of B succeeding and then A succeeding is given by $(8000)(0.7) + (3000)(0.3) = 6500$, which is the expected value of decision point 3. Then the expected value of only B succeeding is $(6500)(0.4) +$

$$(-2000)(0.6) = 1400.$$

Although, success in both investments yields the same pay-out, which path should we take? At decision point 1, we can compare the expected values of 2060 and 1400, at which point we follow the largest which is to invest in A first. At this juncture, the expected value of continuing versus stopping is 3800 versus 3000, so we decide to continue with investment B . This is the strategy offered in *iii*. Hence, taking A and if successful taking B is the best course of action since it returns the greatest EMV.

Chapter 4

Theory of Games

4.1 Types of Games

Game theory comes into practice when there is a well understood problem and the action of all persons of interest are known beforehand. For example, actions taken by company *A* would have a direct impact on company *B*, and thus company *B* must react to the new situation, and must react optimally. Once company *B* has adjusted, company *A* could readjust to the new environment and alter its own actions once again. Thus, game theory is mainly used in figuring out the decisions and options given a set of conflicting outputs and the motives of competing parties. Game theory applies when a game is finite, that is, there exists a limited number of choices, and a finite number of moves at each choice, and most importantly, each choice follows a rational set of rules and behaviors. This will ensure that an acceptable outcome can be predefined for each game. Since game theory deals with human thoughts, a player must take into account its environment. Each player is an independent decision maker, but the outcome of the game would depend on the action and strategies taken by all players. Since each player may have a different goal in mind, this would be categorized as *decision making under conflict*. Players may not all be motivated by maximum profit, but rather by optimal dominating decisions over other players which may help in that particular player dominating the game in the long run. Thus each player must be prepared with strategies and counter-strategies for any given outcome. Game theory is then an interactive process that must be analyzed and evaluated at each step, not only must we criticize our own play, but must ultimately

decipher our opponents goals.

There are many types of games, but can be reduced to the following:

1. 2-Person Zero-Sum Game, a game where the gain of one player is the loss of the other.
2. 2-Person Non-Zero-Sum Game, a game where gain and loss are not equal, and thus the outcome is not obvious.
3. N-Persons Game, a game where there exists a large number of players all trying to maximize profits. There could be multiple winners, each with a different amount of profit.

In order to represent a game with fixed strategies for two players, we can draw a simple matrix like the next table below.

Table 4.1: Standard Matrix

	B_1	B_2	B_3	\cdots	B_m
A_1	C_{11}	C_{12}	C_{13}	\cdots	C_{1m}
A_2	C_{21}	C_{22}	C_{23}	\cdots	C_{2m}
A_3	C_{31}	C_{32}	C_{33}	\cdots	C_{3m}
\vdots	\vdots	\vdots	\vdots	\cdots	\vdots
A_n	C_{n1}	C_{n2}	C_{n3}	\cdots	C_{nm}

The matrix indicates the expected pay-offs as values obtained from using strategy A and strategy B . For example, if player A adopts strategy A_2 and player B counters with strategy B_3 , then the resulting pay-off will be C_{23} . It is standard practice to show the pay-offs from the columns player, that is in this case, from player A 's perspective, where positive numbers indicate gains and negative numbers indicate losses. For the other player, these values would be reversed.

For the remainder of this chapter we focus on 2-person games in which Sharma assumes the following rules:

1. Players act rationally and intelligently.
2. Each player has all relevant information.
3. Each player can use the information in a finite number of moves with finite choices for each move.

4. Players make independent decisions of courses of actions without consultation.
5. Players play the game for optimization.
6. The pay-off is fixed and known in advanced.

Example 4.1.1 Solve the game [Sha09].

Table 4.2: Game 1

A's Strategy	B's Strategy		
	b_1	b_2	b_3
a_1	12	-8	-2
a_2	6	7	3
a_3	-10	-6	2

We want to find the optimal strategy pair for both players A and B . Player A wants to maximize his profits, while player B wants to minimize his losses. We can solve the game by finding the maximin, the maximum of the minimums for player A , and the minimax, the minimum of the maximums for player B . From the matrix we can derive that the minimum for each of player A strategies is $\{-8, 3, -10\}$. From this set, player A will attempt to secure the maximum pay-off of 3. Similarly, player B maximums from each of his strategies is the set $\{12, 7, 3\}$ from which we must give up the minimum of 3. Thus, the solution to the game is given by the strategies a_2b_3 . In this game, by following the same strategy, player A can gain 3 units from player B . The solution a_2b_3 is said to be optimal, since it is the only strategy that will yield the safest return for player A , and at the same time, player B gives up the least amount of equity. When a unique solution to a game exists, it is said to be a saddle point. A *saddle point* is the point of equilibrium for both strategies. At this point there exists a maximum gain for player A and minimum loss for player B .

Let's look at what the previous example solution means. Player A has three strategies open to him. If he chooses either a_1 or a_3 he risks losing as indicated by the negative numbers. Only a_2 offers a profit every time, and at worst, player A can only win 3 units, but his gains could be as high as 7. Although, the 12 from strategy a_1 may

seem like the maximum pay-off, and thus the one that we should be striving to obtain, we run the risk of losing -8 and -2 when player B chooses b_2 and b_3 respectively. Even if we assume that each strategy is as likely to be chosen, by Laplace criterion we have $\frac{1}{3}(12) + \frac{1}{3}(-8) + \frac{1}{3}(-2) = 0.66$, which makes choosing this strategy not ideal. Another point to consider is, when player B realizes that player A is constantly choosing a_1 , player B can switch to playing b_2 exclusively, and thus he will win 8 units from player A every time.

By following the assumptions above, we can define a **strategy** as an alternative course of action available to all players in advance. There are two types of strategies. A **pure strategy** is where a player selects the same strategy each time without variation. In this manner, each player knows exactly what the other player would do against his own selected strategy. The other type of strategy is a **mixed strategy**. Here each of the players use a combination of strategies, and each player leaves the other one guessing as to what he would do next. Thus, opposing players must bring their own counter-strategy with the aim of optimization. Regardless of which type of strategy is used, an **optimal strategy** is the preferred course of action. Such a strategy will put the player in a superior position regardless of his competitor's actions. This will lead to the player obtaining a maximum **pay-off**, a measurement of achievement, or a gain in profit. The pay-off as previously discussed is the expected outcome, or the expected value, which we will call the **value of the game**, which is obtained when players follow their optimal strategies.

2-Person zero-sum games can then be either purely strategic, where a player follows one particular action to victory, or a mixed strategy game, where a player adapts different strategies depending on the present situation. When both players use the best strategy and said strategy is optimal, a saddle point exists. In Example 4.1.1, the saddle point is a_2b_3 which has a value of 3.

4.2 Domination

A game with multiple strategies will lead to a large matrix. However, some of those strategies could be dominated by another strategy, and thus the matrix can be simplified until we obtain a more manageable, smaller matrix. This is what we will term as the **concept of dominance**. Dominance occurs when one strategy is clearly superior to another available strategy. For instance, if a strategy has better pay-offs than

another strategy regardless of which counter-strategy is used against it, such a strategy will dominate, and that strategy will be the preferred course of action to use. Domination is applied as follows: [Sha09].

1. The rule of rows applies when all elements in a row are less than or equal to the corresponding elements in another row. Such row is then said to be dominated and can be removed from future play.
2. The rule of columns, applies when all elements in a column are greater than or equal to the corresponding elements in another column. That column is said to be dominated and can be removed from future play.
3. The rule of averages is when a pure strategy may well be dominated by the average of two or more pure strategies. This will reduce the matrix faster since we can remove several rows and, or columns simultaneously, if these are found to be dominated by one of the previously mentioned rules.

Example 4.2.1 Solve the game [Sha09].

Table 4.3: Game 2

A's Strategy	B's Strategy		
	b_1	b_2	b_3
a_1	9	8	-7
a_2	3	-6	4
a_3	6	7	7

Before we attempt to solve this game we should look for saddle points. If a saddle point exists, then there exists a pure strategy available to both players. The minimums from each of A 's strategies is $\{-7, -6, 6\}$ which has a maximum of 6. Now, the maximums from B are $\{9, 8, 7\}$ whose minimum is 7. Since $6 \neq 7$, there is no saddle point, and a pure strategy does not exist. However, we may now turn our attention to removing dominated strategies from the matrix. By the rule of rows we can see that all values from a_2 are less than those from a_3 . Thus we say that a_3 dominates a_2 . Player A will always achieve a better result by choosing a_3 over a_2 regardless of what option player B chooses. By removing a_2 from the matrix we obtain the row reduced matrix shown on the next page.

Table 4.4: Game 2, Row Reduced

A's Strategy	B's Strategy		
	b_1	b_2	b_3
a_1	9	8	-7
a_3	6	7	7

Now, by rule of columns, player B can remove b_2 since it is dominated by b_3 , because values in b_2 are greater than or equal to those in b_3 . Thus, the new reduced game is given by the matrix as:

Table 4.5: Game 2, Row/Column Reduced

A's Strategy	B's Strategy	
	b_1	b_3
a_1	9	-7
a_3	6	7

Player A minimums from each strategy are $\{-7, 6\}$ whose maximum is 6, and player B maximums are $\{9, 7\}$ whose minimum is 7. Since $6 \neq 7$, no pure strategy exists. Hence, player A must play a mixed strategy from a_1 and a_3 , while player B reacts by playing a mixed strategy from b_1 and b_3 . It is important to note, that it could be possible for a saddle point to appear once dominated strategies have been removed.

In the example above, both players must choose mixed strategies, but how often should they play each strategy? Furthermore, why does player A not simply choose a_3 as his sole strategy since he can at worst get 6 units from player B ?

Perhaps it is not obvious from the preceding example, but player A should still play a_1 from time to time, even at the risk of losing 7 units to player B . In order to illustrate this point, we can look at a simplified game with all positive values.

Example 4.2.2 In Game 3, player A can choose a_2 every time since this will secure him at a minimum a pay-off of 2 units. Player B also loses the minimum by choosing b_1 . However, player A notices that player B always chooses b_1 , so player A switches to a_1 in order to get a better profit of 3. Player B follows the same argument and now switches to b_2 in order to lose less. Thus, both players enter a cycle where

Table 4.6: Game 3

A's Strategy	B's Strategy	
	b_1	b_2
a_1	3	1
a_2	2	4

they switch their strategies in order to maximize and minimize their respective winnings and losses. Assuming player A alternates between his two options equally, his expected value from b_1 is $\frac{1}{2}(3) + \frac{1}{2}(2) = 2.5$ which is preferable over strictly choosing a_2 which will only give him 2. Similarly, the expected value from b_2 is $\frac{1}{2}(1) + \frac{1}{2}(4) = 2.5$. Thus player A must play a mixed strategy since his expected value for both strategies is greater when using mixed strategies than by using a pure strategy alone.

Is it possible that player A can actually increase his expected value by playing one strategy more than half the time, and if so, how often should he play each strategy? Sharma provides three methods for solving games, algebraic, graphical, and linear programming. For the purpose of this project we will focus on the algebraic method. The algebraic method for solving a 2 by 2 game is derived by the following general formula:

Table 4.7: Algebraic 2 by 2 Matrix

	B_1	B_2
A_1	C_{11}	C_{12}
A_2	C_{21}	C_{22}

Suppose player A selects strategy A_1 with probability p and A_2 with probability $1 - p$. Suppose player B selects strategy B_1 with probability q and B_2 with probability $1 - q$. Then, the expected value of A versus B_1 is

$$C_{11}(p) + C_{21}(1 - p).$$

Now, using the values from Example 4.2.2 and the general formula, we can algebraically solve the game to find the correct mixed strategy. We have the following when player A

plays against b_1 :

$$\begin{aligned} 3(p) + 2(1 - p) &= 3p + 2 - 2p \\ &= p + 2. \end{aligned}$$

When player A plays against b_2 we have:

$$\begin{aligned} 1(p) + 4(1 - p) &= p + 4 - 4p \\ &= -3p + 4. \end{aligned}$$

The solution is given when these equations equal each other,

$$\begin{aligned} p + 2 &= -3p + 4 \\ 4p &= 2 \\ p &= \frac{1}{2}. \end{aligned}$$

How often should player A select each strategy? Since $p = \frac{1}{2}$, player A must play a_1 50% of the time, and he must balance his play by selecting a_2 the other 50% of the time. In this example it turns out that A can maximize his expected value by exactly playing each strategy one half of the time! But this is not the case for B . When player B plays against a_1 we have:

$$\begin{aligned} 3(q) + 1(1 - q) &= 3q + 1 - q \\ &= 2q + 1. \end{aligned}$$

And when B plays against a_2 we have:

$$\begin{aligned} 2(q) + 4(1 - q) &= 2q + 4 - 4q \\ &= -2q + 4. \end{aligned}$$

The solution is given when these equations equal each other,

$$\begin{aligned} 2q + 1 &= -2q + 4 \\ 4q &= 3 \\ q &= \frac{3}{4}. \end{aligned}$$

In order for player B to minimize his losses, he must play b_1 75% of the time, and b_2 the remaining 25% of the time.

We can now revisit Example 4.2.1, Game 2, to find out how often each player must play each strategy. We have, A 's strategy versus b_1 as $9(p) + 6(1 - p)$ and versus b_3 as $-7(p) + 7(1 - p)$. Solving these equations and setting them equal to each other gives $p = \frac{1}{17}$. Similarly, for B we have, against a_1 , $9(q) + (-7)(1 - q)$ and against a_3 we have $6(q) + 7(1 - q)$. Solving these equations and setting them equal to each other obtain $q = \frac{14}{17}$. Thus, in Game 2, player A must play a mixed strategy that contains a_1 $1/17$ of the time, a_2 zero percent of the time (since it is dominated), and a_3 the remaining $16/17$ of the time. Player B must play b_1 $14/17$ of the time, b_2 zero percent of the time (since it is also dominated), and b_3 the remaining $3/17$ of the time. The expected value for the game from player A 's perspective is $9(\frac{1}{17}) + 6(1 - \frac{1}{17}) = \frac{105}{17}$. And from player B 's perspective $9(\frac{14}{17}) - 7(1 - \frac{14}{17}) = \frac{105}{17}$. Hence, by using the mixed strategies above, player A stands to win approximately 6.18 units while player B stands to lose the same amount. Thus, this is the optimal strategy for each player to follow.

As stated earlier, when dealing with game theory, we are assuming that the gain of one player is the loss of another player. However, it is highly unlikely that a player will have access to all of the required information ahead of time. The lack of information will leave us guessing as to whether a specific strategy is optimal or not. Therefore, we must restrict our assumptions to a basic standard set of rules that are well defined and understood by all players. In real life, competitors will not divulge their strategies, and as such, pay-offs will not be known, which implies that courses of actions cannot be predicted. Thus, a decision maker is left with the option of taking risk since the outcomes are uncertain. Furthermore in real life, there could be more than two players, all competing for maximum profits, with different strategies and different motives.

Chapter 5

Poker Models

5.1 Uniform Poker Models

We now turn our attention to solving poker models. These are simplified poker games that can be solved algebraically. We will now assume that we are playing two-person zero-sum games. The participants will be called player 1 and player 2. In this type of game, a player will gain some amount from the other player, and this sum will always be zero. We will also assume that hands are randomly and independently distributed. For the models, player 1 is dealt a random hand $X \in [0, 1]$ where X has a uniform distribution over the interval $[0, 1]$. To be precise, of all minimum and maximum values for each member of the family, all intervals of the same length on the distribution have equal probabilities. In this case, in a standard deck of 52 cards, the probability of getting dealt one specific card is exactly $1/52$, the probability of getting dealt a second specific card is exactly $1/51$, and so on. Getting dealt a specific card does not influence the outcome of the second card. Similarly, player 2 is also dealt an independently random hand $Y \in [0, 1]$ with equal uniform distribution on $[0, 1]$ as X . Both players are well aware of the value of their own hands, but have no information on their opponent's hand strength. We can think of it as each player has a number between 0 and 1, but they have no idea if their opponent's number is higher or lower. On each model there exists a betting structure. Both players must pay a "unit" worth, (decided amongst the players in advance) that is paid before the hands are dealt, this is called an *ante*. The antes and any money wagered afterwards is placed in the *pot*, which refers to the sum of all antes

and bets. Each player will have a predefined set of actions available to them. Player 1 may decide to bet or not to bet his hand, and player 2 reacts with his own actions, whether to call or fold. For all models we are ignoring other options found in poker, which could be for player 1 to check-raise, check-call, or for player 2 to bet his own hand or to raise when faced with a bet by player 1, amongst some common poker strategies. At the end of the game, the hands are compared at what is known as *showdown*, and the highest valued hand wins the pot.

If we disregard the randomly distributed hands, given that players' options are known and announced, will make this game a game of perfect information. This type of game can be solved using one or more techniques as described in the previous chapters about decision analysis and game theory. However, since we are ignorant of our opponent's holdings, this is then called a game of *almost perfect information*, where we may deduce the strength of our opponent's cards by what we have. For instance, if I have an ace it is less likely that my opponent has an ace of his own, or, since I hold two spades, it is less likely that my opponent has any spades at all. Thus, we may study and derive the correct strategy to play using decision trees which we will be calling *betting trees*.

5.2 Borel's Poker

Borel's poker model is referred to as *La Relance* (Figure 5.1). In this model each player pays one unit in ante and then they are dealt their respective randomly independent uniformly distributed hands. Player 1 acting first, will have two options, one to *fold* his hand, that is, to discard one's hand and forfeit interest in the current pot. In this case, player 1 surrenders the hand to player 2, giving up his claim to the pot, and conceding the pot to player 2. When player 1 folds, he loses his ante, and player 2 wins one unit. The second option is for player 1 to make a wager, to *bet* some value B where $B > 0$ and B may not necessarily be equal to an ante's worth but could be higher. In the case that player 1 bets, if player 2 folds, then player 1 wins the two antes, and gains one unit worth. If player 2 calls the bet, then hands are compared at showdown, and the best hand will take the pot and in this case will gain both antes, plus whatever the bet B was, for a net worth of $B + 1$ ante. The model disregards the case where both hands are equal in value since the probability of $X = Y$ is 0. However, it is important to note that

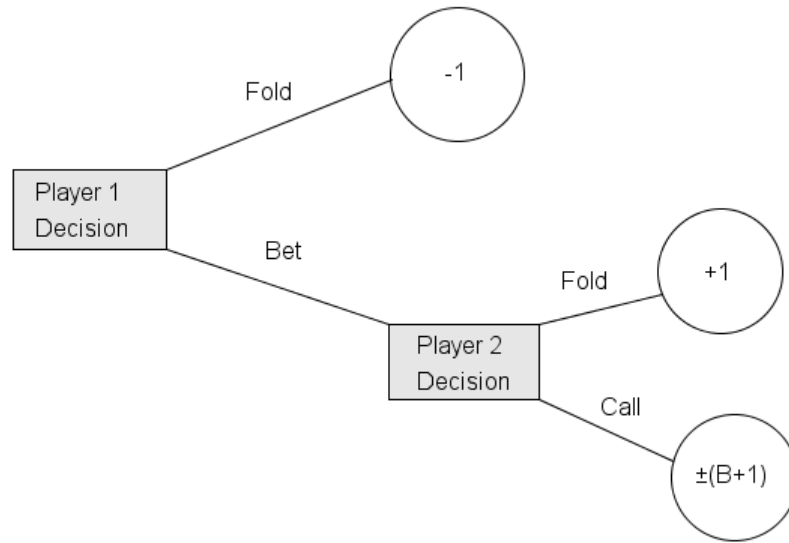


Figure 5.1: La Relance.

when dealing with real cards, as opposed to random number distributions, such cases do occur in real life. In such a scenario, both bets are returned and there is no exchange of money between the players. Borel's poker game favors player 2, given that player 1 must decide to either bet or fold, puts him at a starting disadvantage, he will bet with his strong hands and will only lose to a better hand with a small probability, but at the same time he will not get called often by a worse holding. As a result, player 1 will only gain just the ante [FF03].

The following result is attributed to Borel.

Borel's Theorem: The value of La Relance is:

$$V(B) = -\frac{B^2}{(B+2)^2}.$$

The unique optimal strategy for player 2 is to call if $Y > c$ and to fold otherwise, where

$$c = \frac{B}{B+2}$$

An optimal strategy for player 1 is to bet if $X > c^2$ and to fold otherwise.

Showing that these strategies are optimal can be done using the *principle of indifference*. This is a point where the expected value of given choices is at equilibrium.

Players can choose either course of action without affecting the overall value of the game. Suppose then, that player 2 chooses the point c in such a way as to make player 1 indifferent between folding and betting. If player 1 bets with a hand where $X < c$ he will win 2 units when $Y < c$ (since player 2 is playing optimally and he will fold), and will lose B when player 2 has a hand $Y > c$. Player 1 will have a hand $X > c$ some percent of the time, the remaining time his hand will be in the interval $1 - c$. Now, if player 1 folds, he will win nothing. Player 1 is then indifferent at the point c when $2c - B(1 - c) = 0$. Then,

$$\begin{aligned} 2c - B(1 - c) &= 0 \\ 2c - B + BC &= 0 \\ 2c + BC &= B \\ c(2 + B) &= B \\ c &= \frac{B}{B + 2}. \end{aligned}$$

Example 5.2.1 Suppose two players play with \$1 antes and \$5 bets, who does this game favor by Borel's Theorem?

The value of the game is

$$V(5) = -\frac{5^2}{(5 + 2)^2} = -\frac{25}{49} = -0.51.$$

Since the game is shown from player 1's perspective, and the value of the game is negative, this game favors player 2. With these specific antes and bets, player 1 will lose 51 cents on average, that is, the expected value for player 1 is $-\$0.51$.

In the game above, as stated before, player 1 starts at a disadvantage. By the rules of the game, player 1 must choose whether to bet or to fold his hand. The pot consists of \$2 worth of antes, and player 1 must risk \$5 in order to win \$1, since he will only gain his opponent's ante. If player 2 calls, player 1 stands to win \$6 from his opponent, but if he loses, player 1 will lose his \$1 ante plus his \$5 bet. Player 1 has the option to fold at the beginning of the game, giving up his ante to player 2, which means he still loses \$1. Regardless of how much the antes and the bets are worth, the value of the game for player 1 is always negative, thus, this is a game that we should not play as player 1. For the given values in Example 5.2.1, the optimal strategy for player 1 is to

bet if $X > c^2$, or to fold otherwise. So when $B = 5$

$$c = \frac{B}{B+2} = \frac{5}{7} = 0.71.$$

Then

$$c^2 = \left(\frac{5}{7}\right)^2 = \frac{25}{49} = 0.51.$$

The optimal strategy for player 2 is to call if $Y > c = 0.71$ and to fold otherwise. Let's take a closer look at the implications that these numbers have on our game. Player 1 will receive a hand from the interval $[0,1]$, if his number is lower than 0.51 he will fold automatically, losing his ante to player 2. Now, there will be many instances when player 2's number is less than 0.51, if we assume equal distribution, 51% of the time it will be lower and 49% of the time it will be higher.

For an extreme example, suppose player 1 draws a 0.50, he has to fold to player 2, even knowing that player 2 will have half of the time a number even lower than 0.50. Now, if player 1 draws a number higher than 0.51 he will bet, and now player 2 must call if and only if his own number is greater than 0.71. Thus player 1 will still lose, say when he draws anything between 0.52 and 0.71, but also when both players have numbers greater than 0.72 but player 2's number is greater than player 1's own number. Now consider what happens when player 1's number is greater than 0.51, he will bet, and since player 2 is folding anything below 0.71, player 1 will only gain \$1 from the ante. The best case scenario will be for player 1 to bet a number greater than 0.72, to have player 2 call, with his own number greater than 0.72, and to have player 1's number be greater than player 2's number, in which case, player 1 will win \$6. However, since we are talking about the top 28% of the hands, this will not frequently happen.

If we slightly change Borel's game, say to \$1 antes and \$4 bets, we obtain that the value of the game is -0.44 , $c = 0.66$ and $c^2 = 0.44$. Now player 1 will bet when his number is higher than 0.44, and player 2 will call only when his number is higher than 0.66. Player 1's expected value is -0.44 which means he will lose less overall than when the bet sizing was \$5. From this simple adjustment we can derive the following conclusion:

$$\lim_{B \rightarrow \infty} \frac{B}{B+2} = 1.$$

The importance of this statement is that the lower the bet sizing, the more risk we can take with our poker hand, since we will lose less money overall. We can

introduce at this point the concept of *bluffing*, the act of making a bet with a hand that is mathematically unlikely to be the best at showdown in the hopes of winning the pot by making our opponent fold a better hand. In particular, assume $B = 20$, then $c = 0.90$, and $c^2 = 0.82$. As player 1 we should be betting when our hand is greater than 0.82, but since we know that player 2 can only call with the top 10% of his hands, we can bluff, that is, we could bet a hand significantly lower than 0.82, knowing that player 2 can only call with the very best of his hands. This is one way that we can balance a game that may be unfavorable to us. However, Borel's poker is not as suitable to bluffing as real poker is. The key idea from this argument is that as there is more money on the line, we can bluff less, as we will see in the following model, and yet, when a bluff is successful, it will have a greater expected value. It is important to bluff, even if we lose, to keep our opponents guessing as to the strength of our hands, otherwise, we become predictable, and by game theory, we can become exploitable if we do not change our strategy.

5.3 von Neumann's Poker

In von Neumann's poker model, there exists a small difference that actually has a huge impact on the way the game is played. For this model, if player 1 does not bet the pot, he does not surrender his ante, instead, the hands are compared just like in Borel's model after player 1 bets and player 2 called. In this case, player 1's options are to check his hand or to bet his hand (Figure 5.2).

It is now player 1 who has the unique advantage in this game, and his optimal strategy is to bet for some $X < a$ or $X > b$ where $a < b$ and to check otherwise. That is we will bet our strongest hands and will bluff with our weakest, checking everything in between and still making a profit. The optimal strategy for player 2 is now to call if and only if $Y > c$ for some number c . Thus by observation we conclude that $0 < a < c < b < 1$ (Figure 5.3) [FF03].

The following result is attributed to von Neumann:

von Neumann's Theorem: The value of the von Neumann poker model is

$$V(B) = \frac{B}{(B+1)(B+4)}.$$

An optimal strategy for player 1 is to check if $a < X < b$ and to bet otherwise

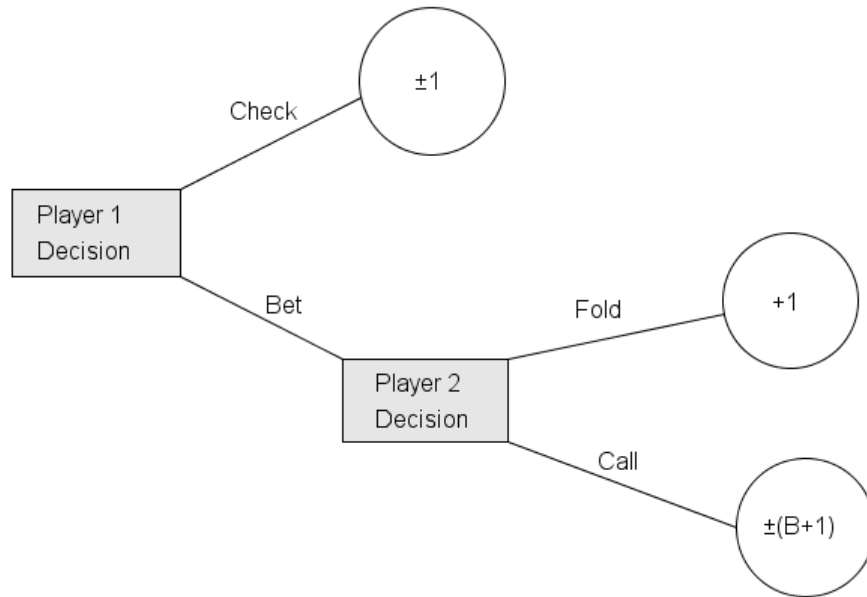


Figure 5.2: von Neumann's Poker Model.

where,

$$a = \frac{B}{(B+1)(B+4)} \quad \text{and} \quad b = \frac{B^2 + 4B + 2}{(B+1)(B+4)}.$$

An optimal strategy for player 2 is to call if $Y > c$ and to fold otherwise where,

$$c = \frac{B(B+3)}{(B+1)(B+4)}.$$

Example 5.3.1 Suppose two players play with \$1 antes and \$2 bets, who does this game

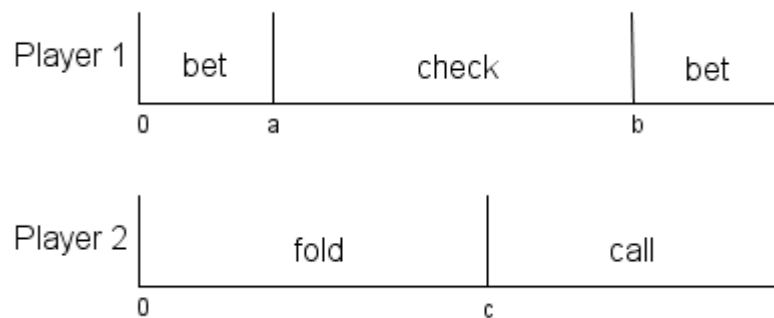


Figure 5.3: von Neumann's Poker Intervals.

favor by von Neumann’s Theorem?

The value of the game is

$$V(2) = \frac{2}{(2 + 1)(2 + 4)} = \frac{2}{18} = 0.11.$$

Since the game is shown from player 1’s perspective, this game favors player 1. As a matter of fact, the value of the game is always positive for player 1 regardless of the bet sizing.

The poker model from von Neumann reflects real poker more closely than Borel’s poker model. Since player 1 is allowed to check his hand, without betting, and player 2 checks back, then both hands are compared. Essentially, player 1 is not penalized for not betting and can still win the hand. The next question to answer for the example above is, what are the ranges for a , b , and c ? Since $B = 2$ we have,

$$a = \frac{B}{(B + 1)(B + 4)} = \frac{2}{(2 + 1)(2 + 4)} = \frac{2}{18} = 0.11,$$

$$b = \frac{B^2 + 4B + 2}{(B + 1)(B + 4)} = \frac{2^2 + 4(2) + 2}{(2 + 1)(2 + 4)} = \frac{14}{18} = 0.77,$$

$$c = \frac{B(B + 3)}{(B + 1)(B + 4)} = \frac{2(2 + 3)}{(2 + 1)(2 + 4)} = \frac{10}{18} = 0.55.$$

Hence, the optimal strategies for player 1 is to bet as a bluff when his number is lower than 0.11, and to bet for value when his number is greater than 0.77. Player 1 must also check every hand when $0.11 < X < 0.77$ as those hands could win on their own some percentage of the time without having to risk a bet of B . The optimal strategy for player 2 is to call a bet if and only if his number is greater than 0.55 and to fold everything else.

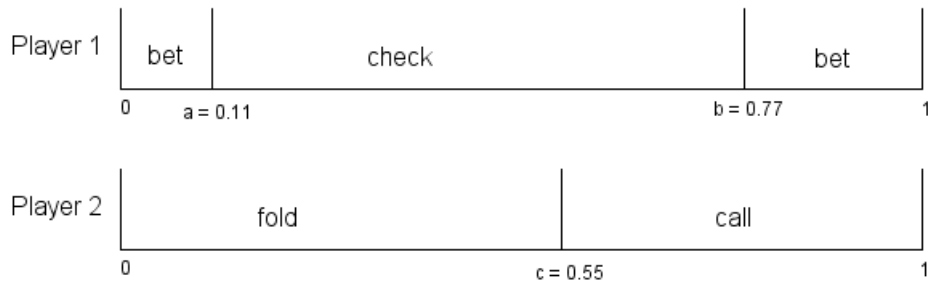


Figure 5.4: Example 5.3.1 Intervals.

Ferguson proves optimal ranges using the principle of indifference, the point where a player is indifferent between his available choices. That is to say, the expected value of both choices is in equilibrium. We have that player 2 should be indifferent between calling and folding when $Y = c$. At this point, if player 2 folds, he wins nothing. However, when calling, player 2 will win $B + 2$ if $X < a$ and will lose B if $X > b$. Thus the expected value of calling versus folding at the point c is $a(B + 2) - B(1 - b) = 0$. Now, player 1 should be indifferent between betting and checking at the point $X = a$. If $Y < a$, player 1 wins 2, when he checks, and nothing otherwise, thus the return at this point is simply $2a$. If player 1 bets at $X = a$, he wins 2 when $Y < c$ and loses B when $Y > c$. The expected value at point a when indifferent between betting and checking is $2c - B(1 - c) = 2a$. Similarly, player 1 is indifferent between checking and betting at $X = b$. If player 1 checks, he will win 2 when $Y < b$. If player 1 bets, he will win 2 if $Y < c$ and will also win $B + 2$ if $c < Y < b$. Lastly, player 1 will lose B when $Y > b$. Then the expected value at point b accounting for the indifference factor is given by $2c + (B + 2)(b - c) - B(1 - b) = 2b$. The last indifference equation can be simplified by solving for B .

$$\begin{aligned}
 2c + (B + 2)(b - c) - B(1 - b) &= 2b \\
 2c + Bb - Bc + 2b - 2c - B + Bb &= 2b \\
 2Bb - Bc - B &= 0 \\
 B(2b - c - 1) &= 0 \\
 2b - c - 1 &= 0.
 \end{aligned}$$

The last line takes into consideration that B is a constant and we divided both sides of the equation by B . Thus we have created a system of three equations in three unknowns, a, b, c . Here a, b, c will give us the indifference points. Then,

$$\left\{ \begin{array}{l} a(B + 2) - B(1 - b) = 0, \\ 2c - B(1 - c) = 2a, \\ 2b - c - 1 = 0. \end{array} \right.$$

Solving the systems of equations for a, b , and c respectively yields the indifference

equations:

$$\begin{aligned} a &= \frac{B}{(B+1)(B+4)}, \\ b &= \frac{B^2 + 4B + 2}{(B+1)(B+4)}, \\ c &= \frac{B(B+3)}{(B+1)(B+4)}, \end{aligned}$$

which are exactly the equations given on von Neumann's Theorem as the optimal strategies for players 1 and 2 [FF03].

5.4 von Neumann's Poker Extension 1

There is a reason why in Example 5.3.1 we let $B = 2$. By previous analysis we know that as B grows large, the bluffing range shrinks towards 0, and the value betting range approaches 1. Since we said that antes are \$1, and each player must pay the ante before the game, this type of game is called *pot limit poker*, where the size of the bet cannot exceed what is already in the pot. So for pot limit poker, there is initially \$2 worth in antes, and the maximum allowable bet is \$2. Pot limit poker allows us to calculate easy, small numbers, and still get all of the benefits of analyzing the game by game theory.

We extend von Neumann's poker by allowing player 2 to bet his own hand in case that player 1 checks first. In this game, player 1 has two options available to him, bet some amount $B > 0$, or check. If player 1 bets, player 2 responds by folding or by calling. If player 1 checks, player 2 can either bet or check his hand. If player 2 bets, player 1 now may call or fold. In the case that either player folds, the pot is awarded to the other player, in the case that the actions are completed, then the hands are compared and the highest hand is declared the winner and awarded the pot. This will happen when either both players check or when there is a bet from either player, followed by a call from the opposing player. Optimal strategies are found by subdividing the intervals $X \in [0, 1]$ and $Y \in [0, 1]$ for all of the appropriate actions taken by each player. From the diagram (Figure 5.6) we may assume the following relations hold: [FFG07].

$$0 < a < b < c < 1, \quad 0 < e < f < 1, \quad a < e < b < f < c, \quad \text{and} \quad a < d < c.$$

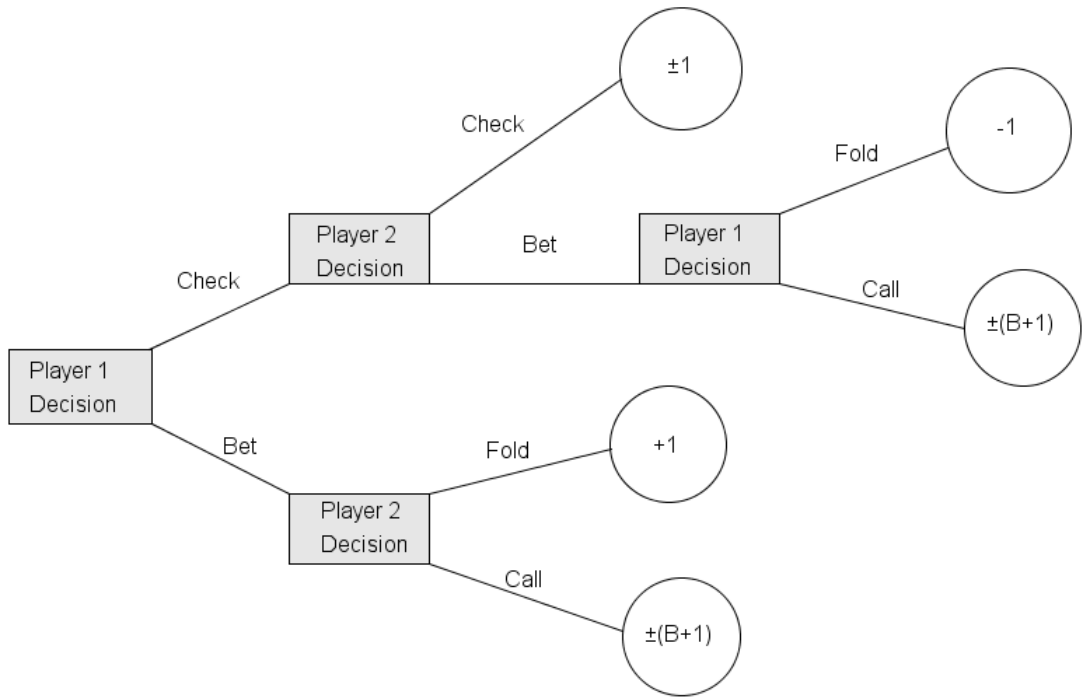


Figure 5.5: von Neumann's Poker Extension 1.

We continue to assume that both players will play an optimal sound, game theoretical strategy. Depending on the action, both players will bet with their respective low hands as a bluff, and will also bet their respective top hands for value, checking,

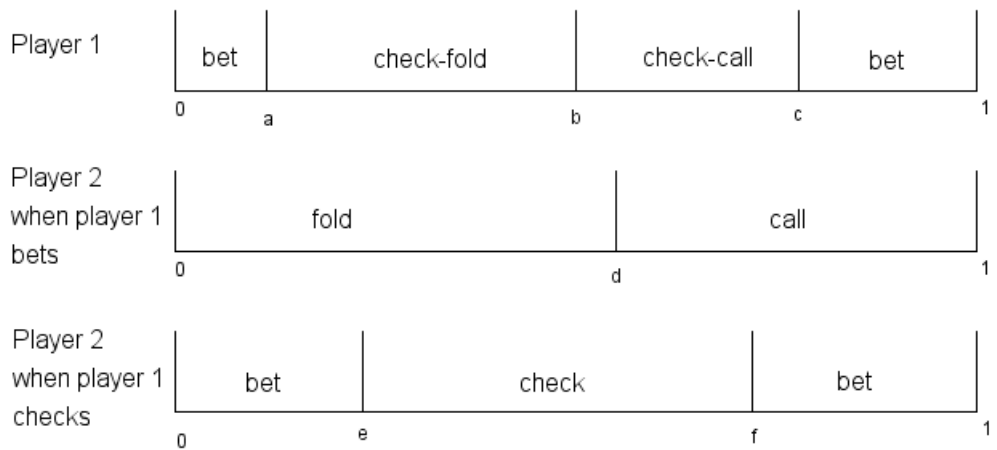


Figure 5.6: von Neumann's Poker Extension 1 Intervals.

calling, or folding everything else in between, as any other strategy would be a mistake. Ferguson finds the indifference equations for this game, the point where any option has the same value, is exactly at the boundary of all of these intervals. That is, for our pot limit poker game where $B = 2$:

1. For player 1 to be indifferent at a : If player 1 bets at $X = a$, he wins 2 with probability d and loses B with probability $1 - d$. If player 1 check-folds at $X = a$, he wins 0. Player 1 is indifferent if these two equations are equal, namely if $2d - B(1 - d) = 0$, we find $d = \frac{B}{B+2}$. This requires that $a < d$ and $a < e$.
2. For player 1 to be indifferent at $X = b$: If player 1 check-folds at $X = b$, he wins 2 with probability $b - e$ and nothing otherwise. If player 1 check-calls, he wins $B + 2$ with probability e , he wins 2 with probability $b - e$ and loses B with probability $1 - f$. Equating expectations gives $2(b - e) = e(B + 2) + 2(b - e) - B(1 - f)$ or simplifying to $e(B + 2) + Bf = B$. This requires $e < b$ and $b < f$.
3. For player 1 to be indifferent at $X = c$: If player 1 check-calls at $X = c$, he wins $B + 2$ with probability $e + (c - f)$, he wins 2 with probability $f - e$ and loses B with probability $1 - c$. If player 1 bets, he wins 2 with probability d , he wins $B + 2$ with probability $c - d$ and loses B with probability $1 - c$. Equating expectations gives $d + e - f = 0$. This requires $f < c$ and $d < c$.
4. For player 2 to be indifferent at $Y = d$: If player 2 folds to a bet at $Y = d$, he wins nothing. If player 2 calls a bet at $Y = d$, he wins $B + 2$ with probability $\frac{a}{a+1-c}$ and loses B with probability $\frac{1-c}{a+1-c}$. Equating the expectation to zero gives $a(B + 2) + Bc = B$. This requires $a < d < c$.
5. For player 2 to be indifferent at $Y = e$: If player 2 bets at $Y = e$, he wins 2 with probability $\frac{b-a}{c-a}$ and loses B with probability $\frac{c-b}{c-a}$. If player 2 checks at $Y = e$, he wins 2 with probability $\frac{e-a}{c-a}$. Equating expectations gives $b(B + 2) - Bc - 2e = 0$. This requires $a < e < b$.
6. For player 2 to be indifferent at $Y = f$: If player 2 checks at $Y = f$, he wins 2 with probability $\frac{f-a}{c-a}$. If player 2 bets at $Y = f$, he wins 2 with probability $\frac{b-a}{c-a}$, he wins $B + 2$ with probability $\frac{f-b}{c-a}$ and loses B with probability $\frac{c-f}{c-a}$. Equating expectations gives $b + c - 2f = 0$. This requires $b < f < c$.

Solving these six equations in six unknowns gives:

$$a = \frac{2B}{(B+2)^2(B+1)} \quad b = \frac{B}{B+2} \quad c = \frac{B(B+3)}{(B+2)(B+1)}$$

$$d = \frac{B}{B+2} \quad e = \frac{B}{(B+1)(B+2)} \quad f = \frac{B}{B+1}$$

[FFG07].

Example 5.4.1 Using von Neumann's poker extension 1, calculate the optimal strategies for each hand range given that two players are playing with \$1 antes and the game is pot limit poker.

Since the game is pot limit poker with \$1 antes, we have that $B = 2$. We must find the optimal strategies of the six equations provided in the extension. Then,

$$a = \frac{2(2)}{(2+2)^2(2+1)} = \frac{4}{(16)(3)} = \frac{1}{12} = 0.08.$$

$$b = \frac{2}{2+2} = \frac{2}{4} = 0.5.$$

$$c = \frac{2(2+3)}{(2+2)(2+1)} = \frac{10}{12} = 0.83.$$

$$d = \frac{2}{2+2} = \frac{2}{4} = 0.5.$$

$$e = \frac{2}{(2+1)(2+2)} = \frac{2}{12} = 0.16.$$

$$f = \frac{2}{2+1} = \frac{2}{3} = 0.66.$$

The optimal strategies are given for all hands that fall into these intervals. Player 1 must bet when his number is lower than 0.08 as a bluff, player 2 can only call him when his number is greater than 0.5. In this scenario, when player 1 receives a hand in the bottom of his range, he can get away with running a bluff half of the time, and even if he gets called and lose, it will show player 2 that he is not just playing top premium hands, and may induce a mistake from player 2 in the future. Player 1 must also bet when his number is greater than 0.83, as those hands are way too strong for him not to bet. Player 2 will still fold half of his hands and call with the other half. There could be instances in real life when player 2 beats player 1, but here, we have assumed that $d < c$. Player 1 will check everything between the interval $a < c$ and will await for player 2 to make his choice. When player 1 checks, player 2 must bet the bottom 0.16 of his range as a bluff. Now player 1 can only call when his number is between 0.5 and 0.83, and will fold when

his number is between 0.08 and 0.5. Also, when player 1 checks, player 2 must bet when his number is greater than 0.66, with player 1 still reacting as before. Lastly, if player 1 checks, and player 2's number is between 0.16 and 0.66, player will check as well, and the hands will be compared.

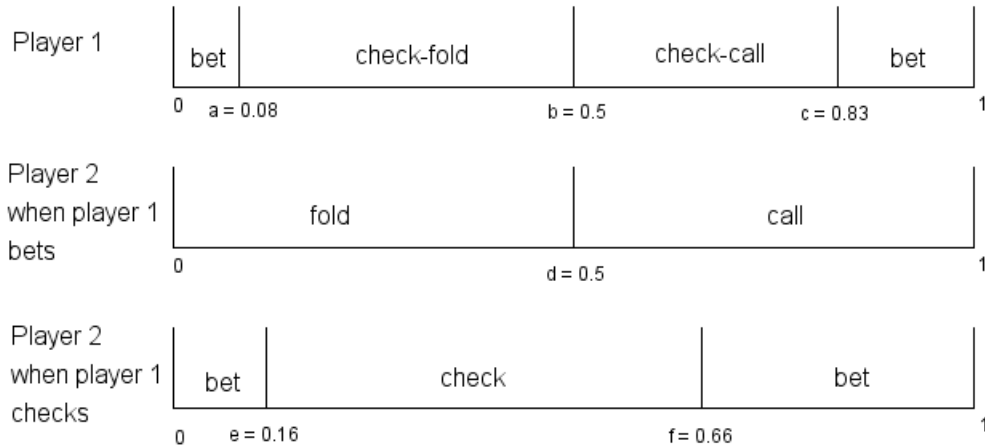


Figure 5.7: Example 5.4.1 Solutions.

The last question to answer is if this new game is favorable to player 1 or to player 2? The value for this extension is given by:

$$V(B) = -\frac{B^2}{(B + 1)(B + 2)^2}$$

Since the value of the game is negative, this game is unfavorable to player 1. In the case of pot limit poker, the value of the game to player 1 is -0.08 .

5.5 von Neumann's Poker Extension 2

In this next extension we will remove player 2's ability to bet his hand if player 1 checks into him. However, we will give player 2 a third option if player 1 bets into him. Instead of simply folding or calling, we will also allow player 2 to raise by some value $R > 0$. In the case of pot limit poker analysis, the antes are still \$1, the bets are \$2, and the raise R will equal the maximum pot size at the moment of \$6. This takes into account that the pot is has \$2 in antes, plus \$2 each from each player, (per pot limit poker rules, the raising player must put in the call first, and then the raise is allowed).

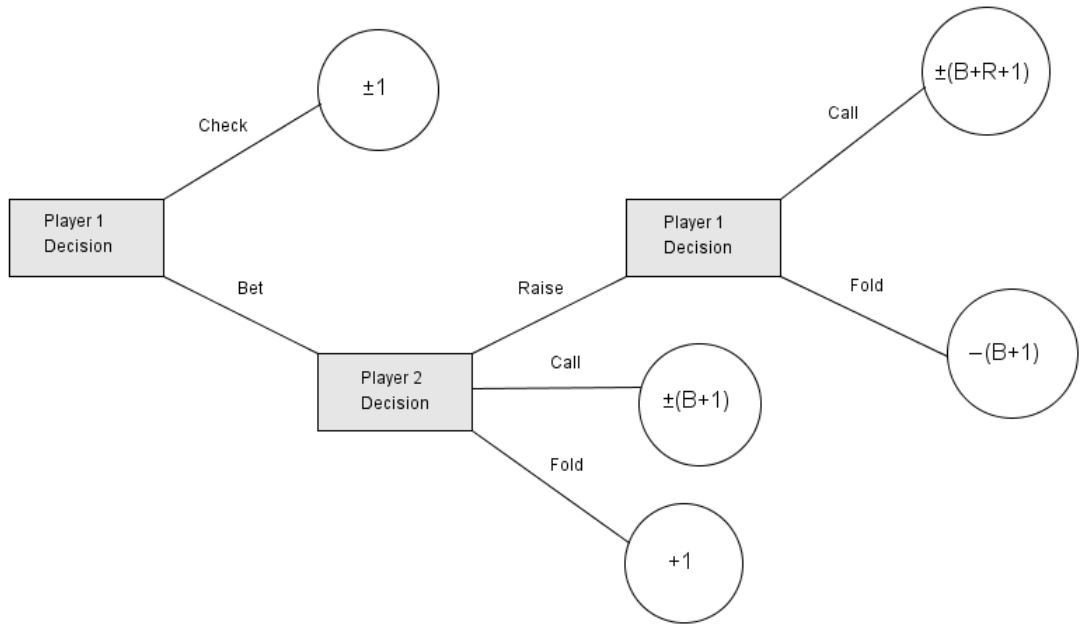


Figure 5.8: von Neumann's Poker Extension 2.

Taking away the betting option from player 2, and allowing him to raise, gives us now different ranges for the optimal strategies. It is now possible for player 1 to check, bet, bet-call, and bet-fold. For player 2, besides the previous option to call or fold, he can now raise. Thus, the new interval ranges look as follow:

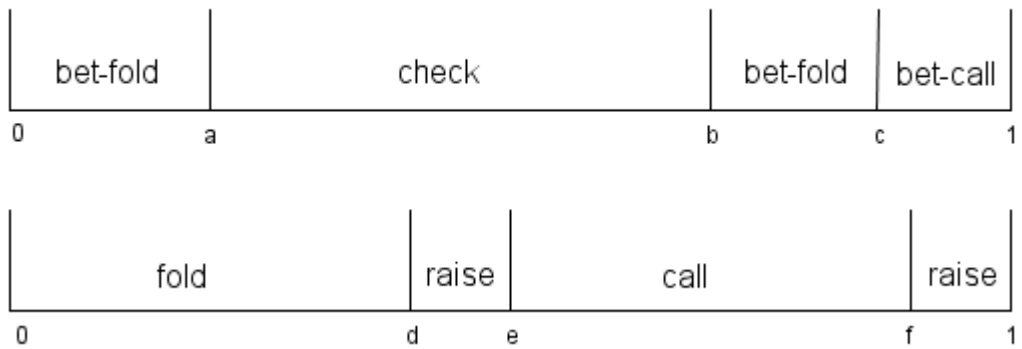


Figure 5.9: von Neumann's Poker Extension 2 Intervals.

In obtaining the indifference equations we must assume that

$0 < a < e < b < c < f < 1$ and $0 < d < e$. In this case, we can allow $d > a$ or $d < a$. We can find the indifference equations as done previously in the first poker extension of von Neumann's poker model, by analyzing what can happen at each indifference point and calculating the expected value of both actions and setting them equal to each other. In short we have the following results from Ferguson's work: [FFG07].

1. Player 1 is indifferent at a when $-2a - d(B + 2) = B$.
2. Player 1 is indifferent at b when $2Bb + d(B + 2) - 2e(B + 1) = B$.
3. Player 1 is indifferent at c when $-d(2B + R + 2) + e(2B + R + 2) + Rf = R$.
4. Player 2 is indifferent at d when $a(B + 2) - b(B + 2) + c(2B + R + 2) = B + R$.
5. Player 2 is indifferent at e when $-2b(B + 1) + c(2B + R + 2) = R$.
6. Player 2 is indifferent at f when $f = \frac{c+1}{2}$.

Solving these six equations in six unknowns gives:

$$a = \frac{B^2(2B + R + 2)^2}{(B + 1)\Delta}, \quad b = 1 - a\frac{B + 2}{B}, \quad c = 1 - \frac{2B(B + 2)(2B + R + 2)}{\Delta},$$

$$d = \frac{a2 + B}{B + 2}, \quad e = \frac{B}{B + 1} - a, \quad f = 1 - \frac{B(B + 2)(2B + R + 2)}{\Delta},$$

here

$$\Delta = B(B + 4)(2B + R + 2)^2 + R(B + 1)(B + 2)^2.$$

The value of the game is given by

$$V(B, R) = \frac{B^2(2B + R + 2)^2}{(B + 1)(B(B + 4)(2B + R + 2)^2 + R(B + 1)(B + 2)^2)}.$$

Since the value of the game is positive, this game favors player 1. For pot limit poker we have $B = 2$ and $R = 6$ which gives an expected value of \$0.09 to player 1. The solutions to the six equations above are:

$$\begin{aligned} a &= 0.9, & b &= 0.81, \\ c &= 0.90, & d &= 0.54, \\ e &= 0.57, & f &= 0.95. \end{aligned}$$

Optimal strategies will depend on our hand number, and actions will be carried as outlined in Figure 5.9 for this game.

Chapter 6

Game Theory of Poker

6.1 Half-Street Games

We are now ready to analyze poker games with actual cards as opposed to random distributions. Half-street games are, for all intents and purposes, poker models. Chen and Ankenman define the characteristics of these games as follow:

1. The first player will check in the dark. No betting from him. This also implies that player 1 will not know the strength of his hand until after player 2 has acted.
2. The second player then will have the option to either check or bet. If both players check, there is a showdown and the hands are compared to determine a winner. Player 2 can bet any amount, there are no restrictions on the betting size.
3. If the second player bets, the first player has the option to call or fold. If player 1 calls, then the cards are compared at showdown, and the best hand determines the winner.

At this point, we will also introduce the concept of *ex-showdown*. This is the value of the game where money exchanges hands as a result of a bet only. Antes are considered sunk cost, since each player must pay in order to play, we will not consider them as part of the calculations. Betting, on the other hand, is done voluntarily, just like calling, hence, that money will be taken into consideration. Also, ex-showdown values include the swing of a pot that exchanges owner as a result of a successful bluff. In this case, the pay-off will be considered as a bonus (the antes), since the player was not supposed to win that pot regularly by just hand strength alone.

6.2 The Clairvoyance Game

The first game we will explore is known as the Clairvoyance Game. In this game, player 2 will be clairvoyant, that is, not only does he know the value of his hand, but he also knows the value of his opponent's hand. Per the rules of the game, player 1 will automatically check, and player 2 will then decide whether to bet or check. In this game player 2 has a tremendous advantage over player 1. When player 2's hand is stronger than player 1's hand, player 2 can bet for value, extracting an extra bet from player 1 when he calls. If player 1's hand is better, player 2 can simply check and lose just the ante. Most importantly, player 2 should be able to beat player 1 out of some medium strength hands, either with a bluff, or with a bet that player 1 can call. In this game player 2 should never make a losing bet.

The pay-off matrix for this game is as follows:

Table 6.1: The Clairvoyance Game

	Player 1	Check-Call	Check-Fold
Player 2			
Strong	Bet	+1	0
	Check	0	0
Bluff	Bet	-1	+p
	Check	0	0

In the Clairvoyance Game, player 2 can only lose when he tries to bluff player 1 but gets called. Since player 2 can see what player 1 has at all times, player 2 should try to induce player 1 to fold. If player 1 calls with frequency c , and folds with frequency $1 - c$, then player 2 is indifferent between checking and bluffing. When player 2 bluffs successfully he will win the pot p when player 1 folds, and will lose his bet when player 1 calls. The expected value at this indifference point is then given by (pot size)(frequency

player 1 folds)=(bluff bet)(frequency player 1 calls). Then

$$\begin{aligned} p(1 - c) &= 1(c) \\ p - pc &= c \\ p &= c + pc \\ p &= c(1 + p) \\ c &= \frac{p}{p + 1}. \end{aligned}$$

Similarly, player 1 must be indifferent between calling and folding. When player 1 calls a value bet from player 2 he loses his bet, but gains $p + 1$ when he calls a bluff. This equilibrium point is given by

$$\begin{aligned} 1 &= b(p + 1) \\ b &= \frac{1}{p + 1}, \end{aligned}$$

where b is the ratio of player 2 bluffing [CA06].

Comparing the bluffing and calling ratios of $b = \frac{1}{p+1}$ and $c = \frac{p}{p+1}$ we observe that as the pot grows larger, the bluffing ratio approaches 0, while the calling ratio approaches 1. This result reinforces the idea that when there is more money in the pot, a successful bluff is more profitable. Although bluffing is a common strategy, it must be used sparingly. On the other side of the game, as the pot grows larger, player 1 must call more and more of his hands. This serves two purposes, first, player 1 cannot willingly give up his equity on the pot, specially if such a pot is large, and two, it will keep player 2 from bluffing in the future since he knows that player 1 is not afraid to call.

Example 6.2.1 What are the calling and bluffing ranges for both players if there are \$1 antes and bet sizes of p ?

The pot initially consists of \$2 worth of antes, one from each player. This is considered sunk cost and should not be part of the equations. Instead, the units in the equation are considered to be 1 ante, but the ante can be worth more than \$1. We only need to concern ourselves with the value of p , which can be chosen at any moment by player 2. Suppose then, that after receiving their hands player 2 decides to bet \$10. The bluffing range is $b = \frac{1}{p+1} = \frac{1}{11} = 0.09$ and the calling range is $c = \frac{p}{p+1} = \frac{10}{11} = 0.90$. In this example, player 2 will bluff with the bottom 9% of hands, and player 1 will call with 90% of the hands.

Let's explain what the game above demonstrates. Player 2 will bluff with the bottom of the hand spectrum, but player 1 will call him with almost anything. It seems that player 2 will be making a bad bet almost always. However, let's remind ourselves that player 2 is clairvoyant, player 2 will not make a bad bet at all. Player 2 can simply check when he is beat, and lose the ante. Now, player 2 can run some exciting bluffs by having this information, say he bets \$100, the calling and bluffing ranges are now 99% and $< 1\%$. Since player 2 knows the strength of player 1's hand, he can tailor his bet sizing in order to confuse player 1. A large bet surely signifies that player 2 wants to make more money, otherwise, why would he bet so much? Unless, the bet is specifically designed to make player 1 fold a better hand. Here we have a classical example of game theory applied to poker. The intentions of both players is to make money, the means of how they achieve the result is entirely up to them. Players can make honest bets, that is, only betting strong hands. However, players often must employ the bluff, so that their strategies do not become exploitable in the future. Of course, if players fold to a bet, they are left guessing whether the bettor was strong or weak. Player 2 should now find his own mixed strategy to use against player 1, he should bet big as a bluff some percent of the time, and should also bet big as a value bet the remaining percent of the time. It will be player 1's job to decipher this strategy in order to develop his own counter-strategy. Player 2 should also bet small as a bluff some percent of the time, and as value bet the remaining percent of the time. This is the safest manner in which to keep the opponent guessing, since bet sizing could be in itself a telling of a certain strategy.

It is important to note that the numbers above are of actual hand values, and not of uniform distributions like those covered in Chapter 5. Recall that Figure 2.1 displays all possible combinations of 5-card hands of poker. From this table we can see that over half of the possible combinations are made up of "high card" only, and 42% are one pair type hands. Less than 1% of the hands are straights or higher, with the rarest of combinations, four-of-a-kind or higher barely making up the top 0.02%. Since this chapter deals with actual cards and combinations found in poker, how can we possibly subdivide an interval into 2,598,960 pieces, one for each hand? Furthermore, we know from previous work that some hands can be equal in worth, a royal flush of spades does not beat a royal flush of hearts.

6.3 The AKQ Game

One of the principal ideas in solving mathematics is to reduce a problem to a simpler, easier problem. In this manner, we can reduce the game of poker from having 52 cards, which will give us almost 2.6 million combinations, into a simpler 3-card deck, in which we can only receive one card and therefore, only one combination is possible. This was the game proposed by Mike Caro, a pioneer poker theorist, in 1995 [CA06]. In this game suppose that there are only 3 cards, the Ace, the King, and the Queen (AKQ), where $A > K > Q$. Players will randomly receive one card from this deck without replacement. Players are not clairvoyant, but can deduce the strength of their opponents hand by judging their own hand. This is the first time so far that we have encountered *card removal*, an idea not explored in any of the previous games. For example, I will be dealt the K one-third of the time, which implies that my opponent will have either the A or the Q , but not another K . If we assume random distribution, my opponent will have each card exactly one-half of the time. In this game, player 1 will either check or bet, and player 2 will react by calling or folding. The ex-showdown pay-off matrix illustrates this game, where the grayed-out areas indicate impossible match-ups, since, for example, we cannot have an ace vs. ace confrontation.

Table 6.2: AKQ Game

		Player 2		Ace		King		Queen	
Player 1		Call	Fold	Call	Fold	Call	Fold	Call	Fold
	Ace	Bet			+1	0	+1	0	
Check				0	0	0	0		
King	Bet	-1	+2			+1	0		
	Check	0	0			0	0		
Queen	Bet	-1	+2	-1	+2				
	Check	0	0	0	0				

In this game, each player will receive exactly one card from the deck. As an example, suppose player 1 receives the king and player 2 receives the ace. Player 1 has two options available to him, bet or check. If player 1 bets, then player 2 can either call or fold. If player 2 calls the bet, then player 1 losses one unit, however, if player

2 folds, player 1 will gain 2 units. This is due to fact that $A > K$, and since we are dealing exclusively with ex-showdown values, we have successfully bluffed our opponent into folding a better hand. In the case that player 1 checks, then the hands are compared immediately. In this particular scenario, player 1 will lose, but ex-showdown value only takes into consideration money that exchanges hands as a result of betting, and therefore, checking has a zero expected value for player 1.

As with the previous games, we must find a proper strategy to play in each situation. Before we proceed, we should note that this game contains dominated strategies and those strategies should be removed. For player 1, when he is dealt the ace, his options are to bet or check. In this case betting dominates checking, since each strategy has a greater or equal expected value to every single counter-strategy from player 2. Hence, checking with an ace can be removed from play.

Table 6.3: AKQ Game, Row Reduced

		Player 2		Ace		King		Queen	
Player 1		Call	Fold	Call	Fold	Call	Fold	Call	Fold
Ace	Bet			+1	0	+1	0		
King	Bet	-1	+2			+1	0		
	Check	0	0			0	0		
Queen	Bet	-1	+2	-1	+2				
	Check	0	0	0	0				

Similarly, player 2 can eliminate some options from his strategies. Folding an ace is dominated by calling with an ace. Also, calling with a queen is dominated by folding with a queen. Both of those strategies can be removed from play (Table 6.4).

In the final step, we can now see that for player 1, checking with a king dominates betting with the king. So we remove this strategy. Note that this play was not obvious before (Table 6.5).

The new matrix should resemble a real poker game. We know that we need to maximize our expected value, as player 1, it should be clear that betting an ace is always a better option than checking with the ace. Player 2 will fold every queen that he has when faced with a bet by player 1 since the queen does not beat anything. Player 2 should never fold an ace of his own, since it beats everything. But what about the other

Table 6.4: AKQ Game, Row/Column Reduced

	Player 2	Ace	King		Queen
Player 1		Call	Call	Fold	Fold
Ace	Bet		+1	0	0
King	Bet	-1			0
	Check	0			0
Queen	Bet	-1	-1	+2	
	Check	0	0	0	

cards? What is the best way to play them?

Player 1 will always bet with an ace, and he also needs to keep player 2 indifferent between calling and folding, so player 1 needs to bluff with his queen some percent of the time. The king for player 1 has also an automatic decision. If we bet, we will only get called by the ace, since player 2 is playing optimally he will fold all of his queens, in which case we lose to the ace every time. We already know by the matrix that checking with the king is the best option, so we may disregard this decision and omit it from the matrix. Likewise, folding a queen for player 2 is an automatic decision, and we may remove this strategy as well from play. This leads to the matrix solution on the following page (Table 6.6).

The non-dominated strategies are for player 1 to bluff with queens some percent of the time, and for player 2 call with kings some percent of the time. Optimal strategies can be found by analyzing the frequency of which these match-ups occur, and calculating the expected value for each. We can label the strategies as follows:

Table 6.5: AKQ Game, Simplified

	Player 2	Ace	King		Queen
Player 1		Call	Call	Fold	Fold
Ace	Bet		+1	0	0
King	Check	0			0
Queen	Bet	-1	-1	+2	
	Check	0	0	0	

Table 6.6: AKQ Game, Matrix Solution

	Player 2	Ace	King	
Player 1		Call	Call	Fold
Ace	Bet		+1	0
Queen	Bet	-1	-1	+2
	Check	0	0	0

1. Player 1 bluffs with queens.
2. Player 1 checks with queens.
3. Player 2 calls with kings.
4. Player 2 folds with kings.

Then the possible match-ups are:

$$1 \text{ vs. } 3, \quad 1 \text{ vs. } 4, \quad 2 \text{ vs. } 3, \quad 2 \text{ vs. } 3.$$

We can find the expected value of each confrontation as follows:

- 1 vs 3: When player 1 has an ace, he bets and gets a call from the king, winning 1 unit, and nothing from the queen. When player 1 has the king no betting occurs, and there is no exchange of money. When player 1 has the queen, he bluffs and gets a call from both the ace and the king, losing one unit in each case. Since player 1 holds each card $\frac{1}{3}$ of the time, player 2 will have the other cards $\frac{1}{2}$ of the time. Then the expected value for this match-up is

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(1) + \frac{1}{3}\left(\frac{1}{2}(-1) + \frac{1}{2}(-1)\right) = -\frac{1}{6}.$$

- 1 vs. 4 When player 1 has an ace, he bets but gains nothing from the king and nothing from the queen. When player 1 has the king no betting occurs, and there is no exchange of money. When player 1 has the queen, he bluffs and gets a call from the ace, losing one unit, but he gets the king to fold, and wins 2 units as a result of a successful bluff. Since player 1 holds each card $\frac{1}{3}$ of the time, player 2 will have the other cards $\frac{1}{2}$ of the time. Then the expected value for this match-up is

$$\frac{1}{3}\left(\frac{1}{2}(-1) + \frac{1}{2}(2)\right) = \frac{1}{6}.$$

- 2 vs 3: When player 1 has an ace, he bets and gets a call from the king, winning 1 unit, and nothing from the queen. When player 1 has the king no betting occurs, and there is no exchange of money. When player 1 has the queen, no betting occurs. Since player 1 holds each card $1/3$ of the time, player 2 will have the other cards $1/2$ of the time. Then the expected value for this match-up is

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(1) = \frac{1}{6}.$$

- 2 vs 4: When player 1 has an ace, he bets but gets nothing from the king and nothing from the queen. When player 1 has the king no betting occurs, and there is no exchange of money. When player 1 has the queen, no betting occurs. In this case there is no exchange of money at all, and the expected value from these strategies is 0.

Using the information above we can create a 2 by 2 matrix to display our results.

Table 6.7: AKQ Game, Expected Values

	Player 2 calls kings	Player 2 folds kings
Player 1 bets queens	-1/6	1/6
Player 1 checks queens	1/6	0

We can always multiply by a constant without changing the expected value. We do this primarily to obtain easier, nicer numbers to calculate. In this case let's multiply each entry by -6. The resulting matrix is shown below.

Table 6.8: AKQ Game, Expected Values, Simplified

	Player 2 calls kings	Player 2 folds kings
Player 1 bets queens	1	-1
Player 1 checks queens	-1	0

As before, we can calculate optimal strategies by finding the indifference equations and indifference points for each strategy. Player 1 must make player 2 indifferent between calling and folding with the kings. We let c be the calling ratio and $1 - c$ the folding ratio. Now, when player 1 bluffs, he will get called by the ace every time, where

player 2 holds the ace 1/2 of the time, in which case player 1 loses 1 unit. Player 1 will also get called by the king with frequency c , in which case player 1 losses 1 unit. Player 2 holds the king 1/2 of the time. Also, player 2 will fold the king with frequency $1 - c$ in which case player 1 wins 2 units. The indifference equation when player 1 bluffs is then given by:

$$\begin{aligned} \langle EV \rangle &= \left(\frac{1}{2}\right)(-1) + \left(\frac{1}{2}\right)(c)(-1) + \left(\frac{1}{2}\right)(1 - c)(2) \\ \langle EV \rangle &= -\frac{1}{2} - \frac{1}{2}c + 1 - c \\ \langle EV \rangle &= \frac{1}{2} - \frac{3}{2}c. \end{aligned}$$

The expected value of player 1 checking with the queen is simply 0. Setting these two expected values equal to each other gives:

$$\begin{aligned} \frac{1}{2} - \frac{3}{2}c &= 0 \\ \frac{1}{2} &= \frac{3}{2}c \\ c &= \frac{1}{3}. \end{aligned}$$

Now, player 2 must make player 1 indifferent between bluffing and folding with queen. Let b be the bluffing frequency of player 1, and $1 - b$ the folding frequency. Since player 2 is calling with the king, he will lose 1 unit when player 1 bets the ace, but will win 1 unit when player 1 bluffs with the queen with frequency b . Remembering that if I have any card, my opponent will have either of the other two cards exactly 1/2 of the time, the expected value of this strategy when player 2 calls is:

$$\begin{aligned} \langle EV \rangle &= \left(\frac{1}{2}\right)(-1) + \left(\frac{1}{2}\right)(b)(1) \\ \langle EV \rangle &= -\frac{1}{2} + \frac{1}{2}b. \end{aligned}$$

The expected value of player 2 folding is -2, since he has allowed player 1 to bluff him out of the pot is:

$$\begin{aligned} \langle EV \rangle &= \frac{1}{2}(b)(-2) \\ \langle EV \rangle &= -b. \end{aligned}$$

Setting these two equations equal to each other gives:

$$\begin{aligned} -\frac{1}{2} + \frac{1}{2}b &= -b \\ \frac{3}{2}b &= \frac{1}{2} \\ b &= \frac{1}{3}. \end{aligned}$$

Then the optimal strategy for player 1 is to bluff with the queen 1/3 of the time, and for player 2 to call with the king 1/3 of the time [CA06].

The AKQ game is a prime example of how a complicated game can be solved by breaking it down to its most basic components that still retain the spirit of the real game. For the card playing enthusiast, given the rules of the game, it should be common sense that if I am player 1 and I have the ace, I should always bet since this is the only play that will maximize my expected value. If I have the king, I should never bet since I can never beat a better hand and I will never be called by a worse hand. Lastly, if I have the queen, I will have to bluff sometimes, since I could get my opponent to fold a king on occasions. The question was, how often should I bluff with the queen? Likewise, if I am playing as player 2, given the rules of the game, I should never fold an ace and I should never call with the queen, but how often should I call with the king? As we have demonstrated, the answer for both questions is one-third of the time. If I am player 1 and never bluff, only betting with the ace, then player 2 will quickly realize that he should never call, since a bet will indicate that player 1 has the ace, which he cannot beat. Any other strategy will be counter-productive and exploitable. In this manner, both players play the game optimally from a game theoretical stand point that maximizes expected value for both. The next logical extension to this game is to add a fourth card and perform the same analysis to find out if optimal strategies still exist.

6.4 The AKQJ Game

In this original game extension we add a fourth card to the deck, namely a jack, where $A > K > Q > J$. The rules are the same as in the AKQ game, and the ex-showdown pay-off matrix is shown on the following page.

Before we attempt to solve this game, let's try to give an educated guess at what we expect the solutions to look like. Since we have added another card to the deck,

Table 6.9: AKQJ Game

	Player 2	Ace		King		Queen		Jack	
Player 1		Call	Fold	Call	Fold	Call	Fold	Call	Fold
Ace	Bet			+1	0	+1	0	+1	0
	Check			0	0	0	0	0	0
King	Bet	-1	+2			+1	0	+1	0
	Check	0	0			0	0	0	0
Queen	Bet	-1	+2	-1	+2			+1	0
	Check	0	0	0	0			0	0
Jack	Bet	-1	+2	-1	+2	-1	+2		
	Check	0	0	0	0	0	0		

the strength of the cards will be changed. For example, the king, which we previously discussed as being stuck in the middle of the ace and queen, is now just a bit more powerful. The king can now beat two cards, the queen and the jack, does this imply that we can now bet the king from time to time instead of automatically checking? We also have the queen, which needed to bluff in order to make a profit, however, the queen is not the weakest card any longer, how will the addition of the jack change the bluffing frequency of the queen? Lastly, how often must the jack bluff? Similar questions arise for player 2, he should still call with all the aces, but is folding a queen still automatic? The obvious answer should be no, folding the jack is now automatic, so how often should the queen fold or call? What about the king, should he still call with one-third of the kings, or will this ratio increase or decrease given the new added card?

The first step as always is to find saddle points, and to remove dominated strategies if they exist. Saddle points are not apparent given that the values are repeated for different strategies. Domination, however, exists, and strategies can be removed. For player 1, betting an ace dominates checking with the ace, so we can remove that strategy. Player 2 can remove folding an ace, and calling with a jack. This leads to the simplified matrix on the next page (Table 6.10).

In this game player 1 has an automatic decision, always bet the ace. Player 2 also has automatic decisions, call with all aces and fold all jacks. Thus the non-dominated strategies are for player 1 to bet or check with kings, bet or check with queens, and bet

Table 6.10: AKQJ Game, Reduced

	Player 2	Ace	King		Queen		Jack
Player 1		Call	Call	Fold	Call	Fold	Fold
Ace	Bet		+1	0	+1	0	0
King	Bet	-1			+1	0	0
	Check	0			0	0	0
Queen	Bet	-1	-1	+2			0
	Check	0	0	0			0
Jack	Bet	-1	-1	+2	-1	+2	
	Check	0	0	0	0	0	

or check with jacks. The non-dominated strategies for player 2 are to call or fold kings, and to call or fold queens. It is our job to find the optimal strategies for this game.

Table 6.11: AKQJ Game, Simplified

	Player 2	King		Queen	
Player 1		Call	Fold	Call	Fold
King	Bet			+1	0
	Check			0	0
Queen	Bet	-1	+2		
	Check	0	0		
Jack	Bet	-1	+2	-1	+2
	Check	0	0	0	0

We will define the non-dominated strategies as follows:

- (a) player 1 bets kings, as k_b ,
- (b) player 1 checks kings, as k_{ch} ,
- (c) player 1 bets queens, as q_b ,
- (d) player 1 checks queens, as q_{ch} ,
- (e) player 1 bets jacks, as j_b ,

- (f) player 1 checks jacks, as j_{ch} ,
- (g) player 2 calls kings, as k_{ca} ,
- (h) player 2 folds kings, as k_f ,
- (i) player 2 calls queens, as q_{ca} ,
- (j) player 2 folds queens, as q_f .

There will be 16 different match-ups, and we must find the expected value of each by analyzing the strategies employed by each player. We must keep in mind that checking, and our opponent folding a weaker hand than our own, have an expected value of zero, as such, we can disregard those numbers from our computations. Also, playing optimally dictates that the jack from player 2's hand will never call.

- k_b vs q_{ca} : When player 1 has the ace he bets and gains one unit from both the king and the queen. When player 1 has the king, he bets and loses one unit to the ace but wins one unit from the queen. When player 1 has the queen, he bets losing one unit to the ace and one unit to the king when player 2 calls with the king. If the king folds, then player 1 wins two units. When player 1 has the jack, he bets and loses one unit to every card, but he could win two units if the king folds. In this strategy the queen is always calling. We can write the equation given all probabilities as:

$$\begin{aligned}
 & p(A)[p(K)(EV_{k_{ca}}) + p(Q)(EV_{q_{ca}})] \\
 & + p(K)[p(A)(EV) + p(Q)(EV_{q_{ca}})] \\
 & + p(Q)[p(A)(EV) + p(K)((EV_{k_{ca}}) + (EV_{k_f}))] \\
 & + p(J)[p(A)(EV) + p(K)((EV_{k_{ca}}) + (EV_{k_f})) + p(Q)(EV_{q_{ca}})].
 \end{aligned}$$

Since there are four cards in the deck, player 1 will hold each card one-fourth of the time, which implies that player 2 will hold any other card one-third of the time. Using this information and the expected values from the simplified matrix (Table

6.11), we substitute into the equation to obtain:

$$\begin{aligned} & \frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(-1)] = \frac{1}{12}. \end{aligned}$$

Thus the expected value of this match-up is 1/12. We will use similar reasoning for the remainder of the match-ups, as such, we will skip the logical equation and will proceed to the numerical equation.

- k_b vs q_f : When player 1 has the ace he bets and gains one unit from the king and nothing from the queen since player 2 folds. When player 1 has the king, he bets and loses one unit to the ace. When player 1 has the queen, he bets losing one unit to the ace and one unit to the king when player 2 calls with the king. If the king folds, then player 1 wins two units. When player 1 has the jack, he bets and loses one unit to the ace and one unit to the king. Player 1 could win two units if the king folds, and he also wins two units from the queen folding. In this strategy the queen is always folding. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(2)] = \frac{2}{12}.$$

- k_{ch} vs q_{ca} : When player 1 has the ace he bets and gains one unit from the king and one unit from the queen. When player 1 has the king no betting occurs. When player 1 has the queen, he bets losing one unit to the ace and one unit to the king when player 2 calls with the king. If the king folds, then player 1 wins two units. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king, and one unit to the queen. Player 1 could win two units if the king folds. In this strategy the queen always calls. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(-1)] = \frac{1}{12}.$$

- k_{ch} vs q_f : When player 1 has the ace he bets and gains one unit from the king. When player 1 has the king no betting occurs. When player 1 has the queen, he bets losing one unit to the ace and one unit to the king when player 2 calls with the

king. If the king folds, then player 1 wins two units. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king. Player 1 could win two units if the king folds. Player 1 also wins two units from the queen folding. In this strategy the queen always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(2)] = \frac{3}{12}.$$

- q_b vs k_{ca} : When player 1 has the ace he bets and gains one unit from the king and one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen, he bets losing one unit to the ace and one unit to the king. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king, and one unit to the queen. Player 1 could win two units if the queen folds. In this strategy the king always calls. The expected value of this match-up is:

$$\begin{aligned} & \frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] = -\frac{1}{12}. \end{aligned}$$

- q_b vs k_f : When player 1 has the ace he bets and gains one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen, he bets losing one unit to the ace but winning two units from the king folding. When player 1 has the jack, he bets and loses one unit to the ace, and one unit to the queen. Player 1 wins two units from the king folding and could win two units if the queen folds. In this strategy the king always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2) + \frac{1}{3}(-1 + 2)] = \frac{4}{12}.$$

- q_{ch} vs k_{ca} : When player 1 has the ace he bets and gains one unit from the king and one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen no betting occurs. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king, and one unit to the queen. Player 1 could win two units if the queen folds. In this strategy the king always calls. The expected value of this match-up

is:

$$\frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] = \frac{1}{12}.$$

- q_{ch} vs k_f : When player 1 has the ace he bets and gains one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen no betting occurs. When player 1 has the jack, he bets and loses one unit to the ace, and one unit to the queen. Player 1 wins two units from the king folding. In this strategy the king always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2) + \frac{1}{3}(-1 + 2)] = \frac{3}{12}.$$

- j_b vs k_{ca} : When player 1 has the ace he bets and gains one unit from the king and one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen he loses a unit each to the ace and to the king. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king, and one unit to the queen. Player 1 could win two units if the queen folds. In this strategy the king always calls. The expected value of this match-up is:

$$\begin{aligned} & \frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] = -\frac{1}{12}. \end{aligned}$$

- j_b vs k_f : When player 1 has the ace he bets and gains one unit from the queen. When player 1 has the king he bets and loses to the ace, but beats the queen for one unit each. When player 1 has the queen he loses a unit to the ace, but wins two units from the king folding. When player 1 has the jack, he bets and loses one unit to the ace, and one unit to the queen. Player 1 wins two units from the king, and could also win two units if the queen folds. In this strategy the king always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2) + \frac{1}{3}(-1 + 2)] = \frac{4}{12}.$$

- j_b vs q_{ca} : When player 1 has the ace he bets and gains one unit each from the king and the queen. When player 1 has the king he bets and loses to the ace, but beats

the queen for one unit each. When player 1 has the queen he loses a unit to the ace, and one unit to the king. If the king folds, he could win two units. When player 1 has the jack, he bets and loses one unit to the ace, one unit to the king and one unit to the queen. Player 1 could win two units if the king folds. In this strategy the queen always calls. The expected value of this match-up is:

$$\begin{aligned} & \frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] \\ & + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(-1)] = \frac{1}{12}. \end{aligned}$$

- j_b vs q_f : When player 1 has the ace he bets and gains one unit from the king. When player 1 has the king he bets and loses to the ace. When player 1 has the queen he loses a unit to the ace, and one unit to the king, but if the king folds, he could win two units. When player 1 has the jack, he bets and loses one unit to the ace, and one unit to the king. If the king folds, he wins two units. Player 1 wins two units from the queen folding. In this strategy the queen always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2) + \frac{1}{3}(2)] = \frac{2}{12}.$$

- j_{ch} vs k_{ca} : When player 1 has the ace he bets and gains one unit from the king and one unit from the queen. When player 1 has the king he bets and loses one unit to the ace but gains one unit from the queen. When player 1 has the queen he loses a unit to the ace, and one unit to the king. When player 1 has the jack no betting occurs. In this strategy the king always calls. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1)] = 0.$$

- j_{ch} vs k_f : When player 1 has the ace he bets and gains one unit from the queen. When player 1 has the king he bets and loses one unit to the ace but gains one unit from the queen. When player 1 has the queen he loses a unit to the ace, but gains two units from the king folding. When player 1 has the jack no betting occurs. In this strategy the king always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(2)] = \frac{2}{12}.$$

- j_{ch} vs q_{ca} : When player 1 has the ace he bets and gains one unit each from the king and the queen. When player 1 has the king he bets and loses one unit to the ace but gains one unit from the queen. When player 1 has the queen he loses a unit to the ace, and one unit to the king, but he could gain two units if the king folds. When player 1 has the jack no betting occurs. In this strategy the queen always calls. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] = \frac{2}{12}.$$

- j_{ch} vs q_f : When player 1 has the ace he bets and gains one unit from the king. When player 1 has the king he bets and loses one unit to the ace. When player 1 has the queen he loses a unit to the ace, and one unit to the king, but he could gain two units if the king folds. When player 1 has the jack no betting occurs. In this strategy the queen always folds. The expected value of this match-up is:

$$\frac{1}{4}[\frac{1}{3}(1)] + \frac{1}{4}[\frac{1}{3}(-1)] + \frac{1}{4}[\frac{1}{3}(-1) + \frac{1}{3}(-1 + 2)] = 0.$$

The expected values of all match-ups are compiled on the table below.

Table 6.12: AKQJ Game, Expected Values

	Player 2 k_{ca}	Player 2 k_f	Player 2 q_{ca}	Player 2 q_f
Player 1 k_b			1/12	2/12
Player 1 k_{ch}			1/12	3/12
Player 1 q_b	-1/12	4/12		
Player 1 q_{ch}	1/12	3/12		
Player 1 j_b	-1/12	4/12	1/12	2/12
Player 1 j_{ch}	0	2/12	2/12	0

Multiplying each entry by 12 gives us the simplified table on the following page.

It is important to note that calculating the expected values has revealed more dominated strategies. Now, we can see that for player 1 checking with the king dominates betting with the king. This was not apparent from any of the previous matrices, however, if we take a look back at all the strategy match-up analyses, we should note that for the most part the king lost to the ace but won against the queen. Betting the king seemed

Table 6.13: AKQJ Game, Expected Values, Simplified

	Player 2 k_{ca}	Player 2 k_f	Player 2 q_{ca}	Player 2 q_f
Player 1 k_b			1	2
Player 1 k_{ch}			1	3
Player 1 q_b	-1	4		
Player 1 q_{ch}	1	3		
Player 1 j_b	-1	4	1	2
Player 1 j_{ch}	0	2	2	0

to always give us an expected value of zero. In some occasions, when the queen folded, the king always lost to the ace, which gave betting with a king a negative expected value. Since the expected value from checking is always zero, checking dominates betting. In this case, we are better off checking with the king every single time. Player 2 can also remove folding with a king from his strategies since it is dominated by calling with the king.

From the new matrix (Table 6.14) we can deduce a few points that have surfaced from our analysis. Player 1 has a new automatic decision, that is to check with all of his kings. Similarly, player 2 should always call with all kings. Furthermore, for player 1, checking with the queen dominates betting with the queen. Thus, player 1 should always check with all of his queens. In fact, if we look at only the quadrant obtained from the match-ups between player 1 options with the queens, and player 2 responses with the king (Table 6.13), it turns out that we had a saddle point all along at checking with the queen and calling with the king.

All of our previous work has revealed only two non-dominated strategies, how

Table 6.14: AKQJ Game, Expected Values, Re-Simplified

	Player 2 k_{ca}	Player 2 q_{ca}	Player 2 q_f	
Player 1 k_{ch}			1	3
Player 1 q_b			-1	
Player 1 q_{ch}	1			
Player 1 j_b	-1	1	2	
Player 1 j_{ch}	0	2	0	

often should player 1 bluff with the jack, and how often should player 2 call with the queen? This goes along with the AKQ Game where player 1 needed to figure out how often to bluff with his bottom card, the queen, and now player 1 needs to figure out how often to bluff with his new bottom card, the jack. Almost identical reasoning holds for player 2, he needs to figure out how often to call with the queen, where in the AKQ Game, folding a queen was automatic, it is now automatic to fold a jack. Then, how often should he call with the queen in order to keep player 1 indifferent between bluffing and checking with the jack? The two non-dominated strategies are represented on the table below:

Table 6.15: AKQJ Game, Non-Dominated Strategies

	Player 2 calls queens	Player 2 folds queens
Player 1 bets jacks	1	2
Player 1 checks jacks	2	0

We can solve for optimal strategies by deductive reasoning as we did for the AKQ Game, or we can take a simpler approach and solve using the fact that this is a simple 2 by 2 matrix, for which we can obtain a solution via the algebraic equation from Table 4.7. Let player 1 bet jacks with probability p and let player 1 check jacks with probability $1 - p$. Then, when player 2 calls with queens we have $1(p) + 2(1 - p)$. When player 2 folds with queens we have $2(p) + 0(1 - p)$. Setting the equations equal to each other and solving for p gives

$$1(p) + 2(1 - p) = 2(p)$$

$$p + 2 - 2p = 2p$$

$$2 = 3p$$

$$\frac{2}{3} = p.$$

Player 2 calls with probability q and folds with probability $1 - q$. When player 1 bets with jacks we have $1(q) + 2(1 - q)$. When player 1 checks with jacks we have $2(q) + 0(1 - q)$. This are actually the same equations as player 1, only with a different variable. Thus, the answer is $q = \frac{2}{3}$. Then, the optimal strategies are for player 1 to bluff with the jack two-thirds of the time, folding the rest of the time, and for player 2 to call with the queen

two-thirds of the time, folding the rest of the time. In this manner, both players keep each other indifferent between their respective choices.

It seems that adding a fourth card to our game has increased both the bluffing ratio and calling ratio for each player. In the AKQ Game, each non-dominated strategy had a weight of $1/3$, and now, in the AKQJ Game, the weight for each non-dominated strategy is higher at $2/3$. If we take a step back and reflect on the solutions, we should come to the conclusion that they are correct. In the AKQJ Game, there are more cards to bluff at, namely the king and the queen, so our bluffing ratio should increase to reflect this change. Player 1 should strive to have player 2 fold more of his middle and bottom cards. On the other hand, for player 2, folding a queen is no longer automatic, since he knows that player 1 will be bluffing, player 2 needs to increase his calling ratio, in order to catch player 1 when he bluffs and increase his own expected value.

At this point, we should recap the optimal solutions for each player.

- Player 1 bets aces all the time.
- Player 1 checks kings all the time.
- Player 1 checks queens all the time.
- Player 1 bluffs jacks $2/3$ of the time.
- Player 1 checks jacks $1/3$ of the time.

Player 1 maximizes his expected value only from betting his aces. When he has the king, he will never beat the ace, and can only beat the queen if it calls. The expected value for this strategy is either zero or negative. The expected value for checking is zero. Player 1's best course of action is to check with the kings. When player 1 has the queen, he will never beat the ace. He could bluff the king out of the pot, but if he gets called he will lose. In short, the queen seldom beats better and will never be called by worst (actually, we know that player 2 calls with kings all the time). Thus, player 1 checks with all queens. Finally, player 1 must bluff with jacks two-thirds of the time and folds the rest.

- Player 2 calls with aces all the time.
- Player 2 calls with kings all the time.
- Player 2 calls with queens $2/3$ of the time.

- Player 2 folds queens 1/3 of the time.
- Player 2 folds jacks all of the time.

Player 2 should never fold his aces, since it beats everything. The king is too powerful to fold, it will lose to a betting ace, but it will win against a betting queen and a bluffing jack (we know that the queen never bets in this game). Player 2 cannot fold a king knowing that he can catch player 1 bluffing with the jack. Player 2 must call with queens two-thirds of the time and fold the rest. It will show player 1 that he is not afraid of calling with a weaker hand, and should keep him from bluffing in the future. Lastly, player 2 folds all jacks since it does not beat anything.

In this manner, we have solved an original game extension, the AKQJ Game, and given proof that optimal strategies exist. Hence, by adding one card at a time, we can solve the whole game of poker, and give optimal strategies for all 52 cards, or any combinations thereof. We should also be able to group certain combinations of cards into betting, folding, checking, calling, and or bluffing ranges. As demonstrated in this paper, there are also other options available to us such as check-folding, check calling, or raising. In real poker, we have yet to explore even more combinations of play, for example the check raise, or the three-bet, or raising a raise. All of those, are possible with the work that we have shown here.

The next logical extension to this game would be the AKQJ10 Game, which I believe will play similar to the original AKQ Game. Here the ace and the king will be played in similar fashion, probably betting all of the time. The queen will most likely have its own strategy never betting but calling bets. The jack and ten being the bottom cards will need to bluff but never call. This seems reasonable, but until we do the analysis of the game we will not know for sure if there are optimal strategies for the king and the jack other than from what we may guess.

Chapter 7

Conclusion

At first glance, the game of poker may seem random and based on luck, if the cards fall in our favor or not dictates how well we do. Although, this is partially true, it is not due to luck, but to probability. Even a 100-to-1 underdog has to win sometimes, specifically, one time out of one-hundred. Throughout this paper we have attempted to show optimal strategies and best courses of action for any given example. This project has only scratched the surface of an immense field of study. Every problem and every example presented here could potentially lead to more questions and other what-if scenarios.

As stated before, this paper was intended as an introduction to poker problem solving through game theory. We could extend Borel's and von Neumann's poker models to reflect any specific poker play or strategic point that we may want. Some of the most natural extensions are for us to allow two rounds of betting, in those cases, how will the hand ranges increase or decrease as the pot grows? What about if instead of limit or pot-limit poker, we played no-limit poker, where the pot can grow to infinity, how will the hand ranges change?

Just like there are 2-person zero-sum and non-zero-sum games, there are n-person zero-sum and non-zero-sum games available to study. Playing the AKQ Game with three players is probably not a great idea, someone will always have the ace and will win always. But what about the AKQJ Game? That game could potentially be played by three players. It is true that in most games someone will have the ace, but in those cases where the ace is not dealt to any of the players, what will optimal strategies look like? Will the most aggressive player win? Is it possible for the jack to bluff all of the

cards out of the pot, or perhaps the king could re-raise to show strength? If we are able to solve the AKQJ10 Game for two players, can we then extend this game to allow for three players? Another modification that we can make to the AKQ Game is to allow player 2 to either bet or to raise depending on the situation. For example, if player 1 holds the king, we know that he will always check, but what happens if player 2 wants to bet his own ace, or bluff with the queen? There are many other strategies that can be explored in the AKQ Game if we expand the options given to the players. Additionally, the majority of the games explored in this paper are from a specific player's perspective. We must keep in mind, that in real life, players will alternate turns. Just because a game favors player 1, all is not lost to player 2, the next hand will be dealt and the players will switch roles. It is up to each player to maximize their own expected value by playing optimally, that implies winning the most when possible, and losing the least when faced with an adverse game situation.

In conclusion, we have attempted to give a thorough look at how to think about poker using game theory. This paper is by no means exhaustive nor extensive. The natural extensions to all models, and the games that we can create, represent the whole game of poker with all of its nuances. Much more work needs to be completed before we can attempt to give an definite proof on how to play the game optimally. There are many poker theorists, poker professionals, mathematicians, and others who may claim to have solved the game. It may be true, however, to truly understand the game and the mathematics behind the game, one must put in the hours to solve the problems. Research is the only way that we can truly learn and discover, as well as being the most rewarding when we can finally solve and answer our questions.

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