12-2015

CONSTRUCTIONS AND ISOMORPHISM TYPES OF IMAGES

Jessica Luna Ramirez
California State University - San Bernardino

Constructions and Isomorphism Types of Images

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Jessica Luna Ramirez

December 2015

Constructions and Isomorphism Types of Images

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

by

Jessica Luna Ramirez

December 2015

Approved by:

Dr. Zahid Hasan, Committee Chair

Date

Dr. Belisario Ventura, Committee Member

Dr. J. Paul Vicknair, Committee Member

Dr. Charles Stanton, Chair,
Department of Mathematics

Dr. Corey Dunn
Graduate Coordinator,
Department of Mathematics

Abstract

In this thesis, we have presented our discovery of true finite homomorphic images of various permutation and monomial progenitors, such as $2^{*7} : D_{14}$, $2^{*7} : (7 : 2)$, $2^{*6} : (S_3 \times 2)$, $2^{*8} : S_4$, $2^{*72} : (3^2 : (2^{\cdot}S_4))$, and $11^{*2} :_m D_{10}$. We have given delightful symmetric presentations and very nice permutation representations of these images which include, the Mathieu groups M_{11}, M_{12}, the 4-fold cover of the Mathieu group M_{22}, $2 \times L_2(8)$, and $L_2(13)$. Moreover, we have given constructions, by using the technique of double coset enumeration, for some of the images, including M_{11} and M_{12}. We have given proofs, either by hand or computer-based, of the isomorphism type of each image. In addition, we use Iwasawa's Lemma to prove that $L_2(13)$ over A_5, $L_2(8)$ over D_{14}, $L_2(13)$ over D_{14}, $L_2(27)$ over $2^{\cdot}D_{14}$, and M_{11} over $2^{\cdot}S_4$ are simple groups. All of the work presented in this thesis is original to the best of our knowledge.

Acknowledgements

First and foremost, I would like to thank my advisor and Professor, Dr. Zahid Hasan for his great patience, motivation, and encouragement through the study and research of my thesis. I thank you for helping me fully understand the concepts of Group Theory. Also, I thank you for always believing in me and for all of your support.

Furthermore, I would like to thank Dr. Belisario Ventura and Dr. J. Paul Vicknair for being part of my thesis committee and motivating me to reach all my educational goals. I also thank Dr. Corey Dunn and Dr. Charles Stanton for guiding me through my graduate studies.

Finally and most importantly, I want to thank my parents, Hector Luna and Maria Luna for all of their love, support, and for never giving up on me. Next, I thank my siblings for taking care of my daughter while I was working on my graduate studies. Further, I am grateful with my husband, Eduardo Ramirez (*alma gemela*) for his patience and love towards my daughter and myself. Lastly, I dedicate this thesis to my daughter, Luna Cuevas for always being on my side despite the time of day.

I love you! ∞ and beyond.

Table of Contents

List of Tables

List of Figures

Introduction

Group theory is the study of symmetry of objects. Symmetric presentations provide a uniform way of constructing finite groups. In this thesis, we are particularly interested in symmetric presentations of finite simple groups, since these can be used to obtain all finite groups. The process to obtain finite homomorphic images is through the use of a progenitor, $m^{*n} : N$, where N is transitive on n letters. The objective is to factor the progenitor by relations, that equate elements of N to the product of t_is, that give finite homomorphic images. An isomorphism is a homomorphism that is also a bijection. We say that group G is isomorphic to group H, denoted by $G \cong H$, if there exists an isomorphism $f : G \to H$.

In Chapter 1, we describe the process of creating permutation progenitors and monomial progenitors. In addition, we factor these progenitors by all first order relations and suitable relations. In Chapter 2, we apply the technique of double coset enumeration, resulting in the construction of Cayley diagrams, and give by hand or computer-based proofs for the isomorphism type of each group. We also, explain the technique of factoring by the center. In Chapter 3, we use Iwasawa's lemma and the transitive action of a group on the set of single cosets to prove that a group is simple. Similarly, in Chapter 4, we apply the technique of double coset enumeration over a maximal subgroup and apply Iwasawa's lemma to prove that a group simple. In Chapter 5, we compute an extension problem, by looking at the composition factors to find the isomorphic type. In Chapter 6, we construct M_{11} over the subgroup S_4 with an imprimitivite action. We then construct this group over the maximal subgroup $2^{\cdot}S_4$ with a primitive action and thus apply Iwasawa's lemma to prove that this group is simple. Similarly, in Chapter 7, we construct M_{12} by performing the double coset enumeration, and our goal is to show that the group is simple, however, at the time of writing our proof, we did not have time to apply Iwasawa's lemma, it is still in progress. Finally, in Chapter 8, we give progenitors tables with homomorphic images.

Chapter 1

Writing Progenitors

1.1 Writing Progenitors Preliminaries

Definition 1.1. ***(Permutation).*** *If X is a nonemty set, a* ***permutation*** *of X is a bijection $\alpha : X \to X$. We denote the set of all permutations of X by S_X. [Rot12]*

Definition 1.2. ***(Operation).*** *Let G be a set. A (binary)* ***operation*** *on G is a function that assigns each ordered pair of elements of G an element on G. [Rot12]*

Definition 1.3. ***(Semigroup).*** *A* ***semigroup*** *$(G,*)$ is a nonempty set G equipped with an associative operation $*$. [Rot12]*

Definition 1.4. ***(Group).*** *A* ***group*** *is a semigroup G containing an element e such that*
(i) $e * a = a = a * e$ for all $a \in G$
(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$. [Rot12]

Definition 1.5. ***(Abelian).*** *A pair of elements a and b in a semigroup* ***commutes*** *if $a*b = b*a$. A group (or a semigroup) is* ***abelian*** *if every pair of its elements commutes. [Rot12]*

Theorem 1.6. *If $K \leq H$ and $[H : K] = n$, then there is a homomorphism $\phi : H \to S_n$ with $ker\phi \leq K$. [Rot12]*

Definition 1.7. ***(Free Group).*** *If X is a nonempty subset of a group F, then F is a* ***free group*** *with basis X if, for every group G and every function $f : X \to G$, there* *exists a unique homomorphism $\phi : F \to G$ extending f. Moreover, X generates F. [Rot12]*

Definition 1.8. ***(Presentation).*** *Let X be a nonempty set and let Δ be a family of words on X. A group G has* ***generators*** *X and* ***relations*** *Δ if $G \cong F/R$, where F is the free group with basis X and R is the normal subgroup of F generated by Δ. The ordered pair $(X|\Delta)$ is called a* ***presentation*** *of G. [Rot12]*

Definition 1.9. ***(Progenitor).*** *Let G be a group and $T = \{t_1, t_2, ..., t_n\}$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the* ***progenitor*** *to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_m. [Cur07]*

Definition 1.10. ***(Normalizer).*** *If $H \leq G$, then the* ***normalizer*** *of H in G, denoted by $N_G(H)$, is*

$$N_G(H) = \{a \in G : aHa^{-1} = H\}. \text{ [Rot12]}$$

Definition 1.11. ***(Centralizer).*** *If $a \in G$, then the* ***centralizer*** *of a in G, denoted by $C_G(a)$, is the set of all $x \in G$ which commute with a. [Rot12]*

Note 1.12. An **isomorphism** is a homomorphism that is also bijection. We say that G is **isomorphic** to H denoted by $G \cong H$, if there exist an isomorphism $\phi : G \to H$. [Rot12]

Definition 1.13. ***(Homomorphism).*** *Let G and H be groups. A map $\phi : G \to H$ is a* ***homomorphism*** *if, for all $\alpha, \beta \in G$,*

$$\phi(\alpha\beta) = \phi(\alpha)\phi(\beta). \text{ [Rot12]}$$

Theorem 1.14. ***(First Isomorphism Theorem (F.I.T)).*** *Let $\phi : G \to H$ be a homomorphism with $ker\phi$. Then*

- $ker\phi \trianglelefteq G$
- $G/ker\phi \cong img\phi$. *[Rot12]*

Theorem 1.15. ***(Second Isomorphism Theorem).***
Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot12]

Theorem 1.16. ***(Third Isomorphism Theorem).***
Let $K \leq H \leq G$, where both K and H are normal subgroups of G. Then H/K is a normal subgroup of G/K and

$$(G/K)(H/K) \cong G/H. \text{ [Rot12]}$$

Definition 1.17. ***(Monomial Character).*** *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H. [Led87]*

Definition 1.18. ***(Character).*** *Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m. We consider the characteristic polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{pmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & \lambda - a_{mm}(x) \end{pmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + .. + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ as a function on G with values in K, and we call it the ***character*** *of $A(x)$. [Led87]*

Theorem 1.19. *The number of irreducible character of G is equal to the number of conjugacy classes of G. [Led87]*

Definition 1.20. ***(Degree of a Character).*** *The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The* ***degree of a character*** *χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character. [Led87]*

Definition 1.21. ***(Lifting Process).*** *Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m$, $\phi(u) = m = \phi(1)$. The lifting process preserves irreducibility. [Led87]*

Definition 1.22. ***(Induced Character)***

Let $H \leq G$ and $\phi(u)$ be a character of H and define $\phi(x) = 0$ if $x \in H$, then

$$\phi^G(x) = \begin{cases} \phi(x), x \in H \\ 0, x \notin H \end{cases}$$

is an induced character of G. [Led87]

Definition 1.23. ***(Formula for Induced Character).***

Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let C_α, $\alpha = 1, 2, \cdots m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha$, $\alpha = 1, 2, \cdots m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:

$$\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w), \alpha = 1, 2, 3, \cdots, m. \text{ [Led87]}$$

1.2 Permutation Progenitor of A_5

We want to write a permutation progenitor of $2^{*5} : A_5$, where our control group is $N = A_5$. First, we write a presentation of A_5 which is $G < x, y >=< x, y | x^2 = y^3 = (xy)^5 = 1 >$. We check in Magma if the above presentation gives A_5.

```
> G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1 >;
>  f,G1,k:=CosetAction(G,sub<G|Id(G)>);
>  s,t:=IsIsomorphic(G1,Alt(5));s;
true
```

Moreover, the corresponding permutation representation is $N =< x, y >$, where $x = (1,2)(3,4)$ and $y = (1,3,5)$. Now we add the free product 2^{*5} ($|t_i's| = 2$) to this group to form our progenitor. Hence, a presentation for the progenitor $2^{*5} : N$ is given by $< x, y, t | x^2, y^3, (xy)^5, t^2, (t, N^1) >$, where $t \sim t_1$. N^1 is the point stabilizer of 1, and $(t, N^1) = 1$ means that $1^g = 1 \ \forall \ g \in N^1$. Note that $1^g = 1 \ \forall \ g \in N^1$ implies t has $[N : N^1]$ conjugates in N. Using Magma, we can see that the point stabilizer of 1 in N is equal to $< (2,3,4), (3,4,5)$. Now we use the Schreier System to convert the permutations into words. Thus, $N^1 =< yxy^{-1}xy^{-1}, y^{-1}xy^{-1}xyxy^{-1} >$ (see below).

```
> G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1>;
> S:=Alt(5);
> xx:=S!(1,2)(3,4);
> yy:=S!(1,3,5);
> N:=sub<S|xx,yy>;
> N1:=Stabiliser(N,1);
> N1;
Permutation group N1 acting on a set of cardinality 5
Order = 12 = 2^2 * 3
    (2, 3, 4)
    (3, 4, 5)
> Sch:=SchreierSystem(G,sub<G|Id(G)>);
> ArrayP:=[Id(N): i in [1..60]];
> for i in [2..60] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..60]do if ArrayP[i] eq N!(2,3,4) then Sch[i];
end if;
for> end for;
y * x * y^-1 * x * y^-1
> for i in [1..60]do if ArrayP[i] eq N!(3,4,5) then Sch[i];
end if;
for> end for;
y^-1 * x * y^-1 * x * y * x * y^-1
```

So, a presentation of the progenitor $G = 2^{*4} : N$ is given by $G < x, y, t >:= Group < x, y, t|x^2, y^3, (xy)^5, t^2, (t, yxy^{-1}xy^{-1}), (t, y^{-1}xy^{-1}xyxy^{-1}) >$. This progenitor is infinite. In order to find find finite images of $2^{*5} : N$ we must factor it by the first order relations.

1.2.1 Factoring $2^{*5} : A_5$ by First Order Relations

The first order relations are written of the form $(\pi t_i^a)^b = 1$, where $\pi \in N$ and w is word in the $t_i's$. In order to find these relations, we compute the conjugacy classes of $N = A_5$, as show below:

Table 1.1: Conjugacy Classes of A_5

Class Number	Order	Class Representative	Length
[1]	1	e	1
[2]	2	(1,2)(3,4)	15
[3]	3	(1,2,3)	20
[4]	5	(1,2,3,4,5)	12
[4]	5	(1,3,4,5,2)	12

Next, we need to compute the centralizer of each class representative in $N = A_5$ and then find the orbits of the corresponding centralizer. The centralizer and their orbits on $\{1, 2, 3, 4, 5\}$ are given below.

Table 1.2: Centralizer of A_5

Class Num	Class Rep	Cent(N,Class Rep)	Orbits
[2]	(1,2)(3,4)	$< (1, 2)(3, 4), (1, 3)(2, 4) >$	{1,2,3,4}, {4}, {5}
[3]	(1,2,3)	$< (1, 2, 3) >$	{1,2,3} {4}, {5}
[4]	(1,2,3,4,5)	$< (1, 2, 3, 4, 5) >$	{1,2,3,4,5}
[5]	(1,3,4,5,2)	$< (1, 3, 4, 5, 2) >$	{1,3,4,5,2}

Thus, we use the table above to obtain the following relations of $N = A_5$:

$$(1, 2)(3, 4)t_1 = xt,$$

$$(1, 2)(3, 4)t_4 = xt^{yx},$$

$$(1, 2)(3, 4)t_5 = xt^{y^{-1}},$$

$$(1, 2, 3)t_1 = yxyxy^{-1}xt,$$

$$(1,2,3)t_4 = yxyxy^{-1}xt^{yx},$$

$$(1,2,3)t_5 = yxyxy^{-1}xt^{y^{-1}},$$

$$(1,2,3,4,5)t_1 = xyt,$$

$$(1,3,4,5,2)t_1 = xy^{-1}xyxy^{-1}xt.$$

Thus, a presentation of the progenitor of $G = 2^{*5} : A_5$ factored by all relations of the first order is
$G < x, y, t >:= Group < x, y, t|x^2, y^3, (x * y)^5, t^2, (t, yxy^-1xy^{-1}), (t, y^{-1}xy^{-1}xyxy^{-1}),$
$(xt)^a, (xt^{y^{-1}})^b, (yxyxy^{-1}xt)^c, (yxyxy^{-1}xt^{(yx)})^d, (yxyxy^-1xt^{(y^{-1})})^e, (xyt)^f,$
$(xy^{-1}xyxy^{-1}xt)^g >$

Hence, the table below shows some finite images of the progenitor $2^{*5} : A_5$ factored by all relations of the first order.

Table 1.3: Some Finite Images of the Progenitor $2^{*5} : A_5$

a	b	c	d	e	f	g	**Order of** G	**Shape of** G
0	0	0	0	0	0	**5**	1920	$2^{\cdot}(A_5 : 2^4)$
0	0	0	0	0	0	**6**	720	$A_6 : 2$
0	0	0	0	0	0	**7**	175560	J_1

1.3 Monomial Progenitor $11^{*2} :_m D_{10}$

Given that D_{10} has a monomial irreducible representation in dimension 2, write a progenitor for $11^{*2} :_m D_{10}$. Lets show that a presentation for D_{10} is given as $< x, y|x^5 = y^2 = (xy)^2 = 1 >$.

Proof. Given $D_{10} =< (1,2,3,4,5),(1,5)(2,4) >$. Let F be a free group with basis $X = \{x, y\}$. Define a homomorphism

$$\phi\ : F \longrightarrow D_{10}$$

by $\phi(x) = (1,2,3,4,5)$ and $\phi(y) = (1,5)(2,4)$.

From Theorem 1.6, we have ϕ is an onto homomorphism. Let $G = F/R$, where $R =< x^5, y^2, (xy)^2 >$ (in $G, x^5 = 1, y^2 = 1, (xy)^2 = 1$). Thus,

$$\phi\ : F \xrightarrow[onto]{homo.} D_{10}.$$

We know $F/ker\phi \cong D_{10}$. Now we want to show that $R \leq ker\phi$. We compute the following:

$$\phi(x^5) = (\phi(x))^5 = ((1,2,3,4,5))^5 = 1$$
$$\implies x^5 \in ker\phi,$$
$$\phi(y^2) = (\phi(y))^2 = ((1,5)(2,4))^2 = 1$$
$$\implies y^2 \in ker\phi,$$
$$\phi((xy)^2) = (\phi(x)\phi(y))^2 = ((1,2,3,4,5)(1,5)(2,4))^2 = 1$$
$$\implies (xy)^2 \in ker\phi.$$

So, $x^5, y^2, (xy)^2 \in ker\phi$. Thus, $< x^5, y^2, (xy)^2 >= R \cong ker\phi$,

$$\begin{aligned} &|F/R| \geq |F/ker\phi| \quad \text{(since R and } ker\phi \text{ are normal)} \\ \implies &|F/R| \geq |D_{10}| = 10 \\ \implies &|G| \geq 10. \end{aligned}$$

Now we need to show $|G| \leq 10$. That is to show

$$G = F/R \leq \{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}.$$

We need to show that the above set is closed under right multiplication by x and y.

(i) Show $\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}$ is closed under right multiplication by y.

$$\begin{aligned} (1) \quad (Rxy)y = Rxy^2 &= xRy^2 \quad \text{(since R is normal)} \\ &= xR \quad (\text{since } y^2 \in R) \\ &= Rx \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (2) \quad (Rx^2y)y = Rx^2y^2 &= x^2Ry^2 \quad \text{(since R is normal)} \\ &= x^2R \quad (\text{since } y^2 \in R) \\ &= Rx^2 \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned}(3)\quad (Rx^3y)y = Rx^3y^2 &= x^3Ry^2 \quad \text{(since R is normal)}\\ &= x^3R \quad (\text{since } y^2 \in R)\\ &= Rx^3 \quad \text{belongs to the set above.}\end{aligned}$$

$$\begin{aligned}(4)\quad (Rx^4y)y = Rx^4y^2 &= x^4Ry^2 \quad \text{(since R is normal)}\\ &= x^4R \quad (\text{since } y^2 \in R)\\ &= Rx^4 \quad \text{belongs to the set above.}\end{aligned}$$

So $\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}$ is closed under right multiplication by y.

(ii) Show $\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}$ is closed under right multiplication by x.

$$\begin{aligned}&(1)\quad \text{Note: } R(xy)^2 = R \quad (\text{since } (xy)^2 \in R)\\ &\implies Rxyxy = R\\ &\implies xRyx = Ry^{-1}\\ &\implies Ryx = x^{-1}Ry^{-1}\\ &\implies Ryx = Rx^{-1}y^{-1}.\end{aligned}$$

Then $Ryx = Rx^4Ry$ since $Rx^5 = R \implies Rx^4 = Rx^{-1}$ and $Ry^2 = R \implies Ry = Ry^{-1}$. Thus, $Ryx = Rx^4y$ belongs to the set above.

$$\begin{aligned}(2)\quad Rxyx &= xRyx \quad \text{(since R is normal)}\\ &= xRx^4y\\ &= Rx^5y\\ &= Ry \text{ belongs to the set above.}\end{aligned}$$

$$\begin{aligned}(3)\quad (Rx^2y)x &= Rx^2yx \quad \text{(since R is normal)}\\ &= x^2Ryx\\ &= x^2Rx^4y\\ &= Rx^6y\\ &= Rxy \text{ belongs to the set above.}\end{aligned}$$

$$\begin{aligned}(4)\quad (Rx^3y)x &= Rx^3yx \quad \text{(since R is normal)}\\ &= x^3Ryx\\ &= x^3Rx^4y\\ &= Rx^7y\\ &= Rx^2y \text{ belongs to the set above.}\end{aligned}$$

$$\begin{aligned}(5)\quad (Rx^4y)x &= Rx^4yx \quad \text{(since R is normal)}\\ &= x^4Ryx\\ &= x^4Rx^4y\\ &= Rx^8y\\ &= Rx^3y \text{ belongs to the set above.}\end{aligned}$$

So $\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}$ is closed under right multiplication by x.

Hence $|G| = 10$. So $G \cong D_{10}$,

$$\phi\ : F \xrightarrow[onto]{homo} D_{10}$$
$$F/ker\phi \cong D_{10}.$$

By Third Isomorphism Theorem, we have an onto homomorphism

$$\psi\ : F/R \xrightarrow[onto]{homo} F/ker\phi$$
$$R \leq ker\phi \leq F$$

$$|F/R| \leq |F/ker\phi|$$

$$F/R/ker\psi \cong F/ker\phi$$

with $F/R = 10$ and $F/ker\phi = 10$. So $ker\psi = 1$. Thus, $F/R \cong F/ker\phi \cong D_{10}$.

So D_{10} has a presentation of $\{x, y|x^5 = y^2 = (xy)^2 = 1\}$. $\square$

Given a presentation for D_{10} is $G < x, y >=< x, y|x^5 = y^2 = (xy)^2 = 1 >$ and a corresponding permutation representation is $N =< x, y >$, where $x = (1, 2, 3, 4, 5)$ and $y = (1, 5)(2, 4)$. By using Magma we get the character table of $G = D_{10}$.

Table 1.4: Character Table of $G = D_{10}$

Conjugacy Classes $= C_\alpha$	1	(1,5)(2,4)	(1,2,3,4,5)	(1,3,5,2,4)
Order $= h_\alpha$	1	5	2	2
λ_1	1	1	1	1
λ_2	1	-1	1	1
λ_3	2	0	$W1$	$W1\#2$
λ_4	2	0	$W1\#2$	$W1$

Explanation of Character Value Symbol

$\#$ denotes algebraic conjugation, that is,

$\#k$ indicates replacing the root of unity w by w^k

$$W1 = zeta(5)_5^3 + zeta(5)_5^2$$

$$W1\#2 = zeta(5)_5 + zeta(5)_5^4$$

Now we find a subgroup H of D_{10} of index $n = 2$. We use the following formula, $n = \frac{|D_5|}{|H|} = \frac{10}{|H|} = 2$, that implies $|H| = 5$. Let $H = \langle 1, (1, 2, 3, 4, 5) \rangle$. The character table of $H = \mathbb{Z}_5$ is given below.

Table 1.5: Character Table of $H = \mathbb{Z}_5$

Conjugacy Classes	1	(1,2,3,4,5)	(1,3,5,2,4)	(1,4,2,5,3)	(1,5,4,3,2)
Order	1	1	1	1	1
ϕ_1	1	1	1	1	
ϕ_2	1	$Z1$	$Z1\#2$	$Z1\#3$	$Z1\#4$
ϕ_3	1	$Z1\#2$	$Z1\#4$	$Z1$	$Z1\#3$
ϕ_4	1	$Z1\#3$	$Z1$	$Z1\#4$	$Z1\#2$
ϕ_5	1	$Z1\#4$	$Z1\#3$	$Z1\#2$	$Z1$

where $Z1 = zeta(5)_5$.

Now we look at the finite smallest field that has fifth roots of unity, which is

$$\mathbb{Z}_{11}\backslash\{0\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Let $Z1 = zeta(5)_5 = 4$.

So $Z1\#2 = (zeta(5)_5)^2 \equiv 5 \pmod{11}$,

$Z1\#3 = (zeta(5)_5)^3 \equiv 9 \pmod{11}$, and

$Z1\#4 = (zeta(5)_5)^4 \equiv 3 \pmod{11}$.

Next we use the following formula

$$\phi_\alpha^G = \tfrac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w),$$

to induce the character ϕ_2 of H up to D_{10}. We get

$$\begin{aligned}
\phi_2^G(1) = \tfrac{n}{h_1} \sum_{w \in C_1 \cap H} \phi_2(w) = \tfrac{2}{1} \sum_{w \in 1 \cap H} \phi_2(w) \\
&= 2 \sum_{w \in \{1\}} \phi_2(w) = 2\phi_2(1) = 2(1) \\
&= 2.
\end{aligned}$$

$$\begin{aligned}
\phi_2^G((1,5)(2,4)) = \tfrac{n}{h_2} \sum_{w \in C_2 \cap H} \phi_2(w) = \tfrac{2}{5} \sum_{w \in (1,5)(2,4) \cap H} \phi_2(w) \\
&= \tfrac{2}{5} \sum_{w \in (1,5)(2,4) \cap H = \emptyset} \phi_2(w) = \tfrac{2}{5}(0) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\phi_2^G((1,2,3,4,5)) = \tfrac{n}{h_3} \sum_{w \in C_3 \cap H} \phi_2(w) = \tfrac{2}{2} \sum_{w \in (1,2,3,4,5) \cap H} \phi_2(w) \\
&= 1 \sum_{w \in \{(1,2,3,4,5),(1,5,4,3,2)\}} \phi_2(w) \\
&= 1(\phi_2(1,2,3,4,5) + \phi_2(1,5,4,3,2)) \\
&= Z1 + Z1\#4 \\
&= zeta(5)_5 + zeta(5)_5^4 \\
&= W1\#2.
\end{aligned}$$

$$\begin{aligned}
\phi_2^G((1,3,5,2,4)) = \tfrac{n}{h_4} \sum_{w \in C_4 \cap H} \phi_2(w) = \tfrac{2}{2} \sum_{w \in (1,3,5,2,4) \cap H} \phi_2(w) \\
&= 1 \sum_{w \in \{(1,3,5,2,4),(1,4,2,5,3)\}} \phi_2(w) \\
&= 1(\phi_2(1,3,5,2,4) + \phi_2(1,4,2,5,3)) \\
&= Z_1\#2 + Z_1\#3 \\
&= zeta(5)_5^3 + zeta(5)_5^2 \\
&= W1.
\end{aligned}$$

Thus, $\phi_2^G = \lambda_4$.

Hence the representation of H (respect to ϕ_2) yields:

$$\begin{aligned} B(1) &= 1 \\ B(1,2,3,4,5) &= 4 \\ B(1,3,5,2,4) &= 5 \\ B(1,4,2,5,3) &= 9 \\ B(1,5,4,3,2) &= 3 \\ B(g) &= 0 \qquad\qquad \text{if } g \notin G. \end{aligned}$$

Since $H \leq G$, the right transversals of H in G (or a complete set of right coset representatives) are $t_1 = e$ and $t_2 = (1,5)(2,4)$

$$\implies G = He \cup H(1,5)(2,4).$$

Now we use the formula for monomial representation to find $A(x)$ and $A(y)$, where $x = (1,2,3,4,5)$ and $y = (1,5)(2,4)$:

$$A(x) = \begin{bmatrix} B(t_1xt_1^{-1}) & B(t_1xt_2^{-1}) \\ B(t_2xt_1^{-1}) & B(t_2xt_2^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} B(1,2,3,4,5) & B((1,2,3,4,5)(1,5)(2,4)) \\ B((1,5)(2,4)(1,2,3,4,5)) & B((1,5)(2,4)(1,2,3,4,5)(1,5)(2,4)) \end{bmatrix}$$

$$= \begin{bmatrix} B(1,2,3,4,5) & B((1,4)(3,2)) \\ B((5,2)(3,4)) & B(1,5,4,3,2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus, $A(x) = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

This matrix has 2 columns: label the columns 1, and 2 as t_1, and t_2, respectively. The entries of the matrix are in $\mathbb{Z}_{11}$. Hence, $t_i's$ are of order 11.

We label t_1^2 and t_2^2 as 3 and 4; t_1^3 and t_2^3 as 5 and 6; t_1^4 and t_2^4 as 7 and 8; t_1^5 and t_2^5 as 9 and 10; t_1^6 and t_2^6 as 11 and 12 ; t_1^7 and t_2^7 as 13 and 14; t_1^8 and t_2^8 as 15 and 16; t_1^9 and

t_2^9 as 17 and 18; t_1^{10} and t_2^{10} as 19 and 20. As shown below:

Table 1.6: Labeling t_is of Order 11

1. t_1	2. t_2	3. t_1^2	4. t_2^2	5. t_1^3
6. t_2^3	7. t_1^4	8. t_2^4	9. t_1^5	10. t_2^5
11. t_1^6	12. t_2^6	13. t_1^7	14. t_2^7	15. t_1^8
16. t_2^8	17. t_1^9	18. t_2^9	19. t_1^{10}	20. t_2^{10}

Now $A(x)$ is a monomial automorphism of $< t_1 > * < t_2 >$ given by $a_{ij} = a \iff t_i \to t_j^a$ Thus, $a_{11} = 4$ or $t_1 \to t_1^4$ and $a_{22} = 3$ or $t_2 \to t_2^3$. We use the chart above, to write down the permutation representation for $A(x)$.

Table 1.7: Permutations of the $t_i's$ using $A(x)$

t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5
1	2	3	4	5	6	7	8	9	10
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
7	6	15	12	1	18	9	2	17	8
t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}
11	12	13	14	15	16	17	18	19	20
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
3	14	11	20	19	4	5	10	13	16

So, $A(x) = (t_1, t_1^4, t_1^5, t_1^9, t_1^3)(t_1^2, t_1^8, t_1^{10}, t_1^7, t_1^6)(t_2, t_2^3, t_2^9, t_2^5, t_2^4)(t_2^2, t_2^6, t_2^7, t_2^{10}, t_2^8)$.
Then $A(x) = (1, 7, 9, 17, 5)(3, 15, 19, 13, 11)(2, 6, 18, 10, 8)(4, 12, 14, 20, 16)$.

$$\text{Now } A(y) = \begin{bmatrix} B(t_1 y t_1^{-1}) & B(t_1 y t_2^{-1}) \\ B(t_2 y t_1^{-1}) & B(t_2 y t_2^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} B((1,5)(2,4)) & B((1,5)(2,4)(1,5)(2,4)) \\ B((1,5)(2,4)(1,5)(2,4)) & B((1,5)(2,4)(1,5)(2,4)(1,5)(2,4)) \end{bmatrix}$$

$$= \begin{bmatrix} B((1,5)(2,4)) & B(1) \\ B(1) & B((1,5)(2,4)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $A(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Now $A(y)$ is an automorphism (permutation) of $< t_1 > * < t_2 >$ given by $a_{ij} = a \iff t_i \to t_j^a$ Thus, $a_{12} = 1$ or $t_1 \to t_2$ and $a_{21} = 1$ or $t_2 \to t_1$.

Table 1.8: Permutations of the $t_i's$ using $A(y)$

t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5
1	2	3	4	5	6	7	8	9	10
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
2	1	4	3	6	5	8	7	10	9
t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}
11	12	13	14	15	16	17	18	19	20
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
12	11	14	13	16	15	18	17	20	19

Using the chart above, we get the following permutation representation for $A(y) = (t_1, t_2)(t_1^2, t_2^2)(t_1^3, t_2^3)(t_1^4, t_2^4)(t_1^5, t_2^5)(t_1^6, t_2^6)(t_1^7, t_2^7)(t_1^8, t_2^8)(t_1^9, t_2^9)(t_1^{10}, t_2^{10})$.

Then

$$A(y) = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20).$$

Thus, $D_{10} =< A(x), A(y) >$

$$=< (1,7,9,17,5)(3,15,19,13,11)(2,6,18,10,8)(4,12,14,20,16),$$
$$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20) >.$$

We are now in a position to give a monomial presentation of the progenitor $11^{\star 2} :_m D_{10}$. A presentation of D_{10} is $< x, y | x^5 = y^2 = (x * y)^2 = 1 >$. We fix one of the two $t_i's$, say t_1 and call it t. Next, we compute the normalizer of the subgroup $< t_1 >$ in D_{10}. Therefore we compute the set stabilizer in D_{10} of the set $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^9, t_1^{10}\} = \{1,3,5,7,9,11,13,15,17,19\}$ which is $<$ $(1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)(4,14,16,12,20) >$ and that
$x^2 = (1,9,5,7,17)(2,18,8,6,10)(3,19,11,15,13)(4,14,16,12,20)$. Hence, a presentation for the monomial progenitor $11^{\star 2} :_m D_{10}$ is given by
$G < x, y, t > := Group < x, y, t | x^5 = y^2 = (x * y)^2 = 1, t^{11}, t^{x^2} = t^5 >$.

Next we add the relation, $t * t^y = t^y * t$ to the progenitor of D_{10}, to verify if the monomial progenitor $11^{\star 2} :_m D_{10}$ is correct. By using MAMGA we verified that the monomial progenitor, $G < x, y, t > := Group < x, y, t | x^5, y^2, (x * y)^2, t^1 1, t^{x^2} =$

$t^5, (t, t^y) >$ is correct :

```
> G< x,y,t >:= Group< x,y,t|x^5 = y^2 = (x*y)^2 = 1,t^11,
 t^(x^2)=t^5, t*t^y=t^y*t>;
> f, G1,k:=CosetAction(G,sub<G|x,y>);
> #G;
1210
> #k;
1
> IN:=sub<G1|f(x),f(y)>;
> T:=sub<G1|f(t)>;
> #T;
11
> Normaliser(IN,T);
Permutation group acting on a set of cardinality 121
Order = 5
> Index(IN,Normaliser(IN,T));
2
```

Hence, we have a presentation of the progenitor $11^{*2} :_m D_{10}$

```
G< x,y,t >:= Group< x,y,t|x^5 = y^2 = (x*y)^2 = 1,t^11,
t^(x^2)=t^5>
```

Next, we apply the first order relation to the progenitor $11^{*2} :_m D_{10}$.

1.3.1 $11^{*2} :_m D_{10}$ Factor by First Order Relations

Given a progenitor of the form $m^{*n} : N$ and $p^{*n} : N$.
All relations of the first order that $m^{*n} : N$ and $p^{*n} : N$ can be factored by are obtain as follows. Compute the conjugacy classes on N. Now we compute the centralizers of the representatives of each non-identity class. Then, we determine the orbits of the centralizer. Once we have the orbits we take the representative from each class and we right multiply by a t_i.

Consider the monomial progenitor of $11^{*2} :_m D_{10}$ that has the following presentation

```
G< x,y,t >:= Group< x,y,t|x^5 = y^2 = (x*y)^2 = 1,t^11,
t^(x^2)=t^5>.
```

To compute all first order relations for the monomial progenitor $11^{*2} :_m D_{10}$, we run the following code in MAGMA.

```
C:= Classes(N);
C;
C2:=Centraliser(N,N!(1,5)(2,4));
C2;
C3:=Centraliser(N,N!(1,2,3,4,5));
C3;
C4:=Centraliser(N,N!(1,3,5,2,4));
C4;
Set(C2);
Orbits(C2);
Set(C3);
Orbits(C3);
Set(C4);
Orbits(C4);
```

Then, we summarize the result in the table below.

Table 1.9: Conjugacy Classes of D_{10}

Classes	Centralizer	Orbits
$C_2 = (1,5)(2,4)$	$< y >$	{1,5},{2,4},{3}
$C_3 = (1,2,3,4,5)$	$< x >$	{1,2,3,4,5}
$C_4 = (1,3,5,2,4)$	$< x^2 >$	{1,3,5,2,4}

Next, we pick a representative from each orbit and we multiply by the representative from each class. Thus, the all first order relations are

$(yt)^a,\ (yt^x)^b,\ (yt^{x^2})^c,\ (xt)^d$, and $(x^2t)^e$, where $t \sim t_1$.

Hence, we factor the monomial progenitor $11^{*2} :_m D_{10}$ by the relations

$$(yt)^a,\ (yt^x)^b,\ (yt^{x^2})^c,\ (xt)^d, \text{ and } (x^2t)^e,$$

to obtained the following homomorphic images:

Table 1.10: Some Finite Images of the Progenitor $11^{*2} :_m D_{10}$

a	b	c	d	e	**Order of G**	**Shape of G**
0	0	**3**	0	0	660	$L_2(11)$
0	0	**4**	0	0	6600	$(5 \times L_2(11)) : 2$
0	**5**	**6**	0	0	435600	$L_2(11) \times L_2(11)$
0	**6**	**5**	0	0	1351680	$2^{\cdot}(2^{10} : L_2(11))$

1.4 Progenitor of $2^{*7} : D_{14}$

We want to write a permutation progenitor of $2^{*7} : D_{14}$ where our control group is $N = D_{14}$. A presentation of D_{14} is $\{x, y | x^7 = y^2 = (xy)^2 = 1\}$. We are going to prove the presentation of D_{14}.

Proof. Given $D_{14} =< (1,2,3,4,5,6,7), (1,6)(2,5)(3,4) >$. Let F be a free group with basis $X = \{x, y\}$. Define a homomorphism

$$\phi \ : F \longrightarrow D_{14}$$

by $\phi(x) = (1,2,3,4,5,6,7)$ and $\phi(y) = (1,6)(2,5)(3,4)$.

From Theorem 1.6, we have ϕ is an onto homomorphism. Let $G = F/R$, where $R =< x^7, y^2, (xy)^2 >$ (in $G, x^7 = 1, y^2 = 1, (xy)^2 = 1$). Thus,

$$\phi \ : F \xrightarrow[onto]{homo.} D_{14}.$$

We know $F/ker\phi \cong D_{14}$. Now we want to show that $R \leq ker\phi$. We compute the following:

$\phi(x^7) = (\phi(x))^7 = ((1,2,3,4,5,6,7))^5 = 1$
$\implies x^7 \in ker\phi,$
$\phi(y^2) = (\phi(y))^2 = ((1,6)(2,5)(3,4))^2 = 1$
$\implies y^2 \in ker\phi,$
$\phi((xy)^2) = (\phi(x)\phi(y))^2 = ((1,2,3,4,5,6,7)(1,6)(2,5)(3,4))^2 = 1$
$\implies (xy)^2 \in ker\phi.$

So $x^7, y^2, (xy)^2 \in ker\phi$. Thus, $< x^7, y^2, (xy)^2 >= R \cong ker\phi$,

$$|F/R| \geq |F/ker\phi| \ \text{(since R and } ker\phi \text{ are normal)}$$
$$\implies |F/R| \geq |D_{14}| = 14$$
$$\implies |G| \geq 14.$$

Now we need to show $|G| \leq 14$. That is to show

$$G = F/R \leq \{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y,$$
$$Rx^3y, Rx^4y, Rx^5y, Rx^6y\}.$$

We need to show that the above set is closed under right multiplication by x and y.

(i) Show
$\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}$
is closed under right multiplication by y.

$$\begin{aligned} (1)\quad (Rxy)y = Rxy^2 &= xRy^2 \quad \text{(since R is normal)} \\ &= xR \quad (\text{since } y^2 \in R) \\ &= Rx \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (2)\quad (Rx^2y)y = Rx^2y^2 &= x^2Ry^2 \quad \text{(since R is normal)} \\ &= x^2R \quad (\text{since } y^2 \in R) \\ &= Rx^2 \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (3)\quad (Rx^3y)y = Rx^3y^2 &= x^3Ry^2 \quad \text{(since R is normal)} \\ &= x^3R \quad (\text{since } y^2 \in R) \\ &= Rx^3 \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (4)\quad (Rx^4y)y = Rx^4y^2 &= x^4Ry^2 \quad \text{(since R is normal)} \\ &= x^4R \quad (\text{since } y^2 \in R) \\ &= Rx^4 \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (5)\quad (Rx^5y)y = Rx^5y^2 &= x^5Ry^2 \quad \text{(since R is normal)} \\ &= x^5R \quad (\text{since } y^2 \in R) \\ &= Rx^5 \quad \text{belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (6)\quad (Rx^6y)y = Rx^6y^2 &= x^6Ry^2 \quad \text{(since R is normal)} \\ &= x^6R \quad (\text{since } y^2 \in R) \\ &= Rx^6 \quad \text{belongs to the set above.} \end{aligned}$$

So $\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}$
is closed under right multiplication by y.

(ii) Show
$\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}$ is closed under right multiplication by x.

$$\begin{aligned} &(1)\quad \text{Note: } R(xy)^2 = R \quad (\text{since } (xy)^2 \in R) \\ &\implies Rxyxy = R \\ &\implies xRyx = Ry^{-1} \\ &\implies Ryx = x^{-1}Ry^{-1} \\ &\implies Ryx = Rx^{-1}y^{-1}. \end{aligned}$$

Then $Ryx = Rx^6Ry$ since $Rx^6 = R \implies Rx^6 = Rx^{-1}$ and $Ry^2 = R \implies Ry = Ry^{-1}$. Thus, $Ryx = Rx^6y$ belongs to the set above.

$$\begin{aligned} (2)\quad Rxyx &= xRyx \quad (\text{since R is normal}) \\ &= xRx^6y \\ &= Rx^7y \\ &= Ry \text{ belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (3)\quad (Rx^2y)x &= Rx^2yx \quad (\text{since R is normal}) \\ &= x^2Ryx \\ &= x^2Rx^6y \\ &= Rx^8y \\ &= Rxy \text{ belongs to the set above.} \end{aligned}$$

$$\begin{aligned} (4)\quad (Rx^3y)x &= Rx^3yx \quad (\text{since R is normal}) \\ &= x^3Ryx \\ &= x^3Rx^6y \\ &= Rx^9y \\ &= Rx^2y \text{ belongs to the set above.} \end{aligned}$$

$$
\begin{aligned}
(5)\quad (Rx^4y)x &= Rx^4yx \quad \text{(since R is normal)}\\
&= x^4Ryx\\
&= x^4Rx^6y\\
&= Rx^{10}y\\
&= Rx^3y \text{ belongs to the set above.}
\end{aligned}
$$

$$
\begin{aligned}
(6)\quad (Rx^5y)x &= Rx^5yx \quad \text{(since R is normal)}\\
&= x^5Ryx\\
&= x^5Rx^6y\\
&= Rx^{11}y\\
&= Rx^4y \text{ belongs to the set above.}
\end{aligned}
$$

$$
\begin{aligned}
(7)\quad (Rx^6y)x &= Rx^6yx \quad \text{(since R is normal)}\\
&= x^6Ryx\\
&= x^6Rx^6y\\
&= Rx^{12}y\\
&= Rx^5y \text{ belongs to the set above.}
\end{aligned}
$$

So $\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}$ is closed under right multiplication by x.

Hence $|G| = 14$. So $G \cong D_{14}$,

$$\phi \ : F \xrightarrow[onto]{homo} D_{14}$$
$$F/ker\phi \cong D_{14}.$$

By Third Isomorphism Theorem, we have an onto homomorphism

$$\psi \ : F/R \xrightarrow[onto]{homo} F/ker\phi$$
$$R \leq ker\phi \leq F$$
$$|F/R| \leq |F/ker\phi|$$
$$F/R\big/ker\psi \cong F/ker\phi$$

with $F/R = 14$ and $F/ker\phi = 14$. So $ker\psi = 1$. Thus,

$$F/R \cong F/ker\phi \cong D_{14}.$$

So D_{14} has a presentation of $\{x, y|x^7 = y^2 = (xy)^2 = 1\}$.

$\square$

Moreover, the corresponding permutation representation is $N = D_{14} =< x, y >$, where $x = (1, 2, 3, 4, 5, 6, 7)$ and $y = (1, 6)(2, 5)(3, 4)$. Now we add the free product 2^{*7} ($|t_i's| = 2$) to this group to form our progenitor. Hence, a presentation for the progenitor $2^{*7} : N$ is given by $< x, y, t|x^7, y^2, (xy)^2, t^2, (t, N^7) >$, where $t \sim t_7$. N^7 is the point stabilizer of 7, and $(t, N^7) = 1$ means that $7^g = 7 \; \forall \; g \in N^7$. Note that $7^g = 7 \; \forall \; g \in N^7$ implies t has $[N : N^7]$ conjugates in N. Using Magma, we can see that the point stabilizer of 7 in N is equal to $N^7 =< (1, 6)(2, 5)(3, 4) >=< y >$.

Hence, a presentation of the progenitor $2^{*7} : D_{14}$ is given by $G < x, y, t >:= Group < x, y, t|x^7, y^2, (xy)^2, t^2, (t, y) >$. Now we factor this progenitor by the following relations: $(xt)^a, (xtt^x)^b, (xyt^xt)^c, (ttxt)^d$

In conclusion we obtained the following progenitor:
$< x, y, t|x^7, y^2, (xy)^2, t^2, (t, y), (xt)^a, (xtt^x)^b, (xyt^xt)^c, (ttxt)^d >$. The table below shows some finite images of the progenitor $2^{*7} : D_{14}$.

Table 1.11: Some Finite Images of the Progenitor $2^{*7} : D_{14}$

a	b	c	d	**Order of** G	**Shape of** G
0	7	**2**	0	196	$((7 \times 7) : 2)\ 2$
6	0	**3**	**6**	79464	$PGL_2(43)$
0	7	**3**	**7**	9828	$L_7(27)$
0	5	**3**	0	68880	$PGL_2(41)$
3	8	**6**	**9**	336	$PGL_2(7)$
3	0	**7**	**3**	1092	$L_2(13)$
3	**9**	**9**	**3**	504	$L_2(8)$
3	**0**	**10**	**0**	24360	$PGL_2(29)$
4	**3**	**8**	8	672	$PGL_2(7) \times 2$
0	**2**	**9**	0	1008	$L_2(8) \times 2$
3	7	0	**9**	2184	$L_2(13) \times 2$
3	0	**9**	3	5040	$A_7 : 2$
3	**8**	0	**3**	21504	$(2^6 \bullet L_2(7)) : 2$
0	**2**	10	0	48720	$PGL_2(29) \times 2$
0	3	**4**	0	4368	$PGL_2(13) \times 2$

In later chapter, we will construct some of the simple groups, given above, using the technique of double coset enumeration and prove their simplicity.

Chapter 2

Double Coset Enumeration and Factoring by the Center

2.1 Double Coset Enumeration Preliminaries

Definition 2.1. ***(Normal Subgroup).*** *A subgroup $H \leq G$ is a **normal subgroup**, denoted by $H \trianglelefteq G$, if $gHg^{-1} = H$ for every $g \in G$. [Rot12]*

Definition 2.2. ***(Right Coset).*** *If $H \leq G$ and if $k \in G$, then a **right coset** of H in G is the subset of G*

$$Hk = \{hk : h \in H\},$$

*where k is a **representative** of Hk. [Rot12]*

Definition 2.3. ***(Index).*** *If $H \leq G$, then the **index** of $H \in G$, denoted by $[G : H]$, is the number of single cosets of H in G. [Rot12]*

Definition 2.4. ***(Order).*** *If G is a group, then the **order** of G, denoted by $[G]$, is the number of elements in G. [Rot12]*

Theorem 2.5. ***(Lagrance).*** *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. [Rot12]*

Definition 2.6. ***(Double Coset).*** *Let H and K be subgroups of the group G and define a relation on G as follows:*

$$x \sim y \iff \exists\ h \in H \text{ and } k \in K \text{ such that } y = hxk$$

where $\sim$ is an equivalence relation and the equivalence classes are sets of the following form

$$HxK = \{hxk | h \in H, k \in K\} = \cup_{k\in K} Hxk = \cup_{h\in H} hxK$$

Such a subset of G is called a ***double coset****. [Cur07]*

Definition 2.7. ***(Point Stabilizer).*** *Let G be a group of permutations of a set S. For each $g, s \in S$, let $g^s = g$, then we call the set of $s \in S$ the* ***point stabilizer*** *of $g \in G$. [Cur07]*

Definition 2.8. ***(Coset Stabilizing Group).*** *The* ***coset stabilizing group*** *of a coset Nw is defined as*

$$N^{(w)} = \{\pi \in N | Nw\pi = Nw\}$$

where $n \in N$ and w is a reduced word in the t_i's. [Cur07]

Theorem 2.9. ***(Number of single cosets in*** NwN). *From above we see that,*

$$\begin{aligned} N^{(w)} &= \{\pi \in N | Nw\pi = Nw\} = \{\pi \in N | Nw\pi w^{-1} = N\} \\ &= \{\pi \in N | (Nw)^\pi = Nw\} \end{aligned}$$

and the number of single cosets in NwN is given by $[N : N^{(w)}]$. [Cur07]

Definition 2.10. ***(Orbits).*** *Let G be a group of permutations of a set S. For each $s \in S$, let $orb_G(s) = \{\phi(s) | \phi \in G\}$. The set $orb_G(s)$ is a subset of S called the* ***orbits*** *of s under G. We use $|orb_G(s)|$ to denote the number of elements in $orb_G(s)$. [Rot12]*

Definition 2.11. ***(Transversal).*** *If $K \leq G$, then a* ***(right) transversal*** *of K in G (or a complete set of right coset representatives) is a subset T of G consisting of one element from each right coset of K in G. [Rot12]*

Definition 2.12. ***(Center).*** *The* ***center*** *of a group G, denoted by $Z(G)$, is the set of all $a \in G$ that commute with every element of G. [Rot12]*

2.2 $2 \times A_5$ as a Homomorphic Image of $2^{*3} : S_3$

2.2.1 Construction of $2 \times A_5$ over S_3

Consider the group $G = 2^{*3} : S_3$ factored by the relator $[(0,1,2)t_3]^5$. Note: $N = S_3 = \{e, (1,2), (1,0), (2,0), (1,2,0), (1,0,2)\}$, where $x \sim (0,1,2)$ and $y \sim (1,2)$. Let $t \sim t_3 \sim t_0$. Let us expand the relator:

$$[(0,1,2)t_0]^5 = 1 \text{ with } \pi = (0,1,2) \text{ becomes}$$

$$1 = [\pi t_0]^5 = \pi^5 {t_0}^{\pi^4} {t_0}^{\pi^3} {t_0}^{\pi^2} {t_0}^{\pi} t_0$$

$$= (0,2,1) t_0^{(0,1,2)} t_0 t_0^{(0,2,1)} t_0^{(0,1,2)} t_0$$

$$= (3,2,1) t_1 t_0 t_2 t_1 t_0$$

$$\implies 1 = (0,2,1) t_1 t_0 t_2 t_1 t_0$$

$$\implies t_0 t_1 = (0,2,1) t_1 t_0 t_2$$

$$\implies N t_0 t_1 = N t_1 t_0 t_2.$$

Moreover if we conjugate the previous relation by all elements of S_3, we otbain the following relations:

$$(t_0t_1)^{(1,2)} = (0,2,1)^{(1,2)}(t_1t_0t_2)^{(1,2)} \implies t_0t_2 = (0,1,2)t_2t_0t_1$$
$$(t_0t_1)^{(1,0)} = (0,2,1)^{(1,0)}(t_1t_0t_2)^{(1,0)} \implies t_1t_0 = (1,2,0)t_0t_1t_2$$
$$(t_0t_1)^{(2,0)} = (0,2,1)^{(2,0)}(t_1t_0t_2)^{(2,0)} \implies t_2t_1 = (2,0,1)t_1t_2t_0$$
$$(t_0t_1)^{(1,2,0)} = (0,2,1)^{(1,2,0)}(t_1t_0t_2)^{(1,2,0)} \implies t_1t_2 = (1,0,2)t_2t_1t_0$$
$$(t_0t_1)^{(1,0,2)} = (0,2,1)^{(1,0,2)}(t_1t_0t_2)^{(1,0,2)} \implies t_2t_0 = (2,1,0)t_0t_2t_1.$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$ where g_i's words in t_i's.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [Nw^n | n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by t_i's. We need to identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilising group $N^{(w)}$

NeN

First, the double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{t_0, t_1, t_2\}$, the orbit of N on $\{0, 1, 2\}$ is:

$$\mathbb{O} = \{0, 1, 2\}.$$

We choose t_0 as our symmetric generator from $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by [0]. Hence, three symmetric generators will go the new double coset [0].

Nt₀N

In order to find how many single cosets [0] contains, we must first find the coset stabilizer $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 =< e, (1, 2) >$ so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{6}{2} = 3$. These three single cosets in [0] are $\{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2\}$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_1, t_2\}$ are:

$$\mathbb{O} = \{0\} \text{ and } \{1, 2\}.$$

We take t_0 and t_1 from each orbit, respectively, and to see which double coset Nt_0t_0 and Nt_0t_1 belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to NeN and two symmetric generators will go to a new double coset Nt_0t_1N, denoted by [01].

Nt₀t₁N

Now Nt_0t_1N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(01)} = N^{01} =< e >$. Only identity (e) will fix 0 and 1. Hence the number of single cosets in Nt_0t_1N is $\frac{|N|}{|N^{(01)}|} = \frac{6}{1} = 6$. These six single cosets in [01] are $\{Nt_0t_1, Nt_1t_0, Nt_0t_2, Nt_2t_0, Nt_1t_2, Nt_2t_1\}$. The orbits of $N^{(01)}$ on $\{t_0, t_1, t_2\}$ are:

$$\mathbb{O} = \{0\}, \ \{1\}, \text{ and } \{2\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_i$ belongs to.

We have:

$$
\begin{aligned}
Nt_0t_1t_1 &= Nt_0 \in [0] \\
\text{relation: } (0,2,1)^{(0,1)}(t_1t_0t_2)^{(0,1)} &= (t_0t_1)^{(0,1)} \\
\implies (1,2,0)t_0t_1t_2 &= t_1t_0 \\
\implies Nt_0t_1t_2 &= Nt_1t_0 \in [01] \\
Nt_0t_1t_0 &\in [010].
\end{aligned}
$$

The new double cosets have single coset representatives $Nt_0t_1t_0$, which is denoted by [010].

$Nt_0t_1t_0N$

Now $Nt_0t_1t_0N$ in N is a new double coset. However, $N^{(010)} = N^{010} =< e >$. Only identity ($e$) will fix 0, and 1. Hence the number of single cosets contained in $Nt_0t_1t_0N$ is $\frac{|N|}{|N^{(010)}|} = \frac{6}{1} = 6$. These six single cosets in [010] are $\{Nt_0t_1t_0, Nt_1t_0t_1, Nt_0t_2t_0, Nt_2t_0t_2, Nt_1t_2t_1, Nt_2t_1t_2\}$. The orbits of $N^{(010)}$ on $\{0,1,2\}$ are:

$$\mathbb{O} = \{0\},\ \{1\},\ \text{and } \{2\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_i$ belongs to. We have:

$$
\begin{aligned}
Nt_0t_1t_0t_0 &= N \in [01] \\
\text{relation: } t_1(0,2,1)t_1t_0t_2 &= t_1t_0t_1 \\
\implies (0,2,1)\underline{(0,2,1)^{-1}t_1(0,2,1)}t_1t_0t_2 &= t_1t_0t_1 \\
\implies (0,2,1)t_1^{(0,2,1)}t_1t_0t_2 &= t_1t_0t_1 \\
\implies (0,2,1)t_0t_1t_0t_2 &= t_1t_0t_1 \\
\implies Nt_0t_1t_0t_2 &= Nt_1t_0t_1 \in [010] \\
Nt_0t_1t_0t_1 &\in [0101].
\end{aligned}
$$

The new double coset is $Nt_0t_1t_0t_1N$, denoted by [0101].

$Nt_0t_1t_0t_1N$

Now $Nt_0t_1t_0t_1N$ is a new double coset. We determine how many single cosets are in this double coset. We have $N^{(0101)} = N^{0101} =< e >$. But $Nt_0t_1t_0t_1$ is not distinct. Using the retlation: $t_0t_1 = (0,2,1)t_1t_0t_2 \implies t_0t_1t_0t_1 = (0,2,1)t_1t_0\underline{t_2t_0t_1} = (0,2,1)t_1t_0(2,1,0)t_0t_2 = (1,2,0)t_0t_2t_0t_2$. Now $t_0t_1t_0t_1 = (1,2,0)t_0t_2t_0t_2$ $\implies Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2$. Thus $N(t_0t_1t_0t_1)^n = Nt_0t_2t_0t_2$. Then $N(t_0t_1t_0t_1)^{(1,2)} = Nt_0t_2t_0t_2$. But $Nt_0t_2t_0t_2 = Nt_0t_1t_0t_1 \implies (1,2) \in N^{(0101)}$ since $N(t_0t_1t_0t_1)^{(1,2)} = Nt_0t_2t_0t_2$. We conclude:

$$N^{(0101)} \geq < e, (1,2) > .$$

Hence $|N^{(0101)}| = 2$ so the number of single cosets in $N^{(0101)}$ is $\frac{|N|}{|N^{(0101)}|} = \frac{6}{2} = 3$. These three single cosets in [010] are $Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2$, $Nt_1t_0t_1t_0 = Nt_1t_2t_1t_2$, and $Nt_2t_1t_2t_1 = Nt_2t_0t_2t_0$. The orbits of $N^{(0101)}$ on $\{0,1,2\}$ are:

$$\mathbb{O} = \{0\} \text{ and } \{1,2\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_1t_i$ belongs to. We have:

$$Nt_0t_1t_0t_1t_0 \in [01010]$$
$$Nt_0t_1t_0t_1t_1 = Nt_0t_1t_0 \in [010].$$

$Nt_0t_1t_0t_1t_0N$

Now $Nt_0t_1t_0t_1t_0N$ is indeed a new double coset. We determine how many single cosets are in this double coset. We have $N^{(01010)} = N^{01010} =< e >$. $Nt_0t_1t_0t_1t_0$ has six names. We have the following:

$$Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 \implies Nt_0t_1t_0t_1t_0 = Nt_0t_2t_0t_2t_0$$
$$Nt_1t_0t_1t_0 = Nt_1t_2t_1t_2 \implies Nt_1t_0t_1t_0t_1 = Nt_1t_2t_1t_2t_1$$
$$Nt_2t_1t_2t_1 = Nt_2t_0t_2t_0 \implies Nt_2t_1t_2t_1t_2 = Nt_2t_0t_2t_0t_2.$$

Now we want to show that the six names are the same. We have the following:

$$t_0t_1t_0t_1t_0 = t_0t_1t_0\underline{t_1t_0t_2}t_2$$
$$= t_0t_1t_0(2,0,1)t_0t_1t_2 = t_1\underline{t_2t_1t_0}t_1t_2$$
$$= t_1(2,0,1)t_1t_2t_1t_2 = (2,0,1)t_2t_1t_2t_1t_2$$
$$\implies t_0t_1t_0t_1t_0 = (2,0,1)t_2t_1t_2t_1t_2 \quad (1)$$

$$\implies Nt_0t_1t_0t_1t_0 = Nt_2t_1t_2t_1t_2,$$

$$t_0t_2t_0t_2t_0 = t_0t_2t_0\underline{t_2t_0t_1}t_1$$
$$= t_0t_2t_0(2,1,0)t_0t_2t_1 = t_2\underline{t_1t_2t_0}t_2t_1$$
$$= t_2(1,0,2)t_2t_1t_2t_1 = (1,0,2)t_1t_2t_1t_2t_1$$
$$\implies t_0t_2t_0t_2t_0 = (1,0,2)t_1t_2t_1t_2t_1 \quad (2)$$
$$\implies Nt_0t_2t_0t_2t_0 = Nt_1t_2t_1t_2t_1.$$

Now by (1) and (2) we have that the six names are equal. Hence,

$$t_0t_1t_0t_1t_0 \sim t_0t_2t_0t_2t_0 \sim t_1t_0t_1t_0t_1 \sim t_1t_2t_1t_2t_1 \sim t_2t_1t_2t_1t_2 \sim t_2t_0t_2t_0t_2$$

Therefore, $N^{(01010)} = n \in N|N(01010)^n = N(01010)$. Thus, $N^{(01010)} \geq < (1,2),(0,2,1) >$ then $N^{(01010)} = N$. Hence $|N^{(01010)}| = 6$, so the number of single cosets in $N^{(01010)}$ is $\frac{|N|}{|N^{(01010)}|} = \frac{6}{6} = 1$. The orbit of $N^{(01010)}$ on $\{1,2,0\}$ is $\{1,2,0\}$. Take a representative from this orbit, say t_0. Hence $Nt_0t_1t_0t_1t_0t_0 \in [0101]$. Therefore, three symmetric generators will go back to $Nt_0t_1t_0t_1N$.

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of N in G is 20. We conclude:

$G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N \cup Nt_0t_1t_0t_1N \cup Nt_0t_1t_0t_1t_0N$, where

$$G = \frac{2^{*3} : S_3}{t_0t_1 = (0,2,1)t_1t_0t_2}$$

$|G| \leq \left(|N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(010)}} + \frac{|N|}{N^{(0101)}} + \frac{|N|}{N^{(01010)}}\right) \times |N|$

$|G| \leq (1+3+6+6+3+1) \times 6$

$|G| \leq 20 \times 6$

$|G| \leq 120.$

A Cayley diagram that summarizes the above information is given below:

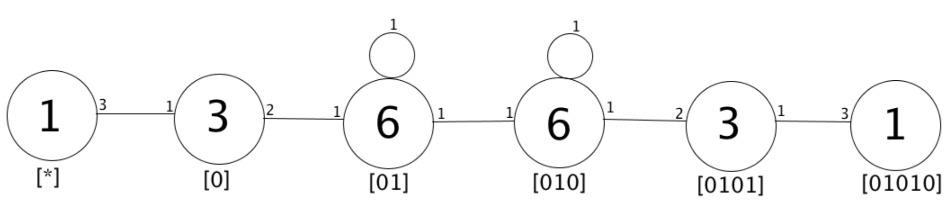

Figure 2.1: Cayley Diagram of $2 \times A_5$ over S_3

2.2.2 Permutation Representation of $2 \times A_5$ over S_3

In order to find the permutation representation of $G = 2^{*3} : S_3$, in terms of x, y, and t_0, we create a table in which we conjugate the twenty single cosets by x and y and we right multiply them by t_0.

Table 2.1: Permutation Representation of $2 \times A_5$ over S_3

Cosets	$x \sim (0,1,2)$	$y \sim (1,2)$	$t \sim t_0$
1. N	1. N	1. N	2. Nt_0
2. Nt_0	3. Nt_1	2. Nt_0	1. N
3. Nt_1	4. Nt_2	4. Nt_2	6. Nt_1t_0
4. Nt_2	2. Nt_0	3. Nt_1	8. Nt_2t_0
5. Nt_0t_1	9. Nt_1t_2	7. Nt_0t_2	11. $Nt_0t_1t_0$
6. Nt_1t_0	10. Nt_2t_1	8. Nt_2t_0	3. Nt_1
7. Nt_0t_2	6. Nt_1t_0	5. Nt_0t_1	13. $Nt_0t_2t_0$
8. Nt_2t_0	5. Nt_0t_1	6. Nt_1t_0	4. Nt_2
9. Nt_1t_2	8. Nt_2t_0	10. Nt_2t_1	10. $Nt_1t_2t_0$
10. Nt_2t_1	7. Nt_0t_2	9. Nt_1t_2	9. $Nt_2t_1t_0$
11. $Nt_0t_1t_0$	15. $Nt_1t_2t_1$	13. $Nt_0t_2t_0$	5. Nt_0t_1
12. $Nt_1t_0t_1$	16. $Nt_2t_1t_2$	14. $Nt_2t_0t_2$	18. $Nt_1t_0t_1t_0$
13. $Nt_0t_2t_0$	12. $Nt_1t_0t_1$	11. $Nt_0t_1t_0$	7. Nt_0t_2
14. $Nt_2t_0t_2$	11. $Nt_0t_1t_0$	12. $Nt_1t_0t_1$	19. $Nt_2t_0t_2t_0$
15. $Nt_1t_2t_1$	14. $Nt_2t_0t_2$	16. $Nt_2t_1t_2$	16. $Nt_1t_2t_1t_0$
16. $Nt_2t_1t_2$	13. $Nt_0t_2t_0$	15. $Nt_1t_2t_1$	15. $Nt_2t_1t_2t_0$
17. $Nt_0t_1t_0t_1$	18. $Nt_1t_2t_1t_2$	17. $Nt_0t_2t_0t_2$	20. $Nt_0t_1t_0t_1t_0$
18. $Nt_1t_0t_1t_0$	19. $Nt_2t_1t_2t_1$	19. $Nt_2t_0t_2t_0$	12. $Nt_1t_0t_1$
19. $Nt_2t_1t_2t_1$	17. $Nt_0t_2t_0t_2$	18. $Nt_1t_2t_1t_2$	14. $Nt_2t_1t_2t_1$
20. $Nt_0t_1t_0t_1t_0$	20. $Nt_1t_2t_1t_2t_1$	20. $Nt_0t_2t_0t_2t_0$	17. $Nt_0t_1t_0t_1$

We have:

$$\phi(x) = (2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)$$

$$\phi(y) = (3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)$$

$$\phi(t) = (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20).$$

Thus, we have a homomorphism $\phi : 2^{*3} : S_3 \longrightarrow S_{20}$. Then $\phi(G) =< \phi(x), \phi(y), \phi(t) >$. In order for us to prove that $\phi(G) =< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of

$G = 2^{*3} : S_3$, we must have the the following conditions met:

(1) $\phi(N) \cong S_3$

(2) $\phi(t)$ has three conjugates under conjugation by $\phi(N)$

(3) $\phi(N)$ acts as S_3 on the three congugates of $\phi(t)$ by conjugates.

Proof. We have $\phi(N) =< \phi(x), \phi(y) >$.

$$\begin{aligned}
(1)\ \phi(N) &=< \phi(x), \phi(y) > \\
&=< (2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19), \\
&\quad (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20) > \\
&\cong S_3 \text{ since } |\phi(x)\phi(y)| = |(2,4)(5,10)(6,9)(7,8)(11,16)(12,15)(13,14)(17,19)| \\
&= 2.
\end{aligned}$$

Thus, $\phi(N) \cong S_3$.
(2) We need to compute $\phi(t)^{\phi(N)}$:

$$\begin{aligned}
\phi(t)^{\phi(x)} &= \{(1,2)(3,6)\dots(15,16)(17,20)^{(2,3,4)(5,9,8)\dots(12,16,13)(17,18,19)}\} \\
&= \{(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)\} \\
&= t_1. \\
\phi(t)^{\phi(x^2)} &= \{(1,2)(3,6)\dots(15,16)(17,20)^{(2,4,3)(5,9,8)\dots(12,13,16)(17,19,18)}\} \\
&= \{(1,4)(2,7)(3,9)(8,14)(10,16)(5,6)(13,17)(15,18)(11,12)(19,20)\} \\
&= t_2. \\
\phi(t)^{\phi(x^3)} &= \{(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)^e\} \\
&= \{(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)\} \\
&= t_0.
\end{aligned}$$

Thus, $\phi(t)^{\phi(N)} = \{t_0, t_1, t_2\}$.
(3) We need to show that $\phi(N)$ acts as S_3 on the three conjugates of $\phi(t)$ by conjugates.
First, we have to conjugate by $\phi(x)$:

$t_0^{\phi(x)} = t_1$
$t_1^{\phi(x)} = (t_0^{\phi(x)})^{\phi(x)} = t_0^{\phi(x^2)} = t_2$
$t_2^{\phi(x)} = (t_1^{\phi(x)})^{\phi(x)} = t_1^{\phi(x^2)} = (t_0^{\phi(x)})^{\phi(x^2)} = t_0^{\phi(x^3)} = t_0.$

Thus, $\phi(x) = (t_0, t_1, t_2)$.

Next, we have to conjugate by $\phi(y)$:

$$\begin{aligned}
t_1^{\phi(y)} &= \{(1,3)(4,10)\dots(14,13)(18,20)^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}\} \\
&= \{(1,4)(2,7)(3,9)(8,14)(10,16)(5,6)(13,17)(15,18)(11,12)(19,20)\} \\
&= t_2 \\
t_2^{\phi(y)} &= (t_1^{\phi(y)})^{\phi(y)} = t_1^{\phi(y^2)} = t_1.
\end{aligned}$$

Thus, $\phi(y) = (t_1, t_2)$.

Hence, $\phi(G) =< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of $G = 2^{*3} : S_3$. $\square$

We have:

$$G = \frac{2^{*3} : S_3}{(0,2,1)t_1t_0t_2 = t_0t_1}.$$

Now, we want to verify if $\phi(0,2,1) = \phi(t_0t_1t_2t_0t_1)$ then $< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of G.

Verify: $\phi(x^{-1}) = \phi(t_0t_1t_2t_0t_1)$:

$$\begin{aligned}
\phi(t_0t_1t_2t_0t_1) =&(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20) \\
&(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20) \\
&(1,4)(2,7)(3,9)(8,14)(10,16)(5,6)(13,17)(15,18)(11,12)(19,20) \\
&(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20) \\
&(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20) \\
=&(2,4,3)(5,8,9)(6,7,10)(11,14,15)(12,13,16)(17,19,18) \\
=&\phi(x^{-1}).
\end{aligned}$$

Hence, $\phi : G \overset{homo.}{\longrightarrow} S_{20}$ with $\phi(G) =< \phi(x), \phi(y), \phi(t) >$. By FIT we have:

$G/ker\phi \cong \phi(G)$

$\implies |G/ker\phi| \cong |\phi(G)|$
$\implies |G| = |ker\phi||\phi(G)|$.
By completing the double coset enumeration we know $|G| \leq 120$. Moreover, by Magma, $|\phi(G)| = | < \phi(x), \phi(y), \phi(t_0) > | = 120$.
So, $|G| = |ker\phi|120$
$\implies |G| \geq 120$.
Hence, $|G| = 120$.

2.2.3 Prove $G \cong 2 \times A_5$

We use two different methods to prove that $G \cong 2 \times A_5$.
(1) We will prove by hand that $G \cong 2 \times A_5$.

Proof. Given:

$$\begin{aligned}
\phi(x) &= (2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)\\
\phi(y) &= (3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)\\
t_0 &= (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)\\
t_1 &= (1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)
\end{aligned}$$

Note:

$$\begin{aligned}
\bullet|\phi(y)\phi(x)t_0t_1t_0t_1t_0| = |&(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)\\
&(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)\\
&(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)\\
&(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)\\
&(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)\\
&(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)\\
&(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)|\\
=&|(1,20)(2,17)(3,18)(4,19)(5,11)(6,12)(7,13)(8,14)(9,15)(10,16)|
\end{aligned}$$

Thus, $|\phi(y)\phi(x)t_0t_1t_0t_1t_0| = 2$, which is the center of order two.

$$\begin{aligned}\bullet|\phi(x)t_0| = &|(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)\\ &(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)|\\ &= |(2,6,9,4,1),(3,8,11,16,7)(5,10,13,18,14)(12,15,19,20,17)|\\ &= 5.\end{aligned}$$

$$\bullet|t_0| = (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20) = 2.$$

$$\begin{aligned}\bullet|\phi(x)t_0t_0| &= |\phi(x)|\\ &= |(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)|\\ &= 3.\end{aligned}$$

Hence, $< \phi(x)t_0, t_0 >= A_5$.

Now, $< \phi(x)t_0, t_0, \phi(y)\phi(x)t_0t_1t_0t_1t_0 >\leq \phi(G) =< \phi(x), \phi(y), \phi(t) >$

$\implies 2 \times A_5 \leq \phi(G)$. But $|\phi(G)| = 120$ and $|2 \times A_5| = 120$.

Thus, $\phi(G) = 2 \times A_5$. □

(2) We use the composition factors of G to construct a computer based proof to show that $G \cong 2 \times A_5$.

Proof. Given:

$$G =< x, y, t|x^3, y^2, (xy)^2, t^2, (t, y), tt^x = x^{-1}t^x tt^{x^2} >= \frac{2^{*3}:S_3}{(0,2,1)t_1t_0t_2=t_0t_1}.$$

We use Magma, to obtain the following composition factors:

```
> CompositionFactors(G1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    1
```

Hence, G has the following composition series $G \supset G_1 \supset 1$, where $G = (G/G_1)(G_1/1) = A_5C_2$. The normal lattice of G is:

```
> NL:=NormalLattice(G1);
> NL;
```

```
Normal subgroup lattice
-----------------------

[4]  Order 120  Length 1  Maximal Subgroups: 2 3
---
[3]  Order 60   Length 1  Maximal Subgroups: 1
---
[2]  Order 2    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

First, we look at the center of G and we find it is of order 2. In addition, by looking at the normal lattice, we find that the normal subgroup NL[2] is of order 2. Hence, we might have a direct product of NL[2] by the Alternating group NL[3]= A_5. Note: $NL[2] \trianglelefteq G_1$ and $A_5 \trianglelefteq G_1$ then $A_5 \cap NL[2] = 1$.

```
> D:=DirectProduct(CyclicGroup(2),NL[3]);
> s:=IsIsomorphic(D,G1);s;
true
```

We use the above loops to confirm that G is a direct product of a cyclic group of order 2 by A_5. By using ATLAS, the presentation of the Alternating group (A_5) is:
$< a, b|a^2, b^3, (a * b)^5 >$.
Now, the element of NL[2] commutes with the element of NL[3]= A_5, since G is a direct product extension. Thus, we have the following presentation for G:

```
> H<a,b,c>:=Group<a,b,c|a^2,b^3,(a*b)^5,c^2,(c,a),(c,b)>; #H;
120
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,G1);s;
true
```

Hence, $G \cong 2 \times A_5$. □

2.3 Finding and Factoring by the Center ($Z(G)$) of $2 \times A_5$ over S_3

Let G acts on $X = \{N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N \cup Nt_0t_1t_0t_1N \cup Nt_0t_1t_0t_1t_0N\}$ where $|X| = 20$. From the Cayley Diagram of $2 \times A_5$ over S_3 we see that G is transitive. Moreover, from the DCE and the Cayley diagram it is clear that the double coset $Nt_0t_1t_0t_1t_0N$ contains one single coset. We stabilize the coset N then another coset $Nt_0t_1t_0t_1t_0N$ at the maximal distance from N also stabilize. This means $\{N, Nt_0t_1t_0t_1t_0N\}$ is a nontrivial block of size 2. Let B be a nontrivial block and $N \in B$. If $Nt_0t_1t_0t_1t_0 \in B$. Then

$$\begin{aligned}
B &= \{N, Nt_0t_1t_0t_1t_0\} = \{1, 20\} \\
Bt_0 &= \{Nt_0, Nt_0t_1t_0t_1\} = \{2, 17\} \\
Bt_1 &= \{Nt_1, \underline{Nt_0t_1t_0t_1t_0}t_1\} = \{Nt_1, Nt_1t_0t_1t_0t_1t_1\} \\
&= \{Nt_1, Nt_1t_0t_1t_0\} = \{3, 18\} \\
Bt_2 &= \{Nt_2, \underline{Nt_0t_1t_0t_1t_0}t_2\} = \{Nt_2, Nt_2t_1t_2t_1t_2t_2\} \\
&= \{Nt_2, Nt_2t_1t_2t_1\} = \{4, 19\} \\
Bt_0t_1 &= \{Nt_0t_1, Nt_0t_1t_0\} = \{5, 11\} \\
Bt_1t_0 &= \{Nt_1t_0, \underline{Nt_0t_1t_0t_1t_0}t_1t_0\} = \{Nt_1t_0, Nt_1t_0t_1t_0t_1t_1t_0\} \\
&= \{Nt_1t_0, Nt_1t_0t_1\} = \{6, 12\} \\
Bt_2t_0 &= \{Nt_2t_0, \underline{Nt_0t_1t_0t_1t_0}t_2t_0\} = \{Nt_2t_0, Nt_2t_0t_2t_0t_2t_2t_0\} \\
&= \{Nt_2t_0, Nt_2t_0t_2\} = \{8, 14\} \\
Bt_0t_2 &= \{Nt_0t_2, \underline{Nt_0t_1t_0t_1}t_2\} = \{Nt_0t_2, Nt_0t_2t_0t_2t_2\} \\
&= \{Nt_0t_2, Nt_0t_2t_0\} = \{7, 13\} \\
Bt_1t_2 &= \{Nt_1t_2, \underline{Nt_0t_1t_0t_1t_0}t_1t_2\} = \{Nt_1t_2, Nt_1t_0t_1t_0t_1t_1t_2\} \\
&= \{Nt_1t_2, \underline{Nt_1t_0t_1t_0}t_2\} = \{Nt_1t_2, Nt_1t_2t_1t_2t_2\} \\
&= \{Nt_1t_2, Nt_1t_2t_1\} = \{9, 15\} \\
Bt_2t_1 &= \{Nt_2t_1, \underline{Nt_0t_1t_0t_1t_0}t_2t_1\} = \{Nt_2t_1, Nt_2t_1t_2t_1t_2t_2t_1\} \\
&= \{Nt_2t_1, Nt_2t_1t_2\} = \{10, 16\}
\end{aligned}$$

We can see that $\{Bt_0, Bt_1, Bt_2, Bt_0t_1, Bt_1t_0, Bt_2t_0, Bt_0t_2, Bt_1t_2, Bt_2t_1\} \cap B = \emptyset$ and $\{Bt_0, Bt_1, Bt_2, Bt_0t_1, Bt_1t_0, Bt_2t_0, Bt_0t_2, Bt_1t_2, Bt_2t_1\} \neq B$. Hence, we have blocks of imprimitive of size two. Therefore, $|Z(G)|$=2 where

$Z(G) =< nw >$ (central elements permute the elements of each block of imprimitive)

$= \{(1, 20)(2, 17)(3, 18)(4, 19)(5, 11)(6, 12)(7, 13)(8, 14)(9, 15)(10, 16)\}$.

Now, we are going to find the central element of order 2 in G not in its homomorphic image G_1. Consider $nt_0t_1t_0t_1t_0 = 1 \in G$;
In addition, $t_0t_1t_0t_1t_0 = n^{-1}$. Let $r = n^{-1}$. Then $t_0t_1t_0t_1t_0 = r$. We now compute r by its action on the cosets $\{Nt_0, Nt_1, Nt_2\}$. Recall that our relation is $t_0t_1 = (0, 2, 1)t_1t_0t_2$. In addition, if we conjugate this relation by the elements of $N = S_3$, we obtain the following relations:

$$t_0t_1 = (0, 2, 1)t_1t_0t_2$$
$$t_0t_2 = (0, 1, 2)t_2t_0t_1$$
$$t_1t_0 = (1, 2, 0)t_0t_1t_2$$
$$t_2t_1 = (2, 0, 1)t_1t_2t_0$$
$$t_1t_2 = (1, 0, 2)t_2t_1t_0$$
$$t_2t_0 = (2, 1, 0)t_0t_2t_1.$$

Compute r by its action on the coset Nt_0:

$$\begin{aligned}
Nt_0^r &= Nt_0^{t_0t_1t_0t_1t_0} = N(t_0t_1t_0t_1t_0)^{-1}t_0t_0t_1t_0t_1t_0 \\
&= N\underline{t_0t_1}t_0t_1t_0t_1t_0t_1t_0 = Nt_1t_0\underline{t_2t_0t_1}t_0t_1t_0t_1t_0 \\
&= Nt_1t_0(2, 1, 0)t_0t_2t_0t_1t_0t_1t_0 \\
&= Nt_0t_2t_0\underline{t_2t_0t_1}t_0t_1t_0 = Nt_0t_2t_0(2, 1, 0)t_0t_2t_0t_1t_0 \\
&= Nt_2\underline{t_1t_2t_0}t_2t_0t_1t_0 = Nt_2(1, 0, 2)t_2t_1t_2t_0t_1t_0 \\
&= Nt_1t_2\underline{t_1t_2t_0}t_1t_0 = Nt_1t_2(1, 0, 2)t_2t_1t_1t_0 \\
&= N\underline{t_0t_1t_2}t_0 = Nt_1t_0t_0 \\
&= Nt_1.
\end{aligned}$$

Now, compute r by its action on the coset Nt_1:

$$\begin{aligned}
Nt_1^r &= Nt_1^{t_0t_1t_0t_1t_0} = N(t_0t_1t_0t_1t_0)^{-1}t_1t_0t_1t_0t_1t_0 \\
&= Nt_0t_1t_0t_1t_0t_1t_0t_1t_0\underline{t_1t_0} = Nt_0t_1t_0t_1t_0t_1t_0t_1t_0(1, 2, 0)t_0t_1t_2 \\
&= Nt_1t_2t_1t_2t_1t_2t_1\underline{t_2t_1t_0}t_1t_2 = Nt_1t_2t_1t_2t_1t_2t_1(2, 0, 1)t_1t_2t_1t_2 \\
&= Nt_2t_0t_2t_0t_2\underline{t_0t_2t_1}t_2t_1t_2 = Nt_2t_0t_2t_0t_2(0, 1, 2)t_2t_0t_2t_1t_2 \\
&= Nt_0t_1t_0\underline{t_1t_0t_2}t_0t_2t_1t_2 = Nt_0t_1t_0(1, 2, 0)t_0t_1t_0t_2t_1t_2
\end{aligned}$$

$$
\begin{aligned}
&= Nt_1\underline{t_2t_1t_0}t_1t_0t_2t_1t_2 = Nt_1(2,0,1)t_1t_2t_1t_0t_2t_1t_2\\
&= Nt_2t_1\underline{t_2t_1t_0}t_2t_1t_2 = Nt_2t_1(2,0,1)t_1t_2t_2t_1t_2\\
&= Nt_0t_2t_2\\
&= Nt_0.
\end{aligned}
$$

Next, compute r by its action on the coset Nt_2:

$$
\begin{aligned}
Nt_2^r &= Nt_2^{t_0t_1t_0t_1t_0}\\
&= N(t_0t_1t_0t_1t_0)^{-1}t_2t_0t_1t_0t_1t_0\\
&= Nt_0t_1t_0\underline{t_1t_0t_2}t_0t_1t_0t_1t_0\\
&= Nt_0t_1t_0(1,2,0)t_0t_1t_0t_1t_0t_1t_0\\
&= Nt_1\underline{t_2t_1t_0}t_1t_0t_1t_0t_1t_0\\
&= Nt_1(2,0,1)t_1t_2t_1t_0t_1t_0t_1t_0\\
&= Nt_2t_1\underline{t_2t_1t_0}t_1t_0t_1t_0\\
&= Nt_2t_1(2,0,1)t_1t_2t_1t_0t_1t_0\\
&= N\underline{t_0t_2t_1}t_2t_1t_0t_1t_0\\
&= Nt_2t_0\underline{t_2t_1t_0}t_1t_0\\
&= Nt_2t_0(2,0,1)t_1t_2t_1t_0\\
&= Nt_0t_1t_1t_2t_1t_0\\
&= Nt_0\underline{t_2t_1t_0}\\
&= Nt_0(2,0,1)t_1t_2\\
&= Nt_1t_1t_2\\
&= Nt_2.
\end{aligned}
$$

Thus, $Nt_0^r = t_1$, $Nt_1^r = t_0$, and $Nt_2^r = t_2$. Therefore, $r = (0,1)$, and the generator of the center is $t_0t_1t_0t_1t_0 = (0,1)$. Hence, the center of G is $Z(G) =< (0,1)t_0t_1t_0t_1t_0 >$.

Now, we factor

$$G \cong \frac{2^{*3} : S_3}{(0,2,1)t_1t_0t_2 = t_0t_1}$$

by the center $Z(G)$, that is:

$$G \cong \frac{2^{*3} : S_3}{(0,2,1)t_1t_0t_2 = t_0t_1,\ (0,1)t_0t_1t_0t_1t_0}.$$

The Cayley diagram of G over N shown below illustrates that $[*]$ consists of N only. Moreover, $[0]$ consists of three cosets, $Nt_0N = \{Nt_0, Nt_1, Nt_2\}$ and the orbits of $N^{(0)}$ on $\{0,1,2\}$ are: $\mathbb{O} = \{0\}$ and $\{1,2\}$. Take a representative t_i from each orbit and see which double coset Nt_0t_i belongs to. We have: $Nt_0t_0 \in [*]$ and $Nt_0t_1 \in [01]$. In addition, $[01]$ consists of six cosets,
$Nt_0t_1N = \{Nt_0t_1, Nt_1t_0, Nt_2t_0, Nt_0t_2, Nt_1t_2, Nt_2t_1\}$ and the orbits $N^{(01)}$ on $\{0,1,2\}$ are: $\mathbb{O} = \{0\}$, $\{1\}$, and $\{2\}$. Now we take a representative t_i from each orbit and see which double coset $Nt_0t_1t_i$ belongs to. We have: $Nt_0t_1t_1 \in [0]$, by using the main relation $(0,2,1)t_1t_0t_2 = t_0t_1 \implies Nt_1t_0 = Nt_0t_1t_2$, thus, $Nt_0t_1t_2 \in [01]$. Moreover, by using the center $(0,1)t_0t_1 = t_0t_1t_0 \implies Nt_0t_1 = Nt_0t_1t_0$, hence, $Nt_0t_1t_0 \in [01]$. This completes our double coset enumeration of G factor by the center $Z(G)$ and our Cayley diagram is as follows:

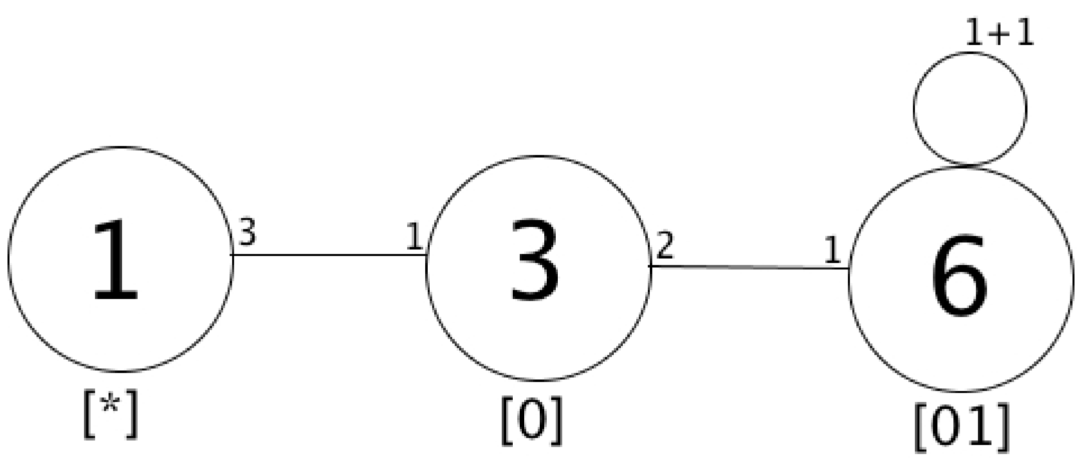

Figure 2.2: Cayley Diagram of A_5 over S_3

2.4 Converting Symmetric and Permutation Representation of $2 \times A_5$ over S_3

Now we want to convert symmetric representation of $G = 2 \times A_5$ over S_3 to permutation representation. Note: every element of G is of the form nw where n is a permutation of S_3 on three letters and w is a word in $\{t_0, t_1, t_2\}$ of length at most four and every element of G is also a permutation of the set of 20 cosets of N in G.

The following examples are converting symmetric representation to its permutation representation of G.

Example 2.13. $(0,1,2)t_0t_1t_0$

$= \phi(x)\phi(t)\phi(t^{\phi(x)})\phi(t)$

$= (2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)$

$(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$

$(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)$

$(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$

$= (1,11,14)(2,18,7)(3,13,17)(4,6,16)(5,8,20)(10,19,12)$

Thus, $(0,1,2)t_0t_1t_0 = (1,11,14)(2,18,7)(3,13,17)(4,6,16)(5,8,20)(10,19,12)$.

Example 2.14. $(2,0)t_0t_1t_2$

$= \phi(y)^{\phi(x)}\phi(t)\phi(t^{\phi(x)})\phi(t^{\phi(x^2)})$

$= (3,4)(5,7)\ldots(15,16)(18,19)^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$

$(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)$

$(1,4)(2,7)(3,9)(8,14)(10,16)(5,6)(13,17)(15,18)(11,12)(19,20)$

$= (1,6)(3,11)(4,9)(5,18)(7,16)(10,13)(12,20)(15,19)$

Hence, $(2,0)t_0t_1t_2 = (1,6)(3,11)(4,9)(5,18)(7,16)(10,13)(12,20)(15,19)$.

Next, we want to convert permutation representation of $G = 2 \times A_5$ over S_3 to symmetric representation. Let p be a permutation on twenty letters. We write it in the form nw, where $n \in S_3$ and w is a word in at most three $t_i's$. Note: $Np = 1^p = Nw$, where

$p \in Np$

$\implies p \in Nw$

$\implies p = nw$ for some $n \in N$

$\implies n = pw^{-1}$.

The following examples are converting permutation representation to its symmetric representation of G.

Example 2.15. Let $p = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12)$. Note: $p = nw \implies n = pw^{-1}$.

$Np = 1^p = 11 = Nt_0t_1t_0$

$\implies Np = Nt_0t_1t_0$

$\implies p = nt_0t_1t_0$

$\implies n = pt_0t_1t_0$

$n = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12)$

$(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)$

$(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)$

$(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)$

$\implies n = (2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)$

Next, we compute n in the actions on $\{Nt_0, Nt_1, Nt_2\}$:

$Nt_0^n = Nt_0^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 2^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 3$

$= Nt_1$.

$Nt_1^n = Nt_1^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 3^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 4$

$= Nt_2$.

$Nt_2^n = Nt_2^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 4^{(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)}$

$= 2$

$= Nt_0$.

Thus, $n = (0, 1, 2)$. We know that $p = nt_0t_1t_0 \implies p = (0, 1, 2)t_0t_1t_0$. Hence, $p = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12) = (0, 1, 2)t_0t_1t_0$.

Example 2.16. Let $p = (1,5,14,20,11,8)(2,3,7,17,18,13)(4,12,16,19,6,10)(9,15)$.
Note: $p = nw \implies n = pw^{-1}$.

$$Np = 1^p = 5 = Nt_0t_1$$
$$\implies N^p = Nt_0t_1$$
$$\implies p = nt_0t_1$$
$$\implies n = pt_1t_0$$
$$n = (1,5,14,20,11,8)(2,3,7,17,18,13)(4,12,16,19,6,10)(9,15)$$
$$(1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)$$
$$(1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$$
$$\implies n = (3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)$$

Next, we compute n in the actions on $\{Nt_0, Nt_1, Nt_2\}$:

$$Nt_0^n = Nt_0^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 2^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 2$$
$$= Nt_0.$$
$$Nt_1^n = Nt_1^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 3^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 4$$
$$= Nt_2.$$
$$Nt_2^n = Nt_2^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 4^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}$$
$$= 3$$
$$= Nt_1.$$

Thus, $n = (1,2)$. We know that $p = nt_0t_1 \implies p = (1,2)t_0t_1$. Hence,
$p = (1,5,14,20,11,8)(2,3,7,17,18,13)(4,12,16,19,6,10)(9,15) = (1,2)t_0t_1$.

2.5 $2 \times S_5$ as a Homomorphic Image of $2^{*4} : S_4$

2.5.1 Construction of $2 \times S_5$ over S_4

Consider the group $G = 2^{*4} : S_4$ factored by the relator $[(0,1,2) = t_0t_1t_2t_0$.
Note: $N = S_4 = \{e, (3,0), (1,3,0,2), (1,3,0), (1,0,3), (1,0)(2,3), (1,2), (1,0,2), (1,2,3,0),$
$(1,3), (1,0), (2,3), (2,0,3), (1,3,2,0), (2,3,0), (1,2,3), (1,2,0,3), (1,0,2,3), (1,2,0),$

$(1,3)(2,0),(1,0,3,2),(2,0),(1,2)(3,0),(1,3,2)\}$, where $x \sim (0,1,2,3)$ and $y \sim (0,1)$. Let $N =< (0,1,2,3),(0,1) >$. and $t \sim t_4 \sim t_0$.

We will begin the manual double coset enumeration by looking at our first double coset. Note the definition of a double coset is as follows: $NwN = \{Nwn | n \in N\}$.

$\boldsymbol{NeN}$

First, the double coset NeN, is denoted by [*]. This double coset contains only the single coset, namely N. Since N is transitive on $\{t_0, t_1, t_2, t_3\}$, the orbit of N on $\{0,1,2,3\}$ is:

$$\mathbb{O} = \{0,1,2,3\}.$$

We choose t_0 as our symmetric generator from $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by [0]. Hence, four symmetric generators will go the new double coset [0].

$\boldsymbol{Nt_0N}$

In order to find how many single cosets [0] contains, we must first find the coset stabilizer $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0$$
$$=< (1,3,2),(1,2) >$$

so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{24}{6} = 4$. These four single cosets in [0] are $\{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_1, t_2, t_3\}$ are:

$$\mathbb{O} = \{0\} \text{ and } \{1,2,3\}.$$

We take t_0 and t_1 from each orbit, respectively, and to see which double coset Nt_0t_0 and Nt_0t_1 belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to NeN and three symmetric generators will go to a new double coset Nt_0t_1N, denoted by [01].

$\boldsymbol{Nt_0t_1N}$

Now Nt_0t_1N is a new double coset. We determine how many single cosets are in this double coset. We have $N^{(01)} = N^{01} =< e >$. But Nt_0t_1 is not distinct. If we conjugate the relation by N we get that:

$$Nt_0t_1 = Nt_0t_2 = Nt_0t_3$$
$$Nt_1t_0 = Nt_1t_2 = Nt_1t_3.$$

$$Nt_2t_1 = Nt_2t_0 = Nt_2t_3.$$

$$Nt_3t_1 = Nt_3t_2 = Nt_3t_0.$$

Thus, there exist $\{n \in N | N(t_0t_1)^n = Nt_0t_1\}$ such that

$$N(t_0t_1)^{(1,2,3)} = Nt_0t_2 = Nt_0t_1 \implies (1,2,3) \in N^{(01)}$$
$$Nt_0t_1 = Nt_0t_2 = N\underline{t_0t_2t_3}t_3 = N(0,2,3)t_0t_3 = Nt_0t_3$$

$$Nt_0t_2^{(2,3)} = Nt_0t_3 = Nt_0t_2 = Nt_0t_1 \implies (2,3) \in N^{(01)}$$

So, $N^{(01)} =< (1,2,3),(2,3) >$. The number of the single cosets in Nt_0t_1N is $\frac{|N|}{|N^{(01)}|} = \frac{24}{6} = 4$. These four single cosets in [01] are $\{Nt_01^n | n \in N\} = \{Nt_0t_1, Nt_1t_0, Nt_2t_1, Nt_3t_1\}$. Furthermore, the orbits of $N^{(01)}$ on $\{t_0,t_1,t_2,t_3\}$ are:

$$\mathbb{O} = \{0\} \text{ and } \{1,2,3\}.$$

$Nt_0t_1t_1 \in [0]$ (three symmetric generators will go back to the coset [0])

$Nt_0t_1t_0 \in [010]$ (one symmetric generator go to the double coset [010])

$Nt_0t_1t_0N$

Consider $Nt_0t_1t_0N$ denoted by [010]. We determine how many single cosets are in this double coset. We have $N^{(010)} = N^{010} =< e >$. But $Nt_0t_1t_0$ is not distinct. From the relation we know that:

$$Nt_0t_1t_0 = Nt_0t_2t_0^{(2,3)} = Nt_0t_3t_0 = Nt_0t_2t_0 = Nt_0t_1t_0 \implies (2,3) \in N^{(010)}$$
$$Nt_0t_1t_0 = Nt_0t_2t_0 = Nt_0\underline{t_2t_0t_1}t_1 = Nt_0(2,0,1)t_2t_1 = Nt_1t_2t_1$$

$$Nt_0t_1t_0^{(0,1,2)} = Nt_1t_2t_1 = Nt_0t_1t_0 \implies (0,1,2) \in N^{(010)}$$

So, $N^{(010)} =< (0,1,2),(2,3) >$. The number of the single cosets in $Nt_0t_1t_0N$ is $\frac{|N|}{|N^{(010)}|} = \frac{24}{24} = 4$. The single coset in [01] are $\{Nt_01^n | n \in N\} = \{Nt_0t_1t_0\}$. Furthermore, the orbits of $N^{(011)}$ on $\{t_0,t_1,t_2,t_3\}$ are:

$$\mathbb{O} = \{0,1,2,3\}.$$

Take a representative from this orbit, say t_0. Hence $Nt_0t_1t_0t_0 \in [01]$. Therefore, four symmetric generators will go back to Nt_0t_1N.

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of N in G is 10. We conclude:

$G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N$, where

$$G = \frac{2^{*4} : S_4}{(0,1,2) = t_0t_1t_2t_0}$$

$|G| \leq \left(|N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(010)}}\right) \times |N|$

$|G| \leq (1+4+4+1) \times 24$

$|G| \leq 10 \times 24$

$|G| \leq 240.$

A Cayley diagram that summarizes the above information is given below:

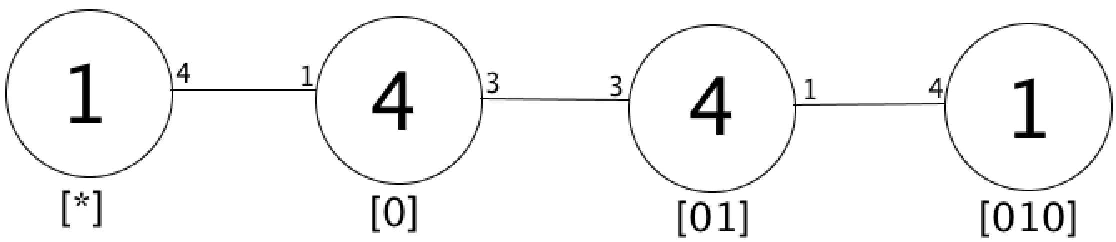

Figure 2.3: Cayley Diagram of $2 \times S_5$ over S_4

2.5.2 Permutation Representation of $2 \times S_5$ over S_4

In order to find the permutation representation of $G = 2^{*4} : S_4$, in terms of x, y, and t_0, we create a table in which we conjugate the twenty single cosets by x and y and we right multiply them by t_0.

Table 2.2: Permutation Representation of $2 \times S_5$ over S_4

Cosets	$x \sim (0,1,2,3)$	$y \sim (0,1)$	$t \sim t_0$
1. N	1. N	1. N	2. Nt_0
2. Nt_0	3. Nt_1	3. Nt_1	1. N
3. Nt_1	4. Nt_2	2. Nt_0	7. Nt_1t_0
4. Nt_2	5. Nt_3	4. Nt_2	8. Nt_2t_0
5. Nt_3	2. Nt_0	5. Nt_3	9. Nt_3t_0
6. Nt_0t_1	7. Nt_1t_2	7. Nt_1t_0	10. $Nt_0t_1t_0$
7. Nt_1t_0	8. Nt_2t_1	6. Nt_0t_1	3. Nt_1
8. Nt_2t_1	9. Nt_3t_2	8. Nt_2t_0	4. $Nt_2t_1t_0$
9. Nt_3t_1	6. Nt_0t_2	9. Nt_3t_0	5. $Nt_3t_1t_0$
10. $Nt_0t_1t_0$	10. $Nt_1t_2t_1$	10. $Nt_1t_0t_1$	6. Nt_0t_1

We have:

$$\phi(x) = (2,3,4,5)(6,7,8,9)$$

$$\phi(y) = (2,3)(6,7)$$

$$\phi(t) = (1,2)(3,7)(4,8)(5,9)(6,10).$$

Thus, we have a homomorphism $\phi : 2^{*4} : S_4 \longrightarrow S_{10}$. Then $\phi(G) =< \phi(x), \phi(y), \phi(t) >$. In order for us to prove that $\phi(G) =< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of $G = 2^{*4} : S_4$, we must have the following conditions met:

(1) $\phi(N) \cong S_4$

(2) $\phi(t)$ has four conjugates under conjugation by $\phi(N)$

(3) $\phi(N)$ acts as S_4 on the three conjugates of $\phi(t)$ by conjugates.

Proof. We have $\phi(N) =< \phi(x), \phi(y) >$.

$$\begin{aligned}(1)\ \phi(N) &=< \phi(x), \phi(y) > \\ &=< (2,3,4,5)(6,7,8,9), (2,3)(6,7) > \\ &\cong S_4 \text{ since } |\phi(x)\phi(y)| = |(3,4,5)(7,8,9)| = 3.\end{aligned}$$

Thus, $\phi(N) \cong S_4$.

(2) We need to compute $\phi(t)^{\phi(N)}$:

$$\begin{aligned}
\phi(t)^{\phi(x)} &= \{(1,2)(3,7)(4,8)(5,9)(6,10)^{(2,3,4,5)(6,7,8,9)}\} \\
&= \{(1,3)(4,8)(5,9)(2,6)(7,10)\} \\
&= t_1 \\
\phi(t)^{\phi(x^2)} &= \{(1,2)(3,7)(4,8)(5,9)(6,10)^{(2,4)(3,5)(6,8)(7,9)}\} \\
&= \{(1,4)(5,9)(2,6)(3,7)(8,10)\} \\
&= t_2 \\
\phi(t)^{\phi(x^3)} &= \{(1,2)(3,7)(4,8)(5,9)(6,10)^{(2,5,4,3)(6,9,8,7)}\} \\
&= \{(1,5)(2,6)(3,7)(4,8)(9,10)\} \\
&= t_3 \\
\phi(t)^{\phi(x^4)} &= \{(1,2)(3,7)(4,8)(5,9)(6,10)^{e}\} \\
&= \{(1,2)(3,7)(4,8)(5,9)(6,10)\} \\
&= t_0
\end{aligned}$$

Thus, $\phi(t)^{\phi(N)} = \{t_0, t_1, t_2, t_3\}$.

(3) We need to show that $\phi(N)$ acts as S_4 on the four conjugates of $\phi(t)$ by conjugates.

First, we have to conjugate by $\phi(x)$:

$$\begin{aligned}
&t_0^{\phi(x)} = t_1 \\
&t_1^{\phi(x)} = (t_0^{\phi(x)})^{\phi(x)} = t_0^{\phi(x^2)} = t_2 \\
&t_2^{\phi(x)} = (t_1^{\phi(x)})^{\phi(x)} = t_1^{\phi(x^2)} = (t_0^{\phi(x)})^{\phi(x^2)} = t_0^{\phi(x^3)} = t_3 \\
&t_3^{\phi(x)} = (t_2^{\phi(x)})^{\phi(x)} = t_2^{\phi(x^2)} = (t_1^{\phi(x)})^{\phi(x^2)} = t_1^{\phi(x^3)} = (t_0^{\phi(x)})^{\phi(x^3)} = t_0^{\phi(x^4)} = t_0.
\end{aligned}$$

Thus, $\phi(x) = (t_0, t_1, t_2, t_3)$.

Next, we have to conjugate by $\phi(y)$:

$$\begin{aligned}
t_0^{\phi(y)} &= \{(1,2)(3,7)(4,8)(5,9)(6,10)^{(2,3)(6,7)}\} \\
&= \{(1,3)(2,6)(4,8)(5,9)(7,10)\} \\
&= t_1 \\
t_1^{\phi(y)} &= (t_0^{\phi(y)})^{\phi(y)} = t_0^{\phi(y^2)} = t_0.
\end{aligned}$$

Thus, $\phi(y) = (t_0, t_1)$.

Hence, $\phi(G) =< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of $G = 2^{*4} : S_4$. $\square$

We have:

$$G = \frac{2^{*4} : S_4}{(0,1,2) = t_0t_1t_2t_0}.$$

Now, we want to verify if $\phi(0,1,2) = \phi(t_0t_1t_2t_0)$ then $< \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of G.

Verify: $\phi(x)\phi(y) = \phi(t_0t_1t_2t_0)$:

$$\begin{aligned}\phi(t_0t_1t_2t_0) =& (1,2)(3,7)(4,8)(5,9)(6,10)\\ & (1,3)(4,8)(5,9)(2,6)(7,10)\\ & (1,4)(5,9)(2,6)(3,7)(8,10)\\ & (1,2)(3,7)(4,8)(5,9)(6,10)\\ =& \phi(x)\phi(y).\end{aligned}$$

Hence, $\phi : G \overset{homo.}{\longrightarrow} S_{10}$ with $\phi(G) =< \phi(x), \phi(y), \phi(t) >$. By FIT we have:

$G/ker\phi \cong \phi(G)$

$\implies |G/ker\phi| \cong |\phi(G)|$

$\implies |G| = |ker\phi||\phi(G)|$.

By completing the double coset enumeration we know $|G| \leq 240$. Moreover, by Magma, $|\phi(G)| = | < \phi(x), \phi(y), \phi(t_0) > | = 240$.

So, $|G| = |ker\phi|240$

$\implies |G| \geq 240$.

Hence, $|G| = 240$.

2.5.3 Prove $G \cong 2 \times S_5$

We use the composition factors of G to construct a computer based proof to show that $G \cong 2 \times S_5$.

Proof. Given:

$$< x, y, t | x^4, y^2, (xy)^3, t^2, (t, y^x), (t, xy), (xy)^{y^{x^3}} = tt^x t^{x^2} t >= \frac{2^{*4}:S_4}{(0,1,2)=t_0t_1t_2t_0}.$$

We use Magma, to obtain the following composition factors:

```
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
```

```
|  Alternating(5)
*
|  Cyclic(2)
1
```

Hence, G has the following composition series $G \supset G_1 \supset G_2 \supset 1$, where $(G/G_1)(G_1/G_2)(G_2/1) = C_2A_5C_2$. The normal lattice of G is:

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------

[7]  Order 240  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 120  Length 1  Maximal Subgroups: 3
[5]  Order 120  Length 1  Maximal Subgroups: 2 3
[4]  Order 120  Length 1  Maximal Subgroups: 3
---
[3]  Order 60   Length 1  Maximal Subgroups: 1
---
[2]  Order 2    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:
```

First, we look at the center of G and we find it is of order 2. In addition, by looking at the normal lattice, we find that the normal subgroup NL[2] is of order 2. Now, we factor G by $NL[2]$ to find H such that $G/NL[2] \cong H$.

```
> q,ff:=quo<G1|NL[2]>;
> s:=IsIsomorphic(q,Sym(5));s;
true
```

We show that $H \cong S_5$ and a presentation for H is $< a, b|a^2, b^4, (a * b)^5, (a, b)^3 >$.

```
> H<a,b>:=Group< a,b| a^2,b^4,(a*b)^5,(a,b)^3>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,q);s;
true
```

Thus, we might have a direct product of a cyclic group of order 2 by S_5.

```
> D:=DirectProduct(CyclicGroup(2),q);
>  s:=IsIsomorphic(D,G1);s;
```

```
true
```

Now, the element of NL[2] commutes with the element of S_5, since G is a direct product extension. Thus, we have the following presentation for G:

```
> HH<a,b,c>:=Group< a,b,c| a^2,b^4,(a*b)^5,(a,b)^3,c^2,(c,a),(c,b)>;
> f2,H2,k2:=CosetAction(HH,sub<HH|Id(HH)>);
> s:=IsIsomorphic(H2,G1);s;
true
```

Hence, $G \cong 2 \times S_5$.

$\square$

2.6 Finding and Factoring by the Center ($Z(G)$) of $2 \times S_5$ over S_4

Let G acts on $X = \{N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N\}$ where $|X| = 10$. From the Cayley Diagram of $2 \times S_5$ over S_4 we see that G is transitive. Moreover, from the DCE and the Cayley diagram it is clear that the double coset $Nt_0t_1t_0N$ contains one single coset. We stabilize the coset N then another coset $Nt_0t_1t_0N$ at the maximal distance from N also stabilize. This means $\{N, Nt_0t_1t_0N\}$ is a nontrivial block of size 2. Let B be a nontrivial block and $N \in B$. If $Nt_0t_1t_0 \in B$. Then

$B = \{N, Nt_0t_1t_0\} = \{1, 10\}$

$Bt_0 = \{Nt_0, Nt_0t_1\} = \{2, 6\}$

$Bt_1 = \{Nt_1, N\underline{t_0t_1t_0}t_1\} = \{Nt_1, Nt_1t_0t_1t_1\} = \{Nt_1, Nt_1t_0\} = \{3, 7\}$

$Bt_2 = \{Nt_2, N\underline{t_0t_1t_0}t_2\} = \{Nt_2, Nt_2t_0t_2t_2\} = \{Nt_2, Nt_2t_0\} = \{Nt_2, Nt_2t_1\} = \{4, 8\}$

$Bt_3 = \{Nt_3, N\underline{t_0t_1t_0}t_3\} = \{Nt_3, Nt_3t_1t_3t_3\} = \{Nt_3, Nt_3t_1\} = \{5, 9\}$.

We can see that $\{Bt_0, Bt_1, Bt_2, Bt_3\} \cap B = \emptyset$ and $\{Bt_0, Bt_1, Bt_2, Bt_3\} \neq B$.Hence, we have blocks of imprimitive of size two. Therefore, $|Z(G)|$=2 where

$Z(G) =< nw >$ (central elements permute the elements of each block of imprimitive)

$= \{(1, 10)(2, 6)(3, 7)(4, 8)(5, 9)\}$.

Now, we are going to find the central element of order 2 in G, not in its homomorphic image G_1. Consider $nt_0t_1t_0 = 1 \in G$;

in addition, $t_0t_1t_0 = n^{-1}$. Let $r = n^{-1}$. Then $t_0t_1t_0 = r$. We now compute r by its action on the cosets $\{Nt_0, Nt_1, Nt_2, Nt_3\}$.

Compute r by its action on the coset Nt_0:

$$\begin{aligned}
Nt_0^r &= Nt_0^{t_0t_1t_0} = N(t_0t_1t_0)^{-1}t_0t_0t_1t_0 \\
&= Nt_0t_1t_0t_1t_0 \\
&= Nt_0t_1t_0\underline{t_1t_0t_2}t_2 = Nt_0t_1t_0(1,0,2)t_1t_2 \\
&= Nt_2\underline{t_0t_2t_1}t_2 = Nt_2(0,2,1)t_0t_2 \\
&= Nt_1t_0t_2 \\
&= Nt_1.
\end{aligned}$$

Now, compute r by its action on the coset Nt_1:

$$\begin{aligned}
Nt_1^r &= Nt_1^{t_0t_1t_0} \\
&= N(t_0t_1t_0)^{-1}t_1t_0t_1t_0 \\
&= Nt_0t_1t_0t_1t_0\underline{t_1t_0} \\
&= Nt_0t_1t_0t_1t_0(0,2,1)t_1t_2 \\
&= Nt_2t_0t_2\underline{t_0t_2t_1}t_2 \\
&= Nt_2t_0t_2(0,2,1)t_0t_2 \\
&= Nt_1\underline{t_2t_1t_0}t_2 \\
&= Nt_1(2,1,0)t_2t_2 = Nt_0.
\end{aligned}$$

Next, compute r by its action on the coset Nt_2:

$$\begin{aligned}
Nt_2^r &= Nt_2^{t_0t_1t_0} \\
&= N(t_0t_1t_0)^{-1}t_2t_0t_1t_0 \\
&= Nt_0t_1t_0\underline{t_2t_0t_1}t_0 \\
&= Nt_0t_1t_0(2,0,1)t_2t_0 \\
&= Nt_1t_2\underline{t_1t_2t_0} \\
&= Nt_1t_2(1,2,0)t_1 \\
&= Nt_2t_0t_1 \\
&= N(2,0,1)t_2 \\
&= Nt_2.
\end{aligned}$$

Finally, compute r by its action on the coset Nt_3:

$$\begin{aligned} Nt_3^r &= Nt_3^{t_0t_1t_0} \\ &= N(t_0t_1t_0)^{-1}t_3t_0t_1t_0 \\ &= Nt_0t_1t_0\underline{t_3t_0t_1}t_0 \\ &= Nt_0t_1t_0(3,0,1)t_3t_0 \\ &= Nt_1t_3\underline{t_1t_3t_0} \\ &= Nt_1t_3(1,3,0)t_1 \\ &= Nt_3t_0t_1 \\ &= Nt_3. \end{aligned}$$

Thus, $Nt_0^r = t_1$, $Nt_1^r = t_0$, $Nt_2^r = t_2$, and $Nt_3^r = t_3$. Therefore, $r = (0,1)$, and the generator of the center is $t_0t_1t_0 = (0,1)$. Hence, the center of G is $Z(G) =< (0,1)t_0t_1t_0 >$.

Now, we factor

$$G \cong \frac{2^{*4} : S_4}{(0,1,2) = t_0t_1t_2t_0}$$

by $Z(G) =< (0,1)t_0t_1t_0 >$. Now the question is whether the new relation $(0,1)t_0t_1t_0$ implies the original relation, $(0,1,2) = t_0t_1t_2t_0$.

We have $t_0t_1t_2 = t_0\underline{t_1t_2t_1}t_1 = t_0(1,2)t_1 = (1,2)\underline{t_0t_1t_0}t_0 = (1,2)(1,0)t_0 = (0,1,2)t_0$. Hence, $(0,1)t_0t_1t_0 \implies t_0t_1t_0 = (0,1,2)t_0$.

Thus, G factor by the center $Z(G)$ is

$$G \cong \frac{2^{*4} : S_4}{(0,1,2) = t_0t_1t_2t_0, (0,1) = t_0t_1t_0} \cong \frac{2^{*4} : S_4}{(0,1) = t_0t_1t_0}.$$

Now we construct a double coset enumeration of $G \cong \frac{2^{*4}:S_4}{(0,1)=t_0t_1t_0}$. Our first double coset, $Ne = \{Nen|n \in N\} = \{N\}$ denoted by $[*]$ contains one single coset. Since N is transitive on $\{t_0, t_1, t_2, t_4\}$, the orbit of N on $\{0,1,2,4\}$ is: $\mathbb{O} = \{0,1,2,4\}$. We take a representative from the orbit, say t_0, and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by $[0]$. Hence, four symmetric generators will go the new double coset $[0]$.

In order to find how many single cosets $[0]$ contains, we must first find the coset stabilizer

$N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 =<$ $(1,2,3),(1,2)>$, so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{24}{6} = 4$. These four single cosets in [0] are $\{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_1, t_2\}$ are: $\mathbb{O} = \{0\}$ and $\{1,2,3\}$. We take t_0 and t_1 from each orbit, respectively, and to see which double coset Nt_0t_0 and Nt_0t_1 belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to NeN and by our relation we have $(0,1)t_0t_1t_0 = e \implies (0,1)t_0t_1 = t_0 \implies Nt_0t_1 = Nt_0$. Hence, $Nt_0t_1 \in [0]$, so three elements will loop back to [0]. A Cayley diagram that summarizes the above information is given below:

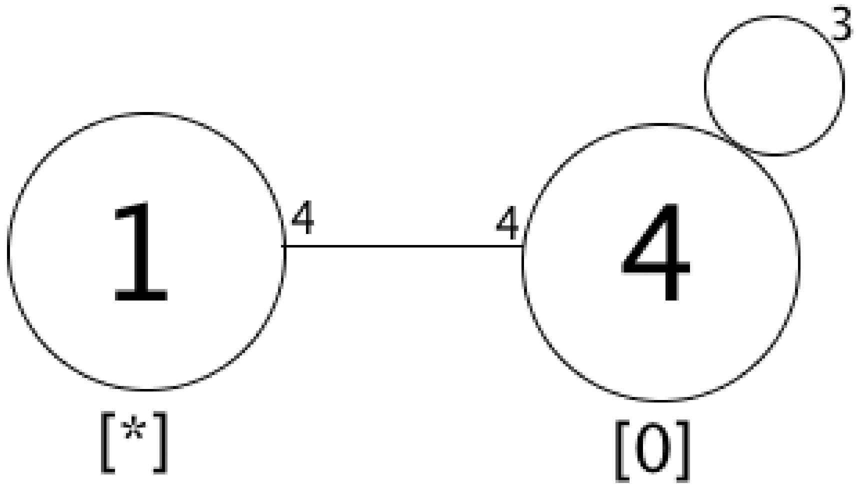

Figure 2.4: Cayley Diagram of S_5 over S_4

Chapter 3

Iwasawa's Lemma

In this chapter, we will apply Iwasawa's lemma to prove that a group G is simple.

3.1 Iwasawa's Lemma Preliminaries

Definition 3.1. ***(G-set).***
*If X is a set and G is a group, then X is a **G-set** if there is a function $\alpha : G \times X \to X$ (called an **action**), denoted by $\alpha : (g, x) \mapsto gx$, such that:*
(i) $1x = x$ for all $x \in X$ and
*(ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. We also say that G **acts** on X. If $|X| = n$, then n is called the **degree** of the G-set X. [Rot12]*

Definition 3.2. ***(Transitive G-set).***
A $G-set$ X is transitive if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot12]

Definition 3.3. *If X is a transitive G-set of degree n, and if $x \in X$, then $|G| = n|G^x|$. [Rot12]*

Definition 3.4. *A $G - set$ X is transitive if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot12]*

Definition 3.5. ***(Block).***
*If X is a $G - set$, then a **block** is a subset B of X such that, for each $g \in G$, either*

$gB = B$ or $gB \cap B = \emptyset$. Note $gB = \{gx : x \in B\}$. ***Trivial*** *blocks are $\emptyset, X$, and one-point subsets; any other other block is called* ***nontrivial***. *[Rot12]*

Definition 3.6. *A $G-set$ X with action α is* ***faithful*** *if $\bar{\alpha} : G \to S_x$ is injective. [Rot12]*

Definition 3.7. ***(Primitive).***
A transitive $G-set$ X is **primitive** *if it contains no nontrivial block; otherwise, it is* ***imprimitive***. *[Rot12]*

Definition 3.8. *Let X be a finite $G-set$, and let $H \leq G$ act transitively on X. Then $G = HG^x$ for each $x \in X$. [Rot12]*

Definition 3.9. *Let X be a $G-set$ and $x, y \in X$.*
(i) If $H \leq G$, then $H_x \cap H_y \neq \emptyset \implies H_x = H_y$
(ii) If H is normal in G, then the subsets Hx are blocks of X. [Rot12]

Definition 3.10. *If X is a faithful primitive $G-set$ of degree $n \geq 2$. If H is normal in G and if $H \neq 1$, then X is a transitive $H-set$. [Rot12]*

Theorem 3.11. *Let X be a transitive $G-set$. Then X is primitive if and only if, for each $x \in X$, the stabilizer G^x is a maximal subgroup. [Rot12]*

Definition 3.12. ***(Commutator).*** *If $a, b \in G$, the* **commutator** *of a and b, denoted by$[a, b]$, is*

$$[a, b] = aba^{-1}b^{-1}.[Rot12]$$

Definition 3.13. ***(Derived Group).*** *The* **commutator subgroup** *(or derived group) of G, denoted by G', is the subgroups of G generated by all the commutators. [Rot12]*

Definition 3.14. ***(Simple).*** *A group $G \neq 1$ is* ***simple*** *if it has no normal subgroups other than G and 1. [Rot12]*

Theorem 3.15. ***(Iwasawa's Lemma).*** *Let $G' = G$ (such a group is called* **perfect***) and let X be a faithful primitive $G-set$. If there is $x \in X$ and an abelian normal subgroup K of G_x whose conjugates $\{ghg^{-1}\}$ generate G, then G is simple. [Rot12]*

3.2 Iwasawa's Lemma to Prove $L_2(13)$ over A_4 is Simple

We consider
$$G \cong \frac{2^{*4}\, A_4}{[(0,1,2)t_0]^7, [(0,1)(2,3)t_0]^7, [(0,1,2)t_0t_1t_2t_3]^2} \cong L_2(13),$$
where A_4 is maximal in $L_2(13)$ and the index of A_4 in G equals 91.

Note $N = A_4 \cong < x, y | x^3, y^3, (x*y)^2 >$, where $x = (1,2,3)$ and $y = (1,2,0)$. Let $t_4 \sim t_0$.

The manual double coset enumeration and the Cayley diagram was done by Maria de la Luz Torres [dlLT05]. The Cayley diagram is shown below:

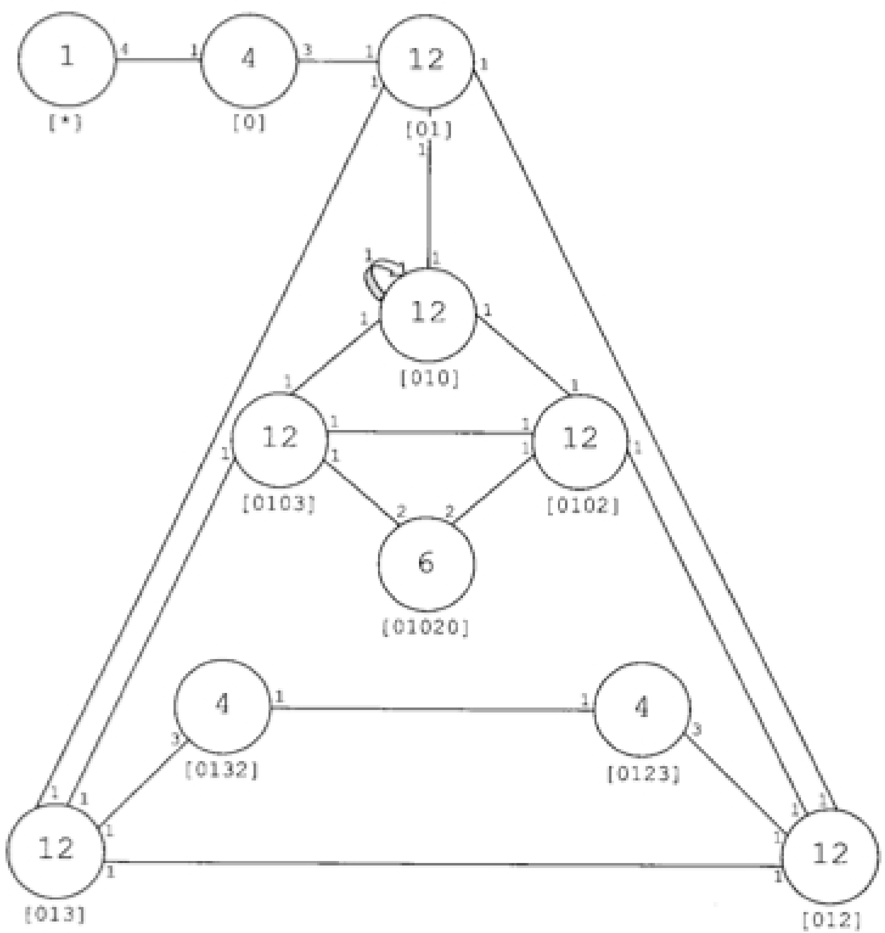

Figure 3.1: Cayley Diagram of $L_2(13)$ over A_4

We use *Iwasawa's lemma* to prove $G \cong L_2(13)$ is **simple**. *Iwasawa's lemma* has three sufficient conditions that we must satisfied:

(1) G acts on X faithfully and primitively

(2) G is perfect $(G = G')$

(3) There exist $x \in X$ and a normal abelian subgroup K of G^x such that the conjugates of K generate G.

Proof. **3.2.1** $\quad G = L_2(13)$ **acts on** X **Faithfully**

Let G acts on $X = \{N, Nt_0N, Nt_0t_1N, Nt_0t_1t_0N, Nt_0t_1t_0t_3N, Nt_0t_1t_0t_2N, Nt_0t_1t_2N, Nt_0t_1t_3N, Nt_0t_1t_3t_2N, Nt_0t_1t_2t_3N, Nt_0t_1t_0t_2t_0N\}$, where $|X| = 91$. G acts on X implies there exist homomorphism

$$f \ : G \longrightarrow S_{91} \qquad\qquad (|X| = 91).$$

By First Isomorphic Theorem we have:

$$G/kerf \cong f(G).$$

If $kerf = 1$ then $G \cong f(G)$. Only elements of N fix N implies $G^1 = N$. Since X is a transitive $G - set$ of degree 91, we have:

$$\begin{aligned} |G| &= 91 \times |G^1| \\ &= 91 \times |N| \\ &= 91 \times 12 \\ &= 1092 \\ \implies |G| &= 1092. \end{aligned}$$

From Cayley diagram, $|G| \leq 1092$. However, from above $|G| = 1092$ implying $ker(f) = 1$. Since $kerf = 1$ then G acts faithfully on X.

3.2.2 $\quad G = L_2(13)$ acts on X Primitively

In order to show that G is primitive, we must show that $G = L_2(13)$ is transitive on $X = 91$ and there exists no nontrivial blocks of X. From the Cayley diagram of $G = L_2(13)$ over A_4, we see that G is transitive. Let B be a nontrivial block, then $|B|||X|$. Note if we had a nontrivial block it would have to be of size 7 or 13. Let B be a nontrivial block and $N \in B$ since G is transitive on the coset of N. Now we look at different cases:

Case(1): Assume $Nt_0 \in B$, then

$$B = \{N, Nt_0\}$$

$$B = \{N, Nt_0N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, Nt_0, Nt_1, Nt_2, Nt_3\}$$

$$Bt_1 = \{Nt_1, Nt_0t_1, N, Nt_2t_1, Nt_3t_1\}$$

$$\implies N \in B \cap Bt_1$$

$$\implies B = Bt_1.$$

Now $B = \{N, Nt_0N, Nt_0t_1N\}$, where $|B| = 17$ (passed size 7 and 13). Note if $Nt_0 \in B$ then $B = X$. So B is a trivial block.

Case(2): Assume $Nt_0t_1 \in B$, then

$$B = \{N, Nt_0t_1\}$$

$$B = \{N, Nt_0t_1N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1, Nt_2t_1, Nt_3t_1, \ldots, Nt_2t_0, Nt_1t_0, Nt_3t_0\} \text{ where } |B| = 13$$

$$Bt_0t_2 = \{Nt_0t_2, Nt_0t_1t_0t_2, Nt_2t_1t_0t_2, \ldots, N, Nt_1t_2, Nt_3t_2\}$$

$$\implies N \in B \cap Bt_0t_2$$

$$\implies B = Bt_0t_2.$$

So $B = \{N, Nt_0t_1N, Nt_0t_1t_0t_2N\}$, where $|B| = 25$ (passed size 7 and 13). Hence, B is a no nontrivial block of X under the action G.

Case(3): Assume $Nt_0t_1t_0 \in B$, then

$$B = \{N, Nt_0t_1t_0\}$$

$$B = \{N, Nt_0t_1t_0N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_0, Nt_2t_1t_2, Nt_3t_1t_3, \ldots, Nt_2t_0t_2, Nt_1t_0t_1, Nt_3t_0t_3\} \text{ where } |B| = 13$$

$$Bt_1 = \{Nt_1, \underline{Nt_0t_1t_0t_1}, Nt_2t_1t_2t_1, \ldots, Nt_2t_0t_2t_1, Nt_1t_0, Nt_3t_0t_3t_1\}$$

$$= \{Nt_1, Nt_0t_1t_0, Nt_2t_1t_2t_1, \ldots, Nt_2t_0t_2t_1, Nt_1t_0, Nt_3t_0t_3t_1\}$$

$$\implies Nt_0t_1t_0 \in B \cap Bt_1$$

$$\implies B = Bt_1.$$

So $B = \{N, Nt_1N, Nt_0t_1N, Nt_0t_1t_0N, Nt_0t_1t_0t_2N, Nt_0t_1t_0t_3N\}$, where $|B| = 52$. Hence, B is a no nontrivial block of X under the action G.

Case(4): Assume $Nt_0t_1t_2 \in B$, then

$$B = \{N, Nt_0t_1t_2\}$$

$$B = \{N, Nt_0t_1t_2N\} \text{ (since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_2, Nt_2t_1t_3, Nt_3t_1t_0, \ldots, Nt_0t_3t_1, Nt_1t_0t_3, Nt_3t_0t_2\} \text{ where } |B| = 13$$

$$Bt_2t_1t_0 = \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, \underline{Nt_0t_3t_1t_2}t_1t_0, Nt_1t_0t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$= \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, Nt_2t_3t_0t_1t_1t_0, Nt_1t_0t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$= \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, Nt_2t_3, Nt_1t_0t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$\implies N \in B \cap Bt_2t_1t_0$$

$$\implies B = Bt_2t_1t_0.$$

So $B = \{N, Nt_0t_1N, Nt_0t_1t_2N\}$, where $|B| = 25$. Hence, B is a no nontrivial block of X under the action G.

Case(5): Assume $Nt_0t_1t_3 \in B$, then

$$B = \{N, Nt_0t_1t_3\}$$

$$B = \{N, Nt_0t_1t_3N\} \text{ (since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_3, Nt_2t_1t_0, Nt_3t_1t_2, \ldots, Nt_2t_0t_3, Nt_1t_0t_2, Nt_3t_0t_1\} \text{ where } |B| = 13$$

$$Bt_3t_1t_0 = \{Nt_3t_1t_0, N, \underline{Nt_2t_1t_0t_3}t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_1t_0t_2t_3t_1t_0, Nt_3t_0t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, \underline{Nt_0t_1t_3t_2}t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_1t_0t_2t_3t_1t_0, Nt_3t_0t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, Nt_0t_1t_2t_3t_1t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_1t_0t_2t_3t_1t_0, Nt_3t_0t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, \underline{Nt_0t_1t_2t_3t_0}, \ldots, Nt_2t_0t_1t_0, Nt_1t_0t_2t_3t_1t_0, Nt_3t_0t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, Nt_0t_1t_2, \ldots, Nt_2t_0t_1t_0, Nt_1t_0t_2t_3t_1t_0, Nt_3t_0t_1t_3t_1t_0\}$$

$$\implies N \in B \cap Bt_3t_1t_0$$

$$\implies B = Bt_3t_1t_0.$$

So $B = \{N, Nt_3t_1t_0N, Nt_0t_1t_2N\}$, where $|B| = 25$. Hence, B is a no nontrivial block of X under the action G.

Case(6): Assume $Nt_0t_1t_0t_3 \in B$, then

$$B = \{N, Nt_0t_1t_0t_3\}$$

$$B = \{N, Nt_0t_1t_0t_3N\} \quad (\text{since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_0t_3, Nt_2t_1t_2t_0, Nt_3t_1t_3t_2, \ldots, Nt_2t_0t_2t_3, Nt_1t_0t_1t_2, Nt_2t_3t_2t_1\}$$

$$\begin{aligned} Bt_1t_2t_0t_1t_0 &= \{\underline{Nt_1t_2t_0t_1t_0}, \underline{Nt_0t_1t_0t_3}t_1t_2t_0t_1t_0, \ldots, \underline{Nt_2t_3t_0t_1}t_0\} \\ &= \{Nt_0t_1t_0t_3, Nt_0t_1t_0t_2t_1t_1t_2t_0t_1t_0, \ldots, Nt_1t_3t_2t_0t_0\} \\ &= \{Nt_0t_1t_0t_3, N, \ldots, Nt_1t_3t_2\} \end{aligned}$$

$$\Longrightarrow N \in B \cap Bt_1t_2t_0t_1t_0$$

$$\Longrightarrow B = Bt_1t_2t_0t_1t_0.$$

So $B = \{N, Nt_0t_1t_0N, Nt_0t_1t_0t_3N\}$, where $|B| = 25$. Hence, B is a no nontrivial block of X under the action G.

Case(7): Assume $Nt_0t_1t_0t_2 \in B$, then

$$B = \{N, Nt_0t_1t_0t_2\}$$

$$B = \{N, Nt_0t_1t_0t_2N\} \quad (\text{since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_0t_2, Nt_2t_1t_2t_3, \ldots, Nt_3t_2t_3t_1\}$$

$$\begin{aligned} Bt_1t_3t_0t_1t_0 &= \{\underline{Nt_1t_3t_0t_1t_0}, \underline{Nt_0t_1t_0t_2}t_1t_3t_0t_1t_0, \ldots, \underline{Nt_3t_2t_0t_1}t_0\} \\ &= \{Nt_0t_1t_0t_2, Nt_0t_1t_0t_3t_1t_1t_3t_0t_1t_0, \ldots, Nt_1t_2t_3t_0t_0\} \\ &= \{Nt_0t_1t_0t_2, N, \ldots, Nt_1t_2t_3\} \end{aligned}$$

$$\Longrightarrow N \in B \cap Bt_1t_3t_0t_1t_0$$

$$\Longrightarrow B = Bt_1t_3t_0t_1t_0.$$

So $B = \{N, Nt_0t_1t_0N, Nt_0t_1t_0t_2N\}$, where $|B| = 25$. Hence, B is a no nontrivial block of X under the action G.

Case(8): Assume $Nt_0t_1t_0t_2t_0 \in B$, then

$$B = \{N, Nt_0t_1t_0t_2t_0\}$$

$$B = \{N, Nt_0t_1t_0t_2t_0N\} \;\; (\text{since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_0t_2t_0, Nt_2t_1t_2t_3t_2, \ldots, Nt_3t_2t_3t_1t_3\} \text{ where } |B| = 7$$

$$Bt_0 = \{Nt_0, Nt_0t_1t_0t_2, Nt_2t_1t_2t_3t_2t_0, \ldots, Nt_3t_2t_3t_1t_3t_0\}$$

$$Bt_1 = \{Nt_1, \underline{Nt_0t_1t_0t_2t_0t_1}, Nt_2t_1t_2t_3t_2t_1, \ldots, Nt_3t_2t_3t_1t_3t_1\}$$

$$= \{Nt_1, Nt_0t_1t_0t_2, Nt_2t_1t_2t_3t_2t_1, \ldots, Nt_3t_2t_3t_1t_3t_1\}$$

Since $B \cap Bt_0 = \emptyset$. Then B is a block and Bt_0 is also a block. Moreover, $B \cap Bt_1 = \emptyset$ so Bt_1 is a block. But $Nt_0t_1t_0t_2 \in Bt_0 \cap Bt_1$. So $B = \{N, Nt_0N, Nt_0t_1t_0t_2N, Nt_0t_1t_0t_2t_0\}$. Hence, B is a no nontrivial block of X under the action G.

Case(9): Assume $Nt_0t_1t_3t_2 \in B$, then

$$B = \{N, Nt_0t_1t_3t_2\}$$

$$B = \{N, Nt_0t_1t_3t_2N\} \;\; (\text{since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_3t_2, Nt_0t_2t_1t_3, Nt_0t_3t_2t_1, Nt_3t_0t_1t_2\} \text{ where } |B| = 5$$

Hence B is not a block since size of $|B| = 5$ does not divide $|X| = 91$.

Case(10): Assume $Nt_0t_1t_2t_3 \in B$, then

$$B = \{N, Nt_0t_1t_2t_3\}$$

$$B = \{N, Nt_0t_1t_2t_3N\} \;\; (\text{since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_2t_3, Nt_0t_2t_3t_1, Nt_0t_3t_1t_2, Nt_3t_0t_2t_1\} \text{ where } |B| = 5$$

Hence B is not a block since size of $|B| = 5$ does not divide $|X| = 91$. In conclusion, from the cases above we show that we cannot create nontrivial block of size 7 or 13. Thus G acts primitively on X.

3.2.3 $G = L_2(13)$ is Perfect

Next we want to show that $G = G'$. Now $A_4 \subseteq G \implies {A_4}' \subseteq G'$.

$A_4{}' =< [a, b] | a, b \in A_4 >$. Now the derived group,

$$A_4{}' =< (1,3)(20), (1,2)(3,0) >$$

$$\implies \{e, (1,3)(2,0), (1,2)(3,0), (1,0)(2,3)\} \subseteq G'.$$

Now $x = (1,2,3)$ and $y = (0,1,2)$.

Then $[x, y] = x^{-1}y^{-1}xy = (1,3,2)(0,2,1)(1,2,3)(0,1,2) = (1,2)(3,0) \in G'$. If we conjugate $(1,2)(3,0)$ by $(1,2,3)$ we get $(2,3)(1,0) \in G'$.

Now by expanding the relation $[(0,1,2)t_0]^7 = 1$, we get

$$(0,1,2)t_0t_2t_1t_0t_2t_1t_0 = 1$$

$$\implies y = t_0t_1t_2t_0t_1t_2t_0.$$

Also by expanding the relation $[(0,1)(2,3)t_0]^7 = 1$, we get

$$(0,1)(2,3)t_0t_1t_0t_1t_0t_1t_0 = 1$$

$$\implies xy = t_0t_1t_0t_1t_0t_1t_0$$

Now we use the second relation to solve for x. We replace $y = t_0t_1t_2t_0t_1t_2t_0$.

$$xy = t_0t_1t_0t_1t_0t_1t_0$$

$$xt_0t_1t_2t_0t_1t_2t_0 = t_0t_1t_0t_1t_0t_1t_0$$

$$\implies x = t_0t_1t_0t_1t_0t_1t_2t_1t_0t_2t_1t_0.$$

So $G =< x, y, t >=< t_0, t_1, t_2, t_3 >$. Our goal is to show that one of the $t_i's \in G'$, then we can conjugate. Since $(0,1)(2,3) \in G'$. Then

$$\begin{aligned}
(0,1)(2,3) &= t_0t_1t_0t_1t_0t_1t_0 \in G' \\
(0,1)(2,3)^{(t_0t_1)} &= (t_0t_1t_0t_1t_0t_1t_0)^{(t_0t_1)} \in G' \text{ (since } G' \trianglelefteq G) \\
&= (t_0t_1)^{-1}(t_0t_1t_0t_1t_0t_1t_0)(t_0t_1) \\
&= (t_1t_0)(t_0t_1t_0t_1t_0t_1t_0)(t_0t_1) \\
&= t_0t_1t_0 \\
&= t_0t_1t_0t_1t_1t_1 \\
&= [t_0, t_1]t_1 \in G' \text{ (since } [t_0, t_1] \in G') \\
&\implies t_1 \in G.'
\end{aligned}$$

So $t_1 \in G'$

$$\implies t_1^y, t_1^{y^{-1}} \in G' \text{ (since } y,\ y^{-1} \in G \text{ and } G' \trianglelefteq G)$$

$$t_1^y = t_1^{(0,1,2)} = t_2 \in G'$$

$$t_1^{y^{-1}} = t_1^{(1,0,2)} = t_0 \in G'$$

$\implies t_2, t_0 \in G'$.

So $t_2^x \in G'$ (since $x \in G$, $t_2 \in G'$, and $G' \trianglelefteq G$)

$$t_2^x = t_2^{(1,2,3)} = t_3 \in G'.$$

Thus $G \supseteq G' \supseteq < t_0, t_1, t_2, t_3 >= G$

$\implies G' = G$. Hence G is perfect.

3.2.4 Conjugates of a Normal Abelian K Generate $G = L_2(13)$ over A_4

Now we require $x \in X$ and a $K \trianglelefteq G^x$, where K is a normal abelian subgroup such that the conjugates of K in G generate G. Recall, $G^1 = N = A_4$. Let $K =< (0,1)(2,3), (0,2)(1,3) >$. Since K is normal abelian subgroup in G then for any $s \in K$ and for all $g \in G$ implies $s^g \in K$. Since $(0,1)(2,3) = t_0t_1t_0t_1t_0t_1t_0 \in K$. Now

$t_0t_1t_0 \in G$ and $t_0t_1t_0t_1t_0t_1t_0 \in K$

$\implies (t_0t_1t_0t_1t_0t_1t_0)^{(t_0t_1t_0)} \in K^G$

$\implies (t_0t_1t_0)^{-1}(t_0t_1t_0t_1t_0t_1t_0)(t_0t_1t_0) \in K^G$

$\implies t_0t_1t_0t_0t_1t_0t_1t_0t_1t_0t_0t_1t_0 \in K^G$

$\implies t_1 \in K^G$

$\implies t_1^G \in K^G$

$\implies K^G \supseteq \{t_1, t_1^{y^{-1}}, t_1^y, (t_1^y)^x\}$

$\implies K^G \supseteq \{t_1, t_1^{y^{-1}}, t_1^y, (t_1^y)^x\} =< t_1, t_0, t_2, t_3 >= G$

Hence, the conjugates of K generate G. Therefore, by Iwasawa's lemma, $G \cong L_2(13)$ is simple.

$\square$

3.3 $2 \times L_2(8)$ as a Homomorphic Image of $2^{*7} : D_{14}$

3.3.1 Construction of $2 \times L_2(8)$ over D_{14}

Consider the group $G = 2^{*7} : D_{14}$ factored by the relators $(xtt^x)^2$ and $(ttxt)^9$. Note $N = D_{14}$, where $x \sim (1,2,3,4,5,6,7)$ and $y \sim (1,6)(2,5)(3,4)$. Let $t \sim t_7 \sim t_0$. Let us expand the relators:

$$
\begin{aligned}
1 &= (xtt^x)^2 = (xt_0t_1)^2 \\
&= x^2(t_0t_1)^x t_0t_1 = x^2t_1t_2t_0t_1 \\
\implies 1 &= x^2t_1t_2t_0t_1 \\
\implies t_1t_0 &= x^2t_1t_2 \implies Nt_1t_0 = Nt_1t_2
\end{aligned}
$$

and

$$
\begin{aligned}
e &= (ttxt)^9 \\
&= (t_0t_0xt_0)^9 \\
&= (xt_0)^9 \\
&= x^9 t_0^{x^8} t_0^{x^7} t_0^{x^6} t_0^{x^5} t_0^{x^4} t_0^{x^3} t_0^{x^2} t_0^{x} t_0 \\
&= x^2t_1t_0t_6t_5t_4t_3t_2t_1t_0 \\
\implies e &= x^2t_1t_0t_6t_5t_4t_3t_2t_1t_0 \\
\implies t_7t_1t_2t_3 &= x^2t_1t_7t_6t_5t_4 \\
\implies Nt_7t_1t_2t_3 &= Nt_1t_7t_6t_5t_4
\end{aligned}
$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N.

NeN

First, the double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{0, 1, 2, 3, 4, 5, 6\}$, the orbit of N on $\{0, 1, 2, 3, 4, 5, 6\}$ is: $\mathbb{O} = \{0, 1, 2, 3, 4, 5, 6\}$. We choose t_0 as our symmetric generator from this orbit $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by [0], so seven symmetric generators will go to [0].

$\boldsymbol{Nt_0N}$

In order to find how many single cosets [0] contains, we must first find $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0$$

$$=< e, (1,6)(2,5)(3,4) >$$

so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\}, \{1,6\}, \{2,5\},$ and $\{3,4\}$. We take t_0, t_1, t_2 and t_3 from each orbit respectively and find to which double coset Nt_0t_0, Nt_0t_1, Nt_0t_2, and Nt_0t_3 belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to $[*]$. Two symmetric generators will go to new double cosets Nt_0t_1, denoted by [01], Nt_0t_2, denoted by [02], and Nt_0t_3, denoted by [03].

$\boldsymbol{Nt_0t_1N}$

Now Nt_0t_1N in N is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(01)} = N^{01} =< Id(N) >$. But Nt_0t_1 is not distinct. Now $Nt_0t_6 \in [01]$ since $(1,6)(2,5)(3,4) \in N$ and $N(t_0t_1)^{(1,6)(2,5)(3,4)} = Nt_0t_6$. Thus, $(1,6)(2,5)(3,4) \in N^{(01)}$. We conclude:

$$N^{(01)} \geq< (1,6)(2,5)(3,4) > .$$

Hence $|N^{(01)}| = 2$ so the number of single cosets in $N^{(01)}$ is $\frac{|N|}{|N^{(01)}|} = \frac{14}{2} = 7$. The orbits of $N^{(01)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\}, \{1,6\}, \{2,5\}, \{3,4\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_i$ belongs to. We have:

$$Nt_0t_1t_1 \in [0]$$

$$Nt_0t_1t_2 \in [012]$$

$$Nt_0t_1t_3 \in [013]$$

$$Nt_0t_1t_0 \in [010].$$

The new double cosets have single coset representatives $Nt_0t_1t_2, Nt_0t_1t_3, Nt_0t_1t_0$, we represent them as [012], [013], [010], respectively.

$\boldsymbol{Nt_0t_2N}$

Now Nt_0t_2N in N is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(02)} = N^{02} =< Id(N) >$. But Nt_0t_2 is not distinct. Now $Nt_3t_1 \in [02]$ since $(1,2)(3,0)(4,6) \in N$ and $N(t_0t_2)^{(1,2)(3,0)(4,6)} = Nt_3t_1$. Thus, $(1,2)(3,0)(4,6) \in N^{(02)}$. We conclude:

$$N^{(02)} \geq< (1,2)(3,0)(4,6) > .$$

Hence $|N^{(02)}| = 2$ so the number of single cosets in $N^{(02)}$ is $\frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7$. The orbits of $N^{(02)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0,3\}, \{1,2\}, \{4,6\}, \{5\}$. Take a representative t_i

from each orbit and see which double cosets $Nt_0t_2t_i$ belongs to. We have

$$\begin{aligned}
t_0t_2t_3 &= x^{-3}t_5t_4t_2 \implies Nt_0t_2t_3 = Nt_5t_4t_2 \in [013]\\
Nt_0t_2t_2 &= Nt_0 \in [0]\\
t_0t_2t_4 &= xt_4t_1 \implies Nt_0t_2t_4 = Nt_4t_1 \in [03]\\
Nt_0t_2t_5 &\in [025].
\end{aligned}$$

The new double coset have single coset representative $Nt_0t_2t_5$, denoted by [025].

$\boldsymbol{Nt_0t_3N}$

Now Nt_0t_3N in N is a new double coset. However, $N^{(03)} = N^{03} =< Id(N) >$. Only identity e will fix 0 and 3. Hence the number of single cosets living in Nt_0t_3 is $\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_i$ belongs to. We have:

$$\begin{aligned}
t_0t_3t_1 &= x^{-1}t_2t_6 \implies Nt_0t_3t_1 = Nt_2t_6 \in [03]\\
t_0t_3t_2 &= x^2t_2t_3t_4 \implies Nt_0t_3t_2 = Nt_2t_3t_4 \in [012]\\
Nt_0t_3t_3 &= Nt_0 \in [0]\\
t_0t_3t_4 &= xt_0t_1t_3 \implies Nt_0t_3t_4 = Nt_0t_1t_3 \in [013]\\
t_0t_3t_5 &= et_3t_5t_1 \implies Nt_0t_3t_5 = Nt_3t_5t_1 \in [025]\\
Nt_0t_3t_6 &\in [036]\\
t_0t_3t_0 &= xt_4t_2 \implies Nt_0t_3t_0 = Nt_4t_2 \in [02].
\end{aligned}$$

The new double coset have single coset representative $Nt_0t_3t_6$, denoted by [036].

$\boldsymbol{Nt_0t_1t_2N}$

Consider $Nt_0t_1t_2N$ in N is a new double coset. We determined how many single cosets are in the double coset. Well $N^{(012)} = N^{012} =< e >$. But $Nt_0t_1t_2$ is not distinct. Now $Nt_3t_2t_1 \in [012]$ since $(1,2)(3,7)(4,6) \in N$ and $N(t_0t_1t_2)^{(1,2)(3,0)(4,6)} = t_3t_2t_1$. Thus, $(1,2)(3,0)(4,6) \in N^{(012)}$. we conclude:

$$N^{(012)} \geq< (1,2)(3,0)(4,6) > .$$

Hence $|N^{(012)}| = 2$ so the number of single cosets in $N^{(012)}$ is $\frac{|N|}{|N^{(012)}|} = \frac{14}{2} = 7$. The orbits of $N^{(012)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0,3\},\{1,2\},\{4,6\},\{5\}$ Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_2t_i$ belongs to.

$$t_0t_1t_2t_0 = x^{-2}t_5t_1 \implies Nt_0t_1t_2t_0 = Nt_5t_1 \in [03]$$

$$Nt_0t_1t_2t_2 = Nt_0t_1 \in [01]$$

$$t_0t_1t_2t_4 = x^{-2}t_4t_3t_1 \implies Nt_0t_1t_2t_4 = Nt_4t_3t_1 \in [013]$$

$$t_0t_1t_2t_5 = et_5t_1t_4 \implies Nt_0t_1t_2t_5 = Nt_5t_1t_4 \in [036].$$

$Nt_0t_1t_3N$

Consider $Nt_0t_1t_3N$ in N is a new double coset. However, $N^{(013)} = N^{013} =< Id(N) >$. Only identity e will fix 0,1, and 3. Hence the number of single cosets living in $Nt_0t_1t_3$ is $\frac{|N|}{|N^{(013)}|} = \frac{14}{1} = 14$. The orbits of $N^{(013)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_3t_i$ belongs to.

$$t_0t_1t_3t_0 = x^{-2}t_4t_3t_2 \implies Nt_0t_1t_3t_0 = Nt_4t_3t_2 \in [012]$$

$$t_0t_1t_3t_1 = x^{-4}t_2t_1t_6 \implies Nt_0t_1t_3t_1 = Nt_2t_1t_6 \in [013]$$

$$t_0t_1t_3t_2 = x^{2}t_2t_4 \implies Nt_0t_1t_3t_2 = Nt_2t_4 \in [02]$$

$$Nt_0t_1t_3t_3 = Nt_0t_1 \in [01]$$

$$t_0t_1t_3t_4 = x^{-1}t_0t_3 \implies Nt_0t_1t_3t_4 = Nt_0t_3 \in [03]$$

$$t_0t_1t_3t_5 = x^{-1}t_1t_4t_0 \implies Nt_0t_1t_3t_5 = Nt_1t_4t_0 \in [036]$$

$$t_0t_1t_3t_6 = x^{-1}t_5t_3t_0 \implies Nt_0t_1t_3t_6 = Nt_5t_3t_0 \in [02].$$

$Nt_0t_1t_0N$

Now $Nt_0t_1t_0N$ is indeed a new double coset. We determine how many single cosets are in this double coset. Well $N^{(010)} = N^{010} =< Id(N) >$. Well $N^{(010)} = N^{010} =< Id(N) >$. We have these two relations $t_0t_1t_0 = (0,5,3,1,6,4,2)t_5t_6t_5$ and $t_0t_1t_0 = (1,0)(2,6)(3,5)t_0t_6t_0$. Since $(0,5,3,1,6,4,2) \in N$ and $N(t_0t_1t_0)^{(0,5,3,1,6,4,2)} = Nt_5t_6t_5$. Thus, $(0,5,3,1,6,4,2) \in N^{(010)}$ and $N^{(010)} \geq< (0,5,3,1,6,4,2) >$. Since

$(1,0)(2,6)(3,5) \in N$ and $N(t_0t_1t_0)^{(1,0)(2,6)(3,5)} = t_1t_0t_1$. Thus, $(1,0)(2,6)(3,5) \in N^{(010)}$. We conclude:

$$N^{(010)} \geq < (1,0)(2,6)(3,5), (0,1,2,3,4,5,6) > .$$

Then $N^{(010)} = N$. Hence $|N^{(010)}| = 14$, so the number of single cosets in $N^{(010)}$ is $\frac{|N|}{|N^{(010)}|} = \frac{14}{14} = 1$. The orbit of $N^{(010)}$ on $\{0,1,2,3,4,5,6\}$ is $\mathbb{O} = \{0,1,2,3,4,5,6\}$. Take a representative from this orbit, say t_0. Hence $Nt_0t_1t_0t_0 \in [01]$. Therefore, seven symmetric generators will go back to Nt_0t_1N.

$Nt_0t_2t_5N$

Now consider $Nt_0t_2t_5N$ in N is a new double coset. We determined how many single cosets are in the double coset. Well $N^{(025)} = N^{025} =< Id(N) >$. But $Nt_0t_2t_5$ is not distinct. Now $Nt_3t_1t_5 \in [025]$ since $(0,3)(1,2)(4,6) \in N$ and $N(t_0t_2t_5)^{(0,3)(1,2)(4,6)} = t_3t_1t_5$. Thus, $(3,0)(1,2)(4,6) \in N^{(025)}$. We conclude:

$$N^{(025)} \geq < (0,3)(1,2)(4,6) > .$$

Hence $|N^{(025)}| = 2$ so the number of single cosets in $N^{(025)}$ is $\frac{|N|}{|N^{(025)}|} = \frac{14}{2} = 7$. The orbits of $N^{(025)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0,3\}, \{1,2\}, \{4,6\}, \{5\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_5t_i$ belongs to.

$$t_0t_2t_5t_0 = x^{-4}t_4t_6t_2 \implies Nt_0t_2t_5t_0 = Nt_4t_6t_2 \in [025]$$

$$t_0t_2t_5t_1 = x^2t_6t_3 \implies Nt_0t_2t_5t_1 = Nt_6t_3 \in [03]$$

$$t_0t_2t_5t_4 = x^{-4}t_5t_6t_1 \implies Nt_0t_2t_5t_4 = Nt_5t_6t_1 \in [013]$$

$Nt_0t_2t_5t_5 \in [02]$.

$Nt_0t_3t_6N$

Now consider $Nt_0t_3t_6N$ in N is a new double coset. We determined how many single cosets are in the double coset. Well $N^{(036)} = N^{036} =< Id(N) >$. But $Nt_0t_3t_6$ is not distinct. Now $Nt_0t_4t_1 \in [036]$ since $(1,6)(2,5)(3,4) \in N$ and $N(t_0t_3t_6)^{(1,6)(3,4)(2,5)} = t_0t_4t_1$. Thus, $(1,6)(3,4)(2,5) \in N^{(036)}$. We conclude:

$$N^{(036)} \geq < (1,6)(3,4)(2,5) > .$$

Hence $|N^{(036)}| = 2$ so the number of single cosets in $N^{(036)}$ is $\frac{|N|}{|N^{(036)}|} = \frac{14}{2} = 7$. The orbits of $N^{(036)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{1,6\}, \{3,4\}, \{2,5\}, \{0\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_6t_i$ belongs to.

$$Nt_0t_3t_6t_6 \in [03]$$

$$t_0t_3t_6t_3 = xt_1t_0t_5 \implies Nt_0t_3t_6t_3 = Nt_1t_0t_5 \in [013]$$

$$t_0t_3t_6t_5 = x^2t_3t_6t_2 \implies Nt_0t_3t_6t_5 = Nt_3t_6t_2 \in [036]$$

$$t_0t_3t_6t_0 = et_2t_3t_4 \implies Nt_0t_3t_6t_0 = Nt_2t_3t_4 \in [036].$$

We have completed the double coset enumeration since the right coset is closed under multiplication, hence, the index of N in G is 72 single cosets. We conclude: $G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_2N \cup Nt_0t_3N \cup Nt_0t_1t_2N \cup Nt_0t_1t_3N \cup Nt_0t_1t_0N \cup Nt_0t_2t_5N \cup Nt_0t_3t_6N$, where

$$G = \frac{2^{*7} : D_{14}}{(xtt^x)^2, (ttxt)^9 = 1}$$

$$|G| \leq |N| + \tfrac{|N|}{N^{(0)}} + \tfrac{|N|}{N^{(01)}} + \tfrac{|N|}{N^{(02)}} + \tfrac{|N|}{N^{(03)}} + \tfrac{|N|}{N^{(012)}} + \tfrac{|N|}{N^{(013)}} + \tfrac{|N|}{N^{(010)}} + \tfrac{|N|}{N^{(025)}} + \tfrac{|N|}{N^{(036)}} \times |N|$$

$$|G| \leq (1 + 7 + 7 + 7 + 14 + 7 + 14 + 1 + 7 + 7) \times 14$$

$$|G| \leq 72 \times 14$$

$$|G| \leq 1008$$

The Cayley diagram that summarizes the above information is given below:

3.3.2 Factoring $2 \times L_2(8)$ by the Center

Consider $G = \frac{2^{*7}:D_{14}}{(xtt^x)^2,(ttxt)^9=1} \cong 2 \times L_2(8)$. We are going to factor G by the center, to do so, we use the following loops in Magma:

```
> D:=DihedralGroup(7);
> xx:=D!(1,2,3,4,5,6,7);
> yy:=D!(1, 6)(2, 5)(3, 4);
> N:=sub<D|xx,yy>;
> G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
(x*t*t^x)^2,(t*t*x*t)^9>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> Center(G1);
Permutation group acting on a set of cardinality 72
Order = 2
    (1, 20)(2, 12)(3, 6)(4, 7)(5, 11)(8, 13)(9, 18)(10, 19)
```

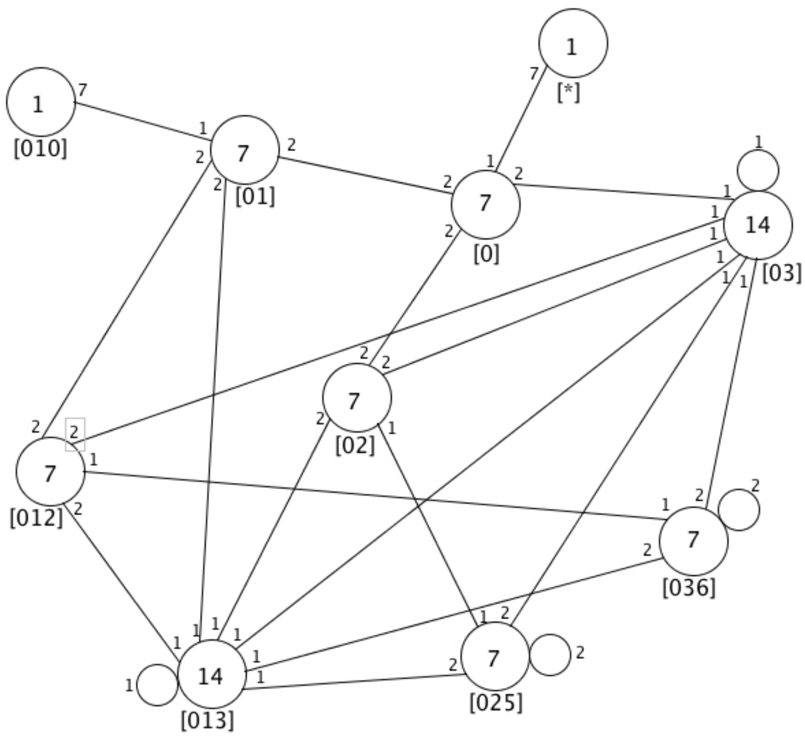

Figure 3.2: Cayley Diagram of $2 \times L_2(8)$ over D_{14}

```
(14, 21)(15, 22)(16, 29)(17, 30)(23, 31)(24, 32)
(25, 38)(26, 33)(27, 43)(28, 35)(34, 44)(36,45)
(37, 52)(39, 46)(40, 56)(41, 55)(42, 48)(47, 57)
(49, 58)(50, 62)(51,63)(53, 68)(54, 67)(59, 69)(60, 65)
(61, 71)(64, 66)(70, 72)
```

By Magma, we know that the center of G is of order 2. We let aa equals to the Center(G1). Now to convert the center in term of word, we use the Schreier System:

```
> A:=f(x);
> B:=f(y);
> C:=f(t);
> N:=sub<G1|A,B,C>;
> NN<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
> (x*t*t^x)^2, (t*t*x*t)^9>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..#N]];
> for i in [2..#N] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
```

```
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..#N] do if ArrayP[i] eq aa
then print Sch[i]; end if; end for;
x * y * t * x * t * x^-1 * t
```

Thus, the center of G is $Z(G) =< xytxtx^{-1}t >$. Now we factor G by the center and we obtain the following:

```
> G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
 (x*t*t^x)^2,(t*t*x*t)^9,x*y*t*x*t*x^-1*t>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
    G
    |  A(1, 8)                      = L(2, 8)
    1
```

Now, $G = 2^{*7} : D_{14}$ is factored by the relators $(xtt^x)^2$,$(ttxt)^9$ and the center $Z(G) =< xytxtx^{-1}t >$.

3.3.3 Construction of $L_2(8)$ over D_{14}

Consider the group $G \cong< x^7, y^2, (x*y)^2, t^2, (t, y) >$ factored by $(xtt^x)^2$,$(ttxt)^9$ and the center $Z(G) =< xytxtx^{-1}t >$. Recall, $G = 2^{*7} : D_{14}$, $N = D_{14} =< x, y >=< (0, 1, 2, 3, 4, 5, 6), (1, 6)(2, 5)(3, 4) >$, and $t \sim t_7 \sim t_0$. Also, by expanding the two relations we have:

$$(xtt^x)^2 = x^2t_1t_2t_0t_1 = 1 \implies x^2t_1t_2 = t_1t_0$$

$$(ttxt)^9 = x^2t_1t_0t_6t_5t_4t_3t_2t_1t_0 = 1 \implies x^2t_1t_0t_6t_5 = t_0t_1t_2t_3t_4.$$

Now lets expand the center $Z(G) =< xytxtx^{-1}t >$:

$$xytxtx^{-1}t = xyt_0xt_0x^{-1}t_0 = xyt_0t_6t_0 = 1 \implies xyt_0t_6 = t_0.$$

Now, we begin the construction of $L_2(8)$ over D_{14}.

$\boldsymbol{NeN}$

First, the double coset NeN, is denoted by [$*$]. This double coset contains only the single coset, namely N. Since N is transitive on $\{0,1,2,3,4,5,6\}$, the orbit of N on $\{0,1,2,3,4,5,6\}$ is: $\mathbb{O} = \{0,1,2,3,4,5,6\}$. We choose t_0 as our symmetric generator from this orbit $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by [0], so seven symmetric generators will go to [0].

$\boldsymbol{Nt_0N}$

In order to find how many single cosets [0] contains, we must first find $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0 =< e, (1,6)(2,5)(3,4) >$$

so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\}, \{1,6\}, \{2,5\},$ and $\{3,4\}$. We take t_0, t_6, t_2 and t_3 from each orbit respectively and find to which double coset Nt_0t_0, Nt_0t_6, Nt_0t_2, and Nt_0t_3 belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to [$*$]. We have the relation $t_0t_6 = xyt_0 \implies Nt_0t_6 = Nt_0 \in [0]$, since $xy \in N$ and $Nt_0 \in [0]$. Thus, $Nt_0t_6 = Nt_0 \in [0]$, so two elements will loop back to [0]. On the other hand, two symmetric generators will go to new double cosets Nt_0t_2, denoted by [02], and Nt_0t_3, denoted by [03].

$\boldsymbol{Nt_0t_2N}$

Now Nt_0t_2N in N is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(02)} = N^{02} =< Id(N) >$. But Nt_0t_2 is not distinct. Now $Nt_3t_1 \in [02]$ since $(1,2)(3,0)(4,6) \in N$ and $N(t_0t_2)^{(1,2)(3,0)(4,6)} = Nt_3t_1$. Thus, $(1,2)(3,0)(4,6) \in N^{(02)}$. We conclude, $N^{(02)} \geq< (1,2)(3,0)(4,6) >$. Hence $|N^{(02)}| = 2$ so the number of single cosets in $N^{(02)}$ is $\frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7$. The orbits of $N^{(02)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{1,2\}, \{3,0\}, \{4,6\}, \{5\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_i$ belongs to. We have

$$Nt_0t_2t_2 = Nt_0 \in [0]$$
$$t_0t_2t_3 = x^2yt_5t_2 \implies Nt_0t_2t_3 = Nt_5t_2 \in [03]$$
$$t_0t_2t_4 = xt_4t_1 \implies Nt_0t_2t_4 = Nt_4t_1 \in [03]$$
$$Nt_0t_2t_5 \in [025].$$

The new double coset have single coset representative $Nt_0t_2t_5$, denoted by [025].

Nt_0t_3N

Now Nt_0t_3N in N is a new double coset. However, $N^{(03)} = N^{03} =< Id(N) >$. Only identity e will fix 0 and 3. Hence the number of single cosets living in Nt_0t_3 is $\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_i$ belongs to. We have:

$$t_0t_3t_1 = x^{-1}t_2t_6 \implies Nt_0t_3t_1 = Nt_2t_6 \in [03]$$
$$t_0t_3t_2 = x^2yt_5t_3 \implies Nt_0t_3t_2 = Nt_5t_3 \in [02]$$
$$Nt_0t_3t_3 = Nt_0 \in [0]$$
$$t_0t_3t_4 = yt_0t_3 \implies Nt_0t_3t_4 = Nt_0t_3 \in [03]$$
$$t_0t_3t_5 = et_3t_5t_1 \implies Nt_0t_3t_5 = Nt_3t_5t_1 \in [025]$$
$$t_0t_3t_6 = yt_5t_3t_0 \implies Nt_0t_3t_6 = Nt_5t_3t_0 \in [025]$$
$$t_0t_3t_0 = xt_4t_2 \implies Nt_0t_3t_0 = Nt_4t_2 \in [02].$$

$Nt_0t_2t_5N$

Now consider $Nt_0t_2t_5N$ in N is a new double coset. We determined how many single cosets are in the double coset. Well $N^{(025)} = N^{025} =< Id(N) >$. But $Nt_0t_2t_5$ is not distinct. Now $Nt_3t_1t_5 \in [025]$ since $(0,3)(1,2)(4,6) \in N$ and $N(t_0t_2t_5)^{(0,3)(1,2)(4,6)} = t_3t_1t_5$. Thus, $(3,0)(1,2)(4,6) \in N^{(025)}$. We conclude, $N^{(025)} \geq< (0,3)(1,2)(4,6) >$. Hence $|N^{(025)}| = 2$ so the number of single cosets in $N^{(025)}$ is $\frac{|N|}{|N^{(025)}|} = \frac{14}{2} = 7$. The orbits of $N^{(025)}$ on $\{0,1,2,3,4,5,6\}$ are: $\mathbb{O} = \{0,3\},\{1,2\},\{4,6\},\{5\}$. Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_5t_i$ belongs to.

$$t_0t_2t_5t_1 = x^2t_6t_3 \implies Nt_0t_2t_5t_1 = Nt_6t_3 \in [03]$$

$$t_0t_2t_5t_3 = x^6t_6t_4t_1 \implies Nt_0t_2t_5t_3 = Nt_6t_3 \in [025]$$

$$t_0t_2t_5t_4 = x^6yt_5t_1 \implies Nt_0t_2t_5t_4 = Nt_5t_1 \in [013]$$

$$Nt_0t_2t_5t_5 \in [02].$$

We have completed the double coset enumeration since the right coset is closed under multiplication, hence, the index of N in G is 36 single cosets. We conclude: $G = N \cup Nt_0N \cup Nt_0t_2N \cup Nt_0t_3N \cup Nt_0t_2t_5N$, where

$$G = \frac{2^{*7} : D_{14}}{(xtt^x)^2, (ttxt)^9, xytxtx^{-1}t = 1}$$

$$|G| \leq |N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(02)}} + \frac{|N|}{N^{(03)}} + \frac{|N|}{N^{(025)}}$$

$$|G| \leq (1 + 7 + 7 + 14 + 7) \times 14$$

$$|G| \leq 36 \times 14$$

$$|G| \leq 504$$

The Cayley diagram that summarizes the above information is given below:

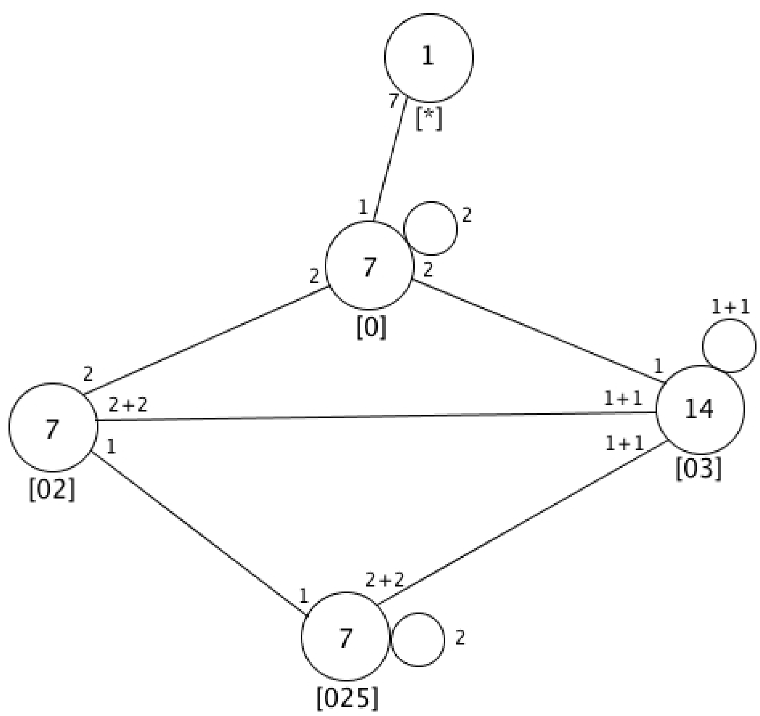

Figure 3.3: Cayley Diagram of $L_2(8)$ over D_{14}

3.4 Iwasawa's Lemma to Prove $L_2(8)$ over D_{14} is Simple

We consider
$$G \cong \frac{2^{*7}\ D_{14}}{[(xt_0t_1]^2, [t_0t_0xt_0]^9, xyt_0xt_0x^{-1}t_0} \cong L_2(8),$$

Now we use *Iwasawa's lemma* to prove $G \cong L_2(8)$ is simple. We use *Iwasawa's lemma* to prove $G \cong L_2(8)$ is **simple**. *Iwasawa's lemma* has three sufficient conditions that we must satisfied:

(1) G acts on X faithfully and primitively

(2) G is perfect ($G = G'$)

(3) There exist $x \in X$ and a normal abelian subgroup K of G^x such that the conjugates of K generate G.

Proof. **3.4.1 $G = L_2(8)$ acts on X Faithfully**

Let G acts on $X = \{N, Nt_0N, Nt_0t_2N, Nt_0t_3N, Nt_0t_2t_5N$, where $|X| = 36$. G acts on X implies there exist homomorphism
$$f\ : G \longrightarrow S_{36} \qquad\qquad (|X| = 36).$$
By First Isomorphic Theorem we have:
$$G/kerf \cong f(G).$$
If $kerf = 1$ then $G \cong f(G)$. Only elements of N fix N implies $G^1 = N$. Since X is a transitive $G - set$ of degree 36, we have:
$$\begin{aligned} |G| &= 36 \times |G^1| \\ &= 36 \times |N| \\ &= 36 \times 14 \\ &= 508 \\ \implies |G| &= 508. \end{aligned}$$
From Cayley diagram, $|G| \leq 508$. However, from above $|G| = 508$ implying $ker(f) = 1$. Since $kerf = 1$ then G acts faithfully on X.

3.4.2 $G = L_2(8)$ acts on X Primitively

In order to show that G is primitive, we must show that $G = L_2(8)$ is transitive on $X = |36|$ and there exists no nontrivial blocks of X. From the Cayley diagram of $G = L_2(8)$ over D_{14}, we see that G is transitive. Let B be a nontrivial block, then $|B|||X|$. Note if we had a nontrivial block it would have to be of size 2, 3, 4, 6, 9, 12, or 18. By inspection, of our Cayley diagram we ca see that we cannot create a nontrivial block of these sizes.

3.4.3 $G = L_2(8)$ is Perfect

Next we want to show that $G = G'$. Now $D_{14} \subseteq G \implies {D_{14}}' \subseteq G'$.

${D_{14}}' =< [a,b]|a,b \in D_{14} >$. Now the derived group,

$$
\begin{aligned}
&{D_{14}}' =< (0,1,2,3,4,5) >=< x > \\
&\implies \{e, x, x^2, x^3, x^4, x^5, x^6)\} \subseteq G'.
\end{aligned}
$$

Now $x = (0,1,2,3,4,5,6)$ and $y = (1,6)(2,5)(3,4)$.

Then $[x,y] = x^{-1}y^{-1}xy$

$$
\begin{aligned}
&= (0,1,2,3,4,5,6)(1,6)(2,5)(3,4)(0,6,5,4,3,2,1)(1,6)(2,5)(3,4) \\
&= (0,2,4,6,1,3,5) \in G'.
\end{aligned}
$$

If we conjugate $(0,2,4,6,1,3,5)$ by (0,1,2,3,4,5,6) we get $(1,3,5,0,2,4,6) \in G'$.

Main relation:

$$
\begin{aligned}
x^2 &= t_1t_0t_2t_1 \quad (*) \\
(x^2)^3 &= (t_1t_0t_2t_1)^3 \\
x^6 &= t_1t_0t_2t_0t_2t_0t_2t_1 \\
x^{-1} &= t_1t_0t_2t_0t_2t_0t_2t_1 \\
x &= t_1t_2t_0t_2t_0t_2t_0t_1
\end{aligned}
$$

Now, we use the relation that we obtained by factoring by the center:

$$
\begin{aligned}
xy &= t_0t_6t_0 \\
x^{-1}xy &= t_1t_0t_2t_0t_2t_0t_2t_1t_0t_6t_0 \\
y &= t_1t_0t_2t_0t_2t_0t_2t_1t_0t_6t_0
\end{aligned}
$$

So $G =< x, y, t >=< t_0, t_1, t_2, t_3, t_4, t_5, t_6 >$. Our goal is to show that one of the $t_i's \in G'$, then we can conjugate. Since $x \in G'$. Then from our double coset relation we have:

$$\begin{aligned} x &= t_0t_2t_4t_1t_4 \in G' \\ &= t_0t_2t_1t_1t_4t_1t_4 \in G' \\ &= t_0t_2t_1[t_1, t_4] \in G' \text{ (since } [t_1, t_4] \in G') \\ &= t_0t_2t_1 \in G' \end{aligned}$$

Now, we multiply $t_0t_2t_1$ by the inverse of (*):

$$t_0t_2t_1t_1t_2t_0t_1 = t_1 \in G' \text{ (since } x \in G' \text{ and } (x^2)^{-1} \in G').$$

So $t_1 \in G'$

$\implies t_1^x, t_1^{x^2}, t_1^{x^3}, t_1^{x^4}, t_1^{x^5}, t_1^{x^6}, t_1^{x^7} \in G'$ (since $x \in G$ and $G' \trianglelefteq G$)

$\implies < t_2, t_3, t_4, t_5, t_6, t_0, t_1 > \in G'$

Thus $G \subseteq G' \subseteq < t_2, t_3, t_4, t_5, t_6, t_0, t_1 > \in G \implies G' = G$.

Hence G is perfect.

3.4.4 Conjugates of a Normal Abelian K Generate $G = L_2(8)$ over D_{14}

Now we require $x \in X$ and a $K \trianglelefteq G^x$, where K is a normal abelian subgroup such as the conjugates of K in G generate G. Recall, $G^1 = N = D_{14}$. Let $K =< x >$. Since K is normal abelian subgroup in G' then for any $s \in K$ and for all $g \in G$ implies $s^g \in K$. Since $x \in K \implies x^2 \in K$. Now from (*) we have:

$$\begin{aligned} x^2 &= t_1t_0t_2t_1 \in K \\ (x^2)^{t_1} &= (t_1t_0t_2t_1)^{t_1} \in K^G \\ t_1(x^2)t_1 &= t_1(t_1t_0t_2t_1)t_1 \in K^G \\ x^2t_3t_1 &= t_0t_2 \in K^G \\ &\implies t_0t_2 \in K^G \end{aligned}$$

So, the inverse $t_2t_0 \in K^G$. Moreover, from the double coset we have the following relation:

$$x = t_0t_2t_4t_1t_4 \in K.$$

Now, we multiply the above relation by t_2t_0:

$$t_0t_2t_4t_1t_4t_2t_0 \in K$$

$$(t_0t_2t_4t_1t_4t_2t_0)^{t_0t_2t_4} \in K^G$$

$$= t_4t_2t_0(t_0t_2t_4t_1t_4t_2t_0)t_0t_2t_4 = t_1 \in K^G$$

Thus, $t_1 \in K^G$
$\implies t_1^G \in K^G$
$\implies K^G \supseteq \{t_1^x, t_1^{x^2}, t_1^{x^3}, t_1^{x^4}, t_1^{x^5}, t_1^{x^6}\}$
$\implies K^G \supseteq \{t_1^x, t_1^{x^2}, t_1^{x^3}, t_1^{x^4}, t_1^{x^5}, t_1^{x^6}\} =< t_1, t_0, t_2, t_3, t_4, t_5, t_6 >= G$

Hence, the conjugates of K generate G. Therefore, by Iwasawa's lemma, $G \cong L_2(8)$ is simple.

□

3.5 $L_2(13)$ as a Homomorphic Image of $2^{*7} : D_{14}$

3.5.1 Construction of $L_2(13)$ over D_{14}

Factoring the progenitor $2^{*7} : D_{14}$ by the following relations

$$[(1,5)(2,4)(6,0)t_1t_0]^7 \text{ and } [(1,2,3,4,5,6,0)t_0]^3$$

yields the finite homomorphic image:

$$G \cong \frac{2^{*7} : D_{14}}{[(1,5)(2,4)(6,0)t_1t_0]^7,\ [t_0t_0(1,2,3,4,5,6,0)t_0]^3},$$

where D_{14} is a maximal in $L_2(13)$ and the index of D_{14} in G equals 78. $G \cong L_2(13)$, the projective special linear group.

A symmetric representation for the above image is given by:

$$< x, y, t | x^7, y^2, (x*y)^2, t^2, (t,y), (x*y*t^x*t)^7, (t*t*x*t)^3 >,$$

where $N = D_{14} \cong < x^7, y^2, (x*y)^2 >$, and the action of x, y on the symmetric generators is given by

$$x \sim (1,2,3,4,5,6,0),$$
$$y \sim (1,6)(2,5)(3,4).$$

The relation

$((1,5)(2,4)(6,0)t_1t_0)^7 = 1$ with $(1,5)(2,4)(6,0) = \pi$ becomes

$$(\pi t_1 t_0)^7 = 1$$

$\implies \pi t_1t_0\pi t_1t_0\pi t_1t_0\pi t_1t_0\pi t_1t_0\pi t_1t_0\pi t_1t_0 = 1$

$\implies \pi^7(t_1t_0)^{\pi^6}(t_1t_0)^{\pi^5}(t_1t_0)^{\pi^5}(t_1t_0)^{\pi^3}(t_1t_0)^{\pi^2}(t_1t_0)^{\pi}(t_1t_0) = 1$

$\implies \pi(t_1t_0)(t_1t_0)^{\pi}(t_1t_0)(t_1t_0)^{\pi}(t_1t_0)^{(}t_1t_0)^{\pi}(t_1t_0) = 1$

$\implies (1,5)(2,4)(6,0)t_1t_0t_5t_6t_1t_0t_5t_6t_1t_0t_5t_6t_1t_0 = 1$

$\implies (1,5)(2,4)(6,0)t_1t_0t_5t_6t_1t_0t_5t_6 = t_0t_1t_6t_5t_0t_1$

$$\implies Nt_1t_0t_5t_6t_1t_0t_5t_6 = Nt_0t_1t_6t_5t_0t_1.$$

The relation

$((1,2,3,4,5,6,0)t_0)^3 = ((1,2,3,4,5,6,0)t_0)^3 = 1$

$\implies x^3 t_0^{x^2} t_0^{x} t_0 = 1$

$\implies (1,4,0,3,5,2,5)t_2t_1t_0 = 1$

$\implies (1,4,0,3,5,2,5)t_2 = t_0t_1$

$\implies Nt_2 = Nt_0t_1.$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$ where g_i's words in t_i's.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [Nw^n | n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by t_i's. We will identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilising group $N^{(w)}$

NeN

First, the double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$, the orbit of N on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ is:

$$\mathbb{O} = \{\{1, 2, 3, 4, 5, 6, 0\}\}.$$

We choose t_0 as our symmetric generator from $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denote it [0].

Nt_0N

In order to find how many single cosets [0] contains, we must first find the coset stabiliser $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0$$
$$=< (1, 6)(2, 5)(3, 4) >$$

so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7$. Furthermore, the orbits of $N^{(0)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}.$$

We take t_0, t_1, t_2 and t_3 from each orbit, respectively, and to see which double coset Nt_0t_0, Nt_0t_1, Nt_0t_2, and Nt_0t_3 belong to. We have:

$$Nt_0t_0 = N \in [*]$$
$$t_0t_1 = x^3t_2 \implies Nt_0t_1 = Nt_2 \in [0]$$
$$Nt_0t_2 \in [02]$$
$$Nt_0t_3 \in [03].$$

The new double cosets have single coset representatives Nt_0t_2 and Nt_0t_3, which we represent them as [02] and [03] respectively.

Nt_0t_2N

Now Nt_0t_2N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(02)} = N^{02} =< e >$. But Nt_0t_2 is not distinct. We have $Nt_5t_3 \in [02]$ since $(1, 4)(3, 2)(0, 5) \in N$ and $N(t_0t_2)^{(1,4)(3,2)(0,5)} = Nt_5t_3$. Thus, $(1, 4)(3, 2)(0, 5) \in N^{(02)}$. We conclude:

$$N^{(02)} \geq< (1, 4)(3, 2)(0, 5) > .$$

Hence $|N^{(02)}| = 2$. So the number of single cosets in $N^{(02)}N$ is $\frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7$. The orbits of $N^{(02)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1, 4\}, \{3, 2\}, \{0, 5\}, \{6\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_i$ belongs to. We have:

$$t_0t_2t_4 = x^2t_1t_5 \implies Nt_0t_2t_4 = Nt_1t_5 \in [03]$$
$$Nt_0t_2t_2 = Nt_0 \in [0]$$
$$Nt_0t_2t_0 \in [020]$$
$$Nt_0t_2t_6 \in [026].$$

The new double coset are $Nt_0t_2t_0$ and $Nt_0t_2t_6$, which we represent them as [020] and [026] respectively.

Nt_0t_3N

Consider Nt_0t_3N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(03)} = N^{03} =< e >$. Only identity (e) will fix 0 and 3. Hence the number of single cosets living in Nt_0t_3N is $\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_i$ belongs to. We have:

$$Nt_0t_3t_1 \in [031]$$
$$t_0t_3t_2 = x^4t_4t_1 \implies Nt_0t_3t_2 = Nt_4t_1 \in [03]$$
$$Nt_0t_3t_3 = Nt_0 \in [0]$$
$$t_0t_3t_4 = x^2t_1t_6 \implies Nt_0t_3t_4 = Nt_1t_6 \in [02]$$
$$t_0t_3t_5 = x^6t_6t_1t_6 \implies Nt_0t_3t_5 = Nt_6t_1t_6 \in [020]$$
$$Nt_0t_3t_6 \in [036]$$
$$Nt_0t_3t_0 \in [030].$$

The new double coset are $Nt_0t_3t_1$, $Nt_0t_3t_6$, and $Nt_0t_3t_0$, which we represent them as [031], [036], and [030] respectively.

$Nt_0t_2t_0N$

Now consider $Nt_0t_2t_0N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(020)} = N^{020} =< e >$. Only identity (e) will fix 0 and 2. Hence the number of single cosets living in $Nt_0t_2t_0N$ is $\frac{|N|}{|N^{(020)}|} = \frac{14}{1} = 14$. The orbits of $N^{(020)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_0t_i$ belongs to. We have:

$$t_0t_2t_0t_1 = x^2t_1t_6t_2 \implies Nt_0t_2t_0t_1 = Nt_1t_6t_2 \in [026]$$

$$t_0t_2t_0t_2 = x^5yt_0t_2t_0 \implies Nt_0t_2t_0t_2 = Nt_0t_2t_0 \in [020]$$

$$t_0t_2t_0t_3 = yt_4t_2t_5 \implies Nt_0t_2t_0t_3 = Nt_4t_2t_5 \in [026]$$

$$t_0t_2t_0t_4 = xyt_2t_5t_1 \implies Nt_0t_2t_0t_4 = Nt_2t_5t_1 \in [036]$$

$$t_0t_2t_0t_5 = (x^2)^{-1}t_5t_1t_4 \implies Nt_0t_2t_0t_5 = Nt_5t_1t_4 \in [036]$$

$$t_0t_2t_0t_6 = xt_1t_4 \implies Nt_0t_2t_0t_6 = Nt_1t_4 \in [03]$$

$$Nt_0t_2t_0t_0 = Nt_0t_2 \in [02].$$

$Nt_0t_2t_6N$

Now $Nt_0t_2t_6N$ is a new double coset. We determine how many single cosets are in the double coset. Now $N^{(026)} = N^{026} =< e >$. But $Nt_0t_2t_6$ is not distinct. Now $Nt_5t_3t_6 \in [026]$ since $(1,4)(3,2)(0,5) \in N$ and $N(t_0t_2t_6)^{(1,4)(3,2)(0,5)} = Nt_5t_3t_6$. Thus, $(1,4)(3,2)(0,5) \in N^{(026)}$. We conclude:

$$N^{(026)} \geq < (1,4)(3,2)(0,5) > .$$

Hence $|N^{(026)}| = 2$. So the number of single cosets in $N^{(026)}N$ is $\frac{|N|}{|N^{(026)}|} = \frac{14}{2} = 7$. The orbits of $N^{(026)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1,4\}, \{3,2\}, \{0,5\}, \{6\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_6t_i$ belongs

to. We have:

$$t_0t_2t_6t_1 = x^6yt_4t_2t_4 \implies Nt_0t_2t_6t_1 = Nt_4t_2t_4 \in [020]$$
$$t_0t_2t_6t_3 = x^2t_0t_3t_6 \implies Nt_0t_2t_6t_3 = Nt_0t_3t_6 \in [036]$$
$$t_0t_2t_6t_0 = x^2t_1t_6t_1 \implies Nt_0t_2t_6t_0 = Nt_1t_6t_1 \in [020]$$
$$Nt_0t_2t_6t_6 = Nt_0t_2 \in [02].$$

$\boldsymbol{t_0t_3t_1N}$

Now $Nt_0t_3t_1N$ is a new double coset. We determine how many single cosets are in the double coset. Now $N^{(031)} = N^{031} =< e >$. But $Nt_0t_3t_1$ is not distinct. Now $Nt_1t_5t_0 \in [031]$ since $(1,0)(5,3)(2,6) \in N$ and $N(t_0t_3t_1)^{(1,0)(5,3)(2,6)} = Nt_1t_5t_0$. Thus, $(1,0)(5,3)(2,6) \in N^{(031)}$. We conclude:

$$N^{(031)} \geq< (1,0)(5,3)(2,6) > .$$

Hence $|N^{(031)}| = 2$. So the number of single cosets in $N^{(031)}N$ is $\frac{|N|}{|N^{(031)}|} = \frac{14}{2} = 7$. The orbits of $N^{(031)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1,0\}, \{5,3\}, \{2,6\}, \{4\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_1t_i$ belongs to. We have:

$$Nt_0t_3t_1t_1 = Nt_0t_3 \in [03]$$
$$t_0t_3t_1t_5 = x^5yt_0t_3t_1 \implies Nt_0t_3t_1t_5 = Nt_0t_3t_1 \in [031]$$
$$t_0t_3t_1t_2 = x^3t_3t_6t_3 \implies Nt_0t_3t_1t_2 = Nt_3t_6t_3 \in [030]$$
$$t_0t_3t_1t_4 = x^3yt_6t_2t_6 \implies Nt_0t_3t_1t_4 = Nt_6t_2t_6 \in [030].$$

$\boldsymbol{Nt_0t_3t_6N}$

Now consider $Nt_0t_3t_6N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(036)} = N^{036} =< e >$. Only identity (e) will fix 0 and 3. Hence the number of single cosets living in $Nt_0t_3t_6N$ is $\frac{|N|}{|N^{(036)}|} = \frac{14}{1} = 14$. The orbits of $N^{(036)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_6t_i$ belongs

to. We have:

$$t_0t_3t_6t_1 = x^3yt_0t_3t_6 \implies Nt_0t_3t_6t_1 = Nt_0t_3t_6 \in [036]$$

$$t_0t_3t_6t_2 = x^5yt_5t_0t_5 \implies Nt_0t_3t_6t_2 = Nt_5t_0t_5 \in [020]$$

$$t_0t_3t_6t_3 = (x^2)^{-1}t_0t_2t_6 \implies Nt_0t_3t_6t_3 = Nt_0t_2t_6 \in [026]$$

$$t_0t_3t_6t_4 = (x^2)^{-1}t_1t_5t_2 \implies Nt_0t_3t_6t_4 = Nt_1t_5t_2 \in [036]$$

$$t_0t_3t_6t_5 = x^4t_4t_0t_4 \implies Nt_0t_3t_6t_5 = Nt_4t_0t_4 \in [030]$$

$$Nt_0t_3t_6t_6 = Nt_0t_3 \in [03]$$

$$t_0t_3t_6t_0 = x^2t_2t_4t_2 \implies Nt_0t_3t_6t_0 = Nt_2t_4t_2 \in [020].$$

$\boldsymbol{Nt_0t_3t_0N}$

Now $Nt_0t_3t_0N$ is a new double coset. We determine how many single cosets are in the double coset. Now $N^{(030)} = N^{030} =< e >$. But $Nt_0t_3t_0$ is not distinct. Now $Nt_3t_0t_3 \in [030]$ since $(0,3)(1,2)(6,4) \in N$ and $N(t_0t_3t_0)^{(0,3)(1,2)(6,4)} = Nt_3t_0t_3$. Thus, $(0,3)(1,2)(6,4) \in N^{(030)}$. We conclude:

$$N^{(030)} \geq< (0,3)(1,2)(6,4) > .$$

Hence $|N^{(030)}| = 2$. So the number of single cosets in $N^{(030)}N$ is $\frac{|N|}{|N^{(030)}|} = \frac{14}{2} = 7$. The orbits of $N^{(030)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathbb{O} = \{\{0,3\}, \{1,2\}, \{6,4\}, \{5\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_3t_0t_i$ belongs to. We have:

$$Nt_0t_3t_0t_0 = Nt_0t_3 \in [03]$$

$$t_0t_3t_0t_1 = x^3t_3t_6t_2 \implies Nt_0t_3t_0t_1 = Nt_3t_6t_2 \in [036]$$

$$t_0t_3t_0t_6 = (x^2)^{-1}t_5t_2t_4 \implies Nt_0t_3t_0t_6 = Nt_5t_2t_4 \in [031]$$

$$t_0t_3t_0t_5 = yt)2t_6t_1 \implies Nt_0t_3t_0t_5 = Nt_2t_6t_1 \in [031].$$

We have completed the double coset enumeration since the right coset is closed under multiplication, hence, the index of N in G is 78 single cosets. We conclude:
$G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_2N \cup Nt_0t_3N \cup Nt_0t_1t_2N \cup Nt_0t_1t_3N \cup Nt_0t_1t_0N \cup Nt_0t_2t_5N \cup Nt_0t_3t_6N$, where

$$G = \frac{2^{*7} : D_{14}}{(xtt^x)^2, (ttxt)^9 = 1}$$

$$|G| \leq |N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(02)}} + \frac{|N|}{N^{(03)}} + \frac{|N|}{N^{(020)}} + \frac{|N|}{N^{(026)}} + \frac{|N|}{N^{(031)}} + \frac{|N|}{N^{(036)}}$$
$$+ \frac{|N|}{N^{(030)}} \times |N|$$

$$|G| \leq (1+7+7+14+14+7+7+14+7) \times 14$$
$$|G| \leq 78 \times 14$$
$$|G| \leq 1092.$$

The Cayley diagram that summarizes the above information is given below:

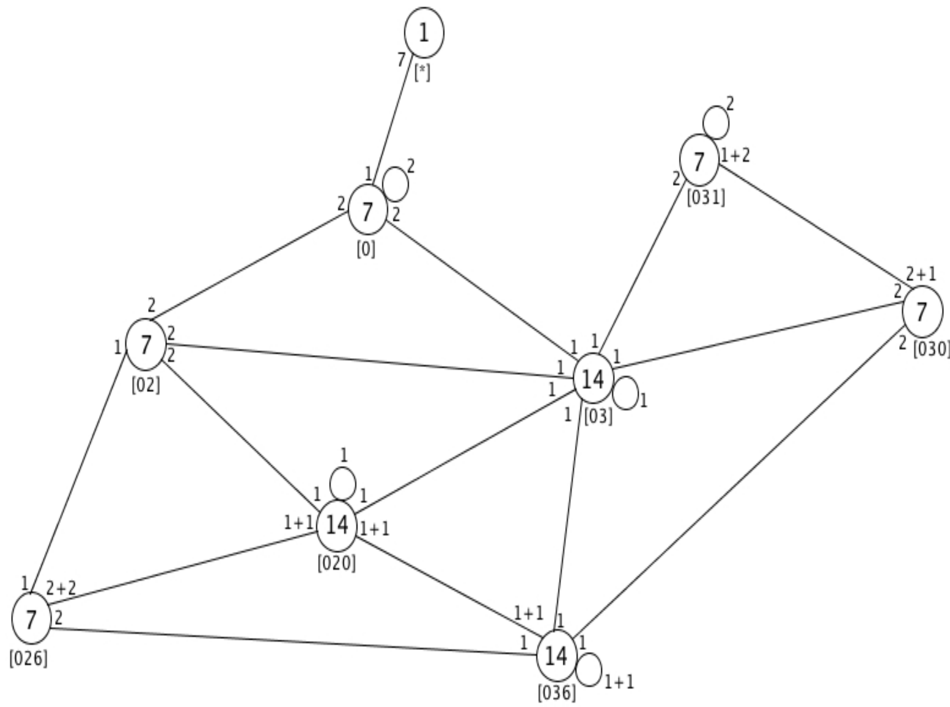

Figure 3.4: Cayley Diagram of $L_2(13)$ over D_{14}

3.6 Iwasawa's Lemma to Prove $L_2(13)$ over D_{14} is Simple

We use *Iwasawa's lemma* and the transitive action of G on the set of single cosets to prove $G \cong L_2(13)$ over D_{14} is a simple group. *Iwasawa's lemma* has three sufficient conditions that we must satisfied:

(1) G acts on X faithfully and primitively

(2) G is perfect ($G = G'$)

(3) There exist $x \in X$ and a normal abelian subgroup K of G^x such that the conjugates of K generate G.

Proof. **3.6.1** $G = L_2(13)$ **over** D_{14} **acts on** X **Faithfully**

Let G acts on $X = \{N, Nt_0N, Nt_0t_2N, Nt_0t_3N, Nt_0t_2t_0N, Nt_0t_2t_6N, Nt_0t_3t_1N, Nt_0t_3t_6N, Nt_0t_3t_0N\}$, where $X = 78$. G acts on X implies there exist homomorphism

$$f\ : G \longrightarrow S_{78} \qquad\qquad (|X| = 78).$$

By First Isomorphic Theorem we have:

$$G/kerf \cong f(G).$$

If $kerf = 1$ then $G \cong f(G)$. Only elements of N fix N implies $G^1 = N$. Since X is transitive $G - set$ of degree 78, we have:

$$\begin{aligned} |G| &= 78 \times |G^1| \\ &= 78 \times |N| \\ &= 78 \times 12 \\ &= 1092 \\ \implies |G| &= 1092. \end{aligned}$$

From Cayley diagram, $|G| \leq 1092$. However, from above $|G| = 1092$ implying $ker(f) = 1$. Since $kerf = 1$ then G acts faithfully on X.

3.6.2 $G = L_2(13)$ over D_{14} acts on X Primitively

Since $G = L_2(13)$ is transitive on $|X| = 78$, if B is a nontrivial block then we may assume that $N \in B$. However, $|B|$ must divide $|X| = 78$. The only nontrivial blocks must be of size 2,3,6,13,26, or 39. Note if $Bt_0 \in B$ then $B = X$. So B is a trivial block. By inspection, we can see from the Cayley diagram that we cannot create a nontrivial block of size 2, 3, 6, 13, 26, or 13. Thus, G acts primitively on X.

3.6.3 $G = L_2(13)$ over D_{14} is Perfect

.

Next we want to show that $G = G'$. Since $G =< N, t >$, we have that $N \leq G'$. Now $D_{14} \leq G \implies {D_{14}}' \leq G'$. The commutators subgroup of D_{14} is:

$$\begin{aligned} {D_{14}}' &=< (1,2,3,4,5,6,0) >=< x > \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'. \end{aligned}$$

Now by expanding the relation $[t_0 t_0 (1,2,3,4,5,6,0) t_0]^3 = 1$, we get:

$$\begin{aligned} (1,4,0,3,5,2,5)t_2 &= t_0 t_1 \\ \implies x^3 &= t_0 t_1 t_2 \\ \implies x^6 &= t_0 t_1 t_2 t_0 t_1 t_2 \\ \implies x^{-1} &= t_0 t_1 t_2 t_0 t_1 t_2 \\ \implies x &= t_2 t_1 t_0 t_2 t_1 t_0. \end{aligned}$$

Also by expanding the relation $[(1,5)(2,4)(6,0)t_1 t_0]^7 = 1$, we get:

$$\begin{aligned} (1,5)(2,4)(6,0) &= t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 \\ \implies xy &= t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 \end{aligned}$$

Now we use the above relation to solve for y. We multiply by $x^{-1} = t_0 t_1 t_2 t_0 t_1 t_2$.

$$\begin{aligned} xy &= t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 \\ x^{-1}xy &= t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 \\ \implies y &= t_0 t_1 t_2 t_0 t_1 t_2 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1 t_6 t_5 t_0 t_1. \end{aligned}$$

Now $D_{14} \leq G \implies {D_{14}}' \leq G'$. ${D_{14}}' =< (1,2,3,4,5,6,0) >=< x >$ $= \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'$. Note $G =< x, y, t >=< t_1, t_2, t_3, t_4, t_5, t_6, t_0 >$. Our goal is to show that one of the $t_i's \in G'$, then we can conjugate by $< x, y >$ to obtain all of the $t_i's$ in G'. Consider, the relation:

$$\begin{aligned} x^2 &= t_0 t_2 t_6 t_0 t_1 t_6 t_1 \\ &= t_0 t_2 t_6 t_0 t_6 t_6 t_1 t_6 t_1 \\ &= t_0 t_2 t_6 t_0 t_6 [6,1] \\ &= t_0 t_2 t_0 t_0 t_6 t_0 t_6 [6,1] \\ &= t_0 t_2 t_0 [0,6][6,1] \\ &= t_2 t_2 t_0 t_2 t_0 [0,6][6,1] \\ &= t_2 [2,0][0,6][6,1] \in G'. \end{aligned}$$

We see that $t_2 \in G'$. So $G' \geq < x, t_2 >=< t_1, t_2, t_3, t_4, t_5, t_6, t_0 >= G$. But $G \geq G'$. We conclude that $G = G'$ and G is perfect.

3.6.4 Conjugates of a Normal Abelian K Generate $G = L_2(13)$ over D_{14}

Now we require $x \in X$ and a normal abelian subgroup K of G^x, the point stabilizer of x in G, such that the conjugates of K in G generate G.
Now $G^1 = N = D_{14}$ possesses a normal abelian subgroup $K =< x >$. We have the relation

$$\begin{aligned} x^3 &= t_0t_1t_2 \in K \\ (x^3)^y &= (t_0t_1t_2)^y \in K^G \\ (x^3)^{-1} &= t_0t_6t_5 \in K. \end{aligned} \tag{3.1}$$

Now conjugate the relation $x^3 = t_0t_1t_2$ by t_0 yields:

$$\begin{aligned} (x^3)^{t_0} &= (t_0t_1t_2)^{t_0} \in K^G \\ t_0(x^3)t_0 &= t_0t_0t_1t_2t_0 \in K \\ x^3t_3t_0 &= t_1t_2t_0 \in k \\ x^3 &= t_1t_2t_3 \in K. \end{aligned} \tag{3.2}$$

Now conjugate the relation $x^3 = t_0t_1t_2$ by t_1 yields:

$$\begin{aligned} (x^3)^{t_1} &= (t_0t_1t_2)^{t_1} \in K^G \\ t_1(x^3)t_1 &= t_1t_0t_1t_2t_1 \in K \\ x^3t_4t_1 &= t_1t_0t_1t_2t_1 \in K \\ (x^3t_4t_1)^{t_1t_2t_1} &= (t_1t_0t_1t_2t_1)^{t_1t_2t_1} \in K^G \\ t_1t_2t_1(x^3t_4t_1)t_1t_2t_1 &= t_1t_2t_1(t_1t_0t_1t_2t_1)t_1t_2t_1 \in K \\ x^3t_4t_5t_2t_1 &= t_1t_2t_0 \in k. \end{aligned}$$

We have $t_1t_2t_0 \in K$ so the inverse is in K. Thus

$$t_0t_2t_1 \in K. \tag{3.3}$$

Multiplying (3.3) and (3.2) yields:

$$\begin{aligned} & t_0t_2t_1t_1t_2t_3 \in K \\ &= t_0t_3 \in K. \end{aligned} \tag{3.4}$$

Now we use the relation $t_0t_3t_0t_1 = x^3t_3t_6t_2$ that we obtained from the double coset enumeration.

$$t_0t_3t_0t_1 = x^3t_3t_6t_2$$
$$x^3 = t_0t_3t_0t_1t_2t_6t_3 \in K$$
$$x^3 = t_3t_0t_3t_1t_2t_6t_3 \in K \text{ (since } t_0t_3t_0 = t_3t_0t_3). \tag{3.5}$$

Multiplying (3.5) and (3.4) yields:

$$x^3t_0t_3 = t_3t_0t_3t_1t_2t_6t_3t_0t_3 \in K$$
$$(x^3t_0t_3)^{t_3t_0t_3} = (t_3t_0t_3t_1t_2t_6t_3t_0t_3)^{t_3t_0t_3} \in K^G$$
$$t_3t_0t_3(x^3t_0t_3)t_3t_0t_3 = t_3t_0t_3(t_3t_0t_3t_1t_2t_6t_3t_0t_3)t_3t_0t_3 \in K$$
$$x^3t_6t_3t_6t_3 = t_1t_2t_6 \in K. \tag{3.6}$$

Multiplying (3.6) and (3.3) yields:

$$t_1t_2t_6t_0t_2t_1 \in K$$
$$(t_1t_2t_6t_0t_2t_1)^{t_1t_2} \in K^G$$
$$t_2t_1(t_1t_2t_6t_0t_2t_1)t_1t_2 \in K$$
$$t_6t_0 \in k. \tag{3.7}$$

Multiplying (3.1) and (3.7) yields:

$$t_0t_6t_5t_6t_0 \in K$$
$$(t_0t_6t_5t_6t_0)^{t_0t_6} \in K^G$$
$$t_6t_0(t_0t_6t_5t_6t_0)t_0t_6 \in K$$
$$t_5 \in K.$$

Thus $t_5 \in K$

$\implies t_5^G \in K^G$

$\implies K^G \leq \{t_5, t_5^x, t_5^{x^2}, t_5^{x^3}, t_5^{x^4}, t_5^{x^5}, t_5^{x^6}\}$

$\implies K^G \leq \{t_5, t_5^x, t_5^{x^2}, t_5^{x^3}, t_5^{x^4}, t_5^{x^5}, t_5^{x^6}\} =< t_5, t_6, t_0, t_1, t_2, t_3, t_4 >= G$

So $G = K^G$.

Hence, the conjugates of K generate G. Therefore, by *Iwasawa's lemma*, $G \cong L_2(13)$ is simple.

$\square$

Chapter 4

Double Coset Enumeration over a Maximal Subgroup

In this chapter, we will construct a double coset enumeration over a maximal subgroup and apply *Iwasawa's lemma* to prove $G \cong L_2(27)$ is a simple group.

4.1 Construction of $L_2(27)$ over $M = 2^{\cdot}D_{14}$

Definition 4.1. ***(Maximal Subgroup).*** *A subgroup $M \neq 1 \leq G$ is a **maximal normal subgroup** of G if there is no normal subgroup N of G with $M < N < G$. [Rot12]*

We start by factoring the progenitor $2^{*7} : D_{14}$ by the relations $(xyt^x t)^3, (xt)^7$ to obtain the homomorphic image:

$$G \cong \frac{2^{*7} : D_{14}}{(xyt^x t)^3, (xt)^7} \cong L_2(27),$$

where $x \sim (0,1,2,3,4,5,6)$, $y \sim (1,6)(2,5)(3,4)$, and $t \sim t_0 \sim t_7$.
Let $\pi = xy = (1,5)(2,4)(6,0)$, then $(xyt^x t)^3 = 1$ can be written as
$1 = (\pi t_1 t_0)^3$, which yields the following calculation:

$$\begin{aligned} 1 &= (\pi t_1 t_0)^3 \\ &= \pi^3 (t_1 t_0)^{\pi^2} (t_1 t_0)^{\pi} t_1 t_0 \\ &= \pi t_1 t_0 t_5 t_6 t_1 t_0. \end{aligned}$$

Thus we have the relation:

$$\pi t_1 t_0 t_5 = t_0 t_1 t_6.$$

Now $(xt)^7 = 1$ can be written as $1 = (xt_0)^7 = x^7 t_0^{x^6} t_0^{x^5} t_0^{x^4} t_0^{x^3} t_0^{x^2} t_0^{x} t_0$. Then

$$t_6 t_5 t_4 = t_0 t_1 t_2 t_3.$$

Let M be the maximal subgroup generated by the control group $N = D_{14}$ and $t_2 t_4 t_5 t_4 t_2 = t^{x^2} t^{x^4} t^{x^5} t^{x^4} t^{x^2}$. That is,

$$M = \left\langle N, t^{x^2} t^{x^4} t^{x^5} t^{x^4} t^{x^2} \right\rangle, = 2^{\cdot} D_{14} \text{ where } |M| = 28.$$

Then M is the maximal subgroup.

We proceed to do a manual double coset enumeration of G over M and N. Denote $[w]$ to be the double coset MwN, where w is a word in the $t_i's$.

MeN

We begin with the double coset MeN, denote $[*]$. This double coset contains only one single coset, namely M. The single coset stabilizer of M is N, which is transitive on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ and therefore, has a single orbit,

$$\mathcal{O} = \{\{0, 1, 2, 3, 4, 5, 6\}\}.$$

Take an element from $\mathcal{O}$ say t_0 and multiply the single coset representative M by it to obtain Mt_0. This is a new double coset Mt_0N, denote it [0].

$\boldsymbol{Mt_0N}$

Continuing with the double coset Mt_0N, we find the point stabilizer N^0. This is

$$N^0 = \left\langle (1, 6)(2, 5)(3, 4) \right\rangle.$$

The coset stabiliser:

$$N^{(0)} \geq \left\langle (1, 6)(2, 5)(3, 4) \right\rangle.$$

Since $\left|N^{(0)}\right| = 2$, the number of single cosets in [0] is $\frac{\left|N\right|}{\left|N^{(0)}\right|} = \frac{14}{2} = 7$. The orbits of $N^{(0)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}.$$

Take an element from each orbit and multiply on the right by the single coset represen-

tative Mt_0 of the double coset Mt_0N. We have:

$$Mt_0t_0 = M \in [*],$$

$$Mt_0t_1 \in [01],$$

$$Mt_0t_2 \in [02],$$

$$Mt_0t_3 \in [03].$$

The new double cosets have single coset representatives Mt_0t_1, Mt_0t_2, and Mt_0t_3, we represent them as [01], [02], and [03], respectively.

Mt_0t_1N

Continuing with the double coset Mt_0t_1N, we find the coset stabilizer $N^{(01)} = N^{01} = \langle e \rangle$. Only e will fix 0 and 1. Hence the number of single cosets in [01] is $\frac{|N|}{|N^{(01)}|} = \frac{14}{1} = 14$. The orbits of $N^{(01)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative Mt_0t_1 of the double coset Mt_0t_1N. We have:

$$\begin{aligned}
& Mt_0t_1t_1 = Mt_0 \in [0], \\
& Mt_0t_1t_0 \in [010], \\
& Mt_0t_1t_2 \in [012], \\
& Mt_0t_1t_3 \in [013], \\
& Mt_0t_1t_4 \in [014], \\
& Mt_0t_1t_5 \in [015], \\
& \quad t_0t_1t_6 = xyt_1t_0t_5 \\
\implies & Mt_0t_1t_6 = Mt_1t_0t_5 \in [013] = \{N(t_0t_1t_3)^n | n \in N\}.
\end{aligned}$$

The new double cosets have single coset representatives $Mt_0t_1t_0$, $Mt_0t_1t_2$, $Mt_0t_1t_3$,$Mt_0t_1t_4$, and $Mt_0t_1t_5$, we represent them as [010], [012], [013], [014], and [015], respectively.

Mt_0t_2N

Continuing with the double coset Mt_0t_2N, we find the coset stabilizer $N^{(02)} = N^{02} =$

$\langle e\rangle$. Only e will fix 0 and 2. Hence the number of single cosets in [02] is $\frac{|N|}{|N^{(02)}|} = \frac{14}{1} =$ 14. The orbits of $N^{(02)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative Mt_0t_2 of the double coset Mt_0t_2N. We have:

$Mt_0t_2t_2 = Mt_0 \in [0]$,

$t_0t_2t_1 = x^2t_2t_0t_1t_2t_0t_2t_0 \implies Mt_0t_2t_1 = Mt_2t_0 \in [02] = \{N(t_0t_2)^n | n \in N\}$,

$t_0t_2t_3 = t_0t_2t_3t_2t_0t_0t_2 \implies Mt_0t_2t_3 = Mt_0t_2 \in [02]$,

$Mt_0t_2t_4 \in [024]$,

$Mt_0t_2t_5 \in [025]$,

$Mt_0t_2t_6 \in [026]$,

$Mt_0t_2t_0 \in [020]$.

The new double cosets have single coset representatives $Mt_0t_2t_4$, $Mt_0t_2t_5$, $Mt_0t_2t_6$, and $Mt_0t_2t_0$, we represent them as [024], [025], [026], and [020], respectively.

$\boldsymbol{Mt_0t_3N}$

Continuing with the double coset Mt_0t_3N, we find the coset stabilizer $N^{(03)} = N^{03} = \langle e\rangle$. Only e will fix 0 and 3. Hence the number of single cosets in [03] is $\frac{|N|}{|N^{(03)}|} = \frac{14}{1} =$ 14. The orbits of $N^{(03)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative Mt_0t_3 of the double coset Mt_0t_3N. We have:

$Mt_0t_3t_3 = Mt_0 \in [0]$,

$t_0t_3t_1 = t_0t_2t_3t_2t_0t_6t_0t_6 \implies Mt_0t_3t_1 = Mt_6t_0t_6 \in [010] = \{N(t_0t_1t_0)^n | n \in N\}$,

$Mt_0t_3t_2 \in [032]$,

$Mt_0t_3t_4 \in [034]$,

$t_0t_3t_5 = t_3t_5t_1 \implies Mt_0t_3t_5 = Mt_3t_5t_1 \in [025] = \{N(t_0t_2t_5)^n | n \in N\}$,

$$t_0t_3t_6 = yx^{-2}t_5t_0t_6t_5t_0t_6t_5t_3 \implies Mt_0t_3t_6 = Mt_6t_5t_3 \in [013]$$
$$\text{(since } \{N(t_0t_1t_3)^n|n \in N\} \text{ and } yx^{-2}t_5t_0t_6t_5t_0 \in M),$$
$$Mt_0t_3t_0 \in [030].$$

The new double cosets have single coset representatives $Mt_0t_3t_2$, $Mt_0t_3t_4$, and $Mt_0t_3t_0$, we represent them as [032], [034], and [030], respectively.

$\boldsymbol{Mt_0t_1t_0N}$

Continuing with the double coset $Mt_0t_1t_0N$, we find the coset stabilizer $N^{(010)} = N^{010} = \langle e \rangle$. Only e will fix 0 and 1. Hence the number of single cosets in [010] is $\frac{|N|}{|N^{(010)}|} = \frac{14}{1} = 14$. The orbits of $N^{(010)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_0$ of the double coset $Mt_0t_1t_0N$. We have:

$$Mt_0t_1t_0t_0 = Mt_0t_1 \in [01],$$
$$t_0t_1t_0t_1 = yxt_4t_2t_4 \implies Mt_0t_1t_0t_1 = Mt_4t_2t_4 \in [020] = \{(Nt_0t_2t_0)^n|n \in N\},$$
$$t_0t_1t_0t_2 = xt_0t_5t_4t_5t_0t_1t_4 \implies Mt_0t_1t_0t_2 = Mt_1t_4 \in [03]$$
$$\text{(since } \{N(t_0t_3)^n|n \in N\} \text{ and } xt_0t_5t_4t_5t_0 \in M),$$
$$t_0t_1t_0t_3 = x^2t_6t_3t_4t_5t_2t_4t_6 \implies Mt_0t_1t_0t_3 = Mt_6t_3t_4t_5t_2t_4t_6 \in [0321420]$$
$$\text{(since } \{N(t_0t_3t_2t_1t_4t_2t_0)^n|n \in N\}),$$
$$t_0t_1t_0t_4 = yxt_0t_5t_4t_5t_0t_1t_0t_1 \implies Mt_0t_1t_0t_4 = Mt_1t_0t_1 \in [010]$$
$$\text{(since } \{N(t_0t_1t_0)^n|n \in N\} \text{ and } yxt_0t_5t_4t_5 \in M),$$
$$t_0t_1t_0t_5 = xt_3t_0t_3t_1t_0 \implies Mt_0t_1t_0t_5 = Mt_3t_0t_3t_1t_0 \in [03023]$$
$$= \{N(t_0t_3t_0t_2t_3)^n|n \in N\},$$
$$t_0t_1t_0t_6 = x^2t_1t_4t_5t_1t_0 \implies Mt_0t_1t_0t_6 = Mt_1t_4t_5t_1t_0 \in [03406]$$
$$= \{N(t_0t_3t_4t_0t_6)^n|n \in N\}.$$

$\boldsymbol{Mt_0t_1t_2N}$

Continuing with the double coset $Mt_0t_1t_2N$, we find the coset stabilizer $N^{(012)} = N^{012} =$

$\langle e \rangle$. Only e will fix 0, 1, and 2. Hence the number of single cosets in [012] is $\frac{|N|}{|N^{(012)}|} = \frac{14}{1} = 14$. The orbits of $N^{(012)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_2$ of the double coset $Mt_0t_1t_2N$. We have:

$Mt_0t_1t_2t_2 = Mt_0t_1 \in [01]$,

$t_0t_1t_2t_1 = xt_0t_5t_4t_5t_0t_6t_2t_1t_0t_3t_1t_6t_5$

$\implies Mt_0t_1t_2t_1 = Mt_6t_2t_1t_0t_3t_1t_6t_5 \in [03214206]$

(since $\{N(t_0t_3t_2t_1t_4t_2t_0t_6)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$t_0t_1t_2t_3 = t_6t_5t_4 \implies Mt_0t_1t_2t_3 = Mt_6t_5t_4 \in [012] = \{N(t_0t_1t_2)^n | n \in N\}$,

$t_0t_1t_2t_4 = xt_0t_5t_4t_5t_0t_3t_6t_0t_3 \implies Mt_0t_1t_2t_4 = Mt_3t_6t_0t_3 \in [0340]$

(since $\{N(t_0t_3t_4t_0)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$t_0t_1t_2t_5 = xt_1t_3t_4t_3t_1t_0t_5t_0 \implies Mt_0t_1t_2t_5 = Mt_0t_5t_0 \in [020]$

(since $\{N(t_0t_2t_0)^n | n \in N\}$ and $xt_1t_3t_4t_3t_1 \in M$),

$t_0t_1t_2t_6 = xt_0t_2t_3t_2t_0t_4t_3t_0 \implies Mt_0t_1t_2t_6 = Mt_4t_3t_0 \in [014]$

(since $\{N(t_0t_1t_4)^n | n \in N\}$ and $xt_0t_2t_3t_2t_0 \in M$),

$t_0t_1t_2t_0 = yx^2t_4t_1t_4t_2t_1 \implies Mt_0t_1t_2t_0 = Mt_4t_1t_4t_2t_1 \in [03023]$

$= \{N(t_0t_3t_0t_2t_3)^n | n \in N\}$.

$\boldsymbol{Mt_0t_1t_3N}$

Continuing with the double coset $Mt_0t_1t_3N$, we find the coset stabiliser $N^{(013)} = N^{013} = \langle e \rangle$. Only e will fix 0, 1, and 3. Hence the number of single cosets in [013] is $\frac{|N|}{|N^{(013)}|} = \frac{14}{1} = 14$. The orbits of $N^{(013)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_3$ of the double coset $Mt_0t_1t_3N$. We have:

$Mt_0t_1t_3t_3 = Mt_0t_1 \in [01]$,

$t_0t_1t_3t_1 = xt_6t_2t_6t_1t_0 \implies Mt_0t_1t_3t_1 = Mt_6t_2t_6t_1t_0 \in [03021]$

$= \{N(t_0t_3t_0t_2t_1)^n | n \in N\}$,

$t_0t_1t_3t_2 = yx^3t_1t_0 \implies Mt_0t_1t_3t_2 = Mt_1t_0 \in [01] = \{N(t_0t_1)^n | n \in N\}$,

$t_0t_1t_3t_4 = xt_0t_2t_3t_2t_0t_0t_4t_3t_6t_0 \implies Mt_0t_1t_3t_4 = Mt_0t_4t_3t_6t_0 \in [03410]$

(since $\{N(t_0t_3t_4t_1t_0)^n | n \in N\}$ and $xt_0t_2t_3t_2t_0 \in M$),

$t_0t_1t_3t_5 = t_2t_6t_2t_0t_1t_3 \implies Mt_0t_1t_3t_5 = Mt_2t_6t_2t_0t_1t_3 \in [030216]$

$= \{N(t_0t_3t_0t_2t_1t_6)^n | n \in N\}$,

$t_0t_1t_3t_6 = x^3t_4t_1t_2t_1 \implies Mt_0t_1t_3t_6 = Mt_4t_1t_2t_1 \in [0323]$

$= \{N(t_0t_3t_2t_3)^n | n \in N\}$,

$t_0t_1t_3t_0 = yx^{-2}t_5t_7t_6t_5t_6t_3 \implies Mt_0t_1t_3t_0 = Mt_6t_3 \in [03]$

(since $\{N(t_0t_3)^n | n \in N\}$ and $yx^{-2}t_5t_7t_6t_5 \in M$).

$\boldsymbol{Mt_0t_1t_4N}$

Continuing with the double coset $Mt_0t_1t_4N$, we find the coset stabiliser $N^{(014)} = N^{014} = \langle e \rangle$. Only e will fix 0, 1, and 4. Hence the number of single cosets in [014] is $\frac{|N|}{|N^{(014)}|} = \frac{14}{1} = 14$. The orbits of $N^{(014)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_4$ of the double coset $Mt_0t_1t_4N$. We have:

$Mt_0t_1t_4t_4 = Mt_0t_1 \in [01]$,

$t_0t_1t_4t_1 = x^{-1}t_5t_0t_6t_5t_0t_2t_6t_0t_1t_5 \implies Mt_0t_1t_4t_1 = Mt_2t_6t_0t_1t_5 \in [03214]$,

(since $\{N(t_0t_3t_2t_1t_4)^n | n \in N\}$ and $x^{-1}t_5t_0t_6t_5t_0 \in M$)

$t_0t_1t_4t_2 = yxt_0t_1t_4 \implies Mt_0t_1t_4t_2 = Mt_0t_1t_4 \in [014]$

$t_0t_1t_4t_3 = t_0t_5t_4t_5t_0t_0t_6t_2t_5 \implies Mt_0t_1t_4t_3 = Mt_0t_6t_2t_5 \in [0152]$,

(since $\{N(t_0t_1t_5t_2)^n | n \in N\}$ and $t_0t_5t_4t_5t_0 \in M$),

$t_0t_1t_4t_5 = x^2t_0t_2t_3t_2t_0t_4t_3t_2 \implies Mt_0t_1t_4t_5 = Mt_4t_3t_2 \in [012]$

(since $\{N(t_0t_1t_2)^n|n \in N\}$ and $x^2t_0t_2t_3t_2t_0 \in M$),

$t_0t_1t_4t_6 = xt_1t_3t_4t_3t_1t_0t_4t_3t_0t_1 \implies Mt_0t_1t_4t_6 = Mt_0t_4t_3t_0t_1 \in [03406]$

(since $\{N(t_0t_3t_4t_0t_6)^n|n \in N\}$ and $xt_1t_3t_4t_3t_1 \in M$),

$t_0t_1t_4t_0 = x^2yt_3t_0t_6t_2 \implies Mt_0t_1t_4t_0 = Mt_3t_0t_6t_2 \in [0341]$

$= \{N(t_0t_3t_4t_1)^n|n \in N\}$.

$\boldsymbol{Mt_0t_1t_5N}$

Continuing with the double coset $Mt_0t_1t_5N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_1t_5 = yx^{-2}t_5t_0t_6t_5t_0t_6t_5t_1$$
$$\implies Mt_0t_1t_5 = Mt_6t_5t_1 \text{ since } yx^{-2}t_5t_0t_6t_5t_0 \in M$$

Then $M(t_0t_1t_5)^{(0,6)(1,5)(2,4)} = Mt_6t_5t_1$. But $Mt_6t_5t_1 = Mt_0t_1t_5 \implies (0,6)(1,5)(2,4) \in N^{(015)}$ since $M(t_0t_1t_5)^{(0,6)(1,5)(2,4)} = Mt_6t_5t_1$

$$\implies N^{(015)} \geq \langle(0,6)(1,5)(2,4)\rangle.$$

Since $\left|N^{(015)}\right| = 2$, the number of single cosets in [015] is $\frac{|N|}{|N^{(015)}|} = \frac{14}{2} = 7$. The orbits of $N^{(015)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{3\}, \{0,6\}, \{1,5\}, \{2,4\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_5$ of the double coset $Mt_0t_1t_5N$. We have:

$Mt_0t_1t_5t_5 = Mt_0t_1 \in [01]$,

$t_0t_1t_5t_3 = x^3t_6t_5t_1 \implies Mt_0t_1t_5t_3 = Mt_6t_5t_1 \in [015] = \{N(t_0t_5)^n|n \in N\}$,

$t_0t_1t_5t_0 = xt_1t_3t_4t_3t_1t_0t_5t_3 \implies Mt_0t_1t_5t_0 = Mt_0t_5t_3 \in [024]$

(since $\{N(t_0t_2t_4)^n|n \in N\}$ and $xt_1t_3t_4t_3t_1 \in M$),

$Mt_0t_1t_5t_2 \in [0152]$.

The new double coset is $Mt_0t_1t_5t_2N$, which we represent by [0152], respectively.

$\boldsymbol{Mt_0t_1t_5t_2N}$

Now with the double coset $Mt_0t_1t_5t_2N$, we find the coset stabilizer $N^{(0152)} = N^{0152} =$

$\langle e \rangle$. Only e will fix 0, 1, 5, and 2. Hence the number of single cosets in [0152] is $\frac{|N|}{|N^{(0152)}|} = \frac{14}{1} = 14$. The orbits of $N^{(0152)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1t_5t_2$ of the double coset $Mt_0t_1t_5t_2N$. We have:

$Mt_0t_1t_5t_2t_2 = Mt_0t_1t_5 \in [015]$,

$t_0t_1t_5t_2t_1 = xt_0t_2t_3t_2t_0t_0t_4t_5t_4 \implies Mt_0t_1t_5t_2t_1 = Mt_0t_4t_5t_4 \in [0323]$

(since $\{N(t_0t_3t_2t_3)^n | n \in N\}$ and $xt_0t_2t_3t_2t_0 \in M$),

$t_0t_1t_5t_2t_3 = x^2t_2t_0t_1t_2t_0t_2t_4t_1 \implies Mt_0t_1t_5t_2t_3 = Mt_2t_4t_1 \in [026]$

(since $\{N(t_0t_2t_6)^n | n \in N\}$ and $x^2t_2t_0t_1t_2t_0 \in M$),

$t_0t_1t_5t_2t_4 = t_0t_2t_3t_2t_0t_0t_6t_3 \implies Mt_0t_1t_5t_2t_4 = Mt_0t_6t_3 \in [014]$

(since $\{N(t_0t_1t_4)^n | n \in N\}$ and $t_0t_2t_3t_2t_0 \in M$),

$t_0t_1t_5t_2t_5 = x^2t_6t_3t_4t_5t_2t_4t_6t_0 \implies Mt_0t_1t_5t_2t_5 = Mt_6t_3t_4t_5t_2t_4t_6t_0 \in [03214206]$

$= \{N(t_0t_3t_2t_1t_4t_2t_0t_6)^n | n \in N\}$,

$t_0t_1t_5t_2t_6 = xt_0t_5t_4t_5t_0t_0t_1t_2t_5 \implies Mt_0t_1t_5t_2t_6 = Mt_0t_1t_2t_5 \in [0152]$

(since $\{N(t_0t_1t_5t_2)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$t_0t_1t_5t_2t_0 = t_0t_2t_3t_2t_0t_5t_1t_2t_6t_5 \implies Mt_0t_1t_5t_2t_0 = Mt_5t_1t_2t_6t_5 \in [03410]$

(since $\{N(t_0t_3t_4t_1t_0)^n | n \in N\}$ and $t_0t_2t_3t_2 \in M$).

$\boldsymbol{Mt_0t_2t_4N}$

Continuing with the double coset $Mt_0t_2t_4N$, we find the coset stabilizer $N^{(024)} = N^{024} = \langle e \rangle$. Only e will fix 0, 2, and 4. Hence the number of single cosets in [024] is $\frac{|N|}{|N^{(024)}|} = \frac{14}{1} = 14$. The orbits of $N^{(024)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_4$ of the double coset $Mt_0t_2t_4N$. We have:

$$Mt_0t_2t_4t_4 = Mt_0t_2 \in [02],$$

$t_0t_2t_4t_1 = x^{-2}t_3t_6t_3t_4 \implies Mt_0t_2t_4t_1 = Mt_3t_6t_3t_4 \in [0301]$

$= \{N(t_0t_3t_0t_1)^n | n \in N\}$,

$t_0t_2t_4t_2 = xt_6t_4t_6 \implies Mt_0t_2t_4t_2 = Mt_6t_4t_6 \in [020]$

$= \{N(t_0t_2t_0)^n | n \in N\}$,

$t_0t_2t_4t_3 = yxt_0t_5t_4t_5t_0t_5t_1t_2t_6t_5 \implies Mt_0t_2t_4t_3 = Mt_5t_1t_2t_6t_5 \in [03410]$

(since $\{N(t_0t_3t_4t_1t_0)^n | n \in N\}$ and $yxt_0t_5t_4t_5t_0 \in M$),

$t_0t_2t_4t_5 = xt_0t_5t_4t_5t_0t_1t_4t_5t_1t_0 \implies Mt_0t_2t_4t_5 = Mt_1t_4t_5t_1t_0 \in [03406]$

(since $\{N(t_0t_3t_4t_0t_6)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$t_0t_2t_4t_6 = xt_6t_1t_4 \implies Mt_0t_2t_4t_6 = Mt_6t_1t_4 \in [025]$

$= \{N(t_0t_2t_5)^n | n \in N\}$,

$t_0t_2t_4t_0 = xt_0t_5t_4t_5t_0t_0t_6t_2 \implies Mt_0t_2t_4t_0 = Mt_0t_6t_2 \in [015]$

(since $\{N(t_0t_1t_5)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$).

$\boldsymbol{Mt_0t_2t_5N}$

Continuing with the double coset $Mt_0t_2t_5N$, we find the coset stabilizer $N^{(025)} = N^{025} = \langle e \rangle$. Only e will fix 0, 2, and 5. Hence the number of single cosets in [025] is $\frac{|N|}{|N^{(025)}|} = \frac{14}{1} = 14$. The orbits of $N^{(025)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_5$ of the double coset $Mt_0t_2t_5N$. We have:

$Mt_0t_2t_5t_5 = Mt_0t_2 \in [02]$,

$t_0t_2t_5t_1 = x^4t_2t_0t_1t_2t_0t_3t_0t_6t_3t_4 \implies Mt_0t_2t_5t_1 = Mt_3t_0t_6t_3t_4 \in [03406]$

(since $\{N(t_0t_3t_4t_0t_6)^n | n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

$t_0t_2t_5t_2 = t_4t_0 \implies Mt_0t_2t_5t_2 = Mt_4t_0 \in [03]$

$= \{N(t_0t_3)^n | n \in N\}$,

$t_0t_2t_5t_3 = xt_6t_2t_1t_0t_3t_1t_6 \implies Mt_0t_2t_5t_3 = Mt_6t_2t_1t_0t_3t_1t_6 \in [0321420]$,

$= \{N(t_0t_3t_2t_1t_4t_2t_0)^n | n \in N\}$,

$t_0t_2t_5t_4 = yt_0t_5t_4t_5t_0t_0t_3t_2 \implies Mt_0t_2t_5t_4 = Mt_0t_3t_2 \in [032]$,

(since $\{N(t_0t_3t_2)^n|n \in N\}$ and $yt_0t_5t_4t_5t_0 \in M$),

$t_0t_2t_5t_6 = x^2t_3t_6t_3t_5t_4t_2 \implies Mt_0t_2t_5t_6 = Mt_3t_6t_3t_5t_4t_2 \in [030216]$

$= \{N(t_0t_3t_0t_2t_1t_6)^n|n| \in N\}$,

$t_0t_2t_5t_0 = x^{-1}t_1t_3t_5 \implies Mt_0t_2t_5t_0 = Mt_1t_3t_5 \in [024]$

$= \{N(t_0t_2t_4)^n|n \in N\}$.

$\boldsymbol{Mt_0t_2t_6N}$

Continuing with the double coset $Mt_0t_2t_6N$, we find the coset stabilizer $N^{(026)} = N^{026} = \langle e \rangle$. Only e will fix 0, 2, and 6. Hence the number of single cosets in [026] is $\frac{|N|}{|N^{(026)}|} = \frac{14}{1} = 14$. The orbits of $N^{(026)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_6$ of the double coset $Mt_0t_2t_6N$. We have:

$Mt_0t_2t_6t_6 = Mt_0t_2 \in [02]$,

$t_0t_2t_6t_1 = x^{-2}t_5t_0t_6t_5t_0t_5t_6t_3t_0 \implies Mt_0t_2t_6t_1 = Mt_5t_6t_3t_0 \in [0152]$

(since $\{N(t_0t_1t_5t_2)^n|n \in N\}$ and $x^{-2}t_5t_0t_6t_5t_0 \in M$),

$t_0t_2t_6t_2 = t_2t_5t_2t_4 \implies Mt_0t_2t_6t_2 = Mt_2t_5t_2t_4 \in [0302]$

$= \{N(t_0t_3t_0t_2)^n|n \in N\}$,

$t_0t_2t_6t_3 = x^2yt_6t_3t_6t_5 \implies Mt_0t_2t_6t_3 = Mt_6t_3t_6t_5 \in [0301]$

$= \{N(t_0t_3t_0t_1)^n|n \in N\}$,

$t_0t_2t_6t_4 = x^{-3}t_3t_6t_5t_6 \implies Mt_0t_2t_6t_4 = Mt_3t_6t_5t_6 \in [0323]$

$= \{N(t_0t_3t_2t_3)^n|n \in N\}$,

$t_0t_2t_6t_5 = t_0t_5t_4t_5t_0t_4t_1t_0 \implies Mt_0t_2t_6t_5 = Mt_4t_1t_0 \in [034]$

(since $\{N(t_0t_3t_4)^n|n \in N\}$ and $t_0t_5t_4t_5t_0 \in M$),

$$t_0t_2t_6t_0 = yx^{-2}t_5t_0t_6t_5t_0t_3t_0t_6t_3 \implies Mt_0t_2t_6t_0 = Mt_3t_0t_6t_3 \in [0340]$$
(since $\{N(t_0t_3t_4t_0)^n|n \in N\}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$).

$Mt_0t_2t_0N$

Continuing with the double coset $Mt_0t_2t_0N$, we find the coset stabilizer $N^{(020)} = N^{020} = \langle e \rangle$. Only e will fix 0, and 2. Hence the number of single cosets in [020] is $\frac{|N|}{|N^{(020)}|} = \frac{14}{1} = 14$. The orbits of $N^{(020)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_0$ of the double coset $Mt_0t_2t_0N$. We have:

$$Mt_0t_2t_0t_0 = Mt_0t_2 \in [02],$$
$$t_0t_2t_0t_1 = yx^{-2}t_5t_0t_6t_5t_0t_0t_2t_0 \implies Mt_0t_2t_0t_1 = Mt_0t_2t_0 \in [020]$$
(since $\{N(t_0t_2t_0)^n|n \in N\}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$),
$$t_0t_2t_0t_2 = xt_0t_5t_4t_5t_0t_0t_6t_5 \implies Mt_0t_2t_0t_2 = Mt_0t_6t_5 \in [012]$$
(since $\{N(t_0t_1t_2)^n|n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),
$$t_0t_2t_0t_3 = yt_4t_3t_4 \implies Mt_0t_2t_0t_3 = Mt_4t_3t_4 \in [010]$$
$$= \{N(t_0t_1t_0)^n|n \in N\},$$
$$t_0t_2t_0t_4 = xt_6t_4t_2 \implies Mt_0t_2t_0t_4 = Mt_6t_4t_2 \in [024]$$
$$= \{N(t_0t_2t_4)^n|n \in N\},$$
$$t_0t_2t_0t_5 = x^{-1}t_3t_1t_3 \implies Mt_0t_2t_0t_5 = Mt_3t_1t_3 \in [020]$$
$$= \{N(t_0t_2t_0)^n|n \in N\},$$
$$t_0t_2t_0t_6 = x^3t_6t_2t_6t_1t_0 \implies Mt_0t_2t_0t_6 = Mt_6t_2t_6t_1t_0 \in [03021]$$
$$= \{N(t_0t_3t_0t_2t_1)^n|n \in N\}.$$

$Mt_0t_3t_2N$

Continuing with the double coset $Mt_0t_3t_2N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2 = x^{-2}t_5t_0t_6t_5t_0t_2t_6t_0$$
$$\implies Mt_0t_3t_2 = Mt_2t_6t_0 \text{ since } x^{-2}t_5t_0t_6t_5t_0 \in M$$

Then $M(t_0t_3t_2)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_0$. But $Mt_2t_6t_0 = Mt_0t_3t_2 \implies (0,2)(3,6)(4,5) \in N^{(032)}$ since $M(t_0t_3t_2)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_0$

$$\implies N^{(032)} \geq \big\langle (0,2)(3,6)(4,5) \big\rangle.$$

Since $\left|N^{(032)}\right| = 2$, the number of single cosets in [032] is $\frac{|N|}{|N^{(032)}|} = \frac{14}{2} = 7$. The orbits of $N^{(032)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{1\}, \{0,2\}, \{3,6\}, \{4,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2$ of the double coset $Mt_0t_3t_2N$. We have:

$Mt_0t_3t_2t_2 = Mt_0t_3 \in [03]$,

$Mt_0t_3t_2t_1 \in [0321]$,

$Mt_0t_3t_2t_3 \in [0323]$,

$t_0t_3t_2t_4 = x^3t_2t_0t_1t_2t_0t_0t_2t_5 \implies Mt_0t_3t_2t_4 = Mt_0t_2t_5 \in [025]$

(since $\{N(t_0t_2t_5)^n | n \in N\}$ and $x^3t_2t_0t_1t_2t_0 \in M$).

The new double cosets have single coset representatives $Mt_0t_3t_2t_1N$ and $Mt_0t_3t_2t_3N$, we represent them as [0321] and [0323], respectively.

$Mt_0t_3t_2t_1N$

Continuing with the double coset $Mt_0t_3t_2t_1N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2t_1 = x^{-2}t_5t_0t_6t_5t_0t_2t_6t_0t_1$$
$$\implies Mt_0t_3t_2t_1 = Mt_2t_6t_0t_1 \text{ since } x^{-2}t_5t_0t_6t_5t_0 \in M.$$

Then $M(t_0t_3t_2t_1)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_0t_1$.

But $Mt_2t_6t_0t_1 = Mt_0t_3t_2t_1 \implies (0,2)(3,6)(4,5) \in N^{(0321)}$

since $M(t_0t_3t_2t_1)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_0t_1$

$$\implies N^{(0321)} \geq \big\langle (0,2)(3,6)(4,5) \big\rangle.$$

Since $\left|N^{(0321)}\right| = 2$, the number of single cosets in [0321] is $\frac{|N|}{|N^{(0321)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0321)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{1\}, \{0,2\}, \{3,6\}, \{4,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset represen-

tative $Mt_0t_3t_2t_1$ of the double coset $Mt_0t_3t_2t_1N$. We have:

$Mt_0t_3t_2t_1t_1 = Mt_0t_3t_2 \in [032]$,

$t_0t_3t_2t_1t_3 = x^3yt_1t_4t_1t_2 \implies Mt_0t_3t_2t_1t_3 = Mt_1t_4t_1t_2 \in [0301]$

$= \{N(t_0t_3t_0t_1)^n | n \in N\}$,

$Mt_0t_3t_2t_1t_4 \in [03214]$,

$t_0t_3t_2t_1t_0 = t_0t_4t_5t_6 \implies Mt_0t_3t_2t_1t_0 = Mt_0t_4t_5t_6 \in [0321]$

$= \{N(t_0t_3t_2t_1)^n | n \in N\}$,

The new double coset is $Mt_0t_3t_2t_1t_4N$, which we represent by [03214], respectively.

$\boldsymbol{Mt_0t_3t_2t_3N}$

Continuing with the double coset $Mt_0t_3t_2t_3N$, we find the coset stabilizer $N^{(0323)} = N^{0323} = \langle e \rangle$. Only e will fix 0, 2, and 3. Hence the number of single cosets in [0323] is $\frac{|N|}{|N^{(0323)}|} = \frac{14}{1} = 14$. The orbits of $N^{(0323)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_3$ of the double coset $Mt_0t_3t_2t_3N$. We have:

$Mt_0t_3t_2t_3t_3 = Mt_0t_3t_2 \in [032]$,

$t_0t_3t_2t_3t_1 = x^3t_4t_6t_3 \implies Mt_0t_3t_2t_3t_1 = Mt_4t_6t_3 \in [026]$

$= \{N(t_0t_2t_6)^n | n \in N\}$,

$t_0t_3t_2t_3t_2 = xyt_4t_0t_4t_6t_5 \implies Mt_0t_3t_2t_3t_2 = Mt_4t_0t_4t_6t_5 \in [03021]$

$= \{N(t_0t_3t_0t_2t_1)^n | n \in N\}$,

$t_0t_3t_2t_3t_4 = x^2t_2t_0t_1t_2t_0t_0t_3t_0t_1 \implies Mt_0t_3t_2t_3t_4 = Mt_0t_3t_0t_1 \in [0301]$

(since $\{N(t_0t_3t_0t_1)^n | n \in N\}$ and $x^2t_2t_0t_1t_2t_0 \in M$),

$t_0t_3t_2t_3t_5 = x^3t_4t_3t_1 \implies Mt_0t_3t_2t_3t_5 = Mt_4t_3t_1 \in [013]$

$= \{N(t_0t_1t_3)^n | n \in N\}$,

$t_0t_3t_2t_3t_6 = yx^{-2}t_5t_0t_6t_5t_0t_0t_6t_2t_5 \implies Mt_0t_3t_2t_3t_6 = Mt_0t_6t_2t_5 \in [0152]$

(since $\{N(t_0t_1t_5t_2)^n | n \in N\}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$),

$t_0t_3t_2t_3t_0 = xt_0t_2t_3t_2t_0t_2t_5t_4t_3t_6t_4 \implies Mt_0t_3t_2t_3t_0 = Mt_2t_5t_4t_3t_6t_4 \in [032142]$
(since $\{N(t_0t_3t_2t_1t_4t_2)^n | n \in N\}$ and $xt_0t_2t_3t_2t_0 \in M$).

$Mt_0t_3t_2t_1t_4N$

Continuing with the double coset $Mt_0t_3t_2t_1t_4N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2t_1t_4 = t_4t_1t_2t_3t_0$$
$$\implies Mt_0t_3t_2t_1t_4 = Mt_4t_1t_2t_3t_0$$

Then $M(t_0t_3t_2t_1t_4)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_2t_3t_0$. But $Mt_4t_1t_2t_3t_0 = Mt_0t_3t_2t_1t_4 \implies (0,4)(3,1)(5,6) \in N^{(03214)}$
since $M(t_0t_3t_2t_1t_4)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_2t_3t_0$

$$\implies N^{(03214)} \geq \langle (0,4)(3,1)(5,6) \rangle.$$

Since $\left|N^{(03214)}\right| = 2$, the number of single cosets in [03214] is $\frac{|N|}{\left|N^{(03214)}\right|} = \frac{14}{2} = 7$. The orbits of $N^{(03214)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{2\}, \{0,4\}, \{3,1\}, \{5,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_1t_4$ of the double coset $Mt_0t_3t_2t_1t_4N$. We have:

$Mt_0t_3t_2t_1t_4t_4 = Mt_0t_3t_2t_1 \in [0321]$,

$Mt_0t_3t_2t_1t_4t_2 \in [032142]$,

$t_0t_3t_2t_1t_4t_3 = x^{-1}t_5t_0t_6t_5t_0t_2t_3t_6 \implies Mt_0t_3t_2t_1t_4t_3 = Mt_2t_3t_6 \in [014]$
(since $\{N(t_0t_1t_4)^n | n \in N\}$ and $x^{-1}t_5t_0t_6t_5t_0 \in M$),

$t_0t_3t_2t_1t_4t_5 = x^4t_2t_0t_1t_2t_0t_1t_5t_1t_6t_0 \implies Mt_0t_3t_2t_1t_4t_5 = Mt_1t_5t_1t_6t_0 \in [03021]$
(since $\{N(t_0t_3t_0t_2t_1)^n | n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

The new double coset is $Mt_0t_3t_2t_1t_4t_2N$, which we represent by [032142], respectively.

$Mt_0t_3t_2t_1t_4t_2N$

Continuing with the double coset $Mt_0t_3t_2t_1t_4t_2N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2t_1t_4t_2 = t_4t_1t_2t_3t_0t_2$$

$$\implies Mt_0t_3t_2t_1t_4t_2 = Mt_4t_1t_2t_3t_0t_2.$$

Then $M(t_0t_3t_2t_1t_4t_2)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_2t_3t_0t_2$.

But $Mt_4t_1t_2t_3t_0t_2 = Mt_0t_3t_2t_1t_4t_2 \implies (0,4)(3,1)(5,6) \in N^{(032142)}$

since $M(t_0t_3t_2t_1t_4t_2)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_2t_3t_0t_2$

$$\implies N^{(032142)} \geq \big\langle (0,4)(3,1)(5,6) \big\rangle.$$

Since $\left|N^{(032142)}\right| = 2$, the number of single cosets in [032142] is $\frac{|N|}{|N^{(032142)}|} = \frac{14}{2} = 7$.

The orbits of $N^{(032142)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{2\}, \{0,4\}, \{3,1\}, \{5,6\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_1t_4t_2$ of the double coset $Mt_0t_3t_2t_1t_4t_2N$. We have:

$Mt_0t_3t_2t_1t_4t_2t_2 = Mt_0t_3t_2t_1t_4 \in [03214]$,

$t_0t_3t_2t_1t_4t_2t_3 = yx^3t_0t_4t_3t_0 \implies Mt_0t_3t_2t_1t_4t_2t_3 = Mt_0t_4t_3t_0 \in [0340]$

$= \{N(t_0t_3t_4t_0)^n | n \in N\}$,

$t_0t_3t_2t_1t_4t_2t_5 = xt_0t_5t_4t_5t_0t_5t_1t_0t_1 \implies Mt_0t_3t_2t_1t_4t_2t_5 = Mt_5t_1t_0t_1 \in [0323]$

(since $\{N(t_0t_3t_2t_3)^n | n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$Mt_0t_3t_2t_1t_4t_2t_0 \in [0321420]$.

The new double coset is $Mt_0t_3t_2t_1t_4t_2t_0N$, which we represent by [0321420], respectively.

$Mt_0t_3t_2t_1t_4t_2t_0N$

Continuing with the double coset $Mt_0t_3t_2t_1t_4t_2t_0N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2t_1t_4t_2t_0 = yt_0t_5t_4t_5t_0t_5t_2t_3t_4t_1t_3t_5$$

$$\implies Mt_0t_3t_2t_1t_4t_2t_0 = Mt_5t_2t_3t_4t_1t_3t_5 \text{ since } yt_0t_5t_4t_5t_0.$$

Now $M(t_0t_3t_2t_1t_4t_2t_0)^n = Mt_5t_2t_3t_4t_1t_3t_5$.

Then $M(t_0t_3t_2t_1t_4t_2t_0)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5$.

But $Mt_5t_2t_3t_4t_1t_3t_5 = Mt_0t_3t_2t_1t_4t_2t_0 \implies (0,5)(2,3)(1,2) \in N^{(0321420)}$

since $M(t_0t_3t_2t_1t_4t_2t_0)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5$

$$\implies N^{(0321420)} \geq \big\langle (0,5)(2,3)(1,2) \big\rangle.$$

Since $\left|N^{(0321420)}\right| = 2$, the number of single cosets in [0321420] is $\frac{|N|}{|N^{(0321420)}|} = \frac{14}{2} = 7$.

The orbits of $N^{(0321420)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{6\}, \{0, 5\}, \{2, 3\}, \{1, 4\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_1t_4t_2t_0$ of the double coset $Mt_0t_3t_2t_1t_4t_2t_0N$. We have:

$Mt_0t_3t_2t_1t_4t_2t_0t_0 = Mt_0t_3t_2t_1t_4t_2 \in [032142]$,

$t_0t_3t_2t_1t_4t_2t_0t_1 = x^{-2}t_5t_0t_6t_5t_0t_4t_2t_6 \implies Mt_0t_3t_2t_1t_4t_2t_0t_1 = Mt_4t_2t_6 \in [025]$

(since $\{N(t_0t_2t_5)^n | n \in N\}$ and $x^{-2}t_5t_0t_6t_5t_0 \in M$),

$t_0t_3t_2t_1t_4t_2t_0t_2 = x^4t_2t_0t_1t_2t_0t_6t_0t_6 \implies Mt_0t_3t_2t_1t_4t_2t_0t_2 = Mt_6t_0t_6 \in [010]$

(since $\{N(t_0t_1t_0)^n | n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

$Mt_0t_3t_2t_1t_4t_2t_0t_6 \in [03214206]$.

The new double coset is $Mt_0t_3t_2t_1t_4t_2t_0t_6N$, which we represent by [03214206], respectively.

$\boldsymbol{Mt_0t_3t_2t_1t_4t_2t_0t_6N}$

Continuing with the double coset $Mt_0t_3t_2t_1t_4t_2t_0t_6N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_2t_1t_4t_2t_0t_6 = yt_0t_5t_4t_5t_0t_5t_2t_3t_4t_1t_3t_5t_6$$

$$\implies Mt_0t_3t_2t_1t_4t_2t_0t_6 = Mt_5t_2t_3t_4t_1t_3t_5t_6 \text{ since } yt_0t_5t_4t_5t_0 \in M$$

Then $M(t_0t_3t_2t_1t_4t_2t_0t_6)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5t_6$.

But $Mt_5t_2t_3t_4t_1t_3t_5t_6 = Mt_0t_3t_2t_1t_4t_2t_0t_6 \implies (0, 5)(2, 3)(1, 2) \in N^{(0321406)}$

since $M(t_0t_3t_2t_1t_4t_2t_0t_6)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5t_6$

$$\implies N^{(03214206)} \geq \langle (0, 5)(2, 3)(1, 2) \rangle.$$

Since $\left|N^{(03214206)}\right| = 2$, the number of single cosets in [03214206] is $\frac{|N|}{\left|N^{(03214206)}\right|} = \frac{14}{2} = 7$.

The orbits of $N^{(03214206)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{6\}, \{0, 5\}, \{2, 3\}, \{1, 4\}\}.$$

Take an element from each orbit and multiply on the right by the single coset represen-

tative $Mt_0t_3t_2t_1t_4t_2t_0t_6$ of the double coset $Mt_0t_3t_2t_1t_4t_2t_0t_6N$. We have:

$Mt_0t_3t_2t_1t_4t_2t_0t_6t_6 = Mt_0t_3t_2t_1t_4t_2t_0 \in [0321420]$,

$t_0t_3t_2t_1t_4t_2t_0t_6t_1 = x^2t_6t_5t_1t_4 \implies Mt_0t_3t_2t_1t_4t_2t_0t_6t_1 = Mt_6t_5t_1t_4 \in [0152]$

$= \{N(t_0t_1t_5t_2)^n | n \in N\}$,

$t_0t_3t_2t_1t_4t_2t_0t_6t_2 = yx^{-2}t_5t_0t_6t_5t_0t_1t_2t_3 \implies Mt_0t_3t_2t_1t_4t_2t_0t_6t_2 = Mt_1t_2t_3 \in [012]$

(since $\{N(t_0t_1t_2)^n | n \in N\}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$),

$t_0t_3t_2t_1t_4t_2t_0t_6t_0 = t_0t_2t_3t_2t_0t_1t_4t_1t_2 \implies Mt_0t_3t_2t_1t_4t_2t_0t_6t_0 = Mt_1t_4t_1t_2 \in [0301]$

(since $\{N(t_0t_3t_0t_1)^n | n \in N\}$ and $t_0t_2t_3t_2t_0 \in M$).

$\boldsymbol{Mt_0t_3t_4N}$

Continuing with the double coset $Mt_0t_3t_4N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_4 = x^2t_2t_0t_1t_2t_0t_2t_6t_5$$
$$\implies Mt_0t_3t_4 = Mt_2t_6t_5 \text{ since } x^2t_2t_0t_1t_2t_0 \in M.$$

Then $M(t_0t_3t_4)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_5$.
But $Mt_0t_6t_5 = Mt_0t_3t_4 \implies (0,2)(3,6)(4,5) \in N^{(034)}$
since $M(t_0t_3t_4)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_5$

$$\implies N^{(034)} \geq \langle (0,2)(3,6)(4,5) \rangle.$$

Since $|N^{(034)}| = 2$, the number of single cosets in [034] is $\frac{|N|}{|N^{(034)}|} = \frac{14}{2} = 7$. The orbits of $N^{(034)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{1\}, \{0,2\}, \{3,6\}, \{4,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4$ of the double coset $Mt_0t_3t_4N$. We have:

$Mt_0t_3t_4t_4 = Mt_0t_3 \in [03]$,

$Mt_0t_3t_4t_1 \in [0341]$,

$Mt_0t_3t_4t_0 \in [0340]$,

$t_0t_3t_4t_6 = xt_1t_3t_4t_3t_1t_4t_2t_5 \implies Mt_0t_3t_4t_6 = Mt_4t_2t_5 \in [026]$

(since $\{N(t_0t_2t_6)^n | n \in N\}$ and $xt_1t_3t_4t_3t_1 \in M$).

The new double cosets have single coset representatives $Mt_0t_3t_4t_1N$ and $Mt_0t_3t_4t_0N$, we represent them as [0341] and [0340], respectively.

$\boldsymbol{Mt_0t_3t_4t_1N}$

Continuing with the double coset $Mt_0t_3t_4t_1N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_4t_1 = x^2t_2t_0t_1t_2t_0t_2t_6t_5t_1$$

$$\implies Mt_0t_3t_4t_1 = Mt_2t_6t_5t_1 \text{ since } x^2t_2t_0t_1t_2t_0 \in M.$$

Then $M(t_0t_3t_4t_1)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_5t_1$.

But $Mt_0t_6t_5t_1 = Mt_0t_3t_4t_1 \implies (0,2)(3,6)(4,5) \in N^{(0341)}$

since $M(t_0t_3t_4t_1)^{(0,2)(3,6)(4,5)} = Mt_2t_6t_5t_1$

$$\implies N^{(0341)} \geq \langle (0,2)(3,6)(4,5) \rangle.$$

Since $\left|N^{(0341)}\right| = 2$, the number of single cosets in [0341] is $\frac{|N|}{\left|N^{(0341)}\right|} = \frac{14}{2} = 7$. The orbits of $N^{(0341)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{1\}, \{0,2\}, \{3,6\}, \{4,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_1$ of the double coset $Mt_0t_3t_4t_1N$. We have:

$Mt_0t_3t_4t_1t_1 = Mt_0t_3t_4 \in [034]$,

$t_0t_3t_4t_1t_3 = yxt_3t_2t_6 \implies Mt_0t_3t_4t_1t_3 = Mt_3t_2t_6 \in [014]$

$= \{N(t_0t_1t_4)^n | n \in N\}$,

$t_0t_3t_4t_1t_4 = x^{-2}t_6t_2t_3t_0 \implies Mt_0t_3t_4t_1t_4 = Mt_6t_2t_3t_0 \in [0341]$

$= \{N(t_0t_3t_4t_1)^n | n \in N\}$,

$Mt_0t_3t_4t_1t_0 \in [03410]$.

The new double coset have single coset representative $Mt_0t_3t_4t_1t_0N$, we represent it as [03410], respectively.

$\boldsymbol{Mt_0t_3t_4t_1t_0N}$

Continuing with the double coset $Mt_0t_3t_4t_1t_0N$, we find the coset stabiliser $N^{(03410)} = N^{03410} = \langle e \rangle$. Only e will fix 0, 1, 3, and 4. Hence the number of single cosets in [03410] is $\frac{|N|}{\left|N^{(03410)}\right|} = \frac{14}{1} = 14$. The orbits of $N^{(03410)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_1t_0$ of the double coset $Mt_0t_3t_4t_1t_0N$. We have:

$$Mt_0t_3t_4t_1t_0t_0 = Mt_0t_3t_4t_1 \in [0341],$$
$$t_0t_3t_4t_1t_0t_1 = yx^3t_5t_1t_2t_5t_4 \implies Mt_0t_3t_4t_1t_0t_1 = Mt_5t_1t_2t_5t_4 \in [03406],$$
$$t_0t_3t_4t_1t_0t_2 = yx^{-2}t_5t_0t_6t_5t_0t_2t_3t_0t_4$$
$$\implies Mt_0t_3t_4t_1t_0t_2 = Mt_2t_3t_0t_4 \in [0152],$$
$$t_0t_3t_4t_1t_0t_3 = yx^{-2}t_5t_0t_6t_5t_0t_0t_6t_4 \implies Mt_0t_3t_4t_1t_0t_3 = Mt_0t_6t_4 \in [013],$$
$$t_0t_3t_4t_1t_0t_4 = yxt_0t_3t_4t_1t_0 \implies Mt_0t_3t_4t_1t_0t_4 = Mt_0t_3t_4t_1t_0 \in [03410],$$
$$t_0t_3t_4t_1t_0t_5 = x^4t_2t_0t_1t_2t_0t_2t_4t_6 \implies Mt_0t_3t_4t_1t_0t_5 = Mt_2t_4t_6 \in [024],$$
$$t_0t_3t_4t_1t_0t_6 = x^3t_2t_0t_1t_2t_0t_0t_3t_0t_2t_3$$
$$\implies Mt_0t_3t_4t_1t_0t_6 = Mt_0t_3t_0t_2t_3 \in [03023].$$

$Mt_0t_3t_4t_0N$

Continuing with the double coset $Mt_0t_3t_4t_0N$, we find the coset stabiliser $N^{(0340)} = N^{0340} = \langle e \rangle$. Only e will fix 0, 3, and 4. Hence the number of single cosets in [0340] is $\frac{|N|}{|N^{(0340)}|} = \frac{14}{1} = 14$. The orbits of $N^{(0340)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_0$ of the double coset $Mt_0t_3t_4t_0N$. We have:

$$Mt_0t_3t_4t_0t_0 = Mt_0t_3t_4 \in [034],$$
$$t_0t_3t_4t_0t_1 = t_0t_5t_4t_5t_0t_4t_5t_6 \implies Mt_0t_3t_4t_0t_1 = Mt_4t_5t_6 \in [012]$$

(since $\{N(t_0t_1t_2)^n | n \in N\}$ and $t_0t_5t_4t_5t_0 \in M$),

$$t_0t_3t_4t_0t_2 = x^{-2}t_6t_3t_6t_4t_5 \implies Mt_0t_3t_4t_0t_2 = Mt_6t_3t_6t_4t_5 \in [03021]$$
$$= \{N(t_0t_3t_0t_2t_1)^n | n \in N\},$$
$$t_0t_3t_4t_0t_3 = x^2t_0t_2t_3t_2t_0t_3t_1t_4 \implies Mt_0t_3t_4t_0t_3 = Mt_3t_1t_4 \in [026]$$

(since $\{N(t_0t_2t_6)^n | n \in N\}$ and $x^2t_0t_2t_3t_2t_0 \in M$),

$t_0t_3t_4t_0t_4 = x^3yt_0t_4t_5t_6t_3t_0 \implies Mt_0t_3t_4t_0t_4 = Mt_0t_4t_5t_6t_3t_0 \in [032142]$

$= \{N(t_0t_3t_2t_1t_4t_2)^n|n \in N\}$,

$t_0t_3t_4t_0t_5 = yt_0t_3t_4t_0 \implies Mt_0t_3t_4t_0t_5 = Mt_0t_3t_4t_0 \in [0340]$

$= \{N(t_0t_3t_4t_0)^n|n \in N\}$,

$Mt_0t_3t_4t_0t_6 \in [03406]$.

The new double coset have single coset representative $Mt_0t_3t_4t_0t_6N$, we represent it as [03406], respectively.

$\boldsymbol{Mt_0t_3t_4t_0t_6N}$

Continuing with the double coset $Mt_0t_3t_4t_0t_6N$, we find the coset stabilizer $N^{(03406)} = N^{03406} = \langle e \rangle$. Only e will fix 0, 3, 4, and 6. Hence the number of single cosets in [03406] is $\frac{|N|}{|N^{(03406)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03406)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_0t_6$ of the double coset $Mt_0t_3t_4t_0t_6N$. We have:

$Mt_0t_3t_4t_0t_6t_6 = Mt_0t_3t_4t_0 \in [0340]$,

$t_0t_3t_4t_0t_6t_1 = xt_0t_5t_4t_5t_0t_0t_6t_3 \implies Mt_0t_3t_4t_0t_6t_1 = Mt_0t_6t_3 \in [014]$

(since $\{N(t_0t_1t_4)^n|n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$),

$t_0t_3t_4t_0t_6t_2 = x^4t_2t_0t_1t_2t_0t_3t_1t_5 \implies Mt_0t_3t_4t_0t_6t_2 = Mt_3t_1t_5 \in [025]$

(since $\{N(t_0t_2t_5)^n|n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

$t_0t_3t_4t_0t_6t_3 = yt_2t_5t_6t_3t_2 \implies Mt_0t_3t_4t_0t_6t_3 = Mt_2t_5t_6t_3t_2 \in [03410]$

$= \{N(t_0t_3t_4t_1t_0)^n|n \in N\}$,

$t_0t_3t_4t_0t_6t_4 = t_0t_2t_3t_2t_0t_6t_1t_3 \implies Mt_0t_3t_4t_0t_6t_4 = Mt_6t_1t_3 \in [024]$

(since $\{N(t_0t_2t_4)^n|n \in N\}$ and $t_0t_2t_3t_2t_0 \in M$),

$t_0t_3t_4t_0t_6t_5 = x^{-2}t_6t_0t_6 \implies Mt_0t_3t_4t_0t_6t_5 = Mt_6t_0t_6 \in [010]$

$= \{N(t_0t_1t_0)^n|n \in N\}$,

$t_0t_3t_4t_0t_6t_0 = x^2yt_0t_4t_0t_6 \implies Mt_0t_3t_4t_0t_6t_0 = Mt_0t_4t_0t_6 \in [0301]$

$= \{N(t_0t_3t_0t_1)^n|n \in N\}$.

$\boldsymbol{Mt_0t_3t_0N}$

Continuing with the double coset $Mt_0t_3t_0N$ we find the single coset stabilizer is trivial. However, the relation

$$\begin{aligned} t_0t_3t_0 &= t_5t_2t_5 \\ \implies Mt_0t_3t_0 &= Mt_5t_2t_5. \end{aligned}$$

Then $M(t_0t_3t_0)^{(0,5)(2,3)(1,4)} = Mt_5t_2t_5$.
But $Mt_5t_2t_5 = Mt_0t_3t_0 \implies (0,5)(2,3)(1,4) \in N^{(030)}$
since $M(t_0t_3t_0)^{(0,5)(2,3)(1,4)} = Mt_5t_2t_5$

$$\implies N^{(030)} \geq \langle(0,5)(2,3)(1,4)\rangle.$$

Since $\left|N^{(030)}\right| = 2$, the number of single cosets in [030] is $\frac{|N|}{\left|N^{(030)}\right|} = \frac{14}{2} = 7$. The orbits of $N^{(030)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{6\}, \{0,5\}, \{2,3\}, \{1,4\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0$ of the double coset $Mt_0t_3t_0N$. We have:

$Mt_0t_3t_0t_0 = Mt_0t_3 \in [03]$,

$Mt_0t_3t_0t_1 \in [0301]$,

$Mt_0t_3t_0t_2 \in [0302]$,

$t_0t_3t_0t_6 = x^{-1}t_5t_0t_6t_5t_0t_0t_3t_0 \implies Mt_0t_3t_0t_6 = Mt_0t_3t_0 \in [030]$

(since $\{N(t_0t_3t_0)^n | n \in N\}$ and $x^{-1}t_5t_0t_6t_5t_0 \in M$).

The new double cosets have single coset representatives $Mt_0t_3t_0t_1N$ and $Mt_0t_3t_0t_2N$, we represent them as [0301] and [0302], respectively.

$\boldsymbol{Mt_0t_3t_0t_1N}$

Continuing with the double coset $Mt_0t_3t_0t_1N$, we find the coset stabiliser $N^{(0301)} = N^{0301} = \langle e\rangle$. Only e will fix 0, 1, and 3. Hence the number of single cosets in [0301] is $\frac{|N|}{\left|N^{(0301)}\right|} = \frac{14}{1} = 14$. The orbits of $N^{(0301)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_1$ of the double coset $Mt_0t_3t_0t_1N$. We have:

$Mt_0t_3t_0t_1t_1 = Mt_0t_3t_0 \in [030]$,

$t_0t_3t_0t_1t_2 = yx^2t_6t_2t_1t_0 \implies Mt_0t_3t_0t_1t_2 = Mt_6t_2t_1t_0 \in [0321]$

$= \{N(t_0t_3t_2t_1)^n | n \in N\}$,

$t_0t_3t_0t_1t_3 = yt_6t_4t_0 \implies Mt_0t_3t_0t_1t_3 = Mt_6t_4t_0 \in [026]$

$= \{N(t_0t_2t_6)^n | n \in N\}$,

$t_0t_3t_0t_1t_4 = x^{-2}t_5t_0t_6t_5t_0t_0t_3t_2t_3 \implies Mt_0t_3t_0t_1t_4 = Mt_0t_3t_2t_3 \in [0323]$

(since $\{N(t_0t_3t_2t_3)^n | n \in N\}$ and $x^{-2}t_5t_0t_6t_5t_0 \in M$),

$t_0t_3t_0t_1t_5 = x^2t_4t_6t_1 \implies Mt_0t_3t_0t_1t_5 = Mt_4t_6t_1 \in [024]$

$= \{N(t_0t_2t_4)^n | n \in N\}$,

$t_0t_3t_0t_1t_6 = x^2t_0t_2t_3t_2t_0t_6t_2t_1t_0t_3t_1t_6t_5$

$\implies Mt_0t_3t_0t_1t_6 = Mt_6t_2t_1t_0t_3t_1t_6t_5 \in [03214206]$

(since $\{N(t_0t_3t_2t_1t_4t_2t_0t_6)^n | n \in N\}$ and $x^2t_0t_2t_3t_2t_0 \in M$),

$t_0t_3t_0t_1t_0 = yx^2t_0t_4t_3t_0t_1 \implies Mt_0t_3t_0t_1t_0 = Mt_0t_4t_3t_0t_1 \in [03406]$

$= \{N(t_0t_3t_4t_0t_6)^n | n \in N\}$.

$\boldsymbol{Mt_0t_3t_0t_2N}$

Continuing with the double coset $Mt_0t_3t_0t_2N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_0t_2 = x^{-2}t_6t_3t_6t_4$$
$$\implies Mt_0t_3t_0t_2 = Mt_6t_3t_6t_4.$$

Then $M(t_0t_3t_0t_2)^{(0,6)(2,4)(1,5)} = Mt_3t_6t_3t_4$.

But $Mt_3t_6t_3t_4 = Mt_0t_3t_0t_2 \implies (0,6)(2,4)(1,5) \in N^{(0302)}$

since $M(t_0t_3t_0t_2)^{(0,6)(2,4)(1,5)} = Mt_3t_6t_3t_4$

$$\implies N^{(0302)} \geq \big\langle (0,6)(2,4)(1,5) \big\rangle.$$

Since $\left|N^{(0302)}\right| = 2$, the number of single cosets in [0302] is $\frac{|N|}{|N^{(0302)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0302)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{3\}, \{0,6\}, \{2,4\}, \{1,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset represen-

tative $Mt_0t_3t_0t_2$ of the double coset $Mt_0t_3t_0t_2N$. We have:

$Mt_0t_3t_0t_2t_2 = Mt_0t_3t_0 \in [030]$,

$Mt_0t_3t_0t_2t_1 \in [03021]$,

$Mt_0t_3t_0t_2t_3 \in [03023]$,

$t_0t_3t_0t_2t_0 = t_5t_0t_4 \implies Mt_0t_3t_0t_2t_0 = Mt_5t_0t_4 \in [026]$

$= \{N(t_0t_2t_6)^n | n \in N\}$.

The new double cosets have single coset representatives $Mt_0t_3t_0t_2t_1N$ and $Mt_0t_3t_0t_2t_3N$, we represent them as [03021] and [03023], respectively.

$\boldsymbol{Mt_0t_3t_0t_2t_1N}$

Continuing with the double coset $Mt_0t_3t_0t_2t_1N$, we find the coset stabiliser $N^{(03021)} = N^{03021} = \langle e \rangle$. Only e will fix 0, 1, 2, and 3. Hence the number of single cosets in [03021] is $\frac{|N|}{|N^{(03021)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03021)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_2t_1$ of the double coset $Mt_0t_3t_0t_2t_1N$. We have:

$Mt_0t_3t_0t_2t_1t_1 = Mt_0t_3t_0t_2 \in [0302]$,

$t_0t_3t_0t_2t_1t_2 = x^{-1}t_1t_2t_4 \implies Mt_0t_3t_0t_2t_1t_2 = Mt_1t_2t_4 \in [013]$

$= \{N(t_0t_1t_3)^n | n \in N\}$,

$t_0t_3t_0t_2t_1t_3 = x^4t_2t_0t_1t_2t_0t_4t_0t_6t_5t_1 \implies Mt_0t_3t_0t_2t_1t_3 = Mt_4t_0t_6t_5t_1 \in [03214]$

(since $\{N(t_0t_3t_2t_1t_4)^n | n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

$t_0t_3t_0t_2t_1t_4 = x^{-2}t_6t_3t_2t_6 \implies Mt_0t_3t_0t_2t_1t_4 = Mt_6t_3t_2t_6 \in [0340]$

$= \{N(t_0t_3t_4t_0)^n | n \in N\}$,

$t_0t_3t_0t_2t_1t_5 = x^2yt_3t_6t_5t_6 \implies Mt_0t_3t_0t_2t_1t_5 = Mt_3t_6t_5t_6 \in [0323]$

$= \{N(t_0t_3t_2t_3)^n | n \in N\}$,

$Mt_0t_3t_0t_2t_1t_6 \in [030216]$,

$t_0t_3t_0t_2t_1t_0 = x^{-3}t_1t_3t_1 \implies Mt_0t_3t_0t_2t_1t_0 = Mt_1t_3t_1 \in [020]$

$= \{N(t_0t_2t_0)^n|n \in N\}$,

The new double coset have single coset representative $Mt_0t_3t_0t_2t_1t_6N$, we represent it as [030216], respectively.

$Mt_0t_3t_0t_2t_1t_6N$

Continuing with the double coset $Mt_0t_3t_0t_2t_1t_6N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_0t_2t_1t_6 = x^3t_2t_0t_1t_2t_0t_4t_1t_4t_2t_3t_5$$

$$\implies Mt_0t_3t_0t_2t_1t_6 = Mt_4t_1t_4t_2t_3t_5 \text{ since } x^3t_2t_0t_1t_2t_0 \in M.$$

Then $M(t_0t_3t_0t_2t_1t_6)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_4t_2t_3t_5$.

But $Mt_4t_1t_4t_2t_3t_5 = Mt_0t_3t_0t_2t_1t_6 \implies (0,4)(3,1)(5,6) \in N^{(030216)}$

since $M(t_0t_3t_0t_2t_1t_6)^{(0,4)(3,1)(5,6)} = Mt_4t_1t_4t_2t_3t_5$

$$\implies N^{(030216)} \geq \langle (0,4)(3,1)(5,6) \rangle.$$

Since $|N^{(030216)}| = 2$, the number of single cosets in [030216] is $\frac{|N|}{|N^{(030216)}|} = \frac{14}{2} = 7$.

The orbits of $N^{(030216)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{2\}, \{0,4\}, \{3,1\}, \{5,6\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_2t_1t_6$ of the double coset $Mt_0t_3t_0t_2t_1t_6N$. We have:

$Mt_0t_3t_0t_2t_1t_6t_6 = Mt_0t_3t_0t_2t_1 \in [03021]$,

$t_0t_3t_0t_2t_1t_6t_2 = yx^2t_0t_3t_0t_2t_1t_6 \implies Mt_0t_3t_0t_2t_1t_6t_2 = Mt_0t_3t_0t_2t_1t_6 \in [030216]$

$= \{N(t_0t_3t_0t_2t_1t_6)^n|n \in N\}$,

$t_0t_3t_0t_2t_1t_6t_3 = x^{-2}t_4t_6t_2 \implies Mt_0t_3t_0t_2t_1t_6t_3 = Mt_4t_6t_2 \in [025]$

$= \{N(t_0t_2t_5)^n|n \in N\}$,

$t_0t_3t_0t_2t_1t_6t_0 = x^3t_2t_0t_1t_2t_0t_2t_3t_5 \implies Mt_0t_3t_0t_2t_1t_6t_0 = Mt_2t_3t_5 \in [013]$

(since $\{N(t_0t_1t_3)^n|n \in N\}$ and $x^3t_2t_0t_1t_2t_0 \in M$).

$Mt_0t_3t_0t_2t_3N$

Continuing with the double coset $Mt_0t_3t_0t_2t_3N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_0t_2t_3 = x^{-2}t_6t_3t_6t_4t_3$$
$$\implies Mt_0t_3t_0t_2t_3 = Mt_6t_3t_6t_4t_3.$$

Then $M(t_0t_3t_0t_2t_3)^{(0,6)(2,4)(1,5)} = Mt_6t_3t_6t_4t_3$.

But $Mt_6t_3t_6t_4t_3 = Mt_0t_3t_0t_2t_3 \implies (0,6)(2,4)(1,5) \in N^{(03023)}$

since $M(t_0t_3t_0t_2t_3)^{(0,6)(2,4)(1,5)} = Mt_3t_6t_6t_4t_3$

$$\implies N^{(03023)} \geq \langle (0,6)(2,4)(1,5) \rangle.$$

Since $|N^{(03023)}| = 2$, the number of single cosets in [03023] is $\frac{|N|}{|N^{(03023)}|} = \frac{14}{2} = 7$. The orbits of $N^{(03023)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{3\}, \{0,6\}, \{2,4\}, \{1,5\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_2t_3$ of the double coset $Mt_0t_3t_0t_2t_3N$. We have:

$Mt_0t_3t_0t_2t_3t_3 = Mt_0t_3t_0t_2 \in [0302]$,

$t_0t_3t_0t_2t_3t_1 = x^{-3}t_3t_4t_3 \implies Mt_0t_3t_0t_2t_3t_1 = Mt_3t_4t_3 \in [010]$

$= \{N(t_0t_1t_0)^n | n \in N\}$,

$t_0t_3t_0t_2t_3t_4 = xyt_4t_3t_2 \implies Mt_0t_3t_0t_2t_3t_4 = Mt_4t_3t_2 \in [012]$

$= \{N(t_0t_1t_2)^n | n \in N\}$,

$t_0t_3t_0t_2t_3t_0 = x^{-1}t_5t_0t_6t_5t_0t_6t_3t_2t_5t_6 \implies Mt_0t_3t_0t_2t_3t_0 = Mt_6t_3t_2t_5t_6 \in [03410]$

(since $\{N(t_0t_3t_4t_1t_0)^n | n \in N\}$ and $x^{-1}t_5t_0t_6t_5t_0 \in M$).

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of M in G is 351. We conclude:

$G = MeN \cup Mt_0N \cup Mt_0t_1N \cup Mt_0t_2N \cup Mt_0t_3N \cup Mt_0t_1t_0N \cup Mt_0t_1t_2N \cup Mt_0t_1t_3N \cup Mt_0t_1t_4N \cup Mt_0t_1t_5N \cup Mt_0t_1t_5t_2N \cup Mt_0t_2t_4N \cup Mt_0t_2t_5N \cup Mt_0t_2t_6N \cup Mt_0t_2t_0N \cup Mt_0t_3t_2N \cup Mt_0t_3t_2t_1N \cup Mt_0t_3t_2t_3N \cup Mt_0t_3t_2t_1t_4N \cup Mt_0t_3t_2t_1t_4t_2N \cup Mt_0t_3t_2t_1t_4t_2t_0N \cup Mt_0t_3t_2t_1t_4t_2t_0t_6N \cup Mt_0t_3t_4N$
$\cup Mt_0t_3t_4t_1N \cup Mt_0t_3t_4t_1t_0N \cup Mt_0t_3t_4t_0N \cup Mt_0t_3t_4t_0t_6N \cup Mt_0t_3t_0N \cup Mt_0t_3t_0t_1N \cup Mt_0t_3t_0t_2N \cup Mt_0t_3t_0t_2t_1N \cup Mt_0t_3t_0t_2t_1t_6N \cup Mt_0t_3t_0t_2t_3N$, where

$$G = \frac{2^{*7} : 2^{\cdot}D_{14}}{(xyt^xt)^3, (xt)^7}$$

$|G| \leq |N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(02)}} + \frac{|N|}{N^{(03)}} + \frac{|N|}{N^{(010)}} + \frac{|N|}{N^{(012)}} + \frac{|N|}{N^{(013)}} + \frac{|N|}{N^{(014)}} + \frac{|N|}{N^{(015)}} +$
$\frac{|N|}{N^{(0152)}} + \frac{|N|}{N^{(024)}} + \frac{|N|}{N^{(025)}} + \frac{|N|}{N^{(026)}} + \frac{|N|}{N^{(020)}} + \frac{|N|}{N^{(032)}} + \frac{|N|}{N^{(0321)}} + \frac{|N|}{N^{(0323)}} + \frac{|N|}{N^{(03214)}} + \frac{|N|}{N^{(032142)}}$
$+ \frac{|N|}{N^{(0321420)}} + \frac{|N|}{N^{(03214206)}} + \frac{|N|}{N^{(034)}} + \frac{|N|}{N^{(0341)}} + \frac{|N|}{N^{(03410)}} + \frac{|N|}{N^{(0340)}} + \frac{|N|}{N^{(03406)}} + \frac{|N|}{N^{(030)}} + \frac{|N|}{N^{(0301)}} +$
$\frac{|N|}{N^{(0302)}} + \frac{|N|}{N^{(03021)}} + \frac{|N|}{N^{(030216)}} + \frac{|N|}{N^{(03023)}} \times |M|$

$|G| \leq (1 + 7 + 14 + 14 + 14 + 14 + 14 + 14 + 14 + 7 + 14 + 14 + 14 + 14 + 14 + 7 + 7$
$+ 14 + 7 + 7 + 7 + 7 + 7 + 7 + 14 + 14 + 14 + 7 + 14 + 7 + 14 + 7 + 7) \times 28$

$|G| \leq 351 \times 28$

$|G| \leq 9828.$

A Cayley diagram that summarizes the above information is given on the next page.

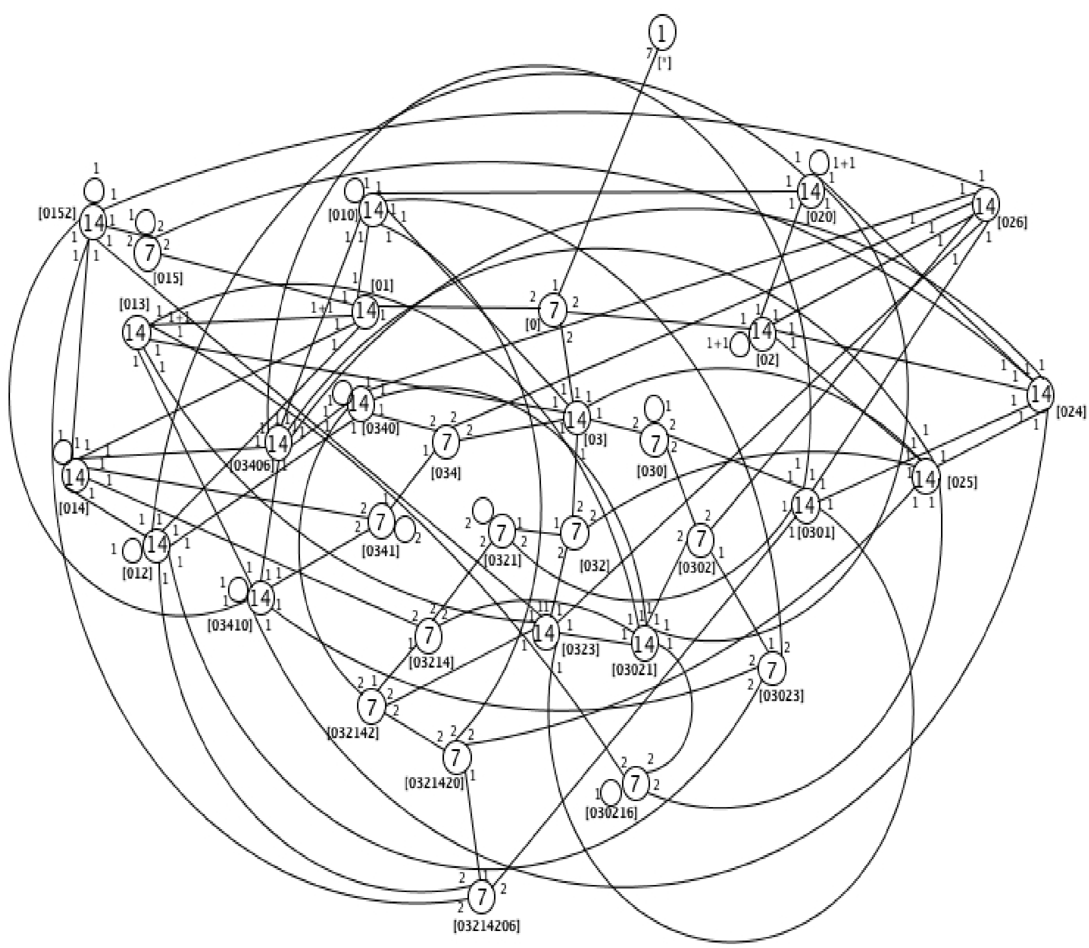

Figure 4.1: Cayley Diagram of $L_2(27)$ over $M = 2^{\cdot}D_{14}$

4.2 Iwasawa's Lemma to Prove $L_2(27)$ over $M = 2^{\cdot}D_{14}$ is Simple

Again, we use *Iwasawa's lemma* and the transitive action of G on the set of single cosets to prove $G \cong L_2(27)$ over $M = 2^{\cdot}D_{14}$ is a simple group. *Iwasawa's lemma* has three sufficient conditions that we must satisfied:

(1) G acts on X faithfully and primitively

(2) G is perfect ($G = G'$)

(3) There exist $x \in X$ and a normal abelian subgroup K of G^x such that the conjugates of K generate G.

Proof. **4.2.1** $G = L_2(27)$ **over** $M = 2^{\cdot}D_{14}$ **acts on** X **Faithfully**

Let G acts on $X = \{M, Mt_0N, Mt_0t_1N, Mt_0t_2N, Mt_0t_3N, \ldots, Mt_0t_3t_0t_2t_3N\}$, where X is a transitive G-set of degree 351. G acts on X implies there exist homomorphism

$$f \ : G \longrightarrow S_{351} \qquad\qquad (|X| = 351).$$

By First Isomorphic Theorem we have:

$$G/kerf \cong f(G).$$

If $kerf = 1$ then $G \cong f(G)$. Only elements of N fix N implies $G^1 = N$. Since X is transitive $G - set$ of degree 351, we have:

$$\begin{aligned} |G| &= 351 \times |G^1| \\ &= 351 \times |M| \\ &= 351 \times 28 \\ &= 9828 \\ \Longrightarrow \ |G| &= 9828. \end{aligned}$$

From Cayley diagram, $|G| \leq 9828$. However, from above $|G| = 9828$ implying $ker(f) = 1$. Since $kerf = 1$ then G acts faithfully on X.

4.2.2 $G = L_2(27)$ over $M = 2^{\cdot}D_{14}$ acts on X Primitively

Since $G = L_2(27)$ is transitive on $|X| = 351$, if B is a nontrivial block then we may assume that $M \in B$. However, $|B|$ must divide $|X| = 351$. The only nontrivial blocks must be of size 3, 9, 13, 27, 39, or 117. Note if $Bt_0 \in B$ then $B = X$. So B is a trivial block. By inspection, we can see from the Cayley diagram that we cannot create a nontrivial block of size 3,9,13,27,39, or 117. Thus, G acts primitively on X.

4.2.3 $G = L_2(27)$ over $M = 2^{\cdot}D_{14}$ is Perfect

.

Next we want to show that $G = G'$. Since $G =< N, t >$, we have that $N \leq G'$. Now $D_{14} \leq G \implies {D_{14}}' \leq G'$. The commutators subgroup of D_{14} is

$$\begin{aligned} {D_{14}}' &=< (1,2,3,4,5,6,0) >=< x > \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'. \end{aligned}$$

Consider the relations obtained through the double coset enumeration, previously given.

$$t_0t_2t_0t_4 = xt_6t_4t_2 \implies x = t_0t_2t_0t_4t_2t_4t_6,$$
$$t_0t_2t_0t_3 = yt_4t_3t_4 \implies y = t_0t_2t_0t_3t_4t_3t_4.$$

Now $D_{14} \leq G \implies {D_{14}}' \leq G'$. ${D_{14}}' =< (1,2,3,4,5,6,0) >=< x >= \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'$. Note $G =< x, y, t >=< t_1, t_2, t_3, t_4, t_5, t_6, t_0 >$. Our goal is to show that one of the $t_i's \in G'$, then we can conjugate by $< x, y >$ to obtain all the $t_i's$ in G'. Consider, the relation:

$$\begin{aligned} x &= t_0t_2t_0t_4t_2t_4t_6 \\ &= t_0t_2t_0t_2t_2t_4t_2t_4t_6 \\ &= [t_0, t_2][t_2, t_4]t_6 \in G' \end{aligned}$$

We see that $t_6 \in G'$. So $G' \geq< x, t_6 >=< t_1, t_2, t_3, t_4, t_5, t_6, t_0 >= G$. But $G \geq G'$. We conclude that $G = G'$ and G is perfect.

4.2.4 Conjugates of a Normal Abelian K Generate $G = L_2(27)$ over $M = 2^{\cdot}D_{14}$

Now we require $x \in X$ and a normal abelian subgroup K of G^x,-the point stabilizer of x in G, such that the conjugates of K in G generate G.

Now $G^1 = M = 2^{\cdot}D_{14}$ possesses a normal abelian subgroup $K =< x >$. Since $x \in K \implies x^{-3}, x^{-1} \in K$. Now we have the following relations:

$$x^{-3} = t_0t_3t_0t_2t_1t_0t_1t_3t_1 \in K \text{ and}$$

$$x^{-1} = t_0t_3t_0t_2t_1t_2t_4t_2t_1 \in K \implies x = t_1t_2t_4t_2t_1t_2t_0t_3t_0 \in K.$$

Now we multiply x^{-3} by x:

$$x^{-3}x = x^{-2} = t_0t_3t_0t_2t_1t_0t_1t_3t_1t_1t_2t_4t_2t_1t_2t_0t_3t_0 \in K.$$

We conjugate both sides by $t_0t_3t_0t_2t_1$:

$$(x^{-2})^{t_0t_3t_0t_2t_1} = (t_0t_3t_0t_2t_1t_0t_1t_3t_2t_4t_2t_1t_2t_0t_3t_0)^{t_0t_3t_0t_2t_1} \in K^G$$

$$t_1t_2t_0t_3t_0x^{-2}t_0t_3t_0t_2t_1 = t_1t_2t_0t_3t_0t_0t_3t_0t_2t_1t_0t_1t_3t_2t_4t_2t_1t_2t_0t_3t_0t_0t_3t_0t_2t_1$$

$$x^{-2}t_6t_0t_5t_1t_5t_0t_3t_0t_2t_1 = t_0t_1t_3t_2t_4t_2 \in K^G$$

$$\text{Thus, } t_0t_1t_3t_2t_4t_2 \in K^G. \tag{4.1}$$

Consider the relation:

$$x = t_0t_2t_4t_2t_6t_4t_6 \in K$$

$$(x)^{t_0} = (t_0t_2t_4t_2t_6t_4t_6)^{t_0} \in K^G$$

$$t_0xt_0 = t_0t_0t_2t_4t_2t_6t_4t_6t_0$$

$$xt_1t_0 = t_2t_4t_2t_6t_4t_6t_0 \in K^G$$

$$\text{Thus, } t_2t_4t_2t_6t_4t_6t_0 \in K^G. \tag{4.2}$$

Now we multiply (4.1) and (4.2):

$$x^{-2}t_6t_0t_5t_1t_5t_0t_3t_0t_2t_1xt_1t_0 = t_0t_1t_3t_2t_4t_2t_2t_4t_2t_6t_4t_6t_0 \in K^G$$

$$x^{-1}t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_0 = t_0t_1t_3t_6t_4t_6t_0 \in K^G$$

$$(x^{-1}t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_0)^{t_0} = (t_0t_1t_3t_6t_4t_6t_0)^{t_0} \in K^G$$

$$t_0x^{-1}t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_0t_0 = t_0t_0t_1t_3t_6t_4t_6t_0t_0 \in K^G$$

$$x^{-1}t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1 = t_1t_3t_6t_4t_6 \in K^G$$

$$\text{Thus, } t_1t_3t_6t_4t_6 \in K^G. \tag{4.3}$$

Consider the relations:

$$x^3 = t_0t_3t_2t_3t_1t_3t_6t_4 \in K \implies x^{-3} = t_4t_6t_3t_1t_3t_2t_3t_0 \in K \text{ and}$$
$$x^3 = t_0t_3t_2t_3t_5t_1t_3t_4 \in K.$$

Now,

$$e = x^3x^{-3} = t_0t_3t_2t_3t_5t_1t_3t_4t_4t_6t_3t_1t_3t_2t_3t_0 \in K$$
$$\implies e = t_0t_3t_2t_3t_5t_1t_3t_6t_3t_1t_3t_2t_3t_0 \in K$$
$$(e)^{t_0t_3t_2t_3} = (t_0t_3t_2t_3t_5t_1t_3t_6t_3t_1t_3t_2t_3t_0)^{t_0t_3t_2t_3} \in K^G$$
$$t_3t_2t_3t_0et_0t_3t_2t_3 = t_3t_2t_3t_0t_0t_3t_2t_3t_5t_1t_3t_6t_3t_1t_3t_2t_3t_0t_0t_3t_2t_3$$
$$e = t_5t_1t_3t_6t_3t_1 \in K^G$$

$$\text{Thus, } t_5t_1t_3t_6t_3t_1 \in K^G. \tag{4.4}$$

Next, we multiply (4.3) & (4.4):

$$x^{-1}t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1 = t_1t_3t_6t_4t_6t_5t_1t_3t_6t_3t_1 \in K$$
$$(x^{-1}t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1)^{t_1t_3t_6} = (t_1t_3t_6t_4t_6t_5t_1t_3t_6t_3t_1)^{t_1t_3t_6} \in K^G$$
$$t_6t_3t_1x^{-1}t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_1t_3t_6 = t_6t_3t_1t_1t_3t_6t_4t_6t_5t_1t_3t_6t_3t_1t_1t_3t_6 \in K^G$$
$$x^{-1}t_5t_2t_0t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_1t_3t_6 = t_4t_6t_5t_1t_3 \in K^G$$

$$\text{Thus, } t_4t_6t_5t_1t_3 \in K^G. \tag{4.5}$$

Consider the relations:

$$x = t_0t_2t_4t_2t_6t_4t_6 \in K \text{ and}$$
$$x = t_0t_2t_4t_6t_4t_1t_6 \in K \implies x^{-1} = t_6t_1t_4t_6t_4t_2t_0 \in K.$$

Now, we multiply both relation:

$$xx^{-1} = e = t_0t_2t_4t_2t_6t_4t_6t_6t_1t_4t_6t_4t_2t_0 \in K$$
$$e = t_0t_2t_4t_2t_6t_4t_1t_4t_6t_2t_0 \in K$$
$$(e)^{t_0t_2t_4} = (t_0t_2t_4t_2t_6t_4t_1t_4t_6t_4t_2t_0)^{t_0t_2t_4} \in K^G$$
$$t_4t_2t_0et_0t_2t_4 = t_4t_2t_0t_0t_2t_4t_2t_6t_4t_1t_4t_6t_4t_2t_0t_0t_2t_4 \in K^G$$
$$e = t_2t_6t_4t_1t_4t_6 \in K^G$$
$$(e)^{t_2t_6} = (t_2t_6t_4t_1t_4t_6)^{t_2t_6} \in K^G$$
$$t_6t_2et_2t_6 = t_6t_2t_2t_6t_4t_1t_4t_6t_2t_6 \in K^G$$
$$e = t_4t_1t_4t_6t_2t_6 \in K^G$$
$$\text{Thus, } t_4t_1t_4t_6t_2t_6 \in K^G. \tag{4.6}$$

Consider the relation:

$$x^{-2} = t_0t_2t_4t_1t_4t_3t_6t_3 \in K$$
$$(x^{-2})^{t_0t_2} = (t_0t_2t_4t_1t_4t_3t_6t_3)^{t_0t_2} \in K^G$$

$$t_2t_0x^{-2}t_0t_2 = t_2t_0t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2 \in K^G$$
$$x^{-2}t_0t_5t_0t_2 = t_4t_1t_4t_3t_6t_3t_0t_2 \in K^G$$
$$\implies t_4t_1t_4t_3t_6t_3t_0t_2 \in K^G$$
$$\implies (t_4t_1t_4t_3t_6t_3t_0t_2)^{-1} \in K^G$$

So,

$$t_2t_0t_3t_6t_3t_4t_1t_4 \in K^G. \tag{4.7}$$

Now, we multiply (4.6) & (4.7):

$$et_2t_0t_5t_0x^2 = t_4t_1t_4t_6t_2t_6t_2t_0t_3t_6t_3t_4t_1t_4 \in K$$
$$(x^2t_4t_2t_0t_2)^{t_4t_1t_4} = (t_4t_1t_4t_6t_2t_6t_2t_0t_3t_6t_3t_4t_1t_4)^{t_4t_1t_4} \in K^G$$
$$t_4t_1t_4x^2t_4t_2t_0t_2t_4t_1t_4 = t_4t_1t_4t_4t_1t_4t_6t_2t_6t_2t_0t_3t_6t_3t_4t_1t_4t_4t_1t_4 \in K^G$$
$$x^2t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4 = t_6t_2t_6t_2t_0t_3t_6t_3 \in K^G$$

$$(x^2t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4)^{t_3t_6t_3} = (t_6t_2t_6t_2t_0t_3t_6t_3)^{t_3t_6t_3} \in K^G$$
$$t_3t_6t_3x^2t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_3t_6t_3t_6t_2t_6t_2t_0t_3t_6t_3t_3t_6t_3 \in K^G$$
$$x^2t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_3t_6t_3t_6t_2t_6t_2t_0 \in K^G$$

$$\text{Thus, } t_3t_6t_3t_6t_2t_6t_2t_0 \in K^G. \tag{4.8}$$

Consider the relation
$$x^{-3} = t_0t_2t_6t_4t_6t_5t_6t_3 \in K. \tag{4.9}$$

Now, we multiply (4.9) & (4.8):
$x^{-3}x^2t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_0t_2t_6t_4t_6t_5t_6t_3t_3t_6t_3t_6t_2t_6t_2t_0 \in K$
$\implies x^{-1}t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_0t_2t_6t_4t_6t_5t_3t_6t_2t_6t_2t_0 \in K$
$(x^{-1}t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3)^{t_0t_2t_6} = (t_0t_2t_6t_4t_6t_5t_3t_6t_2t_6t_2t_0)^{t_0t_2t_6} \in K^G$
$\implies t_6t_2t_0x^{-1}t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2t_6$
$= t_6t_2t_0t_0t_2t_6t_4t_6t_5t_3t_6t_2t_6t_2t_0t_0t_2t_6 \in K^G$
$\implies x^{-1}t_5t_1t_6t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2t_6 = t_4t_6t_5t_3t_6t_2 \in K^G.$
So the inverse
$$t_2t_6t_3t_5t_6t_4 \in K^G \tag{4.10}$$

Now, we multiply (4.5) & (4.10):

$$t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4 \in K.$$

Next, we conjugate by $t_4t_6t_5$:
$$(t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4)^{t_4t_6t_5} = t_5t_6t_4t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4t_4t_6t_5 \in K^G$$
$$= t_1t_3t_2t_6t_3 \in K^G.$$

Now, we conjugate $t_1t_3t_2t_6t_3$ by t_1 :
$$(t_1t_3t_2t_6t_3)^{t_1} = t_1t_1t_3t_2t_6t_3t_1 \in K^G$$

$$\text{Thus, } t_3t_2t_6t_3t_1 \in K^G \tag{4.11}$$

Now, we need to multiply (4.11) & (4.3):
$$t_3t_2t_6t_3t_1t_1t_3t_6t_4t_6 = t_3t_2t_4t_6 \in K.$$

Then

$$(t_3t_2t_4t_6)^{t_3} = t_3t_3t_2t_4t_6t_3 \in K^G$$

$$\text{Thus, } t_2t_4t_6t_3 \in K^G. \tag{4.12}$$

Now we multiply (4.1) & (4.12) to obtain the following:

$$t_0t_1t_3t_2t_4t_2t_2t_4t_6t_3 = t_0t_1t_3t_2t_6t_3 \in K \tag{4.13}$$

By (4.11) we have the following:

$$t_3t_2t_6t_3t_1 \in K$$

$$\begin{aligned}(t_3t_2t_6t_3t_1)^{t_1} &= t_1t_3t_2t_6t_3t_1t_1 \\ &= t_1t_3t_2t_6t_3 \in K^G.\end{aligned}$$

So, $(t_1t_3t_2t_6t_3^{-1} \in K^G$

$$\implies t_3t_6t_2t_3t_1 \in K^G \tag{4.14}$$

Finally, we multiply (4.13) & (4.14) to obtain the following:

$$t_0t_1t_3t_2t_6t_3t_3t_6t_2t_3t_1 = t_0 \in K$$

Thus, $t_0 \in K$

$$\begin{aligned}&\implies t_0^G \in K^G \\ &\implies K^G \leq \{t_0, t_0^x, t_0^{x^2}, t_0^{x^3}, t_0^{x^4}, t_0^{x^5}, t_0^{x^6}\} \\ &\implies K^G \leq \{t_0, t_0^x, t_0^{x^2}, t_0^{x^3}, t_0^{x^4}, t_0^{x^5}, t_0^{x^6}\} =< t_0, t_1, t_2, t_3t_4, t_5, t_6 >= G.\end{aligned}$$

Hence, the conjugates of K generate G. Therefore, by Iwasawa's lemma, $G \cong L_2(27)$ is simple.

$\square$

Chapter 5

Extension Problem

5.1 Extension Problem Preliminaries

Definition 5.1. ***(Extension).*** *G is an **extension** of K by Q if G has a normal subgroup $K_1 \cong K$ such that*

$$G/K_1 \cong Q$$

where G is a product of KQ, $G = KQ$. [Rot12]

Definition 5.2. ***(Normal series).*** *A chain of subgroups of G, $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$ such that*

$$G_i \trianglerighteq G \ \forall\ i,\ 1 \leq i \leq n$$

*is called a **normal series** of G. [Rot12]*

Definition 5.3. ***(Subnormal series).*** *A chain of subgroups of G, $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$ such that*

$$G_{i+1} \trianglerighteq G_i \ \forall\ i,\ 0 \leq i \leq n-1$$

*is called a **subnormal series** of G. [Rot12]*

Definition 5.4. ***(Composition series).*** *A **composition series** is a normals series*

$$G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$$

in which, for all i, either G_{i+1} is a maximal normals subgroup of G_i or $G_{i+1} = G_i$. [Rot12]

Definition 5.5. ***(Composition factors).*** *If G has a composition series, them the factor groups of this series is called the **composition factors** of G. [Rot12]*

Note 5.6. Any two composition series of a group are isomorphic.

Note 5.7. The composition factors G_0/G, G_1/G_2, $\cdots$ G_{n-1}/G_n of the composition series $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_{n-1} \supseteq G_n$ are simple (A group G is simple if G and 1 are the only normal subgroups of G).
There are four possible extensions a group can have namely:

Definition 5.8. ***(Direct Product).*** *A group G is* ***direct product*** *of N by H if N and H are both normal in G, denoted by $G = N \times H$. [Rot12]*

Definition 5.9. ***(Semi-direct Product).*** *A group G is a* ***semi-direct product*** *of N by H if there exist a complement $H_1 \cong H$ and N is normal in G, denoted by $G = N : H$. [Rot12]*

Definition 5.10. ***(Central Extension).*** *A group G is a* ***central extension*** *of N by H if G is perfect, normal, and is also the center of G, denoted $G = N^{\cdot}H$. [Rot12]*

Definition 5.11. ***(Mixed Extension).*** *A group G is a* ***mixed extension*** *if N is an abelian group which is not the center of G, denoted by $G = N^{\cdot} : H$. [Rot12]*

5.2 Mixed Extension ($G \cong \left(2^{6^{\cdot}} : L_2(7)\right) : 2$)

From an original progenitor we have found the following group

$$G = < x, y, t | x^7, y^2, (xy)^2, t^2, (t, y), (xtt^x)^8, (ttxt)^3 > .$$

We will now prove that $\left(2^{6^{\cdot}} : L_2(7)\right) : 2$ is the homomorphic image of the progenitor mentioned above.

Proof. Using MAGMA we get the following **composition factors.**

```
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  A(1, 7)                    = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
```

```
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
```

Therefore we have the following composition series,

$$G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq 1,$$

where $G = (G_1/G_2) \cdot (G_2/G_3) \cdot (G_3/G_4) \cdot (G_4/G_5) \cdot (G_5/G_6) \cdot (G_6/G_7) \cdot (G_7/1)$
$= C_2 \cdot L_2(7) \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2$. The normal lattice of G is

```
> NL:=NormalLattice(G1);
> NL;
Normal subgroup lattice
-----------------------

[4]  Order 21504  Length 1  Maximal Subgroups: 3
---
[3]  Order 10752  Length 1  Maximal Subgroups: 2
---
[2]  Order 64     Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

By inspection we find that the **center** of this group is order 1 which indicates that we do not have a central extension. Next we find that the minimal normal subgroup of G is of order 64. Since the minimal normal subgroup of G is an abelian p-group, it must be elementary abelian. Thus, NL[2] is isomorphic to $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 = (C_2)^6$.

We now have G_2 is isomorphic to $(C_2)^6$. Thus, $G_1/G_2 = L_2(7)$ gives $G_1/(C_2)^6 = L_2(7)$. So $G_1 = (C_2)^6 \cdot L_2(7)$, with $(C_2)^6$= NL[2] normal in G_2. Note that $|L_2(7)| = 168$. Therefore, we must find a normal subgroup of order 168. By inspection we look at the normal lattice of NL[3] to see that it does not have a normal subgroup of order 168. Since $N = (C_2)^6$ is an abelian group and is not the center of G thus G_1 is a mixed extension. Thus, NL[3] is isomorphic to $2^{6^{\cdot}} : L_2(7)$.

Next, we will show that $G_1 = (C_2)^{6^{\cdot}} : L_2(7)$.

Let $N = (C_2)^6$. We note that N$= < k, l, m, n, o, p >=< k > \times < l > \times < m > \times < n > \times < o > \times < p >$ where k, l, m, n, o is of order 2. A presentation for N is

$< k, l, m, n, o, p | k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o)$
$(l, p), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p) >$.

Now we have to write elements of $G_1/N = L_2(7)$ in terms of the generators of N. Note a presentation for $L_2(7)$ is

$$< r, s | r^2, s^4, (rs)^7, (r, s)^4, (rs^2)^3 > .$$

We now find the set of right coset of N in G_2. Let s and t denoted by $NT[i]$ where i goes from 1 to 168. There is an isomorphism from G_1/N to $L_2(7)$. In this isomorphism

$$NT[2] \mapsto r \quad \text{and} \quad NT[3] \mapsto s.$$

Since the permutations are very large, permutation group acting on a set of cardinality 10750, we use the Schreier System in Magma to find the actions. Note in Magma we must store $T[2] = NT[2], T[3] = NT[3]$, and $T[3]^4 = N(T[3])^4$ so they do not change everytime.

```
> N:=sub<G1|A,B,C,D,E,F>;
> NN<k,l,m,n,o,p>:=Group<k,l,m,n,o,p|k^2,l^2,m^2,
n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m),(l,n),
(l,o),(l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p)>;
> #NN;
64
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..64]];
> for i in [2..64] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
for|for> if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
for|for> if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
for|for> if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
```

We let

$$NT[2] = r \text{ and } NT[3] = s.$$

Note that $(NT[3])^4 = N(T[3])^4 = NT34 = N$. We run the following loops to convert the elements of $L_2(7)$ in terms of the generators of N.

```
> for i in [1..64] do if ArrayP[i] eq T34 then Sch[i];
end if; end for;
l * m * p
>  for i in [1..64] do if ArrayP[i] eq (T2,T3)^4 then Sch[i];
end if; end for;
k * l * m * o * p
```

So $(T[3])^4 \in N$; that is;

$$s^4 = lmp.$$

Also $(NT[2], NT[3])^4 = N(T[2], T[3])^4 = N$. So $(T[2], T[3])^4 \in N$; that is;

$$(r, s)^4 = klmop.$$

Thus, the elements of $L_2(7)$ in terms of the generators of N are:

$$s^4 = lmp \text{ and } (r, s)^4 = klmop.$$

Now we need to conjugate the generators of N by the generators of $L_2(7)$ to determine the resulting elements of N. We use the following loops to find the resulting elements of N.

```
> for i in [1..64] do if ArrayP[i] eq A^T2
then Sch[i]; I[2]:=Sch[i]; end if; end for;
k * m * o
> for i in [1..64] do if ArrayP[i] eq B^T2
then Sch[i]; I[3]:=Sch[i]; end if; end for;
n * o * p
> for i in [1..64] do if ArrayP[i] eq C^T2
then Sch[i]; I[4]:=Sch[i]; end if; end for;
m
> for i in [1..64] do if ArrayP[i] eq D^T2
then Sch[i]; I[5]:=Sch[i]; end if; end for;
m * n * o
> for i in [1..64] do if ArrayP[i] eq E^T2
then Sch[i]; I[6]:=Sch[i]; end if; end for;
o
> for i in [1..64] do if ArrayP[i] eq F^T2
then Sch[i]; I[7]:=Sch[i]; end if; end for;
l * m * n
> for i in [1..64] do if ArrayP[i] eq A^T3
```

```
then Sch[i]; I[8]:=Sch[i]; end if; end for;
k * m * n * o * p
> for i in [1..64] do if ArrayP[i] eq B^T3
then Sch[i]; I[9]:=Sch[i]; end if; end for;
n * p
> for i in [1..64] do if ArrayP[i] eq C^T3
then Sch[i]; I[10]:=Sch[i]; end if;end for;
k
> for i in [1..64] do if ArrayP[i] eq D^T3
then Sch[i]; I[11]:=Sch[i]; end if;end for;
k * l * n * p
> for i in [1..64] do if ArrayP[i] eq E^T3
then Sch[i]; I[12]:=Sch[i]; end if;end for;
p
> for i in [1..64] do if ArrayP[i] eq F^T3
 then Sch[i]; I[13]:=Sch[i]; end if;end for;
k * l * m * n
```

Thus, $k^r = kmo, l^r = nop, m^r = m, n^r = mno, o^r = o, p^r = lmn, k^s = kmnop, l^s = np, m^s = k, n^s = klnp, o^s = p$, and $p^s = klmn$. In addition, we check in Magma the presentation of G_1:

```
>  NN<k,l,m,n,o,p,r,s>:=Group<k,l,m,n,o,p,r,s|k^2,l^2,m^2,
n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m), (l,n),(l,o),
(l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p),r^2,
s^4=l*m*p,(r*s)^7,(r,s)^4=k*l*m*o*p,(r*s^2)^3,
k^r=k*m*o,l^r=n*o*p,m^r=m,
n^r=m*n*o,o^r=o,p^r=l*m*n,k^s=k*m*n*o*p,
l^s=n*p,m^s=k,n^s=k*l*n*p,o^s=p,p^s=k*l*m*n>;
>  #NN;
10752
>  f1,g,k1:=CosetAction(NN,sub<NN|Id(NN)>);
>  s,t:=IsIsomorphic(NL[3],g);
>  s;
true
```

Thus, we have G_1 is isomorphic to $2^{6^\cdot} : L_2(7)$. Hence, $G/G_1 = C_2$ gives $G/2^{6^\cdot} : L_2(7) = C_2$. So $G = 2^{6^\cdot} : L_2(7) \cdot C_2$, with $2^{6^\cdot} : L_2(7)$ =NL[3] normal in G_1.

Note C_2 is not a normal subgroup of G, therefore, G cannot be a direct product. By further inspection we find that it must be a semi-direct product. So we find an element of order 2 in G but outside NL[3], say g. So we run the following loop:

```
> for g in G1 do if Order(g) eq 2 and sub<G1|NL[3],g> eq G1
```

```
then Z:=g;break; end if; end for;
> G1 eq sub<G1|NL[3],Z>;
true
```

Now we use the following loops in Magma to find the action of g on the generators k,l,m,n,o,p,r,s of NL[3].

```
> for i in [1..10752] do if ArrayP[i] eq A^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
k
> for i in [1..10752] do if ArrayP[i] eq B^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
k * s * o * s^-1
> for i in [1..10752] do if ArrayP[i] eq C^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
n * o
> for i in [1..10752] do if ArrayP[i] eq D^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
k * m * o * p
> for i in [1..10752] do if ArrayP[i] eq E^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
k * o * p
> for i in [1..10752] do if ArrayP[i] eq F^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
p
> for i in [1..10752] do if ArrayP[i] eq T2^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
k * r * n * s * r * s^-1 * r
> for i in [1..10752] do if ArrayP[i] eq T3^Z
then Sch[i]; I[1]:=Sch[i]; end if; end for;
r * s * r * k * s * r * s^-1 * r * s * r
```

Thus we have:

$$g^2, k^g = ln, l^g = sos^{-1}, m^g = n^s, n^g = ksos^{-1},$$
$$o^g = kmnp, p^g = p^s, r^g = ksrlsrs^{-1}rs^{-1}, s^g = rksrsrs^{-1}r.$$

Hence, we have the following presentation:

$H1 < k, l, m, n, o, p, r, s, g > := Group < k, l, m, n, o, p, r, s, g | k^2, l^2, m^2, n^2, o^2, p^2,$
$(k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o),$
$(n, p), (o, p), r^2, s^4 = lmp, (rs)^7, (r, s)^4 = klmop, (rs^2)^3, k^r = kmo, l^r = nop,$
$m^r = m, n^r = mno, o^r = o, p^r = lmn, k^s = kmnop, l^s = np, m^s = k, n^s = klnp,$
$o^s = p, p^s = klmn, g^2, k^g = ln, l^g = sos^{-}1, m^g = n^s, n^g = ksos^{-}1,$

$o^g = kmnp, p^g = p^s, r^g = ksrlsrs^-1rs^-1, s^g = rksrsrs^-1r > .$

Finally, we check if it is isomorphic to G_1.

```
> #H1;
21504
>  f,h1,k1:=CosetAction(H1,sub<H1|Id(H1)>);
>  s:=IsIsomorphic(h1,G1);
>  s;
true
```

Thus we have solved the extension problem for this group and we can conclude that

$G = \left(2^{6^{\cdot}} : L_2(7)\right) : 2$

$\cong < k, l, m, n, o, p, r, s, g | k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m),$

$(l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p), r^2, s^4 = lmp, (rs)^7,$

$(r, s)^4 = klmop, (rs^2)^3, k^r = kmo, l^r = nop, m^r = m, n^r = mno, o^r = o, p^r = lmn,$

$k^s = kmnop, l^s = np, m^s = k, n^s = klnp, o^s = p, p^s = klmn, g^2, k^g = ln, l^g = sos^-1,$

$m^g = n^s, n^g = ksos^-1, o^g = kmnp, p^g = p^s, r^g = ksrlsrs^-1rs^-1, s^g = rksrsrs^-1r > .$

$\square$

Chapter 6

Double Coset Enumeration of M_{11} over S_4

6.1 G Factor by a Subgroup of Order 12

Consider the group $G \cong < a,b,c,d,t | a^2, b^3, c^4, d^4, b^{-1}aba, c^{-1}ac^{-1}, d^{-1}ad^{-1}, bc^{-1}b^{-1}d^{-1}$, $c^{-1}d^{-1}cd^{-1}, d^{-1}c^{-1}b^{-1}cb, t^2, (t,b), (ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3 >$. Note $N = S_4$, where $a \sim (1,5)(2,8)(3,6)(4,7)$, $b \sim (1,2,4)(5,8,7)$, $c \sim (1,4,5,7)(2,6,8,3)$ and $d \sim (1,3,5,6)(2,4,8,7)$. Now we look at the composition factors of this group:

```
G
|  M11
*
|  Cyclic(3)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
```

Note the center of G is of order 1. Now we look at the normal lattice of G :

```
Normal subgroup lattice
-----------------------

[6]  Order 95040  Length 1  Maximal Subgroups: 3 5
---
[5]  Order 31680  Length 1  Maximal Subgroups: 2 4
```

```
---
[4]  Order 7920   Length 1  Maximal Subgroups: 1
---
[3]  Order 12     Length 1  Maximal Subgroups: 2
---
[2]  Order 4      Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

We can see clearly that [3] is of order 12, therefore we are going to factor G by a subgroup of order 12, to obatin $G \cong M_{11}$:

```
>q,ff:=quo<G1|NL[3]>;
>CompositionFactors(q);
    G
    |  M11
    1
```

Now, we convert the action of the generators of [3] into word, to do so, we use the Shcreier System:

```
x:=NL[3].1;
y:=NL[3].2;
z:=NL[3].3;
A:=f(a);
B:=f(b);
C:=f(c);
D:=f(d);
E:=f(t);
N:=sub<G1|A,B,C,D,E>;
e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=6;l:=0;m:=6;n:=3;o:=0;
NN<a,b,c,d,t> := Group<a,b,c,d,t |a^2,b^3,c^4,d^4, b^-1*a*b*a,
c^-1*a*c^-1, d^-1*a*d^-1, b*c^-1*b^-1*d^-1,
 c^-1*d^-1*c*d^-1, d^-1*c^-1*b^-1*c*b,t^2,(t,b),(c*t^c)^6,
(a*b*t^(c*b^-1))^6,(a*b^-1*t)^3,>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
```

```
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
>for i in [1..#N] do if ArrayP[i] eq x then print Sch[i];
end if; end for;
  b * d * t * c^-1 * t * c * d^-1 * t * d * t * c^-1 * t
>for i in [1..#N] do if ArrayP[i] eq y then print Sch[i];
end if; end for;
b * d * t * d * t * c * t * d^-1 * t * b^-1 * c * t * c
* t * d * t * c^-1 * t
>for i in [1..#N] do if ArrayP[i] eq z then print Sch[i];
end if; end for;
b * c * t * c * t * d^-1 * t * c * t * d^-1 * t * d^-1
* t * b^-1 * c^-1 * t * c^-1 * t
```

Thus, by factoring the group of G by a subgroup of order 12, we obtain the following: $G \cong < a,b,c,d,t | a^2, b^3, c^4, d^4, b^{-1}aba, c^{-1}ac^{-1}, d^{-1}ad^{-1}, bc^{-1}b^{-1}d^{-1}$, $c^{-1}d^{-1}cd^{-1}, d^{-1}c^{-1}b^{-1}cb, t^2, (t,b), (ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3, bdtc^{-1}tcd^{-1}tdtc^{-1}t$, $bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t, bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t > \cong M_{11}$.

6.2 Construction of M_{11} over S_4

We start by factoring the progenitor $2^{*8} : S_4$ by the relations
$(ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3$,
$bdtc^{-1}tcd^{-1}tdtc^{-1}t, bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t, bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t$
to obtain the homomorphic image $G \cong M_{11}$, where
$a \sim (1,5)(2,8)(3,6)(4,7)$, $b \sim (1,2,4)(5,8,7)$, $c \sim (1,4,5,7)(2,6,8,3)$,
$d \sim (1,3,5,6)(2,4,8,7)$, and $t \sim t_0 \sim t_3$. The index of S_4 in G equals 330. Now we expand the relations:

$1 = (ct^c)^6 = (ct_2)^6 = c^6 t_2^{c^5} t_2^{c^4} t_2^{c^3} t_2^{c^2} t_2^{c} t_2 = c^2 t_6 t_2 t_8 t_3 t_2 t_6$

$\implies c^2 t_6 t_2 = t_2 t_6 t_8 t_0$,

$$1 = (abt^{cb^{-1}})^6 = (abt_1)^6 = (ab)^6 t_1^{(ab)^5} t_1^{(ab)^4} t_1^{(ab)^3} t_1^{(ab)^2} t_1^{(ab)} t_1 = t_7t_2t_5t_4t_8t_1$$
$$\implies t_7t_2 = t_1t_8t_4t_5,$$
$$1 = (ab^{-1}t)^3 = (ab^{-1})^3 t_0^{(ab^{-1})^2} t_0(ab^{-1})t_0 = at_0t_6t_0$$
$$\implies at_0 = t_0t_6,$$
$$1 = bdtc^{-1}tcd^{-1}tdtc^{-1}t = bdc^{-1}t_8t_0t_4t_8t_3$$
$$\implies bdc^{-1}t_8t_0 = t_3t_8t_4,$$
$$1 = bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t = dt_8t_1t_0t_7t_1t_4t_8t_0$$
$$\implies dt_8t_1t_0 = t_0t_8t_4t_1,$$
$$1 = bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t = c^{-1}t_4t_2t_1t_0t_7t_6t_8t_0$$
$$\implies c^{-1}t_4t_2t_1t_0 = t_0t_8t_6t_7.$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$ where g_i's words in t_i's.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [Nw^n | n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by t_i's. We will identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilising group $N^{(w)}$

NeN

First, the double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\}$, the orbit of N on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\}$ is:

$$\mathbb{O} = \{\{1, 2, 0, 4, 5, 6, 7, 8\}\}.$$

We choose t_0 as our symmetric generator from $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denote it $[0]$.

Nt₀N

In order to find how many single cosets $[0]$ contains, we must first find the coset stabiliser $N^{(0)}$. Then the number of single coset in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0 =< (1, 2, 4)(5, 8, 7) >$$

so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{24}{3} = 8$. Furthermore, the orbits of $N^{(0)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7 t_8\}$ are:

$$\mathbb{O} = \{\{1,2,4\},\{5,8,7\},\{0\},\{6\}\}.$$

We take t_1, t_5, t_0, and t_6 from each orbit, respectively and to see which double coset $Nt_0t_1, Nt_0t_5, Nt_0t_0$, and Nt_0t_6 belong to. We have:

$$Nt_0t_1 \in [01]$$
$$Nt_0t_5 \in [05]$$
$$Nt_0t_0 = N \in [*]$$
$$at_0t_6 = t_0 \implies Nt_0t_6 = Nt_0 \in [0]$$

The new double cosets have single coset representatives Nt_0t_1 and Nt_0t_5, which we represent them as [01] and [05] respectively.

$\boldsymbol{Nt_0t_1N}$

Consider Nt_0t_1N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(01)} = N^{01} =< e >$. Only identity (e) will fix 0 and 1. Hence the number of single cosets living in Nt_0t_1N is $\frac{|N|}{|N^{(01)}|} = \frac{24}{1} = 24$. The orbits of $N^{(01)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\},\{2\},\{0\},\{4\},\{5\},\{6\},\{7\},\{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_i$ belongs to. We have:

$$Nt_0t_1t_1 \in [0]$$
$$Nt_0t_1t_2 \in [012]$$
$$Nt_0t_1t_0 \in [010]$$
$$Nt_0t_1t_4 \in [014]$$
$$t_0t_1t_5 = at_6t_1 \implies Nt_0t_1t_5 = Nt_6t_1 \in [05]$$
$$Nt_0t_1t_6 \in [016]$$
$$t_0t_1t_7 = bd^{-1}t_1t_0 \implies Nt_0t_1t_7 = Nt_1t_0 \in [05]$$
$$Nt_0t_1t_8 \in [018].$$

The new double coset are $Nt_0t_1t_2N$, $Nt_0t_1t_0N$, $Nt_0t_1t_4N$, $Nt_0t_1t_6N$ and $Nt_0t_1t_8N$, which we represent them as [012], [010], [014], [016], and [018] respectively.

$\boldsymbol{Nt_0t_5N}$

Consider Nt_0t_5N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(05)} = N^{05} =< e >$. Only identity (e) will fix 0 and 5. Hence the number of single cosets living in Nt_0t_5N is $\frac{|N|}{|N^{(05)}|} = \frac{24}{1} = 24$. The orbits of $N^{(05)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_5t_i$ belongs to. We have:

$$t_0t_5t_1 = at_6t_5 \implies Nt_0t_5t_1 = Nt_6t_5 \in [01]$$

$$t_0t_5t_2 = bct_5t_0 \implies Nt_0t_5t_2 = Nt_5t_0 \in [01]$$

$$Nt_0t_5t_0 \in [050]$$

$$t_0t_5t_4 = d^{-1}b^{-1}t_7t_5t_6 \implies Nt_0t_5t_4 = Nt_7t_5t_6 \in [018]$$

$$Nt_0t_5t_5 = Nt_0 \in [0]$$

$$t_0t_5t_6 = at_5t_0t_1 \implies Nt_0t_5t_6 = Nt_5t_0t_1 \in [016]$$

$$t_0t_5t_7 = t_4t_1t_6 \implies Nt_0t_1t_7 = Nt_4t_1t_6 \in [012]$$

$$t_0t_5t_8 = bc^{-1}t_1t_6t_2 \implies Nt_0t_5t_8 = Nt_1t_6t_2 \in [014].$$

The new double coset is $Nt_0t_5t_0$, denoted by [050].

$\boldsymbol{Nt_0t_1t_0N}$

Consider $Nt_0t_1t_0N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(010)} = N^{010} =< e >$. Only identity (e) will fix 0 and 1. Hence the number of single cosets living in $Nt_0t_1t_0N$ is $\frac{|N|}{|N^{(010)}|} = \frac{24}{1} = 24$. The orbits of $N^{(010)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_i$ belongs to. We have:

$$Nt_0t_1t_0t_1 \in [0101]$$

$$Nt_0t_1t_0t_2 \in [0102]$$

$$Nt_0t_1t_0t_0 \in [01]$$

$$Nt_0t_1t_0t_4 \in [0104]$$

$$t_0t_1t_0t_5 = t_6t_1t_6 \implies Nt_0t_1t_0t_5 = Nt_6t_1t_6 \in [050]$$

$$t_0t_1t_0t_6 = at_6t_5t_0 \implies Nt_0t_1t_0t_6 = Nt_6t_5t_0 \in [016]$$

$$t_0t_1t_0t_7 = bd^{-1}t_1t_3t_1 \implies Nt_0t_1t_0t_7 = Nt_1t_3t_1 \in [050]$$

$$Nt_0t_1t_0t_8 \in [0108].$$

The new double coset are $Nt_0t_1t_0t_1N$, $Nt_0t_1t_0t_2N$, $Nt_0t_1t_0t_4N$, and $Nt_0t_1t_0t_8N$, which we represent them as [0101], [0102], [0104], and [0108] respectively.

$\boldsymbol{Nt_0t_1t_2N}$

Consider $Nt_0t_1t_2N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(012)} = N^{012} =< e >$. Only identity (e) will fix 0,1, and 2. Hence the number of single cosets living in $Nt_0t_1t_2N$ is $\frac{|N|}{|N^{(012)}|} = \frac{24}{1} = 24$. The orbits of $N^{(012)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_2t_i$ belongs to. We have:

$$t_0t_1t_2t_1 = bct_5t_0t_5t_8 \implies Nt_0t_1t_2t_1 = Nt_5t_0t_5t_8 \in [0104]$$

$$Nt_0t_1t_2t_2 \in [01]$$

$$t_0t_1t_2t_0 = b^{-1}dt_0t_1t_8 \implies Nt_0t_1t_2t_0 = Nt_0t_1t_8 \in [018]$$

$$t_0t_1t_2t_4 = b^{-1}t_0t_2t_1 \implies Nt_0t_1t_2t_4 = Nt_0t_2t_1 \in [014]$$

$$t_0t_1t_2t_5 = bc^{-1}t_2t_5t_2t_6 \implies Nt_0t_1t_2t_5 = Nt_2t_5t_2t_6 \in [0104]$$

$$t_0t_1t_2t_6 = t_8t_5 \implies Nt_0t_1t_2t_6 = Nt_8t_5 \in [05]$$

$$Nt_0t_1t_2t_7 \in [0127]$$

$$t_0t_1t_2t_8 = at_0t_5t_2 \implies Nt_0t_1t_2t_8 = Nt_0t_5t_2 \in [018].$$

The new double coset is $Nt_0t_1t_2t_7N$, denoted by [0127].

$\boldsymbol{Nt_0t_1t_4N}$

Consider $Nt_0t_1t_4N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(014)} = N^{014} =< e >$. Only identity (e) will fix 0,1, and 4. Hence the number of single cosets living in $Nt_0t_1t_4N$ is $\frac{|N|}{|N^{(014)}|} = \frac{24}{1} = 24$. The orbits of $N^{(014)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_4t_i$ belongs to. We have:

$$t_0t_1t_4t_1 = t_7t_5t_7t_0 \implies Nt_0t_1t_4t_1 = Nt_7t_5t_7t_0 \in [0102]$$

$$t_0t_1t_4t_2 = bt_0t_4t_1 \implies Nt_0t_1t_4t_2 = Nt_0t_4t_1 \in [012]$$

$$t_0t_1t_4t_0 = bd^{-1}t_4t_6t_1 \implies Nt_0t_1t_4t_0 = Nt_4t_6t_1 \in [014]$$

$$Nt_0t_1t_4t_4 \in [01]$$

$$t_0t_1t_4t_5 = db^{-1}t_5t_4t_3 \implies Nt_0t_1t_4t_5 = Nt_5t_4t_3 \in [014]$$

$$t_0t_1t_4t_6 = bdt_1t_6t_1t_7 \implies Nt_0t_1t_4t_6 = Nt_1t_6t_1t_7 \in [0102]$$

$$t_0t_1t_4t_7 = d^{-1}b^{-1}t_5t_6 \implies Nt_0t_1t_4t_7 = Nt_5t_6 \in [05]$$

$$t_0t_1t_4t_8 = bat_6t_7t_5t_2 \implies Nt_0t_1t_4t_8 = Nt_6t_7t_5t_2 \in [0127].$$

$Nt_0t_1t_6N$

Consider $Nt_0t_1t_6N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(016)} = N^{016} =< e >$. Only identity (e) will fix 0,1, and 6. Hence the number of single cosets living in $Nt_0t_1t_6N$ is $\frac{|N|}{|N^{(016)}|} = \frac{24}{1} = 24$. The orbits of $N^{(016)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_6t_i$ belongs to. We have:

$$t_0t_1t_6t_1 = t_5t_6t_5 \implies Nt_0t_1t_6t_1 = Nt_5t_6t_5 \in [050]$$

$$t_0t_1t_6t_2 = bc^{-1}t_6t_5t_0 \implies Nt_0t_1t_6t_2 = Nt_6t_5t_0 \in [016]$$

$$t_0t_1t_6t_0 = at_6t_5t_6 \implies Nt_0t_1t_6t_0 = Nt_6t_5t_6 \in [010]$$

$$t_0t_1t_6t_4 = bd^{-1}t_6t_7t_1 \implies Nt_0t_1t_6t_4 = Nt_6t_7t_1 \in [018]$$

$$t_0t_1t_6t_5 = at_1t_0 \implies Nt_0t_1t_6t_5 = Nt_1t_0 \in [05]$$

$$Nt_0t_1t_6t_6 \in [01]$$

$$t_0t_1t_6t_7 = t_0t_4t_0t_5 \implies Nt_0t_1t_6t_7 = Nt_0t_4t_0t_5 \in [0108]$$

$$t_0t_1t_6t_8 = cb^{-1}t_6t_5t_0 \implies Nt_0t_1t_6t_8 = Nt_6t_5t_0 \in [016].$$

$Nt_0t_1t_8N$

Consider $Nt_0t_1t_8N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(018)} = N^{018} =< e >$. Only identity (e) will fix 0,1, and 8.

Hence the number of single cosets living in $Nt_0t_1t_8N$ is $\frac{|N|}{|N^{(018)}|} = \frac{24}{1} = 24$. The orbits of $N^{(018)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_8t_i$ belongs to. We have:

$$t_0t_1t_8t_1 = cb^{-1}t_0t_1t_0t_8 \implies Nt_0t_1t_8t_1 = Nt_0t_1t_0t_8 \in [0108]$$

$$t_0t_1t_8t_2 = at_6t_5t_8 \implies Nt_0t_1t_8t_2 = Nt_6t_5t_8 \in [012]$$

$$t_0t_1t_8t_0 = bct_0t_1t_2 \implies Nt_0t_1t_8t_0 = Nt_0t_1t_2 \in [012]$$

$$t_0t_1t_8t_4 = b^{-1}ct_8t_7t_6t_5 \implies Nt_0t_1t_8t_4 = Nt_8t_7t_6t_5 \in [0187]$$

$$t_0t_1t_8t_5 = b^{-1}dt_6t_8t_0 \implies Nt_0t_1t_8t_5 = Nt_6t_8t_0 \in [016]$$

$$t_0t_1t_8t_6 = bc^{-1}t_2t_1 \implies Nt_0t_1t_8t_6 = Nt_2t_1 \in [05]$$

$$Nt_0t_1t_8t_7 \in [0187]$$

$$Nt_0t_1t_8t_8 \in [01].$$

The new double coset is $Nt_0t_1t_8t_7N$, denoted by [0187].

$\boldsymbol{Nt_0t_5t_0N}$

Consider $Nt_0t_5t_0N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(050)} = N^{050} =< e >$. Only identity (e) will fix 0, and 5. Hence the number of single cosets living in $Nt_0t_5t_0N$ is $\frac{|N|}{|N^{(050)}|} = \frac{24}{1} = 24$. The orbits of $N^{(050)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_5t_0t_i$ belongs to. We have:

$$t_0t_5t_0t_1 = t_6t_5t_6 \implies Nt_0t_5t_0t_1 = Nt_6t_5t_6 \in [010]$$

$$t_0t_5t_0t_2 = b^{-1}dt_5t_0t_5 \implies Nt_0t_5t_0t_2 = Nt_5t_3t_5 \in [010]$$

$$Nt_0t_5t_0t_0 \in [05]$$

$$t_0t_5t_0t_4 = at_7t_5t_7t_6 \implies Nt_0t_5t_0t_4 = Nt_7t_5t_7t_6 \in [0108]$$

$$t_0t_5t_0t_5 = at_0t_1t_0t_1 \implies Nt_0t_5t_0t_5 = Nt_0t_1t_0t_1 \in [0101]$$

$$t_0t_5t_0t_6 = t_1t_6t_5 \implies Nt_0t_5t_0t_6 = Nt_1t_6t_5 \in [016]$$

$$t_0t_5t_0t_7 = db^{-1}t_4t_1t_4t_6 \implies Nt_0t_5t_0t_7 = Nt_4t_1t_4t_6 \in [0102]$$

$$t_0t_5t_0t_8 = cb^{-1}t_1t_6t_1t_2 \implies Nt_0t_5t_0t_8 = Nt_1t_6t_1t_2 \in [0104].$$

$Nt_0t_1t_0t_1N$

Consider $Nt_0t_1t_0t_1N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0101)} = N^{0101} =< e >$. But $Nt_0t_1t_0t_1$ is not distinct. We have $Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4$. Thus, there exist $\{n \in N | N(t_0t_1t_0t_1)^n = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4\}$ such that

$$Nt_0t_1t_0t_1^{(1,2,4)(5,8,7)} = Nt_0t_2t_0t_2 \implies (1,2,4)(5,8,7) \in N^{(0101)}$$
$$Nt_0t_2t_0t_2^{(1,2,4)(5,8,7)} = Nt_0t_4t_0t_4 \implies (1,2,4)(5,8,7) \in N^{(0101)}$$
$$\implies Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4.$$

Thus, $(1,2,4)(5,8,7) \in N^{(0101)}$. We conclude:

$$N^{(0101)} \geq< (1,2,4)(5,8,7) >$$

so the number of the single cosets in $Nt_0t_1t_0t_1N$ is $\frac{|N|}{|N^{(0101)}|} = \frac{24}{3} = 8$. Furthermore, the orbits of $N^{(0101)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\}$ are:

$$\mathbb{O} = \{\{1,2,4\}, \{5,8,7\}, \{0\}, \{6\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_1t_i$ belongs to. We have:

$$Nt_0t_1t_0t_1t_1 \in [010]$$

$$t_0t_1t_0t_1t_5 = at_0t_5t_0 \implies Nt_0t_1t_0t_1t_5 = Nt_0t_5t_0 \in [050]$$

$$Nt_0t_1t_0t_1t_0 \in [01010]$$

$$t_0t_1t_0t_1t_6 = at_0t_1t_0t_1 \implies Nt_0t_1t_0t_1t_6 = Nt_0t_1t_0t_1 \in [0101]$$

The new double coset is $Nt_0t_1t_0t_1t_0N$, denoted by [01010].

$Nt_0t_1t_0t_2N$

Consider $Nt_0t_1t_0t_2N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0102)} = N^{0102} =< e >$. Only identity (e) will fix 0,1, and 2. Hence the number of single cosets living in $Nt_0t_1t_0t_2N$ is $\frac{|N|}{|N^{(0102)}|} = \frac{24}{1} = 24$. The orbits of $N^{(0102)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_2t_i$ belongs

to. We have:

$$t_0t_1t_0t_2t_1 = cb^{-1}t_5t_0t_8 \implies Nt_0t_1t_0t_2t_1 = Nt_5t_0t_8 \in [014]$$

$$Nt_0t_1t_0t_2t_2 \in [010]$$

$$t_0t_1t_0t_2t_0 = bct_0t_1t_0t_8 \implies Nt_0t_1t_0t_2t_0 = Nt_0t_1t_0t_8 \in [0108]$$

$$t_0t_1t_0t_2t_4 = cd^{-1}t_0t_2t_0t_1 \implies Nt_0t_1t_0t_2t_4 = Nt_0t_2t_0t_1 \in [0104]$$

$$t_0t_1t_0t_2t_5 = t_2t_5t_6 \implies Nt_0t_1t_0t_2t_5 = Nt_2t_5t_6 \in [014]$$

$$t_0t_1t_0t_2t_6 = bct_8t_5t_8 \implies Nt_0t_1t_0t_2t_6 = Nt_8t_5t_8 \in [050]$$

$$t_0t_1t_0t_2t_7 = bt_5t_3t_7t_2 \implies Nt_0t_1t_0t_2t_7 = Nt_5t_3t_7t_2 \in [0187]$$

$$t_0t_1t_0t_2t_8 = at_6t_5t_6t_2 \implies Nt_0t_1t_0t_2t_8 = Nt_6t_5t_6t_2 \in [0108].$$

$\boldsymbol{Nt_0t_1t_0t_4N}$

Consider $Nt_0t_1t_0t_4N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0104)} = N^{0104} =< e >$. Only identity (e) will fix 0,1, and 4. Hence the number of single cosets living in $Nt_0t_1t_0t_4N$ is $\frac{|N|}{|N^{(0104)}|} = \frac{24}{1} = 24$. The orbits of $N^{(0104)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_4t_i$ belongs to. We have:

$$t_0t_1t_0t_4t_1 = d^{-1}b^{-1}t_7t_5t_0 \implies Nt_0t_1t_0t_4t_1 = Nt_7t_5t_0 \in [012]$$

$$t_0t_1t_0t_4t_2 = ct_0t_4t_0t_1 \implies Nt_0t_1t_0t_4t_2 = Nt_0t_4t_0t_1 \in [0102]$$

$$t_0t_1t_0t_4t_0 = t_4t_6t_4t_1 \implies Nt_0t_1t_0t_4t_0 = Nt_4t_6t_4t_1 \in [0104]$$

$$Nt_0t_1t_0t_4t_4 \in [010]$$

$$t_0t_1t_0t_4t_5 = t_5t_4t_5t_3 \implies Nt_0t_1t_0t_4t_5 = Nt_5t_4t_5t_3 \in [0104]$$

$$t_0t_1t_0t_4t_6 = db^{-1}t_1t_6t_7 \implies Nt_0t_1t_0t_4t_6 = Nt_1t_6t_7 \in [012]$$

$$t_0t_1t_0t_4t_7 = bdt_5t_6t_5 \implies Nt_0t_1t_0t_4t_7 = Nt_5t_6t_5 \in [050]$$

$$t_0t_1t_0t_4t_8 = db^{-1}t_7t_2t_5t_6 \implies Nt_0t_1t_0t_4t_8 = Nt_7t_2t_5t_6 \in [0187].$$

$\boldsymbol{Nt_0t_1t_0t_8N}$

Consider $Nt_0t_1t_0t_8N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0108)} = N^{0108} =< e >$. Only identity (e) will fix 0,1, and 8. Hence the number of single cosets living in $Nt_0t_1t_0t_8N$ is $\frac{|N|}{|N^{(0108)}|} = \frac{24}{1} = 24$.

The orbits of $N^{(0108)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_0t_8t_i$ belongs to. We have:

$$t_0t_1t_0t_8t_1 = bc^{-1}t_0t_1t_8 \implies Nt_0t_1t_0t_8t_1 = Nt_0t_1t_8 \in [018]$$

$$t_0t_1t_0t_8t_2 = at_6t_5t_6t_8 \implies Nt_0t_1t_0t_8t_2 = Nt_6t_5t_6t_8 \in [0102]$$

$$t_0t_1t_0t_8t_0 = b^{-1}dt_0t_1t_0t_2 \implies Nt_0t_1t_0t_8t_0 = Nt_0t_1t_0t_2 \in [0102]$$

$$t_0t_1t_0t_8t_4 = db^{-1}t_7t_2t_5t_6 \implies Nt_0t_1t_0t_8t_4 = Nt_7t_2t_5t_6 \in [0127]$$

$$t_0t_1t_0t_8t_5 = t_0t_2t_6 \implies Nt_0t_1t_0t_8t_5 = Nt_0t_2t_6 \in [016]$$

$$t_0t_1t_0t_8t_6 = at_2t_1t_2 \implies Nt_0t_1t_0t_8t_6 = Nt_2t_1t_2 \in [050]$$

$$t_0t_1t_0t_8t_7 = abt_1t_6t_7t_8 \implies Nt_0t_1t_0t_8t_7 = Nt_1t_6t_7t_8 \in [0127]$$

$$Nt_0t_1t_0t_8t_8 \in [010].$$

$\boldsymbol{Nt_0t_1t_2t_7N}$

Consider $Nt_0t_1t_2t_7N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0127)} = N^{0127} =< e >$. Only identity (e) will fix 0,1,7, and 2. Hence the number of single cosets living in $Nt_0t_1t_2t_7N$ is $\frac{|N|}{|N^{(0127)}|} = \frac{24}{1} = 24$. The orbits of $N^{(0127)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_2t_7t_i$ belongs to. We have:

$$t_0t_1t_2t_7t_1 = at_7t_5t_0t_8 \implies Nt_0t_1t_2t_7t_1 = Nt_7t_5t_0t_8 \in [0127]$$

$$t_0t_1t_2t_7t_2 = b^{-1}ct_5t_0t_5t_7 \implies Nt_0t_1t_2t_7t_2 = Nt_5t_0t_5t_7 \in [0108]$$

$$t_0t_1t_2t_7t_0 = d^{-1}b^{-1}t_2t_4t_6t_1 \implies Nt_0t_1t_2t_7t_0 = Nt_2t_4t_6t_1 \in [0127]$$

$$t_0t_1t_2t_7t_4 = b^{-1}at_6t_8t_5 \implies Nt_0t_1t_2t_7t_4 = Nt_6t_8t_5 \in [014]$$

$$t_0t_1t_2t_7t_5 = at_2t_5t_4t_2 \implies Nt_0t_1t_2t_7t_5 = Nt_2t_5t_4t_3 \in [0127]$$

$$t_0t_1t_2t_7t_6 = cbt_4t_8t_4t_5 \implies Nt_0t_1t_2t_7t_6 = Nt_4t_8t_4t_5 \in [0108]$$

$$Nt_0t_1t_2t_7t_7 \in [012]$$

$$t_0t_1t_2t_7t_8 = bat_8t_7t_0t_5 \implies Nt_0t_1t_2t_7t_8 = Nt_8t_7t_0t_5 \in [0127].$$

$\boldsymbol{Nt_0t_1t_8t_7N}$

Consider $Nt_0t_1t_8t_7N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0187)} = N^{0187} =< e >$. Only identity (e) will fix 0, 1, 7, and 8. Hence the number of single cosets living in $Nt_0t_1t_8t_7N$ is $\frac{|N|}{|N^{(0187)}|} = \frac{24}{1} = 24$. The orbits of $N^{(0187)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_1t_8t_7t_i$ belongs to. We have:

$$t_0t_1t_8t_7t_1 = bct_8t_7t_6t_5 \implies Nt_0t_1t_8t_7t_1 = Nt_8t_7t_6t_5 \in [0187]$$

$$t_0t_1t_8t_7t_2 = b^{-1}d^{-1}t_5t_4t_5t_3 \implies Nt_0t_1t_8t_7t_2 = Nt_5t_4t_5t_3 \in [0104]$$

$$t_0t_1t_8t_7t_0 = bd^{-1}t_6t_7t_1t_2 \implies Nt_0t_1t_8t_7t_0 = Nt_6t_7t_1t_2 \in [0187]$$

$$t_0t_1t_8t_7t_4 = b^{-1}c^{-1}t_2t_4t_3t_1 \implies Nt_0t_1t_8t_7t_4 = Nt_2t_4t_3t_1 \in [0187]$$

$$t_0t_1t_8t_7t_5 = bc^{-1}t_2t_4t_0 \implies Nt_0t_1t_8t_7t_5 = Nt_2t_4t_0 \in [018]$$

$$t_0t_1t_8t_7t_6 = b^{-1}dt_6t_8t_4t_1 \implies Nt_0t_1t_8t_7t_6 = Nt_6t_8t_4t_1 \in [0187]$$

$$Nt_0t_1t_8t_7t_7 \in [018]$$

$$Nt_0t_1t_8t_7t_8 = b^{-1}c^{-1}t_1t_6t_1t_7 \implies t_0t_1t_8t_7t_8 = Nt_1t_6t_1t_7 \in [0102].$$

$Nt_0t_1t_0t_1t_0N$

Now $Nt_0t_1t_0t_1t_0N$ is indeed a new double coset. We determine how many single cosets are in this double coset. We have $N^{(01010)} = N^{01010} =< e >$. $Nt_0t_1t_0t_1t_0$ has twenty four names. We have the following:

$$Nt_0t_1t_0t_1^{(1,2,4)(5,8,7)} = Nt_0t_2t_0t_2t_0 \implies (1,2,4)(5,8,7) \in N^{(01010)}$$

$$Nt_0t_2t_0t_2^{(1,4,5,7)(2,6,8,0)} = Nt_2t_6t_2t_6t_2 \implies (1,4,5,7)(2,6,8,0) \in N^{(01010)}$$

Therefore, $N^{(01010)} = n \in N|N(01010)^n = N(01010)$.

Thus, $N^{(01010)} \geq< (1,2,4)(5,8,7), (1,4,5,7)(2,6,8,0) >$ then $N^{(01010)} = N$.

Hence $|N^{(01010)}| = 24$, so the number of single cosets in $N^{(01010)}$ is $\frac{|N|}{|N^{(01010)}|} = \frac{24}{24} = 1$. The orbit of $N^{(01010)}$ on $\{1, 2, 0, 4, 5, 6, 7, 8\}$ is $\{1, 2, 0, 4, 5, 6, 7, 8\}$. Take a representative from this orbit, say t_0. Hence $Nt_0t_1t_0t_1t_0t_0 \in [0101]$. Therefore, eight symmetric generators will go back to $Nt_0t_1t_0t_1N$.

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of N in G is 330. We conclude:

$$G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_5N \cup Nt_0t_1t_0N \cup Nt_0t_1t_2N \cup Nt_0t_1t_4N \cup$$

$Nt_0t_1t_6N \cup Nt_0t_1t_8N \cup Nt_0t_5t_0N \cup Nt_0t_1t_0t_1N \cup Nt_0t_1t_0t_2N \cup Nt_0t_1t_0t_4N \cup Nt_0t_1t_0t_8N \cup Nt_0t_1t_2t_7N \cup Nt_0t_1t_8t_7N \cup Nt_0t_1t_0t_1t_0N$, where

$$G \cong 2^{*8} : S_4/(ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3, bdtc^{-1}tcd^{-1}tdtc^{-1}t,$$
$$bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t, bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t$$

$$|G| \leq \big(|N| + \tfrac{|N|}{N^{(0)}} + \tfrac{|N|}{N^{(01)}} + \tfrac{|N|}{N^{(05)}} + \tfrac{|N|}{N^{(010)}} + \tfrac{|N|}{N^{(012)}} + \tfrac{|N|}{N^{(014)}} + \tfrac{|N|}{N^{(016)}} + \tfrac{|N|}{N^{(018)}} + \tfrac{|N|}{N^{(050)}} + \tfrac{|N|}{N^{(0101)}} + \tfrac{|N|}{N^{(0102)}} + \tfrac{|N|}{N^{(0104)}} + \tfrac{|N|}{N^{(0108)}} + \tfrac{|N|}{N^{(0127)}} + \tfrac{|N|}{N^{(0187)}} + \tfrac{|N|}{N^{(01010)}}\big) \times |N|$$

$$|G| \leq (1+8+24+24+24+24+24+24+24+24+8+24+24+24+24+24+1) \times 24$$

$$|G| \leq 330 \times 24$$

$$|G| \leq 7920.$$

A Cayley diagram that summarizes the above information is given below:

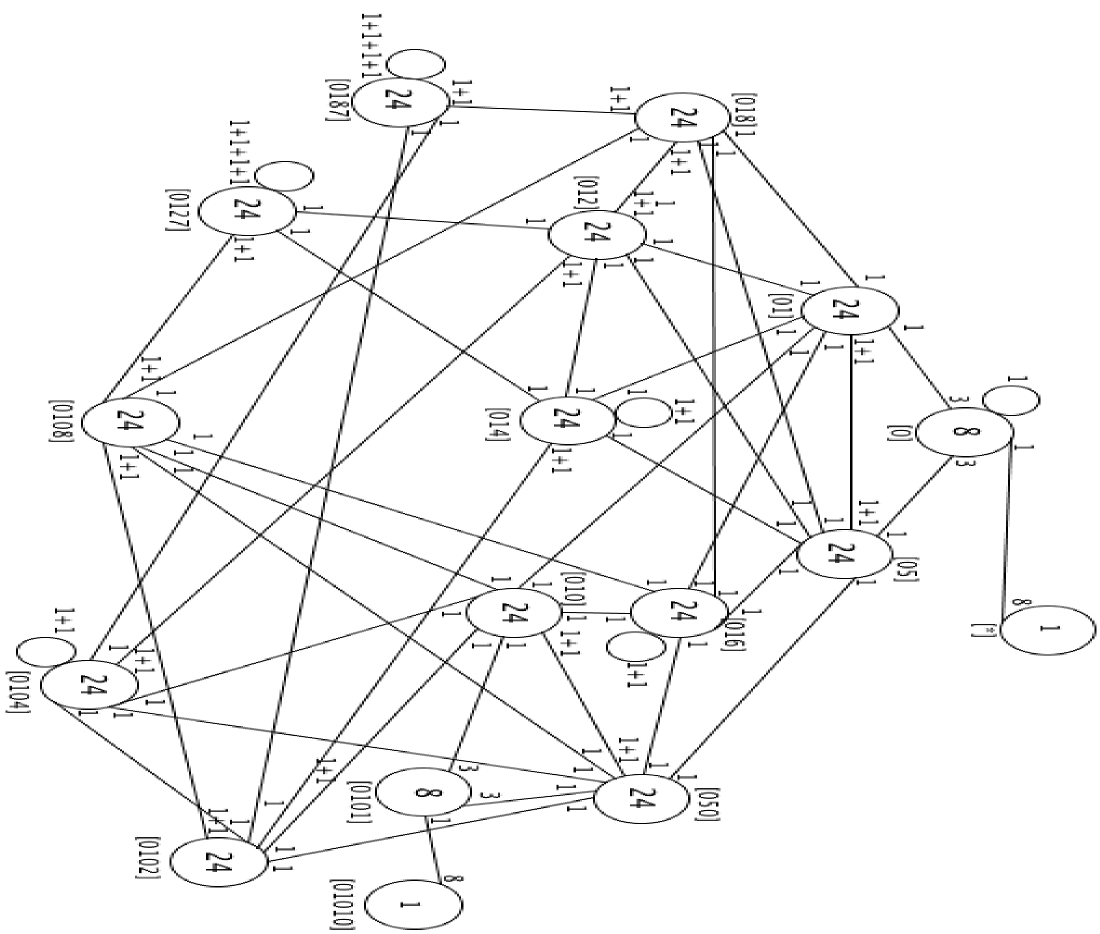

Figure 6.1: Cayley Diagram of M_{11} over S_4

Our goal is to apply *Iwasawa's lemma* to prove that $G \cong M_{11}$ over S_4 is a simple group. However, by inspection, we can see from the Cayley diagram that *Iwasawa's lemma* fails since we have imprimitive blocks of size 2. Thus, $G \cong M_{11}$ over S_4 is not a simple group.

In the next section, we will look at the maximal subgroup of $G \cong M_{11}$ and construct a Cayley diagram of M_{11} over M.

6.3 Construction of M_{11} over $M = 2^{\cdot}S_4$

We start by factoring the progenitor $2^{*8} : S_4$ by the relations
$(ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3,$
$bdtc^{-1}tcd^{-1}tdtc^{-1}t, bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t, bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t$
to obtain the homomorphic image $G \cong M_{11}$, where
$a \sim (1,5)(2,8)(3,6)(4,7)$, $b \sim (1,2,4)(5,8,7)$, $c \sim (1,4,5,7)(2,6,8,3)$,
$d \sim (1,3,5,6)(2,4,8,7)$, and $t \sim t_0 \sim t_3$. In the previous section, we expanded the above relations.

Let M be the group generated by the control group $N = S_4$ and $dt_0t_5t_0t_5t_0 = d * t * d^-1 * t * d * t * d^-1 * t * d * t$. That is,

$$M = \langle N, dt_0t_5t_0t_5t_0 \rangle, = 2^{\cdot}S_4 \text{ where } |M| = 48.$$

Then M is the maximal subgroup.

We decompose G into the double cosets MwN, where w is a word in $t_i's$, via double coset enumeration.

We proceed to do a manual double coset enumeration of G over M and N. Denote $[w]$ to be the double coset MwN, where w is a word in the $t_i's$.

MeN

We begin with the double coset MeN, denote $[*]$. This double coset contains only one single coset, namely M. The single coset stabilizer of M is N, which is transitive on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ and therefore has a single orbit,

$$\mathcal{O} = \{\{1, 2, 0, 4, 5, 6, 7, 8\}\}.$$

Take an element from $\mathcal{O}$ say t_0 and multiply the single coset representative M by it to obtain Mt_0. This is a new double coset Mt_0N, denote it $[0]$.

Mt_0N

In order to find how many single cosets [0] contains, we must first find the coset stabiliser $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now,

$$N^{(0)} = N^0 =< (1,2,4)(5,8,7) >$$

so the number of the single cosets in Mt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{24}{3} = 8$. Furthermore, the orbits of $N^{(0)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\}$ are:

$$\mathbb{O} = \{\{1,2,4\}, \{5,8,7\}, \{0\}, \{6\}\}.$$

We take t_1, t_5, t_0, and t_6 from each orbit, respectively and to see which double coset $Mt_0t_1, Mt_0t_5, Mt_0t_0$, and Mt_0t_6 belong to. We have:

$$Mt_0t_1 \in [01],$$

$$Mt_0t_5 \in [05],$$

$$Mt_0t_0 = M \in [*],$$

$$at_0t_6 = t_0 \implies Mt_0t_6 = Mt_0 \in [0].$$

The new double cosets have single coset representatives Mt_0t_1 and Mt_0t_5, which we represent them as [01] and [05] respectively.

Mt_0t_1N

Consider Mt_0t_1N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(01)} = N^{01} =< e >$. Only identity (e) will fix 0 and 1. Hence the number of single cosets living in Mt_0t_1N is $\frac{|N|}{|N^{(01)}|} = \frac{24}{1} = 24$. The orbits of $N^{(01)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_i$ belongs to. We have:

$$Mt_0t_1t_1 \in [0],$$

$$Mt_0t_1t_2 \in [012],$$

$$t_0t_1t_0 = t_0t_1t_0t_1t_0t_0t_1 \implies Mt_0t_1t_0 = Mt_0t_1t_0 \in [01],$$

$$Mt_0t_1t_4 \in [014]$$

$$t_0t_1t_5 = at_6t_1 \implies Mt_0t_1t_5 = Mt_6t_1 \in [05] = \{N(t_0t_5)^n | n \in N\},$$

$$Mt_0t_1t_6 \in [016],$$

$$t_0t_1t_7 = db^{-1}t_1t_0 \implies Nt_0t_1t_7 = Nt_1t_0 \in [05] = \{N(t_0t_5)^n|n \in N\},$$

$$Mt_0t_1t_8 \in [018].$$

The new double coset are $Mt_0t_1t_2N$, $Mt_0t_1t_4N$, $Mt_0t_1t_6N$ and $Mt_0t_1t_8N$, which we represent them as [012], [014], [016], and [018] respectively.

$\boldsymbol{Mt_0t_5N}$

Consider Mt_0t_5N is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(05)} = N^{05} =< e >$. Only identity (e) will fix 0 and 5. Hence the number of single cosets living in Mt_0t_5N is $\frac{|N|}{|N^{(05)}|} = \frac{24}{1} = 24$. The orbits of $N^{(05)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_5t_i$ belongs to. We have:

$$t_0t_5t_1 = at_6t_5 \implies Mt_0t_5t_1 = Mt_6t_5 \in [01] = \{N(t_0t_5)^n|n \in N\},$$

$$t_0t_5t_2 = bct_5t_0 \implies Mt_0t_5t_2 = Mt_5t_0 \in [01] = \{N(t_0t_5)^n|n \in N\},$$

$$t_0t_5t_0 = t_0t_5t_0t_5t_0t_0t_5 \implies Mt_0t_5t_0 = Mt_0t_5 \in [05]$$

(since $\{N(t_0t_5)^n|n \in N\}$ and $t_0t_5t_0t_5t_0 \in M$),

$$t_0t_5t_4 = d^{-1}b^{-1}t_7t_5t_6 \implies Mt_0t_5t_4 = Mt_7t_5t_6 \in [018] = \{N(t_0t_1t_8)^n|n \in N\},$$

$$Mt_0t_5t_5 = Mt_0 \in [0]$$

$$t_0t_5t_6 = at_5t_0t_1 \implies Mt_0t_5t_6 = Mt_5t_0t_1 \in [016] = \{N(t_0t_1t_6)^n|n \in N\},$$

$$t_0t_5t_7 = t_4t_1t_6 \implies Mt_0t_1t_7 = Mt_4t_1t_6 \in [012] = \{N(t_0t_1t_2)^n|n \in N\},$$

$$t_0t_5t_8 = bc^{-1}t_1t_6t_2 \implies Mt_0t_5t_8 = Mt_1t_6t_2 \in [014] = \{N(t_0t_1t_4)^n|n \in N\}.$$

$\boldsymbol{Mt_0t_1t_2N}$

Consider $Mt_0t_1t_2N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(012)} = N^{012} =< e >$. Only identity (e) will fix 0,1, and 2. Hence the number of single cosets living in $Mt_0t_1t_2N$ is $\frac{|N|}{|N^{(012)}|} = \frac{24}{1} = 24$. The orbits of $N^{(012)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_2t_i$ belongs

to. We have:

$t_0t_1t_2t_1 = t_0t_2t_0t_2t_0t_5t_0t_8 \implies Mt_0t_1t_2t_1 = Mt_0t_5t_8 \in [014]$

(since $\{N(t_0t_1t_4)^n|n \in N\}$ and $t_0t_2t_0t_2 \in M$),

$Mt_0t_1t_2t_2 \in [01]$

$t_0t_1t_2t_0 = b^{-1}dt_0t_1t_8 \implies Mt_0t_1t_2t_0 = Mt_0t_1t_8 \in [018] = \{N(t_0t_1t_8)^n|n \in N\}$,

$t_0t_1t_2t_4 = b^{-1}t_0t_2t_1 \implies Mt_0t_1t_2t_4 = Mt_0t_2t_1 \in [014] = \{N(t_0t_1t_4)^n|n \in N\}$,

$t_0t_1t_2t_5 = t_0t_1t_0t_1t_0t_2t_5t_6 \implies Mt_0t_1t_2t_5 = Mt_2t_5t_6 \in [014]$

(since $\{N(t_0t_1t_4)^n|n \in N\}$ and $t_0t_1t_0t_1t_0 \in M$),

$t_0t_1t_2t_6 = t_8t_5 \implies Mt_0t_1t_2t_6 = Mt_8t_5 \in [05] == \{N(t_0t_5)^n|n \in N\}$,

$Mt_0t_1t_2t_7 \in [0127]$

$t_0t_1t_2t_8 = at_0t_5t_2 \implies Mt_0t_1t_2t_8 = Mt_0t_5t_2 \in [018] = \{N(t_0t_1t_8)^n|n \in N\}$.

The new double coset is $Mt_0t_1t_2t_7N$, denoted by [0127].

$\boldsymbol{Mt_0t_1t_4N}$

Consider $Mt_0t_1t_4N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(014)} = N^{014} =< e >$. Only identity (e) will fix 0,1, and 4. Hence the number of single cosets living in $Mt_0t_1t_4N$ is $\frac{|N|}{|N^{(014)}|} = \frac{24}{1} = 24$. The orbits of $N^{(014)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_4t_i$ belongs to. We have:

$t_0t_1t_4t_1 = ct_0t_8t_0t_8t_0t_7t_5t_0 \implies Mt_0t_1t_4t_1 = Mt_7t_5t_0 \in [012]$

(since $\{N(t_0t_1t_2)^n|n \in N\}$ and $ct_0t_8t_0t_8t_0 \in M$),

$t_0t_1t_4t_2 = bt_0t_4t_1 \implies Mt_0t_1t_4t_2 = Mt_0t_4t_1 \in [012] = \{N(t_0t_1t_2)^n|n \in N\}$,

$t_0t_1t_4t_0 = bd^{-1}t_4t_6t_1 \implies Mt_0t_1t_4t_0 = Mt_4t_6t_1 \in [014]$

$Mt_0t_1t_4t_4 \in [01]$

$t_0t_1t_4t_5 = db^{-1}t_5t_4t_3 \implies Mt_0t_1t_4t_5 = Mt_5t_4t_3 \in [014] = \{N(t_0t_1t_4)^n|n \in N\}$,

$$t_0t_1t_4t_6 = ct_0t_2t_0t_2t_0t_1t_6t_7 \implies Mt_0t_1t_4t_6 = Mt_1t_6t_7 \in [012]$$

(since $\{N(t_0t_1t_2)^n | n \in N\}$ and $ct_0t_2t_0t_2t_0 \in M$),

$$t_0t_1t_4t_7 = d^{-1}b^{-1}t_5t_6 \implies Mt_0t_1t_4t_7 = Mt_5t_6 \in [05] = \{N(t_0t_5)^n | n \in N\},$$

$$t_0t_1t_4t_8 = bat_6t_7t_5t_2 \implies Mt_0t_1t_4t_8 = Mt_6t_7t_5t_2 \in [0127] = \{N(t_0t_1t_2t_7)^n | n \in N\}.$$

$\boldsymbol{Mt_0t_1t_6N}$

Consider $Mt_0t_1t_6N$ is a new double coset. We determine how many single cosets are in the double coset. However,

$$Mt_0t_1t_6 = Mt_6t_5t_0$$

Then $N(t_0t_1t_0)^{(1,5)(2,8)(0,6)} = Nt_6t_5t_1$. But $Nt_6t_5t_1 = Nt_0t_1t_6 \implies (1,5)(2,8)(0,6) \in N^{(015)}$ since $N(t_0t_1t_5)^{(0,6)(1,5)(2,4)} = Nt_6t_5t_1$

$$\implies N^{(015)} \geq \langle (1,5)(2,8)(0,6) \rangle.$$

Hence the number of single cosets living in $Mt_0t_1t_6N$ is $\frac{|N|}{|N^{(016)}|} = \frac{24}{2} = 24$. The orbits of $N^{(016)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1,5\}, \{2,8\}, \{0,6\}, \{4,7\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_6t_i$ belongs to. We have:

$$t_0t_1t_6t_1 = t_0t_1t_0t_1t_0t_5t_6 \implies Mt_0t_1t_6t_1 = Mt_5t_6 \in [05]$$

(since $\{N(t_0t_5)^n | n \in N\}$ and $t_0t_1t_0t_1t_0 \in M$),

$$t_0t_1t_6t_2 = t_0t_8t_0t_8t_0t_0t_6t_1t_6 \implies Mt_0t_1t_6t_2 = Mt_0t_1t_6 \in [016]$$

(since $\{N(t_0t_1t_6)^n | n \in N\}$ and $t_0t_8t_0t_8t_0 \in M$),

$$t_0t_1t_6t_4 = bd^{-1}t_6t_7t_1 \implies Mt_0t_1t_6t_5 = t_6t_7t_1 \in [018] = \{N(t_0t_1t_8)^n | n \in N\},$$

$$Mt_0t_1t_6t_6 \in [01].$$

$\boldsymbol{Mt_0t_1t_8N}$

Consider $Mt_0t_1t_8N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(018)} = N^{018} =< e >$. Only identity (e) will fix 0,1, and 8. Hence the number of single cosets living in $Mt_0t_1t_8N$ is $\frac{|N|}{|N^{(018)}|} = \frac{24}{1} = 24$. The orbits of $N^{(018)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_8t_i$ belongs

to. We have:

$t_0t_1t_8t_1 = bct_0t_2t_0t_2t_0t_0t_1t_8 \implies Mt_0t_1t_8t_1 = Mt_0t_1t_0t_8 \in [018]$

(since $\{N(t_0t_1t_8)^n | n \in N\}$ and $bct_0t_2t_0t_2t_0 \in M$),

$t_0t_1t_8t_2 = at_6t_5t_8 \implies Mt_0t_1t_8t_2 = Mt_6t_5t_8 \in [012] = \{N(t_0t_1t_2)^n | n \in N\}$,

$t_0t_1t_8t_0 = bct_0t_1t_2 \implies Mt_0t_1t_8t_0 = Mt_0t_1t_2 \in [012] = \{N(t_0t_1t_2)^n | n \in N\}$,

$t_0t_1t_8t_4 = ct_0t_2t_0t_2t_0t_7t_2t_5t_6 \implies Mt_0t_1t_8t_4 = Mt_7t_2t_5t_6 \in [0127]$

(since $\{N(t_0t_1t_2t_7)^n | n \in N\}$ and $ct_0t_2t_0t_2t_0 \in M$),

$t_0t_1t_8t_5 = t_0t_1t_0t_1t_0t_0t_2t_6 \implies Mt_0t_1t_8t_5 = Mt_0t_2t_6 \in [016]$

(since $\{N(t_0t_1t_6)^n | n \in N\}$ and $t_0t_1t_0t_1t_0 \in M$),

$t_0t_1t_8t_6 = bc^{-1}t_2t_1 \implies Mt_0t_1t_8t_6 = Mt_2t_1 \in [05] = \{N(t_0t_5)^n | n \in N\}$,

$t_0t_1t_8t_7 = bt_0t_8t_0t_8t_0t_1t_6t_7t_8 \implies Mt_0t_1t_8t_7 = Mt_1t_6t_7t_8 \in [0127]$

(since $\{N(t_0t_1t_2t_7)^n | n \in N\}$ and $bt_0t_8t_0t_8t_0 \in M$),

$Mt_0t_1t_8t_8 \in [01]$.

$\boldsymbol{Mt_0t_1t_2t_7N}$

Consider $Mt_0t_1t_2t_7N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0127)} = N^{0127} =< e >$. Only identity (e) will fix 0,1,7, and 2. Hence the number of single cosets living in $Mt_0t_1t_2t_7N$ is $\frac{|N|}{|N^{(0127)}|} = \frac{24}{1} = 24$. The orbits of $N^{(0127)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative t_i from each orbit and see which double cosets $Mt_0t_1t_2t_7t_i$ belongs to. We have:

$t_0t_1t_2t_7t_1 = at_7t_5t_0t_8 \implies Mt_0t_1t_2t_7t_1 = Mt_7t_5t_0t_8 \in [0127] = \{N(t_0t_1t_2t_7)^n | n \in N\}$,

$t_0t_1t_2t_7t_2 = bt_0t_2t_0t_2t_0t_5t_0t_7 \implies Mt_0t_1t_2t_7t_2 = Mt_5t_0t_7 \in [018]$

(since $\{N(t_0t_1t_8)^n | n \in N\}$ and $bt_0t_2t_0t_2t_0 \in M$),

$t_0t_1t_2t_7t_0 = d^{-1}b^{-1}t_2t_4t_6t_1 \implies Mt_0t_1t_2t_7t_0 = Mt_2t_4t_6t_1 \in [0127]$

$t_0t_1t_2t_7t_4 = b^{-1}at_6t_8t_5 \implies Mt_0t_1t_2t_7t_4 = Mt_6t_8t_5 \in [014] = \{N(t_0t_1t_4)^n | n \in N\}$,

$t_0t_1t_2t_7t_5 = at_2t_5t_4t_2 \implies Mt_0t_1t_2t_7t_5 = Mt_2t_5t_4t_3 \in [0127] = \{N(t_0t_1t_2t_7)^n | n \in N\}$,

$t_0t_1t_2t_7t_6 = t_0t_8t_0t_8t_0t_4t_8t_5 \implies Mt_0t_1t_2t_7t_6 = Mt_4t_8t_5 \in [018]$

(since $\{N(t_0t_1t_8)^n | n \in N\}$ and $t_0t_8t_0t_8t_0 \in M$),

$Mt_0t_1t_2t_7t_7 \in [012]$

$t_0t_1t_2t_7t_8 = bat_8t_7t_0t_5 \implies Mt_0t_1t_2t_7t_8 = Mt_8t_7t_0t_5 \in [0127] = \{N(t_0t_1t_2t_7)^n | n \in N\}$.

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of M in G is 165. We conclude:

$G = MeN \cup Mt_0N \cup Mt_0t_1N \cup Mt_0t_5N \cup Mt_0t_1t_2N \cup Mt_0t_1t_4N \cup Mt_0t_1t_6N \cup Mt_0t_1t_8N \cup Mt_0t_1t_2t_7N$, where

$$G \cong 2^{*8} : S_4/(ct^c)^6, (abt^{cb^{-1}})^6, (ab^{-1}t)^3, bdtc^{-1}tcd^{-1}tdtc^{-1}t,$$
$$bdtdtctd^{-1}tb^{-1}ctctdtc^{-1}t, bctctd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t$$

$|G| \leq |N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(05)}} + \frac{|N|}{N^{(012)}} + \frac{|N|}{N^{(014)}} + \frac{|N|}{N^{(016)}} + \frac{|N|}{N^{(018)}} + \frac{|N|}{N^{(0127)}} \times |M|$

$|G| \leq (1 + 8 + 24 + 24 + 24 + 12 + 24 + 24) \times 28$

$|G| \leq 165 \times 48$

$|G| \leq 7920.$

A Cayley diagram that summarizes the above information is given below:

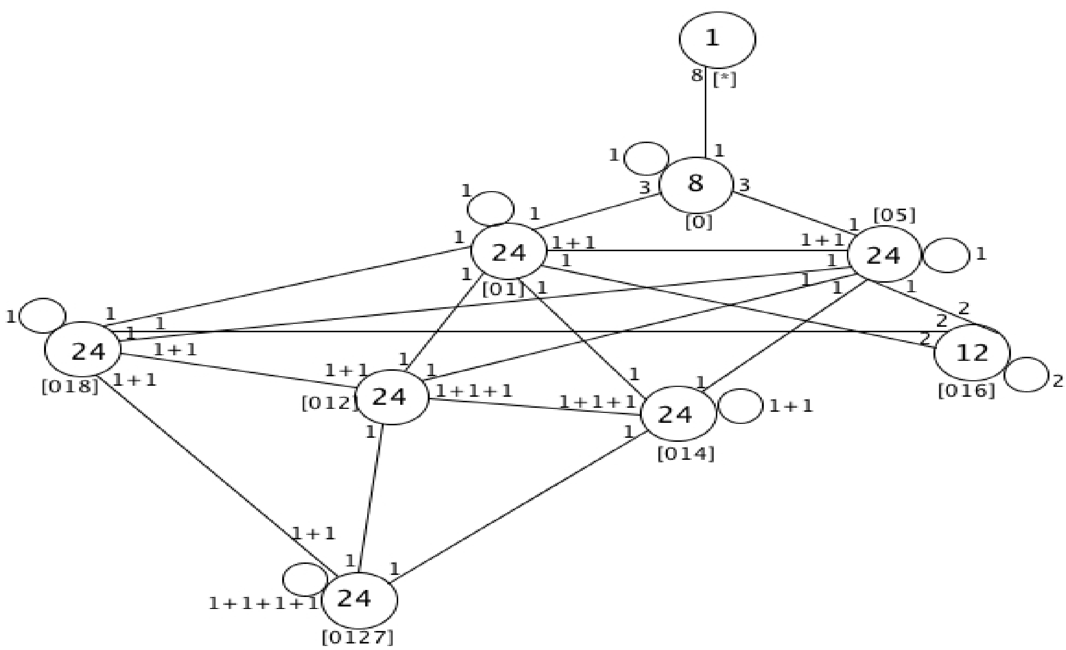

Figure 6.2: Cayley Diagram of M_{11} over $2^{\cdot}S_4$

6.4 Iwasawa's Lemma to Prove M_{11} over $M = 2^{\cdot}S_4$ is Simple

Again, we use *Iwasawa's lemma* and the transitive action of G on the set of single cosets to prove $G \cong M_{11}$ over $M = 2^{\cdot}S_4$ is a simple group. *Iwasawa's lemma* has three sufficient conditions that we must satisfied:

(1) G acts on X faithfully and primitively

(2) G is perfect $(G = G')$

(3) There exist $x \in X$ and a normal abelian subgroup K of G^x such that the conjugates of K generate G.

Proof. **6.4.1 $G = M_{11}$ over $M = 2^{\cdot}S_4$ acts on X Faithfully**

Let G acts on $X = M, Mt_0N, Mt_0t_1N, Mt_0t_5N, Mt_0t_1t_2N, Mt_0t_1t_4, Mt_0t_1t_6, Mt_0t_1t_8,$ $Mt_0t_1t_2t_7N$, where X is a transitive G-set of degree 165. G acts on X implies there exist homomorphism

$$f\ : G \longrightarrow S_{165} \qquad\qquad (|X| = 165).$$

By First Isomorphic Theorem we have:

$$\mathcal{G}/kerf \cong f(\mathcal{G}).$$

If $kerf = 1$ then $G \cong f(G)$. Only elements of N fix N implies $G^1 = N$. Since X is transitive $G - set$ of degree 165, we have:

$$\begin{aligned} |G| &= 165 \times |G^1| \\ &= 165 \times |M| \\ &= 165 \times 48 \\ &= 7920 \\ \implies |G| &= 7920. \end{aligned}$$

From Cayley diagram, $|G| \leq 7920$. However, from above $|G| = 7920$ implying $ker(f) = 1$. Since $kerf = 1$ then G acts faithfully on X.

6.4.2 $G = M_{11}$ over $M = 2^{\cdot}S_4$ acts on X Primitively

Since $G = M_{11}$ is transitive on $|X| = 165$, if B is a nontrivial block then we may assume that $M \in B$. However, $|B|$ must divide $|X| = 165$. The only nontrivial blocks must be of size 3, 5, 11, 15, 33, or 55, since $|B|$ must divide $|X|$.
Case (1): If $Mt_0 \in B$ then $B = \{M, Mt_0\} = \{M, Mt_0N\}$ (since $N \in B, BM = B$)
$\implies B = \{M, Mt_0, Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7, Mt_8\}$
$Bt_1 = \{M, Mt_0t_1, M, Mt_2t_1, Mt_3t_1, Mt_4t_1, Mt_5t_1, Mt_6t_1, Mt_7t_1, Mt_8t_1\}$
$\implies M \in B \cap Bt_1$.
Now $B = \{M, Mt_0N, Mt_0t_1N, Mt_0t_5N\}$, where $|B| = 57$ (passed all possible nontrivial blocks).
Note if $Bt_0 \in B$ then $B = X$.
Case (2): Consider $B = \{M, Mt_0N, Mt_0t_1N\}$, where $|B| = 33$ but if we have $\{M, Mt_0N\}$ we are going to have the entire group $B = X$. Thus, G acts primitively on X.

6.4.3 $G = M_{11}$ over $M = 2^{\cdot}S_4$ is Perfect

Next we want to show that $G = G'$. Since $G =< N, t >$, we have that $N \leq G'$. Now $S_4 \leq G \implies {S_4}' \leq G'$. The commutators subgroup of S_4 is

$${S_4}' =< [a, b]|a, b \in S_4 >=< a, b, c, d >\leq G'.$$

Now by expanding the main relations we get the following:

$$a = t_0t_6t_0$$
$$d = t_0t_8t_4t_1t_7t_0t_1t_8$$
$$c = t_4t_2t_1t_0t_7t_6t_8t_0$$
$$bdc^{-1}t_8t_0t_4t_8t_0 = 1$$

Now we use the above relation and we solve for b by replacing $d = t_0t_8t_4t_1t_7t_0t_1t_8$ and $c^{-1} = t_0t_8t_6t_7t_0t_1t_2t_4$:

$$bdc^{-1}t_8t_0t_4t_8t_0 = 1$$
$$\implies b = t_0t_8t_4t_0t_8t_4t_2t_1t_0t_7t_6t_8t_0t_8t_1t_0t_7t_1t_4t_8t_0$$

So, $G =< a, b, c, d, t >=< t_0, t_1, t_2, t_3, t_5, t_6, t_7, t_8 >$. Our goal is to show that one of the $t_i's \in G'$, then we can conjugate by $< a, b, c, d >$ to obtain all of the $t_i's$ in G'. Since

$a \in G'$. Then

$$a = t_0 t_6 t_0 \in G'$$
$$= t_0 t_6 t_0 t_6 t_6 \in G' \text{ (since } |t_i's|\text{=2)}$$
$$= [t_0, t_6] t_6 \in G'$$
$$\implies t_6 in G'$$

So $t_6 \in G' \implies t_6^c, t_6^{c^2}, t_6^{c^3} \in G'$ (since $c, c^2, c^3 \in G$ and $G' \trianglelefteq G$) also
$t_6^{ad}, t_6^{bd}, t_6^{dc^3}, t_6^{c^3 d} \in G'$ (since $ad, bd, dc^3, c^3 d \in G$ and $G' \trianglelefteq G$)
$\implies G' =< t_6, t_8, t_0, t_2, t_5, t_1, t_7, t_4 >$.
Thus, $G \geq G' \geq < t_0, t_1, t_2 t_4, t_5 t_6 t_7 t_8 >= G$. We conclude that $G' = G$ and G is perfect.

6.4.4 Conjugates of a Normal Abelian K Generate $G = M_{11}$ over $M = 2^{\cdot} S_4$

Now we require $x \in X$ and a normal abelian subgroup K of G^x,-the point stabilizer of x in G, such that the conjugates of K in G generate G.
Now $G^1 = M = 2^{\cdot} S_4$ possesses a normal abelian subgroup $K =< a >$. We use the same relation, as we did in the previous part:

$$a = t_0 t_6 t_0 \in K$$
$$\implies a^{t_0} = (t_0 t_6 t_0)^{t_0} \in K^G$$
$$\implies t_0 a t_0 = t_0 t_0 t_6 t_0 t_0 \in K$$
$$\implies a t_6 t_0 = t_6 \in K$$

So $t_6^G \in K^G$
$\implies K^G \geq \{t_6, t_6^c, t_6^{c^2}, t_6^{c^3}, t_6^{ad}, t_6^{bd}, t_6^{dc^3}, t_6^{c^3 d}\}$
$\implies K^G \geq \{t_6, t_8, t_0, t_2, t_5, t_1, t_7, t_4\} = G$
Hence, the conjugates of K generate G. Therefore, by *Iwasawa's lemma*, $G \cong M_{11}$ is a simple group.

□

Chapter 7

Double Coset Enumeration of M_{12} over $(3^2 : 2^{\cdot}S_4)$

7.1 Factoring by the Center $(Z(G))$ of $2^{*72} : (3^2 : 2^{\cdot}S_4)$

Consider the group $G = 2^{*72} : (3^2 : 2^{\cdot}S_4)$ factored by the relator $[ac^{-1}b^{-1}cb^2t_2]^3$.
Note: $N = (3^2 : 2^{\cdot}S_4) =< a, b, c >$ and $|N| = 432$, where
$a \sim (2, 8)(3, 15)(4, 20) \ldots (59, 65)(60, 64)(68, 72)$,
$b \sim (1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23) \ldots (33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64)$,
and $c \sim (1, 3, 5, 15)(2, 10, 12, 42) \ldots (48, 58, 65, 72)(49, 63, 68, 59)$.
Let $t \sim t_1 \sim t_0$.
Now we look at the composition factors of G given below:

```
G
|  M12
*
|  Cyclic(2)
1
```

Thus, $G \cong 2 \times M12$. Now, we use Magma to factor the group by the center $Z(G)$ and we get that $Z(G) =< ab^3ctbtb^{-1}tbtc >$.
Hence,

$$G = \frac{2^{*72}:(3^2:2^{\cdot}S_4)}{[ac^{-1}b^{-1}cb^2t_2]^3, ab^3ctbtb^{-1}tbtc} \cong M_{12}.$$

7.2 Construction of M_{12} over $(3^2 : 2^{\cdot}S_4)$

Now consider the group

$$G = \frac{2^{*72}:(3^2:2^{\cdot}S_4)}{[ac^{-1}b^{-1}cb^2t_2]^3, ab^3ctbtb^{-1}tbtc.}$$

Note: $N = (3^2 : 2^{\cdot}S_4) =< a, b, c >$ and $|N| = 432$, where

$a \sim (2, 8)(3, 15)(4, 20) \ldots (59, 65)(60, 64)(68, 72)$,

$b \sim (1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23) \ldots (33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64)$,

and $c \sim (1, 3, 5, 15)(2, 10, 12, 42) \ldots (48, 58, 65, 72)(49, 63, 68, 59)$.

Let $t \sim t_1 \sim t_0$.

Let us expand the relations:

$$[ac^{-1}b^{-1}cb^2t_2]^3 = 1 \text{ with } \pi = ac^{-1}b^{-1}cb^2 \text{ becomes}$$

$$1 = [\pi t_2]^3 = \pi^3 t_2^{\pi^2} t_2^{\pi} t_2 = acbt_{23}t_{54}t_2$$

$$\Longrightarrow 1 = acbt_{23}t_{54}t_2$$

$$\Longrightarrow acbt_{23}t_{54} = t_2,$$

$$1 = ab^3ctbtb^{-1}tbtc = ab^3cbct_{10}t_3t_{10}t_3$$

$$\Longrightarrow ab^3cbt_{10}t_3 = t_3t_{10}.$$

We want to find the index of N in G. To do this, we perform a manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$ where g_i's words in t_i's.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [Nw^n|n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by t_i's. We need to identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilising group $N^{(w)}$

NeN

First, the double coset NeN, is denoted by $[*]$. This double coset contains only the single coset, namely N. Since N is transitive on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$, the orbit of N on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ is:

$$\mathbb{O} = \{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$$

We choose t_0 as our symmetric generator from $\mathbb{O}$ and find to which double coset Nt_0 belongs. Nt_0N will be a new double coset, denoted by [0]. Hence, 72 symmetric generators will go the new double coset [0].

Nt_0N

In order to find how many single cosets [0] contains, we must first find the coset stabilizer $N^{(0)}$. Then the number of single coset in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 =< a, acb^{-1}c^{-1}b^{-1}c >$ so the number of the single cosets in Nt_0N is $\frac{|N|}{|N^{(0)}|} = \frac{43}{6} = 72$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ are:
$\mathbb{O} = \{0\}, \{7\}, \{35\}, \{2, 8, 34\}, \{5, 28, 14\}, \{13, 39, 31\}, \{3, 15, 70, 66, 46, 27\}$, $\{4, 20, 11, 71, 44, 52\}, \{6, 9, 32, 12, 26, 37\}, \{10, 19, 72, 57, 68, 24\}, \{16, 30, 67, 51, 50, 41\}$, $\{17, 38, 29, 36, 61, 40\}, \{18, 54, 60, 53, 64, 47\}, \{21, 23, 55, 49, 33, 58\}, \{25, 45, 48, 62, 63, 43\}$, and $\{22, 23, 55, 49, 33, 58\}$.
Take a representative t_i from each orbit and see which double cosets Nt_0t_i belongs to. We have:

$$Nt_0t_0 = N \in [*]$$

$$t_0t_7 = t_{35} \implies Nt_0t_7 = Nt_{35} \in [0] = \{Nt_0^n | n \in N\}$$

$$t_0t_{35} = t_7 \implies Nt_0t_{35} = Nt_7 \in [0] = \{Nt_0^n | n \in N\}$$

$$Nt_0t_2 \in [02]$$

$$Nt_0t_5 \in [05]$$

$$Nt_0t_{13} \in [013]$$

$$t_0t_3 = c^{-1}b^{-1}t_2t_6 \implies Nt_0t_3 = Nt_2t_6 \in [013] = \{N(t_0t_{13})^n | n \in N\}$$

$$t_0t_4 = bcbt_{14}t_{19} \implies Nt_0t_4 = Nt_{14}t_{19} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_6 = abt_{18}t_{55} \implies Nt_0t_6 = Nt_{18}t_{55} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_{10} = b^3t_{15}t_{62} \implies Nt_0t_{10} = Nt_{15}t_{62} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_{16} = ac^{-1}b^{-1}cb^2ct_{28}t_{37} \implies Nt_0t_{16} = Nt_{28}t_{37} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_{17} = abcbt_{15}t_{29} \implies Nt_0t_{17} = Nt_{15}t_{29} \in [013] = \{N(t_0t_{13})^n | n \in N\}$$

$$t_0t_{18} = ab^{-1}c^{-1}b^{-1}cbt_{59}t_{41} \implies Nt_0t_{18} = Nt_{59}t_{41} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_{21} = ab^{-1}c^{-1}b^{-1}cbt_{17} \implies Nt_0t_{21} = Nt_{17} \in [0] = \{N(t_0)^n | n \in N\}$$

$$t_0t_{25} = ac^{-1}bcbt_{53}t_{58} \implies Nt_0t_{25} = Nt_{53}t_{58} \in [05] = \{N(t_0t_5)^n | n \in N\}$$

$$t_0t_{22} = ab^{-1}c^{-1}bcbt_{36} \implies Nt_0t_{22} = Nt_{36} \in [0] = \{N(t_0)^n | n \in N\}.$$

The new double cosets have single coset representatives Nt_0t_2, Nt_0t_5, and Mt_0t_{13}, we represent them as [02], [05], and [013], respectively.

$\boldsymbol{Nt_0t_2N}$

Continuing with the double coset Nt_0t_2N, we find the point stabilizer N^{02}. This is $N^{02} =< acb^{-1}c^{-1}b^{-1}c >$. Also, with the relation $t_0t_2 = abcb^{-2}cbt_{16}t_{40} \implies Nt_0t_2 = Nt_{16}t_{40}$. Then $N(t_0t_2)^{(0,16,6,17,32,57)(2,40,13,44,39,66)\cdots=abcb^{-2}} = Nt_{16}Nt_{40}$. But $Nt_{16}t_{40} = Nt_0t_2 \implies abcb^{-2} \in N^{(02)}$ since $N(t_0t_2)^{abcb^{-2}} = Nt_{16}t_{40}$. Thus the coset stabiliser is

$$N^{(02)} \geq< acb^{-1}c^{-1}b^{-1}c, abcb^{-2} > .$$

Since $\left|N^{(02)}\right| = 36$, the number of single cosets in [02] is $\frac{\left|N\right|}{\left|N^{(02)}\right|} = \frac{432}{36} = 12$.

$\mathbb{O} = \{0, 16, 67, 52, 6, 44, 39, 19, 32, 57, 17, 13, 66, 61, 29, 15, 2, 40\}$,

$\{3, 70, 26, 10, 37, 60, 45, 72, 28, 18, 43, 7, 62, 55, 5, 22, 63, 31\}$,

$\{4, 11, 36, 30, 38, 65, 47, 51, 27, 24, 69, 68, 25, 64, 49, 46, 23, 48\}$,

$\{8, 34, 53, 41, 54, 71, 50, 33, 14, 20, 35, 21, 58, 12, 56, 42, 9, 59.\}$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_2t_i$ belongs to. We have:

$Nt_0t_2t_2 = Nt_0 \in [0]$

$t_0t_2t_3 = cb^{-1}c^{-1}b^{-1}c^{-1}b^{-1}t_{61}t_{33} \implies Nt_0t_2t_3 = Nt_{61}t_{33} \in [05] = \{N(t_0t_5)^n | n \in N\}$

$t_0t_2t_4 = cb^{-2}cbct_{28}t_{37} \implies Nt_0t_2t_4 = Nt_{28}t_{37} \in [05] = \{N(t_0t_5)^n | n \in N\}$

$t_0t_2t_8 = c^{-1}bcbt_{71}t_{43} \implies Nt_0t_2t_8 = Nt_{71}t_{43} \in [013] = \{N(t_0t_{13})^n | n \in N\}$

$\boldsymbol{Nt_0t_5N}$

Continuing with the double coset Nt_0t_5N, we find the point stabilizer N^{05}. This is $N^{05} =< a >$. Also, with the relation $t_0t_5 = at_5t_0 \implies Nt_0t_5 = Nt_5t_0$.

Then $N(t_0t_5)^{(0,5)(2,12)(3,15)\cdots=c^2} = Nt_5Nt_0$. But $Nt_5t_0 = Nt_0t_5 \implies c^2 \in N^{(02)}$ since $N(t_0t_5)^{c^2} = Nt_5t_0$. Thus the coset stabiliser is

$$N^{(05)} \geq< a, c^2 > .$$

Since $\left|N^{(05)}\right| = 4$, the number of single cosets in [05] is $\frac{\left|N\right|}{\left|N^{(05)}\right|} = \frac{432}{4} = 108$. The orbits of $N^{(05)}$ on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ are:

$\mathbb{O} = \{0, 5\}, \{3, 15\}, \{7, 34\}, \{13, 35\}, \{33, 55\}, \{50, 67\}, \{2, 8, 12, 37\}, \{4, 20, 23, 22\}$,

$\{6, 9, 31, 39\}, \{10, 19, 42, 21\}, \{11, 44, 46, 70\}, \{14, 28, 26, 32\}, \{16, 30, 53, 47\}, \{17, 38, 56, 69\}$,

$\{18, 54, 61, 29\}, \{24, 57, 51, 41\}, \{25, 45, 36, 40\}, \{27, 66, 64, 60\}, \{43, 62, 71, 52\}, \{48, 63, 65, 59\}, \{49, 58, 68, 72\}$.

Take a representative t_i from each orbit and see which double cosets $Nt_0t_5t_i$ belongs to. We have:

$$Nt_0t_5t_5 = Nt_0 \in [0]$$

$$\begin{aligned}
t_0t_5t_3 = cb^{-2}t_{63}t_{60} &\implies Nt_0t_5t_3 = Nt_{63}t_{60} \in [02] = \{N(t_0t_2)^n | n \in N\} \\
t_0t_5t_7 = abc^2b^{-1}t_{72}t_{17} &\implies Nt_0t_5t_7 = Nt_{72}t_{17} \in [013] = \{N(t_0t_{13})^n | n \in N\} \\
t_0t_5t_{13} = ac^{-1}b^3c^{-1}t_{68}t_{46} &\implies Nt_0t_5t_{13} = Nt_{68}t_{46} \in [02] = \{N(t_0t_2)^n | n \in N\} \\
t_0t_5t_{33} = ac^2b^{-1}c^{-1}t_2t_{37} &\implies Nt_0t_5t_{33} = Nt_2t_{37} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{50} = b^2cb^{-1}c^{-1}t_{37}t_2 &\implies Nt_0t_5t_{50} = Nt_{37}t_2 \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_2 = acb^{-2}t_{44}t_{59} &\implies Nt_0t_5t_2 = Nt_{44}t_{59} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_4 = bcb^2t_{72}t_{57} &\implies Nt_0t_5t_4 = Nt_{72}t_{57} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_6 = ab^{-1}cbc^2t_{24}t_{68} &\implies Nt_0t_5t_6 = Nt_{24}t_{68} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{10} = b^3cb^{-1}t_{20}t_{53} &\implies Nt_0t_5t_{10} = Nt_{20}t_{53} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{11} = ac^{-1}bcb^{-2}t_{63} &\implies Nt_0t_5t_{11} = Nt_{63} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{14} = ac^{-1}b^3ct_{60} &\implies Nt_0t_5t_{14} = Nt_{60} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{16} = acb^3c^{-1}t_{28} &\implies Nt_0t_5t_{16} = Nt_{28} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{17} = cb^{-1}c^{-1}b^{-1}c^{-1}t_{71}t_{21} &\implies Nt_0t_5t_{17} = Nt_{71}t_{21} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{18} = ab^2cb^{-1}c^{-1}t_{49} &\implies Nt_0t_5t_{18} = Nt_{49} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{24} = acb^{-2}t_{42}t_{50} &\implies Nt_0t_5t_{24} = Nt_{42}t_{50} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{25} = ab^{-1}cb^{-1}t_{64}t_{23} &\implies Nt_0t_5t_{25} = Nt_{64}t_{23} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{27} = acb^2ct_{48} &\implies Nt_0t_5t_{27} = Nt_{48} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{43} = b^2c^{-1}b^{-1}c^{-1}t_{50}t_{42} &\implies Nt_0t_5t_{25} = Nt_{64}t_{23} \in [05] = \{N(t_0t_5)^n | n \in N\} \\
t_0t_5t_{48} = b^ct_{14} &\implies Nt_0t_5t_{48} = Nt_{14} \in [0] = \{N(t_0)^n | n \in N\} \\
t_0t_5t_{49} = ac^{-1}b^{-1}c^{-1}b^2t_{54}t_{42} &\implies Nt_0t_5t_{49} = Nt_{54}t_{42} \in [013] = \{N(t_0t_{13})^n | n \in N.\}
\end{aligned}$$

$\boldsymbol{Nt_0t_{13}N}$

Continuing with the double coset $Nt_0t_{13}N$, we find the point stabilizer N^{013}. This is

$N^{013} =< a >$. Also, with the relation $t_0t_{13} = b^3t_{69}t_{49} \implies Nt_0t_{13} = Nt_{69}t_{49}$. Then $N(t_0t_{13})^{(1,69,13,49)(2,38,6,27)\cdots=b^{-1}cbc} = Nt_{69}Nt_{49}$. But $Nt_{69}t_{49} = Nt_0t_{13} \implies b^{-1}cbc \in N^{(013)}$ since $N(t_0t_{13})^{b^{-1}cbc} = Nt_{69}t_{49}$. Thus the coset stabiliser is

$$N^{(013)} \geq < a, b^{-1}cbc > .$$

Since $\left|N^{(013)}\right| = 16$, the number of single cosets in [013] is $\frac{\left|N\right|}{\left|N^{(013)}\right|} = \frac{432}{16} = 27$.

The orbits of $N^{(013)}$ on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ are:

$\mathbb{O} = \{1, 69, 56, 13, 61, 49, 29, 58\}, \{5, 30, 16, 7, 62, 36, 43, 40\}, \{21, 42, 70, 63, 46, 35, 48, 34\},$
$\{2, 8, 38, 24, 17, 6, 67, 57, 9, 71, 27, 11, 50, 66, 52, 44\},$
$\{3, 15, 22, 60, 23, 10, 31, 64, 19, 32, 45, 72, 39, 25, 26, 68\},$
$\{4, 20, 37, 18, 12, 47, 51, 54, 53, 59, 28, 55, 41, 14, 65, 33\}.$

Take a representative t_i from each orbit and see which double cosets $Nt_0t_{13}t_i$ belongs to. We have:

$Nt_0t_{13}t_{13} = Nt_0 \in [0]$

$t_0t_{13}t_7 = cb^3ct_{66}t_{59} \implies Nt_0t_{13}t_7 = Nt_{66}t_{59} \in [02] = \{N(t_0t_2)^n | n \in N\}$

$t_0t_{13}t_{21} = t_{29}t_{43} \implies Nt_0t_{13}t_{21} = Nt_{29}t_{43} \in [05] = \{N(t_0t_5)^n | n \in N\}$

$t_0t_{13}t_2 = ab^{-1}cbcb^{-1}t_{44} \implies Nt_0t_{13}t_2 = Nt_{44} \in [0] = \{N(t_0)^n | n \in N\}$

$t_0t_{13}t_3 = c^{-1}b^2t_{27} \implies Nt_0t_{13}t_3 = Nt_{27} \in [0] = \{N(t_0)^n | n \in N\}$

$t_0t_{13}t_4 = bcb^{-1}c^{-1}b^{-1}c^{-1}t_{62}t_{15} \implies Nt_0t_{13}t_4 = Nt_{62}t_{15} \in [05] = \{N(t_0t_5)^n | n \in N\}$.

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of N in G is 220. We conclude:

$G = N \cup Nt_0N \cup Nt_0t_2N \cup Nt_0t_5N \cup Nt_0t_{13}N$, where

$$G = \frac{2^{*72} : (3^2 : 2^{\cdot} S_4)}{[ac^{-1}b^{-1}cb^2t_2]^3, ab^3ctbtb^{-1}tbtc.}$$

$|G| \leq \left(|N| + \frac{|N|}{N^{(0)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(05)}} + \frac{|N|}{N^{(013)}}\right) \times |N|$

$|G| \leq (1 + 72 + 12 + 108 + 27) \times 432$

$|G| \leq 220 \times 432$

$|G| \leq 95040.$

A Cayley diagram that summarizes the above information is given on the next page.

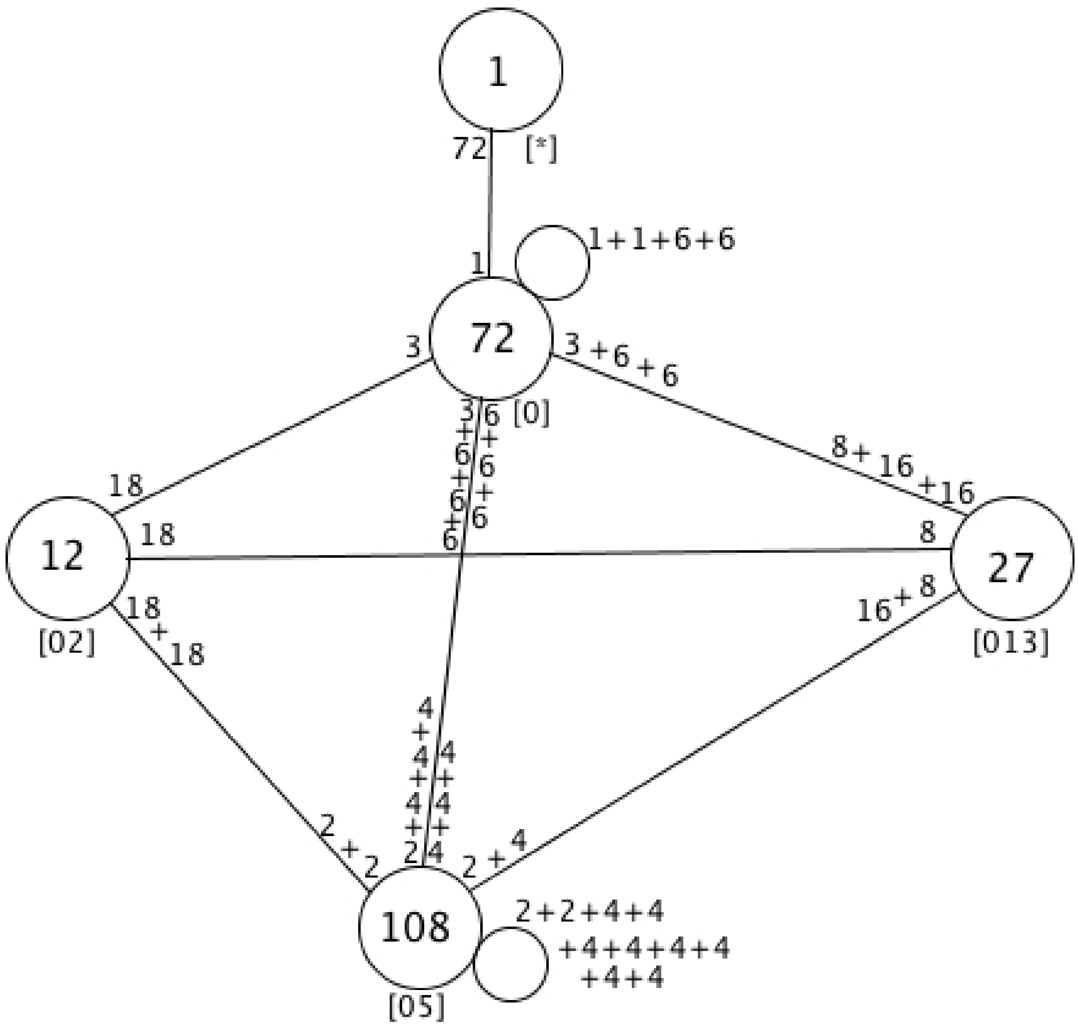

Figure 7.1: Cayley Diagram of M_{12} over $(3^2 : 2^{\cdot}S_4)$

Chapter 8

Tabulated Images

8.1 $2^{*7} : (7 : 2)$

It can be proved that the progenitor given above has M_{23} as a homomorphic image. While looking to find the Mathieu M_{23} group, we ran the following progenitor and what we found is listed below:

```
G<a,b,t> := Group<a,b,t |a^3, b^(-2)*a^-1*b*a,t^2,(t,a),
((a*b^2)*t^(b^-2))^c,((a*b^2)*t^(b^-1))^d,
((a*b^2)*t^(b*a*b))^e,((a*b^2)^-1*t^(b^-2))^f,
((a*b^2)^-1*t^(b^-1))^g,((a*b^2)^-1*t^(b*a*b))^h,
(b*t^(b^-1))^i,(b^3*t^(b^-1))^j>;
```

Table 8.1: Some Finite Images of the Progenitor $2^{*7} : (7 : 2)$

c	d	e	f	g	h	i	j	**Order of G**	**Shape of G**
0	0	0	0	0	5	0	8	1774080	$4^{\cdot}M_{22}$
0	0	0	0	6	10	0	6	15120	$3 : (A_7 : 2)$
0	0	0	0	7	7	0	6	20160	A_8

8.2 $2^{*6} : (S_3 \times 2)$

It can be proved that the progenitor given above has M_{24} as a homomorphic image. While looking to find the Mathieu M_{24} group, we ran the following progenitors and what we found is listed below:

```
G<a,b,c,t> := Group<a,b,c,t |a^2,b^2,c^3, (a*b)^2,(a*c^-1)^2,
b*c^-1*b*c,t^2,(t,a*b),(b*t)^d,(b*t^c)^e,(b*t^(a*c^-1))^f,
(a*t)^g,(a*t^c)^h,(a*t^(a*c))^i,(a*b*t)^j,(a*b*t^a)^k,
(a*b*t^c)^l,(a*b*t^(a*c))^m,(c*t)^n,(c*t^a)^o,
(b*c*t)^p>;
```

Table 8.2: Some Finite Images of the Progenitor $2^{*6} : (S_3 \times 2)$

d	e	...	m	n	o	p	**Order of G**	**Shape of G**
0	0	...	0	0	3	4	240	$(2 \times 4_5) : 2$
0	0	...	0	0	3	4	2184	$PGL_2(13)$
0	0	...	3	0	0	7	24360	$PGL_2(29)$
0	0	...	5	5	0	5	1267200	$2^{\cdot}((2^4)^{\cdot} : L_2(11)) : A_5$

```
for d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u in [0..10] do
G<a,b,c,t> := Group<a,b,c,t |a^2,b^2,c^3, (a*b)^2,(a*c^-1)^2,
b*c^-1*b*c,t^2,(t,a*b),(b*t)^d,(b*t^c)^e,
(b*t^(a*c^-1))^f,(a*t)^g,(a*t^c)^h,
(a*t^(a*c))^i,(a*b*t)^j,(a*b*t^a)^k,(a*b*t^c)^l,
(a*b*t^(a*c))^m,(c*t)^n,(c*t^a)^o,(b*c*t)^p,
(a*t)^q,(b*t)^r,(t*t^a)^s,(t*t^a)^u=a*b>;
```

Table 8.3: Some Finite Images of the Progenitor $2^{*6} : (S_3 \times 2)$

d	e	...	o	p	q	r	s	u	**Order of G**	**Shape of G**
0	0	...	0	7	4	8	8	7	336	$PGL_2(7)$
0	0	...	0	9	0	4	2	3	4896	$PGL_2(17)$
0	0	...	4	6	0	0	6	3	1140480	$6 \times (M_{12} : 2) : 2$
0	0	...	4	10	0	0	4	2	6635520	$(2^{7^{\cdot}} : S_4(3)) : 2$
0	0	...	5	0	8	0	4	9	120	$2 \times A_5$

8.3 $2^{*8} : S_4$

It can be proved that the progenitor given above has M_{11} as a homomorphic image. While looking to find the Mathieu M_{11} group, we ran the following progenitor and what we found is listed below:

```
G<a,b,c,d,t> := Group<a,b,c,d,t |a^2,b^3,c^4,d^4,
```

```
b^-1*a*b*a, c^-1*a*c^-1, d^-1*a*d^-1,
b*c^-1*b^-1*d^-1, c^-1*d^-1*c*d^-1,d^-1*c^-1*b^-1*c*b,
t^2,(t,b),(a*t^(c*b^-1))^e,(c*b*t^(c*b^-1))^f,
(c*b*t^c)^g,(b^-1*c^-1*t^(c*b^-1))^h,(b^-1*c^-1*t^c)^i,
(c*t^(c*b^-1))^j,(c*t^c)^k,(a*b*t)^l,(a*b*t^(c*b^-1))^m,
(a*b^-1*t)^n,(a*b^-1*t^(c*b^-1))^o>;
```

Table 8.4: Some Finite Images of the Progenitor $2^{*8} : S_4$

e	f	...	j	k	l	m	n	o	**Order of G**	**Shape of G**
0	0	...	0	0	0	0	0	4	120	$2 \times A_5$
0	0	...	0	6	0	6	3	0	95040	$(2^2 \times 3)M_{11}$

8.4 $2^{*8} : (2^3 : 2)$

It can be proved that the progenitor given above has M_{12} as a homomorphic image. While looking to find the Mathieu M_{12} group, we ran the following progenitor and what we found is listed below:

```
G<a,b,c,d,t> := Group<a,b,c,d,t |a^2,b^4,c^2,d^2,
b^-2*d, (b^-1*a)^2,(b^-1*c)^2,a*c*b^-1*a*c,t^2,
(t,a), (d*t)^e,(c*t)^f,(c*t^b)^g,(a*t)^h,
(a*t^b)^i,(a*t^c)^j,(b*t)^k,(a*c*t)^l,(a*c*b*t)^m>;
```

Table 8.5: Some Finite Images of the Progenitor $2^{*8} : (2^3 : 2)$

e	f	g	h	i	j	k	l	m	**Order of G**	**Shape of G**
0	0	0	0	0	3	3	0	9	2448	$L_2(17)$
0	0	0	0	0	3	3	0	10	2160	$3 : (A_6 \times 2)$
0	0	0	0	0	4	0	5	6	80640	$2 : (A_8 \times 2)$

8.5 $2^{*72} : (3^2 : (2^{\cdot}S_4))$

It can be proved that the progenitor given above has M_{12} as a homomorphic image. While looking to find the Mathieu M_{12} group, we ran the following progenitor and what we found is listed below:

```
G<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,(b^-1 * a)^2,
(a*c^-1)^2,b^-1*c^-2*b^2*c^2*b^-1,(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),(b^3*t)^d,
(b^3*t^(b^3*c^2))^e,(b^3*t^c)^f,(a*t)^g,(a*t^c)^h,
(a*t^(b^2))^i,(a*t^(a*c^-1*b^-1*c*b))^j,(a*t^(b^2*c))^k,
(a*t^(b^-1*c*b^-1))^l,(b^3*c^2*t)^m,(b^3*c^2*t^b)^n,
(b^3*c^2*t^c)^o,(b^2*t)^p,(b^2*t^c)^q,
(b^2*t^(a*c^-1*b^-1*c*b))^r,(b*c^-1*t)^s,
(b*c^-1*t^(b))^u,(b*c^-1*t^( c))^v,(b*c^-1*t^(c^2))^w,
(b*c^-1*t^(b^-1*c*b^-1))^x,(c*t)^y,
(c*t^b)^z,(c*t^(b^-1*c*b^-1))^hh,
(b*t)^ii,(b*t^c)^jj;(b*t^(b^2))^kk,
(b*t^(b^2*c))^ll,(b*t^(b^-1*c*b^-1))^mm,
(a*b*c^2*t)^nn,(a*b*c^2*t^(b))^oo,
(a*b*c^2*t^(b^2*c))^pp,(a*b*c*t)^qq,
(a*b*c*t^b)^rr,(a*b*c*t^(b^2))^ss,(a*c^-1*b^-1*c*b^2*t)^uu,
(a*c^-1*b^-1*c*b^2*t^(b))^vv,
(a*c^-1*b^-1*c*b^2*t^(b^2*c))^ww>;
```

Table 8.6: Some Finite Images of the Progenitor $2^{*72} : (3^2 : (2^{\cdot}S_4))$

d	e	...	qq	rr	ss	uu	vv	ww	**Order of G**	**Shape of G**
0	0	...	0	0	0	0	3	0	190080	$2^{\cdot}M_{12}$

Appendix A: MAGMA Code for Permutation Progenitor of A_5

```
G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1>;
S:=Alt(5);
xx:=S!(1,2)(3,4);
yy:=S!(1,3,5);
N:=sub<S|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..60]];
for i in [2..60] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..60] do Sch[i], ArrayP[i]; end for;
N1:=Stabiliser(N,1);N1;
C:= Classes(N);C;
C2:=Centraliser(N,N!(1,2)(3,4));C2;
C3:=Centraliser(N,N!(1,2,3));C3;
C4:=Centraliser(N,N!(1,2,3,4,5));C4;
C5:=Centraliser(N,N!(1,3,4,5,2));C5;
Set(C2);Orbits(C2);
Set(C3);Orbits(C3);
Set(C4);Orbits(C4);
Set(C5);Orbits(C5);
```

Appendix B: MAGMA Code for Monomial Progenitor of $11^{*2} :_m D_{10}$

```
D:=DihedralGroup(5);D;
xx:=D! (1,2,3,4,5);
yy:=D!(1,5)(2,4);
G:=sub<D|xx,yy>;
H:=sub<G|(1,2,3,4,5)>;
Set(H);
C:=Classes(G);C;
Cprime:=Classes(H);
Cprime;
CT:=CharacterTable(D);
ch:=CharacterTable(H);
CT;
ch;
I:=Induction(ch[2],D);
Norm(I);
for i in [1..#CT] do if I eq CT[i] then i; end if; end for;
I eq CT[4];
T:=Transversal(G,H);T;
C:=CyclotomicField(5: Sparse := true);
A:=[0: i in [1..4]];
for i in [1..2] do if xx*T[i]^-1 in H
then if ch[2](xx*T[i]^-1) eq C.1
then A[i]:=2; else if ch[2](xx*T[i]^-1) eq C.1^2
then A[i]:=4; else
A[i]:= ch[2](xx*T[i]^-1); end if; end if; end if;
end for;
for i in [1..2] do if T[2]*xx*T[i]^-1 in H
```

```
then if ch[2](T[2]*xx*T[i]^-1) eq C.1
then A[2+i]:=2; else if ch[2](T[2]*xx*T[i]^-1) eq C.1^2
then A[2+i]:=4; else
A[2+i]:= ch[2](T[2]*xx*T[i]^-1); end if; end if; end if;
end for;
B:=[0: i in [1..4]];
for i in [1..2] do if yy*T[i]^-1 in H
then if ch[2](yy*T[i]^-1) eq C.1
then B[i]:=2; else if ch[2](yy*T[i]^-1) eq C.1^2
then B[i]:=4; else
B[i]:= ch[2](yy*T[i]^-1); end if; end if; end if;
end for;
for i in [1..2] do if T[2]*yy*T[i]^-1 in H
then if ch[2](T[2]*yy*T[i]^-1) eq C.1
then B[2+i]:=2; else if ch[2](T[2]*yy*T[i]^-1) eq C.1^2
then B[2+i]:=4; else
B[2+i]:= ch[2](T[2]*yy*T[i]^-1); end if; end if; end if;
end for;
G:=GL(2,11);
A:=G!A;A;
B:=G!B;B;
M:=sub<G|A,B>;
#M;
Order(A);
Order(B);
s:=IsIsomorphic(M,DihedralGroup(5)); s;
/*Monomial Progenitor on 20 letters */
G < x,y >:= Group< x,y|x^5 = y^2 = (x*y)^2 = 1 >;
S:=Sym(20);
xx:=S!(1,7,9,17,5)(3,15,19,13,11)
(2,6,18,10,8)(4,12,14,20,16);
yy:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)
(15,16)(17,18)(19,20);
N:=sub<S|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..10]];
for i in [2..10] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
```

```
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..10] do Sch[i], ArrayP[i]; end for;
Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,17,19});
Normaliser;
Normaliser eq sub<N| (1, 9, 5, 7, 17)(2, 18, 8, 6, 10)
(3, 19, 11, 15, 13)(4, 14, 16, 12, 20)>;
G < x,y,t >:= Group< x,y,t|x^5 = y^2 = (x*y)^2 = 1,
t^11, t^(x^2)=t^5>;
/*Verify*/
G< x,y,t>:= Group< x,y,t|x^5 = y^2 = (x*y)^2 = 1,
 t^11, (t,y*x), t*t^x=t^x*t>;
f, G1,k:=CosetAction(G,sub<G|x,y>);
#G;#k;
IN:=sub<G1|f(x),f(y)>;
T:=sub<G1|f(t)>;#T;
#Normaliser(IN,T);
Index(IN,Normaliser(IN,T));
/* here is the progenitor adding relations with first
order relation */
G< x,y>:= Group< x,y|x^5 = y^2 = (x*y)^2 = 1>;
D:=DihedralGroup(5);D;
xx:=D! (1,2,3,4,5);
yy:=D!(1,5)(2,4);
N:=sub<D|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..10]];
for i in [2..10] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..10] do Sch[i], ArrayP[i]; end for;
C:= Classes(N); C;
C2:=Centraliser(N,N!(1,5)(2,4));C2;
```

```
C3:=Centraliser(N,N!(1,2,3,4,5));C3;
C4:=Centraliser(N,N!(1,3,5,2,4));C4;
Set(C2);Orbits(C2);
Set(C3);Orbits(C3);
Set(C4);Orbits(C4);
```

Appendix C: MAGMA Code for Progenitor of $2^{*7} : D_{14}$

```
G<x,y>:=Group<x,y|x^7, y^2,(x*y)^2>;
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..14]];
for i in [2..14] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..14] do Sch[i], ArrayP[i]; end for;
N7:=Stabiliser(N,7);
N7;
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y)>;
/*  Give all first order relations that this progenitor
can be factored by */
C:= Classes(N);C;
C2:=Centraliser(N,N!(1,6)(2,5)(3,4));C2;
C3:=Centraliser(N,N!(1,2,3,4,5,6,7));C3;
C4:=Centraliser(N,N!(1,3,5,7,2,4,6));C4;
C5:=Centraliser(N,N!(1,4,7,3,6,2,5));C5;
```

```
Set(C2);Orbits(C2);
Set(C3);Orbits(C3);
Set(C4);Orbits(C4);
Set(C5);Orbits(C5);
```

Appendix D: MAGMA Code for DCE of $2^{*3} : S_3$

```
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,
t^2,(t,y),t*t^x=x^-1*t^x*t*t^(x^2)>;
#G;
S:=Sym(3);
xx:=S!(1,2,3);
yy:=S!(1,2);
N:=sub<S|xx,yy>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
ts := [Id(G1): i in [1 .. 3] ];
ts[3]:=f(t); ts[1]:=f(t^x); ts[2]:=f(t^(x^2));
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations
 Q[I] applied sequentially.
*/
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
    for i := 1 to 3 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
```

```
 m:=0;
for i in [1..20] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

N31:=Stabiliser (N,[3,1]);
SSS:={[3,1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N31; #N31;
T31:=Transversal(N,N31);
for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N31);

 for g in IN do for h in IN do
 if ts[3]*ts[1]*ts[2] eq g*(ts[3]*ts[1])^h then g,h;
 end if; end for; end for;
for i in [1..10] do i, cst[i]; end for;
ts[3]*ts[1]*ts[2] eq f(x^-1)*ts[1]*ts[3];

N313:=Stabiliser (N,[3,1,3]);
SSS:={[3,1,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
```

```
N313s:=N313;
T313:=Transversal(N,N313);
for i in [1..#T313] do
ss:=[3,1,3]^T313[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N313);


N3131:=Stabiliser (N,[3,1,3,1]);
SSS:={[3,1,3,1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1]*ts[3]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;

N3131s:=N3131;
for n in N do if 3^n eq 3 and 1^n eq 2
then N3131s:=sub<N|N3131s,n>; end if; end for;
#N3131s;
N3131s;
[3,1,3,1]^N3131s;

N3131:=Stabiliser (N,[3,1,3,1]);
N3131;
N3131:=sub<N| (1,2)>;
#N3131;
[3,1,3,1]^N3131;

T:=Transversal(N,N3131);
 for i in [1..#T] do
{[3,1,3,1]^N3131}^T[i];
end for;
```

```
for n in IN do if ts[3]*ts[2]*ts[3]*ts[2] eq
n*ts[3]*ts[1]*ts[3]*ts[1] then n; end if; end for;
for i in [1..10] do i, cst[i]; end for;
ts[3]*ts[2]*ts[3]*ts[2] eq f(x^-1)*ts[3]*ts[1]*ts[3]*ts[1];

T3131:=Transversal(N,N3131);
for i in [1..#T3131] do
ss:=[3,1,3,1]^T3131[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3131);

N31313:=Stabiliser (N,[3,1,3,1,3]);
SSS:={[3,1,3,1,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1]*ts[3]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]] then print Rep(Seqq[i]);
end if; end for; end for;

N31313s:=N31313;
for n in N do if 3^n eq 3 and 1^n eq 2
then N31313s:=sub<N|N31313s,n>; end if; end for;
#N31313s;
N31313s;
[3,1,3,1,3]^N31313s;

N31313:=Stabiliser (N,[3,1,3,1,3]);
N31313;
N31313:=sub<N| (1,2,3),(1,2)>;
#N31313;
[3,1,3,1,3]^N31313;

T:=Transversal(N,N31313);
 for i in [1..#T] do
```

```
{[3,1,3,1,3]^N31313}^T[i];
end for;

T31313:=Transversal(N,N31313);
for i in [1..#T31313] do
ss:=[3,1,3,1,3]^T31313[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N31313);
/*Relations*/
for n in IN do if ts[1]*ts[2]*ts[1]*ts[2]*ts[1] eq
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
for i in [1..10] do i, cst[i]; end for;
ts[1]*ts[2]*ts[1]*ts[2]*ts[1] eq f((x^-1)^y)
*ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
for n in IN do if ts[2]*ts[3]*ts[2]*ts[3]*ts[2] eq
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
ts[2]*ts[3]*ts[2]*ts[3]*ts[2] eq f(x^-1)*ts[3]
*ts[1]*ts[3]*ts[1]*ts[3];
for n in N do if 3^n eq 3 and 1^n eq 2 then
N31313s:=sub<N|N31313s,n>; end if; end for;
#N31313s;
[3,1,3,1,3]^N31313s;
for n in IN do if ts[3]*ts[2]*ts[3]*ts[2]*ts[3] eq
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
for i in [1..10] do i, cst[i]; end for;
ts[3]*ts[2]*ts[3]*ts[2]*ts[3] eq f(x^-1)*ts[3]
*ts[1]*ts[3]*ts[1]*ts[3];
for n in IN do if ts[2]*ts[1]*ts[2]*ts[1]*ts[2] eq
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
ts[2]*ts[1]*ts[2]*ts[1]*ts[2] eq f((x^-1)^y)*
ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
```

Appendix E: MAGMA Code for DCE of $2 \times L_2(8)$ over D_{14}

```
S:=Sym(7);
xx:=S!(1,2,3,4,5,6,7);
yy:=S!(1, 6)(2, 5)(3, 4);
 N:=sub<S|xx,yy>;
#N;
N7:=Stabiliser(N,7);
N7;
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
(x*t*t^x)^2,(t*t*x*t)^9>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
#G;
IN:=sub<G1|f(x),f(y)>;
ts := [Id(G1) : i in [1 .. 7]];
ts[7] := f(t); ts[1] := f(t^x);
ts[2] := f(t^(x^2)); ts[3] := f(t^(x^3));
ts[4] := f(t^(x^4)); ts[5] := f(t^(x^5));
ts[6] := f(t^(x^6));
f(x^2)*ts[1]*ts[2]*ts[7]*ts[1];
f(x^2)*ts[1]*ts[2] eq ts[1]*ts[7];
f(x^2)*ts[1]*ts[7]*ts[6]*ts[5]*ts[4]
*ts[3]*ts[2]*ts[1]*ts[7];
f(x^2)*ts[1]*ts[7]*ts[6]*ts[5]*ts[4]
*ts[3]*ts[2] eq ts[7]*ts[1];
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
Index(G,sub<G|x,y>);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations
```

```
Q[I] applied sequentially.
*/
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
    for i := 1 to 7 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;
for i in [1..72] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N7 := Stabiliser (N, [7]);
N7; #N7;
Orbits(N7);
N71:=Stabiliser (N,[7,1]);
SSS:={[7,1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N71s := N71;
for n in N do if 7^n eq 7 and 1^n eq 6
then N71s:=sub<N|N71s,n>;
end if;end for;
#N71s;
[7,1]^N71s;
for n in IN do if ts[7]*ts[1] eq n*ts[7]*ts[6]
then n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
ts[7]*ts[1]eq f(((x^2)^-1)^x)*ts[7]*ts[6];
N71:=Stabiliser (N,[7,1]);
N71;
N71:=sub<N| (1,6)(2,5)(3,4)>;
```

```
#N71;
[7,1]^N71;
T:=Transversal(N,N71);
 for i in [1..#T] do
{[7,1]^N71}^T[i];
end for;
T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N71);
N72:=Stabiliser (N,[7,2]);
SSS:={[7,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N72s := N72;
for n in N do if 7^n eq 3 and 2^n eq 1
then N72s:=sub<N|N72s,n>; end if;\
 end for;
#N72s;
[7,2]^N72s;
for n in IN do if ts[7]*ts[2] eq n*ts[3]*ts[1] then
n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
ts[7]*ts[2]eq f((x^2)^((x)^-1))*ts[3]*ts[1];

N72:=Stabiliser (N,[7,2]);
N72;
N72:=sub<N| (1,2)(4,6)(3,7)>;
#N72;
[7,2]^N72;
T:=Transversal(N,N72);
 for i in [1..#T] do
```

```
{[7,2]^N72}^T[i];
end for;
T72:=Transversal(N,N72);
for i in [1..#T72] do
ss:=[7,2]^T72[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N72);

N73:=Stabiliser (N,[7,3]);
SSS:={[7,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73; #N73;
T73:=Transversal(N,N73);
for i in [1..#T73] do
ss:=[7,3]^T73[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73);
/*
[71]
*/
N712:=Stabiliser (N,[7,1,2]);
SSS:={[7,1,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1] *ts[2]eq
```

```
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N712s := N712;

for n in N do if 7^n eq 3 and 1^n eq 2 and 2^n
eq 1 then N712s:=sub<N|N712s,n>; end if;
 end for;
#N712s;
[7,1,2]^N712s;
for n in IN do if ts[7]*ts[1]*ts[2]  eq
n*ts[3]*ts[2]*ts[1] then
n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
ts[7]*ts[1]*ts[2]eq f((x^2)^(x^-1))*ts[3]*ts[2]*ts[1];
N712:=Stabiliser (N,[7,1,2]);
N712;
N712:=sub<N| (1,2)(4,6)(3,7)>;
#N712;
[7,1,2]^N72;
T:=Transversal(N,N712);
 for i in [1..#T] do
{[7,1,2]^N712}^T[i];
end for;
T712:=Transversal(N,N712);
for i in [1..#T712] do
ss:=[7,1,2]^T712[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N712);
N713:=Stabiliser (N,[7,1,3]);
SSS:={[7,1,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1] *ts[3]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
```

```
then print Rep(Seqq[i]);
end if; end for; end for;
N713s := N713;

#N713s;
[7,1,3]^N713s;

T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N713);
N717:=Stabiliser (N,[7,1,7]);
SSS:={[7,1,7]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1] *ts[7]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N717s := N717;
for n in N do if 7^n eq 1 and 1^n eq 7
then N717s:=sub<N|N717s,n>; end if;
 end for;
for n in N do if 7^n eq 6 and 1^n eq 7
then N717s:=sub<N|N717s,n>; end if;  end for;
#N717s;
[7,1,7]^N717s;
N717s;
for n in IN do if ts[7]*ts[1]*ts[7] eq
 n*ts[1]*ts[7]*ts[1]then
n; end if; end for;
for n in IN do if ts[7]*ts[1]*ts[7] eq
n*ts[6]*ts[7]*ts[6]then
n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
```

```
N717:=Stabiliser (N,[7,1,7]);
N717;
N717:=sub<N|(1, 7)(2, 6)(3, 5),
    (1, 7, 6, 5, 4, 3, 2)>;
#N717;
[7,1,7]^N717;
T:=Transversal(N,N717);
 for i in [1..#T] do
{[7,1,7]^N717}^T[i];
end for;
T717:=Transversal(N,N717);
for i in [1..#T717] do
ss:=[7,1,7]^T717[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N717);
/*
[72]
*/
N725:=Stabiliser (N,[7,2,5]);
SSS:={[7,2,5]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[2] *ts[5]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
 *ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
for n in N do if 7^n eq 3 and 2^n eq 1 and
 5^n eq 5 then N725s:=sub<N|N725s,n>; end if;
 end for;
#N725s;
[7,2,5]^N725s;
for n in IN do if ts[7]*ts[2]*ts[5]  eq
n*ts[3]*ts[1]*ts[5] then n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
/*Add relation*/
```

```
N725:=Stabiliser (N,[7,2,5]);
N725;
N725:=sub<N| (1,2)(3,7)(4,6)>;
#N725;
[7,2,5]^N725;
T:=Transversal(N,N72);
 for i in [1..#T] do
{[7,2]^N72}^T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

/*
[73]
*/
N736:=Stabiliser (N,[7,3,6]);
SSS:={[7,3,6]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[3] *ts[6]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N736s := N736;
for n in N do if 7^n eq 7 and 4^n eq 3 and 1^n
eq 6 then N736s:=sub<N|N736s,n>; end if;
 end for;
#N736s;
N736s;
[7,3,6]^N736s;
for n in IN do if ts[7]*ts[3]*ts[6]  eq
n*ts[7]*ts[4]*ts[1] then n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
```

```
N736:=Stabiliser (N,[7,3,6]);
N736;
N736:=sub<N| (1,6)(2,5)(3,4)>;
#N736;
[7,3,6]^N736;
T:=Transversal(N,N736);
 for i in [1..#T] do
{[7,3,6]^N736}^T[i];
end for;
T736:=Transversal(N,N736);
for i in [1..#T736] do
ss:=[7,3,6]^T736[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N736);
/*Relations*/
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[3] eq g*(ts[7]*ts[1]*ts[3] )^h
then g,h;end if; end for; end for;
ts[7]*ts[2]*ts[3] eq f(x^-3)*ts[5]*ts[4]*ts[2];
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[4] eq g*(ts[7]*ts[3] )^h then g,h;
end if; end for; end for;
ts[7]*ts[2]*ts[4] eq f(x)*ts[4]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[1] eq g*(ts[7]*ts[3] )^h then
g,h; end if; end for; end for;
ts[7]*ts[3]*ts[1] eq f(x^-1)*ts[2]*ts[6];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[2] eq g*(ts[7]*ts[1]*ts[2] )^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[2] eq f(x^2)*ts[2]*ts[3]*ts[4];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[4] eq g*(ts[7]*ts[1]*ts[3] )^h
 then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[4] eq f(x)*ts[7]*ts[1]*ts[3];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[5] eq g*(ts[7]*ts[2]*ts[5] )^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[5] eq ts[3]*ts[5]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[7] eq g*(ts[7]*ts[2])^h
```

```
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[7] eq f(x)*ts[4]*ts[2];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[1] eq g*(ts[7]*ts[1]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[1] eq f(x^-4)*ts[2]*ts[1]*ts[6];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[2] eq g*(ts[7]*ts[2])^h
 then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[2] eq f(x^2)*ts[2]*ts[4];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[4] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[4] eq f(x^-1)*ts[7]*ts[3];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[4] eq g*(ts[7]*ts[3]*ts[6])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[5] eq f(x^-1)*ts[1]*ts[4]*ts[7];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[6] eq g*(ts[7]*ts[2]*ts[5])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[6] eq f(x^-1)*ts[5]*ts[3]*ts[7];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[3]*ts[7] eq g*(ts[7]*ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[3]*ts[7] eq f(x^-2)*ts[4]*ts[3]*ts[2];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[2]*ts[4] eq g*(ts[7]*ts[1]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[2]*ts[4] eq f(x^-2)*ts[4]*ts[3]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[2]*ts[5] eq g*(ts[7]*ts[3]*ts[6])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[2]*ts[5] eq ts[5]*ts[1]*ts[4];
for g in IN do for h in IN do
 if ts[7]*ts[1]*ts[2]*ts[7] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[1]*ts[2]*ts[7] eq f(x^-2)*ts[5]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[1] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[5]*ts[1] eq f(x^2)*ts[6]*ts[3];
 if ts[7]*ts[2]*ts[5]*ts[4] eq g*(ts[7]*ts[1]*ts[3])^h
 then g,h; end if; end for; end for;
```

```
ts[7]*ts[2]*ts[5]*ts[4] eq f(x^-4)*ts[5]*ts[6]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[7] eq g*(ts[7]*ts[2]*ts[5])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[5]*ts[7] eq f(x^-4)*ts[4]*ts[6]*ts[2];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[6]*ts[5] eq g*(ts[7]*ts[3]*ts[6])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[6]*ts[5] eq f(x^2)*ts[3]*ts[6]*ts[2];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[6]*ts[3] eq g*(ts[7]*ts[1]*ts[3])^h
 then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[6]*ts[3] eq f(x)*ts[1]*ts[7]*ts[5];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[6]*ts[7] eq g*(ts[7]*ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[6]*ts[7] eq ts[2]*ts[3]*ts[4];
/***********Factor by the center****************/
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;
#N;
Set(N);
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y),(x*t*t^x)^2, (t*t*x*t)^9>;
 #G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
Center(G1);
aa:= G1! (1, 20)(2, 12)(3, 6)(4, 7)(5, 11)(8, 13)
(9, 18)(10, 19)(14, 21)(15, 22)(16, 29)(17, 30)
(23, 31)(24, 32)(25, 38)(26, 33)(27, 43)(28, 35)
(34, 44)(36,45)(37, 52)(39, 46)(40, 56)(41, 55)
(42, 48)(47, 57)(49, 58)(50, 62)(51, 63)(53, 68)
(54, 67)(59, 69)(60, 65)(61, 71)(64, 66)(70, 72);
A:=f(x);
B:=f(y);
C:=f(t);
N:=sub<G1|A,B,C>;
NN<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
(x*t*t^x)^2, (t*t*x*t)^9>;
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq aa
then print Sch[i]; end if; end for;
/* Gives me the center in terms of x, y, and t */
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y), (x*t*t^x)^2,(t*t*x*t)^9,x*y*t*x*t*x^-1*t>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
/***********************************************/
/* Now we need to construct the double
coset enumeration using the above presentation */
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;
#N;
Set(N);
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y),(x*t)^0, (x*t*t^x)^2,(x*y*t^x*t)^0,
 (t*t*x*t)^9,x*y*t*x*t*x^-1*t>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
Index(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts := [Id(G1) : i in [1 .. 7]];
ts[7] := f(t); ts[1] := f(t^x); ts[2] := f(t^(x^2));
 ts[3] := f(t^(x^3)); ts[4] := f(t^(x^4));
```

```
ts[5] := f(t^(x^5)); ts[6] := f(t^(x^6));
f(x^2)*ts[1]*ts[2]*ts[7]*ts[1];
f(x^2)*ts[1]*ts[7]*ts[6]*ts[5]*ts[4]*ts[3]*ts[2]*ts[1]*ts[7];
f(x*y)*ts[7]*ts[6]*ts[7];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations
Q[I] applied sequentially.
*/
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
    for i := 1 to 7 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;
for i in [1..36] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N7 := Stabiliser (N, [7]);
N7; #N7;
T7:=Transversal(N,N7);
for i in [1..#T7] do
ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7);
T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
for g in IN do for h in IN do
 if ts[7]*ts[6] eq g*(ts[7])^h then g,h;
end if; end for; end for;
for i in [1..15] do i, cst[i]; end for;
```

```
/*Relation*/
ts[7]*ts[6] eq f(x*y)*ts[7];
for g in IN do for h in IN do
 if ts[7]*ts[1] eq g*(ts[7])^h
then g,h; end if; end for; end for;
/*Relation*/
ts[7]*ts[1] eq f((x^6)*y)*ts[7];
 N72:=Stabiliser (N,[7,2]);
 SSS:={[7,2]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
 if ts[7]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N72s := N72;
for n in N do if 7^n eq 3 and 2^n eq 1
then N72s:=sub<N|N72s,n>; end if;  end for;
#N72s;
[7,2]^N72s;
 for n in IN do if ts[7]*ts[2] eq n*ts[3]*ts[1] then
 n; end if; end for;
 for n in IN do if ts[7]*ts[2] eq n*ts[3]*ts[1] then
 n; end if; end for;
/*RELATION */
ts[7]*ts[2] eq f(x^2)*ts[3]*ts[1];
N72:=Stabiliser (N,[7,2]);
N72;
N72:=sub<N| (1,2)(3,7)(4,6)>;
#N72;
[7,2]^N72;
T:=Transversal(N,N72);
 for i in [1..#T] do
{[7,2]^N72}^T[i];
end for;
T72:=Transversal(N,N72);
for i in [1..#T72] do
ss:=[7,2]^T72[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
```

```
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N72);
 N73:=Stabiliser (N,[7,3]);
 SSS:={[7,3]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
 if ts[7]*ts[3] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73s := N73;
#N73s;
[7,3]^N73s;
T:=Transversal(N,N73);
 for i in [1..#T] do
{[7,3]^N73}^T[i];
end for;
T73:=Transversal(N,N73);
for i in [1..#T73] do
ss:=[7,3]^T73[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73);
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[1] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[1] eq f(x^-1)*ts[2]*ts[6];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[2] eq g*(ts[7]*ts[2])^h
 then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[2] eq f((x^2)y)*ts[5]*ts[3];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[4] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[4] eq f(y)*ts[7]*ts[3];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[5] eq g*(ts[7]*ts[2]*ts[5])^h
```

```
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[5] eq ts[3]*ts[5]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[6] eq g*(ts[7]*ts[2]*ts[5])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[6] eq f(y)*ts[5]*ts[3]*ts[7];
for g in IN do for h in IN do
 if ts[7]*ts[3]*ts[7] eq g*(ts[7]*ts[2])^h
then g,h; end if; end for; end for;
ts[7]*ts[3]*ts[7] eq f(x)*ts[4]*ts[2];
N723:=Stabiliser (N,[7,2,3]);
T723:=Transversal(N,N723);
for i in [1..#T723] do
ss:=[7,2,3]^T723[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[3] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[3] eq f(x)*ts[5]*ts[2];
N724:=Stabiliser (N,[7,2,4]);
T724:=Transversal(N,N724);
for i in [1..#T724] do
ss:=[7,2,4]^T724[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[4] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[4] eq f(x)*ts[4]*ts[1];

 N725:=Stabiliser (N,[7,2,5]);
 SSS:={[7,2,5]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
 if ts[7]*ts[2]*ts[5] eq
```

```
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
for n in N do if 7^n eq 3 and 2^n eq 1 and
5^n eq 5 then N725s:=sub<N|N725s,n>;
end if;  end for;
#N725s;
[7,2,5]^N725s;
 for n in IN do if ts[7]*ts[2]*ts[5] eq
n*ts[3]*ts[1]*ts[5] then
 n; end if; end for;
ts[7]*ts[2]*ts[5] eq f(x^2)*ts[3]*ts[1]*ts[5];
N725:=Stabiliser (N,[7,2,5]);
N725;
N725:=sub<N| (1,2)(3,7)(4,6)>;
#N725;
[7,2,5]^N725;
T:=Transversal(N,N725);
 for i in [1..#T] do
{[7,2,5]^N725}^T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N725);
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[1] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[5]*ts[1] eq f(x^2)*ts[6]*ts[3];
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[3] eq g*(ts[7]*ts[2]*ts[5])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[5]*ts[3] eq f(x^6)*ts[6]*ts[4]*ts[1];
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[4] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
ts[7]*ts[2]*ts[5]*ts[4] eq f((x^6)*y)*ts[5]*ts[1];
```

Appendix F: MAGMA Code for DCE of $L_2(27)$ over a Maximal Subgroup

```
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2,
 t^2,(t,y),(x*t)^0,(x*t*t^x)^0,(x*y*t^x*t)^3,
(t*t*x*t)^7>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
M:=MaximalSubgroups(G1);
M;
#G1;
#G1/351;
M3:=M[3]`subgroup;
f(x) in M3 and f(y) in M3;
C:=Conjugates(G1,M3);
CC:=SetToSequence(C);
for i in [1..#C] do if f(x) in CC[i] and f(y) in
CC[i] then
i; end if; end for;
H:=sub<G1|CC[124]>;#H;
f(x) in H  and f(y) in H;
for g in G1 do if sub<G1|f(x),f(y)> eq
H then gg=g; end if; end for;
for i in [0..6] do for j in [0..1] do for k,l,m,n,o
in [0..6] do if gg eq
f(x^i*y^j*t^(x^k)*t^(x^l)*t^(x^m)*t^(x^n)*t^(x^o))
then i,j,k,l,m,n,o; end if; end for; end for; end for;
sub<G1|f(t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)
*t^(x^2)),f(x),f(y)> eq H;
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
```

```
(x*t)^0,(x*t*t^x)^0,(x*y*t ^x*t)^3, (t*t*x*t)^7>;
H:=sub<G|x,y,t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)
*t^(x^2)>;#H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
IM:= sub<G1|IN,f(t^(x^2)*t^(x^4)*t^(x^5)
*t^(x^4)*t^(x^2))>;
#IN; #IM;
/***************************************/
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;
#N;
Set(N);
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y),(x*t)^0,(x*t*t^x)^0,(x*y*t^x*t)^3, (t*t*x*t)^7>;
H:=sub<G| x,y,t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)
*t^(x^2)>;
#H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
IM:=sub<G1|IN,f(t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)*t^(x^2))>;
#IN; #IM;
ts := [Id(G1) : i in [1 .. 7]];
ts[7] := f(t); ts[1] := f(t^x); ts[2] := f(t^(x^2));
 ts[3] := f(t^(x^3)); ts[4] := f(t^(x^4));
ts[5] := f(t^(x^5)); ts[6] := f(t^(x^6));
f(x*y)*ts[1]*ts[7]*ts[5] eq ts[7]*ts[1]*ts[6];
ts[6]*ts[5]*ts[4] eq ts[7]*ts[1]*ts[2]*ts[3];
DoubleCosets(G,H,sub<G|x,y>);
#DoubleCosets(G,H,sub<G|x,y>);
Index(G,H);
prodim := function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,H)]]
where null is [Integers() | ];
    for i := 1 to 7 do
       cst[prodim(1, ts, [i])] := [i];
```

```
    end for;
 m:=0;
for i in [1..351] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N7 := Stabiliser (N, [7]);
N7; #N7;
T7:=Transversal(N,N7);
for i in [1..#T7] do
ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7);

N71:=Stabiliser (N,[7,1]);
 SSS:={[7,1]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N71s := N71;
#N71s;
[7,1]^N71s;
T:=Transversal(N,N71);
 for i in [1..#T] do
{[7,1]^N71}^T[i];
end for;
T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N71);

N72:=Stabiliser (N,[7,2]);
```

```
SSS:={[7,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N72s := N72;
#N72s;
[7,2]^N72s;
T:=Transversal(N,N72);
 for i in [1..#T] do
{[7,2]^N72}^T[i];
end for;
T72:=Transversal(N,N72);
for i in [1..#T72] do
ss:=[7,2]^T72[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N72);

 N73:=Stabiliser (N,[7,3]);
 SSS:={[7,3]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73s := N73;
#N73s;
[7,3]^N73s;
T:=Transversal(N,N73);
 for i in [1..#T] do
{[7,3]^N73}^T[i];
end for;
T73:=Transversal(N,N73);
```

```
for i in [1..#T73] do
ss:=[7,3]^T73[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73);

 N717:=Stabiliser (N,[7,1,7]);
 SSS:={[7,1,7]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N717s := N717;
#N717s;
[7,1,7]^N717s;
T:=Transversal(N,N717);
 for i in [1..#T] do
{[7,1,7]^N717}^T[i];
end for;
T717:=Transversal(N,N717);
for i in [1..#T717] do
ss:=[7,1,7]^T717[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N717);

 N712:=Stabiliser (N,[7,1,2]);
 SSS:={[7,1,2]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[2] eq
```

```
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if;
end for; end for;
N712s := N712;
#N712s;
[7,1,2]^N712s;
T:=Transversal(N,N712);
 for i in [1..#T] do
{[7,1,2]^N712}^T[i];
end for;
T712:=Transversal(N,N712);
for i in [1..#T712] do
ss:=[7,1,2]^T712[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N712);

 N713:=Stabiliser (N,[7,1,3]);
 SSS:={[7,1,3]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[3] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N713s := N713;
#N713s;
[7,1,3]^N712s;
T:=Transversal(N,N713);
 for i in [1..#T] do
{[7,1,3]^N713}^T[i];
end for;
T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
```

```
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N713);

 N714:=Stabiliser (N,[7,1,4]);
 SSS:={[7,1,4]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[4] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N714s := N714;
#N714s;
[7,1,4]^N714s;
T:=Transversal(N,N714);
 for i in [1..#T] do
{[7,1,4]^N714}^T[i];
end for;
T714:=Transversal(N,N714);
for i in [1..#T714] do
ss:=[7,1,4]^T714[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N714);

for g in IM do for h in IN do
 if ts[7]*ts[1]*ts[6] eq g*(ts[7]*ts[1]*ts[3])^h
then g,h; end if; end for; end for;
/*Relation */
ts[7]*ts[1]*ts[6] eq f(x*y)*ts[1]*ts[7]*ts[5];
 N715:=Stabiliser (N,[7,1,5]);
 SSS:={[7,1,5]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
```

```
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[5] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N715s := N715;
for n in N do if 7^n eq 6 and 1^n eq 5 and
5^n eq 1 then N715s:=sub<N|N715s,n>; end if;
end for;
#N715s;
[7,1,5]^N715s;
N715:=Stabiliser (N,[7,1,5]);
N715;
N715:=sub<N| (6,7)(1,5)(2,4)>;
#N715;
[7,1,5]^N715;
T:=Transversal(N,N715);
 for i in [1..#T] do
{[7,1,5]^N715}^T[i];
end for;
T715:=Transversal(N,N715);
for i in [1..#T715] do
ss:=[7,1,5]^T715[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N715);

 N726:=Stabiliser (N,[7,2,6]);
 SSS:={[7,2,6]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[2]*ts[6] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
```

```
end if; end for; end for;
N726s := N726;
#N726s;
[7,2,6]^N715s;
T:=Transversal(N,N726);
 for i in [1..#T] do
{[7,2,6]^N726}^T[i];
end for;
T726:=Transversal(N,N726);
for i in [1..#T726] do
ss:=[7,2,6]^T726[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N726);

 N727:=Stabiliser (N,[7,2,7]);
 SSS:={[7,2,7]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[2]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N727s := N727;
#N727s;
[7,2,7]^N727s;
T:=Transversal(N,N727);
 for i in [1..#T] do
{[7,2,7]^N727}^T[i];
end for;
T727:=Transversal(N,N727);
for i in [1..#T727] do
ss:=[7,2,7]^T727[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N727);
```

```
 N724:=Stabiliser (N,[7,2,4]);
 SSS:={[7,2,4]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[2]*ts[4] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N724s := N724;
#N724s;
[7,2,4]^N724s;
T:=Transversal(N,N724);
 for i in [1..#T] do
{[7,2,4]^N724}^T[i];
end for;
T724:=Transversal(N,N724);
for i in [1..#T724] do
ss:=[7,2,4]^T724[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N724);

 N725:=Stabiliser (N,[7,2,5]);
 SSS:={[7,2,5]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[2]*ts[5] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
#N725s;
[7,2,5]^N725s;
```

```
T:=Transversal(N,N725);
 for i in [1..#T] do
{[7,2,5]^N725}^T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N725);
 N737:=Stabiliser (N,[7,3,7]);
 SSS:={[7,3,7]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N737s := N737;
for n in N do if 7^n eq 5 and 3^n eq 2
and 7^n eq 5 then N737s:=sub<N|N737s,n>;
end if;  end for;
#N737s;
[7,3,7]^N737s;
N737:=Stabiliser (N,[7,3,7]);
N737;
N737:=sub<N| (1,4)(2,3)(5,7)>;
#N737;
[7,3,7]^N737;
T:=Transversal(N,N737);
 for i in [1..#T] do
{[7,3,7]^N737}^T[i];
end for;
T737:=Transversal(N,N737);
for i in [1..#T737] do
ss:=[7,3,7]^T737[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
```

```
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N737);

 N732:=Stabiliser (N,[7,3,2]);
 SSS:={[7,3,2]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N732s := N732;
for n in N do if 7^n eq 2 and 3^n eq 6
and 2^n eq 7 then N732s:=sub<N|N732s,n>;
 end if;  end for;
#N732s;
[7,3,2]^N732s;
N732:=Stabiliser (N,[7,3,2]);
N732;
N732:=sub<N| (2,7)(3,6)(4,5)>;
#N732;
[7,3,2]^N732;
T:=Transversal(N,N732);
 for i in [1..#T] do
{[7,3,2]^N732}^T[i];
end for;
T732:=Transversal(N,N732);
for i in [1..#T732] do
ss:=[7,3,2]^T732[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N732);

 N734:=Stabiliser (N,[7,3,4]);
 SSS:={[7,3,4]}; SSS:=SSS^N;
 SSS;
#(SSS);
```

```
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[4] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N734s := N734;
for n in N do if 7^n eq 2 and 3^n eq 6
and 4^n eq 5 then N734s:=sub<N|N734s,n>;
end if;  end for;
#N734s;
[7,3,4]^N734s;
N734:=Stabiliser (N,[7,3,4]);
N734;
N732:=sub<N| (2,7)(3,6)(4,5)>;
#N732;
[7,3,2]^N732;
T:=Transversal(N,N734);
 for i in [1..#T] do
{[7,3,4]^N734}^T[i];
end for;
T734:=Transversal(N,N734);
for i in [1..#T734] do
ss:=[7,3,4]^T734[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N734);

 N7323:=Stabiliser (N,[7,3,2,3]);
 SSS:={[7,3,2,3]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[3] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
```

```
end if; end for; end for;
N7323s := N7323;
#N7323s;
[7,3,2,3]^N7323s;
T:=Transversal(N,N7323);
 for i in [1..#T] do
{[7,3,2,3]^N7323}^T[i];
end for;
T7323:=Transversal(N,N7323);
for i in [1..#T7323] do
ss:=[7,3,2,3]^T7323[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7323);

 N7321:=Stabiliser (N,[7,3,2,1]);
 SSS:={[7,3,2,1]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[1] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7321s := N7321;
for n in N do if 7^n eq 2 and 3^n eq 6
and 2^n eq 7 and 1^n eq 1 then
N7321s:=sub<N|N7321s,n>; end if;  end for;
#N7321s;
[7,3,2,1]^N7321s;
 for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]
eq n*ts[2]*ts[6]*ts[7]*ts[1] then
 n; end if; end for;
N7321:=Stabiliser (N,[7,3,2,1]);
N7321;
N7321:=sub<N| (2,7)(3,6)(4,5)>;
#N7321;
[7,3,2,1]^N7321;
T:=Transversal(N,N7321);
```

```
 for i in [1..#T] do
{[7,3,2,1]^N7321}^T[i];
end for;
T7321:=Transversal(N,N7321);
for i in [1..#T7321] do
ss:=[7,3,2,1]^T7321[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7321);

 N7347:=Stabiliser (N,[7,3,4,7]);
 SSS:={[7,3,4,7]}; SSS:=SSS^N;
 SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[4]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7347s := N7347;
#N7347s;
[7,3,4,7]^N7347s;
T:=Transversal(N,N7347);
 for i in [1..#T] do
{[7,3,4,7]^N7347}^T[i];
end for;
T7347:=Transversal(N,N7347);
for i in [1..#T7347] do
ss:=[7,3,4,7]^T7347[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7347);
N7341:=Stabiliser (N,[7,3,4,1]);
SSS:={[7,3,4,1]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
```

```
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[4]*ts[1] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7341s := N7341;
N7341s := N7341;
for n in N do if 7^n eq 2 and 3^n eq 6
 and 4^n eq 5 and 1^n eq 1 then
N7341s:=sub<N|N7341s,n>; end if;
end for;
#N7341s;
[7,3,4,1]^N7341s;
N7341:=Stabiliser (N,[7,3,4,1]);
N7341;
N7341:=sub<N| (2,7)(3,6)(4,5)>;
#N7341;
[7,3,4,1]^N7341;
T:=Transversal(N,N7341);
 for i in [1..#T] do
{[7,3,4,1]^N7341}^T[i];
end for;
T7341:=Transversal(N,N7341);
for i in [1..#T7341] do
ss:=[7,3,4,1]^T7341[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7341);

 N7371:=Stabiliser (N,[7,3,7,1]);
 SSS:={[7,3,7,1]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[7]*ts[1] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
```

```
then print Rep(Seqq[i]);
end if; end for; end for;
N7371s := N7371;
#N7371s;
[7,3,7,1]^N7371s;
T:=Transversal(N,N7371);
 for i in [1..#T] do
{[7,3,7,1]^N7371}^T[i];
end for;
T7371:=Transversal(N,N7371);
for i in [1..#T7371] do
ss:=[7,3,7,1]^T7371[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7371);

 N7372:=Stabiliser (N,[7,3,7,2]);
 SSS:={[7,3,7,2]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[7]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7372s := N7372;
for n in N do if 7^n eq 6 and 3^n eq 3
and 7^n eq 6 and 2^n eq 4 then
N7372s:=sub<N|N7372s,n>; end if;  end for;
#N7372s;
[7,3,7,2]^N7372s;
N7372:=Stabiliser (N,[7,3,7,2]);
N7372;
N7372:=sub<N| (6,7)(2,4)(1,5)>;
#N7372;
[7,3,7,2]^N7372;
T:=Transversal(N,N7372);
 for i in [1..#T] do
{[7,3,7,2]^N7372}^T[i];
```

```
end for;
T7372:=Transversal(N,N7372);
for i in [1..#T7372] do
ss:=[7,3,7,2]^T7372[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7372);

 N7152:=Stabiliser (N,[7,1,5,2]);
 SSS:={[7,1,5,2]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[1]*ts[5]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7152s := N7152;
#N7152s;
[7,1,5,2]^N7152s;
T:=Transversal(N,N7152);
 for i in [1..#T] do
{[7,1,5,2]^N7152}^T[i];
end for;
T7152:=Transversal(N,N7152);
for i in [1..#T7152] do
ss:=[7,1,5,2]^T7152[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7152);

 N73417:=Stabiliser (N,[7,3,4,1,7]);
 SSS:={[7,3,4,1,7]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
```

```
for n in IM do
 if ts[7]*ts[3]*ts[4]*ts[1]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73417s := N73417;
#N73417s;
[7,3,4,1,7]^N73417s;
T:=Transversal(N,N73417);
 for i in [1..#T] do
{[7,3,4,1,7]^N73417}^T[i];
end for;

T73417:=Transversal(N,N73417);
for i in [1..#T73417] do
ss:=[7,3,4,1,7]^T73417[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73417);

N73214:=Stabiliser (N,[7,3,2,1,4]);
 SSS:={[7,3,2,1,4]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[1]*ts[4] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73214s := N73214;
for n in N do if 7^n eq 4 and 3^n eq 1
and 2^n eq 2 and 1^n eq 3 and 4^n eq 7 then
N73214s:=sub<N|N73214s,n>; end if;  end for;
#N73214s;
[7,3,2,1,4]^N73214s;
 for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]*ts[4] eq
```

```
n*ts[4]*ts[1]*ts[2]*ts[3]*ts[7] then n; end if;
end for;
N73214:=Stabiliser (N,[7,3,2,1,4]);
N73214;
N73214:=sub<N| (4,7)(1,3)(5,6)>;
#N73214;
[7,3,2,1,4]^N73214;
T:=Transversal(N,N73214);
 for i in [1..#T] do
{[7,3,2,1,4]^N73214}^T[i];
end for;
T73214:=Transversal(N,N73214);
for i in [1..#T73214] do
ss:=[7,3,2,1,4]^T73214[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73214);

N73723:=Stabiliser (N,[7,3,7,2,3]);
 SSS:={[7,3,7,2,3]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[7]*ts[3]*ts[7]*ts[2]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73723s := N73723;
for n in N do if 7^n eq 6 and 3^n eq 3 and
 7^n eq 6 and 2^n eq 4 and 3^n eq 3
then N73723s:=sub<N|N73723s,n>;
end if;  end for;
#N73723s;
[7,3,7,2,3]^N73723s;
 for n in IM do if ts[7]*ts[3]*ts[7]*ts[2]*ts[3] eq
 n*ts[6]*ts[3]*ts[6]*ts[4]*ts[3] then n;
end if; end for;
N73723:=Stabiliser (N,[7,3,7,2,3]);
```

```
N73723;
N73723:=sub<N| (6,7)(2,4)(1,5)>;
#N73723;
[7,3,7,2,3]^N73723;
T:=Transversal(N,N73723);
 for i in [1..#T] do
{[7,3,7,2,3]^N73723}^T[i];
end for;
T73723:=Transversal(N,N73723);
for i in [1..#T73723] do
ss:=[7,3,7,2,3]^T73723[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73723);

N732142:=Stabiliser (N,[7,3,2,1,4,2]);
SSS:={[7,3,2,1,4,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[1]*ts[4]*ts[2] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N732142s := N732142;
for n in N do if 7^n eq 4 and 3^n eq 1
and 2^n eq 2 and 1^n eq 3 and 4^n eq 7
and 2^n eq 2 then N732142s:=
sub<N|N732142s,n>; end if;  end for;
#N732142s;
[7,3,2,1,4,2]^N732142s;
 for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]*ts[4]*ts[2]
eq n*ts[4]*ts[1]*ts[2]*ts[3]*ts[7]*ts[2] then n;
end if; end for;
N732142:=Stabiliser (N,[7,3,2,1,4,2]);
N732142;
N732142:=sub<N| (4,7)(1,3)(5,6)>;
#N732142;
```

```
[7,3,2,1,4,2]^N732142;
T:=Transversal(N,N732142);
 for i in [1..#T] do
{[7,3,2,1,4,2]^N732142}^T[i];
end for;
T732142:=Transversal(N,N732142);
for i in [1..#T732142] do
ss:=[7,3,2,1,4,2]^T732142[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N732142);

N7321427:=Stabiliser (N,[7,3,2,1,4,2,7]);
 SSS:={[7,3,2,1,4,2,7]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[1]*ts[4]*ts[2]*ts[7] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*
ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]*
ts[Rep(Seqq[i])[7]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7321427s := N7321427;
for n in N do if 7^n eq 5 and 3^n eq 2 and
2^n eq 3 and 1^n eq 4 and 4^n eq 1
and 2^n eq 3  then
N7321427s:=sub<N|N7321427s,n>;
 end if;  end for;
#N7321427s;
[7,3,2,1,4,2,7]^N7321427s;
 for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]
*ts[4]*ts[2]*ts[7] eq n*ts[5]*ts[2]*ts[3]*ts[4]
*ts[1]*ts[3]*ts[5] then n; end if; end for;
N7321427:=Stabiliser (N,[7,3,2,1,4,2,7]);
N7321427;
N7321427:=sub<N| (5,7)(2,3)(1,4)>;
#N7321427;
[7,3,2,1,4,2,7]^N7321427;
```

```
T:=Transversal(N,N7321427);
 for i in [1..#T] do
{[7,3,2,1,4,2,7]^N7321427}^T[i];
end for;
T7321427:=Transversal(N,N7321427);
for i in [1..#T7321427] do
ss:=[7,3,2,1,4,2,7]^T7321427[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N7321427);

N73214276:=Stabiliser (N,[7,3,2,1,4,2,7,6]);
 SSS:={[7,3,2,1,4,2,7,6]}; SSS:=SSS^N;
 SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[2]*ts[1]*ts[4]*ts[2]*ts[7]*ts[6] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*
ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]*
ts[Rep(Seqq[i])[7]]*ts[Rep(Seqq[i])[8]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73214276s := N73214276;
for n in N do if 7^n eq 5 and 3^n eq 2
and 2^n eq 3 and 1^n eq 4 and 4^n eq 1
 and 2^n eq 3 and 7^n eq 5 and 6^n eq 6
then N73214276s:=sub<N|N73214276s,n>;
 end if;  end for;
#N73214276s;
[7,3,2,1,4,2,7,6]^N73214276s;
 for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]
*ts[4]*ts[2]*ts[7]*ts[6] eq n*ts[5]*ts[2]
*ts[3]*ts[4]*ts[1]*ts[3]*ts[5]*ts[6]
then n; end if; end for;
N73214276:=Stabiliser (N,[7,3,2,1,4,2,7,6]);
N73214276;
N73214276:=sub<N| (5,7)(2,3)(1,4)>;
#N73214276;
[7,3,2,1,4,2,7,6]^N73214276;
```

```
T:=Transversal(N,N73214276);
 for i in [1..#T] do
{[7,3,2,1,4,2,7,6]^N73214276}^T[i];
end for;
T73214276:=Transversal(N,N73214276);
for i in [1..#T73214276] do
ss:=[7,3,2,1,4,2,7,6]^T73214276[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73214276);

N73476:=Stabiliser (N,[7,3,4,7,6]);
SSS:={[7,3,4,7,6]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[4]*ts[7]*ts[6] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73476s := N73476;
#N73476s;
[7,3,4,7,6]^N73476s;
T:=Transversal(N,N73476);
 for i in [1..#T] do
{[7,3,4,7,6]^N73476}^T[i];
end for;
T73476:=Transversal(N,N73476);
for i in [1..#T73476] do
ss:=[7,3,4,7,6]^T73476[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73476);

N73721:=Stabiliser (N,[7,3,7,2,1]);
SSS:={[7,3,7,2,1]}; SSS:=SSS^N;
```

```
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[7]*ts[2]*ts[1] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73721s := N73721;
#N73721s;
[7,3,7,2,1]^N73721s;
T:=Transversal(N,N73721);
 for i in [1..#T] do
{[7,3,7,2,1]^N73721}^T[i];
end for;
T73721:=Transversal(N,N73721);
for i in [1..#T73721] do
ss:=[7,3,7,2,1]^T73721[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N73721);

N737216:=Stabiliser (N,[7,3,7,2,1,6]);
SSS:={[7,3,7,2,1,6]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);Seqq;
for i in [1..#SSS] do
for n in IM do
 if ts[7]*ts[3]*ts[7]*ts[2]*ts[1]*ts[6] eq
 n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N737216s := N737216;
for n in N do if 7^n eq 4 and 3^n eq 1 and 7^n
eq 4 and 2^n eq 2 and 1^n eq 3  and 6^n eq 5
 then N737216s:=sub<N|N737216s,n>; end if;  end for;
#N737216s;
```

```
[7,3,7,2,1,6]^N737216s;
 for n in IM do if ts[7]*ts[3]*ts[7]*ts[2]*ts[1]*ts[6]
eq n*ts[4]*ts[1]*ts[4]*ts[2]*ts[3]*ts[5] then
 n; end if; end for;
N737216:=Stabiliser (N,[7,3,7,2,1,6]);
N737216;
N737216:=sub<N| (4,7)(1,3)(5,6)>;
#N737216;
[7,3,7,2,1,6]^N737216;
T:=Transversal(N,N737216);
 for i in [1..#T] do
{[7,3,7,2,1,6]^N737216}^T[i];
end for;
T737216:=Transversal(N,N737216);
for i in [1..#T737216] do
ss:=[7,3,7,2,1,6]^T737216[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N737216);
/******************************************/
/*We use the Schreier System to convert
permutation into word*/
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;#N;Set(N);
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y),(x*t)^0,(x*t*t^x)^0,(x*y*t^x*t)^3, (t*t*x*t)^7>;
H:=sub<G| x,y,t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)*t^(x^2)>;
#H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
IM:=sub<G1|IN,f(t^(x^2)*t^(x^4)*t^(x^5)*
t^(x^4)*t^(x^2))>;
#IN; #IM;
ts := [Id(G1) : i in [1 .. 7]];
ts[7] := f(t); ts[1] := f(t^x); ts[2] := f(t^(x^2));
 ts[3] := f(t^(x^3)); ts[4] := f(t^(x^4));
 ts[5] := f(t^(x^5)); ts[6] := f(t^(x^6));
A:=f(x);
B:=f(y);
C:=f(t);
```

```
N:=sub<G1|A,B,C>;
NN<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2,
 t^2,(t,y),(x*t)^0,(x*t*t^x)^0,(x*y*t^x*t)^3,
 (t*t*x*t)^7>;
G1:=NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
```

Appendix G: MAGMA Code for Mixed Extension $(2^{6^{\cdot}} : L_2(7)) : 2$

```
a:=0; b:=8;c:=0; d:=3;
G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,
(t,y),(x*t)^a,(x*t*t^x)^b,(x*y*t^x*t)^c,
(t*t*x*t)^d>;#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
Center(G1);
NL:=NormalLattice(G1);
NL;
MinimalNormalSubgroups(G1);
X:=AbelianGroup(GrpPerm,[2,2,2,2,2,2]);
s:=IsIsomorphic(X,NL[2]);s;
q,ff:=quo<NL[3]|NL[2]>;
CompositionFactors(q);
NumberOfGenerators(NL[3]);
T:=Transversal(NL[3],NL[2]);
/*Note we store the generators of NL[2] and the
transversals of NL[3]*/
NumberOfGenerators(NL[2]);
/* now write schreier system, you want the
action of 2^6 so write it with that, you need N and NN,
check their number so you know they equal
64=2^6  */
N:=sub<G1|A,B,C,D,E,F>;#N;
/*presentation for 2^6=64 abelian they commute*/
NN<k,l,m,n,o,p>:=Group<k,l,m,n,o,p|k^2,l^2,m^2,
n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m),(l,n),
(l,o),(l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p)>;
#NN;
/* now you can run the system, [1..64] because 2^6.
```

```
then there are 6 generators so there will be six things,
in this case they're all order 2 (2^6) so you don not
include their inverses */
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..64]];
for i in [2..64] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
/* that's the system, so now you can write T[3]^4 as
elements of 2^6, earlier we called T[3]^4= T34 */
for i in [1..64] do if ArrayP[i] eq T34 then Sch[i];
end if; end for;
T34 eq D;
Order(ff(T2));
q;
#sub<q| ff(T2),ff(T3)>;
Order(ff(T2)*ff(T3));
H<r,s>:= Group<r,s|r^2,s^4,(r*s)^7,(r,s)^4,(r*s^2)^3>;
/*PSL(2,7) Presentation*/
/* H<r,s>:= Group<r,s,|r^2,s^4=n,(r*s)^7,(r,s)^4,
(r*s^2)^3>; */
I:=[Id(NN): i in [1..13]];
for i in [1..64] do if ArrayP[i] eq T34 then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq A^T2 then
Sch[i]; I[2]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq B^T2 then
Sch[i]; I[3]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq C^T2
then Sch[i]; I[4]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq D^T2
then Sch[i]; I[5]:=Sch[i]; end if; end for;
```

```
for i in [1..64] do if ArrayP[i] eq E^T2
then Sch[i]; I[6]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq F^T2
then Sch[i]; I[7]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq A^T3
then Sch[i]; I[8]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq B^T3
then Sch[i]; I[9]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq C^T3
then Sch[i]; I[10]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq D^T3
then Sch[i]; I[11]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq E^T3
then Sch[i]; I[12]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq F^T3
then Sch[i]; I[13]:=Sch[i]; end if; end for;
I;
 Order((T2,T3));
 Order((T2*T3^2));
 I:=[Id(NN): i in [1..14]];
 Order(ff((T2,T3)));
 (T2,T3)^4 in N;
 for i in [1..64] do if ArrayP[i] eq (T2,T3)^4
then Sch[i]; end if; end for;
 NN<k,l,m,n,o,p,r,s>:=Group<k,l,m,n,o,p,r,s|k^2,l^2,
m^2,n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m), (l,
n),(l,o), (l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p),r^2,
s^4=l*m*p,(r*s)^7,(r,s)^4=k*l*m*o*p,(r*s^2)^3,k^r=k*m*o,
l^r=n*o*p,m^r=m,n^r=m*n*o,o^r=o,p^r=l*m*n,
k^s=k*m*n*o*p,l^s=n*p,m^s=k,n^s=k*l*n*p,o^s=p,
p^s=k*l*m*n>;
 #NN;
 f1,g,k1:=CosetAction(NN,sub<NN|Id(NN)>);
 s,t:=IsIsomorphic(NL[3],g);
 s;
/*Now find an element of order 2 in G1 but outside NL[3]*/
for g in G1 do if Order(g) eq 2 and sub<G1|NL[3],g> eq G1
then Z:=g;break; end if; end for;
G1 eq sub<G1|NL[3],Z>;
/*-----------------------------------------*/

/*We find the action of Z on the generators
A,B,C,D,E,F,T2,T3 of NL[3]*/
N:=sub<G1|A,B,C,D,E,F,T2,T3>;#N;
```

```
 NN<k,l,m,n,o,p,r,s>:=Group<k,l,m,n,o,p,r,s|k^2,l^2,
m^2,n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m), (l,n),
(l,o), (l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p),r^2,s^4=
l*m*p,(r*s)^7,(r,s)^4=k*l*m*o*p,
(r*s^2)^3,k^r=k*m*o,l^r=n*o*p,m^r=m,n^r=m*n*o,o^r=o,
p^r=l*m*n,k^s=k*m*n*o*p,l^s=n*p,m^s=k,n^s=k*l*n*p,
o^s=p,p^s=k*l*m*n>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..10752]];
for i in [2..10752] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
if Eltseq(Sch[i])[j] eq 7 then P[j]:=T2; end if;
if Eltseq(Sch[i])[j] eq 8 then P[j]:=T3; end if;
if Eltseq(Sch[i])[j] eq -8 then P[j]:=T3^-1; end if;
end for;
PP:=Id(N);for k in [1..#P] do
PP:=PP*P[k]; end for;ArrayP[i]:=PP;end for;
I:=[Id(NN): i in [1..8]];
for i in [1..10752] do if ArrayP[i] eq A^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq B^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq C^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq D^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq E^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq F^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq T2^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..10752] do if ArrayP[i] eq T3^Z then
Sch[i]; I[1]:=Sch[i]; end if; end for;
/*The presentation of the mixed estension */
 H1<k,l,m,n,o,p,r,s,g>:=Group<k,l,m,n,o,p,r,s,g|
k^2,l^2,m^2,n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),
```

```
(l,m), (l,n),(l,o), (l,p),(m,n),(m,o),(m,p),(n,o),(n,p),
(o,p),r^2,s^4=l*m*p,(r*s)^7,(r,s)^4=k*l*m*o*p,
(r*s^2)^3,k^r=k*m*o,l^r=n*o*p,m^r=m,n^r=
m*n*o,o^r=o,p^r=l*m*n,k^s=k*m*n*o*p,l^s=
n*p,m^s=k,n^s=k*l*n*p,o^s=p,p^s=k*l*m*n,
g^2,k^g=l * n,l^g=s*o*s^-1,m^g=n^s,n^g=
k*s*o*s^-1,o^g=k*m*n*p,p^g=p^s,r^g=
k * s * r * l * s * r * s^-1 * r * s^-1,s^g=
r * k * s * r * s * r * s^-1 * r>;
#H1;
 f,h1,k1:=CosetAction(H1,sub<H1|Id(H1)>);
 s:=IsIsomorphic(h1,G1);s;
```

Appendix H: MAGMA Code for DCE of M_{12}

```
S1:=Sym(72);
aa:=S1!(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)(11, 44)(12, 37)
(14, 28)(16, 30)(17,38)(18, 54)(21, 42)(22, 23)(24, 57)
(25, 45)(26, 32)(27, 66)(29, 61)(31,39)(33, 55)(36, 40)
(41, 51)(43, 62)(46, 70)(47, 53)(48, 63)(49, 58)(50, 67)
(52, 71)(56, 69)(59, 65)(60, 64)(68, 72);
bb:=S1!(1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23)
(4, 21, 53, 47, 42, 20)(5, 26, 39, 7, 31, 32)
(11, 29, 57, 24, 61, 44)(12, 28, 35, 14, 37, 34)
(15, 22, 36,19, 45, 30)(17, 49, 50, 66, 69, 71)
(18, 59, 46, 55, 41, 48)(27, 67, 58,38, 52, 56)
(33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64);
 cc:=S1!(1, 3, 5, 15)(2, 10, 12, 42)(4, 22, 23, 20)
(6, 29, 31, 54)(7, 33, 34, 55)(8, 21, 37, 19)
(9, 18, 39, 61)(11, 45, 46, 40)(13, 50, 35, 67)
(14, 52, 26,62)(16, 38, 53, 69)(17, 30, 56, 47)
(24, 41, 51, 57)(25, 44, 36, 70)(27,60, 64, 66)
(28, 43, 32, 71)(48, 58, 65, 72)(49, 63, 68, 59);
N:=sub<S1|aa,bb,cc>;
G<a,b,c>:=Group<a,b,c| a^2,b^6,c^4,
(b^-1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..432]];
for i in [2..432] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=aa; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=bb; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=bb^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=cc; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=cc^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [50..100] do Sch[i], ArrayP[i]; end for;
G<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^-1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),
(a*c^-1*b^-1*c*b^2*t^(b))^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|a,b,c>);
CompositionFactors(G1);
Center(G1);
/*store center as ccc */
A:=f(a);
B:=f(b);
C:=f(c );
D:=f(t);
N:=sub<G1|A,B,C,D>;
NN<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^-1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),
(a*c^-1*b^-1*c*b^2*t^(b))^3>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
```

```
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq ccc then
print Sch[i]; end if; end for; /* Gives me the center
 in terms of a, b, c, and t */

/*****************************************************/
/* Now we need to construct the double coset
enumeration G=M12=9540*/
S1:=Sym(72);
aa:=S1!(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)(11, 44)(12, 37)
(14, 28)(16, 30)(17,38)(18, 54)(21, 42)(22, 23)(24, 57)
(25, 45)(26, 32)(27, 66)(29, 61)(31,39)(33, 55)(36, 40)
(41, 51)(43, 62)(46, 70)(47, 53)(48, 63)(49, 58)(50, 67)
(52, 71)(56, 69)(59, 65)(60, 64)(68, 72);
bb:=S1!(1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23)
(4, 21, 53, 47, 42, 20)(5, 26, 39, 7, 31, 32)
(11, 29, 57, 24, 61, 44)(12, 28, 35, 14, 37, 34)
(15, 22, 36,19, 45, 30)(17, 49, 50, 66, 69, 71)
(18, 59, 46, 55, 41, 48)(27, 67, 58,38, 52, 56)
(33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64);
 cc:=S1!(1, 3, 5, 15)(2, 10, 12, 42)(4, 22, 23, 20)
(6, 29, 31, 54)(7, 33, 34, 55)(8, 21, 37, 19)
(9, 18, 39, 61)(11, 45, 46, 40)(13, 50, 35, 67)
(14, 52, 26,62)(16, 38, 53, 69)(17, 30, 56, 47)
(24, 41, 51, 57)(25, 44, 36, 70)(27,60, 64, 66)
(28, 43, 32, 71)(48, 58, 65, 72)(49, 63, 68, 59);
N:=sub<S1|aa,bb,cc>;
G<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^-1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
```

```
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),
(a*c^-1*b^-1*c*b^2*t^(b))^3,
a * b^3 * c * t * b * t * b^-1 * t * b * t * c>;
f,G1,k:=CosetAction(G,sub<G|a,b,c>);
CompositionFactors(G1);
#G1;
DoubleCosets(G,sub<G|a,b,c>,sub<G|a,b,c>);
#DoubleCosets(G,sub<G|a,b,c>,sub<G|a,b,c>);
Index(G,sub<G|a,b,c>);
IN:=sub<G1|f(a),f(b), f( c)>;

ts := [Id(G1) : i in [1 .. 72]];
ts[1] := f(t); ts[2]:=f(t^b); ts[3]:=f(t^c);
ts[4]:=f(t^(b^-1*c*b^-1)); ts[5]:=f(t^(c^2));
ts[6]:=f(t^(a*b^-2)); ts[7]:=f(t^(b^2*c^2*b));
ts[8]:=f(t^(a*b^-1)); ts[9]:=f(t^(a*b^2));
ts[10]:=f(t^(a*b*c)); ts[11]:=f(t^((b*c)^2));
ts[12]:=f(t^(b*c^2));ts[13]:=f(t^(b^3));
ts[14]:=f(t^(c*b*c*b*c^-1)); ts[15]:=f(t^(a*c^-1));
ts[16]:=f(t^(a*c*b)); ts[17]:=f(t^(a*c^-1*b^-1*c^-1));
ts[18]:=f(t^(b^2*c)); ts[19]:=f(t^(a*b^-1*c^-1));
ts[20]:=f(t^(b*c^-1*b)); ts[21]:=f(t^(b^-1*c));
ts[22]:=f(t^(c^-1*b)); ts[23]:=f(t^(c*b^-1));
ts[24]:=f(t^(b^2*c^-1*b^-1)); ts[25]:=f(t^(b*c*b^-1));
ts[26]:=f(t^(c^2*b)); ts[27]:=f(t^(b^3*c^-1*b^-1));
ts[28]:=f(t^(b*c^2*b)); ts[29]:=f(t^(a*b^-2*c));
ts[30]:=f(t^(a*c^-1*b^-1)); ts[31]:=f(t^(c^2*b^-2));
ts[32]:=f(t^(c^2*b^-1)); ts[33]:=f(t^(b*c*b^-1*c^-1*b^-1));
ts[34]:=f(t^((b*c*b^-1)^2)); ts[35]:=f(t^(b^3*c^2));
ts[36]:=f(t^(a*c^-1*b^2)); ts[37]:=f(t^(b^-1*c^2));
ts[38]:=f(t^(c*b*c)); ts[39]:=f(t^(b^2*c^2));
ts[40]:=f(t^(b*c*b)); ts[41]:=f(t^(b^2*c^-1*b^-1*c));
ts[42]:=f(t^(b^-1*c*b^3)); ts[43]:=f(t^(c^2*b^-1*c^-1));
ts[44]:=f(t^(b^2*c^-1*b)); ts[45]:=f(t^(a*c^-1*b^-2));
ts[46]:=f(t^(b*c*b*c^-1)); ts[47]:=f(t^(b*c^-1*b^-1));
ts[48]:=f(t^(b^2*c*b^-1)); ts[49]:=f(t^(c^-1*b^-1*c^-1*b));
ts[50]:=f(t^(b^3*c)); ts[51]:=f(t^(b^-2*c^-1*b^2));
ts[52]:=f(t^(c^2*b*c^-1)); ts[53]:=f(t^(a*b^-1*c*b));
ts[54]:=f(t^(a*b^-2*c^-1)); ts[55]:=f(t^(b*c*b*c^-1*b));
ts[56]:=f(t^(c^-1*b^-1*c)); ts[57]:=f(t^(b^-2*c*b));
ts[58]:=f(t^(c*b*c*b^-1)); ts[59]:=f(t^(b^2*c*b));
ts[60]:=f(t^(c^2*b*c*b)); ts[61]:=f(t^(b^2*c^-1));
ts[62]:=f(t^(c^2*b*c)); ts[63]:=f(t^(b^-2*c^-1*b));
ts[64]:=f(t^(c^2*b^-1*c^-1*b^-1));
```

```
ts[65]:=f(t^(a*b^-2*c^-1*b^-1));
ts[66]:=f(t^(a*c*b*c^-1*b^-1)); ts[67]:=f(t^(b^3*c^-1));
ts[68]:=f(t^(b^2*c*b*c^-1)); ts[69]:=f(t^(c*b*c^-1));
ts[70]:=f(t^(b^-1*c^-1*b^-1*c)); ts[71]:=f(t^((c^-1*b^-1)^2));
ts[72]:=f(t^(c^2*b^-1*c^-1*b));

prodim := function(pt, Q, I)
/*
Return the image of pt under
permutations Q[I] applied sequentially.
*/
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|a,b,c>)]]
where null is [Integers() | ];
    for i := 1 to 72 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
 m:=0;
for i in [1..220] do if cst[i] ne [] then
m:=m+1; end if; end for; m;
N1 := Stabiliser (N, [1]);
N1; #N1;
T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N1);

N12:=Stabiliser (N,[1,2]);
N12;
SSS:={[1,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
```

```
if ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s := N12;
for n in N do if 1^n eq 16 and 2^n eq 40
then N12s:=sub<N|N12s,n>; end if;  end for;
#N12s;
N12s;
[1,2]^N12s;
 for n in IN do if ts[1]*ts[2] eq n*ts[16]*ts[40] then
 n; end if; end for;
ts[1]*ts[2] eq f(a*b*c*b^(-2)*c*b)*ts[16]*ts[40];
N12:=Stabiliser (N,[1,2]);
N12;
N12:=sub<N|(3, 70)(4, 11)(5, 28)(6, 32)(8, 34)(9, 12)
(10, 72)(13, 39)(15, 66)(16, 67)(17, 29)(18, 60)(19, 57)
(20, 71)(21, 56)(22, 55)(23, 49)(24, 68)(25, 48)
(26, 37)(27, 46)(30, 51)(33, 58)(36, 38)(40, 61)
(41, 50)(42, 59)(43, 63)(44, 52)(45, 62)(47, 64)
(53, 54)(65, 69),(1, 16, 52, 39, 44, 19)
(2, 40, 29, 32, 17, 15)(3, 26, 60, 43, 5, 22)
(4, 36, 65, 68, 46, 23)(6, 57, 61, 13, 66, 67)
(7, 70, 10, 28, 72, 18)(8, 53, 71,35, 58, 54)
(9, 59, 50, 14, 56, 42)(11, 30, 27, 64, 69, 25)
(12, 20, 21, 34, 41, 33)(24, 48, 38, 47, 49, 51)
(31, 63, 62, 37, 45, 55),(1, 16, 6, 17, 32, 57)
(2, 40, 13, 44, 39, 66)(3, 10, 18, 43, 62, 55)
(4, 30, 24, 46, 64, 49)(5, 72, 28, 22, 31, 63)
(7, 70, 26, 45, 37, 60)(8, 41, 14,56, 34, 53)
(9, 20, 35, 58, 12, 59)(11, 36, 47, 69, 68, 48)
(15, 67, 52,19, 61, 29)(21, 42, 50, 33, 54, 71)
(23, 51, 27)(25, 38, 65)>;
#N12;
[1,2]^N12;
T:=Transversal(N,N12);
 for i in [1..#T] do
{[1,2]^N12}^T[i];
end for;
T12:=Transversal(N,N12);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
```

```
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N12);

N15:=Stabiliser (N,[1,5]);
N15;
SSS:={[1,5]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N15s := N15;
for n in N do if 1^n eq 5 and 5^n eq 1
then N15s:=sub<N|N15s,n>; end if;  end for;
#N15s;
N15s;
[1,5]^N15s;
 for n in IN do if ts[1]*ts[5] eq n*ts[5]*ts[1] then
 n; end if; end for;
ts[1]*ts[5] eq f(a)*ts[5]*ts[1];
N15:=Stabiliser (N,[1,5]);
N15;
N15:=sub<N|(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)
(11, 44)(12, 37)(14, 28)(16, 30)(17, 38)(18, 54)
(21, 42)(22, 23)(24, 57)(25, 45)(26, 32)(27, 66)
(29, 61)(31, 39)(33, 55)(36, 40)(41, 51)(43, 62)
(46, 70)(47, 53)(48, 63)(49, 58)(50, 67)(52, 71)
(56, 69)(59, 65)(60, 64)(68, 72),(1, 5)(2, 12)
(3, 15)(4, 23)(6, 31)(7, 34)(8, 37)(9, 39)(10, 42)
(11, 46)(13, 35)(14, 26)(16, 53)(17, 56)(18, 61)
(19, 21)(20, 22)(24, 51)(25, 36)(27, 64)(28, 32)
(29, 54)(30, 47)(33, 55)(38, 69)(40, 45)(41, 57)
(43, 71)(44, 70)(48, 65)(49, 68)(50, 67)(52, 62)
(58, 72)(59, 63)(60, 66),(1, 5)(2, 37)(4, 22)(6, 39)
(7, 34)(8, 12)(9, 31)(10, 21)(11, 70)(13, 35)(14,
32)(16, 47)(17, 69)(18, 29)(19, 42)(20, 23)(24, 41)
(25, 40)(26, 28)(27, 60)(30, 53)(36, 45)(38, 56)
(43, 52)(44, 46)(48, 59)(49, 72)(51, 57)(54,
61)(58, 68)(62, 71)(63, 65)(64, 66)>;
```

```
#N15;
[1,5]^N15;
T:=Transversal(N,N15);
 for i in [1..#T] do
{[1,5]^N15}^T[i];
end for;
T15:=Transversal(N,N15);
for i in [1..#T15] do
ss:=[1,5]^T15[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N15);

N113:=Stabiliser (N,[1,13]);
N113;
SSS:={[1,13]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[13] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N113s := N113;
for n in N do if 1^n eq 69 and 13^n eq 49
then N113s:=sub<N|N113s,n>; end if;  end for;
#N113s;
N113s;
[1,13]^N113s;
 for n in IN do if ts[1]*ts[13] eq n*ts[69]*ts[49] then
 n; end if; end for;
ts[1]*ts[13] eq f(b^3)*ts[69]*ts[49];
N113:=Stabiliser (N,[1,13]);
N113;
N113:=sub<N|(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)
(11, 44)(12, 37)(14, 28)(16, 30)(17, 38)(18, 54)
(21, 42)(22, 23)(24, 57)(25, 45)(26, 32)(27, 66)
(29, 61)(31,39)(33, 55)(36, 40)(41, 51)(43, 62)
(46, 70)(47, 53)(48, 63)(49, 58)(50, 67)(52, 71)
(56, 69)(59, 65)(60, 64)(68, 72),(1, 69, 13, 49)
```

```
(2, 38, 6, 27)(3, 22, 10, 45)(4, 37, 47, 28)(5, 30, 7, 36)
(8, 24, 9, 11)(12, 51, 14, 65)(15, 60, 19, 72)
(16, 62, 40, 43)(17, 67, 66, 52)(18, 53, 55, 20)
(21, 70, 42, 63)(23, 31, 25, 26)(29, 56, 61, 58)
(32,68, 39, 64)(33, 41, 54, 59)(34, 46, 35, 48)
(44, 50, 57, 71),(1, 69, 61, 56, 13, 49, 29, 58)
(2, 24, 71, 17, 6, 11, 50, 66)(3, 60, 32, 23,
 10, 72, 39, 25)(4, 18, 59, 12, 47, 55, 41, 14)
(5, 30, 62, 16, 7, 36, 43,40)(8, 38, 67, 57, 9, 27,
 52, 44)(15, 22, 31, 64, 19, 45, 26, 68)(20, 37,
51, 54, 53, 28, 65, 33)(21, 63, 34, 46, 42, 70, 35, 48)>;
#N113;
[1,13]^N113;
T:=Transversal(N,N113);
 for i in [1..#T] do
{[1,13]^N113}^T[i];
end for;
T113:=Transversal(N,N113);
for i in [1..#T113] do
ss:=[1,13]^T113[i];
cst[prodim(1, ts, ss)] := ss;
 end for;
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N113);
/*Relations*/
for g in IN do for h in IN do
 if ts[1]*ts[7] eq g*(ts[1])^h then g,h; end if;
end for; end for;
ts[1]*ts[7] eq ts[35];
for g in IN do for h in IN do
 if ts[1]*ts[35] eq g*(ts[1])^h then g,h;
end if; end for; end for;
ts[1]*ts[35] eq ts[7];
for g in IN do for h in IN do
 if ts[1]*ts[21] eq g*(ts[1])^h then g,h;
end if; end for; end for;
ts[1]*ts[21] eq f(a * b^-1 * c^-1 * b^-1 * c * b)
*ts[17];
for g in IN do for h in IN do
 if ts[1]*ts[3] eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[3] eq f(c^-1*b^-1)*ts[2]*ts[6];
for g in IN do for h in IN do
```

```
 if ts[1]*ts[4] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[4] eq f(b*c*b)*ts[14]*ts[9];
for g in IN do for h in IN do
 if ts[1]*ts[6] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[6] eq f(a*b)*ts[18]*ts[55];
for g in IN do for h in IN do
 if ts[1]*ts[10] eq g*(ts[1]*ts[5])^h then g,h;
end if; end for; end for;
ts[1]*ts[10] eq f(b^3)*ts[15]*ts[62];
for g in IN do for h in IN do
 if ts[1]*ts[16] eq g*(ts[1]*ts[5])^h
 then g,h; end if; end for; end for;
ts[1]*ts[16] eq f(a * c^-1 * b^-1 * c * b^2 * c)
*ts[28]*ts[37];
for g in IN do for h in IN do
 if ts[1]*ts[17] eq g*(ts[1]*ts[13])^h then g,h;
end if; end for; end for;
ts[1]*ts[17] eq f(a * b * c * b)*ts[15]*ts[29];
for g in IN do for h in IN do
 if ts[1]*ts[18] eq g*(ts[1]*ts[5])^h then
 g,h; end if; end for; end for;
ts[1]*ts[18] eq f(a * b^-1 * c^-1 * b^-1 * c * b)
*ts[59]*ts[41];
for g in IN do for h in IN do
 if ts[1]*ts[25] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[25] eq f(a * c^-1 * b * c * b)*ts[53]
*ts[58];
for g in IN do for h in IN do
 if ts[1]*ts[22] eq g*(ts[1])^h then g,h; end
 if; end for; end for;
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[3] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[2]*ts[3] eq f(c * b^-1 * c^-1 * b^-1
* c^-1 * b^-1)*ts[61]*ts[33];
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[4] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[2]*ts[4] eq f(c * b^-2 * c * b * c)
*ts[28]*ts[37];
for g in IN do for h in IN do
```

```
if ts[1]*ts[2]*ts[8]  eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[2]*ts[8]  eq f(c^-1 * b * c * b)*
ts[71]*ts[43];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[3]  eq g*(ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[3]  eq f(c * b^-2)*ts[63]*ts[60];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[7]  eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[7]  eq f(a * b * c^2 * b^-1)*
ts[72]*ts[17];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[13]  eq g*(ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[13]  eq f(a * c^-1 * b^3
* c^-1)*ts[68]*ts[46];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[33] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[33] eq f(a * c^2 * b^-1 *
 c^-1)*ts[2]*ts[37];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[50] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[50] eq f(b^2 * c * b^-1 * c^-1)
*ts[37]*ts[2];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[2] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[2] eq f(a * c * b^-2)*ts[44]*ts[59];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[4] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[4] eq f(b * c * b^2)*ts[72]*ts[57];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[6] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[6] eq f(a * b^-1 * c * b * c^2)*ts[24]*ts[68];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[10] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[10] eq f(b^3 * c * b^-1)*ts[20]*ts[53];
```

```
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[11]  eq g*(ts[1])^h then g,h;
end if; end for; end for;
ts[1]*ts[5]*ts[11]  eq f(a * c^-1 * b * c * b^-2)*ts[63];
for g in IN do for h in IN do
 if ts[1]*ts[5]*ts[14] eq g*(ts[1])^h then g,h;
end if; end for; end for;
ts[1]*ts[5]*ts[14] eq f(a * c^-1 * b^3 * c)*ts[60];
for g in IN do for h in IN do
 if ts[1]*ts[5]*ts[16] eq g*(ts[1])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[16] eq f(a * c * b^3 * c^-1)*ts[28];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[17] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[17] eq f(c * b^-1 * c^-1 * b^-1
* c^-1)*ts[71]*ts[21];
for g in IN do for h in IN do
 if ts[1]*ts[5]*ts[18] eq g*(ts[1])^h then
g,h; end if; end for; end for;
ts[1]*ts[5]*ts[18] eq f(a * b^2 * c * b^-1
* c^-1)*ts[49];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[24] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[24] eq f(a * c * b^-2)*ts[42]*ts[50];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[25] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[25] eq f(a * b^-1 * c * b^-1)
*ts[64]*ts[23];
for g in IN do for h in IN do
 if ts[1]*ts[5]*ts[27] eq g*(ts[1])^h then
g,h; end if; end for; end for;
ts[1]*ts[5]*ts[27] eq f(a * c * b^2 * c)*ts[48];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[43] eq g*(ts[1]*ts[5])^h
 then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[43] eq f(b^2 * c^-1 * b^-1
 * c^-1)*ts[50]*ts[42];
for g in IN do for h in IN do
 if ts[1]*ts[5]*ts[48] eq g*(ts[1])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[48] eq f(b^c)*ts[14];
```

```
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[49]  eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[49]  eq f(a * c^-1 * b^-1
 * c^-1 * b^2)*ts[54]*ts[42];
for g in IN do for h in IN do
 if ts[1]*ts[13]*ts[1] eq g*(ts[1])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[1] eq ts[13];
for g in IN do for h in IN do
if ts[1]*ts[13]*ts[5]  eq g*(ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[5]  eq ts[1]*ts[34];
for g in IN do for h in IN do
 if ts[1]*ts[13]*ts[2] eq g*(ts[1])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[2] eq f(a * b^-1 * c *
 b * c * b^-1)*ts[44];
for g in IN do for h in IN do
 if ts[1]*ts[13]*ts[3] eq g*(ts[1])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[3] eq f(c^-1 * b^2)*ts[27];
for g in IN do for h in IN do
if ts[1]*ts[13]*ts[4] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[4] eq f(b * c * b^-1 *
 c^-1 * b^-1 * c^-1)*ts[62]*ts[15];
for g in IN do for h in IN do
if ts[1]*ts[13]*ts[21] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[21] eq ts[29]*ts[43];
for g in IN do for h in IN do
if ts[1]*ts[13]*ts[7]  eq g*(ts[1]*ts[2])^h
 then g,h; end if; end for; end for;
ts[1]*ts[13]*ts[7]  eq f(c * b^3 * c)
*ts[66]*ts[59];
```

Bibliography

[CCN+85] John H Conway, Robert T Curtis, Simon P Norton, Richard A Parker, and Robert A Wilson. *Atlas of Finite Groups*. Clarendon Press, Oxford, 1985.

[CP93] J Cannon and C Playoust. *An introduction to MAGMA*. University of Sydney, Sydney, Australia, 1993.

[Cur07] Robert Curtis. *Symmetric generation of groups: with applications to many of the sporadic finite simple groups*. Number 111. Cambridge University Press, 2007.

[dlLT05] Maria de la Luz Torres. Symmetric generation of finite groups. Master's thesis, CSUSB, 2005.

[HK06] Z Hasan and A Kasouha. Symmetric representation of the elements of finite groups. *arXiv preprint math/0612042*, 2006.

[Led87] Walter Ledermann. *Introduction to group characters*. CUP Archive, 1987.

[Rot12] Joseph Rotman. *An introduction to the theory of groups*, volume 148. Springer Science & Business Media, 2012.

[WWT+05] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Steve Linton, John Bray, and Rachel Abbott. Atlas of finite group representations, version 3. *available at the time of writing from http://brauer. maths. qmul. ac. uk/Atlas/v3*, 2005.