CONSTRUCTIONS AND ISOMORPHISM TYPES OF IMAGES

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Abstract

In this thesis, we have presented our discovery of true finite homomorphic images of various permutation and monomial progenitors, such as $2^*7 : D_{14}$, $2^*7 : (7 : 2)$, $2^*6 : (S_3 \times 2)$, $2^*8 : S_4$, $2^*7^2 : (3^2 : (2 \cdot S_4))$, and $11^*2 : m D_{10}$. We have given delightful symmetric presentations and very nice permutation representations of these images which include, the Mathieu groups $M_{11}$, $M_{12}$, the 4-fold cover of the Mathieu group $M_{22}$, $2 \times L_2(8)$, and $L_2(13)$. Moreover, we have given constructions, by using the technique of double coset enumeration, for some of the images, including $M_{11}$ and $M_{12}$. We have given proofs, either by hand or computer-based, of the isomorphism type of each image. In addition, we use Iwasawa’s Lemma to prove that $L_2(13)$ over $A_5$, $L_2(8)$ over $D_{14}$, $L_2(13)$ over $D_{14}$, $L_2(27)$ over $2 \cdot D_{14}$, and $M_{11}$ over $2 \cdot S_4$ are simple groups. All of the work presented in this thesis is original to the best of our knowledge.
Acknowledgements

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I love you! ∞ and beyond.
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Introduction

Group theory is the study of symmetry of objects. Symmetric presentations provide a uniform way of constructing finite groups. In this thesis, we are particularly interested in symmetric presentations of finite simple groups, since these can be used to obtain all finite groups. The process to obtain finite homomorphic images is through the use of a progenitor, $m^n : N$, where $N$ is transitive on $n$ letters. The objective is to factor the progenitor by relations, that equate elements of $N$ to the product of $t_i$s, that give finite homomorphic images. An isomorphism is a homomorphism that is also a bijection. We say that group $G$ is isomorphic to group $H$, denoted by $G \cong H$, if there exists an isomorphism $f : G \rightarrow H$.

In Chapter 1, we describe the process of creating permutation progenitors and monomial progenitors. In addition, we factor these progenitors by all first order relations and suitable relations. In Chapter 2, we apply the technique of double coset enumeration, resulting in the construction of Cayley diagrams, and give by hand or computer-based proofs for the isomorphism type of each group. We also, explain the technique of factoring by the center. In Chapter 3, we use Iwasawa’s lemma and the transitive action of a group on the set of single cosets to prove that a group is simple. Similarly, in Chapter 4, we apply the technique of double coset enumeration over a maximal subgroup and apply Iwasawa’s lemma to prove that a group simple. In Chapter 5, we compute an extension problem, by looking at the composition factors to find the isomorphic type. In Chapter 6, we construct $M_{11}$ over the subgroup $S_4$ with an imprimitivite action. We then construct this group over the maximal subgroup $2\cdot S_4$ with a primitive action and thus apply Iwasawa’s lemma to prove that this group is simple. Similarly, in Chapter 7, we construct $M_{12}$ by performing the double coset enumeration, and our goal is to show that the group is simple, however, at the time of writing our
proof, we did not have time to apply Iwasawa’s lemma, it is still in progress. Finally, in Chapter 8, we give progenitors tables with homomorphic images.
Chapter 1

Writing Progenitors

1.1 Writing Progenitors Preliminaries

Definition 1.1. (Permutation). If $X$ is a nonempty set, a permutation of $X$ is a bijection $\alpha : X \to X$. We denote the set of all permutations of $X$ by $S_X$. [Rot12]

Definition 1.2. (Operation). Let $G$ be a set. A (binary) operation on $G$ is a function that assigns each ordered pair of elements of $G$ an element on $G$. [Rot12]

Definition 1.3. (Semigroup). A semigroup $(G,\ast)$ is a nonempty set $G$ equipped with an associative operation $\ast$. [Rot12]

Definition 1.4. (Group). A group is a semigroup $G$ containing an element $e$ such that

(i) $e \ast a = a = a \ast e$ for all $a \in G$
(ii) for every $a \in G$, there is an element $b \in G$ with $a \ast b = e = b \ast a$. [Rot12]

Definition 1.5. (Abelian). A pair of elements $a$ and $b$ in a semigroup commutes if $a \ast b = b \ast a$. A group (or a semigroup) is abelian if every pair of its elements commutes. [Rot12]

Theorem 1.6. If $K \leq H$ and $[H : K] = n$, then there is a homomorphism $\phi : H \to S_n$ with $\ker \phi \leq K$. [Rot12]

Definition 1.7. (Free Group). If $X$ is a nonempty subset of a group $F$, then $F$ is a free group with basis $X$ if, for every group $G$ and every function $f : X \to G$, there
exists a unique homomorphism $\phi : F \to G$ extending $f$. Moreover, $X$ generates $F$. [Rot12]

**Definition 1.8. (Presentation)**. Let $X$ be a nonempty set and let $\Delta$ be a family of words on $X$. A group $G$ has **generators** $X$ and **relations** $\Delta$ if $G \cong F/R$, where $F$ is the free group with basis $X$ and $R$ is the normal subgroup of $F$ generated by $\Delta$. The ordered pair $(X|\Delta)$ is called a **presentation** of $G$. [Rot12]

**Definition 1.9. (Progenitor)**. Let $G$ be a group and $T = \{t_1, t_2, ..., t_n\}$ be a symmetric generating set for $G$ with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the **progenitor** to be the semi direct product $m^*n : N$, where $m^*n$ is the free product of $n$ copies of the cyclic group $C_m$. [Cur07]

**Definition 1.10. (Normalizer)**. If $H \leq G$, then the **normalizer** of $H$ in $G$, denoted by $N_G(H)$, is

$$N_G(H) = \{ a \in G : aHa^{-1} = H \}.$$ [Rot12]

**Definition 1.11. (Centralizer)**. If $a \in G$, then the **centralizer** of $a$ in $G$, denoted by $C_G(a)$, is the set of all $x \in G$ which commute with $a$. [Rot12]

**Note 1.12.** An isomorphism is a homomorphism that is also bijection. We say that $G$ is isomorphic to $H$ denoted by $G \cong H$, if there exist an isomorphism $\phi : G \to H$. [Rot12]

**Definition 1.13. (Homomorphism)**. Let $G$ and $H$ be groups. A map $\phi : G \to H$ is a **homomorphism** if, for all $\alpha, \beta \in G$,

$$\phi(\alpha \beta) = \phi(\alpha)\phi(\beta).$$ [Rot12]

**Theorem 1.14. (First Isomorphism Theorem (F.I.T.))**. Let $\phi : G \to H$ be a homomorphism with $\ker \phi$. Then

- $\ker \phi \trianglelefteq G$
- $G/\ker \phi \cong \text{img} \phi$. [Rot12]

**Theorem 1.15. (Second Isomorphism Theorem).**

Let $N$ and $T$ be subgroups of $G$ with $N$ normal. Then $N \cap T$ is normal in $T$ and $T/(N \cap T) \cong NT/N$. [Rot12]
Theorem 1.16. (Third Isomorphism Theorem).
Let $K \leq H \leq G$, where both $K$ and $H$ are normal subgroups of $G$. Then $H/K$ is a normal subgroup of $G/K$ and

$$(G/K)(H/K) \cong G/H.$$ \[Rot12\]

Definition 1.17. (Monomial Character). Let $G$ be a finite group and $H \leq G$. The character $X$ of $G$ is monomial if $X = \lambda^G$, where $\lambda$ is a linear character of $H$. \[Led87\]

Definition 1.18. (Character). Let $A(x) = (a_{ij}(x))$ be a matrix representation of $G$ of degree $m$. We consider the characteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{pmatrix}
\lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\
\lambda - a_{21}(x) & -a_{22}(x) & \cdots & -a_{2m}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & \lambda - a_{mm}(x)
\end{pmatrix}$$

This is a polynomial of degree $m$ in $\lambda$, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + \ldots + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $\text{tr}A(x)$, so that

$$\phi(x) = \text{tr}A(x)$$

We regard $\phi(x)$ as a function on $G$ with values in $K$, and we call it the character of $A(x)$. \[Led87\]

Theorem 1.19. The number of irreducible character of $G$ is equal to the number of conjugacy classes of $G$. \[Led87\]

Definition 1.20. (Degree of a Character). The sum of squares of the degrees of the distinct irreducible characters of $G$ is equal to $|G|$. The degree of a character $\chi$ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character. \[Led87\]
Definition 1.21. (Lifting Process). Let $N$ be a normal subgroup of $G$ and suppose that $A_0(Nx)$ is a representation of degree $m$ of the group $G/N$. Then $A(x) = A_0(Nx)$ defines a representation of $G/N$ lifted from $G/N$. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m$, $\phi(u) = m = \phi(1)$. The lifting process preserves irreducibility. [Led87]

Definition 1.22. (Induced Character)

Let $H \leq G$ and $\phi(u)$ be a character of $H$ and define $\phi(x) = 0$ if $x \in H$, then

$$
\phi^G(x) = \begin{cases} 
\phi(x), & x \in H \\
0, & x \notin H 
\end{cases}
$$

is an induced character of $G$. [Led87]

Definition 1.23. (Formula for Induced Character).

Let $G$ be a finite group and $H$ be a subgroup such that $[G : H] = n$. Let $C_\alpha$, $\alpha = 1, 2, \cdots m$ be the conjugacy classes of $G$ with $|C_\alpha| = h_\alpha$, $\alpha = 1, 2, \cdots m$. Let $\phi$ be a character of $H$ and $\phi^G$ be the character of $G$ induced from the character $\phi$ of $H$ up to $G$. The values of $\phi^G$ on the $m$ classes of $G$ are given by:

$$
\phi^G_\alpha = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w), \alpha = 1, 2, 3, \cdots, m. \quad [Led87]
$$

1.2 Permutation Progenitor of $A_5$

We want to write a permutation progenitor of $2^5 : A_5$, where our control group is $N = A_5$. First, we write a presentation of $A_5$ which is $G < x, y > = < x, y^2 = y^3 = (xy)^5 = 1 >$. We check in Magma if the above presentation gives $A_5$.

```magma
> G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1 >;
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> s,t:=IsIsomorphic(G1,Alt(5));s;
true
```
Moreover, the corresponding permutation representation is \( N = \langle x, y \rangle \), where \( x = (1, 2)(3, 4) \) and \( y = (1, 3, 5) \). Now we add the free product \( 2^*5 \) \( (|t_i^*s| = 2) \) to this group to form our progenitor. Hence, a presentation for the progenitor \( 2^*5 : N \) is given by
\[
< x, y, t | x^2, y^3, (xy)^5, t^2, (t, N^1) >,
\]
where \( t \sim t_1 \). \( N^1 \) is the point stabilizer of 1, and \((t, N^1) = 1\) means that \( 1^g = 1 \ \forall \ g \in N^1 \). Note that \( 1^g = 1 \ \forall \ g \in N^1 \) implies \( t \) has \([N : N^1]\) conjugates in \( N \). Using Magma, we can see that the point stabilizer of 1 in \( N \) is equal to \( < (2, 3, 4), (3, 4, 5) > \). Now we use the Schreier System to convert the permutations into words. Thus, \( N^1 = \langle yxy^\dagger xy^\dagger, y^{-1}xy^{-1}xyxy^{-1} \rangle \) (see below).

```plaintext
> G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1 >;
> S:=Alt(5);
> xx:=S!(1,2)(3,4);
> yy:=S!(1,3,5);
> N:=sub<S|xx,yy>;
> N1:=Stabiliser(N,1);
> N1;
Permutation group N1 acting on a set of cardinality 5
Order = 12 = 2^2 * 3
   (2, 3, 4)
   (3, 4, 5)
> Sch:=SchreierSystem(G,sub<G|Id(G)>);
> ArrayP:=[Id(N): i in [1..60]];
> for i in [2..60] do
    for P:=[Id(N): l in [1..#Sch[i]]] do
      for j in [1..#Sch[i]] do
        if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
        if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
        if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
      end for;
      PP:=Id(N);
      for k in [1..#P] do
        PP:=PP*P[k]; end for;
      ArrayP[i]:=PP;
    end for;
  for i in [1..60]do if ArrayP[i] eq N!(2,3,4) then Sch[i]; end if;
end for;
> y * x * y^-1 * x * y^-1
> for i in [1..60]do if ArrayP[i] eq N!(3,4,5) then Sch[i]; end if;
> y^-1 * x * y^-1 * x * y * x * y^-1
```
So, a presentation of the progenitor $G = 2^4 : N$ is given by $G < x, y, t > : = \langle x, y, t | x^2, y^3, (xy)^5, t^2, (t, yxy^{-1}xy^{-1}), (t, y^{-1}xy^{-1}xy) \rangle$. This progenitor is infinite. In order to find finite images of $2^5 : N$ we must factor it by the first order relations.

### 1.2.1 Factoring $2^5 : A_5$ by First Order Relations

The first order relations are written of the form $(\pi t_i^a)^b = 1$, where $\pi \in N$ and $w$ is word in the $t_i$'s. In order to find these relations, we compute the conjugacy classes of $N = A_5$, as show below:

**Table 1.1: Conjugacy Classes of $A_5$**

<table>
<thead>
<tr>
<th>Class Number</th>
<th>Order</th>
<th>Class Representative</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>e</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(1,2)(3,4)</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(1,2,3)</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>(1,2,3,4,5)</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>(1,2,3,4,5,2)</td>
<td>12</td>
</tr>
</tbody>
</table>

Next, we need to compute the centralizer of each class representative in $N = A_5$ and then find the orbits of the corresponding centralizer. The centralizer and their orbits on $\{1, 2, 3, 4, 5\}$ are given below.

**Table 1.2: Centralizer of $A_5$**

<table>
<thead>
<tr>
<th>Class Num</th>
<th>Class Rep</th>
<th>Cent($N, \text{Class Rep}$)</th>
<th>Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(1,2)(3,4)</td>
<td>$&lt; (1,2)(3,4), (1,3)(2,4) &gt;$</td>
<td>${1,2,3,4}, {4}, {5}$</td>
</tr>
<tr>
<td>3</td>
<td>(1,2,3)</td>
<td>$&lt; (1,2,3) &gt;$</td>
<td>${1,2,3} {4}, {5}$</td>
</tr>
<tr>
<td>4</td>
<td>(1,2,3,4,5)</td>
<td>$&lt; (1,2,3,4,5) &gt;$</td>
<td>${1,2,3,4,5}$</td>
</tr>
<tr>
<td>5</td>
<td>(1,3,4,5,2)</td>
<td>$&lt; (1,3,4,5,2) &gt;$</td>
<td>${1,3,4,5,2}$</td>
</tr>
</tbody>
</table>

Thus, we use the table above to obtain the following relations of $N = A_5$:

\[
(1,2)(3,4)t_1 = xt,
\]
\[
(1,2)(3,4)t_4 = xt^{yx},
\]
\[
(1,2)(3,4)t_5 = xt^{y^{-1}},
\]
\[
(1,2,3)t_1 = yxy^{-1}xt,
\]
(1, 2, 3)t_4 = yxyxy^{-1}xt^{y},

(1, 2, 3)t_5 = yxyxy^{-1}tx^{y},

(1, 2, 3, 4, 5)t_1 = xyt,

(1, 3, 4, 5, 2)t_1 = xy^{-1}xyxy^{-1}xt.

Thus, a presentation of the progenitor of \(G = 2^*5 : A_5\) factored by all relations of the first order is

\[ G < x, y, t > = \text{Group} < x, y, t | x^2, y^2, (x \ast y)^5, t^2, (t, yxy^{-1}1xy^{-1}), (t, y^{-1}xy^{-1}xyxy^{-1}), (xt)^a, (xt^{-1})^b, (yxyxy^{-1}x)^c, (yxyxy^{-1}xt^{(yx)})^d, (yxyxy^{-1}xt^{(yt)})^e, (xyt)^f, \]

\((xy^{-1}xyxy^{-1}xt)^g \rangle \]

Hence, the table below shows some finite images of the progenitor 2*5 : A_5 factored by all relations of the first order.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>1920</td>
<td>2(A_5 : 2^4)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>720</td>
<td>A_5 : 2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>175560</td>
<td>J_1</td>
</tr>
</tbody>
</table>

1.3 Monomial Progenitor 11*2 :_m D_{10}

Given that \(D_{10}\) has a monomial irreducible representation in dimension 2, write a progenitor for 11*2 :_m D_{10}. Let’s show that a presentation for \(D_{10}\) is given as \(< x, y | x^5 = y^2 = (xy)^2 = 1 >\).

Proof. Given \(D_{10} = < (1, 2, 3, 4, 5), (1, 5)(2, 4) \rangle\). Let \(F\) be a free group with basis \(X = \{x, y\}\). Define a homomorphism

\[ \phi : F \rightarrow D_{10} \]

by \(\phi(x) = (1, 2, 3, 4, 5)\) and \(\phi(y) = (1, 5)(2, 4)\).

From Theorem 1.6, we have \(\phi\) is an onto homomorphism. Let \(G = F/R\), where \(R = < x^5, y^2, (xy)^2 >\) (in \(G, x^5 = 1, y^2 = 1, (xy)^2 = 1\)). Thus,

\[ \phi : F_{\text{homo, onto}} \rightarrow D_{10}. \]
We know $F/\ker \phi \cong D_{10}$. Now we want to show that $R \leq \ker \phi$. We compute the following:

$\phi(x^5) = (\phi(x))^5 = ((1, 2, 3, 4, 5))^5 = 1$

$\implies x^5 \in \ker \phi,$

$\phi(y^2) = (\phi(y))^2 = ((1, 5)(2, 4))^2 = 1$

$\implies y^2 \in \ker \phi,$

$\phi((xy)^2) = (\phi(x)\phi(y))^2 = ((1, 2, 3, 4, 5)(1, 5)(2, 4))^2 = 1$

$\implies (xy)^2 \in \ker \phi.$

So, $x^5, y^2, (xy)^2 \in \ker \phi.$ Thus, $<x^5, y^2, (xy)^2 > \cong R \cong \ker \phi,$

$|F/R| \geq |F/\ker \phi|$ (since $R$ and $\ker \phi$ are normal)

$\implies |F/R| \geq |D_{10}| = 10$

$\implies |G| \geq 10.$

Now we need to show $|G| \leq 10$. That is to show

$G = F/R \leq \{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}.$

We need to show that the above set is closed under right multiplication by $x$ and $y$.

(i) Show $\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}$ is closed under right multiplication by $y$.

(1) $(Rxy)y = Rxy^2 = xRy^2$ (since $R$ is normal)

$= xR$ (since $y^2 \in R$)

$= Rx \text{ belongs to the set above.}$

(2) $(Rx^2y)y = Rx^2y^2 = x^2Ry^2$ (since $R$ is normal)

$= x^2R$ (since $y^2 \in R$)

$= Rx^2 \text{ belongs to the set above.}$
(3) \[(Rx^3 y)y = Rx^3 y^2 = x^3 Ry^2 \text{ (since } R \text{ is normal)}
= x^3 R \text{ (since } y^2 \in R)
= Rx^3 \text{ belongs to the set above.}\]

(4) \[(Rx^4 y)y = Rx^4 y^2 = x^4 Ry^2 \text{ (since } R \text{ is normal)}
= x^4 R \text{ (since } y^2 \in R)
= Rx^4 \text{ belongs to the set above.}\]

So \(\{ R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2 y, Rx^3 y, Rx^4 y \} \) is closed under right multiplication by \(y\).

(ii) Show \(\{ R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2 y, Rx^3 y, Rx^4 y \} \) is closed under right multiplication by \(x\).

(1) Note: \(R(xy)^2 = R \text{ (since } (xy)^2 \in R)\)

\[
\begin{align*}
\Rightarrow Rxyxy &= R \\
\Rightarrow xRyx &= Ry^{-1} \\
&\Rightarrow Ryx = x^{-1}Ry^{-1} \\
&\Rightarrow Ryx = Rx^{-1}y^{-1}.
\end{align*}
\]

Then \(Ryx = Rx^4 Ry\) since \(Rx^5 = R \Rightarrow Rx^4 = Rx^{-1}\) and \(Ry^2 = R \Rightarrow Ry = Ry^{-1}\). Thus, \(Ryx = Rx^4 y\) belongs to the set above.

(2) \(Rxyx = xRyx \text{ (since } R \text{ is normal)}
= xRx^4 y
= Rx^5 y
= Ry \text{ belongs to the set above.}\)
(3) \((Rx^2y)x = Rx^2yx\) (since \(R\) is normal)
\[= x^2Ryx\]
\[= x^2Rx^4y\]
\[= Rx^6y\]
\[= Rx^6y \text{ belongs to the set above.}\]

(4) \((Rx^3y)x = Rx^3yx\) (since \(R\) is normal)
\[= x^3Ryx\]
\[= x^3Rx^4y\]
\[= Rx^7y\]
\[= Rx^7y \text{ belongs to the set above.}\]

(5) \((Rx^4y)x = Rx^4yx\) (since \(R\) is normal)
\[= x^4Ryx\]
\[= x^4Rx^4y\]
\[= Rx^8y\]
\[= Rx^8y \text{ belongs to the set above.}\]

So \(\{R, Rx, Rx^2, Rx^3, Rx^4, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y\}\) is closed under right multiplication by \(x\).

Hence \(|G| = 10\). So \(G \cong D_{10}\),
\[
\phi : F \text{ homo} \to D_{10} \\
F/\ker\phi \cong D_{10}.
\]

By Third Isomorphism Theorem, we have an onto homomorphism
\[
\psi : F/R \text{ homo} \to F/\ker\phi \\
R \leq \ker\phi \leq F \\
|F/R| \leq |F/\ker\phi| \\
F/R/\ker\psi \cong F/\ker\phi
\]
with \(F/R = 10\) and \(F/\ker\phi = 10\). So \(\ker\psi = 1\). Thus, \(F/R \cong F/\ker\phi \cong D_{10}\).
So $D_{10}$ has a presentation of \( \{ x, y | x^5 = y^2 = (xy)^2 = 1 \} \).

Given a presentation for $D_{10}$ is $G \langle x, y \rangle = \langle x, y | x^5 = y^2 = (xy)^2 = 1 \rangle$ and a corresponding permutation representation is $N = \langle x, y \rangle$, where $x = (1, 2, 3, 4, 5)$ and $y = (1, 5)(2, 4)$. By using Magma we get the character table of $G = D_{10}$.

Table 1.4: Character Table of $G = D_{10}$

<table>
<thead>
<tr>
<th>Conjugacy Classes $C_\alpha$</th>
<th>Order $h_\alpha$</th>
<th>1</th>
<th>(1,5)(2,4)</th>
<th>(1,2,3,4,5)</th>
<th>(1,3,5,2,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>2</td>
<td>0</td>
<td>$W_1$</td>
<td>$W_1 #2$</td>
<td>$W_1$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>2</td>
<td>0</td>
<td>$W_1 #2$</td>
<td>$W_1$</td>
<td></td>
</tr>
</tbody>
</table>

Explanation of Character Value Symbol

# denotes algebraic conjugation, that is,

$#k$ indicates replacing the root of unity $w$ by $w^k$

\[
W_1 = \text{zeta}(5)_3 + \text{zeta}(5)_2^3 \\
W_1 \#2 = \text{zeta}(5)_5 + \text{zeta}(5)_5^4
\]

Now we find a subgroup $H$ of $D_{10}$ of index $n = 2$. We use the following formula, $n = \frac{|D_{10}|}{|H|} = \frac{10}{|H|} = 2$, that implies $|H| = 5$. Let $H = \langle 1, (1, 2, 3, 4, 5) \rangle$. The character table of $H = \mathbb{Z}_5$ is given below.

Table 1.5: Character Table of $H = \mathbb{Z}_5$

| Conjugacy Classes $\phi_\alpha$ | Order $|\phi_\alpha|$ | 1 | (1,2,3,4,5) | (1,3,5,2,4) | (1,4,2,5,3) | (1,5,4,3,2) |
|-------------------------------|---------------------|---|-------------|-------------|-------------|-------------|
| $\phi_1$                      | 1                   | 1 | 1           | 1           | 1           | 1           |
| $\phi_2$                      | 1                   | $Z_1$ | $Z_1 \#2$ | $Z_1 \#3$ | $Z_1 \#4$ |             |
| $\phi_3$                      | 1                   | $Z_1 \#2$ | $Z_1 \#4$ | $Z_1$       | $Z_1 \#3$ |             |
| $\phi_4$                      | 1                   | $Z_1 \#3$ | $Z_1$     | $Z_1 \#4$ | $Z_1 \#2$ |             |
| $\phi_5$                      | 1                   | $Z_1 \#4$ | $Z_1 \#3$ | $Z_1 \#2$ |             | $Z_1$ |

where $Z_1 = \text{zeta}(5)_5$.

Now we look at the finite smallest field that has fifth roots of unity, which is

$\mathbb{Z}_{11}\setminus\{0\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Let $Z_1 = \text{zeta}(5)_5 = 4$. 

So \( Z1\#2 = (zeta(5))_5^2 \equiv 5 \pmod{11} \),
\( Z1\#3 = (zeta(5))_5^3 \equiv 9 \pmod{11} \), and
\( Z1\#4 = (zeta(5))_5^4 \equiv 3 \pmod{11} \).

Next we use the following formula
\[
\phi^G_{\alpha} = \frac{n}{h} \sum_{w \in C_\alpha \cap H} \phi(w),
\]
to induce the character \( \phi_2 \) of \( H \) up to \( D_{10} \). We get
\[
\phi^G_2(1) = \frac{n}{h_1} \sum_{w \in C_1 \cap H} \phi_2(w) = \sum_{w \in \{1\}} \phi_2(w) = 2 \phi_2(1) = 2(1) = 2.
\]
\[
\phi^G_2((1,5)(2,4)) = \frac{n}{h_2} \sum_{w \in C_2 \cap H} \phi_2(w) = \frac{2}{5} \sum_{w \in \{1,5\}(2,4) \cap H} \phi_2(w) = \frac{2}{5}(0) = 0.
\]
\[
\phi^G_2((1,2,3,4,5)) = \frac{n}{h_3} \sum_{w \in C_3 \cap H} \phi_2(w) = \frac{2}{5} \sum_{w \in \{1,2,3,4,5\}} \phi_2(w) = 1(\phi_2(1,2,3,4,5) + \phi_2(1,5,4,3,2)) = Z1 + Z1\#4 = zeta(5)_5 + zeta(5)^4_5 = W1\#2.
\]
\[
\phi^G_2((1,3,5,2,4)) = \frac{n}{h_4} \sum_{w \in C_4 \cap H} \phi_2(w) = \frac{2}{5} \sum_{w \in \{1,3,5,2,4\}} \phi_2(w) = 1(\phi_2(1,3,5,2,4) + \phi_2(1,4,2,5,3)) = Z1\#2 + Z1\#3 = zeta(5)^2_5 + zeta(5)^2_5 = W1.
\]

Thus, \( \phi^G_2 = \lambda_4 \).
Hence the representation of $H$ (respect to $\phi_2$) yields:

$$
B(1) = 1 \\
B(1, 2, 3, 4, 5) = 4 \\
B(1, 3, 5, 2, 4) = 5 \\
B(1, 4, 2, 5, 3) = 9 \\
B(1, 5, 4, 3, 2) = 3 \\
B(g) = 0 \quad \text{if } g \notin G.
$$

Since $H \leq G$, the right transversals of $H$ in $G$ (or a complete set of right coset representatives) are $t_1 = e$ and $t_2 = (1, 5)(2, 4)$

$$
\implies G = He \cup H(1, 5)(2, 4).
$$

Now we use the formula for monomial representation to find $A(x)$ and $A(y)$, where $x = (1, 2, 3, 4, 5)$ and $y = (1, 5)(2, 4)$:

$$
A(x) = \begin{bmatrix}
B(t_1xt_1^{-1}) & B(t_1xt_2^{-1}) \\
B(t_2xt_1^{-1}) & B(t_2xt_2^{-1})
\end{bmatrix} \\
= \begin{bmatrix}
B(1, 2, 3, 4, 5) & B((1, 2, 3, 4, 5)(1, 5)(2, 4)) \\
B((1, 5)(2, 4)(1, 2, 3, 4, 5)) & B((1, 5)(2, 4)(1, 2, 3, 4, 5)(1, 5)(2, 4))
\end{bmatrix} \\
= \begin{bmatrix}
B(1, 2, 3, 4, 5) & B((1, 4)(3, 2)) \\
B((5, 2)(3, 4)) & B(1, 5, 4, 3, 2)
\end{bmatrix} \\
= \begin{bmatrix}
4 & 0 \\
0 & 3
\end{bmatrix}.
$$

Thus, $A(x) = \begin{bmatrix}
4 & 0 \\
0 & 3
\end{bmatrix}$.

This matrix has 2 columns: label the columns 1, and 2 as $t_1$, and $t_2$, respectively. The entries of the matrix are in $\mathbb{Z}_{11}$. Hence, $t_i's$ are of order 11.

We label $t_1^2$ and $t_2^2$ as 3 and 4; $t_1^3$ and $t_2^3$ as 5 and 6; $t_1^4$ and $t_2^4$ as 7 and 8; $t_1^5$ and $t_2^5$ as 9 and 10; $t_1^6$ and $t_2^6$ as 11 and 12; $t_1^7$ and $t_2^7$ as 13 and 14; $t_1^8$ and $t_2^8$ as 15 and 16; $t_1^9$ and
Now $A(x)$ is a monomial automorphism of $<t_1>*<t_2>$ given by $a_{ij} = a \iff t_i \rightarrow t_j^a$. Thus, $a_{11} = 4$ or $t_1 \rightarrow t_1^4$ and $a_{22} = 3$ or $t_2 \rightarrow t_2^3$. We use the chart above, to write down the permutation representation for $A(x)$.

So, $A(x) = (t_1, t_1^4, t_1^5, t_1^6)(t_2, t_2^4, t_2^5, t_2^6)(t_1, t_1^4, t_1^5, t_1^6)(t_2, t_2^4, t_2^5, t_2^6)(t_1, t_1^4, t_1^5, t_1^6)(t_2, t_2^4, t_2^5, t_2^6)(t_1, t_1^4, t_1^5, t_1^6)(t_2, t_2^4, t_2^5, t_2^6).

Then $A(x) = (1, 7, 9, 17, 5)(3, 15, 19, 13, 11)(2, 6, 18, 10, 8)(4, 12, 14, 20, 16)$.

Now $A(y) = \begin{bmatrix} B(t_1^1y t_1^{-1}) & B(t_1^1y t_2^{-1}) \\ B(t_2^1y t_1^{-1}) & B(t_2^1y t_2^{-1}) \end{bmatrix}$
Thus, $A(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Now $A(y)$ is an automorphism (permutation) of $< t_1 > < t_2 >$ given by
\[ a_{ij} = a \iff t_i \to t_j \]
Thus, $a_{12} = 1$ or $t_1 \to t_2$ and $a_{21} = 1$ or $t_2 \to t_1$.

Table 1.8: Permutations of the $t'_i$s using $A(y)$

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Using the chart above, we get the following permutation representation for $A(y) = (t_1, t_2)(t_1^2, t_2^2)(t_1^3, t_2^3)(t_1^4, t_2^4)(t_1^5, t_2^5)(t_1^6, t_2^6)(t_1^7, t_2^7)(t_1^8, t_2^8)(t_1^9, t_2^9)(t_1^{10}, t_2^{10})$.

Then
\[ A(y) = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20). \]

Thus, $D_{10} = A(x, A(y))$
\[ = (1, 7, 9, 17, 5)(3, 15, 19, 13, 11)(2, 6, 18, 10, 8)(4, 12, 14, 16, 20), \]
\[ (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20) >. \]

We are now in a position to give a monomial presentation of the progenitor $11^{*2} :_m D_{10}$. A presentation of $D_{10}$ is $< x, y | x^5 = y^2 = (x * y)^2 = 1 >$.

We fix one of the two $t'_i$s, say $t_1$ and call it $t$. Next, we compute the normalizer of the subgroup $< t_1 >$ in $D_{10}$. Therefore we compute the set stabilizer in $D_{10}$ of the set $\{ t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7, t_1^8, t_1^{10} \} = \{ 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \}$ which is $< (1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)(4, 14, 16, 12, 20) >$ and that $x^2 = (1, 9, 5, 7, 17)(2, 18, 8, 6, 10)(3, 19, 11, 15, 13)(4, 14, 16, 12, 20)$. Hence, a presentation for the monomial progenitor $11^{*2} :_m D_{10}$ is given by
\[ G < x, y, t > := Group < x, y, t | x^5 = y^2 = (x * y)^2 = 1, t^{11}, t^{x^2} = t^5 >. \]

Next we add the relation, $t * t^y = t^y * t$ to the progenitor of $D_{10}$, to verify if the monomial progenitor $11^{*2} :_m D_{10}$ is correct. By using MAMGA we verified that the monomial progenitor, $G < x, y, t > := Group < x, y, t | x^5, y^2, (x * y)^2, t^{11}, t^{x^2} = t^5 >$.
$t^5, (t, ty) >$ is correct:

```plaintext
g< x, y, t >:= Group< x, y, t | x^5 = y^2 = (x*y)^2 = 1, t^11, 
t^2 = t^5, t*x = y^2 * t; >
f, G, k: = CosetAction(G, sub<G|x, y>);
> #G;
1210
> #k;
1
> IN:= sub<G1| f(x), f(y)>;
> T:= sub<G1| f(t)>;
> #T;
11
> Normaliser(IN, T);
Permutation group acting on a set of cardinality 121
Order = 5
> Index(IN, Normaliser(IN, T));
2
```

Hence, we have a presentation of the progenitor $11^*2 :_m D_{10}$

```plaintext
g< x, y, t >:= Group< x, y, t | x^5 = y^2 = (x*y)^2 = 1, t^11, 
t^2 = t^5 >
```

Next, we apply the first order relation to the progenitor $11^*2 :_m D_{10}$.

### 1.3.1 $11^*2 :_m D_{10}$ Factor by First Order Relations

Given a progenitor of the form $m^n : N$ and $p^n : N$.

All relations of the first order that $m^n : N$ and $p^n : N$ can be factored by are obtained as follows. Compute the conjugacy classes on $N$. Now we compute the centralizers of the representatives of each non-identity class. Then, we determine the orbits of the centralizer. Once we have the orbits we take the representative from each class and we right multiply by a $t_i$.

Consider the monomial progenitor of $11^*2 :_m D_{10}$ that has the following presentation

```plaintext
g< x, y, t >:= Group< x, y, t | x^5 = y^2 = (x*y)^2 = 1, t^11, 
t^2 = t^5 >.
```
To compute all first order relations for the monomial progenitor \(11^2 :_m D_{10}\),
we run the following code in MAGMA.

```magma
C:= Classes(N);
C;
C2:=Centraliser(N,N!(1,5)(2,4));
C2;
C3:=Centraliser(N,N!(1,2,3,4,5));
C3;
C4:=Centraliser(N,N!(1,3,5,2,4));
C4;
Set(C2);
Orbits(C2);
Set(C3);
Orbits(C3);
Set(C4);
Orbits(C4);
```

Then, we summarize the result in the table below.

<table>
<thead>
<tr>
<th>Classes</th>
<th>Centralizer</th>
<th>Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_2 = (1,5)(2,4))</td>
<td>(&lt; y &gt; {1,5}, {2,4}, {3})</td>
<td></td>
</tr>
<tr>
<td>(C_3 = (1,2,3,4,5))</td>
<td>(&lt; x &gt; {1,2,3,4,5})</td>
<td></td>
</tr>
<tr>
<td>(C_4 = (1,3,5,2,4))</td>
<td>(&lt; x^2 &gt; {1,3,5,2,4})</td>
<td></td>
</tr>
</tbody>
</table>

Next, we pick a representative from each orbit and we multiply by the
representative from each class. Thus, the all first order relations are

\[(yt)^a, (yt^x)^b, (yt^{x^2})^c, (xt)^d, \text{ and } (x^{2t})^e, \text{ where } t \sim t_1.\]

Hence, we factor the monomial progenitor \(11^2 :_m D_{10}\) by the relations

\[(yt)^a, (yt^x)^b, (yt^{x^2})^c, (xt)^d, \text{ and } (x^{2t})^e, \]

to obtained the following homomorphic images:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>660</td>
<td>(L_2(11))</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>6600</td>
<td>((5 \times L_2(11)) : 2)</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>6</td>
<td>435600</td>
<td>(L_2(11) \times L_2(11))</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>5</td>
<td>1351680</td>
<td>(2 \times (2^{10} : L_2(11)))</td>
</tr>
</tbody>
</table>
1.4 Progenitor of $2^*7 : D_{14}$

We want to write a permutation progenitor of $2^*7 : D_{14}$ where our control group is $N = D_{14}$. A presentation of $D_{14}$ is \{\(x, y | x^7 = y^2 = (xy)^2 = 1\}\}. We are going to prove the presentation of $D_{14}$.

Proof. Given $D_{14} = < (1, 2, 3, 4, 5, 6, 7), (1, 6)(2, 5)(3, 4) >$. Let $F$ be a free group with basis $X = \{x, y\}$. Define a homomorphism 

$$\phi : F \rightarrow D_{14}$$

by $\phi(x) = (1, 2, 3, 4, 5, 6, 7)$ and $\phi(y) = (1, 6)(2, 5)(3, 4)$. From Theorem 1.6, we have $\phi$ is an onto homomorphism. Let $G = F/R$, where $R = < x^7, y^2, (xy)^2 >$ (in $G$, $x^7 = 1, y^2 = 1, (xy)^2 = 1$). Thus,

$$\phi : F \text{ homo.} \rightarrow \text{onto} D_{14}.$$  

We know $F/\ker\phi \cong D_{14}$. Now we want to show that $R \leq \ker\phi$. We compute the following:

$$\phi(x^7) = (\phi(x))^7 = ((1, 2, 3, 4, 5, 6, 7))^5 = 1$$  

$\implies x^7 \in \ker\phi,$  

$$\phi(y^2) = (\phi(y))^2 = ((1, 6)(2, 5)(3, 4))^2 = 1$$  

$\implies y^2 \in \ker\phi,$  

$$\phi((xy)^2) = (\phi(x)\phi(y))^2 = ((1, 2, 3, 4, 5, 6, 7)(1, 6)(2, 5)(3, 4))^2 = 1$$  

$\implies (xy)^2 \in \ker\phi.$

So $x^7, y^2, (xy)^2 \in \ker\phi$. Thus, $< x^7, y^2, (xy)^2 > \cong R \cong \ker\phi,$

$$|F/R| \geq |F/\ker\phi| \quad (\text{since } R \text{ and } \ker\phi \text{ are normal})$$

$\implies |F/R| \geq |D_{14}| = 14$

$\implies |G| \geq 14.$

Now we need to show $|G| \leq 14$. That is to show

$$G = F/R \leq \{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}.$$  

We need to show that the above set is closed under right multiplication by $x$ and $y$. 
(i) Show
\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}
is closed under right multiplication by \(y\).

1. \((Rxy)y = Rxy^2 = xRy^2 \) (since \(R\) is normal)
   \[= xR \quad \text{(since } y^2 \in R)\]
   \[= Rx \quad \text{belongs to the set above.}\]

2. \((Rx^2y)y = Rx^2y^2 = x^2Ry^2 \) (since \(R\) is normal)
   \[= x^2R \quad \text{(since } y^2 \in R)\]
   \[= Rx^2 \quad \text{belongs to the set above.}\]

3. \((Rx^3y)y = Rx^3y^2 = x^3Ry^2 \) (since \(R\) is normal)
   \[= x^3R \quad \text{(since } y^2 \in R)\]
   \[= Rx^3 \quad \text{belongs to the set above.}\]

4. \((Rx^4y)y = Rx^4y^2 = x^4Ry^2 \) (since \(R\) is normal)
   \[= x^4R \quad \text{(since } y^2 \in R)\]
   \[= Rx^4 \quad \text{belongs to the set above.}\]

5. \((Rx^5y)y = Rx^5y^2 = x^5Ry^2 \) (since \(R\) is normal)
   \[= x^5R \quad \text{(since } y^2 \in R)\]
   \[= Rx^5 \quad \text{belongs to the set above.}\]

6. \((Rx^6y)y = Rx^6y^2 = x^6Ry^2 \) (since \(R\) is normal)
   \[= x^6R \quad \text{(since } y^2 \in R)\]
   \[= Rx^6 \quad \text{belongs to the set above.}\]

So \(\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}\)
is closed under right multiplication by \(y\).
(ii) Show

\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y \} is closed under right multiplication by \( x \).

1. Note: \( R(xy)^2 = R \) (since \( xy \) is in \( R \))
   \( \implies Rxyxy = R \)
   \( \implies xRyx = Ry^{-1} \)
   \( \implies Ryx = x^{-1}Ry^{-1} \)
   \( \implies Ryx = Rx^{-1}y^{-1} \).

Then \( Ryx = Rx^6y \) since \( Rx^6 = R \implies Rx^6 = Rx^{-1} \) and \( Ry^2 = R \implies Ry = Ry^{-1} \).

Thus, \( Rx^2y \) belongs to the set above.

2. \( Rxyx = xRyx \) (since \( R \) is normal)
   \( = xRx^6y \)
   \( = Rx^7y \)
   \( = Ry \) belongs to the set above.

3. \( (Rx^2y)x = Rx^2yx \) (since \( R \) is normal)
   \( = x^2Ryx \)
   \( = x^2Rx^6y \)
   \( = Rx^8y \)
   \( = Rxy \) belongs to the set above.

4. \( (Rx^3y)x = Rx^3yx \) (since \( R \) is normal)
   \( = x^3Ryx \)
   \( = x^3Rx^6y \)
   \( = Rx^9y \)
   \( = Rx^2y \) belongs to the set above.
(5) \((Rx^4y)x = Rx^4yx\) (since \(R\) is normal)
\[= x^4Ryx \]
\[= x^4Rx^6y \]
\[= Rx^{10}y \]
\[= Rx^3y \text{ belongs to the set above.} \]

(6) \((Rx^5y)x = Rx^5yx\) (since \(R\) is normal)
\[= x^5Ryx \]
\[= x^5Rx^6y \]
\[= Rx^{11}y \]
\[= Rx^4y \text{ belongs to the set above.} \]

(7) \((Rx^6y)x = Rx^6yx\) (since \(R\) is normal)
\[= x^6Ryx \]
\[= x^6Rx^6y \]
\[= Rx^{12}y \]
\[= Rx^5y \text{ belongs to the set above.} \]

So \(\{R, Rx, Rx^2, Rx^3, Rx^4, Rx^5, Rx^6, Ry, Rxy, Rx^2y, Rx^3y, Rx^4y, Rx^5y, Rx^6y\}\)
is closed under right multiplication by \(x\).

Hence \(|G| = 14\). So \(G \cong D_{14}\).

\[\phi : F \overset{\text{homo, onto}}{\longrightarrow} D_{14} \]
\[F/\ker \phi \cong D_{14}. \]

By Third Isomorphism Theorem, we have an onto homomorphism

\[\psi : F/R \overset{\text{homo, onto}}{\longrightarrow} F/\ker \phi \]
\[R \leq \ker \phi \leq F \]
\[|F/R| \leq |F/\ker \phi| \]
\[F/R/\ker \psi \cong F/\ker \phi \]

with \(F/R = 14\) and \(F/\ker \phi = 14\). So \(\ker \psi = 1\). Thus,
\[F/R \cong F/\ker \phi \cong D_{14}. \]
So $D_{14}$ has a presentation of $\{x, y | x^7 = y^2 = (xy)^2 = 1\}$. 

Moreover, the corresponding permutation representation is $N = D_{14} = \langle x, y \rangle$, where $x = (1, 2, 3, 4, 5, 6, 7)$ and $y = (1, 6)(2, 5)(3, 4)$. Now we add the free product $2^7$ ($|S^7| = 2$) to this group to form our progenitor. Hence, a presentation for the progenitor $2^7 : N$ is given by $\langle x, y, t | x^7, y^2, (xy)^2, t^2, (t, N^7) \rangle$, where $t \sim t_7$. $N^7$ is the point stabilizer of 7, and $(t, N^7) = 1$ means that $7^g = 7 \forall g \in N^7$. Note that $7^g = 7 \forall g \in N^7$ implies $t$ has $[N:N^7]$ conjugates in $N$. Using Magma, we can see that the point stabilizer of 7 in $N$ is equal to $N^7 = \langle (1, 6)(2, 5)(3, 4) \rangle = \langle y \rangle$. Hence, a presentation of the progenitor $2^7 : D_{14}$ is given by $G = \langle x, y, t | x^7, y^2, (xy)^2, t^2, (t, y), (xt^a), (xt^t)^b, (xtyt)^c, (ttxt)^d \rangle$.

In conclusion we obtained the following progenitor:

\[
\langle x, y, t | x^7, y^2, (xy)^2, t^2, (t, y), (xt^a), (xt^t)^b, (xtyt)^c, (ttxt)^d \rangle.
\]

The table below shows some finite images of the progenitor $2^7 : D_{14}$.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>196</td>
<td>$((7 \times 7) : 2)$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>79464</td>
<td>$PGL_2(43)$</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>3</td>
<td>7</td>
<td>9828</td>
<td>$L_7(27)$</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>68880</td>
<td>$PGL_2(41)$</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>6</td>
<td>9</td>
<td>336</td>
<td>$PGL_2(7)$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>7</td>
<td>3</td>
<td>1092</td>
<td>$L_2(13)$</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>504</td>
<td>$L_2(8)$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>24360</td>
<td>$PGL_2(29)$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>8</td>
<td>8</td>
<td>672</td>
<td>$PGL_2(7) \times 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>9</td>
<td>0</td>
<td>1008</td>
<td>$L_2(8) \times 2$</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>0</td>
<td>9</td>
<td>2184</td>
<td>$L_2(13) \times 2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>5040</td>
<td>$A_7 : 2$</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>21504</td>
<td>$(2^6 \cdot L_2(7)) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>10</td>
<td>0</td>
<td>48720</td>
<td>$PGL_2(29) \times 2$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>4368</td>
<td>$PGL_2(13) \times 2$</td>
</tr>
</tbody>
</table>

In later chapters, we will construct some of the simple groups, given above, using the technique of double coset enumeration and prove their simplicity.
Chapter 2

Double Coset Enumeration and Factoring by the Center

2.1 Double Coset Enumeration Preliminaries

Definition 2.1. (Normal Subgroup). A subgroup $H \leq G$ is a normal subgroup, denoted by $H \trianglelefteq G$, if $gHg^{-1} = H$ for every $g \in G$. [Rot12]

Definition 2.2. (Right Coset). If $H \leq G$ and if $k \in G$, then a right coset of $H$ in $G$ is the subset of $G$

$$Hk = \{hk : h \in H\},$$

where $k$ is a representative of $Hk$. [Rot12]

Definition 2.3. (Index). If $H \leq G$, then the index of $H \in G$, denoted by $[G:H]$, is the number of single cosets of $H$ in $G$. [Rot12]

Definition 2.4. (Order). If $G$ is a group, then the order of $G$, denoted by $|G|$, is the number of elements in $G$. [Rot12]

Theorem 2.5. (Lagrange). If $G$ is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G:H] = |G|/|H|$. [Rot12]

Definition 2.6. (Double Coset). Let $H$ and $K$ be subgroups of the group $G$ and define a relation on $G$ as follows:
\[ x \sim y \iff \exists h \in H \text{ and } k \in K \text{ such that } y = hxk \]

where \( \sim \) is an equivalence relation and the equivalence classes are sets of the following form

\[ HxK = \{ hxk \mid h \in H, k \in K \} = \bigcup_{k \in K} Hxk = \bigcup_{h \in H} hxK \]

Such a subset of \( G \) is called a double coset. \[\text{Cur07}\]

**Definition 2.7. (Point Stabilizer).** Let \( G \) be a group of permutations of a set \( S \). For each \( g, s \in S \), let \( g^s = g \), then we call the set of \( s \in S \) the point stabilizer of \( g \in G \). \[\text{Cur07}\]

**Definition 2.8. (Coset Stabilizing Group).** The coset stabilizing group of a coset \( Nw \) is defined as

\[ N^{(w)} = \{ \pi \in N \mid Nw\pi = Nw \} \]

where \( n \in N \) and \( w \) is a reduced word in the \( t_i \)'s. \[\text{Cur07}\]

**Theorem 2.9. (Number of single cosets in \( NwN \)).** From above we see that,

\[ N^{(w)} = \{ \pi \in N \mid Nw\pi = Nw \} = \{ \pi \in N \mid Nw\pi w^{-1} = N \} \]
\[ = \{ \pi \in N \mid (Nw)^\pi = Nw \} \]

and the number of single cosets in \( NwN \) is given by \([N : N^{(w)}]\). \[\text{Cur07}\]

**Definition 2.10. (Orbits).** Let \( G \) be a group of permutations of a set \( S \). For each \( s \in S \), let \( \text{orb}_G(s) = \{ \phi(s) \mid \phi \in G \} \). The set \( \text{orb}_G(s) \) is a subset of \( S \) called the orbits of \( s \) under \( G \). We use \( |\text{orb}_G(s)| \) to denote the number of elements in \( \text{orb}_G(s) \). \[\text{Rot12}\]

**Definition 2.11. (Transversal).** If \( K \leq G \), then a (right) transversal of \( K \) in \( G \) (or a complete set of right coset representatives) is a subset \( T \) of \( G \) consisting of one element from each right coset of \( K \) in \( G \). \[\text{Rot12}\]

**Definition 2.12. (Center).** The center of a group \( G \), denoted by \( Z(G) \), is the set of all \( a \in G \) that commute with every element of \( G \). \[\text{Rot12}\]
2.2 $2 \times A_5$ as a Homomorphic Image of $2^3 : S_3$

2.2.1 Construction of $2 \times A_5$ over $S_3$

Consider the group $G = 2^3 : S_3$ factored by the relator $[(0,1,2)t_3]^5$. Note: $N = S_3 = \{e, (1,2), (1,0), (2,0), (1,2,0), (1,0,2)\}$, where $x \sim (0,1,2)$ and $y \sim (1,2)$. Let $t \sim t_3 \sim t_0$. Let us expand the relator:

$$[(0,1,2)t_0]^5 = 1$$

with $\pi = (0,1,2)$ becomes

$$1 = [\pi t_0]^5 = \pi^5 t_0^4 \pi^3 t_0^2 \pi^2 t_0 \pi$$

$$= (0,2,1)t_0(0,1,2)t_0(0,2,1)t_0(0,1,2)t_0$$

$$= (3,2,1)t_1t_0t_2t_1t_0$$

$$\implies 1 = (0,2,1)t_1t_0t_2t_1t_0$$

$$\implies t_0t_1 = (0,2,1)t_1t_0t_2$$

$$\implies Nt_0t_1 = Nt_1t_0t_2.$$

Moreover if we conjugate the previous relation by all elements of $S_3$, we obtain the following relations:

$$(t_0t_1)^{(1,2)} = (0,2,1)^{(1,2)}(t_1t_0t_2)^{(1,2)} \implies t_0t_2 = (0,1,2)t_2t_0t_1$$

$$(t_0t_1)^{(1,0)} = (0,2,1)^{(1,0)}(t_1t_0t_2)^{(1,0)} \implies t_1t_0 = (1,2,0)t_0t_1t_2$$

$$(t_0t_1)^{(2,0)} = (0,2,1)^{(2,0)}(t_1t_0t_2)^{(2,0)} \implies t_2t_1 = (2,0,1)t_1t_2t_0$$

$$(t_0t_1)^{(1,2,0)} = (0,2,1)^{(1,2,0)}(t_1t_0t_2)^{(1,2,0)} \implies t_1t_2 = (1,0,2)t_2t_1t_0$$

$$(t_0t_1)^{(1,0,2)} = (0,2,1)^{(1,0,2)}(t_1t_0t_2)^{(1,0,2)} \implies t_2t_0 = (2,1,0)t_0t_2t_1.$$
symmetric generator \( t_i \) from each orbit of the coset stabilising group \( N^{(w)} \)

\[ NeN \]

First, the double coset \( NeN \), is denoted by \([*]\). This double coset contains only the single coset, namely \( N \). Since \( N \) is transitive on \( \{t_0, t_1, t_2\} \), the orbit of \( N \) on \( \{0, 1, 2\} \) is:

\[ O = \{0, 1, 2\}. \]

We choose \( t_0 \) as our symmetric generator from \( O \) and find to which double coset \( Nt_0N \) belongs. \( Nt_0N \) will be a new double coset, denoted by \([0]\). Hence, three symmetric generators will go to the new double coset \([0]\).

\[ Nt_0N \]

In order to find how many single cosets \([0]\) contains, we must first find the coset stabilizer \( N^{(0)} \). Then the number of single coset in \([0]\) is equal to \( \frac{|N|}{|N^{(0)}|} \). Now, \( N^{(0)} = N^0 = e, (1, 2) > \) so the number of the single cosets in \( Nt_0N \) is \( \frac{|N|}{|N^{(0)}|} = \frac{6}{2} = 3 \). These three single cosets in \([0]\) are \( \{Nt_0^n|u \in N\} = \{Nt_0, Nt_1, Nt_2\} \). Furthermore, the orbits of \( N^{(0)} \) on \( \{t_0, t_1, t_2\} \) are:

\[ O = \{0\} \text{ and } \{1, 2\}. \]

We take \( t_0 \) and \( t_1 \) from each orbit, respectively, and to see which double coset \( Nt_0t_0 \) and \( Nt_0t_1 \) belong to. Now \( Nt_0t_0 = N \in [s] \), so one element will go back to \( NeN \) and two symmetric generators will go to a new double coset \( Nt_0t_1N \), denoted by \([01]\).

\[ Nt_0t_1N \]

Now \( Nt_0t_1N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(01)} = N^{01} = \langle e, > \). Only identity \( (e) \) will fix 0 and 1. Hence the number of single cosets in \( Nt_0t_1N \) is \( \frac{|N|}{|N^{(01)}|} = \frac{6}{1} = 6 \). These six single cosets in \([01]\) are \( \{Nt_0t_1, Nt_1t_0, Nt_0t_2, Nt_2t_0, Nt_1t_2, Nt_2t_1\} \). The orbits of \( N^{(01)} \) on \( \{t_0, t_1, t_2\} \) are:

\[ O = \{0\}, \{1\}, \text{ and } \{2\}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_1t_i \) belongs to.
We have:

\[ N_{t_0t_1t_2} = N_{t_0} \in [0] \]

relation: \((0, 2, 1)^{(0,1)}(t_1t_0t_2)^{(0,1)} = (t_0t_1)^{(0,1)} \)

\[ \implies (1, 2, 0)t_0t_1t_2 = t_1t_0 \]

\[ \implies N_{t_0t_1t_2} = N_{t_1t_0} \in [01] \]

\[ N_{t_0t_1t_0} \in [010]. \]

The new double cosets have single coset representatives \(N_{t_0t_1t_0}\), which is denoted by \([010]\).

\(N_{t_0t_1t_0}N\)

Now \(N_{t_0t_1t_0}N\) in \(N\) is a new double coset. However, \(N^{(010)} = N^{010} = < e >.\) Only identity \((e)\) will fix 0, and 1. Hence the number of single cosets contained in \(N_{t_0t_1t_0}N\) is \(\frac{|N|}{|N^{010}|} = \frac{6}{1} = 6.\) These six single cosets in \([010]\) are \(\{N_{t_0t_1t_0}, N_{t_1t_0t_1}, N_{t_0t_2t_0}, N_{t_2t_0t_2}, N_{t_1t_2t_1}, N_{t_2t_1t_2}\}.\) The orbits of \(N^{(010)}\) on \{0, 1, 2\} are:

\[ \emptyset = \{0\}, \{1\}, \text{ and } \{2\}. \]

Take a representative \(t_i\) from each orbit and see which double cosets \(N_{t_0t_1t_0t_i}\) belongs to. We have:

\[ N_{t_0t_1t_0t_0} = N \in [01] \]

relation: \(t_1(0, 2, 1)t_1t_0t_2 = t_1t_0t_1 \)

\[ \implies (0, 2, 1)(0, 2, 1)^{-1}t_1(0, 2, 1)t_1t_0t_2 = t_1t_0t_1 \]

\[ \implies (0, 2, 1)t_1^{(0,2,1)}t_1t_0t_2 = t_1t_0t_1 \]

\[ \implies (0, 2, 1)t_0t_1t_0t_2 = t_1t_0t_1 \]

\[ \implies N_{t_0t_1t_0t_2} = N_{t_1t_0t_1} \in [010] \]

\[ N_{t_0t_1t_0t_1} \in [0101]. \]

The new double coset is \(N_{t_0t_1t_0t_1}N\), denoted by \([0101]\).

\(N_{t_0t_1t_0t_1}N\)
Now $Nt_0t_1t_0t_1N$ is a new double coset. We determine how many single cosets are in this double coset. We have $N^{(0101)} = N^{0101} = e$. But $Nt_0t_1t_0t_1$ is not distinct. Using the relation: $t_0t_1 = (0, 2, 1)t_1t_0t_2 \implies t_0t_1t_0t_1 = (0, 2, 1)t_1t_0t_2t_0t_1 = (0, 2, 1)t_1t_0(2, 1, 0)t_0t_2t_0t_2$. Now $t_0t_1t_0t_1 = (1, 2, 0)t_0t_2t_0t_2$.

$\implies Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2$. Thus $N(t_0t_1t_0t_1)^{n} = Nt_0t_2t_0t_2$. Then $N(t_0t_1t_0t_1)^{(1, 2)} = Nt_0t_2t_0t_2$. But $Nt_0t_2t_0t_2 = Nt_0t_1t_0t_1 \implies (1, 2) \in N^{(0101)}$ since $N(t_0t_1t_0t_1)^{(1, 2)} = Nt_0t_2t_0t_2$. We conclude:

$N^{(0101)} \geq < e, (1, 2) > .$

Hence $|N^{(0101)}| = 2$ so the number of single cosets in $N^{(0101)}$ is $\frac{|N|}{|N^{(0101)}|} = \frac{6}{2} = 3$. These three single cosets in $[010]$ are $Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2, Nt_1t_0t_1t_0 = Nt_1t_2t_1t_2,$ and $Nt_2t_1t_2t_1 = Nt_2t_0t_2t_0$. The orbits of $N^{(0101)}$ on $\{0, 1, 2\}$ are:

$\mathcal{O} = \{0\}$ and $\{1, 2\}$.

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_1t_0t_1t_i$ belongs to. We have:

$Nt_0t_1t_0t_1t_0 \in [01010]$

$Nt_0t_1t_0t_1t_1 = Nt_0t_1t_0 \in [010].$

$Nt_0t_1t_0t_1t_0 N$

Now $Nt_0t_1t_0t_1t_0N$ is indeed a new double coset. We determine how many single cosets are in this double coset. We have $N^{(01010)} = N^{01010} = e$. $Nt_0t_1t_0t_1t_0$ has six names.

We have the following:

$Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 \implies Nt_0t_1t_0t_1t_0 = Nt_0t_2t_0t_2t_0$

$Nt_1t_0t_1t_0 = Nt_1t_2t_1t_2 \implies Nt_1t_0t_1t_0t_1 = Nt_1t_2t_1t_2t_1$

$Nt_2t_1t_2t_1 = Nt_2t_0t_2t_0 \implies Nt_2t_1t_2t_1t_2 = Nt_2t_0t_2t_0t_2$.

Now we want to show that the six names are the same. We have the following:

$t_0t_1t_0t_1t_0 = t_0t_1t_0t_1t_0t_2$  
$= t_0t_1t_0(2, 0, 1)t_0t_1t_2 = t_1t_2t_1t_0t_1t_2$  
$= t_1(2, 0, 1)t_1t_2t_1t_2 = (2, 0, 1)t_1t_2t_1t_2$  
$\implies t_0t_1t_0t_1t_0 = (2, 0, 1)t_1t_2t_1t_2$  (1)
\[ \Rightarrow Nt_0t_1t_0t_1t_0 = Nt_2t_1t_2t_1t_2, \]
\[ t_0t_2t_0t_2t_0 = t_0t_2t_0t_2t_0t_1t_1 \]
\[ = t_0t_2t_0(2, 1, 0)t_0t_2t_1 = t_2t_1t_2t_0t_2t_1 \]
\[ = t_2(1, 0, 2)t_2t_1t_2t_1 = (1, 0, 2)t_1t_2t_1t_2 \]
\[ \Rightarrow t_0t_2t_0t_2t_0 = (1, 0, 2)t_1t_2t_1t_2 \quad (2) \]
\[ \Rightarrow Nt_0t_2t_0t_2t_0 = Nt_1t_2t_1t_2t_1. \]

Now by (1) and (2) we have that the six names are equal. Hence,
\[ t_0t_1t_0t_2t_0 \sim t_0t_2t_0t_2t_0 \sim t_1t_0t_1t_0t_1 \sim t_1t_2t_1t_2t_1 \sim t_2t_1t_2t_1t_2 \sim t_2t_0t_2t_0t_2 \]

Therefore, \( N^{(01010)} = n \in N \middle| N^{(01010)}^n = N^{(01010)} \). Thus, \( N^{(01010)} \geq (1, 2), (0, 2, 1) \) then \( N^{(01010)} = N \). Hence \( |N^{(01010)}| = 6 \), so the number of single cosets in \( N^{(01010)} \) is \( \frac{|N|}{|N^{(01010)}|} = \frac{6}{6} = 1 \). The orbit of \( N^{(01010)} \) on \( \{1, 2, 0\} \) is \( \{1, 2, 0\} \). Take a representative from this orbit, say \( t_0 \). Hence \( Nt_0t_1t_0t_0 \in [0101] \). Therefore, three symmetric generators will go back to \( Nt_0t_1t_0t_1N \).

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of \( N \) in \( G \) is 20. We conclude:
\[ G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N \cup Nt_0t_1t_0N \cup Nt_0t_1t_0t_0N, \]
where
\[ G = \frac{2^*3 : S_3}{t_0t_1 = (0, 2, 1)t_0t_2} \]
\[ |G| \leq (|N| + \frac{|N|}{N^{(010)}} + \frac{|N|}{N^{(01)}} + \frac{|N|}{N^{(0101)}} + \frac{|N|}{N^{(01010)}}) \times |N| \]
\[ |G| \leq (1 + 3 + 6 + 6 + 3 + 1) \times 6 \]
\[ |G| \leq 20 \times 6 \]
\[ |G| \leq 120. \]

A Cayley diagram that summarizes the above information is given below:

![Figure 2.1: Cayley Diagram of \(2 \times A_5\) over \(S_3\)](image)
2.2.2 Permutation Representation of $2 \times A_5$ over $S_3$

In order to find the permutation representation of $G = 2^{3^3} : S_3$, in terms of $x$, $y$, and $t_0$, we create a table in which we conjugate the twenty single cosets by $x$ and $y$ and we right multiply them by $t_0$.

<table>
<thead>
<tr>
<th>Cosets</th>
<th>$x \sim (0, 1, 2)$</th>
<th>$y \sim (1, 2)$</th>
<th>$t \sim t_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $N$</td>
<td>1. $N$</td>
<td>1. $N$</td>
<td>2. $Nt_0$</td>
</tr>
<tr>
<td>2. $Nt_0$</td>
<td>3. $Nt_1$</td>
<td>2. $Nt_0$</td>
<td>1. $N$</td>
</tr>
<tr>
<td>3. $Nt_1$</td>
<td>4. $Nt_2$</td>
<td>4. $Nt_2$</td>
<td>6. $Nt_1t_0$</td>
</tr>
<tr>
<td>4. $Nt_2$</td>
<td>2. $Nt_0$</td>
<td>3. $Nt_1$</td>
<td>8. $Nt_2t_0$</td>
</tr>
<tr>
<td>5. $Nt_0t_1$</td>
<td>9. $Nt_1t_2$</td>
<td>7. $Nt_0t_2$</td>
<td>11. $Nt_0t_1t_0$</td>
</tr>
<tr>
<td>6. $Nt_1t_0$</td>
<td>10. $Nt_2t_1$</td>
<td>8. $Nt_2t_0$</td>
<td>3. $Nt_1$</td>
</tr>
<tr>
<td>7. $Nt_0t_2$</td>
<td>6. $Nt_1t_0$</td>
<td>5. $Nt_0t_1$</td>
<td>13. $Nt_0t_2t_0$</td>
</tr>
<tr>
<td>8. $Nt_2t_0$</td>
<td>5. $Nt_0t_1$</td>
<td>6. $Nt_1t_0$</td>
<td>4. $Nt_2$</td>
</tr>
<tr>
<td>9. $Nt_1t_2$</td>
<td>8. $Nt_2t_0$</td>
<td>10. $Nt_2t_1$</td>
<td>10. $Nt_1t_2t_0$</td>
</tr>
<tr>
<td>10. $Nt_2t_1$</td>
<td>7. $Nt_0t_2$</td>
<td>9. $Nt_1t_2$</td>
<td>9. $Nt_2t_1t_0$</td>
</tr>
<tr>
<td>11. $Nt_0t_1t_0$</td>
<td>15. $Nt_1t_2t_1$</td>
<td>13. $Nt_0t_2t_0$</td>
<td>5. $Nt_0t_1$</td>
</tr>
<tr>
<td>12. $Nt_1t_0t_1$</td>
<td>16. $Nt_2t_1t_2$</td>
<td>14. $Nt_2t_0t_2$</td>
<td>18. $Nt_1t_0t_1t_0$</td>
</tr>
<tr>
<td>13. $Nt_0t_2t_0$</td>
<td>12. $Nt_1t_0t_1$</td>
<td>11. $Nt_0t_1t_0$</td>
<td>7. $Nt_0t_2$</td>
</tr>
<tr>
<td>14. $Nt_2t_0t_2$</td>
<td>11. $Nt_0t_1t_0$</td>
<td>12. $Nt_1t_0t_1$</td>
<td>19. $Nt_2t_0t_2t_0$</td>
</tr>
<tr>
<td>15. $Nt_1t_2t_1$</td>
<td>14. $Nt_2t_0t_2$</td>
<td>16. $Nt_2t_1t_2$</td>
<td>16. $Nt_1t_2t_1t_0$</td>
</tr>
<tr>
<td>16. $Nt_2t_1t_2$</td>
<td>13. $Nt_0t_2t_0$</td>
<td>15. $Nt_1t_2t_1$</td>
<td>15. $Nt_2t_1t_2t_0$</td>
</tr>
<tr>
<td>17. $Nt_0t_1t_0t_1$</td>
<td>18. $Nt_1t_2t_1t_2$</td>
<td>17. $Nt_0t_2t_0t_2$</td>
<td>20. $Nt_0t_1t_0t_1t_0$</td>
</tr>
<tr>
<td>18. $Nt_1t_0t_1t_0$</td>
<td>19. $Nt_2t_1t_2t_1$</td>
<td>19. $Nt_2t_0t_2t_0$</td>
<td>12. $Nt_1t_0t_1$</td>
</tr>
<tr>
<td>19. $Nt_2t_0t_2t_1$</td>
<td>17. $Nt_0t_2t_0t_1$</td>
<td>18. $Nt_1t_2t_1t_2$</td>
<td>14. $Nt_2t_1t_2t_1$</td>
</tr>
<tr>
<td>20. $Nt_0t_1t_0t_1t_0$</td>
<td>20. $Nt_1t_2t_1t_2t_1$</td>
<td>20. $Nt_0t_2t_0t_2t_0$</td>
<td>17. $Nt_0t_1t_0t_1$</td>
</tr>
</tbody>
</table>

We have:

$$
\phi(x) = (2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)
$$

$$
\phi(y) = (3, 4)(5, 7)(6, 8)(9, 10)(11, 13)(12, 14)(15, 16)(18, 19)
$$

$$
\phi(t) = (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20).
$$

Thus, we have a homomorphism $\phi : 2^{3^3} : S_3 \rightarrow S_{20}$. Then $\phi(G) = < \phi(x), \phi(y), \phi(t) >$. In order for us to prove that $\phi(G) = < \phi(x), \phi(y), \phi(t) >$ is a homomorphic image of
\( G = 2^{*3} : S_3 \), we must have the following conditions met:

1. \( \phi(N) \cong S_3 \)
2. \( \phi(t) \) has three conjugates under conjugation by \( \phi(N) \)
3. \( \phi(N) \) acts as \( S_3 \) on the three conjugates of \( \phi(t) \) by conjugates.

**Proof.** We have \( \phi(N) = < \phi(x), \phi(y) >. \)

1. \( \phi(N) = < \phi(x), \phi(y) > 
   = < (2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19), 
   (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20) > 
   \cong S_3 \) since \(|\phi(x)\phi(y)| = |(2, 4)(5, 10)(6, 9)(7, 8)(11, 16)(12, 15)(13, 14)(17, 19)| = 2. 

Thus, \( \phi(N) \cong S_3 \).

2. We need to compute \( \phi(t)^{\phi(N)} : \)

\[
\phi(t)^{\phi(x)} = \{(1, 2)(3, 6) \ldots (15, 16)(17, 20)^{(2,3,4)(5,9,8)\ldots(12,16,13)(17,18,19)}\} \\
= \{(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)\} \\
= t_1.
\]

\[
\phi(t)^{\phi(x^2)} = \{(1, 2)(3, 6) \ldots (15, 16)(17, 20)^{(2,4,3)(5,9,8)\ldots(12,13,16)(17,19,18)}\} \\
= \{(1, 4)(2, 7)(3, 9)(8, 14)(10, 16)(5, 6)(13, 17)(15, 18)(11, 12)(19, 20)\} \\
= t_2.
\]

\[
\phi(t)^{\phi(x^3)} = \{(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)^e\} \\
= \{(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)\} \\
= t_0.
\]

Thus, \( \phi(t)^{\phi(N)} = \{t_0, t_1, t_2\}. \)

3. We need to show that \( \phi(N) \) acts as \( S_3 \) on the three conjugates of \( \phi(t) \) by conjugates.

First, we have to conjugate by \( \phi(x): \)
\[ t_0^\phi(x) = t_1 \]
\[ t_1^\phi(x) = (t_0^\phi(x)) \phi(x) = t_0^{\phi(x^2)} = t_2 \]
\[ t_2^\phi(x) = (t_1^\phi(x)) \phi(x) = t_1^{\phi(x^2)} = (t_0^\phi(x)) \phi(x^2) = t_0^{\phi(x^3)} = t_0. \]

Thus, \( \phi(x) = (t_0, t_1, t_2) \).

Next, we have to conjugate by \( \phi(y) \):

\[ t_1^\phi(y) = \{(1, 3)(4, 10) \ldots (14, 13)(18, 20)^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}\} \]
\[ = \{(1, 4)(2, 7)(3, 9)(8, 14)(10, 16)(5, 6)(13, 17)(15, 18)(11, 12)(19, 20)\} \]
\[ = t_2 \]
\[ t_2^\phi(y) = (t_1^\phi(y)) \phi(y) = t_1^{\phi(y^2)} = t_1. \]

Thus, \( \phi(y) = (t_1, t_2) \).

Hence, \( \phi(G) = < \phi(x), \phi(y), \phi(t) > \) is a homomorphic image of \( G = 2^{*3} : S_3 \).

We have:

\[ G = \frac{2^{*3} : S_3}{(0, 2, 1)t_1t_2t_0} = t_0t_1. \]

Now, we want to verify if \( \phi(0, 2, 1) = \phi(t_0t_1t_2t_0t_1) \) then \( < \phi(x), \phi(y), \phi(t) > \) is a homomorphic image of \( G \).

Verify: \( \phi(x^{-1}) = \phi(t_0t_1t_2t_0t_1) : \)

\[
\phi(t_0t_1t_2t_0t_1) = (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
(1, 4)(2, 7)(3, 9)(8, 14)(10, 16)(5, 6)(13, 17)(15, 18)(11, 12)(19, 20)
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
= (2, 4, 3)(5, 8, 9)(6, 7, 10)(11, 14, 15)(12, 13, 16)(17, 19, 18)
= \phi(x^{-1}).
\]

Hence, \( \phi : G \xrightarrow{\text{homo}} S_{20} \) with \( \phi(G) = < \phi(x), \phi(y), \phi(t) > \). By FIT we have:

\[ G/\ker\phi \cong \phi(G) \]
\[\Rightarrow |G/\ker \phi| \cong |\phi(G)|\]
\[\Rightarrow |G| = |\ker \phi||\phi(G)|.\]

By completing the double coset enumeration we know \(|G| \leq 120\). Moreover, by Magma, 
\(|\phi(G)| = |<\phi(x), \phi(y), \phi(t_0)>| = 120.\)
So, \(|G| = |\ker \phi|120\)
\[\Rightarrow |G| \geq 120.\]
Hence, \(|G| = 120.\)

2.2.3 **Prove** \(G \cong 2 \times A_5\)

We use two different methods to prove that \(G \cong 2 \times A_5\).

(1) We will prove by hand that \(G \cong 2 \times A_5\).

*Proof.* Given:

\[
\phi(x) = (2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)
\]
\[
\phi(y) = (3, 4)(5, 7)(6, 8)(9, 10)(11, 13)(12, 14)(15, 16)(18, 19)
\]
\[
t_0 = (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]
\[
t_1 = (1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
\]

Note:

\[
|\phi(y)\phi(x)t_0t_1t_0t_0| = |(3, 4)(5, 7)(6, 8)(9, 10)(11, 13)(12, 14)(15, 16)(18, 19)
\]
\[
(2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)
\]
\[
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]
\[
(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
\]
\[
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]
\[
(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
\]
\[
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]
\[
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]
\[
\]
Thus, $|\phi(y)\phi(x)t_0t_1t_0t_1| = 2$, which is the center of order two.

- $|\phi(x)t_0| = |(2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)
  (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)|$
  $= |(2, 6, 9, 4, 1), (3, 8, 11, 16, 7)(5, 10, 13, 18, 14)(12, 15, 19, 20, 17)|$
  $= 5.$

- $|t_0| = (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20) = 2.$

- $|\phi(x)t_0t_0| = |\phi(x)|$
  $= |(2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19)|$
  $= 3.$

Hence, $<\phi(x)t_0, t_0> = A_5$.

Now, $<\phi(x)t_0, t_0, \phi(y)\phi(x)t_0t_1t_0t_1, t_0 > \leq \phi(G) = <\phi(x), \phi(y), \phi(t) >$

$\implies 2 \times A_5 \leq \phi(G)$. But $|\phi(G)| = 120$ and $|2 \times A_5| = 120$.

Thus, $\phi(G) = 2 \times A_5$.

(2) We use the composition factors of $G$ to construct a computer based proof to show that $G \cong 2 \times A_5$.

**Proof.** Given:

$G = <x, y, t|x^3, y^2, (xy)^2, t^2, (t, y), tt^x = x^{-1}t^xtx^2 > = \frac{2^3, S_3}{(0, 2, 1)t_1t_0t_2 = t_0t_1}$.

We use Magma, to obtain the following composition factors:

```magma
> CompositionFactors(G1);
G |
  Alternating(5)
*|
  Cyclic(2)
1
```

Hence, $G$ has the following composition series $G \supset G_1 \supset 1$, where $G = (G/G_1)(G_1/1) = A_5C_2$. The normal lattice of $G$ is:

```magma
> NL:=NormalLattice(G1);
> NL;
```
Normal subgroup lattice
-----------------------

[4] Order 120  Length 1  Maximal Subgroups: 2 3
---
[3] Order 60   Length 1  Maximal Subgroups: 1
---
[2] Order 2   Length 1  Maximal Subgroups: 1
---
[1] Order 1   Length 1  Maximal Subgroups:

First, we look at the center of $G$ and we find it is of order 2. In addition, by looking at the normal lattice, we find that the normal subgroup $\text{NL}[2]$ is of order 2. Hence, we might have a direct product of $\text{NL}[2]$ by the Alternating group $\text{NL}[3] = A_5$. Note: $\text{NL}[2] \triangleleft G_1$ and $A_5 \triangleleft G_1$ then $A_5 \cap \text{NL}[2] = 1$.

> D:=DirectProduct(CyclicGroup(2),\text{NL}[3]);
> s:=IsIsomorphic(D,G1);s;
true

We use the above loops to confirm that $G$ is a direct product of a cyclic group of order 2 by $A_5$. By using ATLAS, the presentation of the Alternating group ($A_5$) is:

$$<a,b|a^2,b^3,(a * b)^5>.$$ 

Now, the element of $\text{NL}[2]$ commutes with the element of $\text{NL}[3] = A_5$, since $G$ is a direct product extension. Thus, we have the following presentation for $G$:

> H<a,b,c>:=Group<a,b,c|a^2,b^3,(a*b)^5,c^2,(c,a),(c,b)>; #H;
120
> f1,H1,k1:=CosetAction(H,sub<H|\text{Id}(H)>);
> s:=IsIsomorphic(H1,G1);s;
true

Hence, $G \cong 2 \times A_5$.

2.3 Finding and Factoring by the Center ($Z(G)$) of $2 \times A_5$ over $S_3$

Let $G$ acts on $X = \{N \cup N_{t_0} \cup N_{t_0t_1} N \cup N_{t_0t_1t_0} N \cup N_{t_0t_1t_0t_1} N \}$ where $|X| = 20$. From the Cayley Diagram of $2 \times A_5$ over $S_3$ we see that $G$ is transitive. Moreover, from the DCE and the Cayley diagram it is clear that the dou-
ble coset $N_{t_0t_1t_0t_0}N$ contains one single coset. We stabilize the coset $N$ then another coset $N_{t_0t_1t_0t_0}N$ at the maximal distance from $N$ also stabilize. This means \{$N, N_{t_0t_1t_0t_0}N$\} is a nontrivial block of size 2. Let $B$ be a nontrivial block and $N \in B$. If $N_{t_0t_1t_0t_0} \in B$. Then

$$B = \{N, N_{t_0t_1t_0t_1t_0} = \{1, 20\}$$

$$B_{t_0} = \{N_{t_0}, N_{t_0t_1t_0t_1} \} = \{2, 17\}$$

$$B_{t_1} = \{N_{t_1}, N_{t_0t_0t_0t_1t_0} \} = \{N_{t_1}, N_{t_1t_0t_0t_1t_0} \} \} = \{3, 18\}$$

$$B_{t_2} = \{N_{t_2}, N_{t_0t_0t_0t_1t_0} \} = \{N_{t_2}, N_{t_2t_1t_1t_1t_2} \} \} = \{4, 19\}$$

$$B_{t_0t_1} = \{N_{t_0t_1}, N_{t_0t_1t_0} \} \} = \{5, 11\}$$

$$B_{t_1t_0} = \{N_{t_1t_0}, N_{t_0t_1t_0t_1t_0} \} = \{N_{t_1t_0}, N_{t_1t_0t_1t_0t_1t_0} \} \} = \{6, 12\}$$

$$B_{t_2t_0} = \{N_{t_2t_0}, N_{t_0t_1t_0t_1t_0t_2} \} = \{N_{t_2t_0}, N_{t_2t_0t_2t_0t_2t_2} \} \} = \{8, 14\}$$

$$B_{t_0t_2} = \{N_{t_0t_2}, N_{t_0t_1t_0t_1t_0} \} = \{N_{t_0t_2}, N_{t_0t_2t_0t_2t_2} \} \} = \{7, 13\}$$

$$B_{t_1t_2} = \{N_{t_1t_2}, N_{t_0t_1t_0t_1t_0t_2} \} = \{N_{t_1t_2}, N_{t_1t_0t_1t_0t_1t_2} \} \} = \{9, 15\}$$

$$B_{t_2t_1} = \{N_{t_2t_1}, N_{t_0t_1t_0t_1t_2} \} = \{N_{t_2t_1}, N_{t_2t_1t_2t_2t_1} \} \} = \{10, 16\}$$

We can see that \{B_{t_0}, B_{t_1}, B_{t_2}, B_{t_0t_1}, B_{t_1t_0}, B_{t_0t_2}, B_{t_1t_2}, B_{t_2t_1} \} \cap B = \emptyset and \{B_{t_0}, B_{t_1}, B_{t_2}, B_{t_0t_1}, B_{t_1t_0}, B_{t_0t_2}, B_{t_1t_2}, B_{t_2t_1} \} \neq B$. Hence, we have blocks of imprimitive of size two. Therefore, \|Z(G)\|=2 where $Z(G) = <nw>$ (central elements permute the elements of each block of imprimitive)

Now, we are going to find the central element of order 2 in $G$ not in its homomorphic image $G_1$. Consider $nt_0t_1t_0t_1t_0 = 1 \in G$.

In addition, $t_0t_1t_0t_1t_0 = n^{-1}$. Let $r = n^{-1}$. Then $t_0t_1t_0t_1t_0 = r$. We now compute $r$ by its action on the cosets $\{Nt_0, Nt_1, Nt_2\}$. Recall that our relation is $t_0t_1 = (0, 2, 1)t_1t_0t_2$.

In addition, if we conjugate this relation by the elements of $N = S_3$, we obtain the following relations:

- $t_0t_1 = (0, 2, 1)t_1t_0t_2$
- $t_0t_2 = (0, 1, 2)t_2t_0t_1$
- $t_1t_0 = (1, 2, 0)t_0t_1t_2$
- $t_2t_1 = (2, 0, 1)t_1t_2t_0$
- $t_1t_2 = (1, 0, 2)t_2t_1t_0$
- $t_2t_0 = (2, 1, 0)t_0t_2t_1$.

Compute $r$ by its action on the coset $Nt_0$:

\[
Nt_0^r = Nt_0^t_0t_1t_0t_1t_0 = N(t_0t_1t_0t_1t_0)^{-1}t_0t_0t_1t_0t_1t_0
= Nt_0t_1t_0t_1t_0t_1t_0 = Nt_1t_0t_2t_0t_1t_0t_1t_0
= Nt_1t_0(2, 1, 0)t_0t_2t_0t_1t_0
= Nt_0t_2t_0t_2t_0t_1t_0t_1t_0 = Nt_0t_2t_0(2, 1, 0)t_0t_2t_0t_1t_0
= Nt_2t_1t_2t_0t_2t_1t_0 = Nt_2(1, 0, 2)t_2t_1t_2t_0t_1t_0
= Nt_1t_2t_1t_2t_0t_1t_0 = Nt_1t_2(1, 0, 2)t_2t_1t_1t_0
= Nt_1t_0t_1t_2t_0 = Nt_1t_0t_0
= Nt_1.
\]

Now, compute $r$ by its action on the coset $Nt_1$:

\[
Nt_1^r = Nt_1^t_0t_1t_0t_1t_0 = N(t_0t_1t_0t_1t_0)^{-1}t_1t_0t_1t_0t_1t_0
= Nt_0t_1t_0t_1t_0t_1t_0t_1t_0 = Nt_0t_1t_0t_1t_0t_1t_0t_1t_0(1, 2, 0)t_1t_0t_1t_2
= Nt_1t_2t_1t_2t_1t_2t_1t_2 = Nt_1t_2t_1t_2t_1t_2t_1(2, 0, 1)t_1t_2t_1t_2
= Nt_2t_0t_2t_0t_2t_1t_2t_1t_2 = Nt_2t_0t_2t_0t_2t_0t_2t_1t_2k
= Nt_0t_1t_0t_1t_0t_2t_1t_2 = Nt_0t_1t_0(1, 2, 0)t_0t_1t_0t_2t_1t_2
\]
= \text{N}t_1t_2t_1t_0t_1t_0t_2t_1t_2 = \text{N}t_1(2,0,1)t_1t_2t_1t_0t_2t_1t_2 \\
= \text{N}t_2t_1t_2t_1t_0t_2t_1t_2 = \text{N}t_2t_1(2,0,1)t_1t_2t_1t_2 \\
= \text{N}t_0t_2t_2 \\
= \text{N}t_0.

Next, compute \(r\) by its action on the coset \(\text{N}t_2\):

\[ \text{N}t_2^r = \text{N}t_2^{t_1t_0t_1} \]

\[ = \text{N}(t_0t_1t_0t_1t_0)^{-1}t_2t_0t_1t_0t_1t_0 \]

\[ = \text{N}t_0t_1t_0t_1t_0t_2t_0t_1t_0t_1t_0 \]

\[ = \text{N}t_0t_1t_0(1,2,0)t_0t_1t_0t_1t_0 \]

\[ = \text{N}t_1t_2t_1t_0t_1t_0t_1t_0t_1t_0 \]

\[ = \text{N}t_1(2,0,1)t_1t_2t_1t_0t_1t_0 \]

\[ = \text{N}t_2t_1t_2t_1t_0t_1t_0t_1t_0 \]

\[ = \text{N}t_2t_1(2,0,1)t_1t_2t_1t_0t_1t_0 \]

\[ = \text{N}t_0t_2t_1t_2t_1t_0t_1t_0 \]

\[ = \text{N}t_2t_0t_2t_1t_0t_1t_0 \]

\[ = \text{N}t_2t_0(2,0,1)t_1t_2t_1t_0 \]

\[ = \text{N}t_0t_1t_2t_1t_0 \]

\[ = \text{N}t_0t_2t_1t_0 \]

\[ = \text{N}t_0(2,0,1)t_1t_2 \]

\[ = \text{N}t_1t_1t_2 \]

\[ = \text{N}t_2. \]

Thus, \(\text{N}t_0^r = t_1, \text{N}t_1^r = t_0, \text{and} \text{N}t_2^r = t_2\). Therefore, \(r = (0,1)\), and the generator of the center is \(t_0t_1t_0t_1t_0 = (0,1)\). Hence, the center of \(G\) is \(Z(G) = \langle (0,1)t_0t_1t_0t_1t_0 \rangle\).

Now, we factor

\[ G \cong \frac{2^{*3} : S_3}{(0,2,1)t_1t_0t_2 = t_0t_1} \]
by the center $Z(G)$, that is:

$$G \cong \frac{2^* \cdot 3}{(0,2,1)t_1t_0t_2 = t_0t_1, (0,1)t_0t_1t_0t_1t_0}.$$

The Cayley diagram of $G$ over $N$ shown below illustrates that $[*]$ consists of $N$ only. Moreover, $[0]$ consists of three cosets, $Nt_0N = \{Nt_0, Nt_1, Nt_2\}$ and the orbits of $N^{(0)}$ on $\{0,1,2\}$ are: $\emptyset = \{0\}$ and $\{1,2\}$. Take a representative $t_i$ from each orbit and see which double coset $Nt_0t_i$ belongs to. We have: $Nt_0t_0 \in [*]$ and $Nt_0t_1 \in [01]$. In addition, $[01]$ consists of six cosets, $Nt_0t_1N = \{Nt_0t_1, Nt_1t_0, Nt_2t_0, Nt_0t_2, Nt_1t_2, Nt_2t_1\}$ and the orbits $N^{(01)}$ on $\{0,1,2\}$ are: $\emptyset = \{0\}$, $\{1\}$, and $\{2\}$. Now we take a representative $t_i$ from each orbit and see which double coset $Nt_0t_1t_i$ belongs to. We have: $Nt_0t_1t_1 \in [0]$, by using the main relation $(0,2,1)t_1t_0t_2 = t_0t_1 \implies Nt_1t_0 = Nt_0t_1t_2$, thus, $Nt_0t_1t_2 \in [01]$. Moreover, by using the center $(0,1)t_0t_1 = t_0t_1t_0 \implies Nt_0t_1 = Nt_0t_1t_0$, hence, $Nt_0t_1t_0 \in [01]$. This completes our double coset enumeration of $G$ factor by the center $Z(G)$ and our Cayley diagram is as follows:

![Cayley Diagram of $A_5$ over $S_3$](image-url)
2.4 Converting Symmetric and Permutation Representation of $2 \times A_5$ over $S_3$

Now we want to convert symmetric representation of $G = 2 \times A_5$ over $S_3$ to permutation representation. Note: every element of $G$ is of the form $nw$ where $n$ is a permutation of $S_3$ on three letters and $w$ is a word in $\{t_0, t_1, t_2\}$ of length at most four and every element of $G$ is also a permutation of the set of 20 cosets of $N$ in $G$.

The following examples are converting symmetric representation to its permutation representation of $G$.

Example 2.13. $(0, 1, 2)t_0t_1t_0$

$$= \phi(x)\phi(t)\phi(t^{\phi(x)})\phi(t)$$

$$= (2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)$$

$$= (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$$

$$= (1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)$$

$$= (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$$

Thus, $(0, 1, 2)t_0t_1t_0 = (1,11,14)(2,18,7)(3,13,17)(4,6,16)(5,8,20)(10,19,12)$.

Example 2.14. $(2, 0)t_0t_1t_2$

$$= \phi(y)\phi(t)\phi(t^{\phi(y)})\phi(t^{\phi(y^2)})$$

$$= (3,4)(5,7)\ldots(15,16)(18,19)(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19)$$

$$= (1,2)(3,6)(4,8)(5,11)(7,13)(9,10)(12,18)(14,19)(15,16)(17,20)$$

$$= (1,3)(4,10)(2,5)(9,15)(6,12)(8,7)(16,19)(11,17)(14,13)(18,20)$$

$$= (1,4)(2,7)(3,9)(8,14)(10,16)(5,6)(13,17)(15,18)(11,12)(19,20)$$


Hence, $(2, 0)t_0t_1t_2 = (1,6)(3,11)(4,9)(5,18)(7,16)(10,13)(12,20)(15,19)$.

Next, we want to convert permutation representation of $G = 2 \times A_5$ over $S_3$ to symmetric representation. Let $p$ be a permutation on twenty letters. We write it in the form $nw$, where $n \in S_3$ and $w$ is a word in at most three $t_i$'s. Note: $Np = 1^p = Nw$; where

$p \in Np$

$$\implies p \in Nw$$
\[ \Rightarrow p = nw \text{ for some } n \in N \]
\[ \Rightarrow n = pw^{-1}. \]

The following examples are converting permutation representation to its symmetric representation of \( G \).

**Example 2.15.** Let \( p = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12) \). Note:
\[ p = nw \implies n = pw^{-1}. \]
\[ Np = 1^p = 11 = Nt_0t_1t_0 \]
\[ \implies Np = Nt_0t_1t_0 \]
\[ \implies p = nt_0t_1t_0 \]
\[ \implies n = pt_0t_1t_0 \]
\[ n = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12) \]
\[ (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20) \]
\[ (1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20) \]
\[ (1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20) \]
\[ \implies n = (2, 3, 4)(5, 9, 8)(6, 10, 7)(11, 15, 14)(12, 16, 13)(17, 18, 19) \]

Next, we compute \( n \) in the actions on \( \{Nt_0, Nt_1, Nt_2\} \):
\[ Nt_0^p = Nt_0(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 2(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 3 \]
\[ = Nt_1. \]
\[ Nt_1^p = Nt_1(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 3(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 4 \]
\[ = Nt_2. \]
\[ Nt_2^p = Nt_2(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 4(2,3,4)(5,9,8)(6,10,7)(11,15,14)(12,16,13)(17,18,19) \]
\[ = 2 \]
\[ = Nt_0. \]

Thus, \( n = (0, 1, 2). \) We know that \( p = nt_0t_1t_0 \implies p = (0, 1, 2)t_0t_1t_0. \) Hence,
\[ p = (1, 11, 14)(2, 18, 7)(3, 13, 17)(4, 6, 16)(5, 8, 20)(10, 19, 12) = (0, 1, 2)t_0t_1t_0. \]
Example 2.16. Let \( p = (1, 5, 14, 20, 11, 8)(2, 3, 7, 17, 18, 13)(4, 12, 16, 19, 6, 10)(9, 15) \).

Note: \( p = nw \implies n = pw^{-1} \).

\[
Np = 1^n = 5 = Nt_0 t_1
\]

\[
\implies Np = Nt_0 t_1
\]

\[
\implies p = nt_0 t_1
\]

\[
\implies n = pt_1 t_0
\]

\[
n = (1, 5, 14, 20, 11, 8)(2, 3, 7, 17, 18, 13)(4, 12, 16, 19, 6, 10)(9, 15)
\]

\[
(1, 3)(4, 10)(2, 5)(9, 15)(6, 12)(8, 7)(16, 19)(11, 17)(14, 13)(18, 20)
\]

\[
(1, 2)(3, 6)(4, 8)(5, 11)(7, 13)(9, 10)(12, 18)(14, 19)(15, 16)(17, 20)
\]

\[
\implies n = (3, 4)(5, 7)(6, 8)(9, 10)(11, 13)(12, 14)(15, 16)(18, 19)
\]

Next, we compute \( n \) in the actions on \( \{Nt_0, Nt_1, Nt_2\} \):

\[
Nt_0^{n} = Nt_0^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}
\]

\[
= 2(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)
\]

\[
= 2
\]

\[
= Nt_0.
\]

\[
Nt_1^{n} = Nt_1^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}
\]

\[
= 3(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)
\]

\[
= 3
\]

\[
= Nt_2.
\]

\[
Nt_2^{n} = Nt_2^{(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)}
\]

\[
= 4(3,4)(5,7)(6,8)(9,10)(11,13)(12,14)(15,16)(18,19)
\]

\[
= 4
\]

\[
= Nt_1.
\]

Thus, \( n = (1, 2) \). We know that \( p = nt_0 t_1 \implies p = (1, 2)t_0 t_1 \). Hence,

\[
p = (1, 5, 14, 20, 11, 8)(2, 3, 7, 17, 18, 13)(4, 12, 16, 19, 6, 10)(9, 15) = (1, 2)t_0 t_1.
\]

2.5 \( \mathbb{2} \times S_5 \) as a Homomorphic Image of \( 2^{*4} : S_4 \)

2.5.1 Construction of \( \mathbb{2} \times S_5 \) over \( S_4 \)

Consider the group \( G = 2^{*4} : S_4 \) factored by the relator \([0, 1, 2) = t_0 t_1 t_2 t_0 \).

Note: \( N = S_4 = \{ e, (3, 0), (1, 3, 0, 2), (1, 3, 0), (1, 0, 3), (1, 0, 2)(2, 3), (1, 2), (1, 0, 2), (1, 2, 3, 0), (1, 3), (1, 0), (2, 3), (2, 0, 3), (1, 3, 2, 0), (2, 3, 0), (1, 2, 3), (1, 2, 0, 3), (1, 0, 2, 3), (1, 2, 0), \).
(1, 3)(2, 0), (1, 0, 3, 2), (2, 0), (1, 2)(3, 0), (1, 3, 2)}, where \( x \sim (0, 1, 2, 3) \) and \( y \sim (0, 1) \).
Let \( N = \langle (0, 1, 2, 3), (0, 1) \rangle \) and \( t \sim t_4 \sim t_0 \).

We will begin the manual double coset enumeration by looking at our first double coset. Note the definition of a double coset is as follows: \( NwN = \{ Nwn | n \in N \} \).

\( NeN \)
First, the double coset \( NeN \), is denoted by \([\ast]\). This double coset contains only the single coset, namely \( N \). Since \( N \) is transitive on \( \{ t_0, t_1, t_2, t_3 \} \), the orbit of \( N \) on \( \{ 0, 1, 2, 3 \} \) is:
\[
\mathcal{O} = \{ 0, 1, 2, 3 \}.
\]
We choose \( t_0 \) as our symmetric generator from \( \mathcal{O} \) and find to which double coset \( Nt_0 \) belongs. \( Nt_0N \) will be a new double coset, denoted by \([0]\). Hence, four symmetric generators will go the new double coset \([0]\).

\( Nt_0N \)
In order to find how many single cosets \([0]\) contains, we must first find the coset stabilizer \( N^{(0)} \). Then the number of single coset in \([0]\) is equal to \( \frac{|N|}{|N^{(0)}|} \). Now,
\[
N^{(0)} = N^0 = \langle (1, 3, 2), (1, 2) \rangle
\]
so the number of the single cosets in \( Nt_0N \) is \( \frac{|N|}{|N^{(0)}|} = \frac{24}{6} = 4 \). These four single cosets in \([0]\) are \( \{ Nt_0^n | n \in N \} = \{ Nt_0, Nt_1, Nt_2, Nt_3 \} \). Furthermore, the orbits of \( N^{(0)} \) on \( \{ t_0, t_1, t_2, t_3 \} \) are:
\[
\mathcal{O} = \{ 0 \} \text{ and } \{ 1, 2, 3 \}.
\]
We take \( t_0 \) and \( t_1 \) from each orbit, respectively, and to see which double coset \( Nt_0t_0 \) and \( Nt_0t_1 \) belong to. Now \( Nt_0t_0 = N \in [\ast] \), so one element will go back to \( NeN \) and three symmetric generators will go to a new double coset \( Nt_0t_1N \), denoted by \([01]\).

\( Nt_0t_1N \)
Now \( Nt_0t_1N \) is a new double coset. We determine how many single cosets are in this double coset. We have \( N^{(01)} = N^{01} = \langle e \rangle \). But \( Nt_0t_1 \) is not distinct. If we conjugate
Furthermore, the orbits of $N$ we get that:

$$Nt_0t_1 = Nt_0t_2 = Nt_0t_3$$
$$Nt_1t_0 = Nt_1t_2 = Nt_1t_3.$$  
$$Nt_2t_1 = Nt_2t_0 = Nt_2t_3.$$  
$$Nt_3t_1 = Nt_3t_2 = Nt_3t_0.$$  

Thus, there exist $\{n \in N | N(t_0t_1)^n = Nt_0t_1\}$ such that

$$N(t_0t_1)^{(1,2,3)} = Nt_0t_2 = Nt_0t_1 \implies (1, 2, 3) \in N^{(01)}$$

$$Nt_0t_1 = Nt_0t_2 = Nt_0t_3 = N(0, 2, 3)t_0t_3 = Nt_0t_3$$

$$Nt_0t_2^{(2,3)} = Nt_0t_3 = Nt_0t_2 = Nt_0t_1 \implies (2, 3) \in N^{(01)}$$

So, $N^{(01)} = \langle (1, 2, 3), (2, 3) \rangle$. The number of the single cosets in $Nt_0t_1N$ is $\frac{|N|}{|N^{(01)}|} = \frac{24}{6} = 4$. These four single cosets in $[01]$ are $\{Nt_01^n | n \in N\} = \{Nt_0t_1, Nt_1t_0, Nt_2t_1, Nt_3t_1\}$.

Furthermore, the orbits of $N^{(01)}$ on $\{t_0, t_1, t_2, t_3\}$ are:

$\emptyset = \{0\}$ and $\{1, 2, 3\}$.

$Nt_0t_1t_1 \in [0]$ (three symmetric generators will go back to the coset $[0]$)

$Nt_0t_1t_0 \in [010]$ (one symmetric generator go to the double coset $[010]$)

$Nt_0t_1t_0N$

Consider $Nt_0t_1t_0N$ denoted by $[010]$. We determine how many single cosets are in this double coset. We have $N^{(010)} = N^{010} = \langle e \rangle$. But $Nt_0t_1t_0$ is not distinct. From the relation we know that:

$$Nt_0t_1t_0 = Nt_0t_2t_0^{(2,3)} = Nt_0t_3t_0 = Nt_0t_2t_0 = Nt_0t_1t_0 \implies (2, 3) \in N^{(010)}$$

$$Nt_0t_1t_0 = Nt_0t_2t_0 = Nt_0t_2t_0t_1t_1 = Nt_0(2, 0, 1)t_2t_1 = Nt_1t_2t_1$$

$$Nt_0t_1t_0^{(0,1,2)} = Nt_1t_2t_1 = Nt_0t_1t_0 \implies (0, 1, 2) \in N^{(010)}$$

So, $N^{(010)} = \langle (0, 1, 2), (2, 3) \rangle$. The number of the single cosets in $Nt_0t_1t_0N$ is $\frac{|N|}{|N^{(010)}|} = \frac{24}{24} = 4$. The single coset in $[01]$ are $\{Nt_01^n | n \in N\} = \{Nt_0t_1t_0\}$. Furthermore, the orbits of $N^{(011)}$ on $\{t_0, t_1, t_2, t_3\}$ are:

$\emptyset = \{0, 1, 2, 3\}$.

Take a representative from this orbit, say $t_0$. Hence $Nt_0t_1t_0t_0 \in [01]$. Therefore, four symmetric generators will go back to $Nt_0t_1N$. 
We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of $N$ in $G$ is 10. We conclude:

\[ G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N, \]

where

\[ G = \frac{2^4 : S_4}{(0, 1, 2) = t_0t_1t_2} \]

\[ |G| \leq (|N| + \frac{|N|}{N(0)}) + \frac{|N|}{N(01)} \times |N| \]

\[ |G| \leq (1 + 4 + 4 + 1) \times 24 \]

\[ |G| \leq 10 \times 24 \]

\[ |G| \leq 240. \]

A Cayley diagram that summarizes the above information is given below:

![Cayley Diagram](image)

Figure 2.3: Cayley Diagram of $2 \times S_5$ over $S_4$
2.5.2 Permutation Representation of $2 \times S_5$ over $S_4$

In order to find the permutation representation of $G = 2^4 : S_4$, in terms of $x$, $y$, and $t_0$, we create a table in which we conjugate the twenty single cosets by $x$ and $y$ and we right multiply them by $t_0$.

Table 2.2: Permutation Representation of $2 \times S_5$ over $S_4$

<table>
<thead>
<tr>
<th>Cosets</th>
<th>$x \sim (0,1,2,3)$</th>
<th>$y \sim (0,1)$</th>
<th>$t \sim t_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $N$</td>
<td>1. $N$</td>
<td>1. $N$</td>
<td>2. $Nt_0$</td>
</tr>
<tr>
<td>2. $Nt_0$</td>
<td>3. $Nt_1$</td>
<td>3. $Nt_1$</td>
<td>1. $N$</td>
</tr>
<tr>
<td>3. $Nt_1$</td>
<td>4. $Nt_2$</td>
<td>2. $Nt_0$</td>
<td>7. $Nt_1t_0$</td>
</tr>
<tr>
<td>4. $Nt_2$</td>
<td>5. $Nt_3$</td>
<td>4. $Nt_2$</td>
<td>8. $Nt_2t_0$</td>
</tr>
<tr>
<td>5. $Nt_3$</td>
<td>2. $Nt_0$</td>
<td>5. $Nt_3$</td>
<td>9. $Nt_3t_0$</td>
</tr>
<tr>
<td>6. $Nt_0t_1$</td>
<td>7. $Nt_1t_2$</td>
<td>7. $Nt_1t_0$</td>
<td>10. $Nt_0t_1t_0$</td>
</tr>
<tr>
<td>7. $Nt_1t_0$</td>
<td>8. $Nt_2t_1$</td>
<td>6. $Nt_0t_1$</td>
<td>3. $Nt_1$</td>
</tr>
<tr>
<td>8. $Nt_2t_1$</td>
<td>9. $Nt_3t_2$</td>
<td>8. $Nt_2t_0$</td>
<td>4. $Nt_2t_1t_0$</td>
</tr>
<tr>
<td>9. $Nt_3t_1$</td>
<td>6. $Nt_0t_2$</td>
<td>9. $Nt_3t_0$</td>
<td>5. $Nt_3t_1t_0$</td>
</tr>
<tr>
<td>10. $Nt_0t_1t_0$</td>
<td>10. $Nt_1t_2t_1$</td>
<td>10. $Nt_1t_0t_1$</td>
<td>6. $Nt_0t_1$</td>
</tr>
</tbody>
</table>

We have:

$\phi(x) = (2,3,4,5)(6,7,8,9)$

$\phi(y) = (2,3)(6,7)$

$\phi(t) = (1,2)(3,7)(4,8)(5,9)(6,10)$.

Thus, we have a homomorphism $\phi : 2^4 : S_4 \rightarrow S_{10}$. Then $\phi(G) = \langle \phi(x), \phi(y), \phi(t) \rangle$. In order for us to prove that $\phi(G) = \langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $G = 2^4 : S_4$, we must have the following conditions met:

1. $\phi(N) \cong S_4$

2. $\phi(t)$ has four conjugates under conjugation by $\phi(N)$

3. $\phi(N)$ acts as $S_4$ on the three conjugates of $\phi(t)$ by conjugates.

**Proof.** We have $\phi(N) = \langle \phi(x), \phi(y) \rangle$.

1. $\phi(N) = \langle \phi(x), \phi(y) \rangle$

   $= \langle (2,3,4,5)(6,7,8,9), (2,3)(6,7) \rangle$

   $\cong S_4$ since $|\phi(x)\phi(y)| = |(3,4,5)(7,8,9)| = 3$.

Thus, $\phi(N) \cong S_4$. 

(2) We need to compute $\phi(t)^{\phi(N)}$:

\[
\phi(t)^{\phi(x)} = \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)^{(2,3,4,5)}(6,7,8,9)\} \\
= \{(1, 3)(4, 8)(5, 9)(2, 6)(7, 10)\} \\
= t_1 \\
\phi(t)^{\phi(x^2)} = \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)^{(2,4)}(3,5)(6,8)(7,9)\} \\
= \{(1, 4)(5, 9)(2, 6)(3, 7)(8, 10)\} \\
= t_2 \\
\phi(t)^{\phi(x^3)} = \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)^{(2,5,4,3)}(6,9,8,7)\} \\
= \{(1, 5)(2, 6)(3, 7)(4, 8)(9, 10)\} \\
= t_3 \\
\phi(t)^{\phi(x^4)} = \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)^c\} \\
= \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)\} \\
= t_0 \\
\]

Thus, $\phi(t)^{\phi(N)} = \{t_0, t_1, t_2, t_3\}$.

(3) We need to show that $\phi(N)$ acts as $S_4$ on the four conjugates of $\phi(t)$ by conjugates.

First, we have to conjugate by $\phi(x)$:

\[
\begin{align*}
\phi_0^{\phi(x)} &= t_1 \\
\phi_1^{\phi(x)} &= (t_0^{\phi(x)})^{\phi(x)} = t_0^{\phi(x^2)} = t_2 \\
\phi_2^{\phi(x)} &= (t_1^{\phi(x)})^{\phi(x)} = t_1^{\phi(x^2)} = (t_0^{\phi(x)})^{\phi(x^2)} = t_0^{\phi(x^3)} = t_3 \\
\phi_3^{\phi(x)} &= (t_2^{\phi(x)})^{\phi(x)} = t_2^{\phi(x^2)} = (t_1^{\phi(x)})^{\phi(x^2)} = t_1^{\phi(x^3)} = (t_0^{\phi(x)})^{\phi(x^3)} = t_0^{\phi(x^4)} = t_0.
\end{align*}
\]

Thus, $\phi(x) = (t_0, t_1, t_2, t_3)$.

Next, we have to conjugate by $\phi(y)$:

\[
\begin{align*}
t_0^{\phi(y)} &= \{(1, 2)(3, 7)(4, 8)(5, 9)(6, 10)^{(2,3)}(6,7)\} \\
&= \{(1, 3)(2, 6)(4, 8)(5, 9)(7, 10)\} \\
&= t_1 \\
\phi_0^{\phi(y)} &= (t_0^{\phi(y)})^{\phi(y)} = t_0^{\phi(y^2)} = t_0.
\end{align*}
\]

Thus, $\phi(y) = (t_0, t_1)$.

Hence, $\phi(G) = \langle \phi(x), \phi(y), \phi(t) \rangle$ is a homomorphic image of $G = 2^4 : S_4$. 

\[\square\]
We have:

\[ G = \frac{2^4 : S_4}{(0,1,2) = t_0t_1t_2t_0}. \]

Now, we want to verify if \( \phi(0,1,2) = \phi(t_0t_1t_2t_0) \) then \( \langle \phi(x), \phi(y), \phi(t) \rangle \) is a homomorphic image of \( G \).

Verify: \( \phi(x)\phi(y) = \phi(t_0t_1t_2t_0) : \)

\[
\phi(t_0t_1t_2t_0) = (1,2)(3,7)(4,8)(5,9)(6,10) \\
(1,3)(4,8)(5,9)(2,6)(7,10) \\
(1,4)(5,9)(2,6)(3,7)(8,10) \\
(1,2)(3,7)(4,8)(5,9)(6,10) \\
= \phi(x)\phi(y).
\]

Hence, \( \phi : G \xrightarrow{\text{homo.}} S_{10} \) with \( \phi(G) = \langle \phi(x), \phi(y), \phi(t) \rangle \). By FIT we have:

\[ G/\ker \phi \cong \phi(G) \]

\[ \implies |G/\ker \phi| \cong |\phi(G)| \]

\[ \implies |G| = |\ker \phi|\phi(G)|. \]

By completing the double coset enumeration we know \( |G| \leq 240 \). Moreover, by Magma, \( |\phi(G)| = |\langle \phi(x), \phi(y), \phi(t) \rangle| = 240 \).

So, \( |G| = |\ker \phi|240 \)

\[ \implies |G| \geq 240. \]

Hence, \( |G| = 240 \).

2.5.3 Prove \( G \cong 2 \times S_5 \)

We use the composition factors of \( G \) to construct a computer based proof to show that \( G \cong 2 \times S_5 \).

Proof. Given:

\[ \langle x, y, t|x^4, y^2, (xy)^3, t^2, (t, y^2), (t, xy), (xy)^{2^3} = ttx^2t > = \frac{2^4 : S_4}{(0,1,2) = t_0t_1t_2t_0}. \]

We use Magma, to obtain the following composition factors:

\[ > \text{CompositionFactors(G1)}; \]

\[ \text{G} \]

\[ \mid \text{Cyclic(2)} \]

\[ * \]
Alternating(5)
*  Cyclic(2)

Hence, $G$ has the following composition series $G \supset G_1 \supset G_2 \supset 1$, where $(G/G_1)(G_1/G_2)(G_2/1) = C_2 A_5 C_2$. The normal lattice of $G$ is:

```plaintext
> NL:=NormalLattice(G1);
> NL;
```

Normal subgroup lattice
-----------------------

---
[6] Order 120 Length 1 Maximal Subgroups: 3
[5] Order 120 Length 1 Maximal Subgroups: 2 3
[4] Order 120 Length 1 Maximal Subgroups: 3
---
[3] Order 60 Length 1 Maximal Subgroups: 1
---
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

First, we look at the center of $G$ and we find it is of order 2. In addition, by looking at the normal lattice, we find that the normal subgroup NL[2] is of order 2. Now, we factor $G$ by $NL[2]$ to find $H$ such that $G/NL[2] \cong H$.

```plaintext
> q,ff:=quo<G1|NL[2]>;
> s:=IsIsomorphic(q,Sym(5));s;
true
```

We show that $H \cong S_5$ and a presentation for $H$ is $< a, b \mid a^2, b^4, (a \ast b)^5, (a, b)^3 >$.

```plaintext
> H<a,b>:=Group< a,b\mid a^2,b^4,(a*b)^5,(a,b)^3>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,q);s;
true
```

Thus, we might have a direct product of a cyclic group of order 2 by $S_5$.

```plaintext
> D:=DirectProduct(CyclicGroup(2),q);
> s:=IsIsomorphic(D,G1);s;
```
Now, the element of NL[2] commutes with the element of $S_5$, since $G$ is a direct product extension. Thus, we have the following presentation for $G$:

```plaintext
> HH<a,b,c| a^2,b^4,(a*b)^5,(a,b)^3,c^2,(c,a),(c,b)>;
> f2,H2,k2:=CosetAction(HH,sub<HH|Id(HH)>);
> s:=IsIsomorphic(H2,G1);s;
true
Hence, $G \cong 2 \times S_5$.
```

### 2.6 Finding and Factoring by the Center ($Z(G)$) of $2 \times S_5$ over $S_4$

Let $G$ acts on $X = \{N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_1t_0N\}$ where $|X| = 10$. From the Cayley Diagram of $2 \times S_5$ over $S_4$ we see that $G$ is transitive. Moreover, from the DCE and the Cayley diagram it is clear that the double coset $Nt_0t_1t_0N$ contains one single coset. We stabilize the coset $N$ then another coset $Nt_0t_1t_0N$ at the maximal distance from $N$ also stabilize. This means $\{N,Nt_0t_1t_0N\}$ is a nontrivial block of size 2. Let $B$ be a nontrivial block and $N \in B$. If $Nt_0t_1t_0 \in B$. Then

$B = \{N,Nt_0t_1t_0\} = \{1,10\}$

$Bt_0 = \{Nt_0, Nt_0t_1\} = \{2,6\}$

$Bt_1 = \{Nt_1, Nt_0t_1t_0t_1\} = \{Nt_1, Nt_0t_1t_1\} = \{Nt_1, Nt_1t_0\} = \{3,7\}$

$Bt_2 = \{Nt_2, Nt_0t_1t_0t_2\} = \{Nt_2, Nt_2t_0t_2t_2\} = \{Nt_2, Nt_2t_0\} = \{Nt_2, Nt_2t_1\} = \{4,8\}$

$Bt_3 = \{Nt_3, Nt_0t_1t_0t_3\} = \{Nt_3, Nt_3t_3t_3t_3\} = \{Nt_3, Nt_3t_1\} = \{5,9\}$

We can see that $\{Bt_0, Bt_1, Bt_2, Bt_3\} \cap B = \emptyset$ and $\{Bt_0, Bt_1, Bt_2, Bt_3\} \neq B$. Hence, we have blocks of imprimitive of size two. Therefore, $|Z(G)|=2$ where

$Z(G) = <nw>$ (central elements permute the elements of each block of imprimitive)

$= \{(1,10)(2,6)(3,7)(4,8)(5,9)\}.$

Now, we are going to find the central element of order 2 in $G$, not in its homomorphic image $G_1$. Consider $nt_0t_1t_0 = 1 \in G$;
in addition, $t_0t_1t_0 = n^{-1}$. Let $r = n^{-1}$. Then $t_0t_1t_0 = r$. We now compute $r$ by its action on the cosets $\{Nt_0, Nt_1, Nt_2, Nt_3\}$.

Compute $r$ by its action on the coset $Nt_0$:

$$Nt_0^r = Nt_0^{t_0t_1t_0} = N(t_0t_1t_0)^{-1}t_0t_1t_0$$
$$= Nt_0t_1t_0t_1t_0$$
$$= Nt_0t_1t_0t_1t_0t_2 = Nt_0t_1t_0(1, 0, 2)t_1t_2$$
$$= Nt_2t_0t_2t_1t_2 = Nt_2(0, 2, 1)t_0t_2$$
$$= Nt_1t_0t_2$$
$$= Nt_1.$$

Now, compute $r$ by its action on the coset $Nt_1$:

$$Nt_1^r = Nt_1^{t_0t_1t_0}$$
$$= N(t_0t_1t_0)^{-1}t_1t_0t_1t_0$$
$$= Nt_0t_1t_0t_1t_0t_1t_0$$
$$= Nt_0t_1t_0t_1t_0(0, 2, 1)t_1t_2$$
$$= Nt_2t_0t_2t_1t_2$$
$$= Nt_2t_0t_2(0, 2, 1)t_0t_2$$
$$= Nt_1t_2t_1t_0$$
$$= Nt_1(2, 1, 0)t_2t_2 = Nt_0.$$

Next, compute $r$ by its action on the coset $Nt_2$:

$$Nt_2^r = Nt_2^{t_0t_1t_0}$$
$$= N(t_0t_1t_0)^{-1}t_2t_0t_1t_0$$
$$= Nt_0t_1t_0t_2t_0t_1t_0$$
$$= Nt_0t_1t_0(2, 0, 1)t_2t_0$$
$$= Nt_1t_2t_1t_2t_0$$
$$= Nt_1t_2(1, 2, 0)t_1$$
$$= Nt_2t_0t_1$$
$$= N(2, 0, 1)t_2$$
$$= Nt_2.
Finally, compute \( r \) by its action on the coset \( Nt_3 \):

\[
Nt'_3 = Nt_3^{t_1t_0} = N(t_0t_1t_0)^{-1}t_3t_0t_1t_0 \\
= Nt_0t_1t_0t_3t_0t_1t_0 \\
= Nt_0t_1t_0(3,0,1)t_3t_0 \\
= Nt_1t_3t_1t_3t_0 \\
= Nt_1t_3(1,3,0)t_1 \\
= Nt_3t_0t_1 \\
= Nt_3.
\]

Thus, \( Nt'_0 = t_1 \), \( Nt'_1 = t_0 \), \( Nt'_2 = t_2 \), and \( Nt'_3 = t_3 \). Therefore, \( r = (0,1) \), and the generator of the center is \( t_0t_1t_0 = (0,1) \). Hence, the center of \( G \) is \( Z(G) = \langle (0,1)t_0t_1t_0 \rangle \).

Now, we factor

\[
G \cong \frac{2^*S_4}{(0,1,2) = t_0t_1t_2t_0}
\]

by \( Z(G) = \langle (0,1)t_0t_1t_0 \rangle \). Now the question is whether the new relation \((0,1)t_0t_1t_0\) implies the original relation, \((0,1,2) = t_0t_1t_2t_0\).

We have \( t_0t_1t_2 = t_0t_1t_2t_1t_1 = t_0(1,2)t_1 = (1,2)t_0t_1t_0 = (1,2)(1,0)t_0 = (0,1,2)t_0 \). Hence, \( (0,1)t_0t_1t_0 \implies t_0t_1t_0 = (0,1,2)t_0 \).

Thus, \( G \) factor by the center \( Z(G) \) is

\[
G \cong \frac{2^*S_4}{(0,1,2) = t_0t_1t_2t_0, (0,1) = t_0t_1t_0} \cong \frac{2^*S_4}{(0,1) = t_0t_1t_0}.
\]

Now we construct a double coset enumeration of \( G \cong \frac{2^*S_4}{(0,1) = t_0t_1t_0} \). Our first double coset, \( Ne = \{Nen \mid n \in N \} = \{N\} \) denoted by \([*]\) contains one single coset. Since \( N \) is transitive on \( \{t_0, t_1, t_2, t_4\} \), the orbit of \( N \) on \( \{0,1,2,4\} \) is: \( \Omega = \{0,1,2,4\} \). We take a representative from the orbit, say \( t_0 \), and find to which double coset \( Nt_0 \) belongs. \( Nt_0N \) will be a new double coset, denoted by \([0]\). Hence, four symmetric generators will go the new double coset \([0]\).

In order to find how many single cosets \([0]\) contains, we must first find the coset stabilizer...
$N^{(0)}$. Then the number of single coset in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 = \langle (1,2,3), (1,2) \rangle$, so the number of the single cosets in $N_t0N$ is $\frac{|N|}{|N^{(0)}|} = \frac{24}{6} = 4$. These four single cosets in $[0]$ are $\{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_1, t_2\}$ are: $\varnothing = \{0\}$ and $\{1, 2, 3\}$. We take $t_0$ and $t_1$ from each orbit, respectively, and to see which double coset $Nt_0t_0$ and $Nt_0t_1$ belong to. Now $Nt_0t_0 = N \in [*]$, so one element will go back to $NeN$ and by our relation we have $(0,1)t_0t_1t_0 = e \implies (0,1)t_0t_1 = t_0 \implies Nt_0t_1 = Nt_0$. Hence, $Nt_0t_1 \in [0]$, so three elements will loop back to $[0]$. A Cayley diagram that summarizes the above information is given below:

![Cayley Diagram of $S_5$ over $S_4$](image_url)

Figure 2.4: Cayley Diagram of $S_5$ over $S_4$
Chapter 3

Iwasawa’s Lemma

In this chapter, we will apply Iwasawa’s lemma to prove that a group $G$ is simple.

3.1 Iwasawa’s Lemma Preliminaries

Definition 3.1. *(G-set).*

If $X$ is a set and $G$ is a group, then $X$ is a **G-set** if there is a function $\alpha : G \times X \to X$ (called an **action**), denoted by $\alpha : (g,x) \mapsto gx$, such that:

(i) $1x = x$ for all $x \in X$ and

(ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. We also say that $G$ acts on $X$. If $|X| = n$, then $n$ is called the **degree** of the $G$-set $X$. [Rot12]

Definition 3.2. *(Transitive G-set).*

A $G$-set $X$ is transitive if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot12]

Definition 3.3. If $X$ is a transitive $G$-set of degree $n$, and if $x \in X$, then $|G| = n|G^x|$. [Rot12]

Definition 3.4. A $G$-set $X$ is transitive if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot12]

Definition 3.5. *(Block).*

If $X$ is a $G$-set, then a **block** is a subset $B$ of $X$ such that, for each $g \in G$, either
$gB = B$ or $gB \cap B = \emptyset$. Note $gB = \{gx : x \in B\}$. Trivial blocks are $\emptyset, X,$ and one-point subsets; any other other block is called nontrivial. [Rot12]

**Definition 3.6.** A $G$–set $X$ with action $\alpha$ is faithful if $\bar{\alpha} : G \to S_x$ is injective. [Rot12]

**Definition 3.7.** (Primitive).
A transitive $G$–set $X$ is primitive if it contains no nontrivial block; otherwise, it is imprimitive. [Rot12]

**Definition 3.8.** Let $X$ be a finite $G$–set, and let $H \leq G$ act transitively on $X$. Then $G = HG^x$ for each $x \in X$. [Rot12]

**Definition 3.9.** Let $X$ be a $G$–set and $x, y \in X$.
(i) If $H \leq G$, then $H_x \cap H_y \neq \emptyset \implies H_x = H_y$
(ii) If $H$ is normal in $G$, then the subsets $Hx$ are blocks of $X$. [Rot12]

**Definition 3.10.** If $X$ is a faithful primitive $G$–set of degree $n \geq 2$. If $H$ is normal in $G$ and if $H \neq 1$, then $X$ is a transitive $H$–set. [Rot12]

**Theorem 3.11.** Let $X$ be a transitive $G$–set. Then $X$ is primitive if and only if, for each $x \in X$, the stabilizer $G^x$ is a maximal subgroup. [Rot12]

**Definition 3.12.** (Commutator). If $a, b \in G$, the commutator of $a$ and $b$, denoted by $[a, b]$, is

$$[a, b] = aba^{-1}b^{-1}.$$ [Rot12]

**Definition 3.13.** (Derived Group). The commutator subgroup (or derived group) of $G$, denoted by $G'$, is the subgroups of $G$ generated by all the commutators. [Rot12]

**Definition 3.14.** (Simple). A group $G \neq 1$ is simple if it has no normal subgroups other than $G$ and 1. [Rot12]

**Theorem 3.15.** (Iwasawa’s Lemma). Let $G' = G$ (such a group is called perfect) and let $X$ be a faithful primitive $G$–set. If there is $x \in X$ and an abelian normal subgroup $K$ of $G_x$ whose conjugates $\{ghg^{-1}\}$ generate $G$, then $G$ is simple. [Rot12]
3.2 Iwasawa’s Lemma to Prove $L_2(13)$ over $A_4$ is Simple

We consider

$$G \cong \frac{2^4 A_4}{[(0,1,2)t_0], [(0,1)(2,3)t_0]^7, [(0,1,2)t_0t_1t_2t_3]^2} \cong L_2(13),$$

where $A_4$ is maximal in $L_2(13)$ and the index of $A_4$ in $G$ equals 91.

Note $N = A_4 \cong < x, y | x^3, y^3, (x \ast y)^2 >$, where $x = (1, 2, 3)$ and $y = (1, 2, 0)$. Let $t_4 \sim t_0$.

The manual double coset enumeration and the Cayley diagram was done by Maria de la Luz Torres [dlLT05]. The Cayley diagram is shown below:

![Cayley Diagram of $L_2(13)$ over $A_4$](image)

Figure 3.1: Cayley Diagram of $L_2(13)$ over $A_4$

We use Iwasawa’s lemma to prove $G \cong L_2(13)$ is simple. Iwasawa’s lemma has three sufficient conditions that we must satisfied:
(1) $G$ acts on $X$ faithfully and primitively

(2) $G$ is perfect ($G = G'$)

(3) There exist $x \in X$ and a normal abelian subgroup $K$ of $G^x$ such that the conjugates of $K$ generate $G$.

Proof. 3.2.1 $G = L_2(13)$ acts on $X$ Faithfully

Let $G$ acts on $X = \{N, Nt_0N, Nt_0t_1N, Nt_0t_1t_0N, Nt_0t_1t_0t_3N, Nt_0t_1t_0t_2N, Nt_0t_1t_2N, Nt_0t_1t_3N, Nt_0t_1t_2t_3N, Nt_0t_1t_0t_2t_0N\}$, where $|X| = 91$. $G$ acts on $X$ implies there exist homomorphism

$$f : G \rightarrow S_{91} \quad (|X| = 91).$$

By First Isomorphic Theorem we have:

$$G/\ker f \cong f(G).$$

If $\ker f = 1$ then $G \cong f(G)$. Only elements of $N$ fix $N$ implies $G^1 = N$. Since $X$ is a transitive $G$–set of degree 91, we have:

$$|G| = 91 \times |G^1|$$

$$= 91 \times |N|$$

$$= 91 \times 12$$

$$= 1092$$

$$\Rightarrow |G| = 1092.$$  

From Cayley diagram, $|G| \leq 1092$. However, from above $|G| = 1092$ implying $\ker(f) = 1$. Since $\ker f = 1$ then $G$ acts faithfully on $X$.

3.2.2 $G = L_2(13)$ acts on $X$ Primitively

In order to show that $G$ is primitive, we must show that $G = L_2(13)$ is transitive on $X = 91$ and there exists no nontrivial blocks of $X$. From the Cayley diagram of $G = L_2(13)$ over $A_4$, we see that $G$ is transitive. Let $B$ be a nontrivial block, then $|B|||X|$. Note if we had a nontrivial block it would have to be of size 7 or 13. Let $B$ be a nontrivial block and $N \in B$ since $G$ is transitive on the coset of $N$. Now we look at different cases:
Case(1): Assume $N_{t_0} \in B$, then

$$B = \{N, N_{t_0}\}$$

$$B = \{N, N_{t_0}N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, N_{t_0}, N_{t_1}, N_{t_2}, N_{t_3}\}$$

$$B_{t_1} = \{N_{t_1}, N_{t_0t_1}, N, N_{t_2t_1}, N_{t_3t_1}\}$$

$$\implies N \in B \cap B_{t_1}$$

$$\implies B = B_{t_1}.$$ 

Now $B = \{N, N_{t_0}N, N_{t_0t_1}N\}$, where $|B| = 17$ (passed size 7 and 13). Note if $N_{t_0} \in B$ then $B = X$. So B is a trivial block.

Case(2): Assume $N_{t_0t_1} \in B$, then

$$B = \{N, N_{t_0t_1}\}$$

$$B = \{N, N_{t_0t_1}N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, N_{t_0t_1}, N_{t_2t_1}, N_{t_3t_1}, \ldots, N_{t_2t_0}, N_{t_1t_0}, N_{t_3t_0}\} \text{ where } |B| = 13$$

$$B_{t_0t_2} = \{N_{t_0t_2}, N_{t_0t_1t_0t_2}, N_{t_2t_1t_0t_2}, \ldots, N, N_{t_1t_2}, N_{t_3t_2}\}$$

$$\implies N \in B \cap B_{t_0t_2}$$

$$\implies B = B_{t_0t_2}.$$ 

So $B = \{N, N_{t_0t_1}N, N_{t_0t_1t_0t_2}N\}$, where $|B| = 25$ (passed size 7 and 13). Hence, $B$ is a nontrivial block of $X$ under the action $G$.

Case(3): Assume $N_{t_0t_1t_0} \in B$, then

$$B = \{N, N_{t_0t_1t_0}\}$$

$$B = \{N, N_{t_0t_1t_0}N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, N_{t_0t_1t_0}, N_{t_2t_1t_2}, N_{t_3t_1t_3}, \ldots, N_{t_2t_0t_2}, N_{t_1t_0}, N_{t_3t_0t_3}\} \text{ where } |B| = 13$$

$$B_{t_1} = \{N_{t_1}, N_{t_0t_1t_0t_1}, N_{t_2t_1t_2t_1}, \ldots, N_{t_2t_0t_2t_1}, N_{t_1t_0}, N_{t_3t_0t_3t_1}\}$$

$$= \{N_{t_1}, N_{t_0t_1t_0}, N_{t_2t_1t_2}, N_{t_3t_0t_3t_1}\}$$

$$\implies N_{t_0t_1t_0} \in B \cap B_{t_1}$$

$$\implies B = B_{t_1}.$$ 

So $B = \{N, N_{t_1}N, N_{t_0t_1}N, N_{t_0t_1t_0}N, N_{t_0t_1t_0t_2}N, N_{t_0t_1t_0t_3}N\}$, where $|B| = 52$. Hence, $B$ is a nontrivial block of $X$ under the action $G$. 
Case(4): Assume $Nt_0t_1t_2 \in B$, then
$$B = \{N, Nt_0t_1t_2\}$$

$$B = \{N, Nt_0t_1t_2N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_2, Nt_2t_1t_3, Nt_3t_1t_0, \ldots, Nt_0t_3t_1, Nt_10t_3t_2, Nt_3t_0t_2\} \text{ where } |B| = 13$$

$$Bt_2t_1t_0 = \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, Nt_0t_3t_1t_2t_1t_0, Nt_10t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$= \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, Nt_2t_3t_2t_1t_0, Nt_10t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$= \{Nt_2t_1t_0, N, Nt_2t_1t_3t_2t_1t_0, \ldots, Nt_2t_3t_2t_1t_0, Nt_10t_3t_2t_1t_0, Nt_3t_0t_1t_0\}$$

$$\implies N \in B \cap Bt_2t_1t_0$$

$$\implies B = Bt_2t_1t_0.$$  

So $B = \{N, Nt_0t_1N, Nt_0t_1t_2N\}$, where $|B| = 25$. Hence, $B$ is a no nontrivial block of $X$ under the action $G$.

Case(5): Assume $Nt_0t_1t_3 \in B$, then
$$B = \{N, Nt_0t_1t_3\}$$

$$B = \{N, Nt_0t_1t_3N\} \quad \text{(since } N \in B, BN = B)$$

$$B = \{N, Nt_0t_1t_3, Nt_2t_1t_0, Nt_3t_1t_2, \ldots, Nt_20t_3t_2, Nt_10t_2, Nt_30t_1t_1\} \text{ where } |B| = 13$$

$$Bt_3t_1t_0 = \{Nt_3t_1t_0, N, Nt_3t_1t_3t_0, \ldots, Nt_2t_0t_1t_0, Nt_10t_2t_3t_1t_0, Nt_30t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, Nt_0t_1t_3t_2t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_10t_2t_3t_1t_0, Nt_30t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, Nt_0t_1t_3t_2t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_10t_2t_3t_1t_0, Nt_30t_1t_3t_1t_0\}$$

$$= \{Nt_3t_1t_0, N, Nt_0t_1t_3t_2t_1t_0, \ldots, Nt_2t_0t_1t_0, Nt_10t_2t_3t_1t_0, Nt_30t_1t_3t_1t_0\}$$

$$\implies N \in B \cap Bt_3t_1t_0$$

$$\implies B = Bt_3t_1t_0.$$  

So $B = \{N, Nt_3t_1t_0N, Nt_0t_1t_2N\}$, where $|B| = 25$. Hence, $B$ is a no nontrivial block of $X$ under the action $G$.  

Case(6): Assume $N_{t_0 t_1 t_2 t_3} \in B$, then

\[ B = \{N, N_{t_0 t_1 t_2 t_3}\} \]

\[ B = \{N, N_{t_0 t_1 t_2 t_3} N\} \text{ (since } N \in B, BN = B) \]

\[ B = \{N, N_{t_0 t_1 t_2 t_3}, N_{t_2 t_1 t_2 t_0}, N_{t_3 t_1 t_3 t_2}, \ldots, N_{t_2 t_0 t_2 t_3}, N_{t_1 t_0 t_1 t_2}, N_{t_2 t_3 t_2 t_1}\} \]

\[ B_{t_1 t_2 t_0 t_1 t_0} = \{N_{t_1 t_0 t_1 t_0 t_0}, N_{t_0 t_1 t_0 t_0 t_0}, \ldots, N_{t_2 t_0 t_2 t_0 t_0}\} \]

\[ = \{N_{t_0 t_1 t_0}, N_{t_0 t_2 t_1 t_0 t_0}, \ldots, N_{t_1 t_3 t_2 t_0 t_0}\} \]

\[ = \{N_{t_0 t_1 t_0 t_3}, N, \ldots, N_{t_1 t_3 t_2}\} \]

\[ \implies N \in B \cap B_{t_1 t_2 t_0 t_1 t_0} \]

\[ \implies B = B_{t_1 t_2 t_0 t_1 t_0}. \]

So $B = \{N, N_{t_0 t_1 t_0}, N_{t_0 t_1 t_0 t_3} N\}$, where $|B| = 25$. Hence, $B$ is a nontrivial block of $X$ under the action $G$.

Case(7): Assume $N_{t_0 t_1 t_2} \in B$, then

\[ B = \{N, N_{t_0 t_1 t_2}\} \]

\[ B = \{N, N_{t_0 t_1 t_2} N\} \text{ (since } N \in B, BN = B) \]

\[ B = \{N, N_{t_0 t_1 t_2}, N_{t_2 t_1 t_2}, \ldots, N_{t_3 t_2 t_3 t_1}\} \]

\[ B_{t_1 t_3 t_0 t_1 t_0} = \{N_{t_1 t_2 t_3 t_1 t_0}, N_{t_0 t_1 t_0 t_2 t_3 t_1 t_0}, \ldots, N_{t_3 t_2 t_0 t_1 t_0}\} \]

\[ = \{N_{t_0 t_1 t_2}, N_{t_0 t_2 t_1 t_3 t_2 t_3}, \ldots, N_{t_1 t_3 t_2 t_0 t_1 t_0}\} \]

\[ = \{N_{t_0 t_1 t_2}, N, \ldots, N_{t_1 t_2 t_3}\} \]

\[ \implies N \in B \cap B_{t_1 t_3 t_0 t_1 t_0} \]

\[ \implies B = B_{t_1 t_3 t_0 t_1 t_0}. \]

So $B = \{N, N_{t_0 t_1 t_0 N}, N_{t_0 t_1 t_2} N\}$, where $|B| = 25$. Hence, $B$ is a nontrivial block of $X$ under the action $G$. 
Case(8): Assume $N_{t_0t_1t_0t_2t_0} \in B$, then
\[ B = \{N,N_{t_0t_1t_0t_2t_0}\} \]
\[ B = \{N,N_{t_0t_1t_0t_2t_0}N\} \quad \text{(since } N \in B, BN = B) \]
\[ B = \{N,N_{t_0t_1t_0t_2t_0},N_{t_2t_1t_2t_3t_2},\ldots,N_{t_3t_2t_3t_1t_3}\} \text{ where } |B| = 7 \]
\[ B_{t_0} = \{N_{t_0}N_{t_0t_1t_0t_2},N_{t_2t_1t_2t_3t_2t_0},\ldots,N_{t_3t_2t_3t_1t_3t_0}\} \]
\[ B_{t_1} = \{N_{t_1},N_{t_0t_1t_0t_2t_0t_1},N_{t_2t_1t_2t_3t_2t_1},\ldots,N_{t_3t_2t_3t_1t_3t_1}\} \]
\[ = \{N_{t_1},N_{t_0t_1t_0t_2},N_{t_2t_1t_2t_3t_2t_1},\ldots,N_{t_3t_2t_3t_1t_3t_1}\} \]

Since $B \cap B_{t_0} = \emptyset$. Then $B$ is a block and $B_{t_0}$ is also a block. Moreover, $B \cap B_{t_1} = \emptyset$ so $B_{t_1}$ is a block. But $N_{t_0t_1t_0t_2} \in B_{t_0} \cap B_{t_1}$. So $B = \{N,N_{t_0}N,N_{t_0t_1t_0t_2}N,N_{t_0t_1t_0t_2t_0}\}$.

Hence, $B$ is a no nontrivial block of $X$ under the action $G$.

Case(9): Assume $N_{t_0t_1t_3t_2} \in B$, then
\[ B = \{N,N_{t_0t_1t_3t_2}\} \]
\[ B = \{N,N_{t_0t_1t_3t_2}N\} \quad \text{(since } N \in B, BN = B) \]
\[ B = \{N,N_{t_0t_1t_3t_2},N_{t_0t_2t_1t_3},N_{t_0t_3t_2t_1},N_{t_3t_0t_1t_2}\} \text{ where } |B| = 5 \]

Hence $B$ is not a block since size of $|B| = 5$ does not divide $|X| = 91$.

Case(10): Assume $N_{t_0t_1t_2t_3} \in B$, then
\[ B = \{N,N_{t_0t_1t_2t_3}\} \]
\[ B = \{N,N_{t_0t_1t_2t_3}N\} \quad \text{(since } N \in B, BN = B) \]
\[ B = \{N,N_{t_0t_1t_2t_3},N_{t_0t_2t_3t_1},N_{t_0t_3t_1t_2},N_{t_3t_0t_2t_1}\} \text{ where } |B| = 5 \]

Hence $B$ is not a block since size of $|B| = 5$ does not divide $|X| = 91$. In conclusion, from the cases above we show that we cannot create nontrivial block of size 7 or 13. Thus $G$ acts primitively on $X$.

3.2.3 $G = L_2(13)$ is Perfect

Next we want to show that $G = G'$. Now $A_4 \subseteq G \implies A_4' \subseteq G'$.
\( A_4' = \langle [a, b] | a, b \in A_4 \rangle. \) Now the derived group,
\[
A_4' = \langle (1, 3)(20), (1, 2)(3, 0) \rangle
\]
\( \implies \{e, (1, 3)(2, 0), (1, 2)(3, 0), (1, 0)(2, 3)\} \subseteq G'. \)

Now \( x = (1, 2, 3) \) and \( y = (0, 1, 2). \)
Then \( [x, y] = x^{-1}y^{-1}xy = (1, 3, 2)(0, 2, 1)(1, 2, 3)(0, 1, 2) = (1, 2)(3, 0) \in G'. \) If we conjugate \((1, 2)(3, 0)\) by \((1, 2, 3)\) we get \((2, 3)(1, 0) \in G'.\)

Now by expanding the relation \([0, 1, 2)t_0]^7 = 1, \) we get
\[
(0, 1, 2)t_0t_2t_1t_0t_2t_0t_1t_0 = 1
\]
\( \implies y = t_0t_1t_2t_0t_1t_2t_0. \)

Also by expanding the relation \([0, 1)(2, 3)t_0]^7 = 1, \) we get
\[
(0, 1)(2, 3)t_0t_1t_0t_1t_0t_1t_0t_0 = 1
\]
\( \implies xy = t_0t_1t_0t_1t_0t_1t_0. \)

Now we use the second relation to solve for \( x. \) We replace \( y = t_0t_1t_2t_0t_1t_2t_0. \)
\[
xy = t_0t_1t_0t_1t_0t_1t_0
\]
\[
x = t_0t_1t_0t_0t_1t_2t_0t_2t_0. \]

So \( G = \langle x, t \rangle = \langle t_0, t_1, t_2, t_3 \rangle. \) Our goal is to show that one of the \( t'_1s \in G', \) then we can conjugate. Since \((0, 1)(2, 3) \in G'. \) Then
\[
(0, 1)(2, 3) = t_0t_1t_0t_0t_1t_0 = G'
\]
\[
(0, 1)(2, 3)(t_0t_1) = (t_0t_1t_0t_0t_1t_0)(t_0t_1) \in G' \quad (\text{since } G' \leq G)
\]
\[
= (t_0t_1)^{-1}(t_0t_1t_0t_0t_1t_0)(t_0t_1)
\]
\[
= (t_1t_0)(t_0t_1t_0t_0t_1t_0)(t_0t_1)
\]
\[
= t_0t_1t_0
\]
\[
= t_0t_1t_0t_1t_1t_1
\]
\[
= [t_0, t_1]t_1 \in G' \quad (\text{since } [t_0, t_1] \in G')
\]
\( \implies t_1 \in G'. \)

So \( t_1 \in G' \)
\( \implies t_1^y = t_1^{(0, 1, 2)} = t_2 \in G' \)
\( t_1^{y^{-1}} = t_1^{(1, 0, 2)} = t_0 \in G' \)
\[ \Rightarrow t_2, t_0 \in G'. \]

So \( t_2 \in G' \) (since \( x \in G, t_2 \in G' \), and \( G' \leq G \))

\[ t_2^x = t_2^{(1,2,3)} = t_3 \in G'. \]

Thus \( G \supseteq G' \supseteq \langle t_0, t_1, t_2, t_3 \rangle = G \)

\[ \Rightarrow G' = G. \text{ Hence } G \text{ is perfect.} \]

### 3.2.4 Conjugates of a Normal Abelian \( K \)

**Generate** \( G = L_2(13) \) over \( A_4 \)

Now we require \( x \in X \) and a \( K \leq G^x \), where \( K \) is a normal abelian subgroup such that the conjugates of \( K \) in \( G \) generate \( G \). Recall, \( G^1 = N = A_4 \). Let \( K = \langle (0, 1)(2, 3), (0, 2)(1, 3) \rangle \). Since \( K \) is normal abelian subgroup in \( G \) then for any \( s \in K \) and for all \( g \in G \) implies \( s^g \in K \). Since \( (0, 1)(2, 3) = t_0 t_1 t_0 t_1 t_0 t_1 t_0 \in K \). Now

\[ t_0 t_1 t_0 \in G \text{ and } t_0 t_1 t_0 t_1 t_0 t_1 t_0 \in K. \]

\[ \Rightarrow (t_0 t_1 t_0 t_1 t_0)^{(t_0 t_1 t_0)} \in K^G \]

\[ \Rightarrow (t_0 t_1 t_0)^{-1}(t_0 t_1 t_0 t_1 t_0)(t_0 t_1 t_0) \in K^G \]

\[ \Rightarrow t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_1 t_0 t_0 t_1 t_0 \in K^G \]

\[ \Rightarrow t_1 \in K^G \]

\[ \Rightarrow t_1^G \in K^G \]

\[ \Rightarrow K^G \supseteq \{ t_1, t_1^{y^{-1}}, t_1^{y_x}, (t_1^y)^x \} \]

\[ \Rightarrow K^G \supseteq \{ t_1, t_1^{y^{-1}}, t_1^{y_x}, (t_1^y)^x \} = \langle t_1, t_0, t_2, t_3 \rangle = G \]

Hence, the conjugates of \( K \) generate \( G \). Therefore, by Iwasawa’s lemma, \( G \cong L_2(13) \) is simple.

\[ \square \]

### 3.3 \( 2 \times L_2(8) \) as a Homomorphic Image of \( 2^*7 : D_{14} \)

#### 3.3.1 Construction of \( 2 \times L_2(8) \) over \( D_{14} \)

Consider the group \( G = 2^*7 : D_{14} \) factored by the relators \( (xtt^x)^2 \) and \( (tttx)^9 \). Note \( N = D_{14}, \) where \( x \sim (1, 2, 3, 4, 5, 6, 7) \) and \( y \sim (1, 6)(2, 5)(3, 4). \) Let \( t \sim t_7 \sim t_0. \)

Let us expand the relators:
\[1 = (xt^2)^2 = (xt_0t_1)^2 \]
\[= x^2(t_0t_1)^xt_0t_1 = x^2t_1t_2t_0t_1\]
\[\implies 1 = x^2t_1t_2t_0t_1\]
\[\implies t_1t_0 = x^2t_1t_2 \implies Nt_1t_0 = Nt_1t_2\]

and
\[e = (txtt)^9\]
\[= (t_0t_0xt_0)^9\]
\[= (xt_0)^9\]
\[= x^9t_0^8t_7r_6t_5r_4t_3r_2t_1t_0\]
\[\implies e = x^2t_1t_0t_6t_5t_4t_3t_2t_1t_0\]
\[\implies t_7t_1t_2t_3 = x^2t_1t_7t_6t_5t_4\]
\[\implies Nt_7t_1t_2t_3 = Nt_1t_7t_6t_5t_4\]

We want to find the index of \(N\) in \(G\). To do this, we perform a manual double coset enumeration of \(G\) over \(N\).

\(NeN\)

First, the double coset \(NeN\), is denoted by \([\ast]\). This double coset contains only the single coset, namely \(N\). Since \(N\) is transitive on \(\{0, 1, 2, 3, 4, 5, 6\}\), the orbit of \(N\) on \(\{0, 1, 2, 3, 4, 5, 6\}\) is: \(\mathcal{O} = \{0, 1, 2, 3, 4, 5, 6\}\). We choose \(t_0\) as our symmetric generator from this orbit \(\mathcal{O}\) and find to which double coset \(Nt_0\) belongs. \(Nt_0N\) will be a new double coset, denoted by \([0]\), so seven symmetric generators will go to \([0]\).

\(Nt_0N\)

In order to find how many single cosets \([0]\) contains, we must first find \(N^{(0)}\). Then the number of single coset in \([0]\) is equal to \(\frac{|N|}{|N^{(0)}|}\). Now,
\[N^{(0)} = N^0\]
so the number of the single cosets in \( N_{t_0}N \) is \( \frac{|N|}{|N_{t_0}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(0)} \) on \( \{0, 1, 2, 3, 4, 5, 6\} \) are: \( \emptyset = \{0\}, \{1, 6\}, \{2, 5\}, \) and \( \{3, 4\} \). We take \( t_0, t_1, t_2 \) and \( t_3 \) from each orbit respectively and find to which double coset \( N_{t_0}t_0, N_{t_0}t_1, N_{t_0}t_2, \) and \( N_{t_0}t_3 \) belong to. Now \( N_{t_0}t_0 = N \in [\ast] \), so one element will go back to \([\ast]\). Two symmetric generators will go to new double cosets \( N_{t_0}t_1 \), denoted by \([01]\), \( N_{t_0}t_2 \), denoted by \([02]\), and \( N_{t_0}t_3 \), denoted by \([03]\).

\( N_{t_0}t_1 N \)

Now \( N_{t_0}t_1N \) in \( N \) is a new double coset. We determine how many single cosets are in the double coset. Well \( N^{(01)} = N^{01} = \langle Id(N) \rangle \). But \( N_{t_0}t_1 \) is not distinct. Now \( N_{t_0}t_6 \in [01] \) since \((1,6)(2,5)(3,4) \in N \) and \( N(t_0t_1)^{(1,6)(2,5)(3,4)} = N_{t_0}t_6 \). Thus, \((1,6)(2,5)(3,4) \in N^{(01)} \). We conclude:

\[ N^{(01)} \geq (1,6)(2,5)(3,4) \]

Hence \( |N^{(01)}| = 2 \) so the number of single cosets in \( N^{(01)} \) is \( \frac{|N|}{|N^{(01)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(01)} \) on \( \{0, 1, 2, 3, 4, 5, 6\} \) are: \( \emptyset = \{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\} \). Take a representative \( t_i \) from each orbit and see which double cosets \( N_{t_0}t_1t_i \) belongs to. We have:

\[
\begin{align*}
N_{t_0}t_1t_1 &\in \{0\} \\
N_{t_0}t_1t_2 &\in [012] \\
N_{t_0}t_1t_3 &\in [013] \\
N_{t_0}t_1t_0 &\in [010].
\end{align*}
\]

The new double cosets have single coset representatives \( N_{t_0}t_1t_2, N_{t_0}t_1t_3, N_{t_0}t_1t_0 \), we represent them as \([012], [013], [010]\), respectively.

\( N_{t_0}t_2 N \)

Now \( N_{t_0}t_2N \) in \( N \) is a new double coset. We determine how many single cosets are in the double coset. Well \( N^{(02)} = N^{02} = \langle Id(N) \rangle \). But \( N_{t_0}t_2 \) is not distinct. Now \( N_{t_3}t_1 \in [02] \) since \((1,2)(3,0)(4,6) \in N \) and \( N(t_0t_2)^{(1,2)(3,0)(4,6)} = N_{t_3}t_1 \). Thus, \((1,2)(3,0)(4,6) \in N^{(02)} \). We conclude:

\[ N^{(02)} \geq (1,2)(3,0)(4,6) \]

Hence \( |N^{(02)}| = 2 \) so the number of single cosets in \( N^{(02)} \) is \( \frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(02)} \) on \( \{0, 1, 2, 3, 4, 5, 6\} \) are: \( \emptyset = \{0, 3\}, \{1, 2\}, \{4, 6\}, \{5\} \). Take a representative \( t_i \)
from each orbit and see which double cosets \(Nt_0t_2t_i\) belongs to. We have

\[ t_0t_2t_3 = x^{-3}t_5t_4t_2 \implies Nt_0t_2t_3 = Nt_5t_4t_2 \in [013] \]
\[ Nt_0t_2t_2 = Nt_0 \in [0] \]
\[ t_0t_2t_4 = xt_4t_1 \implies Nt_0t_2t_4 = Nt_4t_1 \in [03] \]
\[ Nt_0t_2t_5 \in [025]. \]

The new double coset have single coset representative \(Nt_0t_2t_5\), denoted by \([025]\).

\[ NT_0t_3N \]

Now \(Nt_0t_3N\) in \(N\) is a new double coset. However, \(N^{(03)} = N^{03} = < Id(N) >\). Only identity \(e\) will fix 0 and 3. Hence the number of single cosets living in \(Nt_0t_3\) is \(\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14\). The orbits of \(N^{(03)}\) on \(\{0, 1, 2, 3, 4, 5, 6\}\) are: \(\emptyset = \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\).

Take a representative \(t_i\) from each orbit and see which double cosets \(Nt_0t_3t_i\) belongs to.

We have:

\[ t_0t_3t_1 = x^{-1}t_2t_6 \implies Nt_0t_3t_1 = Nt_2t_6 \in [03] \]
\[ t_0t_3t_2 = x^2t_2t_3t_4 \implies Nt_0t_3t_2 = Nt_2t_3t_4 \in [012] \]
\[ Nt_0t_3t_3 = Nt_0 \in [0] \]
\[ t_0t_3t_4 = xt_0t_1t_3 \implies Nt_0t_3t_4 = Nt_0t_1t_3 \in [013] \]
\[ t_0t_3t_5 = c_{t_3t_5t_1} \implies Nt_0t_3t_5 = Nt_3t_5t_1 \in [025] \]
\[ Nt_0t_3t_6 \in [036] \]
\[ t_0t_3t_0 = xt_4t_2 \implies Nt_0t_3t_0 = Nt_4t_2 \in [02]. \]

The new double coset have single coset representative \(Nt_0t_3t_6\), denoted by \([036]\).

\[ NT_0t_1t_2N \]

Consider \(Nt_0t_1t_2N\) in \(N\) is a new double coset. We determined how many single cosets are in the double coset. Well \(N^{(012)} = N^{012} = < e >\). But \(Nt_0t_1t_2\) is not distinct. Now \(Nt_3t_2t_1 \in [012]\) since \((1, 2)(3, 7)(4, 6) \in N\) and \(N(t_0t_1t_2)^{(1,2)(3,0)(4,6)} = t_3t_2t_1\). Thus, \((1, 2)(3, 0)(4, 6) \in N^{(012)}\). we conclude:

\[ N^{(012)} \geq < (1, 2)(3, 0)(4, 6) >. \]
Hence \(|N^{(012)}| = 2\) so the number of single cosets in \(N^{(012)}\) is \(\frac{|N|}{|N^{(012)}|} = \frac{14}{2} = 7\). The orbits of \(N^{(012)}\) on \(\{0, 1, 2, 3, 4, 5, 6\}\) are: \(\emptyset = \{0, 3\}, \{1, 2\}, \{4, 6\}, \{5\}\). Take a representative \(t_i\) from each orbit and see which double cosets \(N t_0 t_1 t_2 t_i\) belongs to.

\[
t_0 t_1 t_2 t_0 = x^{-2} t_5 t_1 \implies N t_0 t_1 t_2 t_0 = N t_5 t_1 \in [03]
\]
\[
N t_0 t_1 t_2 = N t_0 t_1 \in [01]
\]
\[
t_0 t_1 t_2 t_4 = x^{-2} t_4 t_3 t_1 \implies N t_0 t_1 t_2 t_4 = N t_4 t_3 t_1 \in [013]
\]
\[
t_0 t_1 t_2 t_5 = t_5 t_1 t_4 \implies N t_0 t_1 t_2 t_5 = N t_5 t_1 t_4 \in [036].
\]

**\(N t_0 t_1 t_3 N\)**

Consider \(N t_0 t_1 t_3 N\) in \(N\) is a new double coset. However, \(N^{(013)} = N^{013} = \langle Id(N) \rangle\). Only identity \(e\) will fix 0, 1, and 3. Hence the number of single cosets living in \(N t_0 t_1 t_3\) is \(\frac{|N|}{|N^{(013)}|} = \frac{14}{1} = 14\). The orbits of \(N^{(013)}\) on \(\{0, 1, 2, 3, 4, 5, 6\}\) are: \(\emptyset = \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\). Take a representative \(t_i\) from each orbit and see which double cosets \(N t_0 t_1 t_3 t_i\) belongs to.

\[
t_0 t_1 t_3 t_0 = x^{-2} t_4 t_3 t_2 \implies N t_0 t_1 t_3 t_0 = N t_4 t_3 t_2 \in [012]
\]
\[
t_0 t_1 t_3 t_1 = x^{-4} t_2 t_1 t_6 \implies N t_0 t_1 t_3 t_1 = N t_2 t_1 t_6 \in [013]
\]
\[
t_0 t_1 t_3 t_2 = x^2 t_2 t_4 \implies N t_0 t_1 t_3 t_2 = N t_2 t_4 \in [02]
\]
\[
N t_0 t_1 t_3 t_3 = N t_0 t_1 \in [01]
\]
\[
t_0 t_1 t_3 t_4 = x^{-1} t_0 t_3 \implies N t_0 t_1 t_3 t_4 = N t_0 t_3 \in [03]
\]
\[
t_0 t_1 t_3 t_5 = x^{-1} t_1 t_4 t_0 \implies N t_0 t_1 t_3 t_5 = N t_1 t_4 t_0 \in [036]
\]
\[
t_0 t_1 t_3 t_6 = x^{-1} t_5 t_3 t_0 \implies N t_0 t_1 t_3 t_6 = N t_5 t_3 t_0 \in [02].
\]

**\(N t_0 t_1 t_0 N\)**

Now \(N t_0 t_1 t_0 N\) is indeed a new double coset. We determine how many single cosets are in this double coset. Well \(N^{(010)} = N^{010} = \langle Id(N) \rangle\). Well \(N^{(010)} = N^{010} = \langle Id(N) \rangle\). We have these two relations \(t_0 t_1 t_0 = (0, 5, 3, 1, 6, 4, 2)_t_5 t_6 t_5\) and \(t_0 t_1 t_0 = (1, 0)(2, 6)(3, 5)_t_0 t_6 t_0\). Since \((0, 5, 3, 1, 6, 4, 2) \in N\) and \(N(t_0 t_1 t_0)^{(0,5,3,1,6,4,2)} = N t_5 t_6 t_5\). Thus, \((0, 5, 3, 1, 6, 4, 2) \in N^{(010)}\) and \(N^{(010)} \geq \langle (0, 5, 3, 1, 6, 4, 2) \rangle\). Since
Take a representative from this orbit, say \( t \). Thus, \((1,0)(2,6)(3,5) \in N \) and \( N(t_0t_1t_0)^{(1,0)(2,6)(3,5)} = t_1t_0t_1. \) We conclude:

\[
N^{(010)} \geq< (1,0)(2,6)(3,5), (0,1,2,3,4,5,6) >.
\]

Then \( N^{(010)} = N \). Hence \( |N^{(010)}| = 14 \), so the number of single cosets in \( N^{(010)} \) is \( \frac{|N|}{|N^{(010)}|} = \frac{14}{14} = 1 \). The orbit of \( N^{(010)} \) on \( \{0,1,2,3,4,5,6\} \) is \( \mathbb{O} = \{0,1,2,3,4,5,6\} \).

Take a representative from this orbit, say \( t_0 \). Hence \( Nt_0t_1t_0t_0 \in [01] \). Therefore, seven symmetric generators will go back to \( Nt_0t_1N \).

\[
Nt_0t_2t_5N
\]

Now consider \( Nt_0t_2t_5N \) in \( N \) is a new double coset. We determined how many single cosets are in the double coset. Well \( N^{(025)} = N^{(025)} =< \text{Id}(N) >. \) But \( Nt_0t_2t_5 \) is not distinct. Now \( Nt_3t_1t_5 \in [025] \) since \( (0,3)(1,2)(4,6) \in N \) and \( N(t_0t_2t_5)^{(0,3)(1,2)(4,6)} = t_3t_1t_5. \) Thus, \( (3,0)(1,2)(4,6) \in N^{(025)}. \) We conclude:

\[
N^{(025)} \geq< (0,3)(1,2)(4,6) >.
\]

Hence \( |N^{(025)}| = 2 \) so the number of single cosets in \( N^{(025)} \) is \( \frac{|N|}{|N^{(025)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(025)} \) on \( \{0,1,2,3,4,5,6\} \) are: \( \mathbb{O} = \{0,3\}, \{1,2\}, \{4,6\}, \{5\} \). Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_2t_5t_i \) belongs to.

\[
t_0t_2t_5t_0 = x^{-4}t_4t_6t_2 \implies Nt_0t_2t_5t_0 = Nt_4t_6t_2 \in [025] \\
t_0t_2t_5t_1 = x^2t_6t_3 \implies Nt_0t_2t_5t_1 = Nt_6t_3 \in [03] \\
t_0t_2t_5t_4 = x^{-4}t_5t_6t_1 \implies Nt_0t_2t_5t_4 = Nt_5t_6t_1 \in [013] \\
Nt_0t_2t_5t_5 \in [02].
\]

\[
Nt_0t_3t_6N
\]

Now consider \( Nt_0t_3t_6N \) in \( N \) is a new double coset. We determined how many single cosets are in the double coset. Well \( N^{(036)} = N^{(036)} =< \text{Id}(N) >. \) But \( Nt_0t_3t_6 \) is not distinct. Now \( Nt_0t_4t_1 \in [036] \) since \( (1,6)(2,5)(3,4) \in N \) and \( N(t_0t_3t_6)^{(1,6)(3,4)(2,5)} = t_0t_4t_1. \) Thus, \( (1,6)(3,4)(2,5) \in N^{(036)}. \) We conclude:

\[
N^{(036)} \geq< (1,6)(3,4)(2,5) >.
\]

Hence \( |N^{(036)}| = 2 \) so the number of single cosets in \( N^{(036)} \) is \( \frac{|N|}{|N^{(036)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(036)} \) on \( \{0,1,2,3,4,5,6\} \) are: \( \mathbb{O} = \{1,6\}, \{3,4\}, \{2,5\}, \{0\} \). Take a represent-
tative $t_i$ from each orbit and see which double cosets $Nt_0t_3t_6t_i$ belongs to.

$$Nt_0t_3t_6t_6 \in [03]$$

$$t_0t_3t_6t_3 = xt_1t_0t_5 \implies Nt_0t_3t_6t_3 = Nt_1t_0t_5 \in [013]$$

$$t_0t_3t_6t_5 = x^2t_3t_6t_2 \implies Nt_0t_3t_6t_5 = Nt_3t_6t_2 \in [036]$$

$$t_0t_3t_6t_0 = et_{2t_3t_4} \implies Nt_0t_3t_6t_0 = Nt_2t_3t_4 \in [036].$$

We have completed the double coset enumeration since the right coset is closed under multiplication, hence, the index of $N$ in $G$ is 72 single cosets. We conclude:

$$G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_2N \cup Nt_0t_3N \cup Nt_0t_1t_3N \cup Nt_0t_1t_0N \cup Nt_0t_2t_5N \cup Nt_0t_3t_6N,$$

where

$$G = \frac{2^{\ast 7} \cdot D_{14}}{(xtt)^2, (txt)^9 = 1}.$$

$$|G| \leq |N| + \frac{|N|}{N_0/t_0} + \frac{|N|}{N_0/t_1} + \frac{|N|}{N_0/t_2} + \frac{|N|}{N_0/t_3} + \frac{|N|}{N_0/t_4} + \frac{|N|}{N_0/t_5} + \frac{|N|}{N_0/t_6} \times |N|$$

$$|G| \leq (1 + 7 + 7 + 7 + 14 + 7 + 14 + 1 + 7 + 7) \times 14$$

$$|G| \leq 72 \times 14$$

$$|G| \leq 1008$$

The Cayley diagram that summarizes the above information is given below:

### 3.3.2 Factoring $2 \times L_2(8)$ by the Center

Consider $G = \frac{2^{\ast 7} \cdot D_{14}}{(xtt)^2, (txt)^9 = 1} \cong 2 \times L_2(8)$. We are going to factor $G$ by the center, to do so, we use the following loops in Magma:

```magma
> D:=DihedralGroup(7);
> xx:=D!(1,2,3,4,5,6,7);
> yy:=D!(1, 6)(2, 5)(3, 4);
> N:=sub<D|xx,yy>;
> G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
(x*t*t*x)^2, (t*t*x*t)^9>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> Center(G1);
Permutation group acting on a set of cardinality 72
Order = 2
(1, 20)(2, 12)(3, 6)(4, 7)(5, 11)(8, 13)(9, 18)(10, 19)
```
Figure 3.2: Cayley Diagram of $2 \times L_2(8)$ over $D_{14}$


$$(25, 38)(26, 33)(27, 43)(28, 35)(34, 44)(36, 45)$$

$$(37, 52)(39, 46)(40, 56)(41, 55)(42, 48)(47, 57)$$


$$(61, 71)(64, 66)(70, 72)$$

By Magma, we know that the center of $G$ is of order 2. We let $a$ equals to the Center($G_1$). Now to convert the center in term of word, we use the Schreier System:

```magma
> A:=f(x);
> B:=f(y);
> C:=f(t);
> N:=sub<G1|A,B,C>;
> NN<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2,(t,y),
> (x*t*t*x)^2, (t*t*x*t)^9>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..#N]];
> for i in [2..#N] do
> for P:=[Id(N): l in [1..Sch[i]]] do
```
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^(-1); end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..#N] do if ArrayP[i] eq aa
then print Sch[i]; end if; end for;
> print x * y * t * x * t * x^(-1) * t;

Thus, the center of $G$ is $Z(G) = <xyttxx^{-1}t>$. Now we factor $G$ by the center and we obtain the following:

> G<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2, t^2, (t,y),
(x*t*t^x) x^2, (t*t*x*t)^9, x*y*t*x*t*x^{-1}*t>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);

$G | A(1, 8) = L(2, 8)$

$1$

Now, $G = 2^*7 : D_{14}$ is factored by the relators $(xttx)^2, (tttx)^9$ and the center $Z(G) = <xyttx^{-1}t>$.

### 3.3.3 Construction of $L_2(8)$ over $D_{14}$

Consider the group $G \cong <x^7, y^2, (x*y)^2, t^2, (t,y)>$ factored by $(xttx)^2, (tttx)^9$ and the center $Z(G) = <xyttx^{-1}t>$. Recall, $G = 2^*7 : D_{14}$, $N = D_{14} =< x, y > =< (0,1,2,3,4,5,6), (1,6)(2,5)(3,4) >$, and $t \sim t_7 \sim t_0$. Also, by expanding the two relations we have:

$$(xttx)^2 = x^2t_1t_2t_0t_1 = 1 \implies x^2t_1t_2 = t_1t_0$$

$$(tttx)^9 = x^2t_1t_0t_6t_5t_4t_3t_2t_1t_0 = 1 \implies x^2t_1t_0t_6t_5 = t_0t_1t_2t_3t_4.$$ 

Now let's expand the center $Z(G) = <xyttx^{-1}t>$:

$$xyttx^{-1}t = xy_0tx_0x^{-1}t_0 = xy_0t_0 = 1 \implies xy_0t_6 = t_0.$$ 

Now, we begin the construction of $L_2(8)$ over $D_{14}$. 

First, the double coset $N_\mathcal{E}N$, is denoted by $[\ast]$. This double coset contains only the single coset, namely $N$. Since $N$ is transitive on $\{0, 1, 2, 3, 4, 5, 6\}$, the orbit of $N$ on $\{0, 1, 2, 3, 4, 5, 6\}$ is: $\mathcal{O} = \{0, 1, 2, 3, 4, 5, 6\}$. We choose $t_0$ as our symmetric generator from this orbit $\mathcal{O}$ and find to which double coset $N t_0 N$ belongs. $N t_0 N$ will be a new double coset, denoted by $[0]$, so seven symmetric generators will go to $[0]$.

In order to find how many single cosets $[0]$ contains, we must first find $N^{(0)}$. Then the number of single coset in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 = < e, (1, 6)(2, 5)(3, 4) >$ so the number of the single cosets in $N t_0 N$ is $\frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0)}$ on $\{0, 1, 2, 3, 4, 5, 6\}$ are: $\mathcal{O} = \{0\}, \{1, 6\}, \{2, 5\},$ and $\{3, 4\}$. We take $t_0$, $t_6$, $t_2$ and $t_3$ from each orbit respectively and find to which double coset $N t_0 t_6$, $N t_0 t_2$, and $N t_0 t_3$ belong to. Now $N t_0 t_0 = N \in [\ast]$, so one element will go back to $[\ast]$. We have the relation $t_0 t_6 = x t_0 = N t_0 t_6 = N t_0 \in [0]$, since $x y \in N$ and $N t_0 \in [0]$. Thus, $N t_0 t_6 = N t_0 \in [0]$, so two elements will loop back to $[0]$. On the other hand, two symmetric generators will go to new double cosets $N t_0 t_2$, denoted by $[02]$, and $N t_0 t_3$, denoted by $[03]$.

Now $N t_0 t_2 N$ in $N$ is a new double coset. We determine how many single cosets are in the double coset. Well $N^{(02)} = N^{02} = < I d(N) >$. But $N t_0 t_2$ is not distinct. Now $N t_3 t_1 \in [02]$ since $(1, 2)(3, 0)(4, 6) \in N$ and $N(t_0 t_2)^{(1,2)(3,0)(4,6)} = N t_3 t_1$. Thus, $(1, 2)(3, 0)(4, 6) \in N^{(02)}$. We conclude, $N^{(02)} \geq < (1, 2)(3, 0)(4, 6) >$. Hence $|N^{(02)}| = 2$ so the number of single cosets in $N^{(02)}$ is $\frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7$. The orbits of $N^{(02)}$ on $\{0, 1, 2, 3, 4, 5, 6\}$ are: $\mathcal{O} = \{1, 2\}, \{3, 0\}, \{4, 6\}, \{5\}$. Take a representative $t_i$ from each orbit and see which double cosets $N t_0 t_2 t_i$ belongs to. We have
\[ N_{t_0 t_2 t_2} = N_{t_0} \in [0] \]

\[ t_0 t_2 t_3 = x^2 y t_5 t_2 \implies N_{t_0 t_2 t_3} = N_{t_5 t_2} \in [03] \]

\[ t_0 t_2 t_4 = x t_4 t_1 \implies N_{t_0 t_2 t_4} = N_{t_4 t_1} \in [03] \]

\[ N_{t_0 t_2 t_5} \in [025]. \]

The new double coset have single coset representative \( N_{t_0 t_2 t_5} \), denoted by \([025]\).

\[ N_{t_0 t_3 N} \]

Now \( N_{t_0 t_3 N} \) in \( N \) is a new double coset. However, \( N^{(03)} = N^{03} = < Id(N) > \). Only identity \( e \) will fix 0 and 3. Hence the number of single cosets living in \( N_{t_0 t_3} \) is \( \frac{|N|}{|N^{03}|} = \frac{14}{2} = 14 \). The orbits of \( N^{(03)} \) on \( \{0, 1, 2, 3, 4, 5, 6\} \) are: \( \emptyset = \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \). Take a representative \( t_i \) from each orbit and see which double cosets \( N_{t_0 t_3 t_i} \) belongs to. We have:

\[ t_0 t_3 t_1 = x^{-1} t_2 t_6 \implies N_{t_0 t_3 t_1} = N_{t_2 t_6} \in [03] \]

\[ t_0 t_3 t_2 = x^2 y t_5 t_3 \implies N_{t_0 t_3 t_2} = N_{t_5 t_3} \in [02] \]

\[ N_{t_0 t_3 t_3} = N_{t_0} \in [0] \]

\[ t_0 t_3 t_4 = y t_0 t_3 \implies N_{t_0 t_3 t_4} = N_{t_0 t_3} \in [03] \]

\[ t_0 t_3 t_5 = e t_3 t_5 t_1 \implies N_{t_0 t_3 t_5} = N_{t_3 t_5 t_1} \in [025] \]

\[ t_0 t_3 t_6 = y t_5 t_3 t_0 \implies N_{t_0 t_3 t_6} = N_{t_5 t_3 t_0} \in [025] \]

\[ t_0 t_3 t_0 = x t_4 t_2 \implies N_{t_0 t_3 t_0} = N_{t_4 t_2} \in [02]. \]

\[ N_{t_0 t_2 t_5 N} \]

Now consider \( N_{t_0 t_2 t_5 N} \) in \( N \) is a new double coset. We determined how many single cosets are in the double coset. Well \( N^{(025)} = N^{025} = < Id(N) > \). But \( N_{t_0 t_2 t_5} \) is not distinct. Now \( N_{t_3 t_1 t_5} \in [025] \) since \( (0, 3)(1, 2)(4, 6) \in N \) and \( N(t_0 t_2 t_5)^{(0,3)(1,2)(4,6)} = t_3 t_1 t_5 \). Thus, \( (3, 0)(1, 2)(4, 6) \in N^{(025)} \). We conclude, \( N^{(025)} \geq < (0, 3)(1, 2)(4, 6) > \). Hence \( |N^{(025)}| = 2 \) so the number of single cosets in \( N^{(025)} \) is \( \frac{|N|}{|N^{(025)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(025)} \) on \( \{0, 1, 2, 3, 4, 5, 6\} \) are: \( \emptyset = \{0, 3\}, \{1, 2\}, \{4, 6\}, \{5\} \). Take a
representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_2t_5t_i \) belongs to.

\[
\begin{align*}
    t_0t_2t_5t_1 &= x^2t_0t_3 \implies Nt_0t_2t_5t_1 = Nt_0t_3 \in [03] \\
t_0t_2t_5t_3 &= x^6t_0t_4t_1 \implies Nt_0t_2t_5t_3 = Nt_0t_3 \in [025] \\
t_0t_2t_5t_4 &= x^6yt_5t_1 \implies Nt_0t_2t_5t_4 = Nt_0t_1 \in [013] \\
    Nt_0t_2t_5t_5 &\in [02].
\end{align*}
\]

We have completed the double coset enumeration since the right coset is closed under multiplication, hence, the index of \( N \) in \( G \) is 36 single cosets. We conclude:

\[
G = N \cup Nt_0N \cup Nt_0t_2N \cup Nt_0t_3N \cup Nt_0t_2t_5N,
\]

where

\[
G = \frac{(xtt^2, (txt)^9, xyttxt^{-1})}{1}
\]

\[
|G| \leq |N| + \frac{|N|}{N(0)} + \frac{|N|}{N(03)} + \frac{|N|}{N(025)}
\]

\[
|G| \leq (1 + 7 + 7 + 14 + 7) \times 14
\]

\[
|G| \leq 36 \times 14
\]

\[
|G| \leq 504
\]

The Cayley diagram that summarizes the above information is given below:

![Cayley Diagram of \( L_2(8) \) over \( D_{14} \)](image-url)
3.4 Iwasawa’s Lemma to Prove $L_2(8)$ over $D_{14}$ is Simple

We consider $G \cong \frac{2^*7 \cdot D_{14}}{[x_t0t_1^2, [t_0t_0xt_0]^9, yxt_0xt_0x^{-1}t_0]} \cong L_2(8)$,

Now we use Iwasawa’s lemma to prove $G \cong L_2(8)$ is simple. We use Iwasawa’s lemma to prove $G \cong L_2(8)$ is simple. Iwasawa’s lemma has three sufficient conditions that we must satisfied:

1. $G$ acts on $X$ faithfully and primitively
2. $G$ is perfect ($G = G'$)
3. There exist $x \in X$ and a normal abelian subgroup $K$ of $G^x$ such that the conjugates of $K$ generate $G$.

Proof. 3.4.1 $G = L_2(8)$ acts on $X$ Faithfully

Let $G$ acts on $X = \{N, Nt_0N, Nt_0t_2N, Nt_0t_3N, Nt_0t_2t_5N\}$, where $|X| = 36$. $G$ acts on $X$ implies there exist homomorphism

$f : G \rightarrow S_{36}$ \quad (|X| = 36).

By First Isomorphic Theorem we have:

$G/kerf \cong f(G)$.

If $kerf = 1$ then $G \cong f(G)$. Only elements of $N$ fix $N$ implies $G^1 = N$. Since $X$ is a transitive $G - set$ of degree 36, we have:

$|G| = 36 \times |G^1|$
$= 36 \times |N|$
$= 36 \times 14$
$= 508$

$\implies |G| = 508$.

From Cayley diagram, $|G| \leq 508$. However, from above $|G| = 508$ implying $ker(f) = 1$. Since $kerf = 1$ then $G$ acts faithfully on $X$. 

3.4.2 \( G = L_2(8) \) acts on \( X \) Primitively

In order to show that \( G \) is primitive, we must show that \( G = L_2(8) \) is transitive on \( X = |36| \) and there exists no nontrivial blocks of \( X \). From the Cayley diagram of \( G = L_2(8) \) over \( D_{14} \), we see that \( G \) is transitive. Let \( B \) be a nontrivial block, then \( |B||X| \). Note if we had a nontrivial block it would have to be of size 2, 3, 4, 6, 9, 12, or 18. By inspection, of our Cayley diagram we ca see that we cannot create a nontrivial block of these sizes.

3.4.3 \( G = L_2(8) \) is Perfect

Next we want to show that \( G = G' \). Now \( D_{14} \subseteq G \implies D_{14}' \subseteq G' \).

\[ D_{14}' = \langle [a, b] | a, b \in D_{14} \rangle \]. Now the derived group,
\[ D_{14}' = \langle (0, 1, 2, 3, 4, 5) \rangle = \langle x \rangle \]
\( \implies \{e, x, x^2, x^3, x^4, x^5\} \subseteq G' \).

Now \( x = (0, 1, 2, 3, 4, 5, 6) \) and \( y = (1, 6)(2, 5)(3, 4) \).

Then \( [x, y] = x^{-1}y^{-1}xy \)
\[ = (0, 1, 2, 3, 4, 5, 6)(1, 6)(2, 5)(3, 4)(0, 6, 5, 4, 3, 2, 1)(1, 6)(2, 5)(3, 4) \]
\[ = (0, 2, 4, 6, 1, 3, 5) \in G' \).

If we conjugate \( (0, 2, 4, 6, 1, 3, 5) \) by \( (0, 1, 2, 3, 4, 5, 6) \) we get \( (1, 3, 5, 0, 2, 4, 6) \in G' \).

Main relation:
\[
\begin{align*}
x^2 &= t_{10}t_{20}t_1 \quad (*) \\
(x^2)^3 &= (t_{10}t_{20}t_1)^3 \\
x^6 &= t_{10}t_{20}t_{020}t_{020}t_1 \\
x^{-1} &= t_{10}t_{20}t_{020}t_{020}t_1 \\
x &= t_{12}t_{020}t_{020}t_{020}t_1 \\
\end{align*}
\]

Now, we use the relation that we obtained by factoring by the center:
\[
\begin{align*}
xy &= t_{0}t_{06}t_0 \\
 x^{-1}xy &= t_{10}t_{20}t_{020}t_{020}t_{10}t_{06}t_0 \\
y &= t_{10}t_{20}t_{020}t_{020}t_{10}t_{06}t_0
\end{align*}
\]
So $G = < x, y, t > = < t_0, t_1, t_2, t_3, t_4, t_5, t_6 >$. Our goal is to show that one of the $t'_i s \in G'$, then we can conjugate. Since $x \in G'$. Then from our double coset relation we have:

$$x = t_0 t_2 t_4 t_1 t_4 \in G'$$

$$= t_0 t_2 t_1 t_4 t_1 t_4 \in G'$$

$$= t_0 t_2 t_1 [t_1, t_4] \in G' \text{ (since } [t_1, t_4] \in G')$$

$$= t_0 t_2 t_1 \in G'$$

Now, we multiply $t_0 t_2 t_1$ by the inverse of (*):

$$t_0 t_2 t_1 t_1 t_2 t_0 t_1 = t_1 \in G' \text{ (since } x \in G' \text{ and } (x^2)^{-1} \in G').$$

So $t_1 \in G'$

$$\implies t_1^2, t_1^3, t_1^4, t_1^5, t_1^6, t_1^7 \in G' \text{ (since } x \in G \text{ and } G' \leq G)$$

$$\implies < t_2, t_3, t_4, t_5, t_6, t_0, t_1 > \in G'$$

Thus $G \subseteq G' \subseteq < t_2, t_3, t_4, t_5, t_6, t_0, t_1 > \in G \implies G' = G$.

Hence $G$ is perfect.

### 3.4.4 Conjugates of a Normal Abelian $K$

**Generate $G = L_2(8)$ over $D_{14}$**

Now we require $x \in X$ and a $K \leq G^x$, where $K$ is a normal abelian subgroup such as the conjugates of $K$ in $G$ generate $G$. Recall, $G^1 = N = D_{14}$. Let $K = < x >$.

Since $K$ is normal abelian subgroup in $G'$ then for any $s \in K$ and for all $g \in G$ implies $s^g \in K$. Since $x \in K \implies x^2 \in K$. Now from (*) we have:

$$x^2 = t_1 t_0 t_2 t_1 \in K$$

$$(x^2)^{t_1} = (t_1 t_0 t_2 t_1)^{t_1} \in K^G$$

$$t_1 (x^2) t_1 = t_1 (t_1 t_0 t_2 t_1) t_1 \in K^G$$

$$x^2 t_3 t_1 = t_0 t_2 \in K^G$$

$$\implies t_0 t_2 \in K^G$$

So, the inverse $t_2 t_0 \in K^G$. Moreover, from the double coset we have the following
relation:

\[ x = t_0 t_2 t_4 t_1 t_4 \in K. \]

Now, we multiply the above relation by \( t_2 t_0 \):

\[
(t_0 t_2 t_4 t_1 t_4 t_2 t_0) (t_0 t_2 t_4 t_1 t_4 t_2 t_0) t_0 t_2 t_4 \in K^G
\]

\[
= t_4 t_2 t_0 (t_0 t_2 t_4 t_1 t_4 t_2 t_0) t_0 t_2 t_4 = t_1 \in K^G
\]

Thus, \( t_1 \in K^G \)

\[
\implies t_1^G \in K^G
\]

\[
\implies K^G \supseteq \{ t_1^2, t_1^3, t_1^4, t_1^5, t_1^6 \}
\]

\[
\implies K^G \supseteq \{ t_1^2, t_1^3, t_1^4, t_1^5, t_1^6 \} = < t_1, t_0, t_2, t_3, t_4, t_5, t_6 >= G
\]

Hence, the conjugates of \( K \) generate \( G \). Therefore, by Iwasawa’s lemma, \( G \cong L_2(8) \) is simple.

3.5 \( L_2(13) \) as a Homomorphic Image of \( 2^*7 : D_{14} \)

3.5.1 Construction of \( L_2(13) \) over \( D_{14} \)

Factoring the progenitor \( 2^*7 : D_{14} \) by the following relations

\[
[(1,5)(2,4)(6,0)t_1 t_0]^7 \text{ and } [(1,2,3,4,5,6,0)t_0]^3
\]

yields the finite homomorphistic image:

\[
G \cong \frac{2^*7 : D_{14}}{[(1,5)(2,4)(6,0)t_1 t_0]^7, [t_0 t_0(1,2,3,4,5,6,0)t_0]^3},
\]

where \( D_{14} \) is a maximal in \( L_2(13) \) and the index of \( D_{14} \) in \( G \) equals 78. \( G \cong L_2(13) \), the projective special linear group.

A symmetric representation for the above image is given by:

\[
< x, y, t | x^7, y^2, (x \ast y)^2, t^2, (t, y), (x \ast y \ast t^x \ast t), (t \ast t \ast x \ast t)^3 >,
\]
where \( N = D_{14} \cong \langle x^7, y^2, (x*y)^2 \rangle \), and the action of \( x, y \) on the symmetric generators is given by
\[
\begin{align*}
x & \sim (1, 2, 3, 4, 5, 6, 0), \\
y & \sim (1, 6)(2, 5)(3, 4).
\end{align*}
\]

The relation
\[
((1, 5)(2, 4)(6, 0)t_1t_0)^7 = 1 \quad \text{with} \quad (1, 5)(2, 4)(6, 0) = \pi \quad \text{becomes}
\]
\[
(\pi t_1t_0)^7 = 1
\]
\[
\implies \pi t_1t_0 \pi t_1t_0 \pi t_1t_0 \pi t_1t_0 \pi t_1t_0 \pi t_1t_0 = 1
\]
\[
\implies \pi^7(t_1t_0)^{\pi^6(t_1t_0)^{\pi^5(t_1t_0)^{\pi^4(t_1t_0)^{\pi^3(t_1t_0)^{\pi^2(t_1t_0)^{\pi(t_1t_0)}(t_1t_0)^{\pi(t_1t_0)}(t_1t_0)^{\pi(t_1t_0)}(t_1t_0)^{\pi(t_1t_0)}(t_1t_0)^{\pi(t_1t_0)} = 1
\]
\[
\implies (1, 5)(2, 4)(6, 0)t_1t_0t_5t_6t_1t_0t_5t_6t_1t_0t_5t_6t_1t_0 = 1
\]
\[
\implies (1, 5)(2, 4)(6, 0)t_1t_0t_5t_6t_1t_0t_5t_6 = t_0t_1t_5t_6t_1t_0
\]
\[
\implies Nt_1t_0t_5t_6t_1t_0t_5t_6 = Nt_0t_1t_5t_6t_0t_1.
\]

The relation
\[
((1, 2, 3, 4, 5, 6, 0)t_0)^3 = ((1, 2, 3, 4, 5, 6, 0)t_0)^3 = 1
\]
\[
\implies x^3t_0^2t_0^x t_0 = 1
\]
\[
\implies (1, 4, 0, 3, 5, 2, 5)t_2t_1t_0 = 1
\]
\[
\implies (1, 4, 0, 3, 5, 2, 5)t_2 = t_0t_1
\]
\[
\implies Nt_2 = Nt_0t_1.
\]

We want to find the index of \( N \) in \( G \). To do this, we perform a manual double coset enumeration of \( G \) over \( N \). We take \( G \) and express it as a union of double cosets \( NgN \), where \( g \) is an element of \( G \). So \( G = NeN \cup Ng_1N \cup Ng_2N \cup ... \) where \( g_i \)'s words in \( t_i \)'s.

We need to find all double cosets \([w]\) and find out how many single cosets each of them contains, where \([w] = [Nw^n | n \in N]\). The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by \( t_i \)'s. We will identify, for each \([w]\), the double coset to which \( Nwt_i \) belongs for one symmetric generator \( t_i \) from each orbit of the coset stabilising group \( N(w) \).

**\( NeN \)**

First, the double coset \( NeN \), is denoted by \([*]\). This double coset contains only the
single coset, namely \( N \). Since \( N \) is transitive on \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_0\} \), the orbit of \( N \) on \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_0\} \) is:

\[ \mathcal{O} = \{1, 2, 3, 4, 5, 6, 0\} \].

We choose \( t_0 \) as our symmetric generator from \( \mathcal{O} \) and find to which double coset \( Nt_0N \) belongs. \( Nt_0N \) will be a new double coset, denote it [0].

**\( Nt_0N \)**

In order to find how many single cosets [0] contains, we must first find the coset stabiliser \( N^{(0)} \). Then the number of single coset in [0] is equal to \( \frac{|N|}{|N^{(0)}|} \). Now,

\[ N^{(0)} = N^0 = < (1, 6)(2, 5)(3, 4) > \]

so the number of the single cosets in \( Nt_0N \) is \( \frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7 \). Furthermore, the orbits of \( N^{(0)} \) on \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_0\} \) are:

\[ \mathcal{O} = \{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\} \].

We take \( t_0, t_1, t_2 \) and \( t_3 \) from each orbit, respectively, and to see which double coset \( Nt_0t_0, Nt_0t_1, Nt_0t_2, \) and \( Nt_0t_3 \) belong to. We have:

\[ Nt_0t_0 = N \in [\ast] \]
\[ t_0t_1 = x^3t_2 \implies Nt_0t_1 = Nt_2 \in [0] \]
\[ Nt_0t_2 \in [02] \]
\[ Nt_0t_3 \in [03] \].

The new double cosets have single coset representatives \( Nt_0t_2 \) and \( Nt_0t_3 \), which we represent them as [02] and [03] respectively.

**\( Nt_0t_2N \)**

Now \( Nt_0t_2N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(02)} = N^{02} = < e > \). But \( Nt_0t_2 \) is not distinct. We have \( Nt_5t_3 \in [02] \) since \( (1, 4)(3, 2)(0, 5) \in N \) and \( N(t_0t_2)(1, 4)(3, 2)(0, 5) = Nt_5t_3 \). Thus, \( (1, 4)(3, 2)(0, 5) \in N^{(02)} \). We conclude:

\[ N^{(02)} \geq < (1, 4)(3, 2)(0, 5) > \].
Hence $|N^{(02)}| = 2$. So the number of single cosets in $N^{(02)}N$ is $\frac{|N|}{|N^{(02)}|} = \frac{14}{2} = 7$. The orbits of $N^{(02)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathcal{O} = \{\{1, 4\}, \{3, 2\}, \{0, 5\}, \{6\}\}.$$  

Take a representative $t_i$ from each orbit and see which double cosets $N t_0 t_2 t_i$ belongs to. We have:

$$t_0 t_2 t_4 = x^2 t_1 t_5 \implies N t_0 t_2 t_4 = N t_1 t_5 \in [03]$$

$$N t_0 t_2 t_2 = N t_0 \in [0]$$

$$N t_0 t_2 t_0 \in [020]$$

$$N t_0 t_2 t_6 \in [026].$$

The new double coset are $N t_0 t_2 t_0$ and $N t_0 t_2 t_6$, which we represent them as $[020]$ and $[026]$ respectively.

$N t_0 t_3 N$

Consider $N t_0 t_3 N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(03)} = N^{03} =< e >$. Only identity (e) will fix 0 and 3. Hence the number of single cosets living in $N t_0 t_3 N$ is $\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03)}$ on $\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}$ are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\}.$$  

Take a representative $t_i$ from each orbit and see which double cosets $N t_0 t_3 t_i$ belongs to. We have:

$$N t_0 t_3 t_1 \in [031]$$

$$t_0 t_3 t_2 = x^4 t_4 t_1 \implies N t_0 t_3 t_2 = N t_4 t_1 \in [03]$$

$$N t_0 t_3 t_3 = N t_0 \in [0]$$

$$t_0 t_3 t_4 = x^2 t_1 t_6 \implies N t_0 t_3 t_4 = N t_1 t_6 \in [02]$$

$$t_0 t_3 t_5 = x^6 t_6 t_1 t_6 \implies N t_0 t_3 t_5 = N t_6 t_1 t_6 \in [020]$$

$$N t_0 t_3 t_6 \in [036]$$

$$N t_0 t_3 t_0 \in [030].$$

The new double coset are $N t_0 t_3 t_1$, $N t_0 t_3 t_6$, and $N t_0 t_3 t_0$, which we represent them as $[031]$, $[036]$, and $[030]$ respectively.
\(Nt_0t_2t_0N\)

Now consider \(Nt_0t_2t_0N\) is a new double coset. We determine how many single cosets are in the double coset. However, \(N^{(020)} = N^{020} =< e >\). Only identity (e) will fix 0 and 2. Hence the number of single cosets living in \(Nt_0t_2t_0N\) is \(\frac{|N|}{|N^{(020)}|} = \frac{14}{1} = 14\). The orbits of \(N^{(020)}\) on \({t_1, t_2, t_3, t_4, t_5, t_6, t_0}\) are:

\[\varnothing = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\} .\]

Take a representative \(t_i\) from each orbit and see which double cosets \(Nt_0t_2t_0t_i\) belongs to. We have:

\[
\begin{align*}
t_0t_0t_0t_1 &= x^2t_1t_6t_2 \implies Nt_0t_2t_0t_1 = Nt_1t_6t_2 \in [026] \\
t_0t_0t_0t_2 &= x^5yt_0t_0t_0 \implies Nt_0t_2t_0t_2 = Nt_0t_2t_0 \in [020] \\
t_0t_0t_0t_3 &= yt_0t_2t_5 \implies Nt_0t_2t_0t_3 = Nt_4t_5\in [026] \\
t_0t_0t_0t_4 &= xyt_2t_5t_1 \implies Nt_0t_2t_0t_4 = Nt_2t_5t_1 \in [036] \\
t_0t_0t_0t_5 &= (x^2)^{-1}t_5t_1t_4 \implies Nt_0t_2t_0t_5 = Nt_5t_1t_4 \in [036] \\
t_0t_0t_0t_6 &= xt_1t_4 \implies Nt_0t_2t_0t_6 = Nt_1t_4 \in [03] \\
Nt_0t_2t_0t_0 &= Nt_0t_2 \in [02].
\]

\(Nt_0t_2t_0N\)

Now \(Nt_0t_2t_0N\) is a new double coset. We determine how many single cosets are in the double coset. Now \(N^{(026)} = N^{026} = < e >\). But \(Nt_0t_2t_6\) is not distinct. Now \(Nt_5t_3t_6 \in [026]\) since \((1, 4)(3, 2)(0, 5) \in N\) and \(N(t_0t_2t_6)^{(1, 4)(3, 2)(0, 5)} = Nt_5t_3t_6\). Thus, \((1, 4)(3, 2)(0, 5) \in N^{(026)}\). We conclude:

\[N^{(026)} \geq < (1, 4)(3, 2)(0, 5) > .\]

Hence \(|N^{(026)}| = 2\). So the number of single cosets in \(N^{(026)}N\) is \(\frac{|N|}{|N^{(026)}|} = \frac{14}{2} = 7\). The orbits of \(N^{(026)}\) on \({t_1, t_2, t_3, t_4, t_5, t_6, t_0}\) are:

\[\varnothing = \{\{1, 4\}, \{3, 2\}, \{0, 5\}, \{6\}\} .\]

Take a representative \(t_i\) from each orbit and see which double cosets \(Nt_0t_2t_0t_i\) belongs
to. We have:

\[ t_0t_2t_6t_1 = x^6yt_4t_2t_4 \implies Nt_0t_2t_6t_1 = Nt_4t_2t_4 \in [02] \]

\[ t_0t_2t_6t_3 = x^2t_0t_3t_6 \implies Nt_0t_2t_6t_3 = Nt_0t_3t_6 \in [036] \]

\[ t_0t_2t_6t_0 = x^2t_1t_6t_1 \implies Nt_0t_2t_6t_0 = Nt_1t_6t_1 \in [020] \]

\[ Nt_0t_2t_6t_6 = Nt_0t_2 \in [02]. \]

**t_0t_3t_1N**

Now \( Nt_0t_3t_1N \) is a new double coset. We determine how many single cosets are in the double coset. Now \( N^{(031)} = N^{031} = \langle e \rangle \). But \( Nt_0t_3t_1 \) is not distinct. Now \( Nt_1t_5t_0 \in [031] \) since \( (1,0)(5,3)(2,6) \in N \) and \( N(t_0t_3t_1)^{(1,0)(5,3)(2,6)} = Nt_1t_5t_0 \). Thus, \( (1,0)(5,3)(2,6) \in N^{(031)} \). We conclude:

\[ N^{(031)} \geq \langle (1,0)(5,3)(2,6) \rangle. \]

Hence \( |N^{(031)}| = 2 \). So the number of single cosets in \( N^{(031)}N \) is \( \frac{|N|}{|N^{(031)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(031)} \) on \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_0\} \) are:

\[ \mathcal{O} = \{\{1,0\}, \{5,3\}, \{2,6\}, \{4\}\}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_3t_1t_i \) belongs to. We have:

\[ Nt_0t_3t_1t_1 = Nt_0t_3 \in [03] \]

\[ t_0t_3t_1t_5 = x^5yt_0t_3t_1 \implies Nt_0t_3t_1t_5 = Nt_0t_3t_1 \in [031] \]

\[ t_0t_3t_1t_2 = x^3t_3t_6t_3 \implies Nt_0t_3t_1t_2 = Nt_3t_6t_3 \in [030] \]

\[ t_0t_3t_1t_4 = x^3yt_6t_2t_6 \implies Nt_0t_3t_1t_4 = Nt_6t_2t_6 \in [030] \]

**Nt_0t_3t_6N**

Now consider \( Nt_0t_3t_6N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(036)} = N^{036} = \langle e \rangle \). Only identity (e) will fix 0 and 3. Hence the number of single cosets living in \( Nt_0t_3t_6N \) is \( \frac{|N|}{|N^{(036)}|} = \frac{14}{1} = 14 \). The orbits of \( N^{(036)} \) on \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_0\} \) are:

\[ \mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0\}\}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_3t_6t_i \) belongs
to. We have:
\[
\begin{align*}
t_0 t_4 t_1 &= x^3 y t_0 t_3 t_6 \implies N t_0 t_4 t_1 = N t_0 t_3 t_6 \in [036] \\
t_0 t_3 t_4 t_2 &= x^5 y t_5 t_0 t_5 \implies N t_0 t_3 t_4 t_2 = N t_3 t_0 t_5 \in [020] \\
t_0 t_3 t_4 t_3 &= (x^2)^{-1} t_0 t_2 t_6 \implies N t_0 t_3 t_4 t_3 = N t_0 t_2 t_6 \in [026] \\
t_0 t_3 t_4 t_4 &= (x^2)^{-1} t_1 t_5 t_2 \implies N t_0 t_3 t_4 t_4 = N t_1 t_5 t_2 \in [036] \\
t_0 t_3 t_4 t_5 &= x^4 t_4 t_0 t_4 \implies N t_0 t_3 t_4 t_5 = N t_4 t_0 t_4 \in [030] \\
n t_0 t_3 t_4 t_6 &= N t_0 t_3 \in [03] \\
t_0 t_2 t_4 t_0 &= x^2 t_2 t_4 t_2 \implies N t_0 t_3 t_0 t_0 = N t_2 t_4 t_2 \in [020].
\end{align*}
\]

\(N t_0 t_3 t_0 N\)

Now \(N t_0 t_3 t_0 N\) is a new double coset. We determine how many single cosets are in
the double coset. Now \(N^{(030)} = N^{030} \ll e \gg\). But \(N t_0 t_3\) is not distinct. Now
\(N t_0 t_3 t_3 \in [030]\) since \((0, 3)(1, 2)(6, 4) \in N\) and \(N(t_0 t_3 t_0)(0, 3)(1, 2)(6, 4) = N t_0 t_3 t_3\). Thus,
\((0, 3)(1, 2)(6, 4) \in N^{(030)}\). We conclude:
\[N^{(030)} \gg (0, 3)(1, 2)(6, 4) \gg .\]

Hence \(|N^{(030)}| = 2\). So the number of single cosets in \(N^{(030)} N\) is
\[|N|/|N^{(030)}| = \frac{14}{2} = 7.\]

The orbits of \(N^{(030)}\) on \(\{t_1, t_2, t_3, t_4, t_5, t_6, t_0\}\) are:
\[\emptyset = \{\{0, 3\}, \{1, 2\}, \{6, 4\}, \{5\}\} .\]

Take a representative \(t_i\) from each orbit and see which double cosets \(N t_0 t_3 t_0 t_i\) belongs
to. We have:
\[
\begin{align*}
n t_0 t_3 t_0 t_0 &= N t_0 t_3 \in [03] \\
t_0 t_3 t_0 t_1 &= x^3 t_3 t_6 t_2 \implies N t_0 t_3 t_0 t_1 = N t_3 t_6 t_2 \in [036] \\
n t_0 t_3 t_0 t_6 &= (x^2)^{-1} t_3 t_0 t_6 \implies N t_0 t_3 t_0 t_6 = N t_3 t_0 t_6 \in [031] \\
n t_0 t_3 t_0 t_5 &= x t_2 t_4 t_0 \implies N t_0 t_3 t_0 t_5 = N t_2 t_4 t_1 \in [031].
\end{align*}
\]

We have completed the double coset enumeration since the right coset is closed
under multiplication, hence, the index of \(N\) in \(G\) is 78 single cosets. We conclude:
\[
G = N \cup N t_0 N \cup N t_0 t_1 N \cup N t_0 t_2 N \cup N t_0 t_3 N \cup N t_0 t_1 t_2 N \cup N t_0 t_1 t_3 N \cup N t_0 t_1 t_0 N \cup N t_0 t_2 t_5 N \cup N t_0 t_3 t_6 N,
\]
where
\[
\begin{align*}
|G| &\leq |N| + \frac{|N|}{N^{(030)}} + \frac{|N|}{N^{(020)}} + \frac{|N|}{N^{(020)}} + \frac{|N|}{N^{(031)}} + \frac{|N|}{N^{(036)}} + \frac{|N|}{N^{(036)}} \\
&\quad + \frac{|N|}{N^{(036)}} \times |N| \\
G &= \frac{2 \times 7 \times D_{14}}{(x t t_x)^2, (t t x t)^9} = 1
\end{align*}
\]
\[ |G| \leq (1 + 7 + 7 + 14 + 14 + 7 + 7 + 14 + 7) \times 14 \]
\[ |G| \leq 78 \times 14 \]
\[ |G| \leq 1092. \]

The Cayley diagram that summarizes the above information is given below:

---

Figure 3.4: Cayley Diagram of $L_2(13)$ over $D_{14}$

### 3.6 Iwasawa’s Lemma to Prove $L_2(13)$ over $D_{14}$ is Simple

We use *Iwasawa’s lemma* and the transitive action of $G$ on the set of single cosets to prove $G \cong L_2(13)$ over $D_{14}$ is a simple group. *Iwasawa’s lemma* has three sufficient conditions that we must satisfied:

1. $G$ acts on $X$ faithfully and primitively
2. $G$ is perfect ($G = G'$)
(3) There exist \( x \in X \) and a normal abelian subgroup \( K \) of \( G^x \) such that the conjugates of \( K \) generate \( G \).

**Proof. 3.6.1** \( G = L_2(13) \) over \( D_{14} \) acts on \( X \) Faithfully

Let \( G \) acts on \( X = \{N, Nt_0N, Nt_0t_2N, Nt_0t_3N, Nt_0t_2t_0N, Nt_0t_2t_6N, Nt_0t_3t_1N, Nt_0t_3t_6N, Nt_0t_3t_0N\} \), where \( X = 78 \). \( G \) acts on \( X \) implies there exist homomorphism

\[
f : G \rightarrow S_{78} \quad (|X| = 78).
\]

By First Isomorphic Theorem we have:

\[
G/\ker f \cong f(G).
\]

If \( \ker f = 1 \) then \( G \cong f(G) \). Only elements of \( N \) fix \( N \) implies \( G^1 = N \). Since \( X \) is transitive \( G - \text{set} \) of degree 78, we have:

\[
|G| = 78 \times |G^1|
\]
\[
= 78 \times |N|
\]
\[
= 78 \times 12
\]
\[
= 1092
\]
\[
\implies |G| = 1092.
\]

From Cayley diagram, \( |G| \leq 1092 \). However, from above \( |G| = 1092 \) implying \( \ker(f) = 1 \). Since \( \ker f = 1 \) then \( G \) acts faithfully on \( X \).

**3.6.2** \( G = L_2(13) \) over \( D_{14} \) acts on \( X \) Primitively

Since \( G = L_2(13) \) is transitive on \( |X| = 78 \), if \( B \) is a nontrivial block then we may assume that \( N \in B \). However, \( |B| \) must divide \( |X| = 78 \). The only nontrivial blocks must be of size 2,3,6,13,26, or 39. Note if \( Bt_0 \in B \) then \( B = X \). So \( B \) is a trivial block. By inspection, we can see from the Cayley diagram that we cannot create a nontrivial block of size 2, 3, 6, 13, 26, or 13. Thus, \( G \) acts primitively on \( X \).

**3.6.3** \( G = L_2(13) \) over \( D_{14} \) is Perfect
Next we want to show that $G = G'$. Since $G = \langle N,t \rangle$, we have that $N \leq G'$.

Now $D_{14} \leq G \implies D_{14}' \leq G'$. The commutators subgroup of $D_{14}$ is:

$$D_{14}' = \langle (1,2,3,4,5,6,0) \rangle = \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'.$$

Now by expanding the relation $[t_0t_0(1,2,3,4,5,6,0)t_0]^3 = 1$, we get:

$$(1,4,0,3,5,2,5)t_2 = t_0t_1$$

$$\implies x^3 = t_0t_1t_2$$

$$\implies x^6 = t_0t_1t_2t_0t_1t_2$$

$$\implies x^{-1} = t_0t_1t_2t_0t_1t_2$$

$$\implies x = t_2t_1t_0t_2t_1t_0.$$  

Also by expanding the relation $[(1,5)(2,4)(6,0)t_1t_0]^5 = 1$, we get:

$$(1,5)(2,4)(6,0) = t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1$$

$$\implies xy = t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1$$

Now we use the above relation to solve for $y$. We multiply by $x^{-1} = t_0t_1t_2t_0t_1t_2$.

$$xy = t_0t_1t_2t_0t_1t_2t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1$$

$$x^{-1}xy = t_0t_1t_2t_0t_1t_2t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1$$

$$\implies y = t_0t_1t_2t_0t_1t_2t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1t_0t_5t_0t_1.$$  

Now $D_{14} \leq G \implies D_{14}' \leq G'$. $D_{14}' = \langle (1,2,3,4,5,6,0) \rangle = \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5, x^6\} \leq G'$. Note $G = \langle x, y, t \rangle = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_0 \rangle$. Our goal is to show that one of the $t'_s \in G'$, then we can conjugate by $< x, y >$ to obtain all of the $t'_s$ in $G'$. Consider, the relation:

$$x^2 = t_0t_2t_0t_0t_1t_0t_1t_6t_1$$

$$= t_0t_2t_0t_0t_0t_1t_0t_1t_6t_1$$

$$= t_0t_2t_0t_0[6,1]$$

$$= t_0t_2t_0t_0t_0t_0[6,1]$$

$$= t_0t_2t_0[0,6][6,1]$$

$$= t_2t_0t_0[0,6][6,1]$$

$$= t_2[2,0][0,6][6,1] \in G'. $$

We see that $t_2 \in G'$. So $G' \geq \langle x, t_2 \rangle = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_0 \rangle = G$. But $G \geq G'$. We conclude that $G = G'$ and $G$ is perfect.
3.6.4 Conjugates of a Normal Abelian $K$


Generate $G = L_2(13)$ over $D_{14}$

Now we require $x \in X$ and a normal abelian subgroup $K$ of $G^x$, the point stabilizer of $x$ in $G$, such that the conjugates of $K$ in $G$ generate $G$.

Now $G^1 = N = D_{14}$ possesses a normal abelian subgroup $K = \langle x \rangle$. We have the relation

$$x^3 = t_0t_1t_2 \in K$$

$$(x^3)^y = (t_0t_1t_2)^y \in K^G$$

$$(x^3)^{-1} = t_0t_0t_5 \in K.$$  \hspace{1cm} (3.1)

Now conjugate the relation $x^3 = t_0t_1t_2$ by $t_0$ yields:

$$x^3t_0 = t_0t_0t_1t_2t_0 \in K$$

$$x^3t_3t_0 = t_1t_2t_0 \in k$$

$$x^3 = t_1t_2t_3 \in K.$$  \hspace{1cm} (3.2)

Now conjugate the relation $x^3 = t_0t_1t_2$ by $t_1$ yields:

$$(x^3)^{t_1} = (t_0t_1t_2)^{t_1} \in K^G$$

$$t_1(x^3)t_1 = t_1t_0t_1t_2t_1 \in K$$

$$x^3t_4t_1 = t_1t_0t_1t_2t_1 \in K$$

$$(x^3t_4t_1)^{t_1t_2t_1} = (t_1t_0t_1t_2t_1)^{t_1t_2t_1} \in K^G$$

$$t_1t_2t_1(x^3t_4t_1)t_1t_2t_1 = t_1t_2t_1(t_1t_0t_1t_2t_1)t_1t_2t_1 \in K$$

$$x^3t_4t_5t_2t_1 = t_1t_2t_0 \in k.$$  \hspace{1cm} (3.3)

We have $t_1t_2t_0 \in K$ so the inverse is in $K$. Thus

$$t_0t_2t_1 \in K.$$  \hspace{1cm} (3.3)

Multiplying (3.3) and (3.2) yields:

$$t_0t_2t_1t_1t_2t_3 \in K$$

$$= t_0t_3 \in K.$$  \hspace{1cm} (3.4)

Now we use the relation $t_0t_3t_0t_1 = x^3t_3t_0t_2$ that we obtained from the double coset
enumeration.

\[ t_0t_3t_0t_1 = x^3t_3t_6t_2 \]

\[ x^3 = t_0t_3t_0t_1t_2t_6t_3 \in K \]

\[ x^3 = t_3t_0t_3t_1t_2t_6t_3 \in K \text{ (since } t_0t_3t_0 = t_3t_0t_3). \tag{3.5} \]

Multiplying (3.5) and (3.4) yields:

\[ x^3t_0t_3 = t_3t_0t_3t_1t_2t_6t_3t_0t_3 \in K \]

\[ (x^3t_0t_3)^t_3t_0t_3 = (t_3t_0t_3t_1t_2t_6t_3t_0t_3)^t_3t_0t_3 \in K^G \]

\[ t_3t_0t_3(x^3t_0t_3)t_3t_0t_3 = t_3t_0t_3(t_3t_0t_3t_1t_2t_6t_3t_0t_3)t_3t_0t_3 \in K \]

\[ x^3t_0t_3t_0t_3 = t_1t_2t_6 \in K. \tag{3.6} \]

Multiplying (3.6) and (3.3) yields:

\[ t_1t_2t_6t_0t_2t_1 \in K \]

\[ (t_1t_2t_6t_0t_2t_1)^t_1t_2 \in K^G \]

\[ t_2t_1(t_1t_2t_6t_0t_2t_1)t_1t_2 \in K \]

\[ t_6t_0 \in K. \tag{3.7} \]

Multiplying (3.1) and (3.7) yields:

\[ t_0t_6t_5t_6t_0 \in K \]

\[ (t_0t_6t_5t_6t_0)^t_6t_0 \in K^G \]

\[ t_6t_0(t_0t_6t_5t_6t_0)t_0t_6 \in K \]

\[ t_5 \in K. \]

Thus \( t_5 \in K \)

\[ \implies t_5^G \in K^G \]

\[ \implies K^G \leq \{ t_5, t_5^2, t_5^3, t_5^4, t_5^5, t_5^6 \} \]

\[ \implies K^G \leq \{ t_5, t_5^2, t_5^3, t_5^4, t_5^5, t_5^6 \} =< t_5, t_6, t_0, t_1, t_2, t_3, t_4 >= G \]

So \( G = K^G \).

Hence, the conjugates of \( K \) generate \( G \). Therefore, by Iwasawa’s lemma, \( G \cong L_2(13) \) is simple.

\[ \square \]
Chapter 4

Double Coset Enumeration over a Maximal Subgroup

In this chapter, we will construct a double coset enumeration over a maximal subgroup and apply Iwasawa’s lemma to prove $G \cong L_2(27)$ is a simple group.

4.1 Construction of $L_2(27)$ over $M = 2 \cdot D_{14}$

Definition 4.1. (Maximal Subgroup). A subgroup $M \neq 1 \leq G$ is a maximal normal subgroup of $G$ if there is no normal subgroup $N$ of $G$ with $M < N < G$.\cite{Rot12}

We start by factoring the progenitor $2^* : D_{14}$ by the relations $(xyt^2t)^3, (xt)^7$ to obtain the homomorphic image:

$$G \cong \frac{2^*: D_{14}}{(xyt^2t)^3, (xt)^7} \cong L_2(27),$$

where $x \sim (0, 1, 2, 3, 4, 5, 6), y \sim (1, 6)(2, 5)(3, 4),$ and $t \sim t_0 \sim t_7$.

Let $\pi = xy = (1, 5)(2, 4)(6, 0)$, then $(xyt^2t)^3 = 1$ can be written as $1 = (\pi t_1t_0)^3$, which yields the following calculation:

$$1 = (\pi t_1t_0)^3$$
$$= \pi^3(t_1t_0)^{\pi^2}(t_1t_0)^{\pi}t_1t_0$$
$$= \pi t_1t_0t_5t_6t_1t_0.$$
Thus we have the relation:
\[ \pi t_1 t_0 t_5 = t_0 t_1 t_6. \]

Now \((xt)^7 = 1\) can be written as \(1 = (xt_0)^7 = x^7 t_0^x t_0^x t_0^x t_0^x t_0^x t_0^x t_0^x t_0^x\). Then \(t_0 t_5 t_4 = t_0 t_1 t_2 t_3\).

Let \(M\) be the maximal subgroup generated by the control group \(N = D_{14}\) and \(t_2 t_4 t_5 t_2 = t^x t^x t^x t^x\). That is,
\[ M = \langle N, t^x t^x t^x t^x \rangle, = 2 \cdot D_{14} \text{ where } |M| = 28. \]

Then \(M\) is the maximal subgroup.

We proceed to do a manual double coset enumeration of \(G\) over \(M\) and \(N\). Denote \([w]\) to be the double coset \(MwN\), where \(w\) is a word in the \(t_i\)'s.

\(MwN\)

We begin with the double coset \(MeN\), denote \([w]\). This double coset contains only one single coset, namely \(M\). The single coset stabilizer of \(M\) is \(N\), which is transitive on \(\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}\) and therefore, has a single orbit,
\[ O = \{\{0, 1, 2, 3, 4, 5, 6\}\}. \]

Take an element from \(O\) say \(t_0\) and multiply the single coset representative \(M\) by it to obtain \(Mt_0\). This is a new double coset \(Mt_0N\), denote it \([0]\).

\(Mt_0N\)

Continuing with the double coset \(Mt_0N\), we find the point stabilizer \(N^0\). This is
\[ N^0 = \langle (1, 6)(2, 5)(3, 4) \rangle. \]

The coset stabiliser:
\[ N^{(0)} \geq \langle (1, 6)(2, 5)(3, 4) \rangle. \]

Since \(|N^{(0)}| = 2\), the number of single cosets in \([0]\) is \(\frac{|N|}{|N^{(0)}|} = \frac{14}{2} = 7\). The orbits of \(N^{(0)}\) on \(\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}\) are:
\[ O = \{\{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}. \]

Take an element from each orbit and multiply on the right by the single coset represen-
tative $Mt_0$ of the double coset $Mt_0N$. We have:

$$Mt_0t_0 = M \in [\ast],$$
$$Mt_0t_1 \in [01],$$
$$Mt_0t_2 \in [02],$$
$$Mt_0t_3 \in [03].$$

The new double cosets have single coset representatives $Mt_0t_1$, $Mt_0t_2$, and $Mt_0t_3$, we represent them as [01], [02], and [03], respectively.

$Mt_0t_1N$

Continuing with the double coset $Mt_0t_1N$, we find the coset stabilizer $N^{(01)} = N^{01} = \langle e \rangle$. Only $e$ will fix 0 and 1. Hence the number of single cosets in [01] is

$$\frac{|N|}{|N^{(01)}|} = \frac{14}{1} = 14.$$ 

The orbits of $N^{(01)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$O = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$ 

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_1$ of the double coset $Mt_0t_1N$. We have:

$$Mt_0t_1t_1 = Mt_0 \in [0],$$
$$Mt_0t_1t_0 \in [010],$$
$$Mt_0t_1t_2 \in [012],$$
$$Mt_0t_1t_3 \in [013],$$
$$Mt_0t_1t_4 \in [014],$$
$$Mt_0t_1t_5 \in [015],$$
$$t_0t_1t_6 = xyt_1t_0t_5$$

$$\implies Mt_0t_1t_6 = Mt_1t_0t_5 \in [013] = \{N(t_0t_1t_3)^n | n \in N\}.$$ 

The new double cosets have single coset representatives $Mt_0t_1t_0$, $Mt_0t_1t_2$, $Mt_0t_1t_3$, $Mt_0t_1t_4$, and $Mt_0t_1t_5$, we represent them as [010], [012], [013], [014], and [015], respectively.

$Mt_0t_2N$

Continuing with the double coset $Mt_0t_2N$, we find the coset stabilizer $N^{(02)} = N^{02} = \langle e \rangle$. Only $e$ will fix 0 and 2. Hence the number of single cosets in [02] is

$$\frac{|N|}{|N^{(02)}|} = \frac{14}{1} = 14.$$ 

The orbits of $N^{(02)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:
Continuing with the double coset \( \langle \varepsilon \rangle \). Only \( \varepsilon \) will fix 0 and 2. Hence the number of single cosets in \([02]\) is
\[
\frac{|N|}{|N^{(02)}|} = \frac{14}{1} = 14.
\]

14. The orbits of \( N^{(02)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\]

We now take an element from each orbit and multiply on the right by the single coset representative \( Mt_0t_2 \) of the double coset \( Mt_0t_2N \). We have:
\[
Mt_0t_2t_2 = Mt_0 \in [0],
\]
\[
t_0t_2t_1 = x^2t_2t_0t_1t_2t_0t_2t_0 \implies Mt_0t_2t_1 = Mt_2t_0 \in [02] = \{N(t_0t_2)^n | n \in N\},
\]
\[
t_0t_2t_3 = t_0t_2t_3t_2t_0t_2 \implies Mt_0t_2t_3 = Mt_0t_2 \in [02],
\]
\[
Mt_0t_2t_4 \in [024],
\]
\[
Mt_0t_2t_5 \in [025],
\]
\[
Mt_0t_2t_6 \in [026],
\]
\[
Mt_0t_2t_0 \in [020].
\]

The new double cosets have single coset representatives \( Mt_0t_2t_4, Mt_0t_2t_5, Mt_0t_2t_6, \) and \( Mt_0t_2t_0 \), we represent them as \([024], [025], [026], \) and \([020], \) respectively.

**\( Mt_0t_3N \)**

Continuing with the double coset \( Mt_0t_3N \), we find the coset stabilizer \( N^{(03)} = N^{03} = \langle \varepsilon \rangle \). Only \( \varepsilon \) will fix 0 and 3. Hence the number of single cosets in \([03]\) is
\[
\frac{|N|}{|N^{(03)}|} = \frac{14}{1} = 14.
\]

14. The orbits of \( N^{(03)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\]

We now take an element from each orbit and multiply on the right by the single coset representative \( Mt_0t_3 \) of the double coset \( Mt_0t_3N \). We have:
\[
Mt_0t_3t_3 = Mt_0 \in [0],
\]
\[
t_0t_3t_1 = t_0t_3t_2t_0t_6t_0t_6 \implies Mt_0t_3t_1 = Mt_6t_0t_6 \in [010] = \{N(t_0t_1t_0)^n | n \in N\},
\]
\[
Mt_0t_3t_2 \in [032],
\]
\[
Mt_0t_3t_4 \in [034],
\]
\[
t_0t_3t_5 = t_3t_5t_1 \implies Mt_0t_3t_5 = Mt_3t_5t_1 \in [025] = \{N(t_0t_2t_5)^n | n \in N\},
\]
Continuing with the double coset representative $M_{t_0t_3t_6} = yx^{-2}t_5t_0t_6t_5t_0t_3t_6$ implies $M_{t_0t_3t_6} = M_{t_0t_3t_6} \in [013]$

(since $\{N(t_0t_1t_3)^n| n \in N \}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$),

$M_{t_0t_3t_0} \in [030]$. The new double cosets have single coset representatives $M_{t_0t_3t_2}$, $M_{t_0t_3t_4}$, and $M_{t_0t_3t_0}$, we represent them as $[032]$, $[034]$, and $[030]$, respectively.

$M_{t_0t_1t_0}N$

Continuing with the double coset $M_{t_0t_1t_0}N$, we find the coset stabilizer $N^{(010)} = N^{010} = \langle e \rangle$. Only $e$ will fix 0 and 1. Hence the number of single cosets in $[010]$ is

$$\frac{|N|}{|N^{(010)}|} = \frac{14}{1} = 14.$$ The orbits of $N^{(010)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$. We now take an element from each orbit and multiply on the right by the single coset representative $M_{t_0t_1t_0}$ of the double coset $M_{t_0t_1t_0}N$. We have:

$M_{t_0t_1t_0} = M_{t_0t_1} \in [01]$,

$t_{01t_0t_1} = yx_{t_4t_2t_4} \implies M_{t_0t_1}t_{01t_0t_1} = M_{t_0t_1}t_{42t_4} \in [020] = \{(N_{t_0t_2t_0})^n| n \in N\}$,

$t_{01t_0t_2} = x_{t_0t_5t_4t_5t_0t_1t_4} \implies M_{t_0t_1}t_{01t_0t_2} = M_{t_1t_4} \in [03]$ (since $\{N(t_0t_3)^n| n \in N \}$ and $x_{t_0t_5t_4t_5t_0} \in M$),

$t_{01t_0t_3} = x_{t_6t_3t_4t_5t_2t_4t_6} \implies M_{t_0t_1}t_{01t_0t_3} = M_{t_0t_1}t_{634524t_6} \in [0324200]$ (since $\{N(t_0t_3t_2t_1t_4t_2t_0)^n| n \in N\}$),

$t_{01t_0t_4} = y_{x_{t_0t_5t_4t_5t_0t_1t_0t_1}} \implies M_{t_0t_1}t_{01t_0t_4} = M_{t_1t_0t_1} \in [010]$ (since $\{N(t_0t_1t_0)^n| n \in N \}$ and $y_{x_{t_0t_5t_4t_5}} \in M$),

$t_{01t_0t_5} = x_{t_3t_0t_3t_1t_0} \implies M_{t_0t_1}t_{01t_0t_5} = M_{t_3t_0t_3t_1t_0} \in [03023]$ = $\{N(t_0t_3t_2t_3)^n| n \in N\}$,

$t_{01t_0t_6} = x_{t_1t_4t_5t_1t_0} \implies M_{t_0t_1}t_{01t_0t_6} = M_{t_1t_4t_5t_1t_0} \in [03406]$ = $\{N(t_0t_3t_4t_0)^n| n \in N\}$. $M_{t_0t_1t_2}N$

Continuing with the double coset $M_{t_0t_1t_2}N$, we find the coset stabilizer $N^{(012)} = N^{012} =$
\[(e)\). Only \(e\) will fix 0, 1, and 2. Hence the number of single cosets in \([012]\) is \(\frac{|N|}{|N^{(012)}|} = \frac{14}{1} = 14\). The orbits of \(N^{(012)}\) on \(\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}\) are:
\[\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.\]

We now take an element from each orbit and multiply on the right by the single coset representative \(Mt_0t_1t_2\) of the double coset \(Mt_0t_1t_2N\). We have:
\[
Mt_0t_1t_2 = Mt_0t_1 \in [01],
\]
\[
t_0t_1t_2t_1 = xt_0t_5t_4t_5t_0t_6t_2t_1t_0t_3t_1t_6t_5
\Rightarrow Mt_0t_1t_2t_1 = Mt_6t_2t_1t_0t_3t_1t_6t_5 \in [03214206]
\]
(since \(N(t_0t_3t_2t_1t_4t_2t_0)^n|n \in N\) and \(xt_0t_5t_4t_5t_0 \in M\),
\[
t_0t_1t_2t_3 = t_6t_5t_4 \Rightarrow Mt_0t_1t_2t_3 = Mt_6t_5t_4 \in [012] = \{N(t_0t_1t_2)^n|n \in N\},
\]
\[
t_0t_1t_2t_4 = xt_0t_5t_4t_5t_0t_3t_6t_0t_3 \Rightarrow Mt_0t_1t_2t_4 = Mt_3t_6t_0t_3 \in [0340]
\]
(since \(N(t_0t_3t_4t_0)^n|n \in N\) and \(xt_0t_5t_4t_5t_0 \in M\),
\[
t_0t_1t_2t_5 = xt_1t_3t_4t_3t_1t_0t_5t_0 \Rightarrow Mt_0t_1t_2t_5 = Mt_0t_5t_0 \in [020]
\]
(since \(N(t_0t_2t_0)^n|n \in N\) and \(xt_1t_3t_4t_1 \in M\),
\[
t_0t_1t_2t_6 = xt_0t_2t_3t_2t_0t_4t_3t_0 \Rightarrow Mt_0t_1t_2t_6 = Mt_4t_3t_0 \in [014]
\]
(since \(N(t_0t_1t_4)^n|n \in N\) and \(xt_0t_2t_3t_2t_0 \in M\),
\[
t_0t_1t_2t_0 = xyt_4t_1t_4t_2t_1 \Rightarrow Mt_0t_1t_2t_0 = Mt_4t_1t_4t_2t_1 \in [03023]
\]
\[= \{N(t_0t_3t_0t_2)^n|n \in N\}.
\]

\[Mt_0t_1t_3N\]

Continuing with the double coset \(Mt_0t_1t_3N\), we find the coset stabiliser \(N^{(013)} = \langle e \rangle\). Only \(e\) will fix 0, 1, and 3. Hence the number of single cosets in \([013]\) is \(\frac{|N|}{|N^{(013)}|} = \frac{14}{1} = 14\). The orbits of \(N^{(013)}\) on \(\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}\) are:
\[\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.\]

We now take an element from each orbit and multiply on the right by the single coset representative \(Mt_0t_1t_3\) of the double coset \(Mt_0t_1t_3N\). We have:
\( Mt_0t_1t_3t_5 = Mt_0t_1 \in [01], \)
\( t_0t_1t_3t_1 = x_0t_2t_0t_1t_0 \implies Mt_0t_1t_3t_1 = Mt_0t_2t_0t_1t_0 \in [03021] \)
\( = \{ N(t_0t_2t_0t_1)^n | n \in N \}, \)
\( t_0t_1t_3t_2 = x_2^3t_1t_0 \implies Mt_0t_1t_3t_2 = Mt_1t_0 \in [01] = \{ N(t_0t_1)^n | n \in N \}, \)
\( t_0t_1t_3t_4 = x_0t_2t_3t_2t_0t_4t_3t_6t_0 \implies Mt_0t_1t_3t_4 = Mt_0t_4t_3t_6t_0 \in [03410] \)
\( (\text{since } \{ N(t_0t_3t_4t_5)^n | n \in N \} \text{ and } x_0t_2t_3t_2t_0 \in M), \)
\( t_0t_1t_3t_5 = t_2t_3t_2t_0t_1t_3 \implies Mt_0t_1t_3t_5 = Mt_2t_3t_2t_0t_1t_3 \in [030216] \)
\( = \{ N(t_0t_3t_2t_4)^n | n \in N \}, \)
\( t_0t_1t_3t_6 = x^3t_4t_1t_2t_1 \implies Mt_0t_1t_3t_6 = Mt_4t_4t_2t_1 \in [0323] \)
\( = \{ N(t_0t_3t_2t_3)^n | n \in N \}, \)
\( t_0t_1t_3t_0 = x_0x_2t_5t_7t_6t_5t_3 \implies Mt_0t_1t_3t_0 = Mt_6t_3 \in [03] \)
\( (\text{since } \{ N(t_0t_3)^n | n \in N \} \text{ and } x_0x_2t_5t_7t_6t_5 \in M). \)

\( Mt_0t_1t_4N \)

Continuing with the double coset \( Mt_0t_1t_4N \), we find the coset stabiliser \( N^{(014)} = N^{014} = \langle e \rangle \). Only \( e \) will fix 0, 1, and 4. Hence the number of single cosets in [014] is \( \frac{|N|}{|N^{(014)}|} = \frac{14}{2} = 14 \). The orbits of \( N^{(014)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:

\( O = \{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \}. \)

We now take an element from each orbit and multiply on the right by the single coset representative \( Mt_0t_1t_4 \) of the double coset \( Mt_0t_1t_4N \). We have:

\( Mt_0t_1t_4 = Mt_0t_1 \in [01], \)
\( t_0t_1t_4t_1 = x^{-1}t_5t_0t_4t_5t_0t_2t_6t_0t_1t_5 \implies Mt_0t_1t_4t_1 = Mt_2t_6t_0t_1t_5 \in [03214], \)
\( (\text{since } \{ N(t_0t_3t_2t_4)^n | n \in N \} \text{ and } x^{-1}t_5t_0t_6t_0t_1t_5 \in M) \)
\( t_0t_1t_4t_2 = x_0xt_0t_1t_4 \implies Mt_0t_1t_4t_2 = Mt_1t_4 \in [014] \)
\( t_0t_1t_4t_3 = t_0t_4t_5t_0t_6t_2t_5 \implies Mt_0t_1t_4t_3 = Mt_0t_6t_2t_5 \in [0152], \)
\( (\text{since } \{ N(t_0t_1t_5)^n | n \in N \} \text{ and } t_0t_5t_4t_5t_0 \in M). \)
\[ t_{0t_1t_4t_5} = x^2t_{0t_2t_3t_2t_0t_4t_3t_2} \implies Mt_{0t_1t_4t_5} = Mt_{4t_3t_2} \in [012] \]

(since \( N(t_{0t_1t_2})^n | n \in N \) and \( x^2t_{0t_2t_3t_2t_0} \in M \),
\[ t_{0t_1t_4t_6} = xt_{1t_3t_4t_3t_0t_4t_3t_0}t_1 \implies Mt_{0t_1t_4t_6} = Mt_{0t_4t_3t_0t_1} \in [03406] \]

(since \( N(t_{0t_3t_4t_0t_6})^n | n \in N \) and \( xt_{1t_3t_4t_3t_1} \in M \),
\[ t_{0t_1t_4t_0} = x^2yt_{3t_0t_6t_2} \implies Mt_{0t_1t_4t_0} = Mt_{3t_0t_6t_2} \in [0341] \]

\[ = \{ N(t_{0t_3t_4t_1})^n | n \in N \} . \]

**\( Mt_{0t_1t_5}N \)**

Continuing with the double coset \( Mt_{0t_1t_5}N \) we find the single coset stabilizer is trivial. However, the relation
\[ t_{0t_1t_5} = xy^{-2}t_{5t_0t_6t_5t_0t_6t_5t_1} \implies Mt_{0t_1t_5} = Mt_{0t_5t_1} \text{ since } xy^{-2}t_{5t_0t_6t_5t_0} \in M \]

Then \( M(t_{0t_1t_5})^{(0,6)(1,5)(2,4)} = Mt_{6t_5t_1} \). But \( Mt_{0t_5t_1} = Mt_{0t_1t_5} \implies (0,6)(1,5)(2,4) \in N^{(015)} \) since \( M(t_{0t_1t_5})^{(0,6)(1,5)(2,4)} = Mt_{0t_5t_1} \)
\[ \implies N^{(015)} \geq \langle (0,6)(1,5)(2,4) \rangle . \]

Since \( |N^{(015)}| = 2 \), the number of single cosets in \([015]\) is \( \frac{|N|}{|N^{(015)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(015)} \) on \( \{ t_0, t_1, t_2, t_3, t_4, t_5, t_6 \} \) are:
\[ \mathcal{O} = \{ \{3\}, \{0, 6\}, \{1, 5\}, \{2, 4\} \} . \]

Take an element from each orbit and multiply on the right by the single coset representative \( Mt_{0t_1t_5} \) of the double coset \( Mt_{0t_1t_5}N \). We have:
\[ Mt_{0t_1t_5t_5} = Mt_{0t_1t_5} \in [01] , \]
\[ t_{0t_1t_5t_3} = x^3t_{0t_5t_1} \implies Mt_{0t_1t_5t_3} = Mt_{0t_5t_1} \in [015] = \{ N(t_{0t_5})^n | n \in N \} , \]
\[ t_{0t_1t_5t_0} = xt_{1t_3t_4t_3t_0t_5t_3} \implies Mt_{0t_1t_5t_0} = Mt_{0t_5t_3} \in [024] \]

(since \( N(t_{0t_2t_4})^n | n \in N \) and \( xt_{1t_3t_4t_3t_1} \in M \),
\[ Mt_{0t_1t_5t_2} \in [0152] . \]

The new double coset is \( Mt_{0t_1t_5t_2}N \), which we represent by \([0152]\), respectively.

**\( Mt_{0t_1t_5t_2}N \)**

Now with the double coset \( Mt_{0t_1t_5t_2}N \), we find the coset stabilizer \( N^{(0152)} = N^{0152} = \)
Continuing with the double coset \( \langle e \rangle \). Only \( e \) will fix 0, 1, 5, and 2. Hence the number of single cosets in \([015] \) is \( \frac{|N|}{|N(015)|} = \frac{14}{1} = 14 \). The orbits of \( N(015) \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\]

We now take an element from each orbit and multiply on the right by the single coset representative \( M_{t_0}t_1t_5t_2 \) of the double coset \( M_{t_0}t_1t_5t_2N \). We have:
\[
M_{t_0}t_1t_5t_2t_2 = M_{t_0}t_1t_5 \in [015],
\]
\[
t_0t_1t_5t_2t_1 = x_0t_2t_3t_2t_0t_4t_5t_4 \implies M_{t_0}t_1t_5t_2t_1 = M_{t_0}t_4t_5t_4 \in [0323]
\]
(since \( \{N(t_0t_3t_2)\} \subseteq N \) and \( x_0t_2t_3t_2t_0 \in M \),
\[
t_0t_1t_5t_2t_3 = x_2t_2t_0t_1t_2t_0t_4t_1 \implies M_{t_0}t_1t_5t_2t_3 = M_{t_2}t_4t_1 \in [026]
\]
(since \( \{N(t_0t_2)\} \subseteq N \) and \( x_2t_2t_0t_1t_2t_0 \in M \),
\[
t_0t_1t_5t_2t_4 = t_0t_2t_3t_2t_0t_6t_3 \implies M_{t_0}t_1t_5t_2t_4 = M_{t_0}t_6t_3 \in [014]
\]
(since \( \{N(t_0t_1t_4)\} \subseteq N \) and \( t_0t_2t_3t_2t_0 \in M \),
\[
t_0t_1t_5t_2t_5 = x_2t_6t_3t_4t_5t_2t_6 \implies M_{t_0}t_1t_5t_2t_5 = M_{t_0}t_3t_4t_5t_2t_4t_6 \in [03214206]
\]
\[
= \{N(t_0t_3t_2t_1t_4t_2t_0t_6)\} \subseteq N,
\]
\[
t_0t_1t_5t_2t_6 = x_0t_5t_4t_5t_0t_1t_2t_5 \implies M_{t_0}t_1t_5t_2t_6 = M_{t_0}t_1t_2t_5 \in [0152]
\]
(since \( \{N(t_0t_1t_5)\} \subseteq N \) and \( x_0t_5t_4t_5t_0 \in M \),
\[
t_0t_1t_5t_2t_0 = t_0t_2t_3t_2t_0t_5t_1t_2t_5 \implies M_{t_0}t_1t_5t_2t_0 = M_{t_5}t_1t_2t_6 \in [03410]
\]
(since \( \{N(t_0t_3t_4t_1)\} \subseteq N \) and \( t_0t_2t_3t_2 \in M \).

\( M_{t_0}t_2t_4N \)

Continuing with the double coset \( M_{t_0}t_2t_4N \), we find the coset stabilizer \( N(024) = N^{024} = \langle e \rangle \). Only \( e \) will fix 0, 2, and 4. Hence the number of single cosets in \([024] \) is \( \frac{|N|}{|N(024)|} = \frac{14}{1} = 14 \). The orbits of \( N(024) \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\]

We now take an element from each orbit and multiply on the right by the single coset representative \( M_{t_0}t_2t_4 \) of the double coset \( M_{t_0}t_2t_4N \). We have:
\[
M_{t_0}t_2t_4 = M_{t_0}t_2 \in [02],
\]
Continuing with the double coset representative, we now take an element from each orbit and multiply on the right by the single coset $14\langle e \rangle$. Only $e$.

\[Mt_0t_2t_1 = x^{-2}t_3t_0t_3t_4 \implies Mt_0t_2t_1 = Mt_3t_0t_3t_4 \in [0301] = \{N(t_0t_3t_0t_1)^n | n \in N\},\]

\[t_0t_2t_4t_1 = x_1t_6t_4t_6 \implies Mt_0t_2t_4t_1 = Mt_6t_4t_6 \in [020] = \{N(t_0t_2t_0)^n | n \in N\},\]

\[t_0t_2t_4t_2 = x_2t_4t_6 \implies Mt_0t_2t_4t_2 = Mt_6t_4t_6 \in [020] = \{N(t_0t_2t_0)^n | n \in N\},\]

\[t_0t_2t_4t_3 = yxt_0t_5t_4t_5t_0t_5t_1t_2t_6t_5 \implies Mt_0t_2t_4t_3 = Mt_5t_1t_2t_6t_5 \in [03410] = \{N(t_0t_3t_4t_0t_6)^n | n \in N\} \text{ and } yxt_0t_5t_4t_5t_0 \in M,\]

\[t_0t_2t_4t_5 = x_1t_0t_4t_5t_0t_1t_4t_5t_1t_0 \implies Mt_0t_2t_4t_5 = Mt_1t_4t_5t_1t_0 \in [03406] = \{N(t_0t_3t_4t_0t_6)^n | n \in N\} \text{ and } x_0t_5t_4t_5t_0 \in M,\]

\[t_0t_2t_4t_6 = x_1t_0t_4t_1t_4 \implies Mt_0t_2t_4t_6 = Mt_6t_1t_4 \in [025] = \{N(t_0t_2t_5)^n | n \in N\},\]

\[t_0t_2t_4t_0 = x_0t_0t_5t_1t_0t_0t_6t_2 \implies Mt_0t_2t_4t_0 = Mt_0t_6t_2 \in [015] = \{N(t_0t_1t_5)^n | n \in N\} \text{ and } x_0t_5t_4t_5t_0 \in M.\]

\[Mt_0t_2t_5N\]

Continuing with the double coset $Mt_0t_2t_5N$, we find the coset stabilizer $N^{(025)} = N^{025} = \langle e \rangle$. Only $e$ will fix 0, 2, and 5. Hence the number of single cosets in $[025]$ is $\frac{|N|}{|N^{(025)}|} = \frac{14}{1} = 14$. The orbits of $N^{(025)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

\[\mathcal{O} = \{|0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_5$ of the double coset $Mt_0t_2t_5N$. We have:

\[Mt_0t_2t_5t_5 = Mt_0t_2 \in [02],\]

\[t_0t_2t_5t_1 = x_4t_2t_0t_1t_2t_0t_3t_0t_3t_4 \implies Mt_0t_2t_5t_1 = Mt_3t_0t_3t_4 \in [03406] = \{N(t_0t_3t_4t_0t_6)^n | n \in N\} \text{ and } x_4t_2t_0t_1t_2t_0 \in M,\]

\[t_0t_2t_5t_2 = t_4t_0 \implies Mt_0t_2t_5t_2 = Mt_4t_0 \in [03] = \{N(t_0t_3)^n | n \in N\},\]

\[t_0t_2t_5t_3 = x_6t_2t_1t_0t_3t_1t_6 \implies Mt_0t_2t_5t_3 = Mt_6t_2t_1t_0t_3t_1t_6 \in [0321420] = \{N(t_0t_3t_2t_1t_4t_2t_0)^n | n \in N\},\]
We now take an element from each orbit and multiply on the right by the single coset \( M_{t_0t_2t_5t_4} \) = \( M_{t_0t_2t_5} \) = \( M_{t_0t_2} \in [032] \),

(since \( \{N(t_0t_3t_2)^n|n \in N\} \) and \( y_{t_0t_5t_4t_5t_0} \in M \),

\[
t_0t_2t_5t_6 = x^2t_3t_6t_3t_4t_2 \implies M_{t_0t_2t_5t_6} = M_{t_3t_6t_3t_4t_2} \in [030216]
\]

\[
= \{N(t_0t_3t_0t_2t_1t_6)^n|n \in N\},
\]

\[
t_0t_2t_5t_0 = x^{-1}t_1t_3t_5 \implies M_{t_0t_2t_5t_0} = M_{t_1t_3t_5} \in [024]
\]

\[
= \{N(t_0t_2t_4)^n|n \in N\}.
\]

\[
M_{t_0t_2t_6t_5}
\]

Continuing with the double coset \( M_{t_0t_2t_6N} \), we find the coset stabilizer \( N^{(026)} = N^{026} = \langle e \rangle \). Only \( e \) will fix 0, 2, and 6. Hence the number of single cosets in \([026] \) is

\[
\frac{14}{1} = 14.
\]

The orbits of \( N^{(026)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:

\[
\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.
\]

We now take an element from each orbit and multiply on the right by the single coset representative \( M_{t_0t_2t_6} \) of the double coset \( M_{t_0t_2t_6N} \). We have:

\[
M_{t_0t_2t_6t_6} = M_{t_0t_2} \in [02],
\]

\[
t_0t_2t_6t_1 = x^{-2}t_5t_0t_6t_5t_0t_5t_6t_3t_0 \implies M_{t_0t_2t_6t_1} = M_{t_5t_6t_3t_0} \in [0152]
\]

(since \( \{N(t_0t_1t_2)^n|n \in N\} \) and \( x^{-2}t_5t_0t_6t_5t_0 \in M \),

\[
t_0t_2t_6t_2 = t_2t_5t_2t_4 \implies M_{t_0t_2t_6t_2} = M_{t_2t_5t_2t_4} \in [0302]
\]

\[
= \{N(t_0t_3t_0t_2)^n|n \in N\},
\]

\[
t_0t_2t_6t_3 = x^2y_0t_3t_6t_5 \implies M_{t_0t_2t_6t_3} = M_{t_6t_3t_6t_5} \in [0301]
\]

\[
= \{N(t_0t_3t_0t_1)^n|n \in N\},
\]

\[
t_0t_2t_6t_4 = x^{-3}t_3t_6t_5t_6 \implies M_{t_0t_2t_6t_4} = M_{t_3t_6t_5t_6} \in [0323]
\]

\[
= \{N(t_0t_3t_2t_3)^n|n \in N\},
\]

\[
t_0t_2t_6t_5 = t_0t_5t_4t_5t_0t_4t_1t_0 \implies M_{t_0t_2t_6t_5} = M_{t_4t_1t_0} \in [034]
\]

(since \( \{N(t_0t_3t_4)^n|n \in N\} \) and \( t_0t_5t_4t_5t_0 \in M \),
Continuing with the double coset $Mt_0t_2t_0N$.

Continuing with the double coset $Mt_0t_2t_0$, we find the coset stabilizer $N^{(020)} = N^{020} = \langle e \rangle$. Only $e$ will fix 0, and 2. Hence the number of single cosets in $[020]$ is $\frac{|N|}{|N^{(020)}|} = \frac{14}{1} = 14$. The orbits of $N^{(020)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$.

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_2t_0$ of the double coset $Mt_0t_2t_0N$. We have:

$Mt_0t_2t_0t_0 = Mt_0t_2 \in [02]$, 
$t_0t_2t_0t_1 = yx^{-2}t_5t_0t_5t_0t_0t_0t_2 = Mt_0t_2t_0t_1 = Mt_0t_2t_0 \in [02]$ 
(since $\{N(t_0t_2t_0)^n|n \in N\}$ and $yx^{-2}t_5t_0t_5t_0t_0t_2 \in M$), 
$t_0t_2t_0t_2 = xt_0t_5t_4t_5t_0t_0t_5 \implies Mt_0t_2t_2 = Mt_0t_2t_0t_2 \in [012]$ 
(since $\{N(t_0t_1t_2)^n|n \in N\}$ and $xt_0t_5t_4t_5t_0 \in M$), 
$t_0t_2t_0t_3 = yt_4t_3t_4 \implies Mt_0t_2t_0t_3 = Mt_4t_3t_4 \in [010]$ 
$= \{N(t_0t_1t_0)^n|n \in N\}$, 
$t_0t_2t_0t_4 = xt_6t_4t_2 \implies Mt_0t_2t_0t_4 = Mt_6t_4t_2 \in [024]$ 
$= \{N(t_0t_2t_4)^n|n \in N\}$, 
$t_0t_2t_0t_5 = x^{-1}t_3t_1t_3 \implies Mt_0t_2t_0t_5 = Mt_3t_1t_3 \in [02]$ 
$= \{N(t_0t_2t_0)^n|n \in N\}$, 
$t_0t_2t_0t_6 = x^3t_6t_0t_1 \implies Mt_0t_2t_0t_6 = Mt_6t_0t_1t_0 \in [03021]$ 
$= \{N(t_0t_3t_0t_2t_1)^n|n \in N\}$.

$Mt_0t_3t_2t_0N$

Continuing with the double coset $Mt_0t_3t_2N$ we find the single coset stabilizer is trivial. However, the relation
Since \( N \) orbits of \( N \) take an element from each orbit and multiply on the right by the single coset representative \( M_t \). Then \( M(t_0t_3t_2)(0,2)(3,6)(4,5) = M_{t_2}t_6 \). But \( M_{t_2}t_6 = M_{t_0}t_3t_2 \implies (0, 2)(3, 6)(4, 5) \in N^{(032)} \) since \( M(t_0t_3t_2)(0,2)(3,6)(4,5) = M_{t_2}t_6 \)
\[
\implies N^{(032)} \geq \langle (0, 2)(3, 6)(4, 5) \rangle.
\]
Since \( |N^{(032)}| = 2 \), the number of single cosets in \([032]\) is \( \frac{|N|}{|N^{(032)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(032)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{1\}, \{0, 2\}, \{3, 6\}, \{4, 5\}\}.
\]
Take an element from each orbit and multiply on the right by the single coset representative \( M_{t_0}t_3t_2 \) of the double coset \( M_{t_0}t_3t_2N \). We have:
\[
M_{t_0}t_3t_2t_2 = M_{t_0}t_3 \in [03],
\]
\[
M_{t_0}t_3t_2t_1 \in [0321],
\]
\[
M_{t_0}t_3t_2t_3 \in [0323],
\]
\[
t_0t_3t_2t_4 = x^3t_2t_0t_1t_2t_0t_1t_2t_5 \implies M_{t_0}t_3t_2t_4 = M_{t_0}t_2t_5 \in [025]
\]
(since \( \{N(t_0t_2t_5)^n|n \in N\} \) and \( x^3t_2t_0t_1t_2t_0 \in M \)).

The new double cosets have single coset representatives \( M_{t_0}t_3t_2t_1N \) and \( M_{t_0}t_3t_2t_3N \); we represent them as \([0321]\) and \([0323]\), respectively.

\[
M_{t_0}t_3t_2t_1N
\]
Continuing with the double coset \( M_{t_0}t_3t_2t_1N \) we find the single coset stabilizer is trivial. However, the relation
\[
t_0t_3t_2t_1 = x^{-2}t_5t_0t_6t_5t_0t_2t_0t_1
\]
\[
\implies M_{t_0}t_3t_2t_1 = M_{t_2}t_6t_0t_1 \text{ since } x^{-2}t_5t_0t_6t_5t_0 \in M.
\]
Then \( M(t_0t_3t_2t_1)(0,2)(3,6)(4,5) = M_{t_2}t_6t_0t_1 \).
But \( M_{t_2}t_6t_0t_1 = M_{t_0}t_3t_2t_1 \implies (0, 2)(3, 6)(4, 5) \in N^{(0321)} \)
since \( M(t_0t_3t_2t_1)(0,2)(3,6)(4,5) = M_{t_2}t_6t_0t_1 \)
\[
\implies N^{(0321)} \geq \langle (0, 2)(3, 6)(4, 5) \rangle.
\]
Since \( |N^{(0321)}| = 2 \), the number of single cosets in \([0321]\) is \( \frac{|N|}{|N^{(0321)}|} = \frac{14}{2} = 7 \). The orbits of \( N^{(0321)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:
\[
\mathcal{O} = \{\{1\}, \{0, 2\}, \{3, 6\}, \{4, 5\}\}.
\]
Take an element from each orbit and multiply on the right by the single coset representati-
tative $Mt_0t_3t_2t_1$ of the double coset $Mt_0t_3t_2t_1N$. We have:

$$Mt_0t_3t_2t_1t_1 = Mt_0t_3t_2 \in [032],$$

$$t_0t_3t_2t_1t_3 = x^3yt_1t_4t_1t_2 \implies Mt_0t_3t_2t_1t_3 = Mt_1t_4t_1t_2 \in [0301]$$

$$= \{N(t_0t_3t_0t_1)^n | n \in N\},$$

$$Mt_0t_3t_2t_1t_4 \in [03214],$$

$$t_0t_3t_2t_1t_0 = t_0t_4t_5t_6 \implies Mt_0t_3t_2t_1t_0 = Mt_0t_4t_5t_6 \in [0321]$$

$$= \{N(t_0t_3t_2t_1)^n | n \in N\},$$

The new double coset is $Mt_0t_3t_2t_1t_4N$, which we represent by $[03214]$, respectively.

**$Mt_0t_3t_2t_3N$**

Continuing with the double coset $Mt_0t_3t_2t_3N$, we find the coset stabilizer $N^{(0323)} = N^{0323} = \langle e \rangle$. Only $e$ will fix 0, 2, and 3. Hence the number of single cosets in $[0323]$ is

$$\frac{|N|}{|N^{(0323)}|} = \frac{14}{1} = 14.$$ 

The orbits of $N^{(0323)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$O = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_3$ of the double coset $Mt_0t_3t_2t_3N$. We have:

$$Mt_0t_3t_2t_3 = Mt_0t_3t_2 \in [032],$$

$$t_0t_3t_2t_3t_1 = x^3t_4t_6t_3 \implies Mt_0t_3t_2t_3t_1 = Mt_4t_6t_3 \in [026]$$

$$= \{N(t_0t_3t_0)^n | n \in N\},$$

$$t_0t_3t_2t_3t_2 = xyt_4t_0t_4t_6t_5 \implies Mt_0t_3t_2t_3t_2 = Mt_4t_0t_4t_6t_5 \in [03021]$$

$$= \{N(t_0t_3t_0t_2)^n | n \in N\},$$

$$t_0t_3t_2t_3t_4 = x^2t_2t_0t_1t_2t_0t_0t_3t_0t_1 \implies Mt_0t_3t_2t_3t_4 = Mt_0t_3t_0t_1 \in [0301]$$

(since $\{N(t_0t_3t_0t_1)^n | n \in N\}$ and $x^2t_2t_0t_1t_2t_0 \in M$),

$$t_0t_3t_2t_3t_5 = x^3t_4t_3t_1 \implies Mt_0t_3t_2t_3t_5 = Mt_4t_3t_1 \in [013]$$

$$= \{N(t_0t_1t_3)^n | n \in N\},$$

$$t_0t_3t_2t_3t_6 = yx^{-2}t_5t_0t_5t_0t_6t_2t_5 \implies Mt_0t_3t_2t_3t_6 = Mt_0t_6t_2t_5 \in [0152]$$

(since $\{N(t_0t_1t_5t_2)^n | n \in N\}$ and $yx^{-2}t_5t_0t_6t_5t_0 \in M$),
orbits of $N$.

Continuing with the double coset $M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}N$ we find the single coset stabilizer is trivial. However, the relation

\[
t_{0}t_{3}t_{2}t_{1}t_{4} = t_{4}t_{1}t_{2}t_{3}t_{0}
\]

\[
\Rightarrow \quad M_{0}t_{0}t_{3}t_{2}t_{1}t_{4} = M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}
\]

Then $M(t_{0}t_{3}t_{2}t_{1}t_{4})^{(0,4)(3,1)(5,6)} = M_{4}t_{4}t_{3}t_{2}t_{1}t_{0}$. But $M_{4}t_{4}t_{3}t_{2}t_{1}t_{4} = M_{0}t_{3}t_{2}t_{1}t_{4} \Rightarrow (0,4)(3,1)(5,6) \in N^{(03214)}$.

Since $M(t_{0}t_{3}t_{2}t_{1}t_{4})^{(0,4)(3,1)(5,6)} = M_{4}t_{4}t_{3}t_{2}t_{1}t_{0}$

\[
\Rightarrow \quad N^{(03214)} \geq \langle (0,4)(3,1)(5,6) \rangle.
\]

Since $|N^{(03214)}| = 2$, the number of single cosets in $[03214]$ is $\frac{|N|}{|N^{(03214)}|} = \frac{14}{2} = 7$. The orbits of $N^{(03214)}$ on $\{t_{0}, t_{1}, t_{2}, t_{3}, t_{4}, t_{5}, t_{6}\}$ are:

\[
O = \{\{2\}, \{0, 4\}, \{3, 1\}, \{5, 5\}\}.
\]

Take an element from each orbit and multiply on the right by the single coset representative $M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}$ of the double coset $M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}N$. We have:

\[
M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{4} = M_{0}t_{0}t_{3}t_{2}t_{1}t_{4} \in [0321],
\]

\[
M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{2} \in [032142],
\]

\[
t_{0}t_{3}t_{2}t_{1}t_{4}t_{3} = x^{-1}t_{0}t_{0}t_{6}t_{5}t_{0}t_{2}t_{3}t_{6} \Rightarrow M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{3} = M_{2}t_{3}t_{6} \in [014]
\]

(since $\{N(t_{0}t_{1}t_{4})^{n}|n \in N\}$ and $x^{-1}t_{0}t_{6}t_{5}t_{0} \in M$),

\[
t_{0}t_{3}t_{2}t_{1}t_{4}t_{5} = x^{4}t_{2}t_{0}t_{1}t_{2}t_{0}t_{5}t_{1}t_{6}t_{0} \Rightarrow M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{5} = M_{1}t_{5}t_{1}t_{0}t_{0} \in [03021]
\]

(since $\{N(t_{0}t_{3}t_{0}t_{2})^{n}|n \in N\}$ and $x^{4}t_{2}t_{0}t_{1}t_{2}t_{0} \in M$),

The new double coset is $M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{2}N$, which we represent by [032142], respectively.

$M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{2}N$

Continuing with the double coset $M_{0}t_{0}t_{3}t_{2}t_{1}t_{4}t_{2}N$ we find the single coset stabilizer is trivial. However, the relation

\[
t_{0}t_{3}t_{2}t_{1}t_{42} = t_{4}t_{1}t_{2}t_{3}t_{0}t_{2}
\]
Continuing with the double coset

Then \( M(t_0t_3t_2t_1t_4t_2) = M(t_1t_2t_3t_0t_2) \).

But \( M(t_4t_1t_2t_3t_0t_2) = M(t_0t_3t_2t_1t_4t_2) \implies (0, 4)(3, 1)(5, 6) \in N^{(032142)} \)

since \( M(t_0t_3t_2t_1t_4t_2) = M(t_1t_2t_3t_0t_2) \)

\( \implies N^{(032142)} \geq \langle (0, 4)(3, 1)(5, 6) \rangle \).

Since \( |N^{(032142)}| = 2 \), the number of single cosets in \([032142]\) is \( \frac{|N|}{|N^{(032142)}|} = \frac{14}{2} = 7 \).

The orbits of \( N^{(032142)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:

\( \mathcal{O} = \{\{2\}, \{0, 4\}, \{3, 1\}, \{5, 6\}\} \).

Take an element from each orbit and multiply on the right by the single coset representative \( M(t_0t_3t_2t_1t_4t_2) \) of the double coset \( M(t_0t_3t_2t_1t_4t_2N) \). We have:

\( M(t_0t_3t_2t_1t_4t_2) = M(t_0t_3t_2t_1t_4t_2) \in [03214] \),

\( t_0t_3t_2t_1t_4t_2 = yx^3t_0t_4t_3t_0 \implies M(t_0t_3t_2t_1t_4t_2) = M(t_0t_4t_3t_0) \in [0340] \)

\( = \{N(t_0t_4t_0)^n | n \in N\} \),

\( t_0t_3t_2t_1t_4t_2t_5 = xt_0t_5t_3t_0t_5t_1t_0t_1 \implies M(t_0t_3t_2t_1t_4t_2t_5) = M(t_5t_1t_0t_1) \in [0323] \)

(since \( \{N(t_0t_3t_2t_3)^n | n \in N\} \) and \( xt_0t_5t_4t_5t_0 \in M \),

\( M(t_0t_3t_2t_1t_4t_2t_0) \in [0321420] \).

The new double coset is \( M(t_0t_3t_2t_1t_4t_2t_0N) \), which we represent by \([0321420]\), respectively.

\( M(t_0t_3t_2t_1t_4t_2t_0N) \)

Continuing with the double coset \( M(t_0t_3t_2t_1t_4t_2t_0N) \) we find the single coset stabilizer is trivial. However, the relation

\( t_0t_3t_2t_1t_4t_2t_0 = yt_0t_5t_4t_5t_0t_5t_2t_3t_4t_1t_5t_3t_5 \)

\( \implies M(t_0t_3t_2t_1t_4t_2t_0) = M(t_5t_2t_3t_4t_1t_5t_3t_5) \) since \( yt_0t_5t_4t_5t_0 \).

Now \( M(t_0t_3t_2t_1t_4t_2t_0) = M(t_5t_2t_3t_4t_1t_5t_3t_5) \).

Then \( M(t_0t_3t_2t_1t_4t_2t_0) = M(t_5t_2t_3t_4t_1t_5t_3t_5) \).

But \( M(t_5t_2t_3t_4t_1t_5t_3t_5) = M(t_0t_3t_2t_1t_4t_2t_0) \implies (0, 5)(2, 3)(1, 2) \in N^{(0321420)} \)

since \( M(t_0t_3t_2t_1t_4t_2t_0) = M(t_5t_2t_3t_4t_1t_5t_3t_5) \)

\( \implies N^{(0321420)} \geq \langle (0, 5)(2, 3)(1, 2) \rangle \).

Since \( |N^{(0321420)}| = 2 \), the number of single cosets in \([0321420]\) is \( \frac{|N|}{|N^{(0321420)}|} = \frac{14}{2} = 7 \).
The orbits of $N^{(0321420)}$ on \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} are:
\[ \mathcal{O} = \{\{6\}, \{0, 5\}, \{2, 3\}, \{1, 4\}\} \]

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_2t_1t_4t_2t_0$ of the double coset $Mt_0t_3t_2t_1t_4t_2t_0N$. We have:
\[ Mt_0t_3t_2t_1t_4t_2t_0 = Mt_0t_3t_2t_1t_4t_2 \in [032142], \]
\[ t_0t_3t_2t_1t_4t_2t_0 = x^{-2}t_5t_0t_3t_0t_4t_5t_2t_0 \implies Mt_0t_3t_2t_1t_4t_2t_0 \in [025] \]
(since $\{N(t_0t_2t_5)^n | n \in N\}$ and $x^{-2}t_5t_0t_3t_0 \in M$),
\[ t_0t_3t_2t_1t_4t_2t_0 = x^4t_2t_0t_1t_2t_0t_6t_0t_6 \implies Mt_0t_3t_2t_1t_4t_2t_2t_0 \in [010] \]
(since $\{N(t_0t_1t_0)^n | n \in N\}$ and $x^4t_2t_0t_1t_2t_0 \in M$),
\[ Mt_0t_3t_2t_1t_4t_2t_0t_6 \in [03214206]. \]

The new double coset is $Mt_0t_3t_2t_1t_4t_2t_0t_6N$, which we represent by $[03214206]$, respectively.

\[ Mt_0t_3t_2t_1t_4t_2t_0t_6N \]

Continuing with the double coset $Mt_0t_3t_2t_1t_4t_2t_0t_6N$ we find the single coset stabilizer is trivial. However, the relation
\[ t_0t_3t_2t_1t_4t_2t_0t_6 = yt_0t_5t_4t_5t_0t_5t_3t_4t_1t_3t_5t_6 \]
\[ \implies Mt_0t_3t_2t_1t_4t_2t_0t_6 = Mt_5t_2t_3t_4t_1t_3t_5t_6 \text{ since } yt_0t_5t_4t_5t_0 = M \]
Then $M(t_0t_3t_2t_1t_4t_2t_0t_6)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5t_6$.

But $Mt_5t_2t_3t_4t_1t_3t_5t_6 = Mt_0t_3t_2t_1t_4t_2t_0t_6 \implies (0,5)(2,3)(1,2) \in N^{(03214206)}$

since $M(t_0t_3t_2t_1t_4t_2t_0t_6)^{(0,5)(2,3)(1,2)} = Mt_5t_2t_3t_4t_1t_3t_5t_6$
\[ \implies N^{(03214206)} \geq \langle (0,5)(2,3)(1,2) \rangle. \]

Since $|N^{(03214206)}| = 2$, the number of single cosets in $[03214206]$ is $\frac{|N|}{|N^{(03214206)}|} = \frac{14}{2} = 7$.

The orbits of $N^{(03214206)}$ on \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} are:
\[ \mathcal{O} = \{\{6\}, \{0, 5\}, \{2, 3\}, \{1, 4\}\} \].

Take an element from each orbit and multiply on the right by the single coset represent-
Continuing with the double coset \( M_{t_0}t_3t_2t_1t_4t_2t_0t_6N \). We have:

\[
M_{t_0}t_3t_2t_1t_4t_2t_0t_6 = M_{t_0}t_3t_2t_1t_4t_2t_0 \in [0321420],
\]

\[
t_{t_0}t_3t_2t_1t_4t_2t_0t_6t_1 = x^2t_5t_4t_1 \implies M_{t_0}t_3t_2t_1t_4t_2t_0t_6t_1 = M_{t_0}t_5t_1t_4 \in [0152]
\]

\[
= \{N(t_0t_1t_2)^n \mid n \in N\},
\]

\[
t_{t_0}t_3t_2t_1t_4t_2t_0t_6t_2 = yx^{-2}t_5t_0t_6t_1t_0t_1t_3 \implies M_{t_0}t_3t_2t_1t_4t_2t_0t_6t_2 = M_{t_1}t_2t_3 \in [012]
\]

(since \( \{N(t_0t_1t_2)^n \mid n \in N\} \) and \( yx^{-2}t_5t_0t_6t_0 \in M \),

\[
t_{t_0}t_3t_2t_1t_4t_2t_0t_6t_0 = t_{t_0}t_3t_2t_1t_4t_2t_0t_6 \implies M_{t_0}t_3t_2t_1t_4t_2t_0t_6t_0 = M_{t_1}t_4t_1t_2 \in [0301]
\]

(since \( \{N(t_0t_3t_0t_1)^n \mid n \in N\} \) and \( t_{t_0}t_3t_2t_0 \in M \).

\[\textbf{M}_{t_0}t_3t_4N\]

Continuing with the double coset \( M_{t_0}t_3t_4N \) we find the single coset stabilizer is trivial. However, the relation

\[
t_{t_0}t_3t_4 = x^2t_2t_0t_1t_2t_0t_6t_5
\]

\[
\implies M_{t_0}t_3t_4 = M_{t_2}t_6t_5 \text{ since } x^2t_2t_0t_1t_2t_0 \in M.
\]

Then \( M(t_{t_0}t_3t_4)^{(0,2)(3,6)(4,5)} = M_{t_2}t_6t_5 \).

But \( M_{t_0}t_6t_5 = M_{t_0}t_3t_4 \implies (0, 2)(3, 6)(4, 5) \in N^{(034)} \)

since \( M(t_{t_0}t_3t_4)^{(0,2)(3,6)(4,5)} = M_{t_2}t_6t_5 \)

\[
\implies N^{(034)} \geq \langle (0, 2)(3, 6)(4, 5) \rangle.
\]

Since \( |N^{(034)}| = 2 \), the number of single cosets in \( [034] \) is

\[
\frac{|N|}{|N^{(034)}|} = \frac{14}{2} = 7.
\]

The orbits of \( N^{(034)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:

\[\mathcal{O} = \{\{1\}, \{0, 2\}, \{3, 6\}, \{4, 5\}\}.\]

Take an element from each orbit and multiply on the right by the single coset representative \( M_{t_0}t_3t_4 \) of the double coset \( M_{t_0}t_3t_4N \). We have:

\[
M_{t_0}t_3t_4t_1 = M_{t_0}t_3 \in [03],
\]

\[
M_{t_0}t_3t_4t_1 \in [0341],
\]

\[
M_{t_0}t_3t_4t_0 \in [0340],
\]

\[
t_{t_0}t_3t_4t_6 = xt_1t_3t_4t_3t_1t_4t_2t_5 \implies M_{t_0}t_3t_4t_6 = M_{t_4}t_2t_5 \in [026]
\]

(since \( \{N(t_{t_0}t_2t_6)^n \mid n \in N\} \) and \( xt_1t_3t_4t_3t_1 \in M \).
The new double cosets have single coset representatives $Mt_0t_3t_4t_1N$ and $Mt_0t_3t_4t_0N$, we represent them as $[0341]$ and $[0340]$, respectively.

$Mt_0t_3t_4t_1N$

Continuing with the double coset $Mt_0t_3t_4t_1N$ we find the single coset stabilizer is trivial. However, the relation

$$t_0t_3t_4t_1 = x^2t_2t_0t_1t_2t_0t_2t_0t_5t_1$$

$$\implies Mt_0t_3t_4t_1 = Mt_2t_0t_5t_1$$

since $x^2t_2t_0t_1t_2t_0 \in M$.

Then $M(t_0t_3t_4t_1)^{(0,2)(3,6)(4,5)} = Mt_2t_0t_5t_1$.

But $Mt_0t_3t_4t_1 = (0,2)(3,6)(4,5) \in N^{(0341)}$

$$\implies N^{(0341)} \geq \langle (0,2)(3,6)(4,5) \rangle.$$

Since $|N^{(0341)}| = 2$, the number of single cosets in $[0341]$ is $\frac{|N|}{|N^{(0341)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0341)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$O = \{\{1\}, \{0, 2\}, \{3, 6\}, \{4, 5\}\}.$$ 

Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_1$ of the double coset $Mt_0t_3t_4t_1N$. We have:

$Mt_0t_3t_4t_1t_1 = Mt_0t_3t_4 \in [034],$

$t_0t_3t_4t_1t_3 = yxt_3t_2t_6 \implies Mt_0t_3t_4t_1t_3 = Mt_2t_2t_6 \in [014]$ 

$$= \{N(t_0t_3t_4)^n | n \in N\},$$

$t_0t_3t_4t_1t_4 = x^{-2}t_2t_3t_0 \implies Mt_0t_3t_4t_1t_4 = Mt_6t_2t_3t_0 \in [0341]$ 

$$= \{N(t_0t_3t_4)^n | n \in N\},$$

$Mt_0t_3t_4t_1t_0 \in [03410].$

The new double coset have single coset representative $Mt_0t_3t_4t_1t_0N$, we represent it as $[03410]$, respectively.

$Mt_0t_3t_4t_1t_0N$

Continuing with the double coset $Mt_0t_3t_4t_1t_0N$, we find the coset stabilizer $N^{(03410)} = \langle e \rangle$. Only $e$ will fix 0, 1, 3, and 4. Hence the number of single cosets in $[03410]$ is $\frac{|N|}{|N^{(03410)}|} = \frac{14}{1} = 14$. The orbits of $N^{(03410)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$O = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$ 

We now take an element from each orbit and multiply on the right by the single coset
representative $Mt_0t_3t_4t_1t_0$ of the double coset $Mt_0t_3t_4t_1t_0N$. We have:

$$Mt_0t_3t_4t_1t_0 = Mt_0t_3t_4t_1 \in [0341],$$

$$t_0t_3t_4t_1t_0t_1 = yx^3t_5t_1t_2t_5t_4 \implies Mt_0t_3t_4t_1t_0t_1 = Mt_5t_1t_2t_5t_4 \in [03406],$$

$$t_0t_3t_4t_1t_0t_2 = yx^{-2}t_5t_0t_6t_5t_0t_2t_3t_0t_4$$

$$\implies Mt_0t_3t_4t_1t_0t_2 = Mt_2t_3t_0t_4 \in [0152],$$

$$t_0t_3t_4t_1t_0t_3 = yx^{-2}t_5t_0t_6t_5t_0t_6t_4 \implies Mt_0t_3t_4t_1t_0t_3 = Mt_0t_6t_4 \in [013],$$

$$t_0t_3t_4t_1t_0t_4 = yx_0t_3t_4t_1t_0 \implies Mt_0t_3t_4t_1t_0t_4 = Mt_0t_3t_4t_1t_0 \in [03410],$$

$$t_0t_3t_4t_1t_0t_5 = x^4t_2t_0t_1t_2t_0t_2t_4t_6 \implies Mt_0t_3t_4t_1t_0t_5 = Mt_2t_4t_6 \in [024],$$

$$t_0t_3t_4t_1t_0t_6 = x^3t_2t_0t_1t_2t_0t_3t_0t_2t_3$$

$$\implies Mt_0t_3t_4t_1t_0t_6 = Mt_0t_3t_0t_2t_3 \in [03023].$$

$Mt_0t_3t_4t_0N$

Continuing with the double coset $Mt_0t_3t_4t_0N$, we find the coset stabiliser $N^{(0340)} = N^{0340} = \langle e \rangle$. Only $e$ will fix 0, 3, and 4. Hence the number of single cosets in $[0340]$ is

$$\frac{|N|}{|N^{(0340)}|} = \frac{14}{1} = 14.$$  

The orbits of $N^{(0340)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$  

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_4t_0$ of the double coset $Mt_0t_3t_4t_0N$. We have:

$$Mt_0t_3t_4t_0t_0 = Mt_0t_3t_4 \in [034],$$

$$t_0t_3t_4t_0t_1 = t_0t_5t_4t_5t_0t_4t_5t_6 \implies Mt_0t_3t_4t_0t_1 = Mt_4t_5t_6 \in [012]$$

(since $\{N(t_0t_1t_2)^n|n \in N\}$ and $t_0t_5t_4t_5t_0 \in M$).

$$t_0t_3t_4t_0t_2 = x^{-2}t_0t_3t_4t_5 \implies Mt_0t_3t_4t_0t_2 = Mt_0t_3t_0t_4t_5 \in [03021]$$

$$= \{N(t_0t_3t_0t_2t_1)^n|n \in N\},$$

$$t_0t_3t_4t_0t_3 = x^2t_0t_2t_3t_0t_3t_4t_4 \implies Mt_0t_3t_4t_0t_3 = Mt_3t_1t_4 \in [026]$$

(since $\{N(t_0t_2t_6)^n|n \in N\}$ and $x^2t_0t_2t_3t_0 \in M$).
We now take an element from each orbit and multiply on the right by the single coset \([03406]\), respectively.

\[ N(03406) \mid \mid t \{ \text{since } t M t \Rightarrow t = 0, \text{ and } 3, 4, \text{ and } 6. \text{ Hence the number of single cosets in } [03406] \text{ is } \frac{|N|}{|N(03406)|} = \frac{14}{1} = 14. \text{ The orbits of } N(03406) \text{ on } \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \text{ are:} \]

\[ \mathcal{O} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}. \]

We now take an element from each orbit and multiply on the right by the single coset representative \(M_{t_0}t_3t_4t_0t_6\) of the double coset \(M_{t_0}t_3t_4t_0t_6 N\). We have:

\[ M_{t_0}t_3t_4t_0t_6 = M_{t_0}t_3t_4t_0 \in [0340], \]

\[ t_0t_3t_4t_0t_6t_1 = x^3t_0t_3t_4t_0t_6t_3t_0 \Rightarrow M_{t_0}t_3t_4t_0t_6t_1 = M_{t_0}t_3t_4t_0t_6t_3 \in [014] \]

(since \(N(t_0t_1)^n|n \in N\) and \(x_{t_0t_3t_4t_0t_6} \in M),\)

\[ t_0t_3t_4t_0t_6t_2 = x^4t_2t_0t_1t_2t_0t_3t_1t_5 \Rightarrow M_{t_0}t_3t_4t_0t_6t_2 = M_{t_0}t_3t_4t_0t_6t_2 \in [025] \]

(since \(N(t_0t_2t_5)^n|n \in N\) and \(x^4t_2t_0t_1t_2t_0 \in M),\)

\[ t_0t_3t_4t_0t_6t_3 = yg_2t_5t_0t_6t_3 \Rightarrow M_{t_0}t_3t_4t_0t_6t_3 = M_{t_0}t_3t_4t_0t_6t_3 \in [03410] \]

\[ = \{N(t_0t_3t_4t_0t_6)^n|n \in N\}, \]

\[ t_0t_3t_4t_0t_6t_4 = t_0t_2t_3t_0t_6t_1t_3 \Rightarrow M_{t_0}t_3t_4t_0t_6t_4 = M_{t_0}t_3t_4t_0t_6t_3 \in [024] \]

(since \(N(t_0t_2t_4)^n|n \in N\) and \(t_0t_2t_3t_2t_0 \in M),\)

\[ t_0t_3t_4t_0t_6t_5 = x^{-2}t_6t_0t_6 \Rightarrow M_{t_0}t_3t_4t_0t_6t_5 = M_{t_0}t_3t_4t_0t_6t_6 \in [010] \]

\[ = \{N(t_0t_1t_0)^n|n \in N\}, \]

\[ t_0t_3t_4t_0t_6t_6 = x^2g_0t_4t_0t_6 \Rightarrow M_{t_0}t_3t_4t_0t_6t_6 = M_{t_0}t_3t_4t_0t_6 \in [0301] \]

\[ = \{N(t_0t_3t_0t_1)^n|n \in N\}. \]

**Mt_0t_3t_0t_1**
Continuing with the double coset $Mt_0t_3t_0 N$ we find the single coset stabilizer is trivial. However, the relation
\[ t_0t_3t_0 = t_5t_2t_5 \]
\[ \implies Mt_0t_3t_0 = Mt_5t_2t_5. \]
Then $M(t_0t_3t_0)(0,5)(2,3)(1,4) = Mt_5t_2t_5$.
But $Mt_5t_2t_5 = Mt_0t_3t_0 \implies (0,5)(2,3)(1,4) \in N^{(030)}$
(since $Mt_0t_3t_0 = Mt_5t_2t_5$
\[ \implies N^{(030)} \geq \langle (0,5)(2,3)(1,4) \rangle. \]
Since $|N^{(030)}| = 2$, the number of single cosets in $[030]$ is $\frac{|N|}{|N^{(030)}|} = \frac{14}{2} = 7$. The orbits of $N^{(030)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:
\[ O = \{\{6\}, \{0, 5\}, \{2, 3\}, \{1, 4\}\}. \]
Take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0$ of the double coset $Mt_0t_3t_0 N$. We have:
\[ Mt_0t_3t_0t_0 = Mt_0t_3 \in [03], \]
\[ Mt_0t_3t_0t_1 \in [0301], \]
\[ Mt_0t_3t_0t_2 \in [0302], \]
\[ t_0t_3t_0t_6 = x^{-1}t_5t_0t_6t_5t_0t_3t_0 \implies Mt_0t_3t_0t_6 = Mt_0t_3t_0 \in [03] \]
(since $\{N(t_0t_3t_0)^n \mid n \in N\}$ and $x^{-1}t_5t_0t_6t_5t_0 \in M$).

The new double cosets have single coset representatives $Mt_0t_3t_0t_1 N$ and $Mt_0t_3t_0t_2 N$, we represent them as $[0301]$ and $[0302]$, respectively.

$Mt_0t_3t_0t_1 N$

Continuing with the double coset $Mt_0t_3t_0t_1 N$, we find the coset stabiliser $N^{(0301)} = N^{0301} = \langle e \rangle$. Only $e$ will fix $0, 1, \text{and} 3$. Hence the number of single cosets in $[0301]$ is
\[ \frac{|N|}{|N^{(0301)}|} = \frac{14}{1} = 14. \]
The orbits of $N^{(0301)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:
\[ O = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}. \]
We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_1$ of the double coset $Mt_0t_3t_0t_1 N$. We have:
Continuing with the double coset $Mt_0t_3t_0t_1t_1 = Mt_0t_3t_0 \in [030]$,

$$t_0t_3t_0t_1t_2 = xy^2t_6t_2t_1t_0 \implies Mt_0t_3t_0t_1t_2 = Mt_6t_2t_1t_0 \in [0321]$$

$$= \{N(t_0t_3t_2t_1)^n|n \in N\},$$

$$t_0t_3t_0t_1t_3 = yt_6t_4t_0 \implies Mt_0t_3t_0t_1t_3 = Mt_6t_4t_0 \in [026]$$

$$= \{N(t_0t_2t_6)^n|n \in N\},$$

$$t_0t_3t_0t_1t_4 = x^{-2}t_5t_0t_5t_0t_0t_3t_2t_3 \implies Mt_0t_3t_0t_1t_4 = Mt_0t_3t_2t_3 \in [0323]$$

(since $\{N(t_0t_3t_2t_3)^n|n \in N\}$ and $x^{-2}t_5t_0t_5t_0t_0 \in M$),

$$t_0t_3t_0t_1t_5 = x^2t_4t_6t_1 \implies Mt_0t_3t_0t_1t_5 = Mt_4t_6t_1 \in [024]$$

$$= \{N(t_0t_2t_4)^n|n \in N\},$$

$$t_0t_3t_0t_1t_6 = x^2t_0t_2t_3t_2t_0t_6t_2t_1t_0t_3t_1t_6t_5$$

$$\implies Mt_0t_3t_0t_1t_6 = Mt_6t_2t_1t_0t_3t_1t_6t_5 \in [03214206]$$

(since $\{N(t_0t_3t_2t_4t_2t_0t_6)^n|n \in N\}$ and $x^2t_0t_2t_3t_2t_0 \in M$),

$$t_0t_3t_0t_1t_0 = xy^2t_0t_4t_3t_0t_1 \implies Mt_0t_3t_0t_1t_0 = Mt_0t_4t_3t_0t_1 \in [03406]$$

$$= \{N(t_0t_3t_4t_0t_6)^n|n \in N\}.$$

$Mt_0t_3t_0t_2N$

Continuing with the double coset $Mt_0t_3t_0t_2N$ we find the single coset stabilizer is trivial.

However, the relation

$$t_0t_3t_0t_2 = x^{-2}t_6t_3t_6t_4$$

$$\implies Mt_0t_3t_0t_2 = Mt_6t_3t_0t_4.$$  

Then $M(t_0t_3t_0t_2)^{\langle 0,6 \rangle(2,4)(1,5)} = Mt_3t_0t_3t_4$.

But $Mt_3t_0t_3t_4 = Mt_0t_3t_0t_2 \implies (0,6)(2,4)(1,5) \in N^{(0302)}$ since $M(t_0t_3t_0t_2)^{\langle 0,6 \rangle(2,4)(1,5)} = Mt_3t_0t_3t_4$

$$\implies N^{(0302)} \geq \langle (0,6)(2,4)(1,5) \rangle.$$  

Since $|N^{(0302)}| = 2$, the number of single cosets in $[0302]$ is $\frac{|N|}{|N^{(0302)}|} = \frac{14}{2} = 7$. The orbits of $N^{(0302)}$ on $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6\}$ are:

$$\mathcal{O} = \{\{3\}, \{0,6\}, \{2,4\}, \{1,5\}\}.$$  

Take an element from each orbit and multiply on the right by the single coset represen-
tative $Mt_0t_3t_0t_2$ of the double coset $Mt_0t_3t_0t_2N$. We have:

\[ Mt_0t_3t_0t_2 = Mt_0t_3t_0 \in [030], \]

\[ Mt_0t_3t_0t_2t_1 \in [03021], \]

\[ Mt_0t_3t_0t_2t_3 \in [03023], \]

\[ t_0t_3t_0t_2t_0 = t_5t_0t_4 \implies Mt_0t_3t_0t_2t_0 = Mt_5t_0t_4 \in [026] \]

\[ = \{ N(t_0t_2t_0)^n | n \in N \}. \]

The new double cosets have single coset representatives $Mt_0t_3t_0t_2t_1N$ and $Mt_0t_3t_0t_2t_3N$, we represent them as $[03021]$ and $[03023]$, respectively.

\[ Mt_0t_3t_0t_2t_1N \]

Continuing with the double coset $Mt_0t_3t_0t_2t_1N$, we find the coset stabiliser $N^{[03021]} = N^{[030]} = \langle e \rangle$. Only $e$ will fix 0, 1, 2, and 3. Hence the number of single cosets in $[03021]$ is $\frac{|N|}{|N^{[03021]}|} = \frac{14}{1} = 14$. The orbits of $N^{[03021]}$ on \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} are:

\[ \mathcal{O} = \{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \}. \]

We now take an element from each orbit and multiply on the right by the single coset representative $Mt_0t_3t_0t_2t_1$ of the double coset $Mt_0t_3t_0t_2t_1N$. We have:

\[ Mt_0t_3t_0t_2t_1t_1 = Mt_0t_3t_0t_2 \in [0302], \]

\[ t_0t_3t_0t_2t_1t_2 = x^{-1}t_1t_2t_4 \implies Mt_0t_3t_0t_2t_1t_2 = Mt_1t_2t_4 \in [013] \]

\[ = \{ N(t_0t_1t_3)^n | n \in N \}, \]

\[ t_0t_3t_0t_2t_1t_3 = x^4t_2t_0t_1t_2t_4t_0t_4t_5t_1 \implies Mt_0t_3t_0t_2t_1t_3 = Mt_4t_0t_4t_5t_1 \in [03214] \]

(since $\{ N(t_0t_3t_2t_1t_4)^n | n \in N \}$ and $x^4t_2t_0t_1t_2t_0 \in M$),

\[ t_0t_3t_0t_2t_1t_4 = x^{-2}t_6t_3t_0t_2 \implies Mt_0t_3t_0t_2t_1t_4 = Mt_6t_3t_0t_2t_6 \in [0340] \]

\[ = \{ N(t_0t_3t_4t_0)^n | n \in N \}, \]

\[ t_0t_3t_0t_2t_1t_5 = x^2yt_3t_6t_5t_6 \implies Mt_0t_3t_0t_2t_1t_5 = Mt_3t_5t_6t_5t_6 \in [0323] \]

\[ = \{ N(t_0t_3t_2t_3)^n | n \in N \}, \]

\[ Mt_0t_3t_0t_2t_1t_6 \in [030216], \]
\[ t_0 t_3 t_0 t_2 t_1 t_0 = x^{-3} t_1 t_3 t_1 \implies M t_0 t_3 t_0 t_2 t_1 t_0 = M t_1 t_3 t_1 \in [020] \]
\[ = \{ N(t_0 t_2 t_0)^n | n \in N \}, \]

The new double coset have single coset representative \( M t_0 t_3 t_0 t_2 t_1 t_0 N \), we represent it as \([030216]\), respectively.

\[ \text{Mt}_0 t_3 t_0 t_2 t_1 t_6 \]

Continuing with the double coset \( M t_0 t_3 t_0 t_2 t_1 t_6 N \) we find the single coset stabilizer is trivial. However, the relation
\[ t_0 t_3 t_0 t_2 t_1 t_6 = x^3 t_2 t_0 t_1 t_2 t_0 t_4 t_1 t_4 t_2 t_3 t_5 \]
\[ \implies M t_0 t_3 t_0 t_2 t_1 t_6 = M t_4 t_1 t_4 t_2 t_3 t_5 \] since \( x^3 t_2 t_0 t_1 t_2 t_0 \in M \).

Then \( M(t_0 t_3 t_0 t_2 t_1 t_6)^{(0,4)(3,1)}(5,6) = M t_4 t_1 t_4 t_2 t_3 t_5 \).

But \( M t_4 t_1 t_4 t_2 t_3 t_5 = M t_0 t_3 t_0 t_2 t_1 t_6 \implies (0,4)(3,1)(5,6) \in N^{[030216]} \)

since \( M(t_0 t_3 t_0 t_2 t_1 t_6)^{(0,4)(3,1)}(5,6) = M t_4 t_1 t_4 t_2 t_3 t_5 \)
\[ \implies N^{[030216]} \geq \langle (0,4)(3,1)(5,6) \rangle. \]

Since \( |N^{[030216]}| = 2 \), the number of single cosets in \([030216]\) is \( \frac{|N|}{N^{[030216]}} = \frac{14}{2} = 7 \).

The orbits of \( N^{[030216]} \) on \( \{ t_0, t_1, t_2, t_3, t_4, t_5, t_6 \} \) are:
\[ \mathcal{O} = \{ \{2\}, \{0,4\}, \{3,1\}, \{5,6\} \}. \]

Take an element from each orbit and multiply on the right by the single coset representative \( M t_0 t_3 t_0 t_2 t_1 t_6 \) of the double coset \( M t_0 t_3 t_0 t_2 t_1 t_6 N \). We have:
\[ M t_0 t_3 t_0 t_2 t_1 t_6 = M t_0 t_3 t_0 t_2 t_1 \in [03021], \]
\[ t_0 t_3 t_0 t_2 t_1 t_0 t_2 = y x^2 t_0 t_3 t_0 t_2 t_1 t_0 \implies M t_0 t_3 t_0 t_2 t_1 t_0 t_2 = M t_0 t_3 t_0 t_2 t_1 t_6 \in [030216] \]
\[ = \{ N(t_0 t_3 t_0 t_2 t_1 t_6)^n | n \in N \}, \]
\[ t_0 t_3 t_0 t_2 t_1 t_0 t_3 = x^{-2} t_4 t_0 t_2 t_1 t_0 t_2 \implies M t_0 t_3 t_0 t_2 t_1 t_6 t_3 = M t_4 t_0 t_2 t_1 t_6 \in [025] \]
\[ = \{ N(t_0 t_2 t_5)^n | n \in N \}, \]
\[ t_0 t_3 t_0 t_2 t_1 t_0 t_0 = x^3 t_2 t_0 t_1 t_2 t_0 t_2 t_3 t_5 \implies M t_0 t_3 t_0 t_2 t_1 t_6 t_0 = M t_2 t_3 t_5 \in [013] \]
(since \( \{ N(t_0 t_1 t_3)^n | n \in N \} \) and \( x^3 t_2 t_0 t_1 t_2 t_0 \in M \)).

\[ \text{Mt}_0 t_3 t_0 t_2 t_3 N \]

Continuing with the double coset \( M t_0 t_3 t_0 t_2 t_3 N \) we find the single coset stabilizer is
trivial. However, the relation

\[ t_0 t_3 t_0 t_2 t_3 = x^{-2} t_0 t_3 t_0 t_2 t_3 \]

\[ \implies M t_0 t_3 t_0 t_2 t_3 = M t_0 t_3 t_0 t_2 t_3. \]

Then \( M(t_0 t_3 t_0 t_2 t_3)^{(0,6)(2,4)(1,5)} = M t_0 t_3 t_0 t_2 t_3. \)

But \( M t_0 t_3 t_0 t_2 t_3 = M t_0 t_3 t_0 t_2 t_3 \implies (0,6)(2,4)(1,5) \in N^{(03023)} \)

since \( M(t_0 t_3 t_0 t_2 t_3)^{(0,6)(2,4)(1,5)} = M t_3 t_6 t_4 t_3 \)

\[ \implies N^{(03023)} \geq ⟨(0,6)(2,4)(1,5)⟩. \]

Since \(|N^{(03023)}| = 2\), the number of single cosets in \([03023]\) is \( \frac{|N|}{|N^{(03023)}|} = \frac{14}{2} = 7 \). The

orbits of \( N^{(03023)} \) on \( \{t_0, t_1, t_2, t_3, t_4, t_5, t_6\} \) are:

\[ O = \{\{3\}, \{0, 6\}, \{2, 4\}, \{1, 5\}\}. \]

Take an element from each orbit and multiply on the right by the single coset representative \( M t_0 t_3 t_0 t_2 t_3 \) of the double coset \( M t_0 t_3 t_0 t_2 t_3 N \). We have:

\[ M t_0 t_3 t_0 t_2 t_3 ∈ [0302], \]

\[ t_0 t_3 t_0 t_2 t_3 = x^{-3} t_3 t_0 t_2 t_3 \implies M t_0 t_3 t_0 t_2 t_3 t_1 = M t_3 t_4 t_3 \in [010] \]

\[ = \{N(t_0 t_1 t_0)^n | n ∈ N\}, \]

\[ t_0 t_3 t_0 t_2 t_3 t_4 = x y t_4 t_3 t_2 \implies M t_0 t_3 t_0 t_2 t_3 t_4 = M t_4 t_3 t_2 \in [012] \]

\[ = \{N(t_0 t_1 t_2)^n | n ∈ N\}, \]

\[ t_0 t_3 t_0 t_2 t_3 t_0 = x^{-1} t_5 t_0 t_6 t_5 t_0 t_3 t_2 t_5 t_6 \implies M t_0 t_3 t_0 t_2 t_3 t_0 = M t_6 t_3 t_2 t_5 t_6 \in [03410] \]

(since \( \{N(t_0 t_3 t_4 t_1 t_0)^n | n ∈ N\} \) and \( x^{-1} t_5 t_0 t_6 t_5 t_0 \in M \)).

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of \( M \) in \( G \) is 351. We conclude:

\[ G = MeN ∪ Mt_0 N \cup M t_0 t_1 N \cup M t_0 t_2 N \cup M t_0 t_3 N \cup M t_0 t_1 t_0 N \cup M t_0 t_1 t_2 N \∪ M t_0 t_3 t_1 t_4 t_0 N \cup M t_0 t_1 t_5 t_2 N \cup M t_0 t_2 t_5 N \cup M t_0 t_2 t_6 N \∪ M t_0 t_3 t_0 N \cup M t_0 t_3 t_0 t_0 t_1 N \cup M t_0 t_3 t_4 t_0 N \cup M t_0 t_3 t_4 t_1 N \cup M t_0 t_3 t_4 t_2 N \∪ M t_0 t_3 t_4 t_3 N \cup M t_0 t_3 t_4 t_0 t_1 N \cup M t_0 t_3 t_4 t_0 t_2 N , \]

\[ G = \frac{2^7 : 2^2 D_14}{(xy t_3 t_4 t_0), (x t_3 t_4 t_0)} \]
\[|G| \leq |N| + \frac{|N|}{N(0)} + \frac{|N|}{N(01)} + \frac{|N|}{N(02)} + \frac{|N|}{N(03)} + \frac{|N|}{N(012)} + \frac{|N|}{N(013)} + \frac{|N|}{N(014)} + \frac{|N|}{N(015)} + \]
\[\frac{|N|}{N(0152)} + \frac{|N|}{N(025)} + \frac{|N|}{N(0257)} + \frac{|N|}{N(026)} + \frac{|N|}{N(0267)} + \frac{|N|}{N(032)} + \frac{|N|}{N(0321)} + \frac{|N|}{N(0323)} + \frac{|N|}{N(032147)} + \frac{|N|}{N(032142)} + \]
\[\frac{|N|}{N(0321420)} + \frac{|N|}{N(03214206)} + \frac{|N|}{N(03214207)} + \frac{|N|}{N(03417)} + \frac{|N|}{N(034170)} + \frac{|N|}{N(034407)} + \frac{|N|}{N(034408)} + \frac{|N|}{N(0340)} + \frac{|N|}{N(0301)} + \]
\[\frac{|N|}{N(0302)} + \frac{|N|}{N(030217)} + \frac{|N|}{N(030216)} + \frac{|N|}{N(03023)} \times |M| \]

\[|G| \leq (1 + 7 + 14 + 14 + 14 + 14 + 14 + 14 + 7 + 14 + 14 + 14 + 14 + 7 + 7 + 14 + 7 + 7 + 7 + 7 + 7 + 7 + 14 + 14 + 14 + 7 + 14 + 7 + 7 + 7) \times 28 \]

\[|G| \leq 351 \times 28 \]

\[|G| \leq 9828.\]

A Cayley diagram that summarizes the above information is given on the next page.

Figure 4.1: Cayley Diagram of \(L_2(27)\) over \(M = 2 \cdot D_{14}\)
4.2 Iwasawa’s Lemma to Prove $L_2(27)$ over $M = 2D_{14}$ is Simple

Again, we use Iwasawa’s lemma and the transitive action of $G$ on the set of single cosets to prove $G \cong L_2(27)$ over $M = 2D_{14}$ is a simple group. Iwasawa’s lemma has three sufficient conditions that we must satisfy:

1. $G$ acts on $X$ faithfully and primitively
2. $G$ is perfect ($G = G'$)
3. There exist $x \in X$ and a normal abelian subgroup $K$ of $G^x$ such that the conjugates of $K$ generate $G$.

Proof. 4.2.1 $G = L_2(27)$ over $M = 2D_{14}$ acts on $X$ Faithfully

Let $G$ acts on $X = \{M, Mt_0N, Mt_0t_1N, Mt_0t_2N, Mt_0t_3N, \ldots, Mt_0t_3t_0t_2t_3N\}$, where $X$ is a transitive $G$-set of degree 351. $G$ acts on $X$ implies there exist homomorphism

\[ f : G \rightarrow S_{351} \quad (|X| = 351). \]

By First Isomorphic Theorem we have:

\[ G/\text{ker}f \cong f(G). \]

If $\text{ker}f = 1$ then $G \cong f(G)$. Only elements of $N$ fix $N$ implies $G^1 = N$. Since $X$ is transitive $G$ set of degree 351, we have:

\[ |G| = 351 \times |G^1| \]
\[ = 351 \times |M| \]
\[ = 351 \times 28 \]
\[ = 9828 \]
\[ \implies |G| = 9828. \]

From Cayley diagram, $|G| \leq 9828$. However, from above $|G| = 9828$ implying $\text{ker}(f) = 1$. Since $\text{ker}f = 1$ then $G$ acts faithfully on $X$. 
4.2.2  \( G = L_2(27) \) over \( M = 2 \cdot D_{14} \) acts on \( X \) Primitively

Since \( G = L_2(27) \) is transitive on \( |X| = 351 \), if \( B \) is a nontrivial block then we may assume that \( M \in B \). However, \(|B|\) must divide \(|X| = 351\). The only nontrivial blocks must be of size 3, 9, 13, 27, 39, or 117. Note if \( B_1 \in B \) then \( B = X \). So \( B \) is a trivial block. By inspection, we can see from the Cayley diagram that we cannot create a nontrivial block of size 3, 9, 13, 27, 39, or 117. Thus, \( G \) acts primitively on \( X \).

4.2.3  \( G = L_2(27) \) over \( M = 2 \cdot D_{14} \) is Perfect

Next we want to show that \( G = G' \). Since \( G = \langle N, t \rangle \), we have that \( N \leq G' \). Now \( D_{14} \leq G \implies D_{14}' \leq G' \). The commutators subgroup of \( D_{14} \) is

\[ D_{14}' = \langle (1, 2, 3, 4, 5, 6, 0) \rangle = \{ e, x, x^2, x^3, x^4, x^5, x^6 \} \leq G' \].

Consider the relations obtained through the double coset enumeration, previously given.

\[ t_0t_2t_0t_4 = xt_6t_4t_2 \implies x = t_0t_2t_0t_4t_2t_4t_6 \]
\[ t_0t_2t_0t_3 = yt_4t_3t_4 \implies y = t_0t_2t_0t_3t_4t_3t_4 \]

Now \( D_{14} \leq G \implies D_{14}' \leq G' \). \( D_{14}' = \langle (1, 2, 3, 4, 5, 6, 0) \rangle = \{ e, x, x^2, x^3, x^4, x^5, x^6 \} \leq G' \). Note \( G = \langle x, y, t \rangle = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_0 \rangle \). Our goal is to show that one of the \( t_i's \in G' \), then we can conjugate by \( \langle x, y \rangle \) to obtain all the \( t_i's \) in \( G' \). Consider, the relation:

\[ x = t_0t_2t_0t_4t_2t_4t_6 \]
\[ = t_0t_2t_0t_2t_4t_2t_4t_6 \]
\[ = [t_0, t_2][t_2, t_4]t_6 \in G' \]

We see that \( t_6 \in G' \). So \( G' = \langle x, t_6 \rangle = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_0 \rangle = G \). But \( G \geq G' \). We conclude that \( G = G' \) and \( G \) is perfect.

4.2.4  Conjugates of a Normal Abelian \( K \)

Generate \( G = L_2(27) \) over \( M = 2 \cdot D_{14} \)

Now we require \( x \in X \) and a normal abelian subgroup \( K \) of \( G^x \), the point stabilizer of \( x \) in \( G \), such that the conjugates of \( K \) in \( G \) generate \( G \).
Now \( G^1 = M = 2^2 D_{14} \) possesses a normal abelian subgroup \( K = \langle x \rangle \). Since \( x \in K \implies x^{-3}, x^{-1} \in K \). Now we have the following relations:

\[
x^{-3} = t_0t_3t_0t_2t_1t_0t_1t_3t_1 \in K \quad \text{and} \quad x^{-1} = t_0t_3t_0t_2t_1t_2t_2t_1t_0t_1t_0t_3t_0 \in K.
\]

Now we multiply \( x^{-3} \) by \( x \):

\[
x^{-3}x = x^{-2} = t_0t_3t_0t_2t_1t_0t_1t_3t_1t_1t_2t_4t_2t_1t_2t_0t_3t_0 \in K.
\]

We conjugate both sides by \( t_0t_3t_0t_2t_1 \):

\[
(x^{-2})^{t_0t_3t_0t_2t_1} = (t_0t_3t_0t_2t_1t_0t_1t_3t_1t_2t_4t_2t_1t_2t_0t_3t_0)^{t_0t_3t_0t_2t_1} \in K^G
\]

\[
t_1t_2t_0t_3t_0x^{-2}t_0t_3t_0t_1 = t_1t_2t_0t_3t_0t_0t_2t_1t_0t_1t_3t_1t_2t_4t_2t_1t_2t_0t_3t_0t_3t_0t_2t_1
\]

\[
x^{-2}t_6t_0t_5t_1t_5t_0t_3t_0t_2t_1 = t_0t_1t_3t_2t_4t_2 \in K^G
\]

Thus, \( t_0t_1t_3t_2t_4t_2 \in K^G \). (4.1)

Consider the relation:

\[
x = t_0t_2t_4t_2t_6t_6t_6 \in K
\]

\[
(x)^{t_0} = (t_0t_2t_4t_2t_6t_6)^{t_0} \in K^G
\]

\[
t_0xt_0 = t_0t_0t_2t_4t_2t_6t_6t_6t_0
\]

\[
xt_1t_0 = t_2t_4t_2t_6t_4t_6t_0 \in K^G
\]

Thus, \( t_2t_4t_2t_6t_4t_6t_0 \in K^G \). (4.2)

Now we multiply (4.1) and (4.2):

\[
x^{-2}t_6t_0t_5t_1t_5t_0t_3t_0t_2t_1xt_1t_0 = t_0t_1t_3t_2t_4t_2t_4t_2t_6t_4t_6t_0 \in K^G
\]

\[
x^{-1}t_0t_1t_6t_0t_1t_1t_3t_2t_1t_0 = t_0t_1t_3t_0t_4t_6t_0 \in K^G
\]

\[
(x^{-1}t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_0)^{t_0} = (t_0t_1t_3t_0t_4t_6t_0)^{t_0} \in K^G
\]

\[
t_0x^{-1}t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1t_0t_0 = t_0t_0t_1t_3t_0t_4t_6t_0t_0 \in K^G
\]

\[
x^{-1}t_6t_0t_1t_6t_2t_6t_1t_4t_1t_3t_2t_1 = t_1t_3t_0t_4t_6 \in K^G
\]
Consider the relations:

\[ x^3 = t_0 t_3 t_2 t_1 t_3 t_6 t_4 \in K \implies x^{-3} = t_4 t_6 t_3 t_1 t_2 t_3 t_0 \in K \text{ and} \]
\[ x^3 = t_0 t_3 t_2 t_5 t_1 t_3 t_4 \in K. \]

Now,

\[ e = x^3 x^{-3} = t_0 t_3 t_2 t_3 t_5 t_1 t_3 t_4 t_4 t_6 t_3 t_1 t_3 t_2 t_3 t_0 \in K \]
\[ \implies e = t_0 t_3 t_2 t_3 t_5 t_1 t_3 t_6 t_3 t_1 t_3 t_2 t_3 t_0 \in K \]
\[ (c)^{t_0 t_3 t_2 t_3} = (t_0 t_3 t_2 t_3 t_5 t_1 t_3 t_6 t_3 t_1 t_3 t_2 t_3 t_0)^{t_0 t_3 t_2 t_3} \in K^G \]
\[ t_3 t_2 t_3 t_0 t_3 t_2 t_3 = t_3 t_2 t_3 t_0 t_0 t_3 t_2 t_3 t_5 t_1 t_3 t_6 t_3 t_1 t_3 t_2 t_3 t_0 t_3 t_2 t_3 \]
\[ e = t_5 t_1 t_3 t_6 t_3 t_1 \in K^G \]

Thus, \( t_5 t_1 t_3 t_6 t_3 t_1 \in K^G \). (4.4)

Next, we multiply (4.3) & (4.4):

\[ x^{-1} t_6 t_0 t_1 t_6 t_0 t_6 t_1 t_4 t_1 t_3 t_2 t_1 = t_1 t_3 t_6 t_4 t_6 t_5 t_1 t_3 t_6 t_3 t_1 \in K \]
\[ (x^{-1} t_6 t_0 t_1 t_6 t_2 t_0 t_1 t_4 t_1 t_3 t_2 t_1)^{t_1 t_3 t_6} = (t_1 t_3 t_6 t_4 t_6 t_5 t_1 t_3 t_6 t_3 t_1)^{t_1 t_3 t_6} \in K^G \]
\[ t_6 t_3 t_1 x^{-1} t_6 t_0 t_1 t_6 t_2 t_0 t_1 t_4 t_1 t_3 t_2 t_1 t_3 t_6 = t_0 t_3 t_1 t_3 t_6 t_4 t_6 t_5 t_1 t_3 t_6 t_3 t_1 t_1 t_3 t_6 \in K^G \]
\[ x^{-1} t_5 t_2 t_0 t_6 t_0 t_1 t_6 t_2 t_0 t_1 t_4 t_1 t_3 t_2 t_1 t_3 t_6 = t_4 t_6 t_5 t_1 t_3 \in K^G \]

Thus, \( t_4 t_6 t_5 t_1 t_3 \in K^G \). (4.5)

Consider the relations:

\[ x = t_0 t_2 t_4 t_6 t_4 t_6 \in K \text{ and} \]
\[ x = t_0 t_2 t_4 t_6 t_4 t_1 t_6 \in K \implies x^{-1} = t_6 t_1 t_4 t_6 t_4 t_2 t_0 \in K. \]
Now, we multiply both relation:

\[ xx^{-1} = e = t_0t_2t_4t_2t_0t_4t_6t_1t_4t_6t_1t_2t_0 \in K \]
\[ e = t_0t_2t_4t_2t_0t_4t_6t_1t_0 \in K \]
\[ (e)^{t_0t_2t_4} = (t_0t_2t_4t_2t_0t_4t_6t_1t_0)^{t_0t_2t_4} \in K^G \]
\[ t_4t_2t_0t_4t_2t_0 = t_4t_2t_0t_4t_2t_0t_4t_6t_1t_0t_4t_2t_0t_4 \in K^G \]
\[ e = t_2t_6t_4t_1t_6 \in K^G \]
\[ (x)^{t_0t_2t_4} = (t_2t_6t_4t_1t_6)^{t_0t_2t_4} \in K^G \]
\[ t_0t_2x^2t_6 = t_0t_2t_0t_2t_4t_1t_4t_6t_3t_0t_2 \in K^G \]
\[ x^{-2}t_0t_3t_0t_2 = t_4t_1t_4t_6t_3t_0t_2 \in K^G \]
\[ \Rightarrow t_4t_1t_4t_3t_6t_3t_0t_2 \in K^G \]
\[ \Rightarrow (t_4t_1t_4t_3t_6t_3t_0t_2)^{-1} \in K^G \]

So,
\[ t_2t_0t_3t_3t_4t_1t_4 \in K^G. \]  \(4.7\)

Now, we multiply (4.6) & (4.7):

\[ et_2t_0t_3t_0x^2 = t_4t_1t_4t_0t_2t_6t_0t_3t_6t_3t_4t_1t_4 \in K \]
\[ (x^2t_4t_2t_0t_2)^{t_4t_1t_4} = (t_4t_1t_4t_6t_0t_2t_0t_3t_6t_3t_4t_1t_4)^{t_4t_1t_4} \in K^G \]
\[ t_4t_1t_4x^2t_4t_2t_0t_2t_4t_1t_4 = t_4t_1t_4t_6t_4t_6t_0t_3t_6t_3t_4t_1t_4 \in K^G \]
\[ x^2t_6t_3t_6t_4t_0t_2t_4t_1t_4 = t_6t_2t_0t_3t_0t_3 \in K^G \]
Thus, \( t_3t_6t_2t_6t_2t_0 \in K^G \).

Consider the relation

\[
x^{-3} = t_0t_2t_6t_4t_6t_5t_6t_3 \in K.
\]  

(4.9) 

Now, we multiply (4.9) & (4.8):
\[
x^{-3}x^2t_5t_1t_5t_6t_3t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_0t_2t_6t_4t_6t_5t_6t_3t_4t_3t_6t_2t_0 \in K \\
\implies x^{-1}t_5t_1t_5t_6t_3t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3 = t_0t_2t_6t_4t_6t_5t_6t_2t_0 \in K \\
(x^{-1}t_5t_1t_5t_6t_3t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3)^{t_0t_2t_6} = (t_0t_2t_6t_4t_6t_5t_6t_2t_0)^{t_0t_2t_6} \in K^G \\
\implies t_6t_2t_0x^{-1}t_5t_1t_5t_6t_3t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2t_6 \\
\quad = t_6t_2t_0x^{-1}t_5t_1t_5t_6t_3t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2t_6 \in K^G \\
\implies x^{-1}t_5t_1t_6t_5t_1t_5t_6t_3t_6t_4t_2t_0t_2t_4t_1t_4t_3t_6t_3t_0t_2t_6 = t_4t_6t_5t_3t_6t_2 \in K^G.
\]

So the inverse

\[
t_2t_6t_3t_5t_6t_4 \in K^G
\]  

(4.10) 

Now, we multiply (4.5) & (4.10):
\[
t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4 \in K.
\]

Next, we conjugate by \( t_4t_6t_5 \):
\[
(t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4)^{t_4t_6t_5} = t_5t_6t_4t_6t_5t_1t_3t_2t_6t_3t_5t_6t_4t_6t_5 \in K^G \\
\quad = t_1t_3t_2t_6t_3 \in K^G.
\]

Now, we conjugate \( t_1t_3t_2t_6t_3 \) by \( t_1 \):
\[
(t_1t_3t_2t_6t_3)^{t_1} = t_1t_1t_3t_2t_6t_3t_1 \in K^G
\]

Thus, \( t_3t_2t_6t_3t_1 \in K^G \)  

(4.11) 

Now, we need to multiply (4.11) & (4.3):
\[
t_3t_2t_6t_3t_1t_3t_6t_4t_6 = t_3t_2t_4t_6 \in K.
\]

(4.12)
Then

\[(t_3t_2t_4t_6)^{t_3} = t_3t_3t_2t_4t_6t_3 \in K^G\]

Thus, \(t_2t_4t_6t_3 \in K^G\). \hfill (4.12)

Now we multiply (4.1) & (4.12) to obtain the following:

\[t_0t_1t_3t_2t_4t_2t_4t_6t_3 = t_0t_1t_3t_2t_6t_3 \in K\] \hfill (4.13)

By (4.11) we have the following:

\[t_3t_2t_6t_3t_1 \in K\]

\[(t_3t_2t_6t_3t_1)^{t_1} = t_1t_3t_2t_6t_3t_1t_1 = t_1t_3t_2t_6t_3 \in K^G.\]

So, \((t_1t_3t_2t_6)^{-1} \in K^G\)

\[\Rightarrow t_3t_6t_2t_3t_1 \in K^G\] \hfill (4.14)

Finally, we multiply (4.13) & (4.14) to obtain the following:

\[t_0t_1t_3t_2t_6t_3t_6t_2t_3t_1 = t_0 \in K\]

Thus, \(t_0 \in K\)

\[\Rightarrow t_0^G \in K^G\]

\[\Rightarrow K^G \leq \{t_0, t_0^2, t_0^3, t_0^4, t_0^5, t_0^6\}\]

\[\Rightarrow K^G \leq \{t_0, t_0^2, t_0^3, t_0^4, t_0^5, t_0^6\} = <t_0, t_1, t_2, t_3t_4, t_5, t_6 > = G.\]

Hence, the conjugates of \(K\) generate \(G\). Therefore, by Iwasawa’s lemma, \(G \cong L_2(27)\) is simple.
Chapter 5

Extension Problem

5.1 Extension Problem Preliminaries

Definition 5.1. (Extension). $G$ is an extension of $K$ by $Q$ if $G$ has a normal subgroup $K_1 \cong K$ such that $G/K_1 \cong Q$ where $G$ is a product of $KQ$, $G = KQ$. [Rot12]

Definition 5.2. (Normal series). A chain of subgroups of $G$, $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$ such that $G_i \supseteq G \forall i, 1 \leq i \leq n$ is called a normal series of $G$. [Rot12]

Definition 5.3. (Subnormal series). A chain of subgroups of $G$, $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$ such that $G_{i+1} \supseteq G_i \forall i, 0 \leq i \leq n - 1$ is called a subnormal series of $G$. [Rot12]

Definition 5.4. (Composition series). A composition series is a normal series $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_n = 1$ in which, for all $i$, either $G_{i+1}$ is a maximal normal subgroup of $G_i$ or $G_{i+1} = G_i$. [Rot12]

Definition 5.5. (Composition factors). If $G$ has a composition series, then the factor groups of this series is called the composition factors of $G$. [Rot12]
Note 5.6. Any two composition series of a group are isomorphic.

Note 5.7. The composition factors $G_0/G, G_1/G_2, \cdots, G_{n-1}/G_n$ of the composition series $G_0 = G \supseteq G_1 \supseteq G_2 \cdots \supseteq G_{n-1} \supseteq G_n$ are simple (A group $G$ is simple if $G$ and 1 are the only normal subgroups of $G$).

There are four possible extensions a group can have namely:

**Definition 5.8. (Direct Product).** A group $G$ is a direct product of $N$ by $H$ if $N$ and $H$ are both normal in $G$, denoted by $G = N \times H$. [Rot12]

**Definition 5.9. (Semi-direct Product).** A group $G$ is a semi-direct product of $N$ by $H$ if there exist a complement $H_1 \cong H$ and $N$ is normal in $G$, denoted by $G = N : H$. [Rot12]

**Definition 5.10. (Central Extension).** A group $G$ is a central extension of $N$ by $H$ if $G$ is perfect, normal, and is also the center of $G$, denoted $G = N : H$. [Rot12]

**Definition 5.11. (Mixed Extension).** A group $G$ is a mixed extension if $N$ is an abelian group which is not the center of $G$, denoted by $G = N : H$. [Rot12]

### 5.2 Mixed Extension $(G \cong (2^6 : L_2(7)) : 2)$

From an original progenitor we have found the following group

$$G = \langle x, y, t | x^7, y^2, (xy)^2, t^2, (t, y), (xtt)^8, (ttxt)^3 \rangle .$$

We will now prove that $(2^6 : L_2(7)) : 2$ is the homomorphic image of the progenitor mentioned above.

**Proof.** Using MAGMA we get the following composition factors.

```plaintext
> CompositionFactors(G1);
G  | Cyclic(2)
  * | A(1, 7)          = L(2, 7)
  * | Cyclic(2)
  * | Cyclic(2)
  *
```
Therefore we have the following composition series,

\[ G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq 1, \]

where \( G = (G_1/G_2) \cdot (G_2/G_3) \cdot (G_3/G_4) \cdot (G_4/G_5) \cdot (G_5/G_6) \cdot (G_6/G_7) \cdot (G_7/1) \)

\[ = C_2 \cdot L_2(7) \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2 \cdot C_2. \] The normal lattice of \( G \) is

```maple
> NL:=NormalLattice(G1);
> NL;
Normal subgroup lattice
-----------------------
---
---
[2] Order 64 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
```

By inspection we find that the center of this group is order 1 which indicates that we do not have a central extension. Next we find that the minimal normal subgroup of \( G \) is of order 64. Since the minimal normal subgroup of \( G \) is an abelian p-group, it must be elementary abelian. Thus, \( \text{NL}[2] \) is isomorphic to \( C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 = (C_2)^6 \).

We now have \( G_2 \) is isomorphic to \( (C_2)^6 \). Thus, \( G_1/G_2 = L_2(7) \) gives \( G_1/(C_2)^6 = L_2(7) \). So \( G_1 = (C_2)^6 \cdot L_2(7) \), with \( (C_2)^6 = \text{NL}[2] \) normal in \( G_2 \). Note that \( |L_2(7)| = 168 \). Therefore, we must find a normal subgroup of order 168. By inspection we look at the normal lattice of \( \text{NL}[3] \) to see that it does not have a normal subgroup of order 168. Since \( N = (C_2)^6 \) is an abelian group and is not the center of \( G \) thus \( G_1 \) is a mixed extension. Thus, \( \text{NL}[3] \) is isomorphic to \( 2^6 : L_2(7) \).
Next, we will show that $G_1 = (C_2)^6 : L_2(7)$.

Let $N = (C_2)^6$. We note that $N = \langle k, l, m, n, o, p \rangle = \langle k \rangle \times \langle l \rangle \times \langle m \rangle \times \langle n \rangle \times \langle o \rangle \times \langle p \rangle$ where $k, l, m, n, o$ is of order 2. A presentation for $N$ is $\langle k, l, m, n, o, p | k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o) \rangle$.

Now we have to write elements of $G_1/N = L_2(7)$ in terms of the generators of $N$. Note a presentation for $L_2(7)$ is $\langle r, s | r^2, s^4, (r s)^7, (r, s)^4, (r s^2)^3 \rangle$.

We now find the set of right coset of $N$ in $G_2$. Let $s$ and $t$ denoted by $NT[i]$ where $i$ goes from 1 to 168. There is an isomorphism from $G_1/N$ to $L_2(7)$. In this isomorphism $NT[2] \mapsto r$ and $NT[3] \mapsto s$.

Since the permutations are very large, permutation group acting on a set of cardinality 10750, we use the Schreier System in Magma to find the actions. Note in Magma we must store $T_2 = NT[2], T_3 = NT[3]$, and $T_3^4 = N(T_3)^4$ so they do not change everytime.

```magma
N:=sub<G1|A,B,C,D,E,F>;
NN<k,l,m,n,o,p>:=Group<k,l,m,n,o,p|k^2,l^2,m^2,n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m),(l,n),(l,o),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p)>;
#NN;
64
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..64]];
for i in [2..64] do
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
    if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
    if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
    if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
    if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
  end for;
P:=Id(N);
for k in [1..#P] do
  PP:=P[k];
end for;
ArrayP[i]:=PP;
for k in [1..#P] do
  PP:=PP*P[k];
end for;
```
We let

\[ NT[2] = r \quad \text{and} \quad NT[3] = s. \]

Note that \((NT[3])^4 = N(T[3])^4 = NT34 = N\). We run the following loops to convert the elements of \(L_2(7)\) in terms of the generators of \(N\).

\[
\begin{align*}
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } T34 \text{ then } \text{Sch}[i];
&\quad \text{end if}; \text{ end for;}
&l * m * p
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } (T2,T3)^4 \text{ then } \text{Sch}[i];
&\quad \text{end if}; \text{ end for;}
&k * l * m * o * p
\end{align*}
\]

So \((T[3])^4 \in N\); that is;

\[ s^4 = lmp. \]

Also \((NT[2],NT[3])^4 = N(T[2],T[3])^4 = N\). So \((T[2],T[3])^4 \in N\); that is;

\[ (r,s)^4 = klmop. \]

Thus, the elements of \(L_2(7)\) in terms of the generators of \(N\) are:

\[ s^4 = lmp \quad \text{and} \quad (r,s)^4 = klmop. \]

Now we need to conjugate the generators of \(N\) by the generators of \(L_2(7)\) to determine the resulting elements of \(N\). We use the following loops to find the resulting elements of \(N\).

\[
\begin{align*}
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } A^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[2]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&k * m * o
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } B^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[3]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&n * o * p
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } C^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[4]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&m
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } D^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[5]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&m * n * o
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } E^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[6]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&o
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } F^T2
&\quad \text{then } \text{Sch}[i]; \text{ I}[7]:=\text{Sch}[i]; \text{ end if}; \text{ end for;}
&l * m * n
&> \text{for } i \text{ in } [1..64] \text{ do if } \text{ArrayP}[i] \text{ eq } A^T3
\end{align*}
\]
then Sch[i]; I[8]:=Sch[i]; end if; end for;
k * m * n * o * p
> for i in [1..64] do if ArrayP[i] eq B^T3
then Sch[i]; I[9]:=Sch[i]; end if; end for;
> for i in [1..64] do if ArrayP[i] eq C^T3
then Sch[i]; I[10]:=Sch[i]; end if;end for;
k
> for i in [1..64] do if ArrayP[i] eq D^T3
then Sch[i]; I[11]:=Sch[i]; end if;end for;
> for i in [1..64] do if ArrayP[i] eq E^T3
then Sch[i]; I[12]:=Sch[i]; end if;end for;
p
> for i in [1..64] do if ArrayP[i] eq F^T3
then Sch[i]; I[13]:=Sch[i]; end if;end for;
k * l * m *
k
Thus, $k^r = kmo, l^r = nop, m^r = m, n^r = mno, o^r = o, p^r = lmn, k^s = kmnop, l^s = np, m^s = k, n^s = klnp, o^s = p, and p^s = klmn$. In addition, we check in Magma the presentation of $G_1$:

```
> NN<k,l,m,n,o,p,r,s>:=Group<k,l,m,n,o,p,r,s|k^2,l^2,m^2,
  n^2,o^2,p^2,(k,l),(k,m),(k,n),(k,o),(k,p),(l,m),(l,n),(l,o),
  (l,p),(m,n),(m,o),(m,p),(n,o),(n,p),(o,p),r^2,
  s^4=l*m*p,(r*s)^7,(r,s)^4=k*l*m*o*p,(r*s^2)^3,
  k^r=k*m*o,l^r=n*o*p,m^r=m,
  n^r=m*n*o,o^r=o,p^r=l*m*n,k^s=k*m*n*o*p,
  l^s=n*p,m^s=k,n^s=k*l*n*p,o^s=p,p^s=k*l*m*n>;
> #NN;
10752
> f1,g,k1:=CosetAction(NN,sub<NN|Id(NN)>);
> s,t:=IsIsomorphic(NL[3],g);
> s;
true
```

Thus, we have $G_1$ is isomorphic to $2^6: L_2(7)$. Hence, $G/G_1 = C_2$ gives $G/2^6: L_2(7) = C_2$. So $G = 2^6: L_2(7) \cdot C_2$, with $2^6: L_2(7) = NL[3]$ normal in $G_1$.

Note $C_2$ is not a normal subgroup of $G$, therefore, $G$ cannot be a direct product. By further inspection we find that it must be a semi-direct product. So we find an element of order 2 in $G$ but outside $NL[3]$, say $g$. So we run the following loop:

```
> for g in G1 do if Order(g) eq 2 and sub<G1|NL[3],g> eq G1
```
then \( Z := g \); break; end if; end for;
> \text{G1 eq sub\textless G1|NL[3],Z\textgreater ; true}

Now we use the following loops in Magma to find the action of \( g \) on the generators \( k, l, m, n, o, p, r, s \) of \( \text{NL[3]} \).

> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq A^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; k} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq B^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; k * s * o * s^{-1}} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq C^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; n * o} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq D^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; k * m * o * p} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq E^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; k * o * p} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq T2^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; p} \)
> for \( i \) in \([1..10752]\) do if \( \text{ArrayP[i] eq T3^Z} \) then \( \text{Sch[i]; I[1]:=Sch[i]; end if; end for; r * s * r * k * s * r * s^{-1} * r * s * r} \)

Thus we have:

\[
g^2, k^g = ln, l^g = sos^{-1}, m^g = n^g, n^g = kso^{-1}, o^g = kmp, p^g = p^g, r^g = ksr^2s^{-1}r^{-1}, s^g = rsr^2s^{-1}r.
\]

Hence, we have the following presentation:

\[
H1 < k, l, m, n, o, p, r, s, g > := \text{Group < k, l, m, n, o, p, r, s, g|k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p), r^2, s^4 = lmp, (rs)^7, (r, s)^4 = klmop, (rs^2)^3, k^r = kmo, l^r = nop, m^r = m, n^r = mno, o^r = o, p^r = lmn, k^s = kmp, l^s = np, m^s = k, n^s = klnp, o^s = p, p^s = klmn, g^2, k^g = ln, l^g = sos^{-1}, m^g = n^g, n^g = kso^{-1},}
\]
\( o^g = mnp, p^g = p^r, r^g = krsrs^{-1}rs^{-1}, s^g = rksrsrs^{-1}r >. \)

Finally, we check if it is isomorphic to \( G_1 \).

```plaintext
> #H1;
21504
> f, h1, k1 := CosetAction(H1, sub<H1|Id(H1)>);
> s := IsIsomorphic(h1, G1);
> s;
true
```

Thus we have solved the extension problem for this group and we can conclude that
\( G = (2^6 : L_2(7)) : 2 \)

\( \cong < k, l, m, n, o, p, r, s, g | k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p), r^2, s^4 = lmp, (rs)^7, (r, s)^4 = klmop, (rs^2)^3, k^r = kmo, l^r = nop, m^r = m, n^r = mno, o^r = o, p^r = lmn, k^s = kmnop, l^s = np, m^s = k, n^s = klnp, o^s = p, p^s = kmnn, g^2, k^g = ln, l^g = sos^{-1}, m^g = n^g = kso^{-1}, o^g = kmp, p^g = p^s, r^g = ksrmlsrs^{-1}rs^{-1}, s^g = rksrsrs^{-1}r >. \)

\( \square \)
Chapter 6

Double Coset Enumeration of $M_{11}$ over $S_4$

6.1 $G$ Factor by a Subgroup of Order 12

Consider the group $G \cong <a, b, c, d, t|a^2, b^3, c^4, d^4, b^{-1}aba, c^{-1}ac^{-1}, d^{-1}ad^{-1}, bc^{-1}b^{-1}d^{-1},
\frac{1}{2}d^{-1}cd^{-1}, d^{-1}c^{-1}b^{-1}cb, t^2, (t, b), (ctc)^6, (abct^{-1})^6, (abt^{-1})^3 >$. Note $N = S_4$, where
$a \sim (1, 5)(2, 8)(3, 6)(4, 7)$, $b \sim (1, 2, 4)(5, 8, 7)$, $c \sim (1, 4, 5, 7)(2, 6, 8, 3)$ and
$d \sim (1, 3, 5, 6)(2, 4, 8, 7)$. Now we look at the composition factors of this group:

\[
\begin{array}{c|c}
G & M_{11} \\
| & Cyclic(3) \\
| & Cyclic(2) \\
| & Cyclic(2) \\
1 & \\
\end{array}
\]

Note the center of $G$ is of order 1. Now we look at the normal lattice of $G$:

Normal subgroup lattice

\[
\begin{array}{c|c}
[5] & Order 95040 Length 1 Maximal Subgroups: 3 5 \\
--- & \\
[6] & Order 31680 Length 1 Maximal Subgroups: 2 4 \\
\end{array}
\]
We can see clearly that [3] is of order 12, therefore we are going to factor $G$ by a subgroup of order 12, to obtain $G \cong M_{11}$:

```plaintext
> q, ff := quo<G1|NL[3]>;
> CompositionFactors(q);

G
| M11
1
```

Now, we convert the action of the generators of [3] into word, to do so, we use the Shreier System:

```plaintext
x := NL[3].1;
y := NL[3].2;
z := NL[3].3;
A := f(a);
B := f(b);
C := f(c);
D := f(d);
E := f(t);
N := sub<G1|A, B, C, D, E>;
e := 0; f := 0; g := 0; h := 0; i := 0; j := 0; k := 6; l := 0; m := 6; n := 3; o := 0;
NN<a, b, c, d, t> := Group<a, b, c, d, t | a^2, b^3, c^4, d^4, b^(-1)*a*b*a,
c^(-1)*a*c^(-1), d^(-1)*a*d^(-1), b*c^(-1)*b^(-1)*d^(-1),
c^(-1)*d^(-1)*c*d^(-1), d^(-1)*c^(-1)*b^(-1)*c*b, t^2, (t, b), (c*t*c)^6,
(a*b*t*(c*b^(-1)))^6, (a*b^(-1)*t)^3,>;
Sch := ShreierSystem(NN, sub<NN|Id(NN)>);
ArrayP := [Id(N): i in [1..#N]];
for i in [2..#N] do
P := [Id(N): i in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j] := A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j] := B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j] := B^(-1); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j] := C; end if;
```

if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
>for i in [1..#N] do if ArrayP[i] eq x then print Sch[i];
end if; end for;
b * d * t * c^-1 * t * c * d^-1 * t * d * t * c^-1 * t
>for i in [1..#N] do if ArrayP[i] eq y then print Sch[i];
end if; end for;
b * d * t * d * t * c * t * d^-1 * t * b^-1 * c * t * c
* t * d * t * c^-1 * t
>for i in [1..#N] do if ArrayP[i] eq z then print Sch[i];
end if; end for;
b * c * t * c * t * d^-1 * t * c * t * d^-1 * t * d^-1
* t * b^-1 * c^-1 * t * c^-1 * t

Thus, by factoring the group of $G$ by a subgroup of order 12, we obtain the following:
$G \cong <a, b, c, d, t|a^2, b^3, c^4, d^4, b^{-1}aba, c^{-1}ac^{-1}, d^{-1}ad^{-1}, bc^{-1}b^{-1}d^{-1},$
c^{-1}d^{-1}cd^{-1}, d^{-1}c^{-1}b^{-1}cb, t^2, (t, b), (ct)^6, (abt^{b^{-1}})^6, (ab^{-1}t)^3, bdte^{-1}ctd^{-1}tdte^{-1}t,$
$bdtdctd^{-1}tb^{-1}ctdtdc^{-1}t, bctc^{-1}ctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t \cong M_{11}.$

6.2 Construction of $M_{11}$ over $S_4$

We start by factoring the progenitor $2^{*8} : S_4$ by the relations
$(ct)^6, (abt^{b^{-1}})^6, (ab^{-1}t)^3,$
$bdte^{-1}ctd^{-1}tdte^{-1}t, bdtdctd^{-1}tb^{-1}ctdtdc^{-1}t, bctc^{-1}ctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t$
to obtain the homomorphic image $G \cong M_{11}$, where
$a \sim (1,5)(2,8)(3,6)(4,7), b \sim (1,2,4)(5,8,7), c \sim (1,4,5,7)(2,6,8,3),$ 
d \sim (1,3,5,6)(2,4,8,7), and $t \sim t_0 \sim t_3$. The index of $S_4$ in $G$ equals 330. Now we expand the relations:
$1 = (ct)^6 = (ct_2)^6 = c^6t_2^3t_2^4t_2^3t_0^2t_2 = c^2t_6t_2t_8t_3t_2t_6$
$\implies c^2t_6t_2 = t_2t_6t_8t_0,$
\[ 1 = (abt^{cb^{-1}})^6 = (abt_1)^6 = (ab)^6t_1(ab)^5t_1(ab)^4t_1(ab)^3t_1(ab)^2t_1 = t_7t_2t_5t_4t_8t_1 \]
\[ \implies t_7t_2 = t_1t_5t_4t_5, \]
\[ 1 = (ab^{-1}t)^3 = (ab^{-1})^3t_0(ab^{-1})t_0 = at_6t_0 \]
\[ \implies at_0 = t_0t_6, \]
\[ 1 = bdtc^{-1}tcd^{-1}tcd^{-1}t = bdtc^{-1}t_8t_0t_4t_8t_3 \]
\[ \implies bdc^{-1}t_8t_0 = t_3t_8t_4, \]
\[ 1 = bdtdctic^{-1}tcd^{-1}tcd^{-1}t = dt_8t_1t_0t_7t_1t_4t_8t_0 \]
\[ \implies dt_8t_1t_0 = t_0t_8t_4t_1, \]
\[ 1 = bctctd^{-1}tcd^{-1}tcd^{-1}t = c^{-1}t_4t_2t_1t_0t_7t_6t_8t_0 \]
\[ \implies c^{-1}t_4t_2t_1t_0 = t_0t_8t_6t_7. \]

We want to find the index of \( N \) in \( G \). To do this, we perform a manual double coset enumeration of \( G \) over \( N \). We take \( G \) and express it as a union of double cosets \( NgN \), where \( g \) is an element of \( G \). So \( G = NeN \cup Ng_1N \cup Ng_2N \cup ... \) where \( g_i \)’s words in \( t_i \)’s.

We need to find all double cosets \([w]\) and find out how many single cosets each of them contains, where \([w] = [Nw^n | n \in N]\). The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by \( t_i \)’s. We will identify, for each \([w]\), the double coset to which \( Nwt_i \) belongs for one symmetric generator \( t_i \) from each orbit of the coset stabilising group \( N^{(w)} \)

\[ NeN \]

First, the double coset \( NeN \), is denoted by \([*]\). This double coset contains only the single coset, namely \( N \). Since \( N \) is transitive on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\} \), the orbit of \( N \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\} \) is:

\[ \mathcal{O} = \{1, 2, 0, 4, 5, 6, 7, 8\}. \]

We choose \( t_0 \) as our symmetric generator from \( \mathcal{O} \) and find to which double coset \( Nt_0 \) belongs. \( Nt_0N \) will be a new double coset, denote it \([0]\).

\[ Nt_0N \]

In order to find how many single cosets \([0]\) contains, we must first find the coset stabiliser \( N^{(0)} \). Then the number of single coset in \([0]\) is equal to \( \frac{|N|}{|N^{(0)}|} \). Now,

\[ N^{(0)} = N^0 = \langle (1, 2, 4)(5, 8, 7) \rangle \]

so the number of the single cosets in \( Nt_0N \) is \( \frac{|N|}{|N^{(0)}|} = \frac{24}{3} = 8 \). Furthermore, the orbits
of $N^{(0)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7 t_8\}$ are:

$$\mathcal{O} = \{\{1, 2, 4\}, \{5, 8, 7\}, \{0\}, \{6\}\}.$$  

We take $t_1, t_5, t_0$, and $t_6$ from each orbit, respectively and to see which double coset $N t_0 t_1, N t_0 t_5, N t_0 t_0$, and $N t_0 t_6$ belong to. We have:

$$N t_0 t_1 \in [01] \quad N t_0 t_5 \in [05] \quad N t_0 t_0 = N \in \star$$

$$at_0 t_6 = t_0 \implies N t_0 t_6 = N t_0 \in [0]$$

The new double cosets have single coset representatives $N t_0 t_1$ and $N t_0 t_5$, which we represent them as $[01]$ and $[05]$ respectively.

**$N t_0 t_1 N$**

Consider $N t_0 t_1 N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(01)} = N^0 = < e >$. Only identity ($e$) will fix 0 and 1. Hence the number of single cosets living in $N t_0 t_1 N$ is $\frac{|N|}{|N^{(01)}|} = 24 \div 1 = 24$. The orbits of $N^{(01)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\mathcal{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$  

Take a representative $t_i$ from each orbit and see which double cosets $N t_0 t_1 t_i$ belongs to. We have:

$$N t_0 t_1 t_1 \in [0] \quad N t_0 t_1 t_2 \in [012] \quad N t_0 t_1 t_0 \in [010] \quad N t_0 t_1 t_4 \in [014] \quad t_0 t_1 t_5 = at_0 t_1 \implies N t_0 t_1 t_5 = N t_6 t_1 \in [05]$$

$$N t_0 t_1 t_6 \in [016] \quad t_0 t_1 t_7 = bd^{-1} t_1 t_0 \implies N t_0 t_1 t_7 = N t_1 t_0 \in [05]$$

$$N t_0 t_1 t_8 \in [018].$$

The new double coset are $N t_0 t_1 t_2 N, N t_0 t_1 t_0 N, N t_0 t_1 t_4 N, N t_0 t_1 t_6 N$ and $N t_0 t_1 t_8 N$, which we represent them as $[012], [010], [014], [016]$, and $[018]$ respectively.

**$N t_0 t_5 N$**

Consider $N t_0 t_5 N$ is a new double coset. We determine how many single cosets are in
the double coset. However, \( N^{(05)} = N^{05} = < e > \). Only identity (e) will fix 0 and 5. Hence the number of single cosets living in \( Nt_{05}t_{5}N \) is \( \frac{|N|}{|N^{(05)}|} = \frac{24}{1} = 24 \). The orbits of \( N^{(05)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:

\[
\{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.
\]

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_{05}t_{5}t_i \) belongs to. We have:

\[
\begin{align*}
t_{05}t_1 &= at_{65} \implies Nt_{05}t_1 = Nt_6t_5 \in [01] \\
t_{05}t_2 &= bct_{50} \implies Nt_{05}t_2 = Nt_5t_0 \in [01] \\
Nt_{05}t_0 \in [050] \\
t_{05}t_4 &= d^{-1}b^{-1}t_7t_5t_6 \implies Nt_{05}t_4 = Nt_7t_5t_6 \in [018] \\
Nt_{05}t_5 &= Nt_0 \in [0] \\
t_{05}t_6 &= at_{50}t_1 \implies Nt_{05}t_6 = Nt_5t_{01} \in [016] \\
t_{05}t_7 &= t_4t_1t_6 \implies Nt_{01}t_7 = Nt_4t_1t_6 \in [012] \\
t_{05}t_8 &= bc^{-1}t_1t_6t_2 \implies Nt_{05}t_8 = Nt_1t_6t_2 \in [014].
\end{align*}
\]

The new double coset is \( Nt_{05}t_{50} \), denoted by [050].

\( Nt_{01}t_0N \)

Consider \( Nt_{01}t_0N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(010)} = N^{010} = < e > \). Only identity (e) will fix 0 and 1. Hence the number of single cosets living in \( Nt_{01}t_0N \) is \( \frac{|N|}{|N^{(010)}|} = \frac{24}{1} = 24 \). The orbits of \( N^{(010)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:

\[
\{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.
\]

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_{01}t_0t_i \) belongs to. We have:

\[
\begin{align*}
Nt_{01}t_0t_1 &\in [0101] \\
Nt_{01}t_0t_2 &\in [0102] \\
Nt_{01}t_0t_0 &\in [01] \\
Nt_{01}t_0t_4 &\in [0104]
\end{align*}
\]
$t_0t_1t_0t_5 = t_0t_1t_6 \implies Nt_0t_1t_0t_5 = Nt_0t_1t_6 \in [050]
$ $t_0t_1t_0t_6 = at_0t_5t_0 \implies Nt_0t_1t_0t_6 = Nt_0t_5t_0 \in [016]
$ $t_0t_1t_0t_7 = bd^{-1}t_1t_3t_1 \implies Nt_0t_1t_0t_7 = Nt_1t_3t_1 \in [050]
$ $Nt_0t_1t_0t_8 \in [0108].$

The new double coset are $Nt_0t_1t_0t_1N$, $Nt_0t_1t_0t_2N$, $Nt_0t_1t_0t_4N$, and $Nt_0t_1t_0t_8N$, which we represent them as $[0101]$, $[0102]$, $[0104]$, and $[0108]$ respectively.

$Nt_0t_1t_2N$

Consider $Nt_0t_1t_2N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(012)} = N^{012} = < e >$. Only identity (e) will fix 0, 1, and 2. Hence the number of single cosets living in $Nt_0t_1t_2N$ is $\frac{|N|}{|N^{(012)}|} = \frac{24}{1} = 24$. The orbits of $N^{(012)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$\bigcup = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_1t_2t_i$ belongs to. We have:

$\begin{align*}
t_0t_1t_2t_1 = bct_5\bar{t}_0t_5t_8 & \implies Nt_0t_1t_2t_1 = Nt_5t_0t_5t_8 \in [0104] \\
Nt_0t_1t_2t_2 \in [01] \\
t_0t_1t_2t_0 = b^{-1}d t_0t_1t_8 & \implies Nt_0t_1t_2t_0 = Nt_0t_1t_8 \in [018] \\
t_0t_1t_2t_4 = b^{-1}t_0t_2t_1 & \implies Nt_0t_1t_2t_4 = Nt_0t_2t_1 \in [014] \\
t_0t_1t_2t_5 = bc^{-1}t_2t_5t_2t_6 & \implies Nt_0t_1t_2t_5 = Nt_2t_5t_2t_6 \in [0104] \\
t_0t_1t_2t_6 = t_8t_5 & \implies Nt_0t_1t_2t_6 = Nt_8t_5 \in [05] \\
Nt_0t_1t_2t_7 \in [0127] \\
t_0t_1t_2t_8 = at_0t_5t_2 & \implies Nt_0t_1t_2t_8 = Nt_0t_5t_2 \in [018].
\end{align*}$

The new double coset is $Nt_0t_1t_2t_7N$, denoted by $[0127]$.

$Nt_0t_1t_4N$

Consider $Nt_0t_1t_4N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(014)} = N^{014} = < e >$. Only identity (e) will fix 0, 1, and 4. Hence the number of single cosets living in $Nt_0t_1t_4N$ is $\frac{|N|}{|N^{(014)}|} = \frac{24}{1} = 24$. The orbits of $N^{(014)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$\bigcup = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$
Take a representative $t_i$ from each orbit and see which double cosets $N_{t_0t_1t_4t_i}$ belongs to. We have:

$$t_0t_1t_4t_1 = t_7t_5t_7t_0 \implies N_{t_0t_1t_4t_1} = N_{t_7t_5t_7t_0} \in [0102]$$

$$t_0t_1t_4t_2 = bt_0t_4t_1 \implies N_{t_0t_1t_4t_2} = N_{t_0t_4t_1} \in [012]$$

$$t_0t_1t_4t_0 = bd^{-1}t_4t_6t_1 \implies N_{t_0t_1t_4t_0} = N_{t_4t_6t_1} \in [014]$$

$$N_{t_0t_1t_4t_1} \in [01]$$

$$t_0t_1t_4t_5 = d{-1}t_5t_4t_3 \implies N_{t_0t_1t_4t_5} = N_{t_5t_4t_3} \in [014]$$

$$t_0t_1t_4t_6 = bdt_1t_6t_1t_7 \implies N_{t_0t_1t_4t_6} = N_{t_1t_6t_1t_7} \in [0102]$$

$$t_0t_1t_4t_7 = d{-1}b{-1}t_5t_6 \implies N_{t_0t_1t_4t_7} = N_{t_5t_6} \in [05]$$

$$t_0t_1t_4t_8 = bat_0t_7t_5t_2 \implies N_{t_0t_1t_4t_8} = N_{t_6t_7t_5t_2} \in [0127].$$

$N_{t_0t_1t_6t}$

Consider $N_{t_0t_1t_6N}$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(016)} = N^{016} = <e>$. Only identity (e) will fix 0,1, and 6. Hence the number of single cosets living in $N_{t_0t_1t_6N}$ is $\frac{|N|}{|N^{(016)}|} = \frac{24}{1} = 24$. The orbits of $N^{(016)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\emptyset = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$

Take a representative $t_i$ from each orbit and see which double cosets $N_{t_0t_1t_6t_i}$ belongs to. We have:

$$t_0t_1t_6t_1 = t_5t_6t_5 \implies N_{t_0t_1t_6t_1} = N_{t_5t_6t_5} \in [050]$$

$$t_0t_1t_6t_2 = bc^{-1}t_6t_5t_0 \implies N_{t_0t_1t_6t_2} = N_{t_6t_5t_0} \in [016]$$

$$t_0t_1t_6t_0 = at_6t_5t_6 \implies N_{t_0t_1t_6t_0} = N_{t_6t_5t_6} \in [010]$$

$$t_0t_1t_6t_4 = bd^{-1}t_6t_7t_1 \implies N_{t_0t_1t_6t_4} = N_{t_6t_7t_1} \in [018]$$

$$t_0t_1t_6t_5 = at_0 \implies N_{t_0t_1t_6t_5} = N_{t_1t_0} \in [05]$$

$$N_{t_0t_1t_6t_6} \in [01]$$

$$t_0t_1t_6t_7 = t_0t_4t_5 \implies N_{t_0t_1t_6t_7} = N_{t_0t_4t_5} \in [0108]$$

$$t_0t_1t_6t_8 = cb^{-1}t_6t_5t_0 \implies N_{t_0t_1t_6t_8} = N_{t_6t_5t_0} \in [016].$$

$N_{t_0t_1t_8N}$

Consider $N_{t_0t_1t_8N}$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(018)} = N^{018} = <e>$. Only identity (e) will fix 0,1, and 8.
Hence the number of single cosets living in $Nt_0t_1t_8N$ is \( \frac{|N|}{|N_{(018)}|} = \frac{24}{1} = 24 \). The orbits of $N^{(018)}$ on \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} are:
\[\varnothing = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.\]

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_1t_8t_i$ belongs to. We have:
\[
\begin{align*}
t_0t_1t_8t_1 &= cb^{-1}t_0t_1t_0t_8 \implies Nt_0t_1t_8t_1 = Nt_0t_1t_0t_8 \in [010] \\
t_0t_1t_8t_2 &= at_6t_5t_8 \implies Nt_0t_1t_8t_2 = Nt_0t_5t_8 \in [012] \\
t_0t_1t_8t_0 &= bct_0t_1t_2 \implies Nt_0t_1t_8t_0 = Nt_0t_1t_2 \in [012] \\
t_0t_1t_8t_4 &= b^{-1}t_8t_7t_6t_5 \implies Nt_0t_1t_8t_4 = Nt_5t_7t_6t_5 \in [0187] \\
t_0t_1t_8t_5 &= b^{-1}dt_6t_8t_0 \implies Nt_0t_1t_8t_5 = Nt_6t_5t_0 \in [016] \\
t_0t_1t_8t_6 &= bc^{-1}t_2t_1 \implies Nt_0t_1t_8t_6 = Nt_2t_1 \in [05] \\
Nt_0t_1t_8t_7 &\in [0187] \\
Nt_0t_1t_8t_8 &\in [01].
\end{align*}
\]

The new double coset is $Nt_0t_1t_8t_7N$, denoted by $[0187]$.

$Nt_0t_5t_0N$

Consider $Nt_0t_5t_0N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(050)} = N^{(050)} = <e>$. Only identity (e) will fix 0, and 5.

Hence the number of single cosets living in $Nt_0t_5t_0N$ is \( \frac{|N|}{|N_{(050)}|} = \frac{24}{1} = 24 \). The orbits of $N^{(050)}$ on \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} are:
\[\varnothing = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.\]

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_5t_0t_i$ belongs to. We have:
\[
\begin{align*}
t_0t_5t_0t_1 &= t_6t_5t_6 \implies Nt_0t_5t_0t_1 = Nt_6t_5t_6 \in [010] \\
t_0t_5t_0t_2 &= b^{-1}dt_5t_0t_5 \implies Nt_0t_5t_0t_2 = Nt_5t_3t_5 \in [010] \\
Nt_0t_5t_0t_0 &\in [05] \\
t_0t_5t_0t_4 &= at_7t_5t_7t_6 \implies Nt_0t_5t_0t_4 = Nt_7t_5t_7t_6 \in [0108] \\
t_0t_5t_0t_5 &= at_0t_1t_0t_1 \implies Nt_0t_5t_0t_5 = Nt_0t_1t_0t_1 \in [0101] \\
t_0t_5t_0t_6 &= t_1t_6t_5 \implies Nt_0t_5t_0t_6 = Nt_1t_6t_5 \in [016]
\end{align*}
\]
\[ t_{0507} = db^{-1}t_{446} \implies Nt_{0t_5}t_{0t_7} = Nt_{446}t_6 \in [0102] \]
\[ t_{0508} = cb^{-1}t_{162} \implies Nt_{0t_5}t_{0t_8} = Nt_{162}t_2 \in [0104]. \]

\[ Nt_0t_4t_1N \]

Consider \( Nt_0t_1t_0t_1N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(0101)} = N^{0101} = <e> \). But \( Nt_0t_1t_0t_1 \) is not distinct. We have \( Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4 \). Thus, there exist \( \{n \in N|N(t_0t_1t_0)^n = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4\} \) such that

\[ Nt_0t_1t_0t_1^{(1,2,4)(5,8,7)} = Nt_0t_2t_0t_2 \implies (1, 2, 4)(5, 8, 7) \in N^{(0101)} \]
\[ Nt_0t_2t_0t_2^{(1,2,4)(5,8,7)} = Nt_0t_4t_0t_4 \implies (1, 2, 4)(5, 8, 7) \in N^{(0101)} \]
\[ \implies Nt_0t_1t_0t_1 = Nt_0t_2t_0t_2 = Nt_0t_4t_0t_4. \]

Thus, \( (1, 2, 4)(5, 8, 7) \in N^{(0101)} \). We conclude:

\[ N^{(0101)} \geq (1, 2, 4)(5, 8, 7) \]

so the number of the single cosets in \( Nt_0t_1t_0t_1N \) is \( \frac{|N|}{|N^{(0101)}|} = \frac{24}{3} = 8 \). Furthermore, the orbits of \( N^{(0101)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7t_8\} \) are:

\( \emptyset = \{\{1, 2, 4\}, \{5, 8, 7\}, \{0\}, \{6\}\} \).

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_1t_0t_1t_i \) belongs to. We have:

\[ Nt_0t_1t_0t_1t_1 \in [010] \]
\[ t_0t_1t_0t_1t_0 = at_0t_5t_0 \implies Nt_0t_1t_0t_1t_0 = Nt_0t_5t_0 \in [050] \]
\[ Nt_0t_1t_0t_1t_0 \in [01010] \]
\[ t_0t_1t_0t_1t_0 = at_0t_5t_0 \implies Nt_0t_1t_0t_1t_0 = Nt_0t_1t_0t_1 \in [0101] \]

The new double coset is \( Nt_0t_1t_0t_0t_0N \), denoted by [01010].

\[ Nt_0t_1t_0t_2N \]

Consider \( Nt_0t_1t_0t_2N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(0102)} = N^{0102} = <e> \). Only identity (e) will fix 0, 1, and 2. Hence the number of single cosets living in \( Nt_0t_1t_0t_2N \) is \( \frac{|N|}{|N^{(0102)}|} = \frac{24}{1} = 24 \). The orbits of \( N^{(0102)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:

\( \emptyset = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\} \).

Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_1t_0t_2t_i \) belongs
to. We have:
\[ t_0t_1t_0t_2t_1 = c b^{-1} t_5 t_0 t_8 \implies N t_0 t_1 t_0 t_2 t_1 = N t_5 t_0 t_8 \in [014] \]
\[ N t_0 t_1 t_0 t_2 t_2 \in [010] \]
\[ t_0 t_1 t_0 t_2 t_0 = b c t_0 t_1 t_0 t_8 \implies N t_0 t_1 t_0 t_2 t_0 = N t_0 t_1 t_0 t_8 \in [0108] \]
\[ t_0 t_1 t_0 t_2 t_4 = c d^{-1} t_0 t_2 t_0 t_1 \implies N t_0 t_1 t_0 t_2 t_4 = N t_0 t_2 t_0 t_1 \in [0104] \]
\[ t_0 t_1 t_0 t_2 t_5 = t_2 t_5 t_6 \implies N t_0 t_1 t_0 t_2 t_5 = N t_2 t_5 t_6 \in [014] \]
\[ t_0 t_1 t_0 t_2 t_6 = b c t_8 t_5 t_8 \implies N t_0 t_1 t_0 t_2 t_6 = N t_8 t_5 t_8 \in [050] \]
\[ t_0 t_1 t_0 t_2 t_7 = b t_5 t_7 t_2 \implies N t_0 t_1 t_0 t_2 t_7 = N t_5 t_7 t_2 \in [0187] \]
\[ t_0 t_1 t_0 t_2 t_8 = a t_6 t_5 t_6 t_2 \implies N t_0 t_1 t_0 t_2 t_8 = N t_6 t_5 t_6 t_2 \in [0108]. \]

\[ N t_0 t_1 t_0 t_4 N \]

Consider \( N t_0 t_1 t_0 t_4 N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(0104)} = N^{0104} = \langle e \rangle \). Only identity (e) will fix 0,1, and 4. Hence the number of single cosets living in \( N t_0 t_1 t_0 t_4 N \) is \( \frac{|N|}{|N^{(0104)}|} = \frac{24}{1} = 24. \)

The orbits of \( N^{(0104)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:
\[ \emptyset = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( N t_0 t_1 t_0 t_4 t_i \) belongs to. We have:
\[ t_0 t_1 t_0 t_4 t_1 = d^{-1} b^{-1} t_7 t_5 t_0 \implies N t_0 t_1 t_0 t_4 t_1 = N t_7 t_5 t_0 \in [012] \]
\[ t_0 t_1 t_0 t_4 t_2 = c t_0 t_4 t_0 t_1 \implies N t_0 t_1 t_0 t_4 t_2 = N t_0 t_4 t_0 t_1 \in [0102] \]
\[ t_0 t_1 t_0 t_4 t_3 = t_4 t_6 t_4 t_1 \implies N t_0 t_1 t_0 t_4 t_3 = N t_4 t_6 t_4 t_1 \in [0104] \]
\[ N t_0 t_1 t_0 t_4 t_4 \in [010] \]
\[ t_0 t_1 t_0 t_4 t_5 = t_5 t_4 t_5 t_3 \implies N t_0 t_1 t_0 t_4 t_5 = N t_5 t_4 t_5 t_3 \in [0104] \]
\[ t_0 t_1 t_0 t_4 t_6 = d b^{-1} t_1 t_6 t_7 \implies N t_0 t_1 t_0 t_4 t_6 = N t_1 t_6 t_7 \in [012] \]
\[ t_0 t_1 t_0 t_4 t_7 = b d t_5 t_6 t_5 \implies N t_0 t_1 t_0 t_4 t_7 = N t_5 t_6 t_5 \in [050] \]
\[ t_0 t_1 t_0 t_4 t_8 = d b^{-1} t_7 t_2 t_5 t_6 \implies N t_0 t_1 t_0 t_4 t_8 = N t_7 t_2 t_5 t_6 \in [0187]. \]

\[ N t_0 t_1 t_0 t_8 N \]

Consider \( N t_0 t_1 t_0 t_8 N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(0108)} = N^{0108} = \langle e \rangle \). Only identity (e) will fix 0,1, and 8. Hence the number of single cosets living in \( N t_0 t_1 t_0 t_8 N \) is \( \frac{|N|}{|N^{(0108)}|} = \frac{24}{1} = 24. \)
The orbits of $N^{(0108)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\varnothing = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$ 

Take a representative $t_i$ from each orbit and see which double cosets $N_{t_0}t_1t_0t_8t_i$ belongs to. We have:

$t_0t_1t_0t_8t_1 = b^{-1}t_0t_1t_8 \implies N_{t_0}t_1t_0t_8t_1 = N_{t_0}t_1t_8 \in [018]$ 
$t_0t_1t_0t_8t_2 = a_{t_0}t_5t_8 \implies N_{t_0}t_1t_0t_8t_2 = N_{t_0}t_5t_8 \in [0102]$ 
$t_0t_1t_0t_8t_0 = b^{-1}d_{t_0}t_1t_0t_2 \implies N_{t_0}t_1t_0t_8t_0 = N_{t_0}t_1t_0t_2 \in [0102]$ 
$t_0t_1t_0t_8t_4 = d^{-1}t_7t_2t_5t_6 \implies N_{t_0}t_1t_0t_8t_4 = N_{t_7}t_2t_5t_6 \in [0127]$ 
$t_0t_1t_0t_8t_5 = t_0t_2t_6 \implies N_{t_0}t_1t_0t_8t_5 = N_{t_0}t_2t_6 \in [016]$ 
$t_0t_1t_0t_8t_6 = a_{t_2}t_1t_2 \implies N_{t_0}t_1t_0t_8t_6 = N_{t_2}t_1t_2 \in [050]$ 
$t_0t_1t_0t_8t_7 = ab_{t_1}t_0t_7t_8 \implies N_{t_0}t_1t_0t_8t_7 = N_{t_1}t_0t_7t_8 \in [0127]$ 

$N_{t_0}t_1t_0t_8t_8 \in [010]$.

$N_{t_0}t_1t_2t_7N$

Consider $N_{t_0}t_1t_2t_7N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(0127)} = N^{0127} = < e >$. Only identity (e) will fix 0,1,7, and 2. Hence the number of single cosets living in $N_{t_0}t_1t_2t_7N$ is $\frac{|N|}{|N^{(0127)}|} = \frac{24}{1} = 24$.

The orbits of $N^{(0127)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$$\varnothing = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.$$ 

Take a representative $t_i$ from each orbit and see which double cosets $N_{t_0}t_1t_2t_7t_i$ belongs to. We have:

$t_0t_1t_2t_7t_1 = a_{t_7}t_5t_0t_8 \implies N_{t_0}t_1t_2t_7t_1 = N_{t_7}t_5t_0t_8 \in [0127]$ 
$t_0t_1t_2t_7t_2 = b^{-1}c_{t_7}t_5t_7t_6 \implies N_{t_0}t_1t_2t_7t_2 = N_{t_7}t_5t_7t_6 \in [0108]$ 
$t_0t_1t_2t_7t_0 = d^{-1}b^{-1}t_2t_4t_1t_0 \implies N_{t_0}t_1t_2t_7t_0 = N_{t_2}t_4t_1t_0 \in [0127]$ 
$t_0t_1t_2t_7t_4 = b^{-1}a_{t_0}t_8t_5 \implies N_{t_0}t_1t_2t_7t_4 = N_{t_0}t_8t_5 \in [014]$ 
$t_0t_1t_2t_7t_5 = a_{t_2}t_5t_4t_2 \implies N_{t_0}t_1t_2t_7t_5 = N_{t_2}t_5t_4t_2 \in [0127]$ 
$t_0t_1t_2t_7t_6 = c_{t_7}t_4t_5t_5 \implies N_{t_0}t_1t_2t_7t_6 = N_{t_7}t_4t_5t_5 \in [0108]$ 

$N_{t_0}t_1t_2t_7t_7 \in [012]$ 
$t_0t_1t_2t_7t_8 = ba_{t_7}t_0t_5 \implies N_{t_0}t_1t_2t_7t_8 = N_{t_7}t_0t_5 \in [0127]$.

$N_{t_0}t_1t_8t_7N$
Consider \(Nt_0t_1t_8t_7N\) is a new double coset. We determine how many single cosets are in the double coset. However, \(N^{(0187)} = N^{0187} = \langle e \rangle\). Only identity \((e)\) will fix 0, 1, 7, and 8. Hence the number of single cosets living in \(Nt_0t_1t_8t_7N\) is \(\frac{|N|}{|N^{(0187)}|} = \frac{24}{1} = 24\). The orbits of \(N^{(0187)}\) on \(\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}\) are:

\[
\bigcup = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.
\]

Take a representative \(t_i\) from each orbit and see which double cosets \(Nt_0t_1t_8t_7t_i\) belongs to. We have:

\[
t_0t_1t_8t_7t_1 = bct_8t_7t_6t_5 \implies Nt_0t_1t_8t_7t_1 = Nt_8t_7t_6t_5 \in [0187]
\]

\[
t_0t_1t_8t_7t_2 = b^{-1}d^{-1}t_5t_4t_3 \implies Nt_0t_1t_8t_7t_2 = Nt_5t_4t_3 \in [0104]
\]

\[
t_0t_1t_8t_7t_0 = bd^{-1}t_6t_7t_2 \implies Nt_0t_1t_8t_7t_0 = Nt_6t_7t_2 \in [0187]
\]

\[
t_0t_1t_8t_7t_4 = b^{-1}-c^{-1}t_2t_4t_3t_1 \implies Nt_0t_1t_8t_7t_4 = Nt_2t_4t_3t_1 \in [0187]
\]

\[
t_0t_1t_8t_7t_5 = bc^{-1}t_2t_4t_0 \implies Nt_0t_1t_8t_7t_5 = Nt_2t_4t_0 \in [018]
\]

\[
t_0t_1t_8t_7t_6 = b^{-1}dt_6t_8t_4t_1 \implies Nt_0t_1t_8t_7t_6 = Nt_6t_8t_4t_1 \in [0187]
\]

\[
Nt_0t_1t_8t_7t_7 \in [018]
\]

\[
Nt_0t_1t_8t_7t_8 = b^{-1}c^{-1}t_1t_6t_1t_7 \implies t_0t_1t_8t_7t_8 = Nt_1t_6t_1t_7 \in [0102].
\]

\[
Nt_0t_1t_0t_1t_0N
\]

Now \(Nt_0t_1t_0t_1t_0N\) is indeed a new double coset. We determine how many single cosets are in this double coset. We have \(N^{(01010)} = N^{01010} = \langle e \rangle\). \(Nt_0t_1t_0t_1t_0\) has twenty four names. We have the following:

\[
Nt_0t_1t_0t_1t_0(1,2,4)(5,8,7) = Nt_0t_2t_0t_2t_0 \implies (1, 2, 4)(5, 8, 7) \in N^{(01010)}
\]

\[
Nt_0t_2t_0t_2(1,4,5,7)(2,6,8,0) = Nt_2t_6t_2t_6t_2 \implies (1, 4, 5, 7)(2, 6, 8, 0) \in N^{(01010)}
\]

Therefore, \(N^{(01010)} = n \in N|N^{(01010)}|^n = N(01010)\).

Thus, \(N^{(01010)} \geq (1, 2, 4)(5, 8, 7), (1, 4, 5, 7)(2, 6, 8, 0) > \text{then} N^{(01010)} = N\).

Hence \(|N^{(01010)}| = 24\), so the number of single cosets in \(N^{(01010)}\) is \(\frac{|N|}{|N^{(01010)}|} = \frac{24}{24} = 1\). The orbit of \(N^{(01010)}\) on \(\{1, 2, 0, 4, 5, 6, 7, 8\}\) is \(\{1, 2, 0, 4, 5, 6, 7, 8\}\). Take a representative from this orbit, say \(t_0\). Hence \(Nt_0t_1t_0t_1t_0 \in [0101]\). Therefore, eight symmetric generators will go back to \(Nt_0t_1t_0t_1N\).

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of \(N\) in \(G\) is 330. We conclude:

\[
G = N \cup Nt_0N \cup Nt_0t_1N \cup Nt_0t_5N \cup Nt_0t_1t_0N \cup Nt_0t_1t_2N \cup Nt_0t_1t_4N \cup ...
\]
\[ Nt_0t_1t_6N \cup Nt_0t_1t_8N \cup Nt_0t_5t_0N \cup Nt_0t_1t_0t_1N \cup Nt_0t_1t_2N \cup Nt_0t_1t_0t_4N \cup Nt_0t_1t_0t_8N \cup Nt_0t_1t_2t_7N \cup Nt_0t_1t_3t_7N \cup Nt_0t_1t_0t_1t_0N, \]

where

\[ G \cong 2^8 : S_4/(ctc)^6, (ab^tcb)^6, (ab^{-1}t)^3, bdte^{-1}tdte^{-1}t, \]

\[ bdcde^{-1}tdte^{-1}t, bctcd^{-1}tdc^{-1}t, \]

\[ |G| \leq (|N| + \frac{|N|}{N_0|O_{01}|} + \frac{|N|}{N_0|O_{02}|} + \frac{|N|}{N_0|O_{03}|} + \frac{|N|}{N_0|O_{04}|} + \frac{|N|}{N_0|O_{05}|} + \frac{|N|}{N_0|O_{06}|} + \frac{|N|}{N_0|O_{07}|} + \frac{|N|}{N_0|O_{08}|}) \times |N| \]

\[ |G| \leq (1 + 8 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 24 + 1) \times 24 \]

\[ |G| \leq 330 \times 24 \]

\[ |G| \leq 7920. \]

A Cayley diagram that summarizes the above information is given below:

Figure 6.1: Cayley Diagram of \( M_11 \) over \( S_4 \)
Our goal is to apply Iwasawa’s lemma to prove that \( G \cong M_{11} \) over \( S_4 \) is a simple group. However, by inspection, we can see from the Cayley diagram that Iwasawa’s lemma fails since we have imprimitive blocks of size 2. Thus, \( G \cong M_{11} \) over \( S_4 \) is not a simple group.

In the next section, we will look at the maximal subgroup of \( G \cong M_{11} \) and construct a Cayley diagram of \( M_{11} \) over \( M = 2S_4 \).

### 6.3 Construction of \( M_{11} \) over \( M = 2S_4 \)

We start by factoring the progenitor \( 2^8 : S_4 \) by the relations

\[
(ce)^6, (abt^{b^{-1}})^6, (ab^{-1}t)^3,
\]

\[
bdtc^{-1}tcd^{-1}tdc^{-1}t, bdtdctcd^{-1}tb^{-1}ctcdtc^{-1}t, bctcd^{-1}tctd^{-1}td^{-1}tb^{-1}c^{-1}tc^{-1}t
\]

to obtain the homomorphic image \( G \cong M_{11} \), where

\[
a \sim (1,5)(2,8)(3,6)(4,7), b \sim (1,2,4)(5,8,7), c \sim (1,4,5,7)(2,6,8,3),
\]

\[
d \sim (1,3,5,6)(2,4,8,7), \text{ and } t \sim t_0 \sim t_3. \] In the previous section, we expanded the above relations.

Let \( M \) be the group generated by the control group \( N = S_4 \) and \( dt_0t_5t_0t_5t_0 = d * t * d^{-1} * t * d * t * d^{-1} * t * d * t. \) That is,

\[
M = \langle N, dt_0t_5t_0t_5t_0 \rangle, = 2S_4 \text{ where } |M| = 48.
\]

Then \( M \) is the maximal subgroup.

We decompose \( G \) into the double cosets \( MwN \), where \( w \) is a word in \( t_i' s \), via double coset enumeration.

We proceed to do a manual double coset enumeration of \( G \) over \( M \) and \( N \).

Denote \([w]\) to be the double coset \( MwN \), where \( w \) is a word in the \( t_i' s \).

**MeN**

We begin with the double coset \( MeN \), denote \([\ast]\). This double coset contains only one single coset, namely \( M \). The single coset stabilizer of \( M \) is \( N \), which is transitive on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) and therefore has a single orbit,

\[
O = \{1, 2, 0, 4, 5, 6, 7, 8\}.
\]

Take an element from \( O \) say \( t_0 \) and multiply the single coset representative \( M \) by it to obtain \( Mt_0 \). This is a new double coset \( Mt_0N \), denote it \([0]\).
Mt_0N

In order to find how many single cosets \([0]\) contains, we must first find the coset stabiliser \(N^{(0)}\). Then the number of single coset in \([0]\) is equal to \(\frac{|N|}{|N^{(0)}|}\). Now, 

\[N^{(0)} = N^0 = \langle (1, 2, 4)(5, 8, 7) \rangle\]

so the number of the single cosets in \(Mt_0N\) is \(\frac{|N|}{|N^{(0)}|} = \frac{24}{3} = 8\). Furthermore, the orbits of \(N^{(0)}\) on \(\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}\) are:

\[\mathbb{O} = \{\{1, 2, 4\}, \{5, 8, 7\}, \{0\}, \{6\}\}\].

We take \(t_1, t_5, t_0,\) and \(t_6\) from each orbit, respectively and to see which double coset \(Mt_0t_1, Mt_0t_5, Mt_0t_0,\) and \(Mt_0t_6\) belong to. We have:

\[Mt_0t_1 \in [01],\]
\[Mt_0t_5 \in [05],\]
\[Mt_0t_0 = M \in [*],\]
\[at_0t_6 = t_0 \implies Mt_0t_6 = Mt_0 \in [0].\]

The new double cosets have single coset representatives \(Mt_0t_1\) and \(Mt_0t_5\), which we represent them as \([01]\) and \([05]\) respectively.

\(Mt_0t_1N\)

Consider \(Mt_0t_1N\) is a new double coset. We determine how many single cosets are in the double coset. However, \(N^{(01)} = N^{01} = \langle e \rangle\). Only identity \((e)\) will fix 0 and 1. Hence the number of single cosets living in \(Mt_0t_1N\) is \(\frac{|N|}{|N^{(01)}|} = \frac{24}{1} = 24\). The orbits of \(N^{(01)}\) on \(\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}\) are:

\[\mathbb{O} = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}\].

Take a representative \(t_i\) from each orbit and see which double cosets \(Mt_0t_1t_i\) belongs to. We have:

\[Mt_0t_1t_1 \in [0],\]
\[Mt_0t_1t_2 \in [012],\]
\[t_0t_1t_0 = t_0t_1t_0t_1t_0t_1 \implies Mt_0t_1t_0 = Mt_0t_1t_0 \in [01],\]
\[Mt_0t_1t_4 \in [014]\]
\[t_0t_1t_5 = at_6t_1 \implies Mt_0t_1t_5 = Mt_0t_1 \in [05] = \{N(t_0t_5)^n | n \in N\},\]
\[Mt_0t_1t_6 \in [016].\]
\[ t_0 t_1 t_7 = d b^{-1} t_1 t_0 \implies N t_0 t_1 t_7 = N t_1 t_0 \in [05] = \{ N(t_0 t_5)^n \mid n \in N \}, \]
\[ M t_0 t_1 t_8 \in [018]. \]

The new double coset are \( M t_0 t_1 t_2 N, M t_0 t_1 t_4 N, M t_0 t_1 t_6 N \) and \( M t_0 t_1 t_8 N \), which we represent them as \([012], [014], [016]\), and \([018]\) respectively.

**\( M t_0 t_5 N \)**

Consider \( M t_0 t_5 N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(05)} = N^{05} = \langle e \rangle \). Only identity \( (e) \) will fix 0 and 5. Hence the number of single cosets living in \( M t_0 t_5 N \) is \( \frac{|N|}{|N^{05}|} = \frac{24}{1} = 24 \). The orbits of \( N^{(05)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:

\[ \bigcirc = \{ \{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( M t_0 t_5 t_i \) belongs to. We have:

\[ t_0 t_5 t_1 = a t_6 t_5 \implies M t_0 t_5 t_1 = M t_6 t_5 \in [01] = \{ N(t_0 t_5)^n \mid n \in N \}, \]
\[ t_0 t_5 t_2 = b c t_5 t_0 \implies M t_0 t_5 t_2 = M t_5 t_0 \in [01] = \{ N(t_0 t_5)^n \mid n \in N \}, \]
\[ t_0 t_5 t_0 = t_0 t_5 t_0 t_5 t_0 t_5 \implies M t_0 t_5 t_0 = M t_0 t_5 \in [05] \]
(since \( \{ N(t_0 t_5)^n \mid n \in N \} \) and \( t_0 t_5 t_0 t_5 t_0 \in M \)),
\[ t_0 t_5 t_1 = d^{-1} b^{-1} t_7 t_5 t_6 \implies M t_0 t_5 t_1 = M t_7 t_5 t_6 \in [018] = \{ N(t_0 t_1 t_8)^n \mid n \in N \}, \]
\[ M t_0 t_5 t_5 = M t_0 \in [0] \]
\[ t_0 t_5 t_6 = a t_5 t_0 t_1 \implies M t_0 t_5 t_6 = M t_5 t_0 t_1 \in [016] = \{ N(t_0 t_1 t_6)^n \mid n \in N \}, \]
\[ t_0 t_5 t_7 = t_4 t_6 t_6 \implies M t_0 t_5 t_7 = M t_4 t_1 t_6 \in [012] = \{ N(t_0 t_1 t_2)^n \mid n \in N \}, \]
\[ t_0 t_5 t_8 = b c^{-1} t_1 t_6 t_2 \implies M t_0 t_5 t_8 = M t_1 t_6 t_2 \in [014] = \{ N(t_0 t_1 t_4)^n \mid n \in N \}. \]

**\( M t_0 t_1 t_2 N \)**

Consider \( M t_0 t_1 t_2 N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(012)} = N^{012} = \langle e \rangle \). Only identity \( (e) \) will fix 0, 1, and 2. Hence the number of single cosets living in \( M t_0 t_1 t_2 N \) is \( \frac{|N|}{|N^{012}|} = \frac{24}{1} = 24 \). The orbits of \( N^{(012)} \) on \( \{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\} \) are:

\[ \bigcirc = \{ \{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( M t_0 t_1 t_2 t_i \) belongs
to. We have:

$$t_0t_1t_2t_1 = t_0t_2t_0t_2t_0t_5t_0t_8 \implies Mt_0t_1t_2t_1 = Mt_0t_5t_8 \in [014]$$

(since \(\{N(t_0t_1t_4)^n | n \in N\}\) and \(t_0t_2t_0t_2 \in M\),

\[Mt_0t_1t_2t_2 \in [01]\]

\[t_0t_1t_2t_0 = b^{-1}dt_0t_1t_8 \implies Mt_0t_1t_2t_0 = Mt_0t_1t_8 \in [018] = \{N(t_0t_1t_8)^n | n \in N\},\]

\[t_0t_1t_2t_4 = b^{-1}t_0t_2t_1 \implies Mt_0t_1t_2t_4 = Mt_0t_2t_1 \in [014] = \{N(t_0t_1t_4)^n | n \in N\},\]

\[t_0t_1t_2t_5 = t_0t_1t_0t_1t_2t_2t_5t_6 \implies Mt_0t_1t_2t_5 = Mt_2t_5t_6 \in [014]

(since \(\{N(t_0t_1t_4)^n | n \in N\}\) and \(t_0t_3t_0t_1t_0 \in M\),

\[t_0t_1t_2t_6 = t_8t_5 \implies Mt_0t_1t_2t_6 = Mt_8t_5 \in [05] \equiv \{N(t_0t_5)^n | n \in N\},\]

\[Mt_0t_1t_2t_7 \in [0127]\]

\[t_0t_1t_2t_8 = at_0t_5t_2 \implies Mt_0t_1t_2t_8 = Mt_0t_5t_2 \in [018] = \{N(t_0t_1t_8)^n | n \in N\}.

The new double coset is \(Mt_0t_1t_2t_7N\), denoted by \([0127]\).

\[Mt_0t_1t_4N\]

Consider \(Mt_0t_1t_4N\) is a new double coset. We determine how many single cosets are in the double coset. However, \(N^{[014]} = N^{014} = \langle e \rangle\). Only identity \((e)\) will fix 0, 1, and 4.

Hence the number of single cosets living in \(Mt_0t_1t_4N\) is \(\frac{|N|}{|N^{[014]}|} = \frac{24}{1} = 24\). The orbits of \(N^{[014]}\) on \(\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}\) are:

\[\bigcup = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}.

Take a representative \(t_i\) from each orbit and see which double cosets \(Mt_0t_1t_4t_i\) belongs to. We have:

\[t_0t_1t_4t_1 = ct_0t_8t_0t_8t_0t_7t_0t_0 \implies Mt_0t_1t_4t_1 = Mt_7t_5t_0 \in [012]

(since \(\{N(t_0t_1t_2)^n | n \in N\}\) and \(ct_0t_8t_0t_8t_0 \in M\),

\[t_0t_1t_4t_2 = bt_0t_4t_1 \implies Mt_0t_1t_4t_2 = Mt_0t_4t_1 \in [012] = \{N(t_0t_1t_2)^n | n \in N\},\]

\[t_0t_1t_4t_0 = bd^{-1}t_4t_0t_1 \implies Mt_0t_1t_4t_0 = Mt_4t_0t_1 \in [014]

\[Mt_0t_1t_4t_4 \in [01]\]

\[t_0t_1t_4t_5 = db^{-1}t_5t_4t_3 \implies Mt_0t_1t_4t_5 = Mt_5t_4t_3 \in [014] = \{N(t_0t_1t_4)^n | n \in N\},\]
$t_0 t_1 t_4 t_6 = c t_0 t_2 t_0 t_1 t_6 t_7 \implies M t_0 t_1 t_4 t_6 = M t_1 t_6 t_7 \in [012]$

(since $\{N(t_0 t_1 t_2)^n|n \in N\}$ and $c t_0 t_2 t_0 t_2 t_0 \in M$),

$t_0 t_1 t_4 t_7 = d^{-1} b^{-1} t_5 t_6 \implies M t_0 t_1 t_4 t_7 = M t_5 t_6 \in [05] = \{N(t_0 t_5)^n|n \in N\}$,

$t_0 t_1 t_4 t_8 = b a t_0 t_7 t_5 t_2 \implies M t_0 t_1 t_4 t_8 = M t_6 t_7 t_5 t_2 \in [0127] = \{N(t_0 t_1 t_2 t_7)^n|n \in N\}$.

$M t_0 t_1 t_6 N$

Consider $M t_0 t_1 t_6 N$ is a new double coset. We determine how many single cosets are in the double coset. However,

$M t_0 t_1 t_6 = M t_6 t_5 t_0$

Then $N(t_0 t_1 t_0)^{(1,5)(2,8)(0,6)} = N t_0 t_5 t_1$. But $N t_0 t_5 t_1 = N t_0 t_1 t_6 \implies (1,5)(2,8)(0,6) \in N^{(015)}$ since $N(t_0 t_1 t_5)^{(0,6)(1,5)(2,4)} = N t_0 t_5 t_1$

$\implies N^{(015)} \geq \langle (1,5)(2,8)(0,6) \rangle$.

Hence the number of single cosets living in $M t_0 t_1 t_6 N$ is $\frac{|N|}{|N^{(015)}|} = \frac{24}{2} = 24$. The orbits of $N^{(016)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$\bigcup = \{\{1,5\}, \{2,8\}, \{0,6\}, \{4,7\}\}$.

Take a representative $t_i$ from each orbit and see which double cosets $M t_0 t_1 t_6 t_i$ belongs to. We have:

$t_0 t_1 t_6 t_1 = t_0 t_1 t_0 t_1 t_0 t_5 t_6 \implies M t_0 t_1 t_6 t_1 = M t_5 t_6 \in [05]$

(since $\{N(t_0 t_5)^n|n \in N\}$ and $t_0 t_1 t_0 t_1 t_0 \in M$),

$t_0 t_1 t_6 t_2 = t_0 t_8 t_0 t_8 t_0 t_6 t_1 t_6 \implies M t_0 t_1 t_6 t_2 = M t_0 t_1 t_6 \in [016]$

(since $\{N(t_0 t_1 t_6)^n|n \in N\}$ and $t_0 t_8 t_0 t_8 t_0 \in M$),

$t_0 t_1 t_6 t_4 = b d^{-1} t_6 t_7 t_1 \implies M t_0 t_1 t_6 t_5 = t_6 t_7 t_1 \in [018] = \{N(t_0 t_1 t_8)^n|n \in N\}$,

$M t_0 t_1 t_6 t_6 \in [01]$.

$M t_0 t_1 t_8 N$

Consider $M t_0 t_1 t_8 N$ is a new double coset. We determine how many single cosets are in the double coset. However, $N^{(018)} = N^{018} = \langle e \rangle$. Only identity (e) will fix 0,1, and 8.

Hence the number of single cosets living in $M t_0 t_1 t_8 N$ is $\frac{|N|}{|N^{(018)}|} = \frac{24}{1} = 24$. The orbits of $N^{(018)}$ on $\{t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8\}$ are:

$\bigcup = \{\{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}$.

Take a representative $t_i$ from each orbit and see which double cosets $M t_0 t_1 t_8 t_i$ belongs
to. We have:
\[ t_0t_1t_8t_1 = bct_0t_2t_0t_1t_8 \implies Mt_0t_1t_8t_1 = Mt_0t_1t_0t_8 \in [018] \]
(since \( \{ N(t_0t_1t_8)^n | n \in N \} \) and \( bct_0t_2t_0t_2 \in M \),
\[ t_0t_1t_8t_2 = at_0t_5t_8 \implies Mt_0t_1t_8t_2 = Mt_0t_5t_8 \in [012] = \{ N(t_0t_1t_2)^n | n \in N \}, \]
\[ t_0t_1t_8t_0 = bct_0t_1t_2 \implies Mt_0t_1t_8t_0 = Mt_0t_1t_2 \in [012] = \{ N(t_0t_1t_2)^n | n \in N \}, \]
\[ t_0t_1t_8t_4 = ct_0t_2t_0t_2t_0t_7t_5t_6 \implies Mt_0t_1t_8t_4 = Mt_0t_1t_8t_6 \in [0127] \]
(since \( \{ N(t_0t_1t_7)^n | n \in N \} \) and \( ct_0t_2t_0t_2t_0 \in M \),
\[ t_0t_1t_8t_5 = t_0t_1t_0t_0t_0t_2t_6 \implies Mt_0t_1t_8t_5 = Mt_0t_2t_6 \in [016] \]
(since \( \{ N(t_0t_1t_6)^n | n \in N \} \) and \( t_0t_1t_0t_1t_0 \in M \),
\[ t_0t_1t_8t_6 = bc^{-1}t_2t_1 \implies Mt_0t_1t_8t_6 = Mt_2t_1 \in [05] = \{ N(t_0t_5)^n | n \in N \}, \]
\[ t_0t_1t_8t_7 = bt_0t_8t_0t_8t_1t_0t_6t_7t_8 \implies Mt_0t_1t_8t_7 = Mt_1t_0t_7t_8 \in [0127] \]
(since \( \{ N(t_0t_1t_7)^n | n \in N \} \) and \( bt_0t_8t_0t_8t_0 \in M \),
\[ Mt_0t_1t_8t_8 \in [01]. \]

\[ Mt_0t_1t_2t_7N \]

Consider \( Mt_0t_1t_2t_7N \) is a new double coset. We determine how many single cosets are in the double coset. However, \( N^{(0127)} = N^{0127} = < e > \). Only identity (e) will fix 0,1,7, and 2. Hence the number of single cosets living in \( Mt_0t_1t_2t_7N \) is \( \frac{|N|}{|N^{(0127)}|} = \frac{24}{1} = 24 \).

The orbits of \( N^{(0127)} \) on \( \{ t_1, t_2, t_0, t_4, t_5, t_6, t_7, t_8 \} \) are:
\[ \emptyset = \{ \{1\}, \{2\}, \{0\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\} \}. \]

Take a representative \( t_i \) from each orbit and see which double cosets \( Mt_0t_1t_2t_7t_i \) belongs to. We have:
\[ t_0t_1t_2t_7t_1 = at_7t_5t_0t_8 \implies Mt_0t_1t_2t_7t_1 = Mt_7t_5t_0t_8 \in [0127] = \{ N(t_0t_1t_2t_7)^n | n \in N \}, \]
\[ t_0t_1t_2t_7t_2 = bt_0t_2t_0t_2t_0t_5t_0t_7 \implies Mt_0t_1t_2t_7t_2 = Mt_5t_0t_7 \in [018] \]
(since \( \{ N(t_0t_1t_8)^n | n \in N \} \) and \( bt_0t_2t_0t_2t_0 \in M \),
\[(503,234)
\[ t_0t_1t_2t_7t_0 = d^{-1}b^{-1}t_2t_4t_0t_1 \implies Mt_0t_1t_2t_7t_0 = Mt_2t_4t_0t_1 \in [0127] \]
\[ t_0t_1t_2t_7t_4 = b^{-1}at_6t_8t_5 \implies Mt_0t_1t_2t_7t_4 = Mt_6t_8t_5 \in [014] = \{ N(t_0t_1t_4)^n | n \in N \},\]
\[ t_0t_1t_2t_7t_5 = at_2t_5t_4t_2 \implies Mt_0t_1t_2t_7t_5 = Mt_2t_5t_4t_3 \in [0127] = \{ N(t_0t_1t_2t_7)^n | n \in N \}, \]
\[ t_0 t_2 t_7 t_6 = t_0 t_8 t_9 t_0 t_4 t_8 t_5 \implies Mt_0 t_1 t_2 t_7 t_6 = Mt_4 t_8 t_5 \in [018] \]

(since \( \{N(t_0 t_1 t_8)^n | n \in N \} \) and \( t_0 t_8 t_9 t_0 \in M \)),

\[ Mt_0 t_1 t_2 t_7 t_7 \in [012] \]

\[ t_0 t_1 t_2 t_7 t_8 = bat_8 t_7 t_0 t_5 \implies Mt_0 t_1 t_2 t_7 t_8 = Mt_8 t_7 t_0 t_5 \in [0127] = \{N(t_0 t_1 t_2 t_7)^n | n \in N \}. \]

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of \( M \) in \( G \) is 165. We conclude:

\[
G = \text{M} \ast \text{N} \cup \text{M} t_0 \text{N} \cup \text{M} t_0 t_1 \text{N} \cup \text{M} t_0 t_2 \text{N} \cup \text{M} t_0 t_4 \text{N} \cup \text{M} t_0 t_6 \text{N} \cup \text{M} t_0 t_8 \text{N} \cup \text{M} t_0 t_1 t_2 t_7 \text{N},
\]

where

\[
G \cong 2^{*8} : S_4 / (c^c) \times 6, (abt^{c-1})^6, (ab^{-1}t)^3, bdte^{-1}tcd^{-1}tdte^{-1}t,
\]

\[
bdtdctde^{-1}tbd^{-1}ctc^{-1}t, bctdte^{-1}tbd^{-1}ctc^{-1}t, cte^{-1}tbd^{-1}ctd^{-1}t,
\]

\[
|G| \leq |N| + \frac{|N|}{N(07)} + \frac{|N|}{N(05)} + \frac{|N|}{N(012)} + \frac{|N|}{N(014)} + \frac{|N|}{N(018)} + \frac{|N|}{N(0127)} \times |M|
\]

\[
|G| \leq (1 + 8 + 24 + 24 + 12 + 24 + 24) \times 28
\]

\[
|G| \leq 165 \times 48
\]

\[
|G| \leq 7920.
\]

A Cayley diagram that summarizes the above information is given below:

Figure 6.2: Cayley Diagram of \( M_{11} \) over \( 2 \cdot S_4 \)
6.4 Iwasawa’s Lemma to Prove $M_{11}$ over $M = 2S_4$ is Simple

Again, we use Iwasawa’s lemma and the transitive action of $G$ on the set of single cosets to prove $G \cong M_{11}$ over $M = 2S_4$ is a simple group. Iwasawa’s lemma has three sufficient conditions that we must satisfy:

1. $G$ acts on $X$ faithfully and primitively
2. $G$ is perfect ($G = G'$)
3. There exist $x \in X$ and a normal abelian subgroup $K$ of $G^x$ such that the conjugates of $K$ generate $G$.

Proof. 6.4.1 $G = M_{11}$ over $M = 2S_4$ acts on $X$ Faithfully

Let $G$ acts on $X = M, Mt_0N, Mt_0t_1N, Mt_0t_5N, Mt_0t_1t_2N, Mt_0t_1t_4, Mt_0t_1t_6, Mt_0t_1t_8, Mt_0t_1t_2t_7N$, where $X$ is a transitive $G$-set of degree 165. $G$ acts on $X$ implies there exist homomorphism

$$ f : G \rightarrow S_{165} \quad (|X| = 165). $$

By First Isomorphic Theorem we have:

$$ G/\ker f \cong f(G). $$

If $\ker f = 1$ then $G \cong f(G)$. Only elements of $N$ fix $N$ implies $G^1 = N$. Since $X$ is transitive $G - set$ of degree 165, we have:

$$ |G| = 165 \times |G^1| $$
$$ = 165 \times |M| $$
$$ = 165 \times 48 $$
$$ = 7920 $$

$$ \Rightarrow |G| = 7920. $$

From Cayley diagram, $|G| \leq 7920$. However, from above $|G| = 7920$ implying $\ker(f) = 1$. Since $\ker f = 1$ then $G$ acts faithfully on $X$. 
6.4.2 \( G = M_{11} \) over \( M = 2S_4 \) acts on \( X \) Primitively

Since \( G = M_{11} \) is transitive on \( |X| = 165 \), if \( B \) is a nontrivial block then we may assume that \( M \in B \). However, \( |B| \) must divide \( |X| = 165 \). The only nontrivial blocks must be of size 3, 5, 11, 15, 33, or 55, since \( |B| \) must divide \( |X| \).

Case (1): If \( Mt_0 \in B \) then \( B = \{ M, Mt_0 \} = \{ M, Mt_0N \} \) (since \( N \in B, B = B \))

\[ B = \{ M, Mt_0, Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7, Mt_8 \} \]

\[ Bt_1 = \{ M, Mt_0t_1, M, Mt_2t_1, Mt_3t_1, Mt_4t_1, Mt_5t_1, Mt_6t_1, Mt_7t_1, Mt_8t_1 \} \]

\[ \Rightarrow M \in B \cap Bt_1. \]

Now \( B = \{ M, Mt_0N, Mt_0t_1N, Mt_0t_2N \} \), where \( |B| = 57 \) (passed all possible nontrivial blocks).

Note if \( Bt_0 \in B \) then \( B = X \).

Case (2): Consider \( B = \{ M, Mt_0N, Mt_0t_1N \} \), where \( |B| = 33 \) but if we have \( \{ M, Mt_0N \} \) we are going to have the entire group \( B = X \). Thus, \( G \) acts primitively on \( X \).

6.4.3 \( G = M_{11} \) over \( M = 2S_4 \) is Perfect

Next we want to show that \( G = G' \). Since \( G =< N, t > \), we have that \( N \leq G' \).

Now \( S_4 \leq G \implies S_4' \leq G' \). The commutators subgroup of \( S_4 \) is

\[ S_4' =< [a,b]|a, b \in S_4 >=< a, b, c, d > \leq G'. \]

Now by expanding the main relations we get the following:

\[ a = t_0t_6t_0 \]

\[ d = t_0t_8t_4t_1t_7t_0t_1t_8 \]

\[ c = t_4t_2t_1t_0t_7t_6t_8t_0 \]

\[ bdc^{-1}t_8t_0t_4t_8t_0 = 1 \]

Now we use the above relation and we solve for \( b \) by replacing \( d = t_0t_8t_4t_1t_7t_0t_1t_8 \) and \( c^{-1} = t_0t_8t_0t_7t_0t_1t_2t_4 \):

\[ bdc^{-1}t_8t_0t_4t_8t_0 = 1 \]

\[ \Rightarrow b = t_0t_8t_4t_0t_8t_4t_2t_1t_0t_7t_6t_8t_0t_8t_1t_0t_7t_1t_4t_8t_0 \]

So, \( G =< a, b, c, d, t > =< t_0, t_1, t_2, t_3, t_5, t_6, t_7, t_8 > \). Our goal is to show that one of the \( t' \)'s \( \in G' \), then we can conjugate by \( < a, b, c, d > \) to obtain all of the \( t' \)'s in \( G' \). Since
\[ a \in G'. \text{ Then} \]
\[ a = t_0t_6t_0 \in G' \]
\[ = t_0t_6t_0t_6 \in G' \text{ (since } |t'_6s| = 2) \]
\[ = [t_0, t_6]t_6 \in G' \]
\[ \implies t_6 \in G' \]
\[ \text{So } t_6 \in G' \implies t_6^2, t_6^3 \in G' \text{ (since } c, c^2, c^3 \in G \text{ and } G' \triangleleft G \text{ also}) \]
\[ t_6^{ad}, t_6^{bd}, t_6^{dc^3}, t_6^{c^3d} \in G' \text{ (since } a, d, b, c, d \in G \text{ and } G' \triangleleft G \]
\[ \implies G' = \langle t_6, t_8, t_0, t_2, t_5, t_1, t_7, t_4 \rangle. \]

Thus, \( G \geq G' \geq \langle t_0, t_1, t_2, t_4, t_5, t_6, t_7, t_8 \rangle = G. \) We conclude that \( G' = G \) and \( G \) is perfect.

### 6.4.4 Conjugates of a Normal Abelian \( K \)

Generate \( G = M_{11} \) over \( M = 2 \cdot S_4 \)

Now we require \( x \in X \) and a normal abelian subgroup \( K \) of \( G^x \), the point stabilizer of \( x \) in \( G \), such that the conjugates of \( K \) in \( G \) generate \( G \).

Now \( G^1 = M = 2 \cdot S_4 \) possesses a normal abelian subgroup \( K = \langle a \rangle \). We use the same relation, as we did in the previous part:
\[ a = t_0t_6t_0 \in K \]
\[ \implies a^{t_0} = (t_0t_6t_0)^{t_0} \in K^G \]
\[ \implies t_0at_0 = t_0t_6t_0t_6 \in K \]
\[ \implies at_6t_0 = t_6 \in K \]

So \( t_6^G \in K^G \)
\[ \implies K^G \supseteq \{t_6, t_6^2, t_6^3, t_6^{ad}, t_6^{bd}, t_6^{dc^3}, t_6^{c^3d}\} \]
\[ \implies K^G \supseteq \{t_6, t_8, t_0, t_2, t_5, t_1, t_7, t_4\} = G \]

Hence, the conjugates of \( K \) generate \( G \). Therefore, by Iwasawa’s lemma, \( G \cong M_{11} \) is a simple group.
Chapter 7

Double Coset Enumeration of $M_{12}$ over $(3^2 : 2 \cdot S_4)$

7.1 Factoring by the Center $(Z(G))$ of $2^{*72} : (3^2 : 2 \cdot S_4)$

Consider the group $G = 2^{*72} : (3^2 : 2 \cdot S_4)$ factored by the relator $[ac^{-1}b^{-1}e^{t_2}]^3$.  

Note: $N = (3^2 : 2 \cdot S_4) = <a, b, c>$ and $|N| = 432$, where  

$a \sim (2, 8)(3, 15)(4, 20) \ldots (59, 65)(60, 64)(68, 72)$,  
b \sim (1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23) \ldots (33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64)$,  
and $c \sim (1, 3, 5, 15)(2, 10, 12, 42) \ldots (48, 58, 65, 72)(49, 63, 68, 59)$.  

Let $t \sim t_1 \sim t_0$.  

Now we look at the composition factors of $G$ given below:

<table>
<thead>
<tr>
<th>$G$</th>
<th>M12</th>
<th>*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cyclic(2)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
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</tbody>
</table>

Thus, $G \cong 2 \times M_{12}$. Now, we use Magma to factor the group by the center $Z(G)$ and we get that $Z(G) = <ab^3ctb^{-1}tbc>$.

Hence,  

$$G = \frac{2^{*72} : (3^2 \cdot 2 \cdot S_4)}{[ac^{-1}b^{-1}e^{t_2}]^3, ab^3ctb^{-1}tbc} \cong M_{12}.$$
7.2 Construction of $M_{12}$ over $(3^2 : 2.S_4)$

Now consider the group $G = \langle a, b, c \rangle$, where $N = (3^2 : 2.S_4) = \langle a, b, c \rangle$ and $|N| = 432$, where

- $a \sim (2,8)(3,15)(4,20)\ldots(59,65)(60,64)(68,72)$,
- $b \sim (1,2,9,13,6,8)(3,16,25,10,40,23)\ldots(33,70,65,54,63,51)(43,72,68,62,60,64)$,
- and $c \sim (1,3,5,15)(2,10,12,42)\ldots(48,58,65,72)(49,63,68,59)$.

Let $t \sim t_1 \sim t_0$.

Let us expand the relations:

$[ac^{-1}b^{-1}cb^2 t_2]^{3} = 1$ with $\pi = ac^{-1}b^{-1}cb^2$ becomes

$1 = [\pi t_2]^3 = \pi^3 t_2^3 t_2 = acbt_5 t_3 t_2$

$\implies 1 = acbt_5 t_3 t_2$

$\implies acbt_5 = t_2$

$1 = ab^3 ctb t_6^{-1} tbc = ab^3 c t_1 t_3 t_3 t_3$

$\implies ab^3 c t_1 t_3 = t_3 t_10.$

We want to find the index of $N$ in $G$. To do this, we perform a manual double coset enumeration of $G$ over $N$. We take $G$ and express it as a union of double cosets $NgN$, where $g$ is an element of $G$. So $G = NeN \cup Ng_1N \cup Ng_2N \cup \ldots$ where $g_i$'s words in $t_i$'s.

We need to find all double cosets $[w]$ and find out how many single cosets each of them contains, where $[w] = [N w^n | n \in N]$. The double cosets enumeration is complete when the set of right cosets obtained is closed under right multiplication by $t_i$'s. We need to identify, for each $[w]$, the double coset to which $N w t_i$ belongs for one symmetric generator $t_i$ from each orbit of the coset stabilising group $N^{(w)}$

$NeN$

First, the double coset $NeN$, is denoted by $[*]$. This double coset contains only the single coset, namely $N$. Since $N$ is transitive on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$, the orbit of $N$ on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ is:

$\mathbb{O} = \{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$

We choose $t_0$ as our symmetric generator from $\mathbb{O}$ and find to which double coset $N t_0$ belongs. $N t_0 N$ will be a new double coset, denoted by $[0]$. Hence, 72 symmetric
Furthermore, the orbits of $Nt_0N$ will go the new double coset $[0]$. 

In order to find how many single cosets $[0]$ contains, we must first find the coset stabilizer $N^{(0)}$. Then the number of single coset in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|}$. Now, $N^{(0)} = N^0 = \langle a, acb^{-1}c^{-1}b^{-1}c \rangle$ so the number of the single cosets in $Nt_0N$ is $\frac{|N|}{|N^{(0)}|} = \frac{43}{6} = 72$. Furthermore, the orbits of $N^{(0)}$ on $\{t_0, t_2, t_3, \ldots, t_70, t_{71}, t_{72}\}$ are:

$\emptyset = \{0\}, \{7\}, \{35\}, \{2, 8, 34\}, \{5, 28, 14\}, \{13, 39, 31\}, \{3, 15, 70, 66, 46, 27\}, \{4, 20, 11, 71, 44, 52\}, \{6, 9, 32, 12, 26, 37\}, \{10, 19, 72, 57, 68, 24\}, \{16, 30, 67, 51, 50, 41\}, \{17, 38, 29, 36, 61, 40\}, \{18, 54, 60, 53, 64, 47\}, \{21, 23, 55, 49, 33, 58\}, \{25, 45, 48, 62, 63, 43\},$ and $\{22, 23, 55, 49, 33, 58\}$.

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_i$ belongs to. We have:

$$Nt_0t_0 = N \in [\ast]$$

$$t_0t_7 = t_{35} \implies Nt_0t_7 = Nt_{35} \in [0] = \{Nt_0^n|n \in N\}$$

$$t_0t_{35} = t_7 \implies Nt_0t_{35} = Nt_7 \in [0] = \{Nt_0^n|n \in N\}$$

$$Nt_0t_2 \in [02]$$

$$Nt_0t_5 \in [05]$$

$$Nt_0t_{13} \in [013]$$

$$t_0t_3 = c^{-1}b^{-1}t_2t_6 \implies Nt_0t_3 = Nt_2t_6 \in [013] = \{N(t_0t_3)^n|n \in N\}$$

$$t_0t_4 = bcbt_{14}t_{19} \implies Nt_0t_4 = Nt_{14}t_{19} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_6 = abt_{18}t_{55} \implies Nt_0t_6 = Nt_{18}t_{55} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_{10} = b^2t_{15}t_{62} \implies Nt_0t_{10} = Nt_{15}t_{62} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_{16} = ac^{-1}b^{-1}cb^2ct_{28}t_{37} \implies Nt_0t_{16} = Nt_{28}t_{37} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_{17} = abct_{15}t_{29} \implies Nt_0t_{17} = Nt_{15}t_{29} \in [013] = \{N(t_0t_13)^n|n \in N\}$$

$$t_0t_{18} = ab^{-1}c^{-1}b^{-1}ct_{59}t_{41} \implies Nt_0t_{18} = Nt_{59}t_{41} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_{21} = ab^{-1}c^{-1}b^{-1}ct_{17} \implies Nt_0t_{21} = Nt_{17} \in [0] = \{N(t_0)^n|n \in N\}$$

$$t_0t_{25} = ac^{-1}bcct_{53}t_{58} \implies Nt_0t_{25} = Nt_{53}t_{58} \in [05] = \{N(t_0t_5)^n|n \in N\}$$

$$t_0t_{22} = ab^{-1}c^{-1}bcbt_{36} \implies Nt_0t_{22} = Nt_{36} \in [0] = \{N(t_0)^n|n \in N\}.$$
The new double cosets have single coset representatives $Nt_0t_2$, $Nt_0t_5$, and $Mt_0t_{13}$, we represent them as $[02]$, $[05]$, and $[013]$, respectively.

$Nt_0t_2 N$

Continuing with the double coset $Nt_0t_2 N$, we find the point stabilizer $N^{02}$. This is $N^{02} =< acb^{-1}c^{-1}b^{-1}c >$. Also, with the relation $t_0t_2 = abcb^{-2}cblt_{16}t_{40} \implies Nt_0t_2 = Nt_{16}t_{40}$. Then $N(t_0t_2)^{(0,16,6,17,32,57)(2,40,13,44,39,66)\cdots} = abcb^{-2} = Nt_{16}t_{40}$. But $Nt_{16}t_{40} = Nt_0t_2 \implies abcb^{-2} \in N^{(02)}$ since $N(t_0t_2)^{abcb^{-2}} = Nt_{16}t_{40}$. Thus the coset stabiliser is $N^{(02)} =< acb^{-1}c^{-1}b^{-1}c, abcb^{-2} >$.

Since $|N^{(02)}| = 36$, the number of single cosets in $[02]$ is $\left|\frac{N}{N^{(02)}}\right| = \frac{432}{36} = 12$.

$\mathcal{O} = \{0, 16, 67, 52, 6, 44, 39, 19, 32, 57, 17, 13, 66, 61, 29, 15, 2, 40\}$,
$\{3, 70, 26, 10, 37, 60, 45, 72, 28, 18, 43, 7, 62, 55, 5, 22, 63, 31\}$,
$\{4, 11, 36, 30, 38, 65, 47, 51, 27, 24, 69, 68, 25, 64, 49, 46, 23, 48\}$,
$\{8, 34, 53, 41, 54, 71, 50, 33, 14, 20, 35, 21, 58, 12, 56, 42, 9, 59\}$

Take a representative $t_i$ from each orbit and see which double cosets $Nt_0t_2t_i$ belongs to.

We have:

$Nt_0t_2t_2 = Nt_0 \in [0]$

$t_0t_2t_3 = cb^{-1}c^{-1}b^{-1}c^{-1}t_{61}t_{33} \implies Nt_0t_2t_3 = Nt_{61}t_{33} \in [05] = \{N(t_0t_3)^{n} | n \in N\}$

$t_0t_2t_4 = cb^{-2}cblt_{28}t_{37} \implies Nt_0t_2t_4 = Nt_{28}t_{37} \in [05] = \{N(t_0t_3)^{n} | n \in N\}$

$t_0t_2t_8 = c^{-1}bcbt_{71}t_{43} \implies Nt_0t_2t_8 = Nt_{71}t_{43} \in [013] = \{N(t_0t_{13})^{n} | n \in N\}$

$Nt_0t_5 N$

Continuing with the double coset $Nt_0t_5 N$, we find the point stabilizer $N^{05}$. This is $N^{05} =< a >$. Also, with the relation $t_0t_5 = at_5t_0 \implies Nt_0t_5 = Nt_5t_0$.

Then $N(t_0t_5)^{(0,5),(2,12),(3,15)\cdots} = a^2 = Nt_5t_0$. But $Nt_5t_0 = Nt_0t_5 \implies a^2 \in N^{(02)}$ since $N(t_0t_5)^{a^2} = Nt_5t_0$. Thus the coset stabiliser is $N^{(05)} =< a, c^2 >$.

Since $|N^{(05)}| = 4$, the number of single cosets in $[05]$ is $\left|\frac{N}{N^{(05)}}\right| = \frac{432}{4} = 108$. The orbits of $N^{(05)}$ on $\{t_0, t_2, t_3, \ldots, t_{70}, t_{71}, t_{72}\}$ are:

$\mathcal{O} = \{0, 5\}, \{3, 15\}, \{7, 34\}, \{13, 35\}, \{33, 55\}, \{50, 67\}, \{2, 8, 12, 37\}, \{4, 20, 23, 22\}$,
$\{6, 9, 31, 39\}, \{10, 19, 42, 21\}, \{11, 44, 46, 70\}, \{14, 28, 26, 32\}, \{16, 30, 53, 47\}, \{17, 38, 56, 69\}$,
Take a representative \( t_i \) from each orbit and see which double cosets \( Nt_0t_5t_i \) belongs to. We have:

\[
Nt_0t_5t_5 = Nt_0 \in [0]
\]

\[
t_0t_5t_3 = cb^{-2}t_{63}t_{60} \implies Nt_0t_5t_3 = Nt_{63}t_{60} \in [02] = \{N(t_0t_2)^n|n \in N\}
\]

\[
t_0t_5t_7 = abc^2b^{-1}t_{72}t_{17} \implies Nt_0t_5t_7 = Nt_{72}t_{17} \in [013] = \{N(t_0t_{13})^n|n \in N\}
\]

\[
t_0t_5t_{13} = ac^{-1}b^3c^{-1}t_{68}t_{46} \implies Nt_0t_5t_{13} = Nt_{68}t_{46} \in [02] = \{N(t_0t_2)^n|n \in N\}
\]

\[
t_0t_5t_{33} = ac^{-2}b^{-1}c^{-1}t_{2}t_{37} \implies Nt_0t_5t_{33} = Nt_{2}t_{37} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{50} = b^2c^2b^{-1}c^{-1}t_{37}t_{2} \implies Nt_0t_5t_{50} = Nt_{37}t_{2} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{52} = acb^{-2}t_{44}t_{59} \implies Nt_0t_5t_{52} = Nt_{44}t_{59} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{54} = bcb^2t_{72}t_{57} \implies Nt_0t_5t_{54} = Nt_{72}t_{57} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{6} = ab^{-1}cbc^2t_{24}t_{68} \implies Nt_0t_5t_{6} = Nt_{24}t_{68} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{10} = b^3cb^{-1}t_{20}t_{53} \implies Nt_0t_5t_{10} = Nt_{20}t_{53} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{11} = ac^{-1}bcb^{-2}t_{63} \implies Nt_0t_5t_{11} = Nt_{63} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{14} = ac^{-1}b^3ct_{60} \implies Nt_0t_5t_{14} = Nt_{60} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{16} = acb^3c^{-1}t_{28} \implies Nt_0t_5t_{16} = Nt_{28} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{17} = cb^{-1}c^{-1}b^{-1}c^{-1}t_{71}t_{21} \implies Nt_0t_5t_{17} = Nt_{71}t_{21} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{18} = ab^2cb^{-1}c^{-1}t_{49} \implies Nt_0t_5t_{18} = Nt_{49} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{24} = ach^{-2}t_{42}t_{50} \implies Nt_0t_5t_{24} = Nt_{42}t_{50} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{25} = ab^{-1}c^3b^{-1}t_{64}t_{23} \implies Nt_0t_5t_{25} = Nt_{64}t_{23} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{27} = abc^2t_{48} \implies Nt_0t_5t_{27} = Nt_{48} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{43} = b^2c^{-1}b^{-1}c^{-1}t_{50}t_{42} \implies Nt_0t_5t_{25} = Nt_{64}t_{23} \in [05] = \{N(t_0t_5)^n|n \in N\}
\]

\[
t_0t_5t_{48} = b^c t_{14} \implies Nt_0t_5t_{48} = Nt_{14} \in [0] = \{N(t_0)^n|n \in N\}
\]

\[
t_0t_5t_{49} = ac^{-1}b^{-1}c^2t_{54}t_{42} \implies Nt_0t_5t_{49} = Nt_{54}t_{42} \in [013] = \{N(t_0t_{13})^n|n \in N\}
\]

\[Nt_0t_{13}N\]

Continuing with the double coset \( Nt_0t_{13}N \), we find the point stabilizer \( N^{013} \). This is
\(N^{013} = \langle a \rangle\). Also, with the relation \(t_0 t_{13} = b^3 t_{69} t_{49} \implies N t_0 t_{13} = N t_{69} t_{49}\). Then 
\[N(t_0 t_{13})^{(1,69,13,49),(2,38,6,27)} = b^{-1} c b c \in N^{(013)}\] since \(N(t_0 t_{13}) b^{-1} c b c = N t_{69} t_{49}\). Thus the coset stabiliser is 
\[N^{(013)} \geq \langle a, b^{-1} c b c \rangle.\]

Since \(|N^{(013)}| = 16\), the number of single cosets in [013] is \(|N| |N^{(013)}|^{-1} = \frac{432}{16} = 27\).

The orbits of \(N^{(013)}\) on \(\{t_0, t_2, t_3, \ldots, t_7, t_{71}, t_{72}\}\) are:
\[\emptyset = \{1, 69, 56, 13, 61, 49, 29, 58\}, \{5, 30, 16, 7, 62, 36, 43, 40\}, \{21, 42, 70, 63, 46, 35, 48, 34\}, \{2, 8, 38, 24, 17, 6, 67, 57, 9, 71, 27, 11, 50, 66, 52, 44\}, \{3, 15, 22, 60, 23, 10, 31, 64, 19, 32, 45, 72, 39, 25, 26, 68\}, \{4, 20, 37, 18, 12, 47, 51, 54, 53, 59, 28, 55, 41, 14, 65, 33\}.

Take a representative \(t_i\) from each orbit and see which double cosets \(N t_0 t_{13} t_i\) belongs to. We have:
\[N t_0 t_{13} t_{13} = N t_0 \in [0]\]
\[t_0 t_{13} t_7 = a b c t_0 t_6 t_5 \implies N t_0 t_{13} t_7 = N t_{66} t_{59} \in [02] = \{N(t_0 t_2)^n | n \in N\}\]
\[t_0 t_{13} t_{21} = t_9 t_4 \implies N t_0 t_{13} t_{21} = N t_{29} \in [05] = \{N(t_0 t_5)^n | n \in N\}\]
\[t_0 t_{13} t_2 = a b^{-1} c b c b^{-1} t_{44} \implies N t_0 t_{13} t_2 = N t_{44} \in [0] = \{N(t_0)^n | n \in N\}\]
\[t_0 t_{13} t_3 = c^{-1} b^2 t_{27} \implies N t_0 t_{13} t_3 = N t_{27} \in [0] = \{N(t_0)^n | n \in N\}\]
\[t_0 t_{13} t_4 = b c b^{-1} e^{-1} b^{-1} c^{-1} t_{02} t_{15} \implies N t_0 t_{13} t_4 = N t_{02} \in [05] = \{N(t_0 t_5)^n | n \in N\}\]

We have completed the double coset enumeration since the set of right cosets is closed under right multiplication, hence, the index of \(N\) in \(G\) is 220. We conclude:
\[G = N \cup N t_0 N \cup N t_0 t_2 N \cup N t_0 t_5 N \cup N t_0 t_{13} N,\]
where
\[G = \frac{\mathfrak{S}(2^7 2 \cdot (3^2 2 \cdot S_4))}{(ac^{-1} b^{-1} c b c b^{-1} t_{b t c})}\]
\[|G| \leq (|N| + \frac{|N|}{N^{013}} + \frac{|N|}{N^{013}} + \frac{|N|}{N^{013}}) \times |N|\]
\[|G| \leq (1 + 72 + 12 + 108 + 27) \times 432\]
\[|G| \leq 220 \times 432\]
\[|G| \leq 95040.\]

A Cayley diagram that summarizes the above information is given on the next page.
Figure 7.1: Cayley Diagram of $M_{12}$ over $(3^2 : 2 \cdot S_4)$
Chapter 8

Tabulated Images

8.1 $2^7 : (7 : 2)$

It can be proved that the progenitor given above has $M_{23}$ as a homomorphic image. While looking to find the Mathieu $M_{23}$ group, we ran the following progenitor and what we found is listed below:

$$G<a,b,t> := \text{Group}<a,b,t | a^3, b^(-2)*a^-1*b*a, t^2, (t,a), ((a*b^2)*t^(-1)*b^-2))^c, ((a*b^2)*t^(-1)*b^-1))^d, ((a*b^2)*t^(-1)*b^-2))^e, ((a*b^2)*t^(-1)*b^-1))^f, ((a*b^2)*t^(-1)*b^-1))^g, ((a*b^2)*t^(-1)*b^-1))^h,(b^3*t^(-1)*b^-1)^i, (b^3*t^(-1)*b^-1)^j>$$

<table>
<thead>
<tr>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>8</td>
<td>1774080</td>
<td>4M_{22}</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>10</td>
<td>0</td>
<td>6</td>
<td>15120</td>
<td>3:(A_7:2)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td>0</td>
<td>6</td>
<td>20160</td>
<td>A_8</td>
<td></td>
</tr>
</tbody>
</table>

8.2 $2^6 : (S_3 \times 2)$

It can be proved that the progenitor given above has $M_{24}$ as a homomorphic image. While looking to find the Mathieu $M_{24}$ group, we ran the following progenitors and what we found is listed below:
G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2, b^2, c^3, (a+b)^2, (a+c^{-1})^2, b^t c^{-1} b^c, t^2, (t, a*b), (b*t \\triangleleft d, (b*t \cdot c)^e, (b*t \cdot (a*c^{-1}))^f, (a*t)^g, (a*t \cdot c)^h, (a*t \cdot (a*c))^i, (a*b*t)^j, (a*b*t \cdot a)^k, (a*b*t \cdot c)^l, (a*b*t \cdot (a*c))^m, (c*t)^n, (c*t \cdot a)^o, (b*c*t)^p >;

Table 8.2: Some Finite Images of the Progenitor $2^6 \times (S_3 \times 2)$

<table>
<thead>
<tr>
<th>d</th>
<th>e</th>
<th>...</th>
<th>m</th>
<th>n</th>
<th>o</th>
<th>p</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>240</td>
<td>$(2 \times 45) : 2$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>2184</td>
<td>$PGL_2(13)$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>24360</td>
<td>$PGL_2(29)$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>1267200</td>
<td>$2 \cdot (2^4 : L_2(11)) : A_5$</td>
<td></td>
</tr>
</tbody>
</table>

for d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, u in [0..10] do
G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2, b^2, c^3, (a+b)^2, (a+c^{-1})^2, b^t c^{-1} b^c, t^2, (t, a*b), (b*t \\triangleleft d, (b*t \cdot c)^e, (b*t \cdot (a*c^{-1}))^f, (a*t)^g, (a*t \cdot c)^h, (a*t \cdot (a*c))^i, (a*b*t)^j, (a*b*t \cdot a)^k, (a*b*t \cdot c)^l, (a*b*t \cdot (a*c))^m, (c*t)^n, (c*t \cdot a)^o, (b*c*t)^p, (a*t)^q, (b*t)^r, (t*t \cdot a)^s, (t*t \cdot a)^u = a*b >;

Table 8.3: Some Finite Images of the Progenitor $2^6 \times (S_3 \times 2)$

<table>
<thead>
<tr>
<th>d</th>
<th>e</th>
<th>...</th>
<th>o</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
<th>u</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>336</td>
<td>$PGL_2(7)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>9</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>4896</td>
<td>$PGL_2(17)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>1140480</td>
<td>$6 \times (M_{12} : 2) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>4</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>6635520</td>
<td>$(2^7 : S_4(3)) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>5</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>4</td>
<td>9</td>
<td>120</td>
<td>$2 \times A_5$</td>
</tr>
</tbody>
</table>

8.3 $2^8 : S_4$

It can be proved that the progenitor given above has $M_{11}$ as a homomorphic image. While looking to find the Mathieu $M_{11}$ group, we ran the following progenitor and what we found is listed below:

G\langle a, b, c, d, t \rangle := \text{Group}\langle a, b, c, d, t \mid a^2, b^3, c^4, d^4, \rangle
\( b^{*-1}a^*b^a, c^{*-1}a^*c^{*-1}, d^{*-1}a^*d^{*-1}, \\
\)
\( b^*c^{*-1}b^{*-1}d^{*-1}, c^{*-1}d^{*-1}c^*d^{*-1}, d^{*-1}c^{*-1}b^{*-1}c*b, \\
\)
\( t^2, (t,b), (a*t^c (c*b^{*-1})), (c*b*t^c (c*b^{*-1})), \\
\)
\( (c*b*t^c)^g, (b^{*-1}c^{*-1}t^c (c*b^{*-1})), (b^{*-1}c^{*-1}t^c (c*b^{*-1})), \\
\)
\( (c*t^c (c*b^{*-1})), (c*t^c)^k, (a*b*t)^l, (a*b*t^c (c*b^{*-1})), \\
\)
\( (a*b^{*-1}t)^m, (a*b^{*-1}t^c (c*b^{*-1})), >; \\
\)

Table 8.4: Some Finite Images of the Progenitor \( 2^8 : S_4 \)

<table>
<thead>
<tr>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>m</th>
<th>n</th>
<th>o</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>120</td>
<td>2 \times A_5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>95040</td>
<td>((2^2 \times 3)M_{11})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8.4 \( 2^8 : (2^3 : 2) \)

It can be proved that the progenitor given above has \( M_{12} \) as a homomorphic image. While looking to find the Mathieu \( M_{12} \) group, we ran the following progenitor and what we found is listed below:

\[ G < a, b, c, d, t > := \text{Group} < a, b, c, d, t \mid a^2, b^4, c^2, d^2, \\
\]
\[ b^{*-2}d, (b^{*-1}a)^2, (b^{*-1}c)^2, a*c*b^{*-1}a*c, t^2, \\
\]
\[ (t,a), (d*t)^e, (c*t)^f, (c*t^b)^g, (a*t)^h, \\
\]
\[ (a*t^b)^i, (a*t^c)^j, (b*t)^k, (a*c*t)^l, (a*c*b*t)^m; \]

Table 8.5: Some Finite Images of the Progenitor \( 2^8 : (2^3 : 2) \)

<table>
<thead>
<tr>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>m</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>9</td>
<td>2448</td>
<td>( L_2(17) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>10</td>
<td>2160</td>
<td>( 3 : (A_6 \times 2) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>6</td>
<td>80640</td>
<td>( 2 : (A_8 \times 2) )</td>
</tr>
</tbody>
</table>

8.5 \( 2^7 : (3^2 : (2 : S_4)) \)

It can be proved that the progenitor given above has \( M_{12} \) as a homomorphic image. While looking to find the Mathieu \( M_{12} \) group, we ran the following progenitor and what we found is listed below:
\[ G_{a,b,c,t} := \text{Group} \langle a,b,c,t \mid a^2, b^6, c^4, (b^{-1} * a)^2, \\
(a^* c^{-1})^2, b^{-1} * c^{-2} * b^2 * c^2 * b^{-1}, (c^{-1} * b)^3, \\
c^{-1} * b^{-1} * c^{-2} * b^{-1} * c^2 * b^{-1} * c^{-1}, \\
t^2, (t,a), (t,a * c * b^{-1} * c^{-1} * b^{-1} * c), (b^3 * t)^d, \\
(b^3 * t^e) (b^3 * c^2)^f, (a * t)^g, (a * t^c)^h, \\
(a * t^2)^i, (a * t^j (a * c^{-1} * b^{-1} * c * b))^k, \\
(a * t^l (b^{-1} * c * b^{-1})^m, (b^{-1} * c^2 * t)^n, \\
(b^3 * c^2 * t^c)^o, (b^2 * t)^p, (b^2 * t^c)^q, \\
(b^2 * t^r (a * c^{-1} * b^{-1} * c * b))^s, (b * c^{-1} * t)^t, \\
(b * c^{-1} * t^u (b)^v, (b * c^{-1} * t^v (c))^w, \\
(b * c^{-1} * t^x (b^{-1} * c * b^{-1})^y, (c * t)^z, \\
(c * t^b)^i, (c * t^2)^j, (c * t^k)^l, \\
(b^2 * t)^m, (b * c * b^{-1})^n, (a * b * c * t)^o, \\
(a * b * c * t^b)^p, (a * b * c * t^p)^q, \\
(a * b * c * t^r)^s, (a * b * c * t^r (b^2)^t, (a * c^{-1} * b^{-1} * c * b^2 * t)^u, \\
(a * c^{-1} * b^{-1} * c * b^2 * t^b)^v, \\
(a * c^{-1} * b^{-1} * c * b^2 * t^b (b^2 * c))^w >; \]

Table 8.6: Some Finite Images of the Progenitor $2^{*72} : (3^2 : (2 S_4))$

<table>
<thead>
<tr>
<th>d</th>
<th>e</th>
<th>...</th>
<th>q</th>
<th>r</th>
<th>s</th>
<th>u</th>
<th>v</th>
<th>v</th>
<th>w</th>
<th>w</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>190080</td>
<td>$2M_{12}$</td>
<td></td>
</tr>
</tbody>
</table>
Appendix A: MAGMA Code for Permutation Progenitor of $A_5$

```magma
G<x,y>:=Group< x,y | x^2 = y^3 = (x*y)^5 = 1 >;
S:=Alt(5);
xx:=S!(1,2)(3,4);
yy:=S!(1,3,5);
N:=sub<S|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..60]];
for i in [2..60] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
for i in [1..60] do Sch[i], ArrayP[i]; end for;
N1:=Stabiliser(N,1);N1;
C:= Classes(N);C;
C2:=Centraliser(N,N!(1,2)(3,4));C2;
C3:=Centraliser(N,N!(1,2,3));C3;
C4:=Centraliser(N,N!(1,2,3,4,5));C4;
C5:=Centraliser(N,N!(1,3,4,5,2));C5;
Set(C2);Orbits(C2);
Set(C3);Orbits(C3);
Set(C4);Orbits(C4);
Set(C5);Orbits(C5);
```
Appendix B: MAGMA Code for Monomial Progenitor of $11^2 : m D_{10}$

D:=DihedralGroup(5);D;
xx:=D!(1,2,3,4,5);
yy:=D!(1,5)(2,4);
G:=sub<D|xx,yy>;
H:=sub<G|(1,2,3,4,5)>;
Set(H);
C:=Classes(G);C;
Cprime:=Classes(H);
CT:=CharacterTable(D);
ch:=CharacterTable(H);
I:=Induction(ch[2],D);
Norm(I);
for i in [1..#CT] do if I eq CT[i] then i; end if; end for;
T:=Transversal(G,H);T;
C:=CyclotomicField(5: Sparse := true);
A:=[0: i in [1..4]];
for i in [1..2] do if xx*T[i]^(-1) in H
then if ch[2](xx*T[i]^(-1)) eq C.1
then A[i]:=2; else if ch[2](xx*T[i]^(-1)) eq C.1^2
then A[i]:=4; else
A[i]:= ch[2](xx*T[i]^(-1); end if; end if; end if;
end for;
for i in [1..2] do if T[2]*xx*T[i]^(-1) in H
then if \( ch[2](T[2] \times x \times T[i]^{-1}) \) eq C.1
then \( A[2+i] := 2 \); else if \( ch[2](T[2] \times x \times T[i]^{-1}) \) eq C.1^2
then \( A[2+i] := 4 \); else
\( A[2+i] := ch[2](T[2] \times x \times T[i]^{-1}) \); end if; end if; end if;
end for;

B := [0: i in [1..4]];
for i in [1..2] do if \( yy \times T[i]^{-1} \) in H
then if \( ch[2](yy \times T[i]^{-1}) \) eq C.1
then \( B[i] := 2 \); else if \( ch[2](yy \times T[i]^{-1}) \) eq C.1^2
then \( B[i] := 4 \); else
\( B[i] := ch[2](yy \times T[i]^{-1}) \); end if; end if; end if;
end for;

for i in [1..2] do if \( T[2] \times yy \times T[i]^{-1} \) in H
then if \( ch[2](T[2] \times yy \times T[i]^{-1}) \) eq C.1
then \( B[2+i] := 2 \); else if \( ch[2](T[2] \times yy \times T[i]^{-1}) \) eq C.1^2
then \( B[2+i] := 4 \); else
\( B[2+i] := ch[2](T[2] \times yy \times T[i]^{-1}) \); end if; end if; end if;
end for;

G := GL(2, 11);
A := G!A; A;
B := G!B; B;
M := sub<G|A, B>;
#M;
Order(A);
Order(B);
s := IsIsomorphic(M, DihedralGroup(5)); s;
/*Monomial Progenitor on 20 letters*/
G := Group<x, y|x^5 = y^2 = (x*y)^2 = 1>;
S := Sym(20);
xx := S!(1, 7, 9, 17, 5)(3, 15, 19, 13, 11)
(2, 6, 18, 10, 8)(4, 12, 14, 20, 16);
yy := S!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)
(15, 16)(17, 18)(19, 20);
N := sub<S|xx, yy>;
Sch := SchreierSystem(G, sub<G|Id(G)>);
ArrayP := [Id(N): i in [1..10]];
for i in [2..10] do
P := [Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j] := xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j] := xx^{-1}; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j] := yy; end if;
end for;
PP := Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..10] do Sch[i], ArrayP[i]; end for;
Normaliser:=Stabiliser(N,{1,3,5,7,9,11,13,15,17,19});
Normaliser;
Normaliser eq sub<N| (1, 9, 5, 7, 17)(2, 18, 8, 6, 10)
(3, 19, 11, 15, 13)(4, 14, 16, 12, 20)>;
G < x, y, t > := Group< x, y, t | x^5 = y^2 = (x*y)^2 = 1,
t^11, t^(x^2)=t^5 >;
/*Verify*/
G < x, y, t > := Group< x, y, t | x^5 = y^2 = (x*y)^2 = 1,
t^11, (t,y*x), t*t^x=t^x*t >;
f, G1, k:=CosetAction(G, sub<G|x,y>);
#G;#k;
IN:=sub<G1|f(x),f(y)>;
T:=sub<G1|f(t)>;#T;
#Normaliser(IN,T);
Index(IN,Normaliser(IN,T));
/* here is the progenitor adding relations with first
order relation */
G < x, y > := Group< x, y | x^5 = y^2 = (x*y)^2 = 1 >;
D:=DihedralGroup(5);D;
xx:=D! (1,2,3,4,5);
yy:=D!(1,5)(2,4);
N:=sub<D|xx,yy>;
Sch:=SchreierSystem(G, sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..10]];
for i in [2..10] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..10] do Sch[i], ArrayP[i]; end for;
C:= Classes(N); C;
C2:=Centraliser(N,N!(1,5)(2,4));C2;
C3:=Centraliser(N,N!(1,2,3,4,5)); C3;
C4:=Centraliser(N,N!(1,3,5,2,4)); C4;
Set(C2); Orbits(C2);
Set(C3); Orbits(C3);
Set(C4); Orbits(C4);
Appendix C: MAGMA Code for Progenitor of $2^7 : D_{14}$

\begin{verbatim}
G<x,y>:=Group<x,y| x^7, y^2, (x*y)^2>;
D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx,yy>;
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..14]];
for i in [2..14] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
end for;
for i in [1..14] do Sch[i], ArrayP[i]; end for;
N7:=Stabiliser(N,7);
N7;
G<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2, t^2, (t,y)>;
/* Give all first order relations that this progenitor can be factored by */
C:= Classes(N);C;
C2:=Centraliser(N,N!(1,6)(2,5)(3,4));C2;
C3:=Centraliser(N,N!(1,2,3,4,5,6,7));C3;
C4:=Centraliser(N,N!(1,3,5,7,2,4,6));C4;
C5:=Centraliser(N,N!(1,4,7,3,6,2,5));C5;
\end{verbatim}
Set($C_2$); Orbits($C_2$);
Set($C_3$); Orbits($C_3$);
Set($C_4$); Orbits($C_4$);
Set($C_5$); Orbits($C_5$);
Appendix D: MAGMA Code for DCE of $2^3 : S_3$

```
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^2,(t,y),t*t*x=x^-1*t^-x*t*t^x*(x^-2)>;
#G;
S:=Sym(3);
xx:=S!(1,2,3);
yy:=S!(1,2);
N:=sub<S|xx,yy>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
ts := [Id(G1): i in [1 .. 3] ];
ts[3]:=f(t); ts[1]:=f(t^x); ts[2]:=f(t*(x^2));
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
prodim := function(pt, Q, I)
  /*
  Return the image of pt under permutations Q[I] applied sequentially.
  */
  v := pt;
  for i in I do
def := v^Q[i];
  end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
for i := 1 to 3 do
cst[prodim(1, ts, [i])]:= [i];
end for;
```
m:=0;
for i in [1..20] do if cst[i] ne [] then m:=m+1; end if; end for; m;

N31:=Stabiliser(N,[3,1]);
SSS:=[(3,1)]; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]); end if; end for; end for;
N31; #N31;
T31:=Transversal(N,N31);
for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne [] then m:=m+1; end if; end for; m;

Orbits(N31);

for g in IN do for h in IN do
if ts[3]*ts[1]*ts[2] eq g*(ts[3]*ts[1])^h then g,h; end if; end for; end for;
for i in [1..10] do i, cst[i]; end for;

N313:=Stabiliser(N,[3,1,3]);
SSS:=[(3,1,3)]; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[3]*ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]); end if; end for; end for;
N313s := N313;
T313 := Transversal(N, N313);
for i in [1..#T313] do
  ss := [3, 1, 3]ˆT313[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..20] do if cst[i] ne []
  then m := m+1; end if; end for; m;
Orbits(N313);

N3131 := Stabiliser (N, [3, 1, 3, 1]);
SSS := ([3, 1, 3, 1]); SSS := SSSˆN;
SSS;
#{SSS};
Seqq := Setseq(SSS);
Seqq;

for i in [1..#SSS] do
  for n in IN do
    if ts[3]*ts[1]*ts[3]*ts[1] eq
      n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      *ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
    then print Rep(Seqq[i]);
    end if; end for; end for;

N3131s := N3131;
for n in N do if 3^n eq 3 and 1^n eq 2
  then N3131s := sub<N|N3131s,n>; end if; end for;
#N3131s;
N3131s;
[3,1,3,1]^N3131s;

N3131 := Stabiliser (N, [3, 1, 3, 1]);
N3131;
N3131 := sub<N| (1,2)>;
#N3131;
[3,1,3,1]^N3131;

T := Transversal(N, N3131);
for i in [1..#T] do
  [{3,1,3,1]^N3131]^T[i];
end for;
for i in [1..10] do i, cst[i]; end for;

T3131:=Transversal(N,N3131);
for i in [1..#T3131] do ss:=[3,1,3,1]^T3131[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..20] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N3131);

N31313:=Stabiliser (N,[3,1,3,1,3]);
SSS:={[3,1,3,1,3]}; SSS:=SSS\N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do for n in IN do if ts[3]*ts[1]*ts[3]*ts[1]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
   ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
   *ts[Rep(Seqq[i])[5]] then print Rep(Seqq[i]); end if; end for; end for;

N31313s:=N31313;
for n in N do if 3\n eq 3 and 1\n eq 2 then N31313s:=sub<N|N31313s,n>; end if; end for;
#N31313s;
N31313s;
[3,1,3,1,3]^N31313s;

N31313:=Stabiliser (N,[3,1,3,1,3]);
N3131;
N31313:=sub<N| (1,2,3),(1,2)>;
#N31313;
[3,1,3,1,3]^N31313;

T:=Transversal(N,N31313);
for i in [1..#T] do
T31313 := Transversal(N, N31313);
for i in [1..#T31313] do
ss := [3, 1, 3, 1, 3]^T31313[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..20] do if cst[i] ne []
then m := m + 1; end if; end for; m;
Orbits(N31313);

/* Relations */
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
for i in [1..10] do i, cst[i]; end for;
*ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
for n in N do if 3^n eq 3 and 1^n eq 2 then
N31313s := sub<N|N31313s,n>; end if; end for;

#N31313s;

[3, 1, 3, 1, 3]^N31313s;
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
for i in [1..10] do i, cst[i]; end for;
ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
n*ts[3]*ts[1]*ts[3]*ts[1]*ts[3] then n; end if; end for;
ts[3]*ts[1]*ts[3]*ts[1]*ts[3];
Appendix E: MAGMA Code for DCE of $2 \times L_2(8)$ over $D_{14}$

\[
\begin{align*}
S &:= \text{Sym}(7); \\
xx &:= S!(1,2,3,4,5,6,7); \\
yy &:= S!(1, 6)(2, 5)(3, 4); \\
N &:= \text{sub}<S|xx, yy>; \\
#N; \\
N7 &:= \text{Stabiliser}(N, 7); \\
\text{IN} &:= \text{sub}<G|x, y>|f(x), f(y)>; \\
\text{ts} &:= [\text{Id}(G1) : i \in [1 .. 7]]; \\
\text{ts}[7] &:= f(t); \text{ts}[1] := f(t^x); \\
\text{ts}[2] &:= f(t^*(x^2)); \text{ts}[3] := f(t^*(x^3)); \\
\text{ts}[4] &:= f(t^*(x^4)); \text{ts}[5] := f(t^*(x^5)); \\
\text{ts}[6] &:= f(t^*(x^6)); \\
f(x^2) &* \text{ts}[1] * \text{ts}[2] * \text{ts}[7] * \text{ts}[1]; \\
f(x^2) &* \text{ts}[1] * \text{ts}[2] \text{ eq ts}[1] * \text{ts}[7]; \\
* \text{ts}[3] * \text{ts}[2] * \text{ts}[1] * \text{ts}[7]; \\
* \text{ts}[3] * \text{ts}[2] \text{ eq ts}[7] * \text{ts}[1]; \\
\text{DoubleCosets}(G, \text{sub}<G|x, y>, \text{sub}<G|x, y>); \\
#\text{DoubleCosets}(G, \text{sub}<G|x, y>, \text{sub}<G|x, y>); \\
\text{Index}(G, \text{sub}<G|x, y>); \\
\text{prodim} &:= \text{function}(pt, Q, I) \\
/** \\
\text{Return the image of pt under permutations}
\end{align*}
\]
Q[I] applied sequentially.

```plaintext
*/
v := pt;
for i in I do
    v := v^(Q[i]);
end for;
return v;
end function;
cst := [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];
for i := 1 to 7 do
    cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..72] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N7 := Stabiliser (N, [7]);
N7; #N7;
Orbits(N7);
N71:=Stabiliser (N,[7,1]);
SSS:=[[7,1]]; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IN do
        if ts[7]*ts[1] eq 
            n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
N71s := N71;
for n in N do if 7^n eq 7 and 1^n eq 6
    then N71s:=sub<N|N71s,n>;
    end if;
end for; #N71s;
[7,1]^N71s;
    then n; end if;
end for;
for i in [1..15] do i, cst[i]; end for;
N71:=Stabiliser (N,[7,1]);
N71;
N71:=sub<N| (1,6)(2,5)(3,4)>;
```
#N71;
[7,1]^N71;
T:=Transversal(N,N71);
for i in [1..#T] do
{[7,1]^N71}^T[i];
end for;
T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N71);
N72:=Stabiliser (N,[7,2]);
SSS:={[7,2]}; SSS:=SSS^N;
SSS;
#{SSS};
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N72s := N72;
for n in N do if 7^n eq 3 and 2^n eq 1
then N72s:=sub<N|N72s,n>; end if;
end for;
#N72s;
[7,2]^N72s;
n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
ts[7]*ts[2] eq f((x^2)^((x)^-1))*ts[3]*ts[1];
N72:=Stabiliser (N,[7,2]);
N72;
N72:=sub<N| (1,2)(4,6)(3,7)>;
#N72;
[7,2]^N72;
T:=Transversal(N,N72);
for i in [1..#T] do
\[ \{[7, 2]^N72\}^T[i]; \]
end for;
T72:=Transversal(N,N72);
for i in [1..#T72] do
ss:=[7, 2]^T72[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N72);

N73:=Stabiliser (N,[7,3]);
SSS:={[7,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for;
end for;
N73; #N73;
T73:=Transversal(N,N73);
for i in [1..#T73] do
ss:=[7, 3]^T73[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N73);
/*
[71]
*/
N712:=Stabiliser (N,[7,1,2]);
SSS:={[7,1,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
for n in N do if \(7^n \equiv 3 \mod 1 \equiv 2 \mod 2^n \) then print \(\text{Rep}(\text{Seqq}[i])\); end if; end for; end for;

\(N_{712s} := N_{712};\)

for n in N do if \(7^n \equiv 3 \mod 1 \equiv 2 \mod 2^n \) then \(N_{712s} := \text{sub}<N|N_{712s},n>\); end if; end for;

\(\#N_{712s};\)

\([7,1,2]^N_{712s};\)

for n in IN do if \(ts[7]*ts[1]*ts[2] \equiv n*ts[3]*ts[2]*ts[1] \) then n; end if; end for;

for i in [1..15] do i, cst[i]; end for;

\(ts[7]*ts[1]*ts[2] \equiv f((x^2)^{(x^2-1)})*ts[3]*ts[2]*ts[1];\)

\(N_{712} := \text{Stabiliser}(N,[7,1,2]);\)

\(N_{712} := \text{sub}<N| (1,2)(4,6)(3,7)>;\)

\(\#N_{712};\)

\([7,1,2]^N_{72};\)

\(T := \text{Transversal}(N,N_{712});\)

for i in [1..\#T] do
\(\{[7,1,2]^N_{712}\}^T[i];\)
end for;

\(T_{712} := \text{Transversal}(N,N_{712});\)

for i in [1..\#T_{712}] do
\(ss := [7,1,2]^T_{712}[i];\)
\(\text{cst}[\text{prodim}(1, ts, ss)] := ss;\)
end for;

\(m := 0; \) for i in [1..72] do if cst[i] ne [] then m := m + 1; end if; end for; m;

\(\text{Orbits}(N_{712});\)

\(N_{713} := \text{Stabiliser}(N,[7,1,3]);\)

\(\text{SSS} := \{[7,1,3]\}; \text{SSS} := \text{SSS}^N;\)

\(\#(\text{SSS});\)

\(\text{Seqq} := \text{Setseq}(\text{SSS});\)

\(\text{Seqq};\)

for i in [1..\#\text{SSS}] do for n in IN do if \(ts[7]*ts[1] *ts[3] \equiv n*ts[\text{Rep}(\text{Seqq}[i])[1]]*ts[\text{Rep}(\text{Seqq}[i])[2]]\)
\(\equiv ts[\text{Rep}(\text{Seqq}[i])[3]]\)
then print Rep(Seqq[i]);
end if; end for; end for;
N713s := N713;

#N713s;
[7,1,3]^N713s;

T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N713);
N717:=Stabiliser (N,[7,1,7]);
SSS:={[7,1,7]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1]*ts[7]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N717s := N717;
for n in N do if 7^n eq 1 and 1^n eq 7
then N717s:=sub<N|N717s,n>; end if;
end for;
for n in N do if 7^n eq 6 and 1^n eq 7
then N717s:=sub<N|N717s,n>; end if; end for;
N717s;
for n in IN do if ts[7]*ts[1]*ts[7] eq
n*ts[1]*ts[7]*ts[1]then
n; end if; end for;
for n in IN do if ts[7]*ts[1]*ts[7] eq
n*ts[6]*ts[7]*ts[6]then
n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
N717:=Stabiliser (N,[7,1,7]);
N717;
N717:=sub\(\langle 1, 7 \rangle (2, 6) (3, 5),
(1, 7, 6, 5, 4, 3, 2)\rangle;
N717;
[7,1,7]^N717;
T:=Transversal(N,N717);
for i in \{1..#T\} do
{[7,1,7]^N717}^T[i];
end for;
T717:=Transversal(N,N717);
for i in \{1..#T717\} do
ss:=[7,1,7]^T717[i];
cst[prodim(1, ts, ss)]:={ss};
end for;
m:=0; for i in \{1..72\} do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N717);
/*
[72]
*/
N725:=Stabiliser (N,[7,2,5]);
SSS:={[7,2,5]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in \{1..#SSS\} do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
for n in N do if 7^n eq 3 and 2^n eq 1 and
5^n eq 5 then N725s:=sub\(\langle N\rangle^N725s, n\rangle; end if; end for;
#N725s;
[7,2,5]^N725s;
n*ts[3]*ts[1]*ts[5] then n; end if; end for;
for i in \{1..15\} do i, cst[i]; end for;
/*Add relation*/
N725:=Stabiliser (N,[7,2,5]);
N725;
N725:=sub<N | (1,2)(3,7)(4,6)>;
#N725;
[7,2,5]^N725;
T:=Transversal(N,N72);
for i in [1..#T] do
{[7,2]^N72}`T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..72] do if cst[i] ne []
then m:=m+1; end if; end for; m;

/*
[73]
*/
N736:=Stabiliser (N,[7,3,6]);
SSS:={[7,3,6]}; SSS:=SSS\N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N736s := N736;
for n in N do if 7\n eq 7 and 4\n eq 3 and 1\n eq 6 then N736s:=sub<N|N736s,n>; end if;
end for;
#N736s;
N736s;
[7,3,6]^N736s;
for n in IN do if ts[7]*ts[3]*ts[6] eq
n*ts[7]*ts[4]*ts[1] then n; end if; end for;
for i in [1..15] do i, cst[i]; end for;
\text{N736}:=\text{Stabiliser} \ (N, [7, 3, 6]);
\text{N736};
\text{N736}:=\text{sub}<N| \ (1, 6)(2, 5)(3, 4)>;
\#\text{N736};
[7, 3, 6]^\text{N736};
\text{T}:=\text{Transversal} \ (N, \text{N736});
\text{for} \ i \ \text{in} \ [1..\#T] \ \text{do}
\{[7, 3, 6]^\text{N736}\}^\text{T}[i];
\text{end for};
\text{T736}:=\text{Transversal} \ (N, \text{N736});
\text{for} \ i \ \text{in} \ [1..\#T736] \ \text{do}
\text{ss}:=\{[7, 3, 6]^\text{T736}[i]\};
\text{cst}[\text{prodim}(1, \text{ts}, \text{ss})]:=\text{ss};
\text{end for};
\text{m}:=0; \ \text{for} \ i \ \text{in} \ [1..72] \ \text{do}
\text{if} \ \text{cst}[i] \neq []
\text{then} \ m:=m+1; \ \text{end if}; \ \text{end for}; \ \text{m};
\text{Orbits} \ (\text{N736});
/\!*\text{Relations}*/
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[2]*\text{ts}[3] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[1]*\text{ts}[3])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[2]*\text{ts}[3] \ \text{eq} \ f(x^-3)*\text{ts}[5]*\text{ts}[4]*\text{ts}[2];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[2]*\text{ts}[4] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[3])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[2]*\text{ts}[4] \ \text{eq} \ f(x)*\text{ts}[4]*\text{ts}[1];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[3]*\text{ts}[1] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[3])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[3]*\text{ts}[1] \ \text{eq} \ f(x^-1)*\text{ts}[2]*\text{ts}[6];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[3]*\text{ts}[2] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[1]*\text{ts}[2])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[3]*\text{ts}[2] \ \text{eq} \ f(x^-2)*\text{ts}[2]*\text{ts}[3]*\text{ts}[4];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[3]*\text{ts}[4] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[1]*\text{ts}[3])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[3]*\text{ts}[4] \ \text{eq} \ f(x)*\text{ts}[7]*\text{ts}[1]*\text{ts}[3];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[3]*\text{ts}[5] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[2]*\text{ts}[5])^\text{h}
\text{then} \ g, h; \ \text{end if}; \ \text{end for};
\text{end for};
\text{ts}[7]*\text{ts}[3]*\text{ts}[5] \ \text{eq} \ \text{ts}[3]*\text{ts}[5]*\text{ts}[1];
\text{for} \ g \ \text{in} \ \text{IN} \ \text{do}
\text{for} \ h \ \text{in} \ \text{IN} \ \text{do}
\text{if} \ \text{ts}[7]*\text{ts}[3]*\text{ts}[7] \ \text{eq} \ g*(\text{ts}[7]*\text{ts}[2])^\text{h}
\begin{verbatim}
then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[1]*ts[3]*ts[1] eq g*(ts[7]*ts[1]*ts[3])^h
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[1]*ts[3]*ts[2] eq g*(ts[7]*ts[2])^h
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[6] eq g*(ts[7]*ts[3]*ts[6])^h
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[2]*ts[5] eq g*(ts[7]*ts[3])^h
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[4] eq g*(ts[7]*ts[3])^h
     then g,h; end if; end for; end for;
for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[4] eq g*(ts[7]*ts[3])^h
     then g,h; end if; end for; end for;
  then g,h; end if; end for; end for;
\end{verbatim}

for g in IN do for h in IN do
  if ts[7]*ts[2]*ts[5]*ts[7] eq g*(ts[7]*ts[2]*ts[5])^h then g,h; end if; end for; end for;


for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[6]*ts[5] eq g*(ts[7]*ts[3]*ts[6])^h then g,h; end if; end for; end for;


for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[6]*ts[3] eq g*(ts[7]*ts[1]*ts[3])^h then g,h; end if; end for; end for;

\[ ts[7]*ts[3]*ts[6]*ts[3] \text{ eq } f(x)*ts[1]*ts[7]*ts[5]; \]

for g in IN do for h in IN do
  if ts[7]*ts[3]*ts[6]*ts[7] eq g*(ts[7]*ts[1]*ts[2])^h then g,h; end if; end for; end for;


/*******Factor by the center*************/

D:=DihedralGroup(7);

xx:=D!(1,2,3,4,5,6,7);

yy:=D!(1, 6)(2, 5)(3, 4);

N:=sub<D|xx,yy>;

#N;

Set(N);

G<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2, (t,y), (x*t*t^x)^2, (t*t*x*t)^9>;

#G;

f,G1,k:=CosetAction(G,sub<G|x,y>);

CompositionFactors(G1);

Center(G1);


A:=f(x);

B:=f(y);

C:=f(t);

N:=sub<G1|A,B,C>;

NN<x,y,t>:=Group<x,y,t|x^7, y^2,(x*y)^2, t^2, (t,y), (x*t*t^x)^2, (t*t*x*t)^9>;
Sch := SchreierSystem(NN, sub<NN|Id(NN)>);
ArrayP := [Id(N): i in [1..#N]];
for i in [2..#N] do
P := [Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j] := A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j] := A^(-1); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j] := B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j] := C; end if;
end for;
PP := Id(N);
for k in [1..#P] do
PP := PP * P[k]; end for;
ArrayP[i] := PP;
end for;

for i in [1..#N] do if ArrayP[i] eq aa then print Sch[i]; end if; end for;

/* Gives me the center in terms of x, y, and t */
G<x,y,t> := Group<x,y,t|x^7, y^2, (x*y)^(^2), t^2,
         (t,y), (x*t*x)^2, (t*x*t*x)^(^9), x*y*t*x*t*x^(-1)*t>;
f, G1, k := CosetAction(G, sub<G|x,y>);
CompositionFactors(G1);

/* Now we need to construct the double coset enumeration using the above presentation */
D := DihedralGroup(7);
xx := D!(1,2,3,4,5,6,7);
yy := D!(1,6)(2,5)(3,4);
N := sub<D|xx,yy>;
#N;
Set(N);
G<x,y,t> := Group<x,y,t|x^7, y^2, (x*y)^(^2), t^2,
         (t,y), (x*t*x)^0, (x*x*t*x)^2, (x*y*t*x*t)^0,
         (t*x*t*x)^9, x*y*t*x*t*x^(-1)*t>;
f, G1, k := CosetAction(G, sub<G|x,y>);
CompositionFactors(G1);
DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
#DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
Index(G, sub<G|x,y>);
IN := sub<G1|f(x), f(y)>;
ts := [Id(G1) : i in [1..7]];
\( ts[5] := f(t^5); \)
\( ts[6] := f(t^6); \)
\( f(x^2)*ts[1]*ts[2]*ts[7]*ts[1]; \)
\( f(x^2)*ts[1]*ts[7]*ts[6]*ts[5]*ts[4]*ts[3]*ts[2]*ts[1]*ts[7]; \)
\( f(x*y)*ts[7]*ts[6]*ts[7]; \)
\( \text{prodim} := \text{function}(\text{pt}, Q, I) \)

/*
Return the image of pt under permutations 
\( Q[I] \) applied sequentially.
*/

\( v := pt; \)
\( \text{for } i \text{ in } I \text{ do } \)
  \( v := v^{Q[i]}; \)
\( \text{end for; } \)
\( \text{return } v; \)
\( \text{end function; } \)

\( \text{cst} := \{ \text{null} : i \text{ in } [1 .. \text{Index}(G, \text{sub}<G|x,y>)] \} \)

\( \text{where null is } \{ \text{Integers()} \mid \}; \)
\( \text{for } i \text{ := 1 to 7 do } \)
  \( \text{cst[prodim(1, ts, [i])]} := [i]; \)
\( \text{end for; } \)
\( m:=0; \)
\( \text{for } i \text{ in } [1..36] \text{ do if cst[i] ne [] then } m:=m+1; \)
\( \text{end if; end for; m; } \)
\( \text{N7 := Stabiliser (N, [7]); } \)
\( \text{N7; } \#N7; \)
\( \text{T7:=Transversal(N,N7); } \)
\( \text{for } i \text{ in } [1..\#T7] \text{ do } \)
  \( \text{ss} := [7]^T7[i]; \)
  \( \text{cst[prodim(1, ts, ss)]} := ss; \)
\( \text{end for; } \)
\( m:=0; \text{ for } i \text{ in } [1..36] \text{ do if cst[i] ne [] then } m:=m+1; \)
\( \text{end if; end for; m; } \)
\( \text{Orbits(N7); } \)
\( \text{T71:=Transversal(N,N71); } \)
\( \text{for } i \text{ in } [1..\#T71] \text{ do } \)
  \( \text{ss} := [7,1]^T71[i]; \)
  \( \text{cst[prodim(1, ts, ss)]} := ss; \)
\( \text{end for; } \)
\( m:=0; \text{ for } i \text{ in } [1..36] \text{ do if cst[i] ne [] then } m:=m+1; \)
\( \text{end if; end for; m; } \)
\( \text{for } g \text{ in } \text{IN do for } h \text{ in } \text{IN do } \)
  \( \text{if ts[7]*ts[6] eq g*(ts[7])^h then } g,h; \)
  \( \text{end if; end for; end for; } \)
\( \text{for } i \text{ in } [1..15] \text{ do } i, \text{cst[i]; end for; } \)
/*Relation*/
ts[7]*ts[6] eq f(x*y)*ts[7];
for g in IN do for h in IN do
  if ts[7]*ts[1] eq g*(ts[7])^h
  then g,h; end if; end for; end for;
/*Relation*/
ts[7]*ts[1] eq f((x^6)*y)*ts[7];
N72:=Stabiliser (N,[7,2]);
SSS:={[7,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
  for n in IN do
    if ts[7]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    then print Rep(Seqq[i]);
    end if; end for;
  end for;
N72s := N72;
for n in N do if 7^n eq 3 and 2^n eq 1
  then N72s:=sub<N|N72s,n>; end if; end for;
#N72s;
[7,2]^N72s;
  n; end if; end for;
  n; end if; end for;
/*RELATION */
ts[7]*ts[2] eq f(x^2)*ts[3]*ts[1];
N72:=Stabiliser (N,[7,2]);
N72;
N72:=sub<N| (1,2)(3,7)(4,6)>;
#N72;
[7,2]^N72;
T:=Transversal(N,N72);
for i in [1..#T] do
  {{[7,2]^N72}^T[i];
end for;
T72:=Transversal(N,N72);
for i in [1..#T72] do
  ss:=[7,2]^T72[i];
  cst[prodim(1, ts, ss)] := ss;
  end for;
m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1; end if; end for; m; Orbits(N72); N73:=Stabiliser (N,[7,3]); SSS:=[{7,3}]; SSS:=SSS^N; SSS; #(SSS); Seqq:=Setseq(SSS); Seqq; for i in [1..#SSS] do for n in IN do if ts[7]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]] then print Rep(Seqq[i]); end if; end for; end for; N73s := N73; #N73s; [7,3]^N73s; T:=Transversal(N,N73); for i in [1..#T] do {[7,3]^N73}^T[i]; end for; T73:=Transversal(N,N73); for i in [1..#T73] do ss:=[7,3]^T73[i]; cst[prodim(1, ts, ss)]:= ss; end for; m:=0; for i in [1..36] do if cst[i] ne [] then m:=m+1; end if; end for; m; Orbits(N73); for g in IN do for h in IN do if ts[7]*ts[3]*ts[1] eq g*(ts[7]*ts[3])^h then g,h; end if; end for; end for; ts[7]*ts[3]*ts[1] eq f(x^-1)*ts[2]*ts[6]; for g in IN do for h in IN do if ts[7]*ts[3]*ts[2] eq g*(ts[7]*ts[2])^h then g,h; end if; end for; end for; ts[7]*ts[3]*ts[2] eq f((x^2)y)*ts[5]*ts[3]; for g in IN do for h in IN do if ts[7]*ts[3]*ts[4] eq g*(ts[7]*ts[3])^h then g,h; end if; end for; end for; ts[7]*ts[3]*ts[4] eq f(y)*ts[7]*ts[3]; for g in IN do for h in IN do if ts[7]*ts[3]*ts[5] eq g*(ts[7]*ts[2]*ts[5])^h then g,h; end if; end for; end for; ts[7]*ts[3]*ts[5] eq f((x^2)y)*ts[5]*ts[3];
then g, h; end if; end for; end for;
for g in IN do for h in IN do
for g in IN do for h in IN do
N723 := Stabiliser (N, [7, 2, 3]);
T723 := Transversal (N, N723);
for i in [1..#T723] do
  ss := [7, 2, 3]^T723[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..36] do if cst[i] ne []
  then m := m + 1; end if; end for; end for;
for g in IN do for h in IN do
N724 := Stabiliser (N, [7, 2, 4]);
T724 := Transversal (N, N724);
for i in [1..#T724] do
  ss := [7, 2, 4]^T724[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..36] do if cst[i] ne []
  then m := m + 1; end if; end for; end for;
for g in IN do for h in IN do
N725 := Stabiliser (N, [7, 2, 5]);
SSS := {[7, 2, 5]}; SSS := SSS^N;
SSS;
#(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
for n in N do if 7^n eq 3 and 2^n eq 1 and
5^n eq 5 then N725s:=sub<N|N725s,n>; end if; end for;
#N725s;
[7,2,5]^N725s;
n*ts[3]*ts[1]*ts[5] then
n; end if; end for;
N725:=Stabiliser (N,[7,2,5]);
N725;
N725:=sub<N| (1,2)(3,7)(4,6)>;
#N725;
[7,2,5]^N725;
T:=Transversal(N,N725);
 for i in [1..#T] do
{[7,2,5]^N725}^T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..36] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N725);
for g in IN do for h in IN do
 if ts[7]*ts[2]*ts[5]*ts[1] eq g*(ts[7]*ts[3])^h
then g,h; end if; end for; end for;
for g in IN do for h in IN do
then g,h; end if; end for; end for;
for g in IN do for h in IN do
then g,h; end if; end for; end for;
Appendix F: MAGMA Code for DCE of $L_2(27)$ over a Maximal Subgroup

G<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2,
   t^2, (t,y), (x*t)^0, (x*t*y*x)^0, (x*y*t^x*t)^3,
   (t^t*x*x)^7>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
M:=MaximalSubgroups(G1);
M;
#G1;
#G1/351;
M3:=M[3] `subgroup;
f(x) in M3 and f(y) in M3;
C:=Conjugates(G1,M3);
CC:=SetToSequence(C);
for i in [1..#C] do if f(x) in CC[i] and f(y) in CC[i] then i; end if; end for;
H:=sub<G1|CC[124]>; #H;
f(x) in H and f(y) in H;
for g in G1 do if sub<G1|f(x),f(y)> eq H then gg=g; end if; end for;
for i in [0..6] do for j in [0..1] do for k,l,m,n,o
in [0..6] do if gg eq f(x^i*y^j*t^k*(x^l)*t^(x^m)*t^(x^n)*t^(x^o))
then i,j,k,l,m,n,o; end if; end for; end for; end for;
sub<G1|f(t^(x^2)*t^(x^4)*t^(x^5)*t^(x^4)*
t^(x^2)),f(x),f(y)> eq H;
G<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2, t^2, (t,y),
(x+t)^0, (x+t+t*x)^0, (x+y*t*x+t)^0, (t+t*x+t)^7>
H:=sub<G|x,y,t*(x^2)*t*(x^4)*t*(x^5)*t*(x^4)
*t*(x^2)>; #H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x), f(y)>;
IM:=sub<G1|IN, f(t*(x^2)*t*(x^4)*t*(x^5)
*t*(x^4)*t*(x^2))>;
#IN; #IM;

D:=DihedralGroup(7);
xx:=D!(1,2,3,4,5,6,7);
yy:=D!(1, 6)(2, 5)(3, 4);
N:=sub<D|xx, yy>;
#N;
Set(N);
G<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2, t^2,
(t,y), (x+t)^0, (x+t+t*x)^0, (x+y*t*x+t)^3, (t+t*x+t)^7>;
H:=sub<G| x,y,t*(x^2)*t*(x^4)*t*(x^5)*t*(x^4)
*t*(x^2)>; #H;
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x), f(y)>;
IM:=sub<G1|IN, f(t*(x^2)*t*(x^4)*t*(x^5)
*t*(x^4)*t*(x^2))>;
#IN; #IM;
ts := [Id(G1) : i in [1 .. 7]];
f(x*y)*ts[1]*ts[7]*ts[5] eq ts[7]*ts[1]*ts[6];
DoubleCosets(G,H,sub<G|x,y>);
#DoubleCosets(G,H,sub<G|x,y>);
Index(G,H);
prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v*(Q[i]);
  end for;
  return v;
end function;
cst := [null : i in [1 .. Index(G,H)]]
where null is [Integers() | ];
for i := 1 to 7 do
  cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..351] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N7 := Stabiliser (N, [7]);
N7; #N7;
T7:=Transversal(N,N7);
for i in [1..#T7] do
  ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
  end for;
m:=0; for i in [1..351] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
Orbits(N7);

N71:=Stabiliser (N,[7,1]);
SSS:={[7,1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
  for n in IM do
    if ts[7]*ts[1] eq
      n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    then print Rep(Seqq[i]);
      end if; end for; end for;
N71s := N71;
#N71s;
[7,1]^N71s;
T:=Transversal(N,N71);
  for i in [1..#T] do
    {[7,1]^N71}^T[i];
  end for;
T71:=Transversal(N,N71);
  for i in [1..#T71] do
    ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
    end for;
m:=0; for i in [1..351] do if cst[i] ne []
    then m:=m+1; end if; end for; m;
Orbits(N71);

N72:=Stabiliser (N,[7,2]);
SSS := ([7, 2]); SSS := SSS^N;
SSS := (#SSS);
Seqq := Setseq(SSS); Seqq;
for i in [1..#SSS] do
  for n in IM do
    if ts[7]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N72s := N72;
#N72s;
[7, 2] ^ N72s;
T := Transversal(N, N72);
  for i in [1..#T] do
    {[7, 2]^N72}^T[i];
  end for;
T72 := Transversal(N, N73);
for i in [1..#T72] do
  ss := [7, 2]^T72[i];
  cst[prodim(1, ts, ss)] := ss;
  end for;
  m := 0; for i in [1..351] do if cst[i] ne []
    then m := m + 1; end if; end for; m;
Orbits(N72);

N73 := Stabiliser(N, [7, 3]);
SSS := ([7, 3]); SSS := SSS^N;
SSS := (#SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
  for n in IM do
    if ts[7]*ts[3] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N73s := N73;
#N73s;
[7, 3] ^ N73s;
T := Transversal(N, N73);
  for i in [1..#T] do
    {[7, 3]^N73}^T[i];
  end for;
T73 := Transversal(N, N73);
for i in [1..#T73] do
    ss:=[7,3]^T73[i];
    cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
    then m:=m+1; end if; end for; m;
Orbits(N73);

N717:=Stabiliser (N,[7,1,7]);
SSS:={[7,1,7]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
        if ts[7]*ts[1]*ts[7] eq
            n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
            *ts[Rep(Seqq[i])[3]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
N717s := N717;
#N717s;
[7,1,7]^N717s;
T:=Transversal(N,N717);
for i in [1..#T] do
    [{7,1,7]^N717}^T[i];
end for;
T717:=Transversal(N,N717);
for i in [1..#T717] do
    ss:=[7,1,7]^T717[i];
    cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
    then m:=m+1; end if; end for; m;
Orbits(N717);

N712:=Stabiliser (N,[7,1,2]);
SSS:={[7,1,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
        if ts[7]*ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])][1])*ts[Rep(Seqq[i])][2])
*ts[Rep(Seqq[i])][3])
then print Rep(Seqq[i]);
end if;
end for; end for;
N712s := N712;
#cN712s;
[7,1,2]^N712s;
T:=Transversal(N,N712);
for i in [1..#T] do
{[7,1,2]^N712}^T[i];
end for;
T712:=Transversal(N,N712);
for i in [1..#T712] do
ss:=[7,1,2]^T712[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N712);

N713:=Stabiliser (N,[7,1,3]);
SSS:={[7,1,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[7]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])][1])*ts[Rep(Seqq[i])][2])
*ts[Rep(Seqq[i])][3])
then print Rep(Seqq[i]);
end if; end for; end for;
N713s := N713;
#cN713s;
[7,1,3]^N712s;
T:=Transversal(N,N713);
for i in [1..#T] do
{[7,1,3]^N713}^T[i];
end for;
T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N713);

N714:=Stabiliser (N,[7,1,4]);
SSS:={[7,1,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[7]*ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N714s := N714;
#N714s;
[7,1,4]^N714s;
T:=Transversal(N,N714);
for i in [1..#T] do
{{7,1,4}^N714}^T[i];
end for;
T714:=Transversal(N,N714);
for i in [1..#T714] do
ss:=[7,1,4]^T714[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N714);

for g in IM do for h in IN do
if ts[7]*ts[1]*ts[6] eq g*(ts[7]*ts[1]*ts[3])^h
then g,h; end if; end for; end for;
/*Relation */
ts[7]*ts[1]*ts[6] eq f(x*y)*ts[1]*ts[7]*ts[5];
N715:=Stabiliser (N,[7,1,5]);
SSS:={[7,1,5]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
  if ts[7]*ts[1]*ts[5] eq
    n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    *ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if;
end for;
end for;
N715s := N715;
for n in N do if 7^n eq 6 and 1^n eq 5 and
5^n eq 1 then N715s:=sub<N|N715s,n>; end if;
end for;
#N715s;
[7,1,5]^N715s;
N715:=Stabiliser (N,[7,1,5]);
N715;
N715:=sub<N| (6,7)(1,5)(2,4)>;
#N715;
[7,1,5]^N715;
T:=Transversal(N,N715);
  for i in [1..#T] do
    {[7,1,5]^N715}^T[i];
  end for;
T715:=Transversal(N,N715);
for i in [1..#T715] do
  ss:=[7,1,5]^T715[i];
  cst[prodim(1, ts, ss)] := ss;
  end for;
  m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for;
  m;
Orbits(N715);

N726:=Stabiliser (N,[7,2,6]);
SSS:={[7,2,6]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
    n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
    *ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if;
end for;
end for;
N726s := N726;
#N726s;
[7,2,6]^N715s;
T:=Transversal(N,N726);
for i in [1..#T] do
{[7,2,6]^N726}^T[i];
end for;
T726:=Transversal(N,N726);
for i in [1..#T726] do
ss:=[7,2,6]^T726[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N726);
N727:=Stabiliser (N,[7,2,7]);
SSS:={[7,2,7]}; SSS:=SSS^N;
SSS; #(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N727s := N727;
#N727s;
[7,2,7]^N727s;
T:=Transversal(N,N727);
for i in [1..#T] do
{[7,2,7]^N727}^T[i];
end for;
T727:=Transversal(N,N727);
for i in [1..#T727] do
ss:=[7,2,7]^T727[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N727);
N724 := Stabiliser (N, [7, 2, 4]);
SSS := {{7, 2, 4}}; SSS := SSS \ N;
SSS; #(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N724s := N724;
#N724s;
[7, 2, 4]^N724s;
T := Transversal(N, N724);
for i in [1..#T] do
{[7, 2, 4]^N724}^T[i];
end for;
T724 := Transversal(N, N724);
for i in [1..#T724] do
ss := [7, 2, 4]^T724[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..351] do if cst[i] ne []
then m := m+1; end if; end for;
m;
Orbits(N724);

N725 := Stabiliser (N, [7, 2, 5]);
SSS := {{7, 2, 5}}; SSS := SSS \ N;
SSS; #(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N725s := N725;
#N725s;
[7, 2, 5]^N725s;
T:=Transversal(N,N725);
for i in [1..#T] do
{[7,2,5]^N725]^T[i];
end for;
T725:=Transversal(N,N725);
for i in [1..#T725] do
ss:=[7,2,5]^T725[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N725);
N737:=Stabiliser (N,[7,3,7]);
SSS:={[7,3,7]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[7]*ts[3]*ts[7] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N737s:=N737;
for n in N do if 7^n eq 5 and 3^n eq 2
and 7^n eq 5 then N737s:=sub<N|N737s,n>
end if; end for;
#N737s;
[7,3,7]^N737s;
N737:=Stabiliser (N,[7,3,7]);
N737;
N737:=sub<N| (1,4)(2,3)(5,7)>;
#N737;
[7,3,7]^N737;
T:=Transversal(N,N737);
for i in [1..#T] do
{[7,3,7]^N737]^T[i];
end for;
T737:=Transversal(N,N737);
for i in [1..#T737] do
ss:=[7,3,7]^T737[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N737);

N732:=Stabiliser (N,[7,3,2]);
SSS:={[7,3,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
  if ts[7]*ts[3]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
  then print Rep(Seqq[i]);
  end if;
end for;
end for;
N732s := N732;
for n in N do if 7^n eq 2 and 3^n eq 6 and 2^n eq 7 then N732s:=sub<N|N732s,n>;
  end if;
end for;
#N732s;
[7,3,2]^N732s;
N732:=Stabiliser (N,[7,3,2]);
N732;
N732:=sub<N| (2,7)(3,6)(4,5)>;
#N732;
[7,3,2]^N732;
T:=Transversal(N,N732);  
  for i in [1..#T] do
    {{[7,3,2]^N732}^T[i];
  end for;
T732:=Transversal(N,N732);
for i in [1..#T732] do 
  ss:=[7,3,2]^T732[i];
  cst[prodim(1, ts, ss)] := ss;
  end for;
m:=0; for i in [1..351] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N732);

N734:=Stabiliser (N,[7,3,4]);
SSS:={[7,3,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N734s := N734;
for n in N do if 7^n eq 2 and 3^n eq 6
and 4^n eq 5 then N734s:=sub<N|N734s,n>;
end if; end for;
#N734s;
[7,3,4]^N734s;
N734:=Stabiliser (N,[7,3,4]);
N734;
N732:=sub<N| (2,7)(3,6)(4,5)>;
#N732;
[7,3,2]^N732;
T:=Transversal(N,N734);
for i in [1..#T] do
{[7,3,4]^N734}^T[i];
end for;
T734:=Transversal(N,N734);
for i in [1..#T734] do
ss:=[7,3,4]^T734[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N734);

N7323:=Stabiliser (N,[7,3,2,3]);
SSS:={[7,3,2,3]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7323s := N7323;
#N7323s;
[7,3,2,3]^N7323s;
T:=Transversal(N,N7323);
for i in [1..#T] do
{[7,3,2,3]^N7323}^T[i];
end for;
T7323:=Transversal(N,N7323);
for i in [1..#T7323] do
ss:=[7,3,2,3]^T7323[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7323);

N7321:=Stabiliser (N,[7,3,2,1]);
SSS:={[7,3,2,1]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7321s := N7321;
for n in N do if 7^n eq 2 and 3^n eq 6 and 2^n eq 7 and 1^n eq 1 then
N7321s:=sub<N|N7321s,n>; end if; end for;
#N7321s;
[7,3,2,1]^N7321s;
n*ts[2]*ts[6]*ts[7]*ts[1] then
n; end if; end for;
N7321:=Stabiliser (N,[7,3,2,1]);
N7321;
N7321:=sub<N| (2,7)(3,6)(4,5)>;
#N7321;
[7,3,2,1]^N7321;
T:=Transversal(N,N7321);
for i in [1..#T] do
{[7, 3, 2, 1]^N7321}^T[i];
end for;
T7321:=Transversal(N,N7321);
for i in [1..#T7321] do
ss:=[7, 3, 2, 1]^T7321[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7321);

N7347:=Stabiliser (N,[7, 3, 4, 7]);
SSS:={[7, 3, 4, 7]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
    n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]*ts[Rep(Seqq[i])][4]
then print Rep(Seqq[i]);
end if; end for; end for;
N7347s := N7347;
#N7347s;
[7, 3, 4, 7]^N7347s;
T:=Transversal(N,N7347);
for i in [1..#T] do
{[7, 3, 4, 7]^N7347}^T[i];
end for;
T7347:=Transversal(N,N7347);
for i in [1..#T7347] do
ss:=[7, 3, 4, 7]^T7347[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7347);
N7341:=Stabiliser (N,[7, 3, 4, 1]);
SSS:={[7, 3, 4, 1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
            n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
            * ts[Rep(Seqq[i])[3]] * ts[Rep(Seqq[i])[4]]
        then print Rep(Seqq[i]);
        end if;
    end for;
end for;
N7341s := N7341;
N7341s := N7341;
for n in N do if 7^n eq 2 and 3^n eq 6
    and 4^n eq 5 and 1^n eq 1 then
        N7341s := sub<N|N\N7341s, n>; end if;
    end for;
#N7341s;
[7, 3, 4, 1] \ N7341s;
N7341 := Stabiliser (N, [7, 3, 4, 1]);
N7341;
N7341 := sub<N| (2, 7)(3, 6)(4, 5)>;
#N7341;
[7, 3, 4, 1] \ N7341;
T := Transversal (N, N7341);
for i in [1..#T] do
    {{[7, 3, 4, 1] \ N7341} \ T[i];
end for;
T7341 := Transversal (N, N7341);
for i in [1..#T7341] do
    ss := [7, 3, 4, 1] \ T7341[i];
    cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..351] do if cst[i] ne []
    then m := m + 1; end if; end for;
m;
Orbits (N7341);

N7371 := Stabiliser (N, [7, 3, 7, 1]);
SSS := {[7, 3, 7, 1]}; SSS := SSS \ N;
SSS; #(SSS);
Seqq := Setseq (SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
            n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
            * ts[Rep(Seqq[i])[3]] * ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]); end if; end for; end for;
N7371s := N7371;
#N7371s;
[7,3,7,1]^N7371s;
T:=Transversal(N,N7371);
for i in [1..#T] do
{[7,3,7,1]^N7371}^T[i];
end for;
T7371:=Transversal(N,N7371);
for i in [1..#T7371] do
ss:=[7,3,7,1]^T7371[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7371);

N7372:=Stabiliser (N,[7,3,7,2]);
SSS:={[7,3,7,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]
*ts[Rep(Seqq[i])][3]*ts[Rep(Seqq[i])][4]
then print Rep(Seqq[i]);
end if; end for; end for;
N7372s := N7372;
for n in N do if 7^n eq 6 and 3^n eq 3
and 7^n eq 6 and 2^n eq 4 then
N7372s:=sub<N|N7372s,n>; end if; end for;
#N7372s;
[7,3,7,2]^N7372s;
N7372:=Stabiliser (N,[7,3,7,2]);
N7372;
N7372:=sub<N| (6,7) (2,4) (1,5)>;
#N7372;
[7,3,7,2]^N7372;
T:=Transversal(N,N7372);
for i in [1..#T] do
{[7,3,7,2]^N7372}^T[i];
end for;
T7372:=Transversal(N,N7372);
for i in [1..#T7372] do
ss:=[7,3,7,2]^T7372[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7372);

N7152:=Stabiliser(N,[7,1,5,2]);
SSS:={[7,1,5,2]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N7152s:=N7152;
#N7152s;
[7,1,5,2]^N7152s;
T:=Transversal(N,N7152);
for i in [1..#T] do
{"[7,1,5,2]^N7152"^T[i]};
end for;
T7152:=Transversal(N,N7152);
for i in [1..#T7152] do
ss:=[7,1,5,2]^T7152[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7152);

N73417:=Stabiliser(N,[7,3,4,1,7]);
SSS:={[7,3,4,1,7]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
        n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        *ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
        *ts[Rep(Seqq[i])[5]]
    then print Rep(Seqq[i]);
    end if; end for; end for;
N73417s := N73417;
#N73417s;
[7,3,4,1,7]ˆN73417s;
T:=Transversal(N,N73417);
for i in [1..#T] do
    {[7,3,4,1,7]ˆN73417}ˆT[i];
end for;
T73417:=Transversal(N,N73417);
for i in [1..#T73417] do
    ss:=[7,3,4,1,7]ˆT73417[i];
cst[prodim(1, ts, ss)] := ss;
    end for;
m:=0; for i in [1..351] do if cst[i] ne []
    then m:=m+1; end if; end for; m;
Orbits(N73417);
N73214:=Stabiliser (N,[7,3,2,1,4]);
    SSS:={[7,3,2,1,4]}; SSS:=SSSˆN;
    SSS; #(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
            n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
            *ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
            *ts[Rep(Seqq[i])[5]]
        then print Rep(Seqq[i]);
        end if; end for; end for;
N73214s := N73214;
for n in N do if 7ˆn eq 4 and 3ˆn eq 1
and 2ˆn eq 2 and 1ˆn eq 3 and 4ˆn eq 7 then
    N73214s:=sub<N|N73214s,n>; end if; end for;
#N73214s;
[7,3,2,1,4]ˆN73214s;
end for;
N73214:=Stabiliser (N,[7,3,2,1,4]);
N73214;
N73214:=sub<N| (4,7)(1,3)(5,6)>;
#N73214;
[7,3,2,1,4]^N73214;
T:=Transversal(N,N73214);
    for i in [1..#T] do
{[7,3,2,1,4]^N73214}^T[i];
end for;
T73214:=Transversal(N,N73214);
for i in [1..#T73214] do
ss:=[7,3,2,1,4]^T73214[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N73214);
N73723:=Stabiliser (N,[7,3,7,2,3]);
SSS:={[7,3,7,2,3]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for;
N73723s := N73723;
for n in N do if 7^n eq 6 and 3^n eq 3 and
 7^n eq 6 and 2^n eq 4 and 3^n eq 3
then N73723s:=sub<N|N73723s,n>;
end if; end for;
#N73723s;
[7,3,7,2,3]^N73723s;
end if; end for;
N73723:=Stabiliser (N,[7,3,7,2,3]);
N73723;
N73723:=sub<N| (6,7) (2,4) (1,5)>;
#N73723;
[7,3,7,2,3]~N73723;
T:=Transversal(N,N73723);
for i in [1..#T] do
{[7,3,7,2,3]~N73723}~T[i];
end for;
T73723:=Transversal(N,N73723);
for i in [1..#T73723] do
ss:=[7,3,7,2,3]~T73723[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N73723);

N732142:=Stabiliser(N,[7,3,2,1,4,2]);
SSS:=[7,3,2,1,4,2]; SS:=SSS~N;
SS;#(SS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*
*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N732142s := N732142;
for n in N do if 7^n eq 4 and 3^n eq 1
and 2^n eq 2 and 1^n eq 3 and 4^n eq 7
and 2^n eq 2 then N732142s:=
sub<N|N732142s,n>; end if; end for;
#N732142s;
[7,3,2,1,4,2]~N732142s;
end if; end for;
N732142:=Stabiliser(N,[7,3,2,1,4,2]);
N732142;
N732142:=sub<N| (4,7) (1,3) (5,6)>;
#N732142;
\[ [7,3,2,1,4,2]^N 732142; \]
\[ T := \text{Transversal}(N, N 732142); \]
\[ \text{for } i \text{ in } [1..\#T] \text{ do} \]
\[ \{ [7,3,2,1,4,2]^N 732142 \}^T[i]; \]
\[ \text{end for}; \]
\[ T 732142 := \text{Transversal}(N, N 732142); \]
\[ \text{for } i \text{ in } [1..\#T 732142] \text{ do} \]
\[ ss := [7,3,2,1,4,2]^T 732142[i]; \]
\[ \text{cst}[\text{prodim}(1, ts, ss)] := ss; \]
\[ \text{end for}; \]
\[ m := 0; \text{for } i \text{ in } [1..351] \text{ do if cst}[i] \neq [] \]
\[ \text{then } m := m + 1; \text{ end if}; \text{ end for}; m; \]
\[ \text{Orbits}(N 732142); \]
\[ N 7321427 := \text{Stabiliser}(N, [7,3,2,1,4,2,7]); \]
\[ \text{SSS} := \{ [7,3,2,1,4,2,7] \}; \]
\[ \text{SSS} := \text{SSS}^N; \]
\[ \text{Seqq} := \text{Setseq}(\text{SSS}); \]
\[ \text{Seqq}; \]
\[ \text{for } i \text{ in } [1..\#\text{SSS}] \text{ do} \]
\[ \text{for } n \text{ in IM do} \]
\[ n * ts[\text{Rep}(\text{Seqq}[i])[1]] * ts[\text{Rep}(\text{Seqq}[i])[2]] \]
\[ * ts[\text{Rep}(\text{Seqq}[i])[3]] * ts[\text{Rep}(\text{Seqq}[i])[4]] * \]
\[ ts[\text{Rep}(\text{Seqq}[i])[5]] * ts[\text{Rep}(\text{Seqq}[i])[6]] * \]
\[ ts[\text{Rep}(\text{Seqq}[i])[7]] \]
\[ \text{then print } \text{Rep}(\text{Seqq}[i]); \]
\[ \text{end if}; \text{ end for}; \]
\[ N 7321427 s := N 7321427; \]
\[ \text{for } n \text{ in N do if } 7^n \text{ eq 5 and } 3^n \text{ eq 2 and } \]
\[ 2^n \text{ eq 3 and } 1^n \text{ eq 4 and } 4^n \text{ eq 1 } \]
\[ \text{and } 2^n \text{ eq 3 then} \]
\[ N 7321427 s := \text{sub}<N|N 7321427 s,n>; \]
\[ \text{end if}; \text{ end for}; \]
\[ \#N 7321427 s; \]
\[ [7,3,2,1,4,2,7]^N 7321427 s; \]
\[ * ts[1] * ts[3] * ts[5] \text{ then } n; \text{ end if}; \text{ end for}; \]
\[ N 7321427 := \text{Stabiliser}(N, [7,3,2,1,4,2,7]); \]
\[ N 7321427; \]
\[ N 7321427 := \text{sub}<N| (5,7)(2,3)(1,4)>; \]
\[ \#N 7321427; \]
\[ [7,3,2,1,4,2,7]^N 7321427; \]
T:=Transversal(N,N7321427);
for i in [1..#T] do
{[7,3,2,1,4,2,7]^N7321427}^T[i];
end for;
T7321427:=Transversal(N,N7321427);
for i in [1..#T7321427] do
ss:=[7,3,2,1,4,2,7]^T7321427[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..351] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N7321427);

N73214276:=Stabiliser(N,[7,3,2,1,4,2,7,6]);
SSS:={{[7,3,2,1,4,2,7,6]}; SSS:=SSS^N;
SSS;#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
*n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*
*ts[Rep(Seqq[i])][3]*ts[Rep(Seqq[i])][4]*
*ts[Rep(Seqq[i])][5]*ts[Rep(Seqq[i])][6]*
*ts[Rep(Seqq[i])][7]*ts[Rep(Seqq[i])][8]
then print Rep(Seqq[i]);
end if; end for; end for;
N73214276s := N73214276;
for n in N do if 7^n eq 5 and 3^n eq 2
and 2^n eq 3 and 1^n eq 4 and 4^n eq 1
and 2^n eq 3 and 7^n eq 5 and 6^n eq 6
then N73214276s:=sub<N|N73214276s,n>;
end if; end for;
#N73214276s;
[7,3,2,1,4,2,7,6]^N73214276s;
for n in IM do if ts[7]*ts[3]*ts[2]*ts[1]*
then n; end if; end for;
N73214276:=Stabiliser(N,[7,3,2,1,4,2,7,6]);
N73214276;
N73214276:=sub<N| (5,7) (2,3) (1,4)>;
#N73214276;
[7,3,2,1,4,2,7,6]^N73214276;
T := Transversal(N, N^73214276);
for i in [1..#T] do
    {N^73214276}^T[i];
end for;
T^73214276 := Transversal(N, N^73214276);
for i in [1..#T^73214276] do
    ss := [N^73214276[i];
    cst[prodim(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..351] do
    if cst[i] ne []
        then m := m + 1;
    end if;
end for;
m;

Orbits(N^73214276);

N^73476 := Stabiliser(N, [7, 3, 4, 7, 6]);
SSS := ([7, 3, 4, 7, 6]);
SSS := SSS^N;
SSS;#
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
    for n in IM do
            n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
            ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*
            ts[Rep(Seqq[i])[5]]
        then print Rep(Seqq[i]);
    end if;
end for;
end for;
N^73476s := N^73476;
#N^73476s;
[7, 3, 4, 7, 6]^N^73476s;
T := Transversal(N, N^73476);
for i in [1..#T] do
    {N^73476}^T[i];
end for;
T^73476 := Transversal(N, N^73476);
for i in [1..#T^73476] do
    ss := [N^73476[i];
    cst[prodim(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..351] do
    if cst[i] ne []
        then m := m + 1;
    end if;
end for;
m;

Orbits(N^73476);

N^73721 := Stabiliser(N, [7, 3, 7, 2, 1]);
SSS := ([7, 3, 7, 2, 1]);
SSS := SSS^N;
SSS; #(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
   n * ts[Rep(Seqq[i]) [1]] * ts[Rep(Seqq[i]) [2]] * 
   ts[Rep(Seqq[i]) [3]] * ts[Rep(Seqq[i]) [4]] * 
   ts[Rep(Seqq[i]) [5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N73721s := N73721;
#N73721s;
[7,3,7,2,1]^N73721s;
T := Transversal(N,N73721);
for i in [1..#T] do
   {[7,3,7,2,1]^N73721}^T[i];
end for;
T73721 := Transversal(N,N73721);
for i in [1..#T73721] do
   ss := [7,3,7,2,1]^T73721[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..351] do if cst[i] ne []
   then m := m + 1; end if; end for; m;
Orbits(N73721);
N737216 := Stabiliser(N,[7,3,7,2,1,6]);
SSS := ([7,3,7,2,1,6]); SSS := SSS \ N;
SSS; #(SSS);
Seqq := Setseq(SSS); Seqq;
for i in [1..#SSS] do
for n in IM do
   n * ts[Rep(Seqq[i]) [1]] * ts[Rep(Seqq[i]) [2]] * 
   ts[Rep(Seqq[i]) [3]] * ts[Rep(Seqq[i]) [4]] * 
   ts[Rep(Seqq[i]) [5]] * ts[Rep(Seqq[i]) [6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N737216s := N737216;
for n in N do if 7^n eq 4 and 3^n eq 1 and 7^n 
   eq 4 and 2^n eq 2 and 1^n eq 3 and 6^n eq 5 
   then N737216s := sub<N|N737216s, n>;
end if; end for;
#N737216s;

N737216 := Stabiliser (N, [7, 3, 7, 2, 1, 6]);

N737216 := sub<N| (4, 7) (1, 3) (5, 6)>;

T := Transversal (N, N737216);
for i in [1..#T] do
{[7, 3, 7, 2, 1, 6] ^ N737216} ^ T[i];
end for;

T737216 := Transversal (N, N737216);
for i in [1..#T737216] do
ss := [7, 3, 7, 2, 1, 6] ^ T737216[i];
cst[prodim(1, ts, ss)] := ss;
end for;

m := 0; for i in [1..351] do if cst[i] ne [] then m := m + 1; end if; end for; m;

Orbits (N737216);

We use the Schreier System to convert permutation into word*/

D := DihedralGroup(7);
xx := D! (1, 2, 3, 4, 5, 6, 7);
yy := D!(1, 6)(2, 5)(3, 4);
N := sub<D| xx, yy>; #N; Set (N);

G<x, y, t> := Group<x, y, t|x^7, y^2, (x*y)^2, t^2, (t*y), (x*t)^0, (x*t*x)^0, (x*y*t*x*t)^3, (t*x*t*x*t)^7>;
H := sub<G| x, y, t^ (x^2) * t^ (x^4) * t^ (x^5) * t^ (x^4) * t^ (x^2)>;

f, G1, k := CosetAction (G, H);
IN := sub<G1| f(x), f(y)>;
IM := sub<G1| IN, f(t^ (x^2) * t^ (x^4) * t^ (x^5) * t^ (x^4) * t^ (x^2))>;

ts := [Id(G1) : i in [1 .. 7]];

A := f(x); B := f(y);
C := f(t);
N:=sub<G1|A,B,C>;
NN<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^2,
t^2, (t,y), (x*t)^0, (x*t*x)^0, (x*y*t*x*t)^3,
(t*t*x*t)^7>;
G1:=NN;
Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
    P:=[Id(N): l in [1..#Sch[i]]];
    for j in [1..#Sch[i]] do
        if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
        if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
        if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
        if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
    end for;
    PP:=Id(N);
    for k in [1..#P] do
        PP:=PP*P[k]; end for;
    ArrayP[i]:=PP;
end for;
Appendix G: MAGMA Code for Mixed Extension \((2^6 : L_2(7)) : 2\)

\[
a := 0; b := 8; c := 0; d := 3;
G<x,y,t> := \text{Group}\langle x, y, t \mid x^7, y^2, (x*y)^2, t^2, \\
(t, y), (x*t)^a, (x*t*x)^b, (x*y*t)^c, (t*t*x)^d \rangle; \#G;
f, G1, k := \text{CosetAction}(G, \text{sub}\langle G \mid x, y \rangle);
\text{CompositionFactors}(G1);
\text{Center}(G1);
\text{NL} := \text{NormalLattice}(G1);
\text{NL};
\text{MinimalNormalSubgroups}(G1);
X := \text{AbelianGroup}(\text{GrpPerm}, [2, 2, 2, 2, 2, 2]);
s := \text{IsIsomorphic}(X, \text{NL}[2]); \#s;
q, ff := \text{quo}\langle \text{NL}[3] \mid \text{NL}[2] \rangle;
\text{CompositionFactors}(q);
\text{NumberOfGenerators}(\text{NL}[3]);
T := \text{Transversal}(\text{NL}[3], \text{NL}[2]);
/* Note we store the generators of \text{NL}[2] and the transversals of \text{NL}[3] */
\text{NumberOfGenerators}(\text{NL}[2]);
/* now write schreier system, you want the action of \(2^6\) so write it with that, you need N and NN, check their number so you know they equal 64 = 2^6 */
N := \text{sub}\langle G1 | A, B, C, D, E, F \rangle; \#N;
/* presentation for \(2^6 = 64\) abelian they commute */
\text{NN}\langle k, l, m, n, o, p \rangle := \text{Group}\langle k, l, m, n, o, p \mid k^2, l^2, m^2, \\
n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p) \rangle;
\#NN;
/* now you can run the system, [1..64] because \(2^6\).
then there are 6 generators so there will be six things, in this case they’re all order 2 (2^6) so you don’t include their inverses */
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..64]];
for i in [2..64] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
/* that’s the system, so now you can write T[3]^4 as elements of 2^6, earlier we called T[3]^4= T34 */
for i in [1..64] do if ArrayP[i] eq T34 then Sch[i];
end if; end for;
T34 eq D;
Order(ff(T2));
q;
#sub<q| ff(T2),ff(T3)>;
Order(ff(T2)*ff(T3));
H<r,s>:= Group<r,s|r^2,s^4,(r*s)^7,(r,s)^4,(r*s^2)^3>;
/!*PSL(2,7) Presentation*/
/!* H<r,s>:= Group<r,s|r^2,s^4=n,(r*s)^7,(r,s)^4,(r*s^2)^3>;
I:=[Id(NN): i in [1..13]];
for i in [1..64] do if ArrayP[i] eq T34 then Sch[i]; I[1]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq A^T2 then Sch[i]; I[2]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq B^T2 then Sch[i]; I[3]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq C^T2 then Sch[i]; I[4]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq D^T2 then Sch[i]; I[5]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq E\^T2
then Sch[i]; I[6]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq F\^T2
then Sch[i]; I[7]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq A\^T3
then Sch[i]; I[8]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq B\^T3
then Sch[i]; I[9]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq C\^T3
then Sch[i]; I[10]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq D\^T3
then Sch[i]; I[11]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq E\^T3
then Sch[i]; I[12]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq F\^T3
then Sch[i]; I[13]:=Sch[i]; end if; end for;
for i in [1..64] do if ArrayP[i] eq (T2,T3)^4
then Sch[i]; end if; end for;
NN<k, l, m, n, o, p, r, s>:=Group<k, l, m, n, o, p, r, s | k^2, l^2,
m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p), r^2,
s^4=l*m*p, (r*s)^7, (r, s)^4=k*l*m*n^*p, (r*s^2)^3, k^r=k*m*n,
l^r=n^*o*p, m^r=m, n^r=m*n^*o, o^r=o, p^r=l*m*n,
k^s=k*m*n^*o*p, l^s=n*p, m^s=k, n^s=k*l*n*p, o^s=p,
p^s=k*l*m*n>;
#NN;
f1, g, k1 := CosetAction(NN, sub<NN | Id(NN)>);
s, t := IsIsomorphic(NL[3], g);
s;
/*Now find an element of order 2 in G1 but outside NL[3]*/
for g in G1 do if Order(g) eq 2 and sub<G1 | NL[3], g> eq G1
then Z:=g; break; end if; end for;
G1 eq sub<G1 | NL[3], Z>;
/*-----------------------------------------*/
/*We find the action of Z on the generators
A, B, C, D, E, F, T2, T3 of NL[3]*/
N:=sub<G1 | A, B, C, D, E, F, T2, T3>; #N;
\[ \text{NN} = \langle k, l, m, n, o, p, r, s \rangle : = \text{Group} \langle k, l, m, n, o, p, r, s \mid k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p), (l, m), (l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o), (n, p), (o, p), r^2, s^2 = l^m*p, (r*s)^7, (r, s)^4 = k^1*l^1*m^1*n^1, (r^s^2)^3, k^r = k^1*m^1*n^1*p^1, m^r = m^1*n^1*p^1, n^r = n^1*p^1, m^s = k^1*l^1*m^1*n^1, o^s = p^1*p^1 = k^1*l^1*m^1*n^1; \]

\[ \text{Sch} = \text{SchreierSystem}(\text{NN}, \text{sub}<\text{NN}|\text{Id}(\text{NN})>); \]

\[ \text{ArrayP}[i] = [\text{Id}(\text{N}) : i \in [1..10752]]; \]

for \( i \) in [2..10752] do
  \( P[j] = [\text{Id}(\text{N}) : l \in [1..\#\text{Sch}[i]]] ; \)
  for \( j \) in [1..\#\text{Sch}[i]] do
    if Eltseq(\text{Sch}[i])[j] eq 1 then \( P[j] = \text{A} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 2 then \( P[j] = \text{B} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 3 then \( P[j] = \text{C} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 4 then \( P[j] = \text{D} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 5 then \( P[j] = \text{E} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 6 then \( P[j] = \text{F} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 7 then \( P[j] = \text{T2} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq 8 then \( P[j] = \text{T3} \); end if;
    if Eltseq(\text{Sch}[i])[j] eq -8 then \( P[j] = \text{T3}^-1 \); end if;
  end for;
  \( \text{PP} = \text{Id}(\text{N}); \)
  for \( k \) in [1..\#P] do
    \( \text{PP} = \text{PP} * P[k] \);
  end for;
  \( \text{ArrayP}[i] = \text{PP} \);
end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{A} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{B} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{C} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{D} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{E} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{F} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{T2} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;
for \( i \) in [1..10752] do if ArrayP[i] eq \text{T3} \( ^Z \) then
  \( \text{Sch}[i]; \text{I}[i] = \text{Sch}[i]; \)
end if; end for;

/* The presentation of the mixed extension */
\[ \text{H1} = \langle k, l, m, n, o, p, r, s, g \rangle : = \text{Group} \langle k, l, m, n, o, p, r, s, g \mid k^2, l^2, m^2, n^2, o^2, p^2, (k, l), (k, m), (k, n), (k, o), (k, p) \rangle, \]
\((l, m), (l, n), (l, o), (l, p), (m, n), (m, o), (m, p), (n, o), (n, p)\),
\((o, p)\),
\(r^2, s^4 = l \times m \times p, (r \times s)^7, (r, s)^4 = k \times l \times m \times o \times p,\)
\((r \times s^2)^3, k \times r = k \times m \times o, l \times r = n \times o \times p, m \times r = m, n \times r = m \times n \times o, o \times r = o, p \times r = l \times m \times n, k \times s = k \times m \times n \times o \times p, l \times s = n \times p, m \times s = k, n \times s = k \times l \times n \times p, o \times s = p, p \times s = k \times l \times m \times n,\)
\(g^2, k \times g = l \times n, l \times g = s \times o \times s^{-1}, m \times g = n \times s, n \times g = k \times s \times o \times s^{-1}, o \times g = k \times m \times n \times p, p \times g = p \times s, r \times g = k \times s \times r \times l \times s \times r \times s^{-1} \times r \times s^{-1} \times r;\)
\#H1;

\(f, h1, k1 := \text{CosetAction}(H1, \text{sub}<H1|\text{Id}(H1)>);\)
\(s := \text{IsIsomorphic}(h1, G1); s;\)
Appendix H: MAGMA Code for DCE of $M_{12}$

```magma
S1:=Sym(72);
bb:=S1!(1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23)(4, 21, 53, 47, 42, 20)(5, 26, 39, 7, 31, 32)(11, 29, 57, 24, 61, 44)(12, 28, 35, 14, 37, 34)(15, 22, 36, 19, 45, 30)(17, 49, 50, 66, 69, 71)(18, 59, 46, 55, 41, 48)(27, 67, 58, 38, 52, 56)(33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64);
N:=sub<S1|aa, bb, cc>;
G<a,b,c>:=Group<a,b,c| a^2, b^6, c^4, (b^-1 * a)^2, (a*c^-1)^2, b^-1*c^-2*b^2*c^2*b^-1, (c^-1*b)^3, c^-1*b^-1*c^-2*b^-1*c^-2*b^-1*c^-1>;
Sch:=SchreierSystem(G, sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..432]];
for i in [2..432] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
```

if Eltseq(Sch[i])[j] eq 1 then P[j]:=aa; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=bb; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=bb^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=cc; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=cc^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
    PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [50..100] do Sch[i], ArrayP[i]; end for;
G<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),
(a*c^-1*b^-1*c*b^2*t^b))^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|a,b,c>);
CompositionFactors(G1);
Center(G1);
/*store center as ccc */
A:=f(a);
B:=f(b);
C:=f(c);
D:=f(t);
N:=sub<G1|A,B,C,D>;
NN<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^-1*c^2*b^-1*c^-1,
t^2,(t,a),(t,a * c * b^-1 * c^-1 * b^-1 * c),
(a*c^-1*b^-1*c*b^2*t^b))^3>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=>[Id(N): i in [1..#N]];
for i in [2..#N] do
    P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;

end for;

PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;

ArrayP[i]:=PP;
end for;

for i in [1..#N] do if ArrayP[i] eq ccc then
print Sch[i]; end if; end for; /* Gives me the center
in terms of a, b, c, and t */

/********************************************************
/* Now we need to construct the double coset
enumeration G=M12=9540*/
S1:=Sym(72);
aa:=S1!(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)(11, 44)(12, 37)
(14, 28)(16, 30)(17, 38)(18, 54)(21, 42)(22, 23)(24, 57)
(52, 71)(56, 69)(59, 65)(60, 64)(68, 72);
bb:=S1!(1, 2, 9, 13, 6, 8)(3, 16, 25, 10, 40, 23)
(4, 21, 53, 47, 42, 20)(5, 26, 39, 7, 31, 32)
(11, 29, 57, 24, 61, 44)(12, 28, 35, 14, 37, 34)
(15, 22, 36, 19, 45, 30)(17, 49, 35, 67)
(18, 59, 46, 55, 41, 48)(27, 67, 58, 38, 52, 56)
(33, 70, 65, 54, 63, 51)(43, 72, 68, 62, 60, 64);
cc:=S1!(1, 3, 5, 15)(2, 10, 12, 42)(4, 22, 23, 20)
(6, 29, 31, 54)(7, 33, 34, 55)(8, 21, 37, 19)
(9, 18, 39, 61)(11, 45, 46, 40)(13, 50, 35, 67)
(14, 52, 26, 62)(16, 38, 53, 69)(17, 30, 56, 47)
(24, 41, 51, 57)(25, 44, 36, 70)(27, 60, 64, 66)
(28, 43, 32, 71)(48, 58, 65, 72)(49, 63, 68, 59);
N:=sub<S1|aa,bb,cc>;
G<a,b,c,t>:=Group<a,b,c,t| a^2,b^6,c^4,
(b^-1 * a)^2,
(a*c^-1)^2,
b^-1*c^-2*b^2*c^2*b^-1,
(c^-1*b)^3,
c^-1*b^-1*c^-2*b^1*c^2*b^-1*c^-1>,
\[ t^2, (t, a), (t, a * c * b^-1 * c^-1 * b^-1 * c), \\
(a*c^-1*b^-1*c*b^2*t^3(b))^3, \\
a * b^3 * c * t * b * t * b^-1 * t * b * t * c>;
\]
\[ f, G_1, k := \text{CosetAction}(G, \text{sub}<G|a,b,c>); \\
\text{CompositionFactors}(G_1); \\
\#G_1; \\
\text{DoubleCosets}(G, \text{sub}<G|a,b,c>, \text{sub}<G|a,b,c>); \\
\#\text{DoubleCosets}(G, \text{sub}<G|a,b,c>, \text{sub}<G|a,b,c>); \\
\text{Index}(G, \text{sub}<G|a,b,c>); \\
\text{IN} := \text{sub}<G|f(a), f(b), f(c)>;
\]
\[
\text{ts} := [\text{Id}(G_1) : i \in [1..72]]; \\
\text{ts}[1] := f(t); \text{ts}[2] := f(t^b); \text{ts}[3] := f(t^c); \\
\text{ts}[4] := f(t^((b^-1*c*b^-1))); \text{ts}[5] := f(t^((c^2))); \\
\text{ts}[6] := f(t^((a*b^-2))); \text{ts}[7] := f(t^((b^2*c^2*b^2))); \\
\text{ts}[8] := f(t^((a*b^-1))); \text{ts}[9] := f(t^((a*b^2))); \\
\text{ts}[10] := f(t^((a*b*c))); \text{ts}[11] := f(t^((b*c)^2)); \\
\text{ts}[12] := f(t^((b^2*c^2))); \text{ts}[13] := f(t^((b^3))); \\
\text{ts}[14] := f(t^((c*b+c*b*c^-1))); \text{ts}[15] := f(t^((a*c^-1))); \\
\text{ts}[16] := f(t^((a*c*b))); \text{ts}[17] := f(t^((a*c^-1*b^-1*c^-1))); \\
\text{ts}[18] := f(t^((b^2*c^2))); \text{ts}[19] := f(t^((a*b^-1*c^-1))); \\
\text{ts}[20] := f(t^((b*c^2))); \text{ts}[21] := f(t^((b^-1*c))); \\
\text{ts}[22] := f(t^((c^-1*b^1))); \text{ts}[23] := f(t^((b^2*c^-1))); \\
\text{ts}[24] := f(t^((b^2*c^-1*b^-1))); \text{ts}[25] := f(t^((b^3*c^-1))); \\
\text{ts}[26] := f(t^((c^2*b))); \text{ts}[27] := f(t^((b^3*c^2))); \\
\text{ts}[28] := f(t^((b^2*c^2))); \text{ts}[29] := f(t^((a*b^-2*c)))); \\
\text{ts}[30] := f(t^((a*c^-1*b^-1))); \text{ts}[31] := f(t^((c^-1*b^-2))); \\
\text{ts}[32] := f(t^((c^-1*b^-1))); \text{ts}[33] := f(t^((b^2*c^2*c^-1*b^-1))); \\
\text{ts}[34] := f(t^((b^2*c^2))); \text{ts}[35] := f(t^((b^3*c^2))); \\
\text{ts}[36] := f(t^((a*c^-1*b^-2))); \text{ts}[37] := f(t^((b^-1*c^-1))); \\
\text{ts}[38] := f(t^((c*b+c))); \text{ts}[39] := f(t^((b^2*c)^2)); \\
\text{ts}[40] := f(t^((b^2*c))); \text{ts}[41] := f(t^((b^-2*c^-1*c^-1))); \\
\text{ts}[42] := f(t^((b^-1*c^-1*c))); \text{ts}[43] := f(t^((c^2*b^-1*c^-1))); \\
\text{ts}[44] := f(t^((b^2*c^-1*b^-1))); \text{ts}[45] := f(t^((a*c^-1*b^-2))); \\
\text{ts}[46] := f(t^((b*c*c*b*c^-1))); \text{ts}[47] := f(t^((b*c^2))); \\
\text{ts}[48] := f(t^((b^2*c^2))); \text{ts}[49] := f(t^((c^-1*b^-1*c^-1*b^-1))); \\
\text{ts}[50] := f(t^((b^-2*c^-1))); \text{ts}[51] := f(t^((b^-2*c^-1))); \\
\text{ts}[52] := f(t^((c^2*b))); \text{ts}[53] := f(t^((a*b^-1*c*b))); \\
\text{ts}[54] := f(t^((a*b^-2*c^-1))); \text{ts}[55] := f(t^((b^2*c*b*c^-1))); \\
\text{ts}[56] := f(t^((c^-1*b^-1*c))); \text{ts}[57] := f(t^((b^-2*c*b))); \\
\text{ts}[58] := f(t^((c*b^2*c*b^-1))); \text{ts}[59] := f(t^((b^2*c*r))); \\
\text{ts}[60] := f(t^((c^2*b*c*b))); \text{ts}[61] := f(t^((b^2*c^2))); \\
\text{ts}[62] := f(t^((c^2*b*c))); \text{ts}[63] := f(t^((b^-2*c^-1*b))); \\
\text{ts}[64] := f(t^((c^2*b^-1*c^-1*b^-1))); \]
\[ ts[65] := f(t^((a \cdot b - 2 \cdot c - 1) \cdot b - 1)) \]
\[ ts[66] := f(t^((a \cdot c \cdot b \cdot c - 1) \cdot b - 1)) \]
\[ ts[67] := f(t^((b \cdot 3 \cdot c - 1))) \]
\[ ts[68] := f(t^((b \cdot 2 \cdot c \cdot b \cdot c - 1))) \]
\[ ts[69] := f(t^((c \cdot b \cdot c - 1))) \]
\[ ts[70] := f(t^((b - 1 \cdot c - 1) \cdot b - 1)) \]
\[ ts[71] := f(t^((c - 1) \cdot b - 1)) \]
\[ ts[72] := f(t^((c \cdot 2 \cdot b - 1) \cdot c - 1) \cdot b)) \]

prodim := function(pt, Q, I)
/*
Return the image of pt under
permutations Q[I] applied sequentially.
*/
v := pt;
for i in I do
  v := v^Q[i];
end for;
return v;
end function;
cst := [null : i in [1 .. Index(G, sub<G|a,b,c>)]]
where null is [Integers()] | ;
for i := 1 to 72 do
  cst[prodim(1, ts, [i])] := [i];
end for;
m := 0;
for i in [1..220] do if cst[i] ne [] then
  m := m + 1; end if; end for; m;
N1 := Stabiliser (N, [1]);
N1; #N1;
T1 := Transversal(N, N1);
for i in [1..#T1] do
  ss := [1]^T1[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..220] do if cst[i] ne [] then
  m := m + 1; end if; end for; m;
Orbits(N1);
N12 := Stabiliser (N, [1,2]);
N12;
SSS := ([1,2]); SSS := SSS^N;
SSS; #(SSS);
Seqq := Setseq(SSS);
Seqq;
for i in [1..#SSS] do
  for n in IN do
if ts[1] * ts[2] eq n * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s := N12;
for n in N do if 1^n eq 16 and 2^n eq 40
then N12s := sub<n|N12s, n>; end if; end for;
#N12s;
N12s;
[1,2]^N12s;

N12 := Stabiliser (N, [1, 2]);
N12;
N12 := sub<n|(3, 70)(4, 11)(5, 28)(6, 32)(8, 34)(9, 12)
(10, 72)(13, 39)(15, 66)(16, 67)(17, 29)(18, 60)(19, 57)
(26, 37)(27, 46)(30, 51)(33, 58)(36, 38)(40, 61)
(41, 50)(42, 59)(43, 63)(44, 52)(45, 62)(47, 64)
(53, 54)(65, 69), (1, 16, 52, 39, 44, 19)
(2, 40, 29, 32, 17, 15)(3, 26, 60, 43, 5, 22)
(4, 36, 65, 68, 46, 23)(6, 57, 61, 13, 66, 67)
(7, 70, 10, 28, 72, 18)(8, 53, 71, 35, 58, 54)
(9, 59, 50, 14, 56, 42)(11, 30, 27, 64, 69, 25)
(12, 20, 21, 34, 41, 33)(24, 48, 38, 47, 49, 51)
(31, 63, 62, 37, 45, 55), (1, 16, 6, 17, 32, 57)
(2, 40, 13, 44, 39, 66)(3, 10, 18, 43, 62, 55)
(4, 30, 24, 46, 64, 49)(5, 72, 28, 22, 31, 63)
(7, 70, 26, 45, 37, 60)(8, 41, 14, 54, 34, 53)
(9, 20, 35, 58, 12, 59)(11, 36, 47, 69, 68, 48)
(15, 67, 52, 19, 61, 29)(21, 42, 50, 33, 54, 71)
(23, 51, 27)(25, 38, 65)>;
#N12;
[1,2]^N12;
T := Transversal (N, N12);
for i in [1..#T] do
{{[1,2]^N12}^T[i]};
end for;
T12 := Transversal (N, N12);
for i in [1..#T12] do
ss := [1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12);

N15:=Stabiliser (N,[1,5]);
N15;
SSS:={[1,5]}; SSS:=SSS^N;
SSS:#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N15s := N15;
for n in N do if 1^n eq 5 and 5^n eq 1
then N15s:=sub<N|N15s,n>; end if; end for;
#N15s;
N15s;
[1,5]^N15s;
then n; end if; end for;
ts[1]*ts[5] eq f(a)*ts[5]*ts[1];
N15:=Stabiliser (N,[1,5]);
N15;
N15:=sub<N|(2, 8)(3, 15)(4, 20)(6, 9)(10, 19)
(11, 44)(12, 37)(14, 28)(16, 30)(17, 38)(18, 54)
(21, 42)(22, 23)(24, 57)(25, 45)(26, 32)(27, 66)
(56, 69)(59, 65)(60, 64)(68, 72), (1, 5)(2, 12)
(3, 15)(4, 23)(6, 31)(7, 34)(8, 37)(9, 39)(10, 42)
(11, 46)(13, 35)(14, 26)(16, 53)(17, 56)(18, 61)
(19, 21)(20, 22)(24, 51)(25, 36)(27, 64)(28, 32)
(29, 54)(30, 47)(33, 55)(38, 69)(40, 45)(41, 57)
(58, 72)(59, 63)(60, 66),(1, 5)(2, 37)(4, 22)(6, 39)
(7, 34)(8, 12)(9, 31)(10, 21)(11, 70)(13, 35)(14,
32)(16, 47)(17, 69)(18, 29)(19, 42)(20, 23)(24, 41)
(25, 40)(26, 28)(27, 60)(30, 53)(36, 45)(38, 56)
(43, 52)(44, 46)(48, 59)(49, 72)(51, 57)(54,
61)(58, 68)(62, 71)(63, 65)(64, 66)>;
\#N15;
[1,5]^N15;
T := Transversal(N, N15);
for i in [1..#T] do
{[1,5]^N15}^T[i];
end for;
T15 := Transversal(N, N15);
for i in [1..#T15] do
ss := [1,5]^T15[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..220] do if cst[i] ne []
then m := m+1; end if; end for;
m;
Orbits(N15);

N113 := Stabiliser(N, [1,13]);
N113;
SSS := {[1,13]}; SSS := SSS^N;
SSS;#(SSS);
Seq := Setseq(SSS);
Seq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[13] eq
n*ts[Rep(Seq[i])[1]]*ts[Rep(Seq[i])[2]]
then print Rep(Seq[i]);
end if; end for; end for;
N113s := N113;
for n in N do if 1^n eq 69 and 13^n eq 49
then N113s := sub<N|N113s,n>; end if; end for;
#N113s;
N113s;
[1,13]^N113s;
n; end if; end for;
ts[1]*ts[13] eq f(b^3)*ts[69]*ts[49];
N113 := Stabiliser(N, [1,13]);
N113;
N113 := sub<N|(2, 8) (3, 15) (4, 20) (6, 9) (10, 19)
(11, 44) (12, 37) (14, 28) (16, 30) (17, 38) (18, 54)
(21, 42) (22, 23) (24, 57) (25, 45) (26, 32) (27, 66)
(29, 61) (31, 39) (33, 55) (36, 40) (41, 51) (43, 62)
(46, 70) (47, 53) (48, 63) (49, 58) (50, 67) (52, 71)
(56, 69) (59, 65) (60, 64) (68, 72), (1, 69, 13, 49)
(2, 38, 6, 27) (3, 22, 10, 45) (4, 37, 47, 28) (5, 30, 7, 36)
(8, 24, 9, 11) (12, 51, 14, 65) (15, 60, 19, 72)
(16, 62, 40, 43) (23, 31, 25, 26) (18, 53, 55, 20)
(21, 70, 42, 63) (23, 31, 25, 26) (29, 56, 61, 58)
(32, 68, 39, 64) (33, 41, 54, 59) (34, 46, 35, 48)
(44, 50, 57, 71) (1, 69, 61, 56, 13, 49, 29, 58)
(2, 24, 71, 17, 6, 11, 50, 66) (3, 60, 32, 23,
10, 72, 39, 25) (4, 18, 59, 12, 47, 55, 41, 14)
(5, 30, 62, 16, 7, 36, 43, 40) (8, 38, 67, 57, 9, 27,
52, 44) (15, 22, 31, 64, 19, 45, 26, 68) (20, 37,
51, 54, 53, 28, 65, 33) (21, 63, 34, 46, 42, 70, 35, 48);
#N113;
[1,13]^N113;
T:=Transversal(N,N113);
for i in [1..#T] do
{[1,13]^N113]^T[i];
end for;
T113:=Transversal(N,N113);
for i in [1..#T113] do
ss:=[1,13]^T113[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..220] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N113);
/*Relations*/
for g in IN do for h in IN do
if ts[1]*ts[7] eq g*(ts[1])^h then g,h; end if;
end for; end for;
ts[1]*ts[7] eq ts[35];
for g in IN do for h in IN do
if ts[1]*ts[35] eq g*(ts[1])^h then g,h;
end if; end for;
ts[1]*ts[35] eq ts[7];
for g in IN do for h in IN do
if ts[1]*ts[21] eq g*(ts[1])^h then g,h;
end if; end for;
ts[1]*ts[21] eq f(a * b^-1 * c^-1 * b^-1 * c * b) *ts[17];
for g in IN do for h in IN do
if ts[1]*ts[3] eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[3] eq f(c^-1*b^-1)*ts[2]*ts[6];
for g in IN do for h in IN do
if ts[1]*ts[4] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[4] eq f(b*c*b)*ts[14]*ts[9];
for g in IN do for h in IN do
  if ts[1]*ts[6] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[6] eq f(a*b)*ts[18]*ts[55];
for g in IN do for h in IN do
  if ts[1]*ts[10] eq g*(ts[1]*ts[5])^h then g,h;
end if; end for; end for;
ts[1]*ts[10] eq f(a*b)*ts[18]*ts[55];
for g in IN do for h in IN do
  if ts[1]*ts[16] eq g*(ts[1]*ts[5])^h then g,h;
end if; end for; end for;
ts[1]*ts[16] eq f(a*b)*ts[18]*ts[55];
for g in IN do for h in IN do
  if ts[1]*ts[17] eq g*(ts[1]*ts[5])^h then g,h;
end if; end for; end for;
ts[1]*ts[17] eq f(a*b)*ts[18]*ts[55];
for g in IN do for h in IN do
  if ts[1]*ts[2] eq g*(ts[1])^h then g,h; end if; end for;
for g in IN do for h in IN do
  if ts[1]*ts[3] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[2]*ts[3] eq f(c*b^-1 * c^-1 * b^-1 * c * b)
*ts[28]*ts[37];
for g in IN do for h in IN do
  if ts[1]*ts[13] eq g*(ts[1]*ts[5])^h then g,h;
end if; end for; end for;
ts[1]*ts[13] eq f(a*b*c*ts[15]*ts[29];
for g in IN do for h in IN do
  if ts[1]*ts[18] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[18] eq f(a*b^-1 * c^-1 * b^-1 * c * b)
*ts[59]*ts[41];
for g in IN do for h in IN do
  if ts[1]*ts[25] eq g*(ts[1]*ts[5])^h then g,h; end if; end for; end for;
ts[1]*ts[25] eq f(a*b^-1 * c^-1 * b^-1 * c * b)*ts[53]
*ts[58];
for g in IN do for h in IN do
  if ts[1]*ts[22] eq g*(ts[1])^h then g,h; end if; end for;
for g in IN do for h in IN do
  if ts[1]*ts[3] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[2]*ts[3] eq f(c*b^-1 * c^-1 * b^-1
* c^-1 * b^-1)*ts[61]*ts[33];
for g in IN do for h in IN do
  if ts[1]*ts[4] eq g*(ts[1]*ts[5])^h then
g,h; end if; end for; end for;
ts[1]*ts[2]*ts[4] eq f(c*b^-2 * c * b * c)
*ts[28]*ts[37];
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[8] eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[2]*ts[8] eq f(c^-1 * b * c * b)*
ts[71]*ts[43];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[3] eq g*(ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[3] eq f(c * b^-2)*ts[63]*ts[60];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[7] eq g*(ts[1]*ts[13])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[7] eq f(a * b * c^2 * b^-1)*
ts[72]*ts[17];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[13] eq g*(ts[1]*ts[2])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[13] eq f(a * c^-1 * b^3
* c^-1)*ts[68]*ts[46];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[33] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[33] eq f(a * c^2 * b^-1*
c^-1)*ts[2]*ts[37];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[2] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[2] eq f(a * c * b^-2)*ts[44]*ts[59];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[4] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[4] eq f(b * c * b^-2)*ts[72]*ts[57];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[6] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[6] eq f(a * b^-1 * c * b * c^2)*ts[24]*ts[68];
for g in IN do for h in IN do
if ts[1]*ts[5]*ts[10] eq g*(ts[1]*ts[5])^h
then g,h; end if; end for; end for;
ts[1]*ts[5]*ts[10] eq f(b^3 * c * b^-1)*ts[20]*ts[53];
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[11] \) eq \( g \times (ts[1])^h \) then g,h; 
end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[11] \) eq \( f(a \times c^{-1} \times b \times c \times b^{-2}) \times ts[63] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[14] \) eq \( g \times (ts[1])^h \) then g,h; 
end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[14] \) eq \( f(a \times c^{-1} \times b^3 \times c) \times ts[60] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[16] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[16] \) eq \( f(a \times c \times b^3 \times c \times b^{-3} \times c^{-1}) \times ts[28] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[17] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[17] \) eq \( f(c \times b^{-1} \times c^{-1} \times b^{-1} \times c^{-1}) \times ts[71] \times ts[21] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[18] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[18] \) eq \( f(a \times b^2 \times c \times b^{-1} \times c^{-1}) \times ts[49] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[24] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[25] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[25] \) eq \( f(a \times b^{-1} \times c \times b^{-1} \times c^{-1}) \times ts[64] \times ts[23] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[27] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[27] \) eq \( f(a \times c \times b^2 \times c) \times ts[48] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[43] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[43] \) eq \( f(b^2 \times c^{-1} \times b^{-1} \times c^{-1}) \times ts[50] \times ts[42] \); 
for g in IN do for h in IN do 
if \( ts[1] \times ts[5] \times ts[48] \) eq \( g \times (ts[1])^h \) then g,h; end if; end for; end for; 

\( ts[1] \times ts[5] \times ts[48] \) eq \( f(b \times c) \times ts[14] \);
for g in IN do for h in IN do
  if ts[1]*ts[5]*ts[49] eq g*(ts[1]*ts[13])^h
  then g,h; end if; end for; end for;
  ts[1]*ts[5]*ts[49] eq f(a * c^-1 * b^-1
  * c^-1 * b^2)*ts[54]*ts[42];
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[1] eq g*(ts[1])^h
    then g,h; end if; end for; end for;
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[5] eq g*(ts[1]*ts[2])^h
    then g,h; end if; end for; end for;
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[2] eq g*(ts[1])^h
    then g,h; end if; end for; end for;
  ts[1]*ts[13]*ts[2] eq f(a * b^-1 * c *
    b * c * b^-1)*ts[44];
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[3] eq g*(ts[1])^h
    then g,h; end if; end for; end for;
  ts[1]*ts[13]*ts[3] eq f(c^-1 * b^2)*ts[27];
  for g in IN do for h in IN do
    then g,h; end if; end for; end for;
  ts[1]*ts[13]*ts[4] eq f(b * c * b^-1 *
    c^-1 * b^-1 * c^-1)*ts[62]*ts[15];
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[21] eq g*(ts[1]*ts[5])^h
    then g,h; end if; end for; end for;
  for g in IN do for h in IN do
    if ts[1]*ts[13]*ts[21] eq g*(ts[1]*ts[5])^h
    then g,h; end if; end for; end for;
  ts[1]*ts[13]*ts[7] eq f(c * b^3 * c)
  *ts[66]*ts[59];
Bibliography


