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Geometric Constructions from an Algebraic Perspective

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Geometric Constructions from an Algebraic Perspective

A Thesis

Presented to the

Faculty of

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In Partial Fulfillment

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Master of Arts

in

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by

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Abstract

Many topics that mathematicians study at times seem so unrelated such as Geometry and Abstract Algebra. These two branches of math would seem unrelated at first glance. I will try to bridge Geometry and Abstract Algebra just a bit with the following topics. We can be sure that after we construct our basic parallel and perpendicular lines, bisected angles, regular polygons, and other basic geometric figures, we are actually constructing what in geometry is simply stated and accepted, because it will be proven using abstract algebra. Also we will look at many classic problems in Geometry that are not possible with only straightedge and compass but need a marked ruler.

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Chapter 1

Introduction

Around Plato's period, geometric constructions were what we would consider now tedious and lengthy. He did not use a compass such as the one we use in modern times. His collapsible compass, did just that with every use, collapsed. Throughout my research I decided to focus on the geometric constructions with non-collapsible compasses because they do the exact constructions without the many steps necessary with collapsible compasses. Since my focus is to show how we can shorten our proofs of geometric constructions by using Abstract Algebra instead of actually constructing, we will assume that when I use a compass it is non-collapsible. Euclid did extensive constructions of the triangle, square, and pentagon. In 1800, Erchinger constructed a regular heptadecagon (17-gon), and in 1832 the construction of a regular 257-gon was done by Richelot and Schwendenwein. Other geometers have constructed regular polygons with even more sides. Although many Greek mathematicians spent years on the construction of regular polygons with many sides, Gauss was the first to prove mathematically which regular polygons were actually constructible. I will show some "simple" constructions and later relate them to abstract algebra.

Chapter 2

Basic Constructions

Let us now begin with some examples of geometric constructions that are very basic. We are only allowed to use straightedge and compass.

Example 1. ***Midpoint Construction*** *Given $\overline{AB}$, we can construct its midpoint C.*

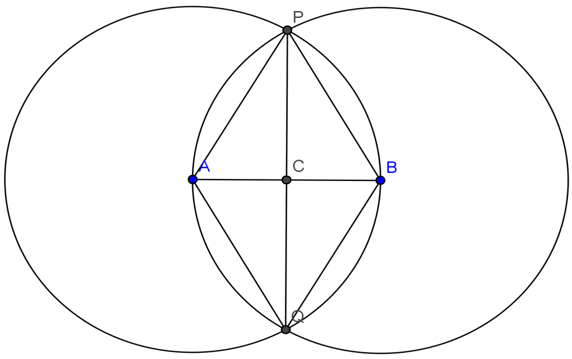


Figure 2.1: Construction of a Midpoint

First construct $\overline{AB}$. Now construct a circle with center A and radius $\overline{AB}$ and circle with center B with radius $\overline{AB}$. The two circles will intersect at two points, label them Q and P. Draw $\overline{QP}$ with straight edge. $\overline{QP} \cap \overline{AB} = C$ which is the midpoint of $\overline{AB}$.

Proof. $\triangle APQ \cong \triangle BPQ$ since $AP = PB = BQ = AQ$ because they are all radii in congruent circles. $\angle APQ \cong \angle BPQ$ because they are corresponding parts of congruent triangles $\Rightarrow \triangle APC \cong \triangle BPC$ by the side-angle-side postulate $\Rightarrow AC = CB \Rightarrow C$ is the midpoint of $\overline{AB}$.

□

Example 2. ***Bisecting an Angle*** *Draw $\angle A$. Construct a circle with center A that*

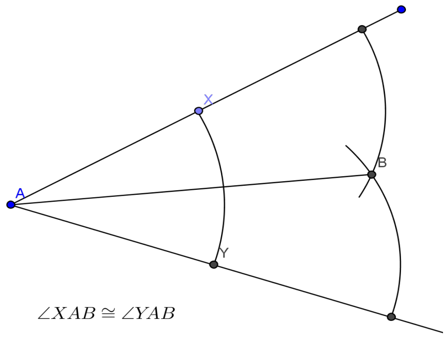


Figure 2.2: Construction of an Angle Bisector

intersects both sides of $\angle A$. Label the intersections X and Y. Construct circle with center X radius $\overline{AX}$. Construct circle with center Y and radius $\overline{AY}$. Mark the intersection of circle X and circle Y, point B. Draw $\overline{AB}$ which is the bisector of $\angle XAY$.

Proof. AX=XB=BY=AY are all congruent because they are radii of congruent circles. $\Rightarrow \triangle AXB \cong \triangle AYB \Rightarrow \angle XAB \cong \angle YAB \Rightarrow \overline{AB}$ is the bisector of $\angle XAY$.

□

We can construct many other figures using a compass and straightedge. To construct the figures there are basic steps that are defined as axiomatic basic constructions.

Definition 3. *Basic operations in the plane used in straightedge and compass constructions:*

1. *Draw a line through two given points.*
2. *Draw a circle with a given center and radius equal to the distance between two other given points.*
3. *Mark the point of the intersection of two straight lines, a line and a circle, and two circles.*

Every construction begins with given points, lines and circles and will be completed with a sequence of the above steps.

Now we begin to relate the construction of real numbers to algebra so we begin by constructing our unit measurement $\overline{OX}$ which has length 1. Although we are restricted to those simple 5 constructions, as we combine them in finite steps we will show that we are able to construct the rationals and also many of the irrational numbers. Also we can construct many of the geometric figures we all know well. We can construct angles of certain magnitudes and many regular polygons.

Definition 4. *A real number α is* ***constructible*** *by straightedge and compass if a segment of length $\mid \alpha \mid$ can be obtained starting with our unit segment by a straightedge and compass construction.*

Definition 5. *A set X has closure under an operation if performance of that operation on members of the set always produces a member of the same set; in this case we say the set is closed under this operation.*

Definition 6. *A nonempty set F is a $field$ if it has the following properties: If a, b, and $c \in F$ then*

1. $a+b \in F$
2. $a+b=b+a$
3. $(a+b)+c=a+(b+c)$
4. There is an element $0 \in F$ such that $a+0=a \; \forall \; a \in F$
5. There exist an element $-a \in F$ such that $a + (-a)=0$
6. $a \cdot b \in F$
7. $a \cdot b=b \cdot a$
8. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. $a \cdot (b+c) = a \cdot b+a \cdot c$ and $(b+c) \cdot a = b \cdot a+c \cdot a$
10. There is a unit element $1 \in F$ such that $a \cdot 1=1 \cdot a=a$

11. There exist a and b, $b \neq 0$, such that, $a \cdot \frac{1}{b} = 1$

Let C be the set of constructible real numbers. We now show that C is closed under the field operations of $\mathbb{R}$. This helps us make the connection between geometric constructions and Field Theory.

Proposition 7. *C is closed under addition and subtraction. If γ and δ are constructible real numbers with $0 \leq \delta \leq \gamma$, then so are $\gamma+\delta$ and γ-δ.*

Proof. We construct a circle with radius γ and label a radius $\overline{AB}$. Then construct a circle with center B and radius δ. The intersection of circle B and circle A give us the desired lengths AD=$\gamma+\delta$ and AC=γ-δ see Figure 3.C is closed under addition and subtraction. □

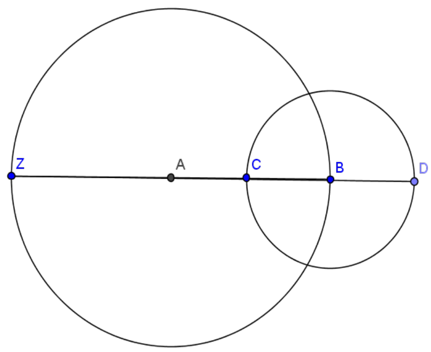


Figure 2.3: Addition and Subtraction Closure

Corollary 7.1. *Let C be the set of constructible real numbers. Then $\mathbb{Z} \subseteq C$.*

Proof. Since $1 \in$ C then, $\mathbb{Z} \in C$ therefore, $\mathbb{Z} \subseteq C$.

□

Proposition 8. *C is closed under multiplication and division.*

If α and β are constructible real numbers, then so is $\alpha \cdot \beta$ and $\frac{\alpha}{\beta}$.

Proof. Construct a segment $\overline{PC}$ and a line perpendicular to $\overline{PC}$ at P. Construct a circle with center P and radius α that intersects the perpendicular line at D. Construct another circle with center P and radius $\delta > \alpha$ that intersects the perpendicular line at Z. Draw

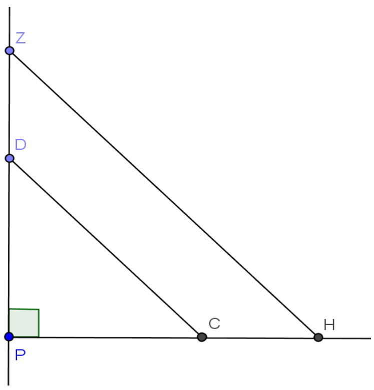


Figure 2.4: Multiplication Closure

$\overline{DC}$. Construct a line parallel to $\overline{DC}$ through point Z and label the intersection with $\overline{PC}$, point H. By the Angle Angle postulate of Similar Triangles we have that $\triangle PDC$ is similar to $\triangle PZH$. Corresponding sides of similar triangles are proportional therefore $\frac{DP}{ZP}$ = $\frac{PC}{PH}$. Let PC = 1, PH=$\beta \Rightarrow \frac{\alpha}{\delta}$=$\frac{1}{\beta}$. So we have $\delta = \alpha \cdot \beta$. This proves that the product of two constructible numbers is constructible. Closure of division is similarly proven.

$\square$

Definition 9. *Let F be a field. The polynomial ring, F[X], in x over the field F is defined as the set of expressions, called polynomials in x, of the form*

$p = p_0 + p_1 x + p_2\ x^2 + \cdots + p_{m-1}\ x^{m-1} + p_m\ x^m$, *where* $p_0,\ p_1,,\ p_m$, *the coefficients of p, are elements of F.*

Definition 10. *Let F be a field, Let* $F(\alpha) = \{p(\alpha) \mid p(x) \in F[x]\}$. *If* $\alpha \notin F$, *then F is a proper subset of* $F(\alpha)$.

Proposition 11. *The set C of constructible real numbers is a field.*

Proof. Let C be the set of constructible numbers and let α, β, $\lambda \in C$.

1. Previously we showed that if α and $\beta \in C$, then α +β,α-β, also $\alpha \cdot \beta$, and $\frac{\alpha}{\beta}$, β $\neq$0, $\in C$.

2. (α+β)+λ=α+(β+λ).

3. $0 \in C$, α+0=α $\forall\ \alpha \in C$.

4. -$\alpha \in C$, α + (-α)=0.

5. $(\alpha \cdot \beta)\cdot \lambda = \alpha \cdot(\beta \cdot \lambda)$.

6. $1 \in C$ such that α ·1=1· α=α.

7. There exist a and b, b≠0,such that, a· $\frac{1}{b}$=1 .

Therefore C is a field.

□

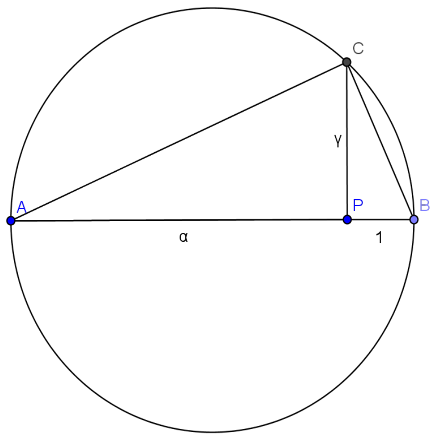


Figure 2.5: Square Root Construction

Proposition 12. *Let α be a constructible real number, $\alpha \geq 0$, the $\sqrt{\alpha}$ is constructible.*

Proof. Construct $AB = AP + PB$, let AP =α and $PB = 1$. Construct circle X with diameter AB. Construct $\overline{C}P \perp \overline{A}B$ with C on circle X.
Let $CP = \gamma \Rightarrow \gamma$ is constructible. $\triangle APC$ is similar to $\triangle CPB \Rightarrow \frac{PC}{PB} = \frac{PA}{PC} \Rightarrow \gamma^2 = \alpha$ $\Rightarrow \gamma = \sqrt{\alpha} \Rightarrow \sqrt{\alpha}$ is constructible.

□

2.1 Examples of Constructible Real Numbers

The set of natural numbers are easily shown to be constructible numbers with our unit segment but the following examples are not so obvious. Properties of C help us show that the following numbers are indeed constructible.

Example 13. *Fourth Root of Three*
Construct circle P with diameter AB=4.
Construct the perpendicular bisector to the radius $\overline{AP}$.
Label the midpoint M. *Label the intersection of circle P and the perpendicular bisector of the radius, point* C.
We know that any triangle inscribed in a semi-circle is a right triangle therefore $\triangle ABC$ *is a right triangle with* $\overline{CM}$ *an altitude from point* C. *This makes* $\overline{CM}$ *the geometric mean between* $\overline{AM}$ *and* $\overline{MB}$. *The radius of Circle P is 2 therefore AM=1 and BM=3. Since* $\overline{CM}$ *is the geometric mean between 1 and 3, then CM* $=\sqrt{3}$. *Now we construct segment ST=1+*$\sqrt{3}$.
We construct circle Q with diameter CM*=ST=1+*$\sqrt{3}$. *Draw point R on radius* $\overline{SQ}$ *so that SR=1 which leaves RT=CM=*$\sqrt{3}$. *Draw the perpendicular line through R to* $\overline{ST}$ *and we have the same properties as the first right triangle inscribed in circle P.* $\Rightarrow$ *The altitude from the right angle is the geometric mean in the right triangle of circle Q and therefore is the* $\sqrt[4]{3}$.

Generalizing this construction above we can repeat the process and construct any root of the form 2^n.

Example 14. *Is* $\sqrt{3+\sqrt[4]{5}}$ *constructible?*
The $\sqrt[4]{5}$ *is constructible by repeating the process of the square root twice. This makes 3 +* $\sqrt[4]{5}$ *the sum of constructible numbers therefore constructible. The square root of any constructible number is constructible therefore* $\sqrt{3+\sqrt[4]{5}}$ *is constructible.*

Example 15. *Are 3 -* $\dfrac{2}{\sqrt[8]{5}}$ *and* $\dfrac{3}{1+\sqrt[8]{5}}$ *constructible?*
They are both constructible because they are sums, differences, and quotients of constructible numbers which are all closed operations under C.

Example 16. *Is (*$\sqrt{2}$*+3*$\sqrt{3}$*) constructible?*
Clearly we can construct the $\sqrt{2}$ *and 3*$\sqrt{3}$ *and the sum of constructible numbers are constructible.*

2.2 Constructible Angles

Trigonometric Identities are very useful in relating our geometric constructions to algebra. The following identities are true for every angle α, β:

Pythagorean Identity

$\sin^2\alpha$+$\cos^2\alpha$=1

Sum of Angles Identity

$\sin(\alpha\pm\beta)$=$\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$

$\cos(\alpha\pm\beta)$=$\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

Double Angle Identities

$\cos 2\alpha$ =$\cos^2\alpha$-$\sin^2\alpha$

$\sin 2\alpha$ =2$\sin\alpha\cos\alpha$

Half Angle Identity

$\cos(\frac{\alpha}{2})$=$\sqrt{\frac{1+\cos\alpha}{2}}$

$\sin(\frac{\alpha}{2})$=$\sqrt{\frac{1-\cos\alpha}{2}}$

Example 17. *For any angle ϕ, the real number $\cos\phi$ is constructible if and only if the real number $\sin\phi$ is constructible.*

Proof. ($\Rightarrow$) Let ϕ be any angle and the $\cos\phi$ be a constructible real number.
By the Trigonometric Pythagorean Identity:

$\sin^2\phi$ + $\cos^2\phi$=1

$\sin^2\phi$=1-$\cos^2\phi$

$\sin\phi$=$\sqrt{1-\cos^2\phi}$.
As previously mentioned $\sqrt{1-\cos^2\phi}$ is a constructible number therefore $\sin\phi$ is constructible.

($\Leftarrow$) Similarly using the same identity and solving for $\cos\phi$, we get that $\cos\phi$=$\sqrt{1-\sin^2\phi}$ is a constructible number.

$\square$

Example 18. *As we constructed the midpoint of a segment in a previous example(1), we can also construct an equilateral triangle with the radii of our congruent circles.* $\triangle PAB$

is equilateral therefore we can construct an angle of magnitude 60°.
Also, we can construct a right angle with the construction of the perpendicular lines.

Example 19. *Since we constructed both* 60° *and* 90° *we can bisect these angles like in example 2 and construct* 30° *and* 45°.

Chapter 3

Algebraic Description of the Field of Constructible Real Numbers C

We will describe our field C of constructible real numbers by using some field theory.
Choosing our origin O =$(0,0)$ and our unit segment point $X=(1,0)$ on the $x-axis$ we have the following:

Definition 20. *Square Root Tower*

Let F be a subfield of $\mathbb{R}$. Then a ***square root tower*** *over F is a sequence of fields K_0,K_1, ... ,K_n such that $F=K_0 \subseteq K_1 \subseteq ... \subseteq K_n \subseteq \mathbb{R}$ and $K_{i+1}=K_i(\sqrt{\rho})$ for some $\rho \in K_i$ with $\sqrt{\rho} \notin K_i$ $\forall$ j satisfying $1 \leq j \leq i$. K_n is the top of the tower.*

Definition 21. *The Polynomial Ring Extension*
For every ring R, the polynomial ring $R[x]$is a ring extension of R. If S is a ring extension of R, and a is an element in S, the set $R[a]$ equals all f(a) such that f(x) is in $R[x]$, is the smallest subring of S containing R and a, and is a ring extension of R.

Proposition 22. *Let F be a subfield of $\mathbb{R}$ and let $f(x)=x^2+ax+b \in$ F[x] be a quadratic polynomial with coefficients in F. If f(x) has a zero in $\mathbb{R}$, then both zeros α_1 and α_2 belong to $\mathbb{R}$ and either α_1, $\alpha_2 \in F$ or α_1, $\alpha_2 \in F(\sqrt{\gamma}) \subseteq \mathbb{R}$.*

Proof. Using the quadratic formula to find the zeros of $f(x)$, α_1=$\frac{-a+\sqrt{a^2-4b}}{2}$ and α_2=$\frac{-a-\sqrt{a^2-4b}}{2}$
Let γ =a^2-4$b \geq 0$ then α_1=$\frac{-a+\sqrt{\gamma}}{2}$ and α_2=$\frac{-a-\sqrt{\gamma}}{2}$ are real and belong to $F[\sqrt{\gamma}]$.

$\square$

Theorem 23. *The Rational Root Theorem states a constraint on rational solutions (or roots) of a polynomial equation.*
If we have an equation $a_n\ x^n + a_{n-1}\ x^{n-1} + \cdots + a_0\ = 0$ *with integer coefficients then rational solution* x*, when written as a fraction* $x\ =\ \frac{p}{q}$ *in lowest terms (i.e., the greatest common divisor of* p *and* q *is 1), satisfies* p *is an integer factor of the constant term* a_0*, and* q *is an integer factor of the leading coefficient* a_n*.*

Theorem 24. *Let* α *be a real number. Then* α *is constructible if and only if* α *belongs to the top of some square root tower over* $\mathbb{Q}$*.*

Proof. ($\Leftarrow$)Let C be the set of constructible real numbers. C is an extension field of $\mathbb{Q}$. C is a subfield of $\mathbb{R}$ because we have shown earlier that the constructible set C is closed under addition, subtraction, multiplication, and division.
Let $F = K_0 \subseteq K_1 \subseteq ... \subseteq K_n \subseteq \mathbb{R}$ be a square root tower, and let α be in the top of the square root tower. We show how α is constructible by inducting on the size of the tower.
If $n = 0$ then $K_0 = \mathbb{Q} \Rightarrow \alpha$ is rational.
Since we proved earlier that addition, subtraction, multiplication, and division of integers are constructible $\Rightarrow$ Any rational number is constructible.
Now we assume $K_i \subseteq C$. By definition of a square root tower,
$K_{i+1} = K_i(\sqrt{\rho})$ for some $\rho \in K_i$ with $\sqrt{\rho} \notin K_i\ \forall\ j$ satisfying $1 \leq j \leq i$.
$\Rightarrow \sqrt{\rho} \in C$ since we can construct $\sqrt{\rho}$ by proposition 12.
With $\sqrt{\rho} \in C \Rightarrow K_i(\sqrt{\rho}) \in C$ because C is closed under addition and multiplication.
$\Rightarrow K_n$ is constructible by induction. $\Rightarrow \alpha$ is constructible.

($\Rightarrow$) Let $\alpha \in C$, beginning with O at the *origin* and marking X on the $x - axis$ to be our unit segment we can construct a line segment of length $|\alpha|$ from the *origin*. We now need to show that α belongs to the top of the square root tower. In fact we can show through any straight edge and compass construction starting from our unit segment there is a square root tower over $\mathbb{Q}$ in which in the course of the construction the following belong to the top of the tower:

1. The coordinate (x, y) of any point marked.
2. The coefficients m,b of the equation of any line drawn $y = mx + b$.
3. The coefficients κ, λ,μ of the equation of any circle drawn $x^2 + \kappa\ x + y^2 + \lambda\ y + \mu = 0$.

The basic operations we defined as our possible constructions drawn with a compass and straight edge previously will be our 5 cases; that is:

1. Draw a line through 2 given points.
2. Draw a circle with center at a given point and radius equal to the distance between 2 other given points.
3. Mark the points of intersection of a straight line and a circle.
4. Mark the point of intersection of two straight lines.
5. Mark the intersection of 2 circles.

We begin with the trivial case (1) of two points with distance between them equal to 1 unit, (our unit segment) so the origin (0,0) and (0,1) which belong to the top of the trivial tower $\mathbb{Q} = K_0 \subseteq \mathbb{R}$. Suppose we perform m operations and that all the coordinates and coefficients above belong to the top of some tower $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$ and perform one more operation considering all basic operations we have the following cases:

(Case 1)

The line through (x_1,y_1) and (x_2,y_2) which we already marked.

Therefore x_1,y_1, x_2,y_2 $\in K_n$.

If we pick any point (x,y) on our line we can compute the slope between (x,y), and the two marked points (x_1,y_1) and (x_2,y_2) and the slopes must be equal. We get the following equations:

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow$ (y-y_1)(x_2-x_1)=(y_2-y_1)(x-x_1)

y(x_2-x_1)-y_1(x_2-x_1)= x(y_2-y_1)-x_1(y_2-y_1)

y=$\frac{x(y_2-y_1)}{x_2-x_1}$-$\frac{x_1(y_2-y_1)}{x_2}$-$x_1$+$\frac{y_1(x_2-x_1)}{x_2-x_1}$

Since K_n is closed under addition, subtraction, multiplication, and division we rewrite our equation as $y = mx + b$

where m=$\frac{(y_2-y_1)}{x_2-x_1}$,

b=$\frac{x_1(y_2-y_1)}{x_2-x_1}$+$\frac{y_1(x_2-x_1)}{x_2-x_1}$ where b, $m \in K_n$.

(Case 2)

For the next construction we want to draw a circle with center (x_0,y_0) that is already marked and its radius is equal to the distance between two other points that are also already marked (x_1,y_1) and (x_2,y_2). All x_i, $y_i \in K_n$, 0≤i≤2 (top of the tower).

Taking any point on the circle (x,y) we find the distance (length of radius) between the center and this point and it is equal to the distance between (x_1,y_1) and (x_2,y_2) We now have the equation:

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$(x-x_0)^2+(y-y_0)^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$x^2\text{-}2x_0\text{x}+y^2\text{-}2y_0\text{y}+(x_0)^2+(y_0)^2\text{-}(x_2-x_1)^2\text{-}(y_2-y_1)^2=0.$$

We can set $\kappa=-2x_0$, $\lambda=-2y_0$, and
$\mu=(x_0)^2+(y_0)^2-(x_2-x_1)^2-(y_2-y_1)^2$
and now we have the equation of a circle
$x^2+\kappa\text{x}+y^2+\lambda\text{y}+\mu$ with all its coefficients in K_n.

(Case 3)

For this case where two lines intersect, we have lines y=m_1x+$b_1$and y=m_2x+b_2 that have already been marked, m_1, m_2, b_1, $b_2 \in K_n$.
We now solve the system of equations:
y=m_1x+b_1
y=m_2x+b_2

and the solution to this system is the point of intersection.
Setting both equations equal to each other we have the following equation
m_1x+b_1=m_2x+b_2.
We now solve for x:
m_1x-m_2x=b_2-b_1

$x = \dfrac{b_2 - b_1}{m_1 - m_2} \in K_n$.
Substituting x into the original system we can solve for y and we have that
$y = \dfrac{b_2 - b_1}{m_1 - m_2} \cdot m_1 + b_1 \in K_n$.

(Case 4)

Now we look at the intersection between a circle and a line.

We begin with the line $y = mx + b$ and the circle $x^2+\kappa\text{x}+y^2+\lambda\text{y}+\mu=0$ already marked.

To solve this system of equations

$y = mx + b$

$x^2+\kappa\text{x}+y^2+\lambda\text{y}+\mu=0$

we substitute $y = mx + b$ and get

$x^2+\kappa\text{x}+(mx+b)^2+\lambda(\text{mx+b})+\mu=0$

$(1+m^2)x^2+(\kappa\text{+2mb+}\lambda\text{m})\text{x}+b^2+\lambda\text{b}+\mu=0$

using the quadratic formula we have

$$x = \frac{-(\kappa + 2mb + \lambda m) \pm \sqrt{(\kappa + 2mb + \lambda m)^2 - 4(1 + m^2)(b^2 + \lambda b + \mu)}}{2(1 + m^2)}$$

let $\rho=(\kappa + 2mb + \lambda m)^2\text{-}4(1+m^2)(b^2+\lambda\text{b}+\mu)$

and $\beta=(\kappa+2mb+\lambda\ m)$

and are contained in K_n,

then, x=-$\beta \pm \sqrt{\rho}$.

We have the case where either the circle intersects at one point or two points.

Either way, $x \in K_n(\sqrt{\rho})$ and $y = mx + b \in K_n(\sqrt{\rho})$.

So now K_{n+1} becomes the new top of our square root tower, where $K_{n+1}=K_n(\sqrt{\rho})$.

(Case 5)

Finally for the case of two intersecting circles we have

$x^2+\kappa_1$x$+y^2+\lambda_1$y$+\mu_1=0$
$x^2+\kappa_2$x$+y^2+\lambda_2$y$+\mu_2=0$
setting both equations equal to each other we get:

$x^2+\kappa_1$x$+4y^2+\lambda_1$y$+\mu_1=x^2+\kappa_2 4$x$+y^2+\lambda_2$y$+\mu_2$

κ_1x$+\lambda_1$ y$+\mu_1=\kappa_2$x$+\lambda_2 4$y$+\mu_2$

$(\kappa_1$-$\kappa_2)$x$+(\lambda_1$-$\lambda_2)$y$=\mu_2$-μ_1

y$=\frac{(\kappa_1-\kappa_2)x}{\lambda_1-\lambda_2}+\frac{\mu_2-\mu_1}{\lambda_1-\lambda_2}$

Let $\gamma=\frac{\kappa_1-\kappa_2}{\lambda_1-\lambda_2}$ and
$\delta=\frac{\mu_2-\mu_1}{\lambda_1-\lambda_2}$

$y=\gamma$x$+\delta$ which reduces to the previous case.

$\square$

The following definitions partnered with the previous theorem will help show which regular polygons are constructible and which need more than a compass and straightedge.

Definition 25. *A number is said to be algebraic if it is a root of a finite , non-zero polynomial in one variable with rational coefficients.*

Definition 26. *A number is said to be transcendental if it is not a root of a non-zero polynomial equation with rational coefficients.*

Corollary 26.1. *Let α be a real number. If α is constructible, then [$\mathbb{Q}(\alpha)$:$\mathbb{Q}$] is a power of 2.*

Proof. Let α be a real constructible number. We know $\alpha \in K_n$ which is the top of a square root tower and $K_{i+1}=K_i(\sqrt{\alpha})$ for some $\rho \in K_i$.
$\Rightarrow [K_{i+1}\colon K_i]=2$, $[K_n:\mathbb{Q}(\alpha)][\mathbb{Q}(\alpha):\mathbb{Q}]=[K_n:\mathbb{Q}]=[K_n:K_{n-1}]\cdots[K_2:K_1][K_1:\mathbb{Q}]=2^n$
$\Rightarrow [\mathbb{Q}(\alpha):\mathbb{Q}]=2^m$ for some $m \leq n$ $\square$

Exercise 27. *Is $3+\sqrt[6]{5}$ constructible?*

$P(x)=x^6-18x^5+135x^4-540x^3+1215x^2-1458x+724$ is irreducible over $\mathbb{Q}$ by The Rational Root Theorem because ±1, ±2, ±4, ±181, ±362, ±724 are not zeros and are the only possible rational zeros of $p(x)$. Let $\alpha=3+\sqrt[6]{5}$. $\Rightarrow$ α is a zero of $p(x)$, $[\mathbb{Q}((3+\sqrt[6]{5}):\mathbb{Q}]=6$, not a power of 2 therefore α is not constructible by Corollary 25.1.

Corollary 27.1. *Transcendental real numbers are not constructible.*

Proof. If α is constructible then $\mathbb{Q}(\alpha)$ is of finite degree over $\mathbb{Q}$ and hence algebraic over $\mathbb{Q}$, and is not transcendental. □

3.1 Construction of Regular Polygons

Proposition 28. *An angle of magnitude ϕ can be constructed with compass and straightedge if and only if $\sin\phi$ and $\cos\phi$ are constructible real numbers.*

Proof. ($\Rightarrow$) Let $\sin\phi$ and $\cos\phi$ be constructible real numbers. Then there exist an angle ϕ such that in a right triangle the ratio of the side opposite of ϕ and the hypotenuse is equal to $\sin\phi$.

($\Leftarrow$) Let $\angle$ABC=ϕ be constructible. Construct a perpendicular line from point A to the side BC of $\angle$ABC, name the point of intersection D. $\overline{AD}$, $\overline{AB}$, and $\overline{BD}$ are all constructible. $\Rightarrow$ $\sin\phi=\frac{AD}{AB}$ and $\cos\phi=\frac{BD}{AB}$ which are both constructible.

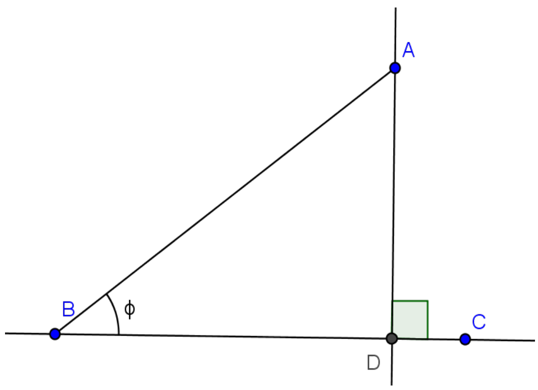


Figure 3.1: Construction of Sine and Cosine

□

Proposition 29. *A regular n-gon for n$\geq$3 is constructible using only compass and straightedge if and only if* $\cos\frac{360}{n}$ *is a constructible real number.*

Proof. ($\Rightarrow$) In any regular polygon, the measure of one exterior angle is $\frac{360}{n}$ and the interior angle of the regular polygon is supplementary to the exterior angle so its measure is 180-$\frac{360}{n}$. If these angles are constructible then so is $\cos(180 - \frac{360}{n})$ = -$\cos\frac{360}{n}$ by Proposition 27.

($\Leftarrow$) Starting with a unit segment OX, we draw a circle O with radius 1. Since $\cos\frac{360}{n}$ is constructible, then the angle of magnitude $\frac{360}{n}$ is constructible too. If an angle of magnitude ϕ=$\frac{360}{n}$ is constructible then such an angle $\angle$ PXQ can be constructed having X as its vertex and the extension of OX as a side. Let XP intersect the unit circle at A. Then XA is one side of the required n-gon. Repeat the process with OA in place of OX to construct the next side and continue this process until the entire n-gon is constructed. $\square$

Below are more examples of constructible angles and the algebraic reasons of their constructibility. Also with constructible angles we can use Algebra to show the regular polygons that are constructible.

Example 30. *Equilateral Triangle*
We constructed an equilateral triangle ealier with compass and straightedge in figure 2.1 but we can also use the fact that $\cos 60$=$\frac{1}{2}$. *Since it is equal to a real constructible number,*$\frac{1}{2}$*, then 60 degrees is constructible and we can say a regular 3-gon, triangle can be constructed by applying proposition 29.*

Example 31. *The* $\sin 90 = 1$*, so of course its constructible, therefore* 90° *is constructible and a regular quadrilateral (Square) is constructible.*

Next we use Algebra to prove the following polygons are not constructible.

Definition 32. *An nth root of unity, where n is a positive integer, is a number* z=$\cos\alpha+i\sin\alpha$ *satisfying the equation* $z^n = 1$.

As we transition from the geometric aspect of constructibility and focus more on the Abstract Algebra, we need to relate degrees to radian measure.

Definition 33. *One radian is a unit angle equal to an angle at the center of a circle whose arc is equal in length to the radius.*

An angle can be measured in degrees as well as radians. When we measure an angle with radian measure, we are actually measuring the distance traveled on the circle's circumference (the arc). So an angle θ measured in radians is the arc length divided by the radius. A circle has 360° or $\frac{2\pi r}{r}$ radians which is simply 2π radians. Thus far, all of the measures of angles have been in degrees but we need to use radian measure to discuss the angles that are not constructible.

Proposition 34. *(De Moivre's Formula)*
$(\cos x + i \sin x)^n = \cos nx + i \sin nx$, *for all real* x *and integers* n.

Setting $x = \frac{2\pi}{n}$ *gives a primitive nth root of unity:*

$(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^n = \cos 2\pi + i \sin 2\pi = 1$,

but for $k = 1, 2, \ldots, n-1$,

$(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^k = \cos \frac{2k}{\pi n} + i\sin \frac{2k}{\pi n} \neq 1$

Theorem 35. *The real number* $\cos\frac{2\pi}{5}$ *is constructible therefore a regular pentagon is constructible.*

Proof. $\cos \frac{2\pi}{5}$+isin $\frac{2\pi}{5}$ is a fifth root of unity hence a root of the equation x^5=1.
We can factor this equation:

x^5-1=$(x-1)(x^4 + x^3 + x^2 + x + 1)$.

Clearly $\cos \frac{2\pi}{5}$+isin $\frac{2\pi}{5} \neq 1$ so it is not a root of $(x-1)$ so it must be a root of $x^4+x^3+x^2+x+1$.

We have that $(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^4 = \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5}$

Let $z = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$.

$(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^4 + \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = 2\cos \frac{2\pi}{5}$
Substituting we have:

$z+z^4=2\cos\frac{2\pi}{5}$

Let $\alpha=2\cos\frac{2\pi}{5}$

$\Rightarrow \alpha^2=(z+z^4)^2$
$\Rightarrow \alpha^2+\alpha=(z+z^4)^2+z+z^4$
$\Rightarrow \alpha^2+\alpha=z^4+z^3+z^2+z+2=1$
$\Rightarrow \alpha^2+\alpha=1$
$\Rightarrow \alpha^2+\alpha\text{-}1=0$
Now we have that α is a zero of the quadratic $x^2+x\text{-}1=0$ and when we solve it using the quadratic formula we get

$x=\frac{-1+\sqrt{5}}{2}$ and $x=\frac{-1-\sqrt{5}}{2}$
$\alpha>0$ therefore $\alpha=2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2}$
$\Rightarrow \cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$
$\Rightarrow \cos\frac{2\pi}{5}$ is constructible. $\Rightarrow$ A regular pentagon is constructible by proposition 28. □

Euclid discovered that a polygon of n sides can only be constructed with compass and straight edge if n=2^k , $2^k\cdot 3$, $2^k\cdot 5$, or $2^k\cdot 3\cdot 5$. Two thousand years later, C.F.Gauss proved that a 17-gon was also constructible with compass and straightedge. So then, the real question arises, exactly which polygons are constructible? This is later answered by Galois theory which is out of the realm of this research paper.

Example 36. *Show algebraically that an angle of magnitude ϕ is constructible if and only if an angle of magnitude $\frac{\phi}{2}$ is constructible.*

Proof. ($\Rightarrow$)Let an angle of magnitude $\frac{\phi}{2}$ be constructible. If $\frac{\phi}{2}$ is constructible then the $\sin\frac{\phi}{2}$ is constructible. By the half angle formula $\sin\frac{\phi}{2}=\pm\sqrt{\frac{1-\cos\phi}{2}} \Rightarrow \cos\phi$ is constructible $\Rightarrow$ an angle of magnitude ϕ is constructible.

($\Leftarrow$)If an angle of magnitude ϕ is constructible then $\cos\phi$ constructible. $\sqrt{\frac{1+\cos\phi}{2}}$ is constructible. But $\sqrt{\frac{1+\cos\phi}{2}}=\cos\frac{\phi}{2} \Rightarrow \cos\frac{\phi}{2}$ is constructible. $\Rightarrow \frac{\phi}{2}$ is constructible. □

In the following examples, we determine in which case an angle of the indicated magnitude is constructible.

Example 37. *Let θ=40°, θ is not constructible.*

Proof. From trigonometric identities we have cos3θ=4cos^3 θ-3cos θ. Since cos120=-$\frac{1}{2}$ the number α=cos40 satisfies the equation $4\alpha^3$-3α=-$\frac{1}{2}$ hence is a zero of polynomial $p(x)$=$8x^3$-$6x$+1 which is irreducible over $\mathbb{Q}$ by the rational root theorem since $\pm\frac{1}{8}$, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, ±1 are not zeros of $p(x)$ and are the only possible rational zeros.
$\Rightarrow$ $[\mathbb{Q}(\alpha):\mathbb{Q}]$=3, not a power of 2. α is not constructible by corollary 26.1.
$\Rightarrow$ 40° is not constructible. □

Example 38. *Is 30° constructible? The* sin*30=*$\frac{1}{2}$ *therefore 30° is constructible by corollary 25.1.*

Example 39. $\sin 90$*=1 therefore 90° is constructible.*$\Rightarrow$ $\frac{90}{2}$ = 45° *is constructible.*

Example 40. *Is 42° constructible?*
By Theorem 35 cos*(*$\frac{2\pi}{5}$*) is constructible. We know that 2π=360° therefore* $\frac{2\pi}{5}$=72° *is constructible by proposition 29. Also 30° is constructible therefore we have that*
$72° - 30° = 42°$ *is constructible.*

Example 41. *Let θ=10°. Is θ constructible? From trigonometric identities we have*
$\sin 3\theta = -4\sin^3\theta + 3\sin\theta$. *Since* $\sin 30$=$\frac{1}{2}$ *the number α=*sin*10 satisfies the equation*
-4α^3+3α=$\frac{1}{2}$ *hence is a zero of polynomial $p(x)$=8x^3-6x+1 which is irreducible over $\mathbb{Q}$ by the rational root theorem because* $\pm\frac{1}{8}$, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, ±1 *are not zeros of $p(x)$ and are the only possible rational zeros.*
$\Rightarrow$ $[\mathbb{Q}(\alpha):\mathbb{Q}]$*=3, which is not a power of 2. α is not constructible by corollary 25.1*
$\Rightarrow$ 10° *is not constructible by proposition 29.*

Example 42. *The angle of magnitude 72° is constructible. If we bisect 72° we have an angle of magnitude 36°. If we bisect an angle of 36° we have an angle of 18°. An angle of measure 18° is constructible.*

In the following exercises we will determine if the regular polygon with n sides is constructible.

Example 43. *An octagon has 8 sides. So for n=8 the* cos*(*$\frac{360}{8}$*)=*cos*45=*$\frac{\sqrt{2}}{2}$ *therefore by proposition 29 a regular polygon with 8 sides is constructible.*

Example 44. *A polygon with 9 sides is not constructible.*
The $\cos(\frac{360}{9})$ *is not a real constructible number because in example 37 we proved that* $40°$ *is not constructible and by proposition 29* $\cos 40$ *is not constructible.*

Example 45. *A regular decagon has 10 sides.* $\cos(\frac{360}{10})$=$\cos$*36. The angle of magnitude* $36°$ *is constructible because* $72°$ *by example 36. Now we can say* $\cos$*36 is constructible by propostion 29.*

Example 46. *A regular polygon with 20 sides is constructible because* $\cos$*18 is constructible since the angle of magnitude* $18°$ *is constructible by example 42.*

Example 47. *A regular polygon with 30 sides is constructible.* $\cos\frac{360}{30}$=$\cos$*12. We know* $72°$ *and* $60°$ *are magnitudes of constructible angles. Their difference is constructible* $\Rightarrow$ $12°$ *is constructible* $\Rightarrow$ $\cos$*12 is constructible* $\Rightarrow$ *a polygon with 30 sides is constructible by proposition 29.*

Example 48. *Show that in a regular 5-gon with sides of length 1 any diagonal has length* α=$\frac{1+\sqrt{5}}{2}$.

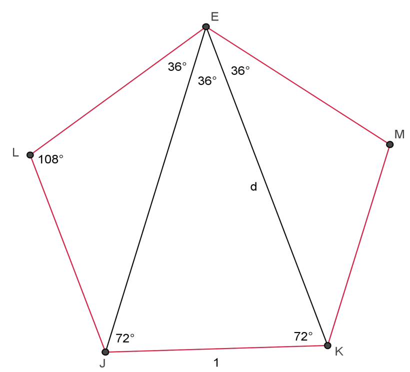


Figure 3.2: Construction of 5-gon Side 1

Proof. By the law of sines we have:
$\frac{\sin 36}{1}$=$\frac{\sin 72}{d}$ where d is the diagonal of a regular pentagon.
d=$\frac{sin72}{\sin 36}$
d=$\frac{1}{2}(1+\sqrt{5})$ ∎

Example 49. *In this example we show that 2*$\cos(\frac{2\pi}{7})$ *is a zero of* x^3+x^2-2x-1*. We know that if* $\cos\frac{2\pi}{7}$ *is not a constructible real number then by proposition 29 a heptagon is not* *constructible.*

Let α=2cos$(\frac{2\pi}{7})$

Let $x^3+x^2-2x-1=(x-\alpha)(ax^2+bx+c)$

Distributing the right side we have: $ax^3+(b-a\alpha)x^2+(c-b\alpha)x-c\alpha$

$\Rightarrow a=1 \; b-\alpha=1 \; c-b\alpha=-2 \; -c\alpha=-1$

$\Rightarrow b=\alpha+1 \; c=\frac{1}{\alpha}$

Therefore

$x^3+x^2-2x-1 = (x-2\cos\frac{2\pi}{7})(x^2+(2\cos\frac{2\pi}{7}+1)\, x+\frac{1}{2\cos\frac{2\pi}{7}}$ *and* $2\cos(\frac{2\pi}{7})$ *is a zero of* $p(x)=x^3+x^2-2x-1$. $[\mathbb{Q}(\alpha):\mathbb{Q}]=3$ *because* $p(x)$ *is irreducible over* $\mathbb{Q}$ *by the rational root theorem, and* α *is not constructible. We recall that* $\frac{2\pi}{7}$ *in radians is the same as* $\frac{360}{7}$ *degrees. By corollary 25.1,* $\frac{360}{7}$ *is not constructible. By proposition 29 a heptagon is not constructible.*

3.2 Classical Problems

There are many constructible geometric figures that cannot be done with only straightedge and compass. We encounter a few below that are considered classical problems. The Greeks made many attempts at three specific problems, the "squaring of the circle ", "doubling of the cube", and the "trisection of an arbitrary angle".

The Babylonians and Egyptians tried to approximate the area of a given circle with a square as well as the Russians and Indians. They all figured it boiled down to the approximation of the number π. Archimedes showed that the value of π was between $3\frac{1}{7}$ and $3\frac{10}{71}$. Anaxagoras worked on the problem in prison. He was probably the first to think of the polygons with a large number of sides inscribed in the circle would work to square the circle. Later Antiphon the Sophist wrote if he inscribed regular polygons inside a circle and kept doubling the number of sides he would eventually cover the whole circle area but Eudemus argued that the lengths of sides could not be divided in halves without limits. Finally, in 1882, Ferdinand von Linderman proved its impossibility.

3.2.1 Squaring the Circle

Given a circle in the plane, construct a square of the same area.
This cannot be done with only a compass and straightedge.

Proof. The area of a circle is πr^2. The area of a square with side s is s^2. Setting these two areas equal to each other we have $\pi r^2 = s^2$. If we divide both sides by r^2 we have that $\pi = (\frac{s}{r})^2$. This would imply that $\sqrt{\pi} = \frac{s}{r}$. If $\frac{s}{r}$ is constructible then $\sqrt{\pi}$ is constructible. This is a contradiction because π is not constructible. □

3.2.2 Duplicating the Cube

The doubling of the cube is also known as the Delian problem. The people of Delos were plagued by Apollo and so asked an oracle to help them get rid of the plague. The oracle advised them to double the size of the altar to Apollo which was a cube. The Delians asked Plato for help and he told them they needed to study geometry. Plato asked Eudoxus, Menaechmus and Archytas to solve this problem but they did not get Plato's approval for their solutions were wrong. This problem was also worked on by many others, the Egyptians, the Indians, and the Greeks to no avail. In 1837 Pierre Wantzel proved it impossible to solve with constructions that only use straight edge and compass. He found that the problem required the construction of the cube root of two which is impossible to construct with only compass and straight edge.[Bog15]

3.2.3 Eisenstein's Criterion

Eisenstein's Criterion gives a sufficient condition for a polynomial with integer coefficients to be irreducible over the rationals. Meaning the polynomial will be unfactorable into the product of nonconstant polynomials with rational coefficients. Given the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where all a_i are integers, for all i between 0 and n and there exist a prime p such that the if following hold:

1. p divides all a_i, $i \leq$ n
2. p^2 does not divide a_0

then $f(x)$ is irreducible.

Example 50. *Given a cube with side s, construct a second cube with double the volume.*

Proof. If the given cube has side s, then the volume of the cube is s^3. The cube with double the volume would have volume $2s^3$. The edge of the duplicated cube would be $\sqrt[3]{2}a$. Taking the side of the given cube as our unit, we would have to construct a segment of length $\sqrt[3]{2}$. But $\sqrt[3]{2}$ is a zero of x^3-2 which is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion with p=2 and therefore $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$=3, not a power of 2. Therefore $\sqrt[3]{2}$ is not constructible with only compass and straight edge. □

3.2.4 The Problem of Trisecting an Angle

The two classical problems above are by far much more popular in the math world. Trisecting an arbitrary angle is the least studied. Perhaps because there is no interesting story about the gods asking for a solution. So the problem arises plainly. Another reason might be because of how different it is from the other two problems. We cannot square a circle or double a cube, but there are specific angles that can be trisected. For instance we can trisect 90°, but the problem is that of trisecting an arbitrary angle. There is no general sequence of steps that would always end with trisecting an arbitrary angle.

Example 51. *Let us look at the angle* 60°*, it cannot be trisected.*

Proof. We can construct an angle of 60°.
Now we use trigonometric identity of the sum angle formula.
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
$\cos 60 = \frac{1}{2}$
Let $\alpha = \cos 20$
$4\cos^3 20 - 3\cos 20 = \frac{1}{2}$
$4\alpha^3$-$3\alpha = \frac{1}{2}$ so now we have that α is a zero of $p(x)=8x^3$-$6x$-1. $p(x)$ is irreducible over the rationals by the Rational Root Theorem because $\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1$ are not zeros of $p(x)$ therefore $[\mathbb{Q}(\alpha):\mathbb{Q}]$=3, not a power of 2. α is not constructible therefore 20° is not constructible by Corollary 25.1.

□

Chapter 4

Constructions with Marked Ruler and Compass

The ancient civilizations firmly believed in the gods and their demands. The gods demanded precise solutions of mathematical problems. The ancient mathematicians insisted on doing mathematics with precision so that the problems of their world would be vanished by the gods. The people believed that the god Apollo was cruel for giving them the impossible problem of doubling the cube.[Bog15]
With a marked ruler and compass the constructions are not impossible. As we see in the following examples we can construct figures and solve problems that were impossible without the ruler.

Definition 52. *A construction with a marked ruler and compass allows (in addition to the operations defined in Definition 11.1.8) the following operation: (6)Given two points* A *and* B*, lines* L *and* M*, and a point* P*, to construct points* C *on line* L *and point* D *on line* M *such that point* P *lies on the line through point* C *and point* D *and the length of* CD *equals the length of* AB*.*

A real number α is constructible using a marked ruler and compass if a segment $|\alpha|$ is constructible.

Lemma 53. *Let* C *be a point outside of a circle. If we draw two secant line segments from* E *then the following products are equal* $EC \cdot BC = CF \cdot DC$

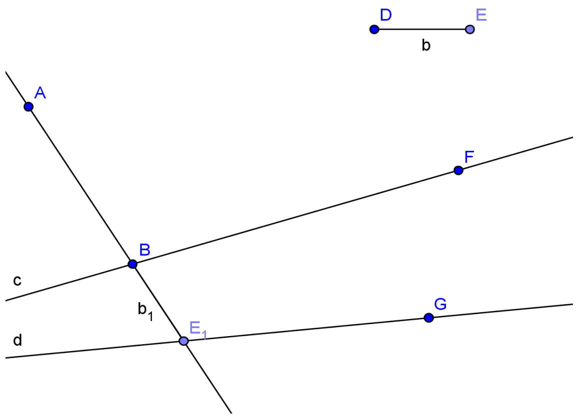
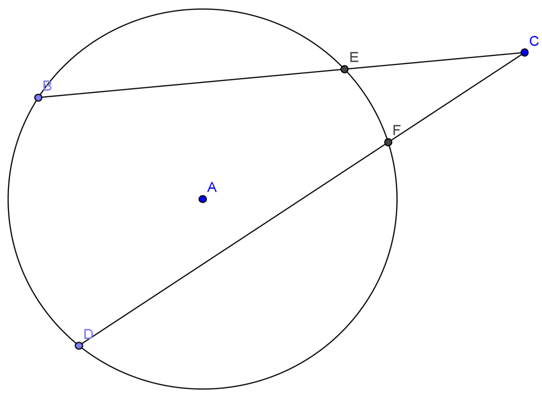


Figure 4.1: Construction with a Marked Ruler



Proof.

Figure 4.2: Secants from a Common Point

We draw the triangles BFC and DEC. By the Angle-Angle theorem of similar triangles △BCF ∼ △DCE. Therefore $\frac{BC}{DC} = \frac{CF}{EC} \Rightarrow$ (BC)(EC)=(DC)(CF) □

Theorem 54. *(Menelaus' Theorem) Given any line that intersects the three sides of a triangle (one side extended), six segments are cut off on the sides. The product of three non-adjacent segments is equal to the product of the other three.*

Let AFC be any triangle and let a line L cut the sides of their extensions at points D,B,E then $\frac{AD}{AB} \cdot \frac{BF}{EF} \cdot \frac{EC}{DC}$*=1.*

Similar to the construction of square roots or roots of the form 2^n, the construction of the cube root of a constructible number helps us construct roots of the for 3^n.

theorem.png

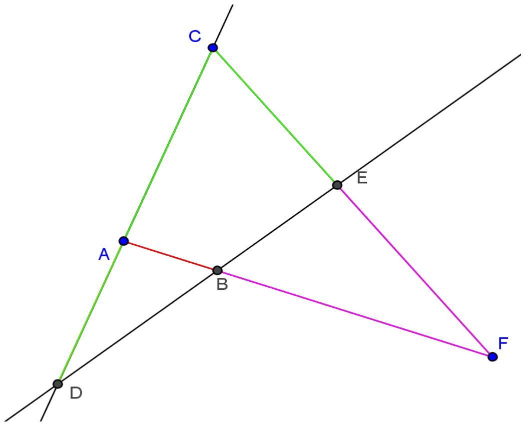


Figure 4.3: Menelaus Theorem

Proposition 55. *Let α be a real number that is constructible with a compass and marked ruler, then the real number $\sqrt[3]{\alpha}$ is constructible with the marked ruler and compass.*

This propositon also helps solve the problem of duplicating the cube.

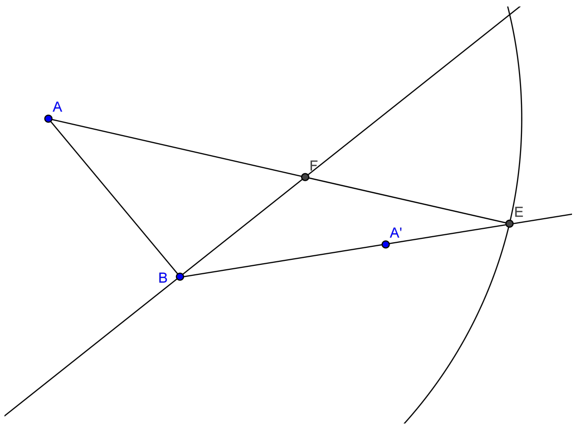


Figure 4.4: Construction of Cube Root

Proof. Let $AB = CD = 1$ ∠ABC = 90° and ∠CBD = 30° AC=x BC=$\sqrt{x^2 - 1}$ AD= x + 1. Using Law of Sines and substituting all the given information we can say:

$\frac{\sin \angle BDC}{BC} = \frac{\sin \angle CBD}{CD}$

$\frac{\sin \angle BDC}{\sqrt{x^2-1}} = \frac{\sin 30}{1}$

$\frac{\sin \angle ADB}{AB} = \frac{\sin \angle ABD}{AD}$

$\frac{\sin \angle ADB}{1} = \frac{\sin 120}{x+1}$

$\angle ADB$ and $\angle BDC$ are the same angle so we can set the two equations equal to each other.

$\sin BDC = \frac{\sqrt{x^2-1}}{2}$
$\sin ADB = \frac{\sqrt{3}}{2(x+1)}$

$\frac{\sqrt{x^2-1}}{2} = \frac{\sqrt{3}}{2(x+1)}$
$4(x^2 + 2x + 1)(x^2 - 1) = 12$
$4x^4 + 8x^3 - 8x - 16 = 0$
$x^4 + 2x^3 - 2x - 4 = 0$
$(x^3 - 2)(x + 2) = 0$
$x = \sqrt[3]{2}$ and x=-2 but we let x=AC so it must be $\sqrt[3]{2}$. □

Corollary 55.1. *The problem of the duplication of the cube can be solved using the marked ruler and compass.*

Proof. This problem amounts to the construction of $\sqrt[3]{\alpha}$ which can be done with a marked ruler and compass by proposition 55. □

Example 56. *Show that $2\cos(\frac{\theta}{3})$ is a zero of $y^3 - 3y - 2\cos\theta$*

Proof. Let h(y)=$y^3 - 3y - 2\cos\theta$ Substituting $2\cos(\frac{\theta}{3})$ into our function $h(y)$ we have $h(2\cos(\frac{\theta}{3})) = (2\cos(\frac{\theta}{3})^3 - 3(2\cos(\frac{\theta}{3}) - 2\cos(\theta)$.
After simplifying the right hand side of the equation we have 8cos$^3(\frac{\theta}{3})$ - 6cos$(\frac{\theta}{3})$ - 2cos θ.
$2\cos(\frac{\theta}{3})$ is a zero of $y^3 - 3y - 2\cos\theta$ □

Example 57. *Another example of how to construct $\sqrt[3]{2}$ as the intersection of a parabola with the hyperbola xy=1.*

Proof. We can use the hyperbola xy=1 and the parabola y=$\frac{1}{2}x^2$ to construct $\sqrt[3]{2}$.

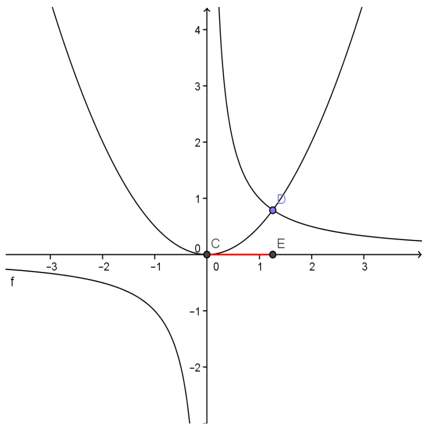


Figure 4.5: Cube Root of Two

If we solve the system:

y=$\frac{1}{x}$

y=$\frac{1}{2}x^2$

Substituting y=$\frac{1}{x}$ into y=$\frac{1}{2}x^2$

we have the equation:

$\frac{1}{x}$=$\frac{1}{2}x^2$

Solving for x we get

1=$\frac{1}{2}x^2\cdot\ x$
2=x^3
x=$\sqrt[3]{2}$
The intersection of the parabola with the hyperbola is the point ($\sqrt[3]{2}$,$\frac{1}{\sqrt[3]{2}}$) and we have line segment CE is the length $\sqrt[3]{2}$.

□

Proposition 58. *The general problem of the trisection of an angle can be solved using a* *compass and a ruler.*

Proof. Archimedes' Trisection
Draw the unit circle with central ∠DOC=θ and radius $\overline{OD}$ and $\overline{OC}$. Extend $\overline{OC}$ and with the distance of 1 marked on a ruler, keeping it in contact with D, slide it until it reaches a position where if B and A are the points of intersection with the circle and the extension of $\overline{OC}$, respectively, then AB=1. Let ∠OAB=α. Show that α=$\frac{\theta}{3}$.

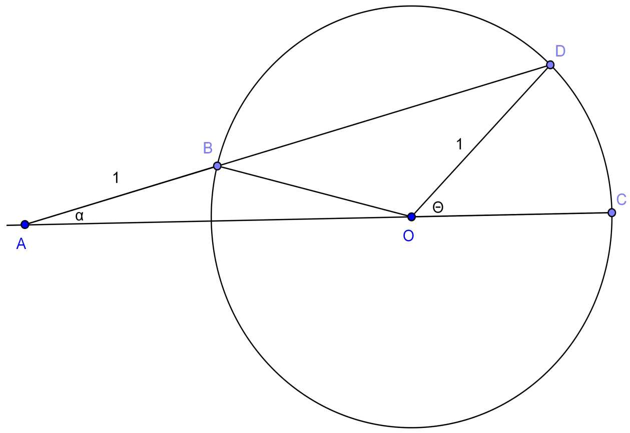


Figure 4.6: Archimedes Trisection of an Angle

△ BOD and △ ABO are both isosceles triangle with base angles equal to 2α and α, respectively.

Using the sum of interior angles of a triangle and angle addition postulate we have the following equation:
$180 - 4\alpha$=$180 - \alpha - \theta$. Subtracting 180 from both sides of the equation we have:
-4α=$-\alpha - \theta$. Combining like terms we have the equation:
-3α=$-\theta$. Finally, dividing by 3 on both sides of the equation we have the desired result:
α=$\frac{\theta}{3}$, the angle α is one third of angle θ. □

Chapter 5

Conclusion

As the ancient civilizations discovered some impossible problems, we then extend our construction with the compass and straight edge to include a marked ruler. We begin with our marked points, lines, circles, and construct most of the basic geometric figures. We can now construct segments of lengths with cube roots if we use a marked ruler. Similar to the constructions of roots of the form 2^k we can repeat cube root construction and construct roots of the form 3^k. The problems of doubling the cube and trisecting an angle were both solved by using a marked ruler. The impossibility of solving these two problems was proven by using algebra thousands of years after they were posed. Lindemann proved the impossibility of squaring the circle. He also went so far as to show that π is transcendental. One problem that is still unsolved is the construction of regular n-gons. Sure, Gauss showed that a regular 17-gon is constructible. He also went on to show that a regular polygon with n sides is constructible if and only if $n{=}2^m \cdot p_1 \cdot p_2 \cdots p_k$ where p_i are Fermat's Primes. This is just a partial answer to the question of constructible regular polygons. Also, can we construct other roots of numbers that are not in the form 2^k or 3^j? This question can also be answered by further research in Galios Theory and Field Theory

Bibliography

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