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Homomorphic Images And Related Topics

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

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Abstract

We will explore progenitors extensively throughout this project. The progenitor, developed by Robert T Curtis, is a special type of infinite group formed by a semidirect product of a free group m^{*n} and a transitive permutation group of degree n. Since progenitors are infinite, we add necessary relations to produce finite homomorphic images. Curtis proved that any non-abelian simple group is a homomorphic image of a progenitor of the form $2^{*n} : N$. In particular, we will investigate progenitors that generate two of the Mathieu sporadic groups, M_{11} and M_{22}, as well as some classical groups. We will prove their existences a variety of different ways, including the process of double coset enumeration, Iwasawa's Lemma, and linear fractional mappings. We will also investigate the various techniques of finding finite images and their corresponding isomorphism types.

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Introduction

Group theory is a study of symmetry of objects and can some times be very complex. There are many different representations of these structures, or as we refer to them, groups. We will focus on permutation groups and symmetric groups. We use progenitors to create new and original presentations of finite groups. We then prove their existence and observe the various properties these groups have.

Though some of the techniques we cover may seem elementary, we use these methods in a way that yields very interesting results. This thesis focuses on groups that have been found using progenitors most of which have been new discoveries. We use double coset enumeration to verify the order of a group, as well as determining if a group is faithful and primitive. If we have a group that is faithful, primitive, perfect, and has a normal abelian subgroup in which the conjugates of itself with G generate G, we have proven that group's simplicity by a 70 year old lemma.

In **Chapter** 1 we give definitions, lemmas, and theorems that will be used throughout this project. In **Chapter** 2 we introduce a few finite progenitors and prove their existences primarily using double coset enumeration and manipulation of relations. We will solve a basic example then give two examples of linear groups, as well as the verification proofs of each group. In **Chapter** 3 we investigate the Mathieu sporadic groups M_{11} and M_{22}. A formal existence proof of M_{11} is given utilizing Iwasawa's Lemma. A partial existence proof for M_{22} over a maximal subgroup M. In **Chapter** 4 we use knowledge of extensions to determine isomorphism types of a few progenitors found. In **Chapter** 5 we discuss the various ways to find progenitors. We find progenitors can be formed very easily using MAGMA, a computational algebra system. However, we can find ways to narrow down which progenitors are worth investigating. In **Chapter** 6 we list some interesting groups found throughout this project.

Chapter 1

Group Theory Preliminaries

1.1 Some Definitions

Definition 1.1. *[Rot95] A* **group** *G $(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*
- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.*

Definition 1.2. *[Rot95] For group G, a* **subgroup** *S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$.*

Definition 1.3. *[Rot95] Let H be a subgroup of group G. H is a* **proper** *subgroup of G if $H \neq G$. We denote this as $H < G$.*

Definition 1.4. *[Rot95] Let G be a group and $H \leq G$. H is a* **maximal subgroup** *of G if there is no normal subgroup $N \leq G$ such that $H < N < G$.*

Definition 1.5. *[Rot95] A* **symmetric group***, S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions.*

Definition 1.6. *[Rot95] If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$.*

Definition 1.7. *[Rot95] If* $x \in X$ *and* $\phi \in S_X$*, then* ϕ **fixes** x *if* $\phi(x) = x$ *and* ϕ **moves** x *if* $\phi(x) \neq x$*.*

Definition 1.8. *[Rot95] For permutations* $\alpha, \beta \in S_X$*,* α *and* β *are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(a) = a \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y.$$

Definition 1.9. *[Rot95] A permutation which interchanges a pair of elements is a* **transposition***.*

Definition 1.10. *[Rot95] In group* G*, if* a*,*$b \in G$*,* a *and* b **commute** *if* $a * b = b * a$*.*

Definition 1.11. *[Rot95] A group* G *is* **abelian** *if every pair of elements in* G *commutes with one another.*

Definition 1.12. *[Rot95] Let* G *be a group. The* **order** *of* G *is the number of elements contained in* G*. We denote the order of* G *by* $|G|$*.*

Definition 1.13. *[Rot95] Let* G *be a group.* G *is* **simple** *if the only normal subgroups of* G *are* 1 *and* G*.*

Definition 1.14. *[Rot95] Let* p *be prime. If* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$*, then we say* G *is* **elementary abelian***.*

Definition 1.15. *[Rot95] Let* $(G, *)$ *and* $(H, \circ)$ *be groups. The function* $\phi : G \to H$ *is a* **homomorphism** *if* $\phi(a * b) = \phi(a) \circ \phi(b)$*, for all* a*,*$b \in G$*. An* **isomorphism** *is a bijective homomorphism. We say* G *is isomorphic to* H*,* $G \cong H$*, if there is exists an isomorphism* $f : G \to H$*.*

Definition 1.16. *[Rot95] Let* $f : G \to H$ *be a homomorphism. The* **kernel of a homomorphism** *is the set* $\{x \in G | f(x) = 1\}$*, where* 1 *is the identity in* H*. We denote the kernel of* f *as* **ker f** *.*

Definition 1.17. *[Rot95] Let* X *be a nonempty subset of a group* G*. Let* $w \in G$ *where* $w = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$*, with* $x_i \in X$ *and* $e_i = \pm 1$*. We say that* w *is a* **word** *on* X*.*

Definition 1.18. *[Rot95] Let* G *be a group such that* $K \leq G$*.* K *is* **normal** *in* G *if* $gKg^{-1} = K$*, for every* $g \in G$*. We will use* $K \lhd G$ *to denote* K *as being normal in* G*.*

Definition 1.19. *[Rot95] Let G be a group. We say G is a* **direct product** *of two subgroups H and K if:*

1. $H \trianglelefteq G$, $K \trianglelefteq G$;
2. $G = HK$;
3. $H \cap K = 1$,

Definition 1.20. *[Rot95] G is a* **semi-direct product** *of two subgroups H and K if:*

1. $K \trianglelefteq G$, $Q \leq G$;
2. $G = KQ$;
3. $K \cap Q = 1$.

Definition 1.21. *[Rot95] Let a, $b \in G$. We denote the* **commutator** *of a and b by $[a, b]$, where $[a, b] = aba^{-1}b^{-1}$.*

Definition 1.22. *[Rot95] Let G be a group. The* **Derived Group** *of G, denoted G', is the subgroup of G formed by all the commutators of G.*

Definition 1.23. *[Rot95] Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a* **representative** *of the coset St.*

Definition 1.24. *[Rot95] Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G.*

Definition 1.25. *[Rot95] Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$.*

Definition 1.26. *[Rot95] Let N be a group. The* **point stabiliser** *of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s.}$$

Definition 1.27. *[Rot95] Let N be a group. The* **coset stabiliser** *of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s.}$$

Definition 1.28. *[Rot95] Let X be a set and G be a group. We say X is a* **G-set** *if there exists a function $\phi : G \times X \to X$ (which we call an* **action***) and the following hold for ϕ:*

- *$1x = x$, for all $x \in X$.*
- *$g(hx) = (gh)x$, for $g,h \in G$ and $x \in X$.*

Definition 1.29. *[Rot95] Let G be a group. The* **center** *of G, $Z(G)$, is the set of all elements in G that commute with all elements of G.*

Definition 1.30. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **central extension** *by H if N is the center of G. We denote this by $G \cong N^{\bullet}H$.*

Definition 1.31. *[Rot95] Let G be a group and H, $N \leq G$ such that $|G| = |N||H|$. G is a* **mixed extension** *by H if it is a combination of both central extensions and semi-direct products, where N is the normal subgroup of G but not central. We denote this by $G \cong N^{\bullet} : H$.*

Definition 1.32. *[Rot95] Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by $N_G(H) = \{a \in G | aHa^{-1} = H\}$*

Definition 1.33. *[Rot95] Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}.$$

Definition 1.34. *[Rot95] Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$*

Definition 1.35. *[Rot95] Let G be a group and X be a G-set. For $x \in X$, the set $x^G = \{x^g | g \in G\}$ is a* **G-Orbit***.*

Definition 1.36. *[Rot95] Let X be a G-set. Let α be an action of G on X. If $\tilde{\alpha} : G \to S_X$ is injective, we say X is* **faithful***.*

Definition 1.37. *[Rot95] Let G be a group and X be a G-set. X is* **transitive** *if for all x,$y \in X$ there exists a $g \in G$ such that $y = gx$.*

Definition 1.38. *[Rot95] Let G be a group. A* **normal series** *G is a sequence of subgroups*

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

with $G_{i+1} \lhd G_i$. Furthermore, the **factors groups** *of G are given by G_i/G_{i+1} for $i = 0, 1, \ldots, n-1$.*

Definition 1.39. *[Rot95] Let G be a group. A* **composition series** *of G given by:*

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

is a normal series where, for all i, either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_i$.

Definition 1.40. *[Rot95] If group G has a composition series, the factor groups of its series are the* **composition factors** *of G.*

Definition 1.41. *[Rot95] Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G.*

Definition 1.42. *[Rot95] The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$.*

Definition 1.43. *[Rot95] A* **general linear group***, $GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$.*

Definition 1.44. *[Rot95] A* **special linear group***, $SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$.*

Definition 1.45. *[Rot95] A* **projective special linear group***, $PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F})}.$$

Definition 1.46. *[Rot95] A* **projective general linear group**, *$PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F})}.$$

Definition 1.47. *[Rot95] Let X be a G-set. Then for $B \subseteq X$, B is a* **block** *if for every $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$.*

Definition 1.48. *[Rot95] Let X and Y be G-sets. The function $f : X \to Y$ is a* **G-map** *if $f(gx) = gf(x)$, for all $x \in X$ and $g \in G$.*

Definition 1.49. *[Rot95] Let X be a G-set. X is* **primitive** *if X has no nontrivial blocks. If X is primitive, the only blocks of X are $B = X$ and $B = \emptyset$.*

1.2 Some Theorems

Many of these theorems can be found in an introductory level group theory text, but for our research purposes we will use the theorems stated by Joseph Rotman [Rot95] .

Theorem 1.50. *[Rot95] Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles.*

Theorem 1.51. *[Rot95] Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*
- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*
- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$.*

Theorem 1.52. *[Rot95] The intersection of any family of subgroups of a group G is again a subgroup of G.*

Theorem 1.53. *[Rot95] If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint.*

Theorem 1.54. *[Rot95] If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$.*

Theorem 1.55. *[Rot95] If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|.$$

Theorem 1.56. *[Rot95] If $N \lhd G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$.*

Theorem 1.57. *[Rot95] The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$.*

Theorem 1.58. *[Rot95] Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$.*

Theorem 1.59. *[Rot95] Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$.*

Theorem 1.60. *[Rot95] Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$.*

Theorem 1.61. *[Rot95] If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite.

Theorem 1.62. *[Rot95] If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$.*

Theorem 1.63. *[Rot95] Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n.*

Theorem 1.64. *[Rot95] If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $ker\rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H.*

Theorem 1.65. *[Rot95] If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set.*

Theorem 1.66. *[Rot95] Every two composition series of a group G are equivalent.*

We will refer to this Theorem as the **Jordan-Hölder Theorem***.*

Theorem 1.67. *[Rot95] Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$.*

1.3 Some Lemmas

Of these lemmas, the first helps show blocks of imprivitity. The second lemma is a powerful tool which we will use to prove the simplicity of groups. Most non-abelian simple groups can be proved using what we will refer to as **Iwasawa's Lemma**.

Lemma 1.68. *[Rot95] Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*
- *If $H \lhd G$, then the subsets Hx are blocks of X.*

Lemma 1.69. *[Rot95] G is simple if the following hold true:*

1. *G is faithful,*
2. *G is primitive,*
3. *G is perfect ($G = G'$),*
4. *There exists an $x \in X$ and an abelian normal subgroup $K \lhd G_x$ whose conjugates $\{gKg^{-1} : g \in G\}$ generate G.*

   We will refer to this lemma as **Iwasawa's Lemma***.*

Chapter 2

Double Coset Enumeration

2.1 S_4 over $N = S_3$

2.1.1 Double Coset Enumeration of G

We factor the progenitor $2^{*3} : S_3$ by the relation $[xt]^3$, where $x = (1, 2, 3)$ and $y = (1, 2)$. Letting t be represented by t_3, we compute the relation:

$$
\begin{aligned}
(xt)^3 &= e \\
(xt_3)^3 &= e \\
x^3[t_3]^{x^2}[t_3]^x t_3 &= e \\
x^3 t_2 t_1 t_3 &= e \\
t_2 &= t_3 t_1.
\end{aligned}
$$

Now, we are able to find $\{t_2 = t_3t_1\}^N$:

$\{t_2 = t_3t_1\}^{Id(N)}$ $\{t_2 = t_3t_1\}^{(1,2)}$ $\{t_2 = t_3t_1\}^{(2,3)}$

$\{t_2 = t_3t_1\}^{(1,3)}$ $\{t_2 = t_3t_1\}^{(1,2,3)}$ $\{t_2 = t_3t_1\}^{(1,3,2)}$.

So we obtain all of the following relations:

$t_2 = t_3t_1$ $t_1 = t_3t_2$ $t_3 = t_2t_1$

$t_2 = t_1t_3$ $t_3 = t_1t_2$ $t_1 = t_2t_3$.

We let G be $2^{*3} : S_3/t_2t_1t_3$, where $N =< (1, 2, 3), (1, 2) >$ and $t \sim t_3$. We will find our index of N in G by manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

We will complete a double coset enumeration of G over N to find the index of N in G. We must find all distinct double cosets $[w]$, where $[w] = \{Nw^n | n \epsilon N\}$, and the number of single cosets contained in each double coset. Our manual double coset enumeration is completed when all potentially new double cosets have previously been accounted for and when the set of right cosets is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n \ in \ N : Nw^n = Nw\}$, where w is a word of t_i's on $\{0, 1, 2, 3, 4, 5\} = X$.

We begin with the double coset NeN, which we denote $[*]$. This double coset consists of the single coset N. Allowing 3 to be 0, the single orbit of N on X is $\{0, 1, 2\}$. We will choose $t_3 = t_0$ as our symmetric generator from the orbit $\{0, 1, 2\}$ and find Nt_0 belongs to Nt_0N which is a new double coset. We denote Nt_0N by [0].

To find out how many single cosets [0] contains, we find the set of coset stabilizers of [0], denoted $N^{(0)}$. The number of single cosets in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. We have the following:

$$\begin{aligned} |N^{(0)}| &\geq |< Id(G), (2,3) >| \\ &\geq 2. \end{aligned}$$

The number of single cosets in $Nt_0N = \frac{|N|}{|N^{(0)}|} = \frac{6}{2} = 3$. Our index is the sum of distinct single cosets in each distinct double coset, such as [*] and [0]. As of now, we have $1 + 3 = 4$ single cosets. We note that the orbits of [0] are $\{0\}$ and $\{1, 2\}$.

We will continue to the next level of potential double cosets by investigating the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$ and $\{1, 2\}$ and we take t_0 and t_1 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1, 2\}$ we find a potentially new double coset Nt_0t_1, which we denote [01]. Since we already have accounted for the double coset $[*]$, we should examine the potentially new double coset [01] and determine the number of new, distinct single cosets contained inside of it.

However, consider the relation $t_2 = t_0t_1$. This implies that the coset Nt_0t_1 is equal to Nt_2, which we have already accounted for in [0]. Therefore, if we right multiplied by a representative from the orbit $\{1, 2\}$, we would return back at [0]. This also implies that [01] is not a new double coset.

Since there are no potentially new double cosets that we can investigate, our Cayley graph is closed under right multiplication and our double coset enumeration of G over N is complete. The index of N in G is 4.

2.1.2 Proof of $G \cong S_4$

First let us show that G acts faithfully on $X = \{N, Nt_0, Nt_1, Nt_2\}$. Since X is a transitive G-set of degree 4, we have:

$$|G| = 4|G^1|,$$

where G^1 is the stabilizer of the single coset N. However, N is only stabilised by elements from N. Therefore $G^1 = N$ and $|G^1| = |N| = 6$. It is then evident that $|G| = 24$. If $|G| > 24$, X would not be faithful.

Hence we see

Now we determine the action of ϕ on x, y, and t. We have the following distinct single cosets: N, Nt_0, Nt_1, and Nt_2. We label our distinct single cosets and permute the t_i's by x to determine $\phi(x)$, permute the t_i's by y to determine $\phi(y)$, and right multiply by t_0 to determine where each would advance in terms of our labeling.

We will first determine $\phi(x)$:

(1) N	$[N]^x = N = (1)$
(2) Nt_0	$[Nt_0]^x = Nt_1 = (3)$
(3) Nt_1	$[Nt_1]^x = Nt_2 = (4)$
(4) Nt_2	$[Nt_2]^x = Nt_0 = (2)$.

Starting with N, which we labeled (1), we see conjugating N by x remains N since elements of N will fix the coset N. So we see that the permutation $\phi(x)$ should send (1) to itself. We then obtain that $\phi(x) = (1)(2, 3, 4) = (2, 3, 4)$.

Continuing this pattern and expressing the actions of x, y, and t_0 in a chart, we obtain:

Table 2.1: Single Coset Action of S_4 Over S_3

Label	Single Cosets	x		y		t_0	
1	N	1	N	1	N	2	Nt_0
2	Nt_0	3	Nt_1	2	Nt_0	1	N
3	Nt_1	4	Nt_2	4	Nt_2	4	Nt_2
4	Nt_2	2	Nt_0	3	Nt_1	3	Nt_1

Hence $\phi(x) = (1)(2,3,4) = (2,3,4)$, $\phi(y) = (1)(2)(3,4) = (3,4)$, and $\phi(t) = (1,2)(3,4)$.

Now observe $f(G) =< \phi(x), \phi(y), \phi(t) >=< (2,3,4),(3,4),(1,2)(3,4) >$, where $f : G \to G/N$ and f is bijective.

Observe that $\phi(x)$ and $\phi(y)$ generate S_3 on the letters 2, 3, and 4. But S_3 is a maximal subgroup of S_4, therefore any element found outside of our S_3 that is contained in S_4 and joined with $\phi(x)$ and $\phi(y)$ would give us S_4. This is the case. $< (1,2)(3,4) >$ is a subgroup of S_4 but is not contained in $< (2,3,4),(3,4) >$. Therefore $f(G) \cong S_4$.

2.1.3 Alternative Proof of $G \cong S_4$

Since we have $f(G) =< \phi(x), \phi(y), \phi(t) >=< (2,3,4),(3,4),(1,2)(3,4) >$, one can observe that $|\phi(y)| = 2$, $|\phi(x)\phi(y)\phi(t)| = 4$, and $|\phi(y)\phi(x)\phi(y)\phi(t)| = 3$. So G has an element of order 2, y, and an element of order 4, xyt, such that the product of the two elements is of order 3. This is an alternative proof that verifies that our G is indeed S_4.

2.2 $PSL(2,11) \times 2$ over $N = D_{12}$ by Method of Factoring by Center

2.2.1 Double Coset Enumeration of $L(2,11) \times 2$ over D_{12}

We factor the progenitor $2^{*6} : D_{12}$ by the two relations $[xtt^x]^3$ and $[xt]^5$, where $x = (1,2,3,4,5,6)$ and $y = (1,5)(2,4)$. Letting t be represented by t_6, we compute the two relations:

$$(xtt^x)^3 = e$$
$$(xt_6t_1)^3 = e$$
$$x^3[t_6t_1]^{x^2}[t_6t_1]^x t_6t_1 = e$$
$$x^3t_2t_3t_1t_2t_6t_1 = e$$
$$(1,4)(2,5)(3,6)t_2t_3t_1 = t_1t_6t_2$$

and

$$(xt)^5 = e$$
$$(xt_6)^5 = e$$
$$x^5t_6^{x^4}t_6^{x^3}t_6^{x^2}t_6^xt_6 = e$$
$$x^5t_4t_3t_2t_1t_6 = e$$
$$(1,6,5,4,3,2)t_4t_3t_2 = t_6t_1.$$

Let G be $2^{*5} : D_{12}/(1,4)(2,5)(3,6)t_2t_3t_1t_2t_6t_1, (1,6,5,4,3,2)t_4t_3t_2t_1t_6$, where $N =< (1,2,3,4,5,6),(1,5)(2,4) >$ and $t \sim t_6$.

We will find our index of N in G by manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

We will complete a double coset enumeration of G over N to find the index of N in G. We must find all distinct double cosets $[w]$, where $[w] = \{Nw^n | n \epsilon N\}$, and the number of single cosets contained in each double coset. Our manual double coset enumeration is completed when all potentially new double cosets have previously been accounted for and when the set of right cosets is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n\ in\ N : Nw^n = Nw\}$, where w is a word of t_i's on $\{0,1,2,3,4,5\} = X$.

We begin with the double coset NeN, which we denote $[*]$. This double coset consists of the single coset N. Allowing 6 to be 0, the single orbit of N on X is $\{0,1,2,3,4,5\}$. We will choose $t_6 = t_0$ as our symmetric generator from the orbit $\{0,1,2,3,4,5\}$ and find Nt_0 belongs to Nt_0N which is a new double coset. We denote Nt_0N by [0].

To find out how many single cosets [0] contains, we find the set of coset stabilizers of [0], denoted $N^{(0)}$. The number of single cosets in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. We have:

$$|N^{(0)}| \geq |< Id(G),(1,5)(2,4) >| \\ \geq 2.$$

The number of single cosets in $Nt_0N = \frac{|N|}{|N^{(0)}|} = \frac{12}{2} = 6$. Our index is the sum of distinct single cosets in each distinct double coset, such as [*] and [0]. As of now, we have $1 + 6 = 7$ single cosets. We note that the orbits of [0] are $\{0\}$, $\{1,5\}$, $\{2,4\}$ and $\{3\}$.

We will continue to the next level of potential double cosets by investigating the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$, $\{1,5\}$, $\{2,4\}$ and $\{3\}$ and we take t_0, t_1, t_2, and t_3 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1,5\}$ we find a potentially new double coset Nt_0t_1, which we denote [01]. From the orbit $\{2,4\}$ we get Nt_0t_2 we find a potentially new double coset Nt_0t_2, which we denote [02]. From the orbit $\{3\}$ we get another potentially new double coset Nt_0t_3, which we will denote [03]. We must now find the number of distinct single cosets in [01], [02] and [03].

Computing $N^{(01)}$ in N, we obtain:

$$|N^{(01)}| \geq |N^{01}| \\ \geq |< Id(G) >| \\ \geq 1.$$

Computing $N^{(02)}$ in N, we obtain:

$$\begin{aligned} |N^{(02)}| &\geq |N^{02}| \\ &\geq |< Id(G) >| \\ &\geq 1. \end{aligned}$$

Computing $N^{(03)}$ in N, we obtain:

$$\begin{aligned} |N^{(03)}| &\geq |N^{03}| \\ &\geq |< Id(G), (1,5)(2,4) >| \\ &\geq 2. \end{aligned}$$

So the number of single cosets in $Nt_0t_1N = \frac{|N|}{|N^{(01)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_2N = \frac{|N|}{|N^{(02)}|} = \frac{12}{1} = 12$. And the number of single cosets in $Nt_0t_3N = \frac{|N|}{|N^{(03)}|} = \frac{12}{2} = 6$.

Hence, our index is now $1 + 6 + 12 + 12 + 6 = 37$.

We now explore the potentially new double cosets coming from representatives from the orbits of $N^{(01)}$ on X. We find [01] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{1\}$ advances back to [0]. The other orbit representatives bring the potentially new double cosets [012], [013], [014], [015], and [010]. However, consider the following relation: $t_0t_1t_2 = (0,5,4,3,2,1)[t_0t_1]^{(0,4)(1,3)}$

Hence in [01], the representative $\{2\}$ loops back to [01] and is already being accounted for by the double coset [01]. So the only new double cosets coming from the orbit representatives of $N^{(01)}$ on X are [013], [014], [015], and [010].

The orbits of $N^{(02)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{2\}$ advances back to [0]. The other representatives bring the potentially new double cosets [021], [023], [024], [025], and [020]. Consider the following relations:

$t_0t_2t_4 = (0,4,2)(1,5,3)[t_0t_2]^{(0,2)(5,3)}$

$t_0t_2t_0 = (0,5,4,3,2,1)[t_0t_1t_4]^{(0,1,2,3,4,5)}$.

Hence in [02], the representative $\{4\}$ will loop back to [02] and the representative $\{0\}$ advances to [014]. However, [021], [023], and [025] are new, distinct double cosets.

Now, the orbits of $N^{(03)}$ on X are $\{0\}$, $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The representative from the orbit $\{3\}$ advances back to [0]. We take t_0, t_1, and t_2 from the other three orbits of $N^{(03)}$ on X. These three orbit representatives advance to the potentially new double cosets [030], [031], and [032]. Consider the following relations:

$t_0t_3t_1 = (0,2,4)(1,3,5)[t_0t_2t_5]^{(054321)}$

$t_0t_3t_2 = (0,4,2)(1,3,4)[t_0t_1t_4]^{(012345)}$.

Hence the representatives from $\{1,5\}$ will actually advance to [025] and the representatives from $\{2,4\}$ advance to [014]. The only new, distinct double coset coming from the orbit representatives of $N^{(03)}$ on X is [030].

The double cosets we must now investigate are [013], [014], [015], [010], [021], [023], [025] and [030].

Consider the relations:

$t_0t_1t_3 = t_4t_3t_1$, which implies $[t_0t_1t_3]^{(1,3)(4,0)} = t_4t_3t_1 \Rightarrow [(1,3)(4,0)]\epsilon N^{013}$.

$t_0t_1t_5 = t_5t_4t_0$, which implies $[t_0t_1t_5]^{(1,4)(2,3)(5,0)} = t_5t_4t_0$
$\Rightarrow [(1,4)(2,3)(5,0)]\epsilon N^{015}$.

$t_0t_1t_0 = t_1t_0t_1$, which implies $[t_0t_1t_0]^{(1,0)(2,5)(3,4)} = t_1t_0t_1$
$\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{010}$.

Computing $N^{(013)}$ in N, we obtain:

$$\begin{aligned}|N^{(013)}| &\geq |N^{013}|\\ &\geq |<Id(G),(1,3)(4,0)>|\\ &\geq 2.\end{aligned}$$

Computing $N^{(014)}$ in N, we obtain:

$$\begin{aligned}|N^{(014)}| &\geq |N^{014}|\\ &\geq |<Id(G)>|\\ &\geq 1.\end{aligned}$$

Computing $N^{(015)}$ in N, we obtain:

$$\begin{aligned}|N^{(015)}| &\geq |N^{015}| \\ &\geq |<Id(G),(1,4)(2,3)(5,0)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(010)}$ in N, we obtain:

$$\begin{aligned}|N^{(010)}| &\geq |N^{010}| \\ &\geq |<Id(G),(1,0)(2,5)(3,4)>| \\ &\geq 2.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_3N = \frac{|N|}{|N^{(013)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_4N = \frac{|N|}{|N^{(014)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_1t_5N = \frac{|N|}{|N^{(015)}|} = \frac{12}{2} = 6$. And the number of single cosets in $Nt_0t_1t_0N = \frac{|N|}{|N^{(010)}|} = \frac{12}{2} = 6$.

Hence our index is increased to $37 + 6 + 12 + 6 + 6 = 67$

Consider the relations:

$t_0t_2t_1 = t_4t_2t_3$, which implies $[t_0t_2t_1]^{(1,3)(4,0)} = t_4t_2t_3 \Rightarrow [(1,3)(4,0)]\epsilon N^{021}$.

$t_0t_2t_3 = t_3t_1t_0$, which implies $[t_0t_2t_3]^{(1,2)(3,0)(4,5)} = t_3t_1t_0$ $\Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{023}$.

Computing $N^{(021)}$ in N, we obtain:

$$\begin{aligned}|N^{(021)}| &\geq |N^{021}| \\ &\geq |<Id(G),(1,3)(4,0)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(023)}$ in N, we obtain:

$$\begin{aligned}|N^{(023)}| &\geq |N^{023}| \\ &\geq |<Id(G),(1,2)(3,0)(4,5)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(025)}$ in N, we obtain:

$$|N^{(025)}| \geq |N^{025}|$$
$$\geq | < Id(G) > |$$
$$\geq 1.$$

The number of single cosets in $Nt_0t_2t_1N = \frac{|N|}{|N^{(021)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_2t_3N = \frac{|N|}{|N^{(023)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_2t_5N = \frac{|N|}{|N^{(025)}|} = \frac{12}{1} = 12$.

Hence our index is increased to $67 + 6 + 6 + 12 = 91$.

Finally, consider the relations:

$t_0t_3t_0 = t_5t_2t_5$, which implies $[t_0t_3t_0]^{(1,4)(2,3)(5,0)} = t_5t_2t_5$
$\Rightarrow [(1,4)(2,3)(5,0)]\epsilon N^{030}$.

$t_0t_3t_0 = t_3t_0t_3$, which implies $[t_0t_3t_0]^{(1,4)(2,5)(3,0)} = t_3t_1t_0$
$\Rightarrow [(1,4)(2,5)(3,0)]\epsilon N^{030}$.

$t_0t_3t_0 = t_1t_4t_1$, which implies $[t_0t_3t_0]^{(1,0)(2,5)(3,4)} = t_1t_4t_1$
$\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{030}$.

Computing $N^{(030)}$ in N, we obtain:

$$|N^{(030)}| \geq |N^{030}|$$
$$\geq | < Id(G), (1,4)(2,3)(5,0), (1,4)(2,5)(3,0), (1,0)(2,5)(3,4) > |$$
$$\geq 12.$$

The number of single cosets in $Nt_0t_3t_0N = \frac{|N|}{|N^{(030)}|} = \frac{12}{12} = 1$.

Hence our index is increased to $91 + 1 = 92$.

We must now find the new level of double cosets coming from each double coset's orbits respectively. The orbits of $N^{(013)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{3\}$ advances back to [01]. The other representatives bring the potentially new double cosets [0131], [0132], [0134], [0135], and [0130]. Consider the following relations:

$t_0t_1t_3t_1 = (0,1,2,3,4,5)[t_0t_1]^{(0,4)(1,3)}$

$t_0t_1t_3t_2 = (0,5,4,3,2,1)[t_0t_2t_1]^{(0,4)(1,3)}$

$t_0t_1t_3t_5 = (0,4,2)(1,5,3)[t_0t_1t_3]^{(0,4)(1,3)}$.

Hence the representative from the {1} advances to [01], the representative from {2} advances to [021], and the representative from {5} loops back to [013]. So [0134] and [0130] are our only new, distinct double cosets.

However, consider the relation $t_0t_1t_3t_0 = (0,1,2,3,4,5)[t_0t_1t_3t_4]^{(0,4)(1,3)}$. Hence, the double coset [0134] is actually [0130]. From the orbits of $N^{(013)}$ on X, the only new distinct double coset is [0134].

The orbits of $N^{(013)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from the orbit {3} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0141], [0142], [0143], [0145], and [0140]. Consider the following relations:

$t_0t_1t_4t_1 = (0,2,4)(1,3,5)[t_0t_3]^{(0,5,4,3,2,1)}$
$t_0t_1t_4t_3 = (0,2,4)(1,5,3)[t_0t_1t_4]^{(1,5)(2,3)}$
$t_0t_1t_4t_5 = (0,1,2,3,4,5)[t_0t_2]^{(0,5,4,3,2,1)}$
$t_0t_1t_4t_0 = (e)[t_0t_2t_1]^{(0,2,4)(1,3,5)}$.

Hence the representative from {1} advances to [03], the representative from {3} loops back to [014], the representative from {5} advances to [02], and the representative from {0} advances to [021]. So [0142] is our only potentially new double coset coming from the orbits of $N^{(014)}$ on X.

The orbits of $N^{(015)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from the orbit {5} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0151], [0152], [0153], [0154], and [0150]. Consider the following relations:

$t_0t_1t_5t_1 = (0,1,2,3,4,5)[t_0t_1t_5]^{(0,2)(5,3)}$
$t_0t_1t_5t_2 = (0,3)(1,4)(5,2)[t_0t_2t_5]^{(0,2,4)(1,3,5)}$
$t_0t_1t_5t_3 = (e)[t_0t_2t_5]^{(0,3)(1,2)(5,4)}$
$t_0t_1t_5t_4 = (0,1,2,3,4,5)[t_0t_1t_5]^{(0,2)(5,3)}$
$t_0t_1t_5t_0 = (0,3)(1,4)(5,2)[t_0t_1]^{(0,5)(1,4)(2,3)}$.

Hence the representative from {1} loops back to [015], the representative from {2} advances to [025], the representative from {3} advances to [025], the representative from {4} loops back to [015], and the representative from {0} advances to [01]. There are no potentially new double cosets coming from the orbits of $N^{(015)}$ on X.

The orbits of $N^{(010)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [01]. The other orbit representatives bring the potentially new double cosets [0101], [0102], [0103], [0104], and [0105]. Consider the following relations:

$t_0t_1t_0t_1 = (e)[t_0t_1]^{(0,1)(5,2)}$
$t_0t_1t_0t_2 = (0,5,4,3,2,1)[t_0t_2t_3]^{(1,5)(2,4)}$
$t_0t_1t_0t_4 = (e)[t_0t_1t_0t_3]^{(0,1)(5,2)}$
$t_0t_1t_0t_5 = (0,1,2,3,4,5)[t_0t_2t_3]^{0,1,2,3,4,5)}$.

Hence the representative from $\{1\}$ advances to [01], the representative from $\{2\}$ advances to [023], the representative from $\{4\}$ advances to [013], and the representative from $\{5\}$ advances to [023]. So [0103] is our only potentially new double coset coming from the orbits of $N^{(010)}$ on X.

The orbits of $N^{(021)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{1\}$ advances back to [02]. The other orbit representatives bring the potentially new double cosets [0212], [0213], [0214], [0215], and [0210]. Consider the following relations:

$t_0t_2t_1t_2 = (0,5,4,3,2,1)[t_0t_1t_3]^{(0,4)(1,3)}$
$t_0t_2t_1t_3 = (0,3)(1,4)(5,2)[t_0t_2]^{(0,4)(1,3)}$
$t_0t_2t_1t_4 = (e)[t_0t_1t_4]^{(0,4,2)(1,5,3)}$
$t_0t_2t_1t_5 = (0,5,4,3,2,1)[t_0t_1t_0t_3]^{(0,4,2)(1,5,3)}$
$t_0t_2t_1t_0 = (0,3)(1,4)(5,2)[t_0t_1t_4]^{(1,5)(2,4)}$.

Hence the representative from $\{2\}$ advances to [013], the representative from $\{3\}$ advances to [02], the representative from $\{4\}$ advances to [014], the representative from $\{5\}$ advances to [0103], and the representative from $\{0\}$ advances to [01]. There are no potentially new double cosets coming from the orbits of $N^{(021)}$ on X.

The orbits of $N^{(023)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{3\}$ advances back to [02]. The other orbit representatives bring the potentially new double cosets [0211], [0212], [0214], [0215], and [0210]. Consider the following relations:

$t_0t_2t_3t_1 = (0,1,2,3,4,5)[t_0t_1t_4t_2]^{(0,3)(1,4)(5,2)}$
$t_0t_2t_3t_2 = (0,1,2,3,4,5)[t_0t_1t_4t_2]^{(0,2)(5,3)}$
$t_0t_2t_3t_4 = (0,5,4,3,2,1)[t_0t_1t_0]^{(0,5,4,3,2,1)}$

$t_0t_2t_3t_5 = (0,2,4)(1,3,5)[t_0t_1t_0]^{(0,3)(1,2)(5,4)}$
$t_0t_2t_3t_0 = (0,1,2,3,4,5)[t_0t_2]^{(0,3)(1,4)(5,2)}$.

Hence the representative from $\{1\}$ advances to [0142], the representative from $\{2\}$ advances to [0142], the representative from $\{4\}$ advances to [010], the representative from $\{5\}$ advances to [010], and the representative from $\{0\}$ advances to [02]. There are no potentially new double cosets coming from the orbits of $N^{(023)}$ on X.

The orbits of $N^{(025)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{5\}$ advances back to [02]. The other representatives will bring the potentially new double cosets [0251], [0252], [0253], [0254], and [0250]. Consider the following relations:

$t_0t_2t_5t_1 = (0,1,2,3,4,5)[t_0t_1t_3t_4]^{(0,3)(1,2)(5,4)}$
$t_0t_2t_5t_2 = (0,4,2)(1,5,3)[t_0t_2]^{(0,1,2,3,4,5)}$
$t_0t_2t_5t_3 = (0,5,4,3,2,1)[t_0t_2t_5]^{(1,5)(2,4)}$
$t_0t_2t_5t_4 = (0,5,4,3,2,1)[t_0t_1t_0t_3]^{(0,2,4)(1,3,5)}$
$t_0t_2t_5t_0 = (e)[t_0t_1t_5]^{(0,3)(1,4)(5,2)}$.

Hence the representative from $\{1\}$ advances to [0134], the representative from $\{2\}$ advances to [02], the representative from $\{3\}$ will loop back to [025], the representative from $\{4\}$ advances to [0103], and the representative from $\{0\}$ advances to [015]. There are no potentially new double cosets coming from the orbits of $N^{(025)}$ on X.

The orbits of $N^{(030)}$ on X are $\{0\}$, $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The representative from $\{0\}$ advances back to [03]. We will take t_1, t_2, and t_3 from the other three orbits of $N^{(030)}$ on X. These representatives bring the potentially new double cosets [0301], [0302], and [0303]. Consider the following relations:

$t_0t_3t_0t_1 = (0,2,4)(1,3,5)[t_0t_3]^{(0,1,2,3,4,5)}$
$t_0t_3t_0t_2 = (0,4,2)(1,5,3)[t_0t_3]^{(0,2,4)(1,3,5)}$
$t_0t_3t_0t_3 = (e)[t_0t_3]^{(0,3)(1,4)(5,2)}$.

Hence the representatives from $\{1,5\}$ advance to [03], the representatives from $\{2,4\}$ advance to [03], and the representative from $\{3\}$ advances to [03]. There are no potentially new double cosets coming from the orbits of $N^{(030)}$ on X.

We now continue to the next level of double cosets. The only new, distinct double cosets are [0134], [0142], and [0103].

Consider the relations:

$t_0t_1t_3t_4 = t_3t_2t_0t_5$, which implies $[t_0t_1t_3t_4]^{(1,2)(3,0)(4,5)} = t_3t_2t_0t_5$
$\Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{0134}$.

$t_0t_1t_4t_2 = t_1t_0t_3t_5$, which implies $[t_0t_1t_4t_2]^{(1,0)(2,5)(3,4)} = t_1t_0t_3t_5$
$\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{0142}$.

$t_0t_1t_0t_3 = t_2t_1t_2t_5$, which implies $[t_0t_1t_4t_2]^{(2,0)(3,5)} = t_2t_1t_2t_5$
$\Rightarrow [(2,0)(3,5)]\epsilon N^{0103}$.

Computing $N^{(0134)}$ in N:

$$\begin{aligned}|N^{(0134)}| &\geq |N^{0134}|\\ &\geq |<Id(G),(1,2)(3,0)(4,5)>|\\ &\geq 2\end{aligned}$$

Computing $N^{(0142)}$ in N, we obtain:

$$\begin{aligned}|N^{(0142)}| &\geq |N^{0142}|\\ &\geq |<Id(G),(1,0)(2,5)(3,4)>|\\ &\geq 2.\end{aligned}$$

Computing $N^{(0103)}$ in N, we obtain:

$$\begin{aligned}|N^{(0103)}| &\geq |N^{0103}|\\ &\geq |<Id(G),(2,0)(3,5)>|\\ &\geq 2.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_3t_4N = \frac{|N|}{|N^{(0134)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_4t_2N = \frac{|N|}{|N^{(0142)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_0t_3N = \frac{|N|}{|N^{(0103)}|} = \frac{12}{2} = 6$.

Hence our index is increased to $92 + 6 + 6 + 6 = 110$

The orbits of $N^{(0134)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{4\}$ advances back to [013]. The other orbit representatives bring the potentially new double cosets [01341], [01342], [01343], [01345] and [01340]. Consider the following relations:

$t_0t_1t_3t_4t_1 = (0,3)(1,4)(5,2)[t_0t_2t_5]^{(e)}$

$t_0t_1t_3t_4t_2 = (0,1,2,3,4,5)[t_0t_2t_5]^{(0,3)(1,2)(5,4)}$

$t_0t_1t_3t_4t_3 = (0,1,2,3,4,5)[t_0t_1t_4t_2]^{(0,5,4,3,2,1)}$

$t_0t_1t_3t_4t_5 = (0,4,2)(1,5,3)[t_0t_1t_3]^{(0,3)(1,2)(5,4)}$

$t_0t_1t_3t_4t_0 = (e)[t_0t_1t_4t_2]^{(0,3)(1,4)(5,2)}.$

Hence the representatives from $\{1\}$ advances to [025], the representative from $\{2\}$ will advance to [025], the representative from $\{3\}$ advances to [0142], the representative from $\{5\}$ advances to [013], and the representative from $\{0\}$ advances to [0142]. There are no new double cosets coming from the orbits of $N^{(0134)}$ on X.

The orbits of $N^{(0142)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{2\}$ advance back to [014]. The other orbit representatives bring the potentially new double cosets [01421], [01423], [01424], [01425] and [01420]. Consider the following relations:

$t_0t_1t_4t_2t_1 = (0,2,4)(1,3,5)[t_0t_2t_3]^{(0,5,4,3,2,1)}$

$t_0t_1t_4t_2t_3 = (e)[t_0t_1t_3t_4]^{(0,3)(1,4)(5,2)}$

$t_0t_1t_4t_2t_4 = (0,5,4,3,2,1)[t_0t_1t_3t_4]^{(0,1,2,3,4,5)}$

$t_0t_1t_4t_2t_5 = (0,3)(1,4)(5,2)[t_0t_1t_4]^{0,1)(5,2)}$

$t_0t_1t_4t_2t_0 = (e)[t_0t_2t_3]^{(0,5,4,3,2,1)}.$

Hence the representatives from $\{1\}$ advances to [023], the representative from $\{3\}$ advances to [0134], the representative from $\{4\}$ advances to [0134], the representative from $\{5\}$ advances to [014], and the representative from $\{0\}$ advances to [023]. There are no new double cosets coming from the orbits of $N^{(0142)}$ on X.

The orbits of $N^{(0103)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{3\}$ advances back to [010]. The other orbit representatives bring the potentially new double cosets [01031], [01032], [01034], [01035] and [01030]. Consider the following relations:

$t_0t_1t_0t_3t_1 = (0,1,2,3,4,5)[t_0t_2t_1]^{(0,2,4)(1,3,5)}$

$t_0t_1t_0t_3t_2 = (0,1,2,3,4,5)[t_0t_2t_5]^{(0,2,4)(1,5,3)}$

$t_0t_1t_0t_3t_4 = (0,2,4)(1,3,5)[t_0t_1t_0t_3]^{(0,2)(5,3)}$

$t_0t_1t_0t_3t_5 = (0,5,4,3,2,1)[t_0t_1t_0]^{(0,1,2,3,4,5)}$

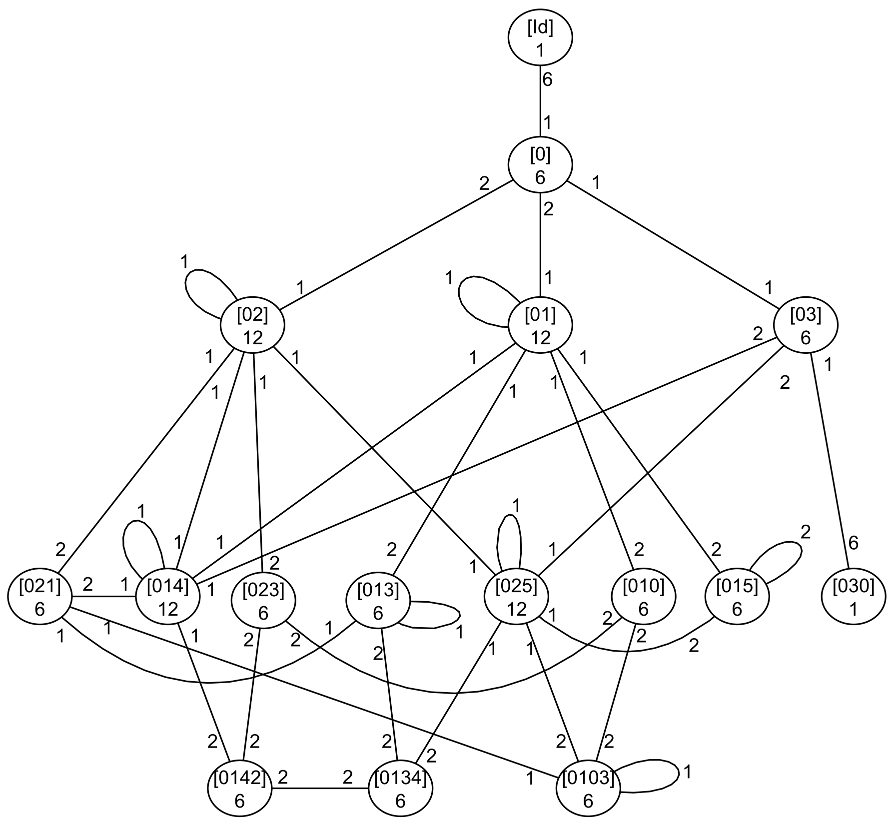
$t_0t_1t_0t_3t_0 = (0,4,2)(1,5,3)[t_0t_2t_5]^{(0,4)(1,3)}.$

Hence the representatives from $\{1\}$ advances to [021], the representative from

$\{2\}$ advances to [025], the representative from $\{4\}$ loops back to [0103], the representative from $\{5\}$ advances to [010], and the representative from $\{0\}$ advances to [025]. There are no new double cosets coming from the orbits of $N^{(0103)}$ on X.

Because there are no new words, we have completed our double coset enumeration of G over N. Our group is closed under right multiplication of t_i's. The index of N in G is 110. The Cayley graph of G is given below.

Figure 2.1: $L(2,11) \times 2$ Cayley Graph



2.2.2 Double Coset Enumeration of $L(2,11)$ over D_{12}

We factor the progenitor $2^{*6} : D_{12}$ by the three relations $[xtt^x]^3$, $[xt]^5$, and $(x^3y)[tt^{x^3}t]$ where $x = (0,1,2,3,4,5)$ and $y = (1,5)(2,4)$. Letting t be represented by t_0, we have the two relations from before as well as the third relation which we will calculate:

$$x^3ytt^{x^3}t = e$$
$$tt^{x^3}t = x^3y$$
$$t_0t_3t_0 = (1,2)(3,0)(4,5)$$
$$t_6t_3t_0t_0 = (1,2)(3,0)(4,5)t_0$$
$$t_0t_3 = (1,2)(3,0)(4,5)t_0.$$

We let G be $2^{*6} : D_{12}/[(1,4)(2,5)(3,0)t_2t_3t_1t_2t_0t_1, (0,5,4,3,2,1)t_4t_3t_2t_1t_0,$ $(1,2)(3,0)(4,5)t_0t_3t_0]$, where $N =< (0,1,2,3,4,5),(1,5)(2,4) >$.

We will find the index of N in G by manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

We will complete a double coset enumeration of G over N to find the index of N in G. We must find all distinct double cosets $[w]$, where $[w] = \{Nw^n | n \epsilon N\}$, and how many single cosets are contained in each double coset. The manual double coset enumeration is finished when all potentially new double cosets have already been accounted for and when the set of right cosets we find is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n \; in \; N : Nw^n = Nw\}$, where w is a word of t_i's on $\{0,1,2,3,4,5\} = X$.

We begin with the double coset NeN, which we denote $[*]$. This double coset consists of the single coset N. For convenience, we will let 6 be 0. The single orbit of N on X is $\{0,1,2,3,4,5\}$. We will choose $t_6 = t_0$ as our symmetric generator from the orbit $\{0,1,2,3,4,5\}$ and find Nt_0 belongs to Nt_0N which is a new double coset. We denote Nt_0N by $[0]$.

To find the number of single cosets contained in $[0]$ we must find the set of coset stabilizers of 0, denoted $N^{(0)}$. This is relevant to us because the number of single cosets in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. We have:

$$\begin{aligned} |N^{(0)}| &\geq | < Id(G), (1,5)(2,4) > | \\ &\geq 2. \end{aligned}$$

So the number of single cosets in $Nt_0N = \frac{|N|}{|N^{(0)}|} = \frac{12}{2} = 6$. When we permutate t_0 by the transversals of [0], we find 6 single cosets are distinct. Our index is the sum of distinct single cosets in the distinct double cosets, such as [*] and $[\bar{0}]$. As of now, we have $1 + 6 = 7$ single cosets since [0] has 6 distinct single cosets and $[*]$ has 1. We note that the orbits of [0] are $\{0\}$, $\{1, 5\}$, $\{2, 4\}$ and $\{3\}$.

We continue to the next level of potential double cosets by working with the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$, $\{1, 5\}$, $\{2, 4\}$ and $\{3\}$ and we take t_0, t_1, t_2, and t_3 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1, 5\}$ we find a potentially new double coset Nt_0t_1, which we will denote [01]. From the orbit $\{2, 4\}$ we get Nt_0t_2 which belongs to [02]. From the orbit $\{3\}$ we get another potentially new double coset Nt_0t_3, which we will denote [03].

Consider the double coset [03]. We have the relation: $t_0t_3 = (1, 2)(3, 6)(4, 5)t_0$. This implies that any representative from the orbit $\{3\}$ will actually loop back to [0].

We will now determine how many distinct single cosets are contained in [01] and [02].

Computing $N^{(01)}$ in N, we obtain:

$$\begin{aligned} |N^{(01)}| &\geq |N^{01}| \\ N^{(01)} &\geq | < Id(G) > | \\ &\geq 1. \end{aligned}$$

And also $N^{(02)}$ in N, we obtain:

$$\begin{aligned} |N^{(02)}| &\geq |N^{02}| \\ &\geq | < Id(G) > | \\ &\geq 1. \end{aligned}$$

The number of single cosets in $Nt_0t_1N = \frac{|N|}{|N^{(01)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_2N = \frac{|N|}{|N^{(02)}|} = \frac{12}{1} = 12$.

Hence, our index is increased to $1 + 6 + 12 + 12 = 31$.

We now explore the potentially new double cosets coming from representatives from the orbits of $N^{(01)}$ on X. We find [01] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{1\}$ will advance to [0]. The other orbit representatives will bring the potentially new double cosets [012], [013], [014], [015], and [010]. However, consider the following relations:

$t_0t_1t_2 = (0,5,4,3,2,1)[t_0t_1]^{(0,4)(1,3)}$

$t_0t_1t_4 = (0,4,2)(1,5,2)[t_0t_2]^{(0,2)(5,3)}$.

Hence in [01], the single representative $\{2\}$ goes to [01] and the single representative $\{4\}$ goes to [02]. So the only new, distinct double cosets are [013], [015], and [010].

The orbits of $N^{(02)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{2\}$ will advance to [0]. The other representatives will bring the potentially new double cosets [021], [023], [024], [025], and [020]. Consider the following relations:

$t_0t_2t_4 = (0,4,2)(1,5,3)[t_0t_2]^{(0,2)(5,3)}$

$t_0t_2t_5 = (0,1)(5,2)[t_0t_1]^{(0,5,4,3,2,1)}$

$t_0t_2t_0 = (0,5,4,3,2,1)[t_0t_1t_4]^{(0,1,2,3,4,5)}$.

Hence in [02], the representative $\{4\}$ will go to [02], the representative $\{5\}$ will go to [01] and the representative $\{0\}$ will go to [014]. We find that [021] and [023] are the only new, distinct double cosets coming from the orbits of $N^{(02)}$ on X.

We must now investigate the double cosets: [013], [015], [010], [021] and [023].

Consider the following relations:

$t_0t_1t_3 = t_4t_3t_1$. Hence $[t_0t_1t_3]^{(1,3)(4,0)} = t_4t_3t_1 \Rightarrow [(1,3)(4,0)]\epsilon N^{013}$.

$t_0t_1t_5 = t_5t_4t_0 = t_2t_1t_3 = t_3t_4t_2$, which implies the following three statements:

$[t_0t_1t_5]^{(2,0)(3,5)} = t_2t_1t_3 \Rightarrow [(2,0)(3,5)]\epsilon N^{015}$

$[t_0t_1t_5]^{(1,4)(2,3)(5,0)} = t_5t_4t_6 \Rightarrow [(1,4)(2,3)(5,0)]\epsilon N^{015}$

$[t_0t_1t_5]^{(1,4)(2,5)(3,0)} = t_3t_4t_2 \Rightarrow [(1,4)(2,5)(3,0)]\epsilon N^{015}$.

$t_0t_1t_0 = t_1t_0t_1$, which implies $[t_0t_1t_0]^{(1,0)(2,5)(3,4)} = t_1t_0t_1$ $\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{010}$.

$t_0t_2t_1 = t_1t_5t_0 = t_4t_2t_3 = t_3t_5t_4$, which implies the following three statements:
$[t_0t_2t_1]^{(1,0)(2,5)(3,4)} = t_1t_5t_0 \Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{021}$
$[t_0t_2t_1]^{(0,4)(1,3)} = t_4t_2t_3 \Rightarrow [(1,4)(2,3)(5,0)]\epsilon N^{021}$
$[t_0t_2t_1]^{(1,4)(2,5)(3,0)} = t_3t_5t_4 \Rightarrow [(1,4)(2,5)(3,0)]\epsilon N^{021}$.

$t_0t_2t_3 = t_3t_1t_0$, which implies $[t_0t_2t_3]^{(1,2)(3,0)(4,5)} = t_4t_3t_1$
$\Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{023}$.

Computing $N^{(013)}$ in N, we obtain:

$$\begin{aligned}|N^{(013)}| &\geq |N^{013}| \\ &\geq |< Id(G), (1,3)(4,0) >| \\ &\geq 2.\end{aligned}$$

Computing $N^{(015)}$ in N, we obtain:

$$\begin{aligned}|N^{(015)}| &\geq |N^{015}| \\ &\geq |< Id(G), (2,0)(3,5), (1,4)(2,3)(5,0), (1,4)(2,5)(3,0) >| \\ &\geq 4.\end{aligned}$$

Computing $N^{(010)}$ in N, we obtain:

$$\begin{aligned}|N^{(010)}| &\geq |N^{010}| \\ &\geq |< Id(G), (1,0)(2,5)(3,4) >| \\ &\geq 2.\end{aligned}$$

Computing $N^{(021)}$ in N, we obtain:

$$\begin{aligned}|N^{(021)}| &\geq |N^{021}| \\ &\geq |< Id(G), (1,0)(2,5)(3,4), (1,4)(2,3)(5,0), (1,4)(2,5)(3,0) >| \\ &\geq 4.\end{aligned}$$

Computing $N^{(023)}$ in N, we obtain:

$$\begin{aligned} |N^{(023)}| &\geq |N^{023}| \\ &\geq |< Id(G), (1,2)(3,0)(4,5) >| \\ &\geq 2. \end{aligned}$$

The number of single cosets in $Nt_0t_1t_3N = \frac{|N|}{|N^{(013)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_5N = \frac{|N|}{|N^{(015)}|} = \frac{12}{4} = 3$. The number of single cosets in $Nt_0t_1t_0N = \frac{|N|}{|N^{(010)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_2t_1N = \frac{|N|}{|N^{(021)}|} = \frac{12}{4} = 3$. The number of single cosets in $Nt_0t_2t_3N = \frac{|N|}{|N^{(023)}|} = \frac{12}{2} = 6$.

Hence our index is increased to $31 + 6 + 3 + 6 + 3 + 6 = 55$.

We now explore any potentially new double cosets coming from representatives from the orbits of $N^{(013)}$ on X, $N^{(015)}$ on X, $N^{(010)}$ on X, $N^{(021)}$ on X, and $N^{(023)}$ on X.

The orbits of $N^{(013)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{3\}$ advances to [01]. The other representatives will be the potentially new double cosets [0131], [0132], [0134], [0135], and [0130]. However, consider the following relations:

$t_0t_1t_3t_1 = (0,1,2,3,4,5)[t_0t_1]^{(0,4)(1,3)}$

$t_0t_1t_3t_2 = (0,5,4,3,2,1)[t_0t_2t_1]^{(0,4)(1,3)}$

$t_0t_1t_3t_4 = (0,4)(1,3)[t_0t_1t_0]^{(0,1,2,3,4,5)}$

$t_0t_1t_3t_5 = (0,4,2)(1,5,3)[t_0t_1t_3]^{(0,4)(1,3)}$

$t_0t_1t_3t_0 = (0,3)(1,2)(5,4)[t_0t_1t_0]^{(0,2,4)(1,3,5)}$.

Hence the representative from the $\{1\}$ will advance to [01], the representative from $\{2\}$ will advance to [021], the representative from $\{4\}$ will advance to [010], the representative from $\{5\}$ will advance to [013], and the representative from $\{0\}$ will advance to [010]. So no new double cosets come from the orbits of $N^{(013)}$ on X.

The orbits of $N^{(015)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{3\}$ will advance to [01]. The other representatives will bring the potentially new double cosets [0141], [0142], [0143], [0145], and [0140]. Consider the following relations:

$t_0t_1t_5t_1 = (0,1,2,3,4,5)[t_0t_1t_5]^{(0,2)(5,3)}$

$t_0t_1t_5t_2 = (0,2)(5,3)[t_0t_1]^{(0,3)(1,4)(5,2)}$
$t_0t_1t_5t_3 = (0,5)(1,4)(2,3)[t_0t_1]^{(0,2)(5,3)}$
$t_0t_1t_5t_4 = (0,1,2,3,4,5)[t_0t_1t_5]^{(0,2)(5,3)}$
$t_0t_1t_5t_0 = (0,3)(1,4)(5,2)[t_0t_1]^{(0,5)(1,4)(2,3)}$.

Hence the representative from {1} will advance to [015], the representative from {2} will advance to [01], the representative from {3} will advance to [01], the representative from {4} will advance to [015], and the representative from {0} will advance to [01]. So no new double cosets come from the orbits of $N^{(014)}$ on X.

The orbits of $N^{(010)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {0} will advance to [01]. The other representatives will bring the potentially new double cosets [0101], [0102], [0103], [0104], and [0105]. Consider the following relations:

$t_0t_1t_0t_1 = (e)[t_0t_1]^{(0,1)(5,2)}$
$t_0t_1t_0t_2 = (0,5,4,3,2,1)[t_0t_2t_3]^{(1,5)(2,4)}$
$t_0t_1t_0t_3 = (0,2)(5,3)[t_0t_1t_0t_3]^{(0,5,4,3,2,1)}$
$t_0t_1t_0t_4 = (1,5)(2,4)[t_0t_1t_0t_3]^{(0,2)(5,3)}$
$t_0t_1t_0t_5 = (0,1,2,3,4,5)[t_0t_2t_3]^{(0,1,2,3,4,5)}$.

Hence the representative from {1} will advance to [01], the representative from {2} will advance to [023], the representative from {3} will advance to [013], the representative from {4} will advance to [013], and the representative from {5} will advance to [023]. So no new double cosets come from the orbits of $N^{(015)}$ on X.

The orbits of $N^{(021)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {1} will advance to [02]. The other representatives will bring the potentially new double cosets [0212], [0213], [0214], [0215], and [0210]. Consider the following relations:

$t_0t_2t_1t_2 = (0,5,4,3,2,1)[t_0t_1t_3]^{(0,4)(1,3)}$
$t_0t_2t_1t_3 = (0,3)(1,4)(5,2)[t_0t_2]^{(0,4)(1,3)}$
$t_0t_2t_1t_4 = (0,1)(5,2)[t_0t_2]^{(0,3)(1,4)(5,2)}$
$t_0t_2t_1t_5 = (1,5)(2,4)[t_0t_1t_3]^{(0,1)(5,3)}$
$t_0t_2t_1t_0 = (0,4)(1,3)[t_0t_2]^{(0,1)(5,2)}$.

Hence the representative from {2} will advance to [013], the representative from {3} will advance to [02], the representative from {4} will advance to [02], the representative from {5} will advance to [013], and the representative from {0} will advance to [02]. So there are no potentially new double cosets coming from the orbits of $N^{(021)}$ on X.

The orbits of $N^{(023)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {3} will advance to [02]. The other representatives will bring the potentially new double cosets [0231], [0232], [0234], [0235], and [0230]. Consider the following relations:

$$t_0t_2t_3t_1 = (0,1)(5,2)[t_0t_2t_3]^{(e)}$$
$$t_0t_2t_3t_2 = (0,4)(1,3)[t_0t_2t_3]^{(0,3)(1,2)(5,4)}$$
$$t_0t_2t_3t_4 = (0,5,4,3,2,1)[t_0t_1t_0]^{(0,5,4,3,2,1)}$$
$$t_0t_2t_3t_5 = (0,2,4)(1,3,5)[t_0t_1t_0]^{(0,3)(1,4)(5,2)}$$
$$t_0t_2t_3t_0 = (0,1,2,3,4,5)[t_0t_2]^{(0,3)(1,2)(5,4)}.$$

Hence the representative from {1} will advance to [023], the representative from {2} will advance to [023], the representative from {4} will advance to [010], the representative from {5} will advance to [010], and the representative from {0} will advance to [02]. So there are no potentially new double cosets coming from the orbits of $N^{(023)}$ on X.

Because there are no new words, we have completed our double coset enumeration of G over N. Our group is closed under right multiplication of t_i's. The index of N in G is 55. The Cayley graph of G is given below. Since we obtained this $L(2,11)$ group by factoring by the center, the Cayley graph is very similar in structure to that of our $L(2,11) \times 2$ Cayley graph we constructed previously.

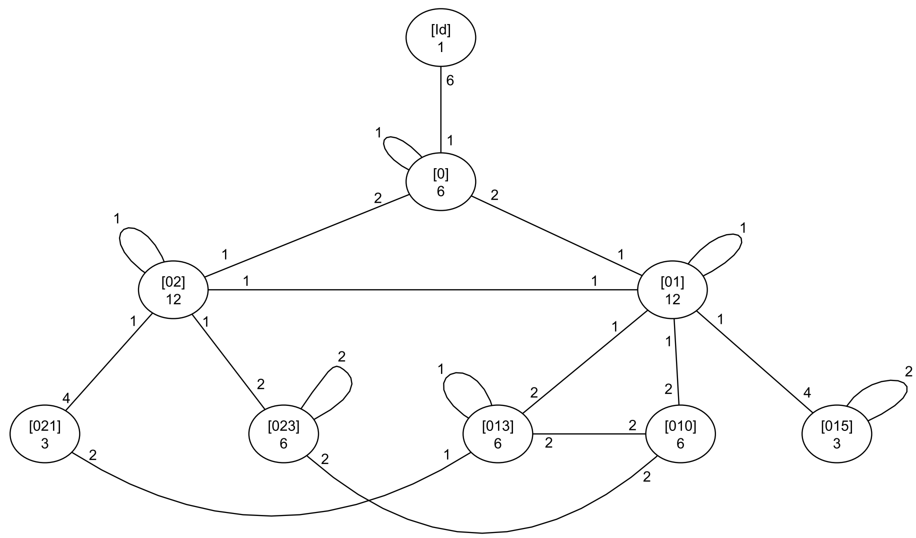


Figure 2.2: $L(2,11)$ Factored by Center Cayley Graph

2.2.3 Proof of $G \cong L(2,11)$

We let $X = \{N\omega\}$ be the set of single cosets of G over N. We will use Iwasawa's Lemma and the transitive action of G on X to prove G is a simple group. If we can show that G is faithful, G acts primitively on X, $G = G'$, and that there exists a normal, abelian subgroup of G such that $< K^G >= G$, we will have shown that G is a non-abelian simple group of order 660.

(i) G acts faithfully on X

Proof. Since X is a transitive G-set of degree 55, we have:

$$|G| = 55|G_1|,$$

where G_1 is the one point stabiliser of the single coset N. However, N is only stabilised by elements of N. Therefore $G_1 = N$ and $|G_1| = |N| = 12$. It is then evident that $|G| = 660$. If $|G| > 660$, X would not be faithful. □

(ii) The group G acts primitively on X

Proof. Since G is transitive, we can assume $N \in B$. However, $|B|$ must divide $|X| = 55 = 5 \times 11$. The only possible nontrivial blocks must be of size 5 or 11. By observation of our Cayley graph, there are no possibilities for a block of either of these sizes. Thus G acts primitively on X. □

(iii) The group G is perfect

Proof. Let us first begin by showing that G is generated by involutions. We have $G =< x, y, t_0, t_1, \ldots, t_5 >$. However, consider the following relations:

$$t_0t_1t_5t_4 = (0, 1, 2, 3, 4, 5)t_2t_1t_3$$
$$t_0t_1t_5t_4t_3t_1t_2 = (0, 1, 2, 3, 4, 5)t_2t_1t_3t_3t_1t_2$$
$$t_0t_1t_5t_4t_3t_1t_2 = (0, 1, 2, 3, 4, 5) = x$$

$$t_0t_1t_0t_4 = (1, 5)(2, 4)t_2t_1t_5$$
$$t_0t_1t_0t_4t_5t_1t_2 = (1, 5)(2, 4)t_2t_1t_5t_5t_1t_2$$
$$t_0t_1t_0t_4t_5t_1t_2 = (1, 5)(2, 4) = y.$$

Since x and y are product of t_i's and $G =< x, y, t_0, t_1, \ldots, t_5 >$, we see $G =< t_0, t_1, \ldots, t_5 >$.

Since $G =< N, t >$, where $N = D_{12}$, we know $(D_{12})' \leq G'$. Hence we have $(D_{12})' =< x^2 >=< (0, 2, 4)(1, 3, 5) >=\leq G'$.

Consider the following relation:

$$
\begin{aligned}
t_0t_2t_4 &= (0,4,2)(1,5,3)t_2t_0 \\
t_0t_2t_4t_0 &= (0,4,2)(1,5,3)t_2t_0t_0 \\
t_0t_2t_4t_0 &= (0,4,2)(1,5,3)t_2 \\
t_0t_2t_4t_0t_2 &= (0,4,2)(1,5,3)t_2t_2 \\
t_0t_2t_4t_0t_2 &= (0,4,2)(1,5,3).
\end{aligned}
$$

So we see $t_0t_2t_4t_0t_2 \in N \leq G'$. Now we conjugate $t_0t_2t_4t_0t_2 \in G'$ by the element $t_0t_2 \in G$ and find:

$$
\begin{gathered}
[t_0t_2t_4t_0t_2]^{t_0t_2} \in G' \\
[t_0t_2]^{-1}[t_0t_2t_4t_0t_2][t_0t_2] \in G' \\
{t_2}^{-1}{t_0}^{-1}t_0t_2t_4t_0t_2t_0t_2 \in G' \\
t_2t_0t_0t_2t_4t_0t_2t_0t_2 \in G' \\
t_2t_2t_4t_0t_2t_0t_2 \in G' \\
t_4t_0t_2t_0t_2 \in G' \\
t_4[t_0,t_2] \in G'.
\end{gathered}
$$

But $[t_0,t_2] \in G'$. Therefore $t_4 \in G'$. So we have $G' \geq < (0,2,4)(1,3,5), t_4 >$. So we have $G' =< (0,2,4)(1,3,5), t_0, t_2, t_4 >$ after conjugating t_4 by x^2 and x^4.

Now consider the relation:

$$t_0t_1t_3t_5 = (0,4,2)(1,5,3)t_4t_3t_1$$
$$t_0t_1t_3t_5 = t_0(0,4,2)(1,5,3)t_3t_1$$
$$t_0t_0t_1t_3t_5 = t_0t_0(0,4,2)(1,5,3)t_3t_1$$
$$t_1t_3t_5 = (0,4,2)(1,5,3)t_3t_1$$
$$t_1t_3t_5t_1 = (0,4,2)(1,5,3)t_3t_1t_1$$
$$t_1t_3t_5t_1 = (0,4,2)(1,5,3)t_3$$
$$t_1t_3t_5t_1t_3 = (0,4,2)(1,5,3)t_3t_3$$
$$t_1t_3t_5t_1t_3 = (0,4,2)(1,5,3).$$

So we see $t_1t_3t_5t_1t_3 \in G'$. Now we conjugate $t_1t_3t_5t_1t_3 \in G'$ by the element $t_1t_3 \in G$ and find:

$$[t_1t_3t_5t_1t_3]^{t_1t_3} \in G'$$
$$[t_1t_3]^{-1}[t_1t_3t_5t_1t_3][t_1t_3] \in G'$$
$$t_3{}^{-1}t_1{}^{-1}t_1t_3t_5t_1t_3t_1t_3 \in G'$$
$$t_3t_1t_1t_3t_5t_1t_3t_1t_3 \in G'$$
$$t_3t_3t_5t_1t_3t_1t_3 \in G'$$
$$t_5t_1t_3t_1t_3 \in G'$$
$$t_5[t_1,t_3] \in G'.$$

But $[t_1,t_3] \in G'$. Therefore $t_5 \in G'$. So we have $G' =< (0,2,4)(1,3,5), t_0, t_2, t_4, t_5 >$. After conjugating t_5 by x^2 and x^4, we see $G' =< (0,2,4)(1,3,5), t_0, t_2, t_4, t_5, t_1, t_3 >$. We have already shown x is generated by t_i's and therefore x^2 would also be generated by t_i's. So we see $G' =< t_0, t_1, \ldots, t_5 >= G$. So $G' = G$.

□

(iv) The point stabiliser of N of G contains a normal abelian subgroup K whose conjugates generate G

Proof. Since $N = D_{12}$, we will take the normal, abelian subgroup K given by $K =< (0,3)(1,2)(4,5) >$. Utilizing the following relation, we have:

$$t_0t_3 = (0,3)(1,2)(4,5)t_0$$
$$t_0t_3t_0 = (0,3)(1,2)(4,5) \in K^G$$
$$t_0t_3t_0 \in K \subseteq K^G.$$

Now conjugating $t_0t_3t_0$ by the element $t_0 \in G$ we see:

$$[t_0t_3t_0]^{t_0} \in K^G$$
$$t_0{}^{-1}t_0t_3t_0t_0 \in K^G$$
$$t_3 \in K^G.$$

Since $N \in G$, we also have $(t_3)^N \in K^G$. Now we have $K^G \geq < t_0, t_1, \ldots, t_5 >= G$ since $G =< t_0, t_1, \ldots, t_5 >$. But $K^G \leq G$, hence $K^G = G$. □

(v) The group G is simple. Furthermore, $G \cong L_2(11)$.

Proof. We have shown that the group G acts faithfully on X, is primitive, is perfect, and contains a normal abelian subgroup whose conjugates generate G. Therefore by Iwasawa's Lemma G is a simple group. Refering to [WB99], $L_2(11)$ is the only non-abelian simple group of order 660. □

2.2.4 Alternative Proof of $G \cong L(2, 11)$

An alternative proof can be used to show $G \cong L(2, 11) = L_2(11)$ utilizing linear fractional mappings.

Let us first define our mappings given by α, β, γ, and δ.

For the linear fractional mapping $L_2(n)$, we have the following which are all in modulo n:

$\alpha : x \mapsto x + 1$

$\beta : x \mapsto kx$, where k is a nonzero, finite square found in the integers $1, 2, \ldots,$ $n-1$ such that the powers of k generate the set of nonzero squares of $1, 2, \ldots, n-1$

$\gamma : x \mapsto -x^{-1}$

$\delta : x \mapsto M$, where $M = \dfrac{ax+b}{cx+d}$, where $ad - bc$ is a nonzero, nonsquare in modulo n.

Furthermore, if $n \equiv 3 \pmod 4$ then the presentation of $L_2(n)$ is given by:

$PSL(2,n) = L_2(n) =< \alpha, \beta, \gamma | \alpha^n, \beta^{\frac{(n-1)}{2}}, \gamma^2, \alpha^\beta \alpha^{-k}, (\beta\gamma)^2, (\alpha\gamma)^3 >$.

Similarly, if $n \equiv 3 \pmod 4$ a presentation for $PGL(2,n)$ is given by:

$PGL(2,n) =< \alpha, \beta, \gamma | \alpha^{n-1}, \beta^{\frac{(n-1)}{2}}, \gamma^2, \alpha^\beta \alpha^{-k}, (\beta\gamma)^2, (\alpha\gamma)^3, \delta^2, \alpha^\delta = \_\_,$ $\beta^\delta = \_\_, \gamma^\delta = \_\_ >$, where the action of $\alpha^\delta, \beta^\delta, \gamma^\delta$ must be determined.

For now, we will only use the formula for $L_2(11)$ which is of the form $n \equiv 3 \pmod 4$. We write our 12-letter permutations on the 12 letters given by $0, 1, 2, \ldots, 10, \infty$. We find the following:

$\alpha : x \mapsto x + 1 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)(\infty) = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

In modulo 11, we calculate the nonzero, squares of $1, 2, \ldots, 10, \infty$:
$\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, \infty^2\} = \{1, 4, 9, 5, 3, 3, 5, 9, 4, 1, \infty\} = \{1, 3, 4, 5, 9\}$.

We find the following:

$4^1 \equiv 4 \pmod{11}$,

$4^2 \equiv 16 \pmod{11} \equiv 5 \pmod{11}$,

$4^3 \equiv 64 \pmod{11} \equiv 9 \pmod{11}$

$4^4 \equiv 256 \pmod{11} \equiv 3 \pmod{11}$

$4^5 \equiv 1024 \pmod{11} \equiv 1 \pmod{11}$.

Therefore the powers of 4 generate the set $\{1, 3, 4, 5, 9\}$ in modulo 11. So $k = 4$ and we obtain:

$\beta : x \mapsto 4x = (0)(\infty)(1, 4, 5, 9, 3)(2, 8, 10, 7, 6) = (1, 4, 5, 9, 3)(2, 8, 10, 7, 6)$.

Finally, to find $\gamma : x \mapsto -x^{-1}$, we will first find x^{-1}, or the multiplicative inverse of each letter, and then multiply that value by -1 and determine its representative value modulo 11.

For instance, the multiplicative inverse of 1 is itself since $1 \times 1 = 1 \equiv 1 \pmod{11}$, therefore $1 \mapsto -(1) \equiv 10 \pmod{11}$. The multiplicative inverse of 2 is 6, since $2 \times 6 = 12 \equiv 1 \pmod{11}$. We define the multiplicative inverse of 0 as ∞ and claim that $-\infty$ corresponds to the letter ∞. Let us first find solve $x = 1$. $1 \mapsto -(1)^{-1} = -1 \equiv 10 \pmod{11}$. So we find that 1 should advance to 10 in permutation γ. Continuing this pattern with the other 11 letters, we obtain the following permutation for γ:

$\gamma : x \mapsto -x^{-1} = (0, \infty)(1, 10)(2, 5)(3, 7)(4, 8)(6, 9)$.

When observing the permutations for α, β, and γ, the order of each permutation follows the order that our presentation should have. α is of order 11, β is of order $\frac{11-1}{2} = 5$, and γ is of order 2. We denote the group $H =< \alpha, \beta, \gamma >$.

We can utilize the presentation formula of $PSL(2,11) = L_2(11)$ from earlier and obtain the following:

$PSL(2,11) = L_2(11) =< \alpha, \beta, \gamma | \alpha^{11}, \beta^{\frac{(11-1)}{2}}, \gamma^2, \alpha^{\beta}\alpha^{-4}, (\beta\gamma)^2, (\alpha\gamma)^3 >$.

After a quick computerized check, we find our group G is isomorphic to this presentation of $L_2(11)$ and to our constructed group H.

2.3 $PGL(2,13)$ over $N = D_{12}$

2.3.1 Double Coset Enumeration of G

We factor the progenitor $2^{*6} : D_{12}$ by the two relations $[xtt^x]^7$ and $[xyt^xt]^3$, where $x = (1,2,3,4,5,6)$ and $y = (1,5)(2,4)$. Letting t be represented by t_6, we compute the two relations:

$$\begin{aligned}
(xtt^x)^3 &= e \\
(xt_6t_1)^3 &= e \\
x^3[t_6t_1]^{x^2}[t_6t_1]^x t_6t_1 &= e \\
x^3 t_2t_3t_1t_2t_6t_1 &= e \\
(1,4)(2,5)(3,6)t_2t_3t_1 &= t_1t_6t_2
\end{aligned}$$

$$\begin{aligned}
(xt)^5 &= e \\
(xt_6)^5 &= e \\
x^5 t_6^{x^4} t_6^{x^3} t_6^{x^2} t_6^{x} t_6 &= e \\
x^5 t_4t_3t_2t_1t_6 &= e \\
(1,6,5,4,3,2)t_4t_3t_2 &= t_6t_1.
\end{aligned}$$

Let G be $2^{*6} : D_{12}/(1,4)(2,5)(3,6)t_2t_3t_1t_2t_6t_1, (1,6,5,4,3,2)t_4t_3t_2t_1t_6$, where $N =< (1,2,3,4,5,6), (1,5)(2,4) >$ and $t \sim t_6$.

We will find the index of N in G by manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

We will complete a double coset enumeration of G over N to find the index of N in G. We must find all distinct double cosets $[w]$, where $[w] = \{Nw^n | n\epsilon N\}$, and the number of single cosets contained in each double coset. Our manual double coset enumeration is completed when all potentially new double cosets have previously been accounted for and when the set of right cosets is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n \ in \ N : Nw^n = Nw\}$, where w is a word of t_i's on $\{0, 1, 2, 3, 4, 5\} = X$.

We begin with the double coset NeN, which we denote $[*]$. This double coset consists of the single coset N. Allowing 6 to be 0, the single orbit of N on X is $\{0, 1, 2, 3, 4, 5\}$. We will choose $t_6 = t_0$ as our symmetric generator from the orbit $\{0, 1, 2, 3, 4, 5\}$ and find Nt_0 belongs to Nt_0N which is a new double coset. We denote Nt_0N by [0].

To find out how many single cosets [0] contains, we find the set of coset stabilizers of [0], denoted $N^{(0)}$. The number of single cosets in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. We have:

$$\begin{aligned} N^{(0)} &\geq < Id(G), (1, 5)(2, 4) > \\ &\geq 2. \end{aligned}$$

The number of single cosets in $Nt_0N = \frac{|N|}{|N^{(0)}|} = \frac{12}{2} = 6$. Our index is the sum of distinct single cosets in each distinct double coset, such as [*] and [0]. As of now, we have $1 + 6 = 7$ single cosets. Note that the orbits of [0] are $\{0\}$, $\{1, 5\}$, $\{2, 4\}$ and $\{3\}$.

We will continue to the next level of potential double cosets by investigating the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$, $\{1, 5\}$, $\{2, 4\}$ and $\{3\}$ and we take t_0, t_1, t_2, and t_3 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1, 5\}$ we find a potentially new double coset Nt_0t_1, which we denote [01]. From the orbit $\{2, 4\}$ we get Nt_0t_2 we find a potentially new double coset Nt_0t_2, which we denote [02]. From the orbit $\{3\}$ we get another potentially new double coset Nt_0t_3, which we will denote [03]. We must now find the number of distinct single cosets in [01], [02] and [03].

Computing $N^{(01)}$ in N, we obtain:

$$\begin{aligned}|N^{(01)}| &\geq |N^{01}| \\ &\geq |< Id(G) >| \\ &\geq 1.\end{aligned}$$

Computing $N^{(02)}$ in N:

$$\begin{aligned}|N^{(02)}| &\geq |N^{02}| \\ &\geq |< Id(G) >| \\ &\geq 1.\end{aligned}$$

Computing $N^{(03)}$ in N:

$$\begin{aligned}|N^{(03)}| &\geq |N^{03}| \\ &\geq |< Id(G), (1,5)(2,4) >| \\ &\geq 2.\end{aligned}$$

So the number of single cosets in $Nt_0t_1N = \frac{|N|}{|N^{(01)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_2N = \frac{|N|}{|N^{(02)}|} = \frac{12}{1} = 12$. And the number of single cosets in $Nt_0t_3N = \frac{|N|}{|N^{(03)}|} = \frac{12}{2} = 6$.

Hence, our index is now $1 + 6 + 12 + 12 + 6 = 37$.

We now explore the potentially new double cosets coming from representatives from the orbits of $N^{(01)}$ on X. We find [01] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{1\}$ advances back to [0]. The other orbit representatives bring the potentially new double cosets [012], [013], [014], [015], and [010]. However, consider the following relation:

$t_0t_1t_5 = (0,5)(1,4)(2,3)[t_0t_1t_3]^{(0,1)(5,2)}$.

Hence in [01], the representative $\{5\}$ advances to [013] and is already being accounted for by the double coset [013]. So the only new double cosets coming from the orbit representatives of $N^{(01)}$ on X are [012], [013], [014], and [010].

The orbits of $N^{(02)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{2\}$ advances back to [0]. The other representatives bring the potentially new double cosets [021], [023], [024], [025], and [020]. Consider the following relations:

$t_0t_2t_3 = (0,3)(1,2)(5,4)[t_0t_2t_1]^{(0,2)(5,3)}$

$t_0t_2t_0 = (0,1,2,3,4,5)[t_0t_1t_4]^{(0,4,2)(1,5,3)}$.

Hence in [02], the representative $\{3\}$ will advance to [021] and the representative $\{0\}$ advances to [014]. However, [023], [024], and [025] are new, distinct double cosets.

Finally, the orbits of $N^{(03)}$ on X are $\{0\}$, $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The representative from the orbit $\{3\}$ advances back to [0]. We take t_0, t_1, and t_2 from the other three orbits of $N^{(03)}$ on X. These three orbit representatives advance to the potentially new double cosets [030], [031], and [032]. Consider the following relation:

$t_0t_3t_2 = (0,2,4)(1,3,5)[t_0t_1t_4]^{(0,3)(1,4)(5,2)}$.

Hence the representatives from $\{2,4\}$ will actually advance to [014]. The other two orbit representatives of $N^{(03)}$ on X will bring the new, distinct double cosets [031] and [030].

The double cosets we must now investigate are [012], [013], [014], [010], [023], [024], [025], [031] and [030].

Consider the relation:

$t_0t_1t_2 = t_1t_0t_5$, which implies $[t_0t_1t_2]^{(1,0)(2,5)(3,4)} = t_1t_0t_5$
$\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{012}$.

Computing $N^{(012)}$ in N, we obtain:

$$\begin{aligned}|N^{(012)}| &\geq |N^{012}| \\ &\geq |<Id(G),(1,0)(2,5)(3,4)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(013)}$ in N, we obtain:

$$\begin{aligned}|N^{(013)}| &\geq |N^{013}| \\ &\geq |<Id(G)>| \\ &\geq 1.\end{aligned}$$

Computing $N^{(014)}$ in N, we obtain:

$$|N^{(014)}| \geq |N^{014}|$$
$$\geq | < Id(G) > |$$
$$\geq 1.$$

Computing $N^{(010)}$ in N, we obtain:

$$|N^{(010)}| \geq |N^{010}|$$
$$\geq < Id(G) >$$
$$\geq 1.$$

The number of single cosets in $Nt_0t_1t_2N = \frac{|N|}{|N^{(012)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_3N = \frac{|N|}{|N^{(013)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_1t_4N = \frac{|N|}{|N^{(014)}|} = \frac{12}{1} = 12$. And the number of single cosets in $Nt_0t_1t_0N = \frac{|N|}{|N^{(010)}|} = \frac{12}{1} = 12$.

Hence our index is increased to $37 + 6 + 12 + 12 + 12 = 79$.

Consider the relation:

$t_0t_2t_5 = t_3t_1t_4$, so $[t_0t_2t_5]^{(1,2)(3,0)(4,5)} = t_3t_1t_4 \Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{025}$.

Computing $N^{(023)}$ in N, we obtain:

$$|N^{(021)}| \geq |N^{021}|$$
$$\geq | < Id(G) > |$$
$$\geq 1.$$

Computing $N^{(024)}$ in N, we obtain:

$$|N^{(023)}| \geq |N^{023}|$$
$$\geq | < Id(G) > |$$
$$\geq 1.$$

Computing $N^{(025)}$ in N, we obtain:

$$\begin{aligned}|N^{(025)}| &\geq |N^{025}| \\ &\geq |< Id(G), (1,2)(3,0)(4,5) >| \\ &\geq 2.\end{aligned}$$

The number of single cosets in $Nt_0t_2t_3N = \frac{|N|}{|N^{(023)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_2t_4N = \frac{|N|}{|N^{(024)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_2t_5N = \frac{|N|}{|N^{(025)}|} = \frac{12}{2} = 6$.

Hence our index is increased to $79 + 12 + 12 + 6 = 109$.

Finally, consider the relations:

$t_0t_3t_1 = t_4t_1t_3$, which implies $[t_0t_3t_1]^{(0,4)(1,3)} = t_4t_1t_3 \Rightarrow [(0,4)(1,3)]\epsilon N^{031}$.

$t_0t_3t_0 = t_3t_0t_3$, so $[t_0t_3t_5]^{(1,4)(2,5)(3,0)} = t_2t_5t_3 \Rightarrow [(1,4)(2,5)(3,0)]\epsilon N^{030}$.

$t_0t_3t_5 = t_2t_5t_3$, so $[t_0t_3t_5]^{(1,2)(3,0)(4,5)} = t_2t_5t_3 \Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{030}$.

Computing $N^{(031)}$ in N, we obtain:

$$\begin{aligned}|N^{(031)}| &\geq |N^{031}| \\ &\geq |< Id(G), (0,4)(1,3) >| \\ &\geq 2.\end{aligned}$$

Computing $N^{(030)}$ in N, we obtain:

$$\begin{aligned}|N^{(030)}| &\geq |N^{030}| \\ &\geq |< Id(G), (1,4)(2,5)(3,0), (1,2)(3,0)(4,5) >| \\ &\geq 4.\end{aligned}$$

The number of single cosets in $Nt_0t_3t_1N = \frac{|N|}{|N^{(031)}|} = \frac{12}{12} = 6$. The number of single cosets in $Nt_0t_3t_0N = \frac{|N|}{|N^{(030)}|} = \frac{12}{4} = 3$.

Hence our index is increased to $109 + 6 + 3 = 118$.

We must now find the new level of double cosets coming from each double coset's orbits respectively. The orbits of $N^{(012)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{2\}$ advances back to [01]. The other representatives bring the potentially new double cosets [0121], [0123], [0124], [0125], and [0120]. Consider the following relations:

$t_0t_1t_2t_4 = (0,4,2)(1,5,3)[t_0t_1t_2t_3]^{(0,1)(5,2)}$

$t_0t_1t_2t_5 = (0,4,2)(1,5,3)[t_0t_1]^{(0,1)(5,2)}$

$t_0t_1t_2t_0 = (0,5,4,3,2,1)[t_0t_1t_2t_1]^{(e)}$.

Hence the representative from the $\{4\}$ advances to [0123], the representative from $\{5\}$ advances to [01], and the representative from $\{0\}$ advances to [0121]. From the orbits of $N^{(013)}$ on X, the only new distinct double cosets are [0121] and [0123].

The orbits of $N^{(013)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{3\}$ advances back to [01]. The other orbit representatives bring the potentially new double cosets [0131], [0132], [0134], [0135], and [0130]. Consider the following relations:

$t_0t_1t_3t_1 = (0,4)(1,3)[t_0t_1t_2t_3]^{(0,3)(1,2)(5,4)}$

$t_0t_1t_3t_2 = (0,3)(1,2)(5,4)[t_0t_1]^{(0,1)(5,2)}$.

Hence the representative from $\{1\}$ advances to [0123] and the representative from $\{2\}$ advances to [014]. So [0134], [0135], and [0130] are the potentially new double coset coming from the orbits of $N^{(013)}$ on X.

The orbits of $N^{(014)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from the orbit $\{4\}$ advances back to [01]. The other orbit representatives bring the potentially new double cosets [0141], [0142], [0143], [0145], and [0140]. Consider the following relations:

$t_0t_1t_4t_1 = (0,1)(5,2)[t_0t_1t_2t_3]^{(0,5)(1,4)(2,3)}$

$t_0t_1t_4t_2 = (0,5,4,3,2,1)[t_0t_2]^{(0,2,4)(1,3,5)}$

$t_0t_1t_4t_3 = (0,2)(5,3)[t_0t_1t_3t_0]^{(e)}$

$t_0t_1t_4t_5 = (0,4,2)(1,5,3)[t_0t_3]^{(0,3)(1,4)(5,2)}$.

Hence the representative from $\{1\}$advances to [0123], the representative from $\{2\}$ advances to [02], the representative from $\{3\}$ advances to [0130], and the representative from $\{5\}$ advances to [03]. There are no potentially new double cosets coming from the orbits of $N^{(014)}$ on X.

The orbits of $N^{(010)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [01]. The other orbit representatives bring the potentially new double cosets [0101], [0102], [0103], [0104], and [0105]. Consider the following relations:

$t_0t_1t_0t_1 = (0,3)(1,2)(5,4)[t_0t_1t_4t_0]^{(0,5,4,3,2,1)}$

$t_0t_1t_0t_2 = (0,3)(1,4)(5,2)[t_0t_1t_3t_0]^{(1,5)(2,4)}$

$t_0t_1t_0t_3 = (0,4)(1,3)[t_0t_1t_2t_1]^{(0,5,4,3,2,1)}$

$t_0t_1t_0t_4 = (0,1,2,3,4,5)[t_0t_2t_3]^{0,1,2,3,4,5)}$.

Hence the representative from {1} advances to [0140], the representative from {2} advances to [0130], the representative from {3} advances to [0121], and the representative from {4} advances to [023]. So [0105] is our only potentially new double coset coming from the orbits of $N^{(010)}$ on X.

The orbits of $N^{(021)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {1} advances back to [02]. The other orbit representatives bring the potentially new double cosets [0212], [0213], [0214], [0215], and [0210]. Consider the following relations:

$t_0t_2t_1t_2 = (0,3)(1,4)(5,2)[t_0t_1t_3t_0]^{(0,4)(1,3)}$

$t_0t_2t_1t_3 = (0,5,4,3,2,1)[t_0t_1t_3t_4]^{(0,3)(1,2)(5,4)}$

$t_0t_2t_1t_4 = (0,3)(1,2)(5,4)[t_0t_1t_2t_3]^{(0,4,2)(1,5,3)}$

$t_0t_2t_1t_5 = (0,1)(5,2)[t_0t_2]^{(0,2)(5,3)}$

$t_0t_2t_1t_0 = (0,1,2,3,4,5)[t_0t_1t_0t_5]^{(0,2,4)(1,3,5)}$.

Hence the representative from {2} advances to [0130], the representative from {3} advances to [0134], the representative from {4} advances to [0123], the representative from {5} advances to [02], and the representative from {0} advances to [0105]. There are no potentially new double cosets coming from the orbits of $N^{(021)}$ on X.

The orbits of $N^{(024)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {4} advances back to [02]. The other orbit representatives bring the potentially new double cosets [0241], [0242], [0243], [0245], and [0240]. Consider the following relations:

$t_0t_2t_4t_1 = (e)[t_0t_1t_2t_1]^{(0,4)(1,3)}$

$t_0t_2t_4t_2 = (0,3)(1,4)(5,2)[t_0t_1t_4t_0]^{(0,1,2,3,4,5)}$

$t_0t_2t_4t_3 = (0,5)(1,4)(2,3)[t_0t_1t_3t_4]^{(0,5)(1,4)(2,3)}$

$t_0t_2t_4t_0 = (0,3)(1,4)(5,2)[t_0t_1t_3t_0]^{(0,5,4,3,2,1)}$.

Hence the representative from {1} advances to [0121], the representative from {2} advances to [0140], the representative from {4} advances to [0134], and the representative from $\{0\}$ advances to [0130]. So [0245] is our only potentially new double coset coming from the orbits of $N^{(024)}$ on X.

The orbits of $N^{(025)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{5\}$ advances back to [02]. The other orbit representatives bring the potentially new double cosets [0251], [0252], [0253], [0254], and [0250]. Consider the following relations:

$t_0t_2t_5t_1 = (0,2,4)(1,3,5)[t_0t_1t_2t_3]^{(e)}$

$t_0t_2t_5t_2 = (0,5,4,3,2,1)[t_0t_1t_2t_3]^{(0,3)(1,2)(5,4)}$

$t_0t_2t_5t_3 = (0,5,4,3,2,1)[t_0t_1t_4t_0]^{(0,5,4,3,2,1)}$

$t_0t_2t_5t_4 = (0,1,2,3,4,5)[t_0t_2]^{(0,3)(1,2)(5,4)}$

$t_0t_2t_5t_0 = (0,2,4)(1,3,5)[t_0t_1t_4t_0]^{(0,4)(1,3)}.$

Hence the representative from $\{1\}$ advances to [0123], the representative from $\{2\}$ advances to [0123], the representative from $\{3\}$ will advances to [0140], the representative from $\{4\}$ advances to [02], and the representative from $\{0\}$ advances to [0140]. There are no potentially new double cosets coming from the orbits of $N^{(025)}$ on X.

The orbits of $N^{(031)}$ on X are $\{0\}$, $\{1\}$,$\{2\}$, $\{3\}$, $\{4\}$, and $\{5\}$. The representative from $\{1\}$ advances back to [03]. The other orbit representatives bring the potentially new double cosets [0312], [0313], [0314], [0315], and [0310]. Consider the following relations:

$t_0t_3t_1t_2 = (e)[t_0t_2t_4t_5]^{(0,2,4)(1,3,5)}$

$t_0t_3t_1t_3 = (0,5,4,3,2,1)[t_0t_3]^{(0,4,2)(1,5,3)}$

$t_0t_3t_1t_4 = (0,5,4,3,2,1)[t_0t_1t_3t_0]^{(0,5,4,3,2,1)}$

$t_0t_3t_1t_5 = (e)[t_0t_1t_0t_5]^{(0,2)(5,3)}$

$t_0t_3t_1t_0 = (e)[t_0t_1t_3t_0]^{(0,5)(1,4))(2,3)}.$

Hence the representative from $\{2\}$ advances to [0245], the representative from $\{3\}$ advances to [03], the representative from $\{4\}$ advances to [0130], the representative from $\{5\}$ advances to [0105], and the representative from $\{0\}$ advances to [0130]. There are no potentially new double cosets coming from the orbits of $N^{(031)}$ on X.

The orbits of $N^{(030)}$ on X are $\{0\}$, $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The representative from $\{0\}$ advances back to [03]. The other orbit representatives bring the potentially new double cosets [0301], [0302], and [0303]. Consider the following relations:

$t_0t_3t_0t_1 = (0,1)(5,2)[t_0t_1t_2t_3]^{(0,2,4)(1,3,5)}$

$t_0t_3t_0t_2 = (0,5,4,3,2,1)[t_0t_1t_2t_3]^{(0,1)(5,2)}$

$t_0t_3t_0t_3 = (1,5)(2,4)[t_0t_3]^{(0,3)(1,4)(5,2)}$.

Hence the representatives from $\{1,5\}$ advance to [0123], the representatives from $\{2,4\}$ advance to [0123], and the representative from $\{3\}$ advances to [03]. There are no potentially new double cosets coming from the orbits of $N^{(030)}$ on X.

We now continue to the next level of double cosets. The only new, distinct double cosets we must investigate are [0121], [0123], [0134], [0135], [0130], [0140], [0105], and [0245].

Consider the relations:

$t_0t_1t_2t_1 = t_1t_0t_5t_0$, which implies $[t_0t_1t_2t_1]^{(1,0)(2,5)(3,4)} = t_1t_0t_5t_0$
$\Rightarrow [(1,0)(2,5)(3,4)]\epsilon N^{0121}$.

$t_0t_1t_3t_4 = t_5t_4t_2t_1$, which implies $[t_0t_1t_3t_4]^{(1,4)(2,3)(5,0)} = t_5t_4t_2t_1$
$\Rightarrow [(1,4)(2,3)(5,0)]$
ϵN^{0134}.

$t_0t_1t_3t_5 = t_0t_5t_3t_1 = t_3t_4t_0t_2 = t_3t_2t_0t_4$, which implies the following three statements:

$[t_0t_1t_3t_5]^{(1,5)(2,4)} = t_0t_5t_3t_1 \Rightarrow [(1,5)(2,4)]\epsilon N^{0135}$

$[t_0t_1t_3t_5]^{(1,4)(2,5)(3,0)} = t_3t_4t_0t_2 \Rightarrow [(1,4)(2,5)(3,0)]\epsilon N^{0135}$

$[t_0t_1t_3t_5]^{(1,2)(3,0)(4,5)} = t_3t_2t_0t_4 \Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{0135}$.

$t_0t_1t_0t_5 = t_0t_5t_0t_1$, which implies $[t_0t_1t_0t_5]^{(1,5)(2,4)} = t_0t_5t_0t_1$
$\Rightarrow [(1,5)(2,4)]\epsilon N^{0105}$.

$t_0t_2t_4t_5 = t_0t_4t_2t_1 = t_3t_1t_5t_4 = t_3t_5t_1t_2$, which implies the following three statements:

$[t_0t_2t_4t_5]^{(1,5)(2,4)} = t_0t_4t_2t_1 \Rightarrow [(1,5)(2,4)]\epsilon N^{0245}$

$[t_0t_2t_4t_5]^{(1,2)(3,0)(4,5)} = t_3t_1t_5t_4 \Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{0245}$

$[t_0t_2t_4t_5]^{(1,4)(2,5)(3,0)} = t_3t_5t_1t_2 \Rightarrow [(1,2)(3,0)(4,5)]\epsilon N^{0245}$.

Computing $N^{(0121)}$ in N, which implies

$$
\begin{aligned}
|N^{(0121)}| &\geq |N^{0121}| \\
&\geq |< Id(G), (1,0)(2,5)(3,4) >| \\
&\geq 2.
\end{aligned}
$$

Computing $N^{(0123)}$ in N, which implies

$$\begin{aligned}|N^{(0123)}| &\geq |N^{0123}| \\ &\geq |< Id(G) >| \\ &\geq 1.\end{aligned}$$

Computing $N^{(0134)}$ in N, which implies

$$\begin{aligned}|N^{(0134)}| &\geq |N^{0134}| \\ &\geq |< Id(G), (1,4)(2,3)(5,0) >| \\ &\geq 2.\end{aligned}$$

Computing $N^{(0135)}$ in N, which implies

$$\begin{aligned}|N^{(0135)}| &\geq |N^{0135}| \\ &\geq |< Id(G), (1,5)(2,4), (1,4)(2,5)(3,0), (1,2)(3,0)(4,5) >| \\ &\geq 4.\end{aligned}$$

Computing $N^{(0130)}$ in N, which implies

$$\begin{aligned}|N^{(0130)}| &\geq |N^{0130}| \\ &\geq |< Id(G) >| \\ &\geq 1.\end{aligned}$$

Computing $N^{(0140)}$ in N, which implies

$$\begin{aligned}|N^{(0140)}| &\geq |N^{0140}| \\ &\geq |< Id(G) >| \\ &\geq 1.\end{aligned}$$

Computing $N^{(0105)}$ in N, which implies

$$\begin{aligned}|N^{(0105)}| &\geq |N^{0105}| \\ &\geq |< Id(G), (1,5)(2,4) >| \\ &\geq 2.\end{aligned}$$

Computing $N^{(0245)}$ in N, which implies

$$\begin{aligned}|N^{(0245)}| &\geq |N^{0245}| \\ &\geq |< Id(G), (1,5)(2,4), (1,2)(3,0)(4,5), (1,4)(2,5)(3,0) >| \\ &\geq 4.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_2t_1N = \frac{|N|}{|N^{(0121)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_2t_3N = \frac{|N|}{|N^{(0123)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_1t_3t_4N = \frac{|N|}{|N^{(0134)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_1t_3t_5N = \frac{|N|}{|N^{(0135)}|} = \frac{12}{4} = 3$. The number of single cosets in $Nt_0t_1t_3t_0N = \frac{|N|}{|N^{(0130)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_1t_4t_0N = \frac{|N|}{|N^{(0140)}|} = \frac{12}{1} = 12$. The number of single cosets in $Nt_0t_1t_0t_5N = \frac{|N|}{|N^{(0105)}|} = \frac{12}{2} = 6$. The number of single cosets in $Nt_0t_2t_4t_5N = \frac{|N|}{|N^{(0245)}|} = \frac{12}{4} = 3$.

Hence our index is increased to $118 + 6 + 12 + 6 + 3 + 12 + 12 + 6 + 3 = 178$.

We must now find the new level of double cosets coming from each double coset's orbits respectively. The orbits of $N^{(0121)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{1\}$ advances back to [012]. The other orbit representatives bring the potentially new double cosets [01212], [01213], [01214], [01215] and [01210]. Consider the following relations:

$t_0t_1t_2t_1t_2 = (0,3)(1,2)(5,4)[t_0t_1t_0]^{(1,5)(2,4)}$

$t_0t_1t_2t_1t_3 = (e)[t_0t_2t_4]^{(0,4)(1,3)}$

$t_0t_1t_2t_1t_4 = (0,5,4,3,2,1)[t_0t_2t_4]^{(0,3)(1,4)(5,2)}$

$t_0t_1t_2t_1t_5 = (1,5)(2,4)[t_0t_1t_0]^{(0,1,2,3,4,5)}$

$t_0t_1t_2t_1t_0 = (0,1,2,3,4,5)[t_0t_1t_2]^{(e)}$.

Hence the representatives from $\{2\}$ advances to [010], the representative from $\{3\}$ advances to [024], the representative from $\{4\}$ advances to [024], the representative from {5} advances to [010], and the representative from {0} advances to [012]. There are no new double cosets coming from the orbits of $N^{(0121)}$ on X.

The orbits of $N^{(0123)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {3} advances back to [012]. The other orbit representatives bring the potentially new double cosets [01231], [01232], [01234], [01235] and [01230]. Consider the following relations:

$$t_0t_1t_2t_3t_1 = (0,5,4,3,2,1)[t_0t_2t_5]^{(0,3)(1,2)(5,4)}$$
$$t_0t_1t_2t_3t_2 = (0,2)(5,3)[t_0t_1t_3]^{(0,3)(1,2)(5,4)}$$
$$t_0t_1t_2t_3t_4 = (0,3)(1,2)(5,4)[t_0t_1t_4]^{(0,5)(1,4)(2,3)}$$
$$t_0t_1t_2t_3t_5 = (0,5,4,3,2,1)[t_0t_3t_0]^{(0,1,2,3,4,5)}$$
$$t_0t_1t_2t_3t_0 = (0,1)(5,2)[t_0t_2t_1]^{(0,2,4)(1,3,5)}.$$

Hence the representatives from {1} advances to [025], the representative from {2} advances to [013], the representative from {4} advances to [014], the representative from {5} advances to [030], and the representative from {0} advances to [021]. There are no new double cosets coming from the orbits of $N^{(0123)}$ on X.

The orbits of $N^{(0134)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {4} advances back to [013]. The other orbit representatives bring the potentially new double cosets [01341], [01342], [01343], [01345] and [01340]. Consider the following relations:

$$t_0t_1t_3t_4t_1 = (0,2,4)(1,3,5)[t_0t_1t_3]^{(0,5)(1,4)(2,3)}$$
$$t_0t_1t_3t_4t_2 = (0,5)(1,4)(2,3)[t_0t_2t_4]^{(0,5)(1,4)(2,3)}$$
$$t_0t_1t_3t_4t_3 = (0,3)(1,2)(5,4)[t_0t_2t_4]^{(e)}$$
$$t_0t_1t_3t_4t_5 = (0,3)(1,4)(5,2)[t_0t_2t_1]^{(0,2,4)(1,3,5)}$$
$$t_0t_1t_3t_4t_0 = (0,5,4,3,2,1)[t_0t_2t_1]^{(0,3)(1,2)(5,4)}.$$

Hence the representatives from {1} advances to [013], the representative from {2} advances to [024], the representative from {3} advances to [024], the representative from {5} advances to [021], and the representative from {0} advances to [021]. There are no new double cosets coming from the orbits of $N^{(0134)}$ on X.

The orbits of $N^{(0135)}$ on X are {0}, {1}, {2}, {3}, {4} and {5}. The representative from {5} advances back to [013]. The other orbit representatives bring the potentially new double cosets [01351], [01352], [01353], [01354] and [01350]. Consider the following relations:

$t_0t_1t_3t_5t_1 = (0,3)(1,2)(5,4)[t_0t_1t_3]^{(1,5)(2,4)}$
$t_0t_1t_3t_5t_2 = (0,4)(1,3)(2,3)[t_0t_1t_3]^{(0,3)(1,4)(5,2)}$
$t_0t_1t_3t_5t_4 = (0,5,4,3,2,1)[t_0t_1t_3]^{(0,3)(1,2)(5,4)}$.

Hence the representatives from $\{1\}$ advances to [013], the representative from $\{2\}$ advances to [013], and the representative from $\{3\}$ also advances to [013]. The other two orbit representatives of $N^{(0135)}$ on X will bring the potentially new, distinct double cosets [01353] and [01350].

Now consider the relation: $t_0t_1t_3t_5t_0 = (0,1,2,3,4,5)[t_0t_1t_3t_5t_3]^{(e)}$. Hence [01353] and [01350] are the same double coset. We will denote this new, distinct double coset as [01353].

The orbits of $N^{(0130)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [013]. The other orbit representatives bring the potentially new double cosets [01301], [01302], [01303], [01304] and [01305]. Consider the following relations:

$t_0t_1t_3t_0t_1 = (0,3)(1,4)(5,2)[t_0t_2t_4]^{(0,1,2,3,4,5)}$
$t_0t_1t_3t_0t_2 = (0,3)(1,4)(5,2)[t_0t_2t_1]^{(0,4)(1,3)}$
$t_0t_1t_3t_0t_3 = (0,2)(5,3)[t_0t_1t_4]^{(e)}$
$t_0t_1t_3t_0t_4 = (0,3)(1,4)(5,2)[t_0t_1t_0]^{(1,5)(2,4)}$
$t_0t_1t_3t_0t_5 = (e)[t_0t_3t_1]^{(0,5)(1,4)(2,3)}$.

Hence the representatives from $\{1\}$ advances to [024], the representative from $\{2\}$ advances to [021], the representative from $\{3\}$ advances to [014], the representative from $\{4\}$ advances to [010], and the representative from $\{5\}$ advances to [031]. There are no new double cosets coming from the orbits of $N^{(0130)}$ on X.

The orbits of $N^{(0140)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [014]. The other orbit representatives bring the potentially new double cosets [01401], [01402], [01403], [01404] and [01405]. Consider the following relations:

$t_0t_1t_4t_0t_1 = (0,3)(1,4)(5,2)[t_0t_2t_4]^{(0,5,4,3,2,1)}$
$t_0t_1t_4t_0t_2 = (0,5)(1,4)(2,3)[t_0t_1t_0]^{(0,1,2,3,4,5)}$
$t_0t_1t_4t_0t_3 = (0,4,2)(1,5,3)[t_0t_1t_0]^{(1,5)(2,4)}$
$t_0t_1t_4t_0t_4 = (0,1,2,3,4,5)[t_0t_2t_5]^{(0,1,2,3,4,5)}$
$t_0t_1t_4t_0t_5 = (0,1,2,3,4,5)[t_0t_1t_3t_5t_3]^{(0,3)(1,4)(5,2)}$.

Hence the representatives from $\{1\}$ advances to [024], the representative from $\{2\}$ advances to [010], the representative from $\{3\}$ advances to [010], the representative from $\{4\}$ advances to [025], and the representative from $\{5\}$ advances to [01353]. There are no new double cosets coming from the orbits of $N^{(0140)}$ on X.

The orbits of $N^{(0105)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [010]. The other orbit representatives bring the potentially new double cosets [01051], [01052], [01053], [01054] and [01050]. Consider the following relations:

$t_0t_1t_0t_5t_1 = (0,1,2,3,4,5)[t_0t_1t_0]^{(1,5)(2,4)}$

$t_0t_1t_0t_5t_2 = (0,2,4)(1,3,5)[t_0t_2t_1]^{(0,2)(5,3)}$

$t_0t_1t_0t_5t_3 = (e)[t_0t_3t_1]^{(0,2)(5,3)}$

$t_0t_1t_0t_5t_4 = (0,5,4,3,2,1)[t_0t_2t_1]^{(0,4,2)(1,5,3)}.$

Hence the representatives from $\{1\}$ advances to [010], the representative from $\{2\}$ advances to [021], the representative from $\{3\}$ advances to [021], and the representative from $\{4\}$ advances to [021]. The other orbit representative of $N^{(0105)}$ on X will bring the potentially new, distinct double coset [01050].

The orbits of $N^{(0245)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [024]. The other orbit representatives bring the potentially new double cosets [02451], [02452], [02453], [02454] and [02450]. Consider the following relations:

$t_0t_2t_4t_5t_1 = (0,5,4,3,2,1)[t_0t_2t_4]^{(1,5)(2,4)}$

$t_0t_2t_4t_5t_2 = (0,1)(5,2)[t_0t_2t_4]^{(0,3)(1,4)(5,2)}$

$t_0t_2t_4t_5t_3 = (1,5)(2,4)[t_0t_3t_1]^{(0,5)(1,4)(2,3)}$

$t_0t_2t_4t_5t_4 = (1,5)(2,4)[t_0t_2t_4]^{(0,3)(1,2)(5,4)}$

$t_0t_2t_4t_5t_0 = (e)[t_0t_3t_1]^{(0,4,2)(1,5,3)}.$

Hence the representatives from $\{1\}$ advances to [024], the representative from $\{2\}$ advances to [024], the representative from $\{3\}$ advances to [031], the representative from $\{4\}$ advances to [024], and the representative from $\{0\}$ advances to [031]. There are no new double cosets coming from the orbits of $N^{(0245)}$ on X.

We now continue to the next level of double cosets. The only new, distinct double cosets we must investigate are [01353] and [01050].

Consider the relations:

$t_0t_1t_3t_5t_3 = t_3t_4t_0t_2t_0 = t_0t_5t_3t_1t_3 = t_3t_2t_0t_4t_0$, which implies the following three statements:

$[t_0t_1t_3t_5t_3]^{(0,3)(1,4)(5,2)} = t_3t_4t_0t_2t_0 \Rightarrow [(0,3)(1,4)(5,2)]\epsilon N^{01353}$

$[t_0t_1t_3t_5t_3]^{(1,5)(2,4)} = t_0t_5t_3t_1t_3 \Rightarrow [(1,5)(2,4)]\epsilon N^{01353}$

$[t_0t_1t_3t_5t_3]^{(0,3)(1,2)(5,4)} = t_3t_2t_0t_4t_0 \Rightarrow [(0,3)(1,2)(5,4)]\epsilon N^{01353}$.

$t_0t_1t_0t_5t_0 = t_2t_1t_2t_3t_2 = t_4t_3t_4t_5t_4 = t_2t_3t_2t_1t_2 = t_3t_4t_3t_2t_3 = t_1t_0t_1t_2t_1 = t_5t_4t_5t_0t_5 = t_4t_5t_4t_3t_4 = t_1t_2t_1t_0t_1 = t_3t_2t_3t_4t_3 = t_0t_5t_0t_1t_0 = t_5t_0t_5t_4t_5$, which implies the following eleven statements:

$[t_0t_1t_0t_5t_0]^{(0,2)(5,3)} = t_2t_1t_2t_3t_2 \Rightarrow [(0,2)(5,3)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,4)(1,3)} = t_4t_3t_4t_5t_4 \Rightarrow [(0,4)(1,3)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,2,4)(1,3,5)} = t_2t_3t_2t_1t_2 \Rightarrow [(0,2,4)(1,3,5)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,3)(1,4)(5,2)} = t_3t_4t_3t_2t_3 \Rightarrow [(0,3)(1,4)(5,2)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,1)(5,2)} = t_1t_0t_1t_2t_1 \Rightarrow [(0,1)(5,2)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,5)(1,4)(2,3)} = t_5t_4t_5t_0t_5 \Rightarrow [(0,5)(1,4)(2,3)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,4,2)(1,5,3)} = t_4t_5t_4t_3t_4 \Rightarrow [(0,4,2)(1,5,3)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,1,2,3,4,5)} = t_1t_2t_1t_0t_1 \Rightarrow [(0,1,2,3,4,5)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(0,3)(1,2)(5,4)} = t_3t_2t_3t_4t_3 \Rightarrow [(0,3)(1,2)(5,4)]\epsilon N^{01050}$

$[t_0t_1t_0t_5t_0]^{(1,5)(2,4)} = t_0t_5t_0t_1t_0 \Rightarrow [(1,5)(2,4)]\epsilon N^{01050}$

and $[t_0t_1t_0t_5t_0]^{(0,5,4,3,2,1)} = t_5t_0t_5t_4t_5 \Rightarrow [(0,5,4,3,2,1)]\epsilon N^{01050}$.

We note that every element of D_{12} creates an equal face of [01050].

Computing $N^{(01353)}$ in N, we obtain:

$$
\begin{aligned}
|N^{(01353)}| &\geq |N^{01353}| \\
&\geq |< Id(G), (0,3)(1,4)(5,2), (1,5)(2,4), (0,3)(1,2)(5,4) >| \\
&\geq 4.
\end{aligned}
$$

Computing $N^{(01050)}$ in N, we obtain:

$$
\begin{aligned}
|N^{(01050)}| &\geq |N^{01050}| \\
&\geq |D_{12}| \\
&\geq 12.
\end{aligned}
$$

The number of single cosets in $Nt_0t_1t_3t_5t_3N = \frac{|N|}{|N^{(01353)}|} = \frac{12}{4} = 3$. The number of single cosets in $Nt_0t_1t_0t_5t_0N = \frac{|N|}{|N^{(01050)}|} = \frac{12}{12} = 1$.

Hence our index is increased to $178 + 3 + 1 = 182$

We must now find the new level of double cosets coming from each double coset's orbits respectively. The orbits of $N^{(01353)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{3\}$ advances back to [0135]. The other orbit representatives bring the potentially new double cosets [013531], [013532], [013534], [013535] and [013530]. Consider the following relations:

$t_0t_1t_3t_5t_3t_1 = (0,5,4,3,2,1)[t_0t_1t_4t_0]^{(1,5)(2,4)}$

$t_0t_1t_3t_5t_3t_2 = (0,5,4,3,2,1)[t_0t_1t_4t_0]^{(0,3)(1,4)(5,2)}$

$t_0t_1t_3t_5t_3t_4 = (0,4)(1,3)[t_0t_1t_4t_0]^{(0,3)(1,2)(5,4)}$

$t_0t_1t_3t_5t_3t_5 = (0,4)(1,3)[t_0t_1t_4t_0]^{(e)}$

$t_0t_1t_3t_5t_3t_0 = (0,5,4,3,2,1)[t_0t_1t_3t_5]^{(e)}$.

Hence the representatives from $\{1\}$ advances to [0140], the representative from $\{2\}$ advances to [0140], the representative from $\{4\}$ advances to [0140], the representative from $\{5\}$ advances to [0140], and the representative from $\{0\}$ advances to [0135]. There are no new double cosets coming from the orbits of $N^{(01353)}$ on X.

The orbits of $N^{(01050)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. The representative from $\{0\}$ advances back to [0105]. The other orbit representatives bring the potentially new double cosets [010501], [010502], [010503], [010504] and [010505]. Consider the following relations:

$t_0t_1t_0t_5t_0t_1 = (e)[t_0t_1t_0t_5]^{(0,1,2,3,4,5)}$

$t_0t_1t_0t_5t_0t_2 = (e)[t_0t_1t_0t_5]^{(0,2,4)(1,3,5)}$

$t_0t_1t_0t_5t_0t_3 = (e)[t_0t_1t_0t_5]^{(0,3)(1,4)(5,2)}$

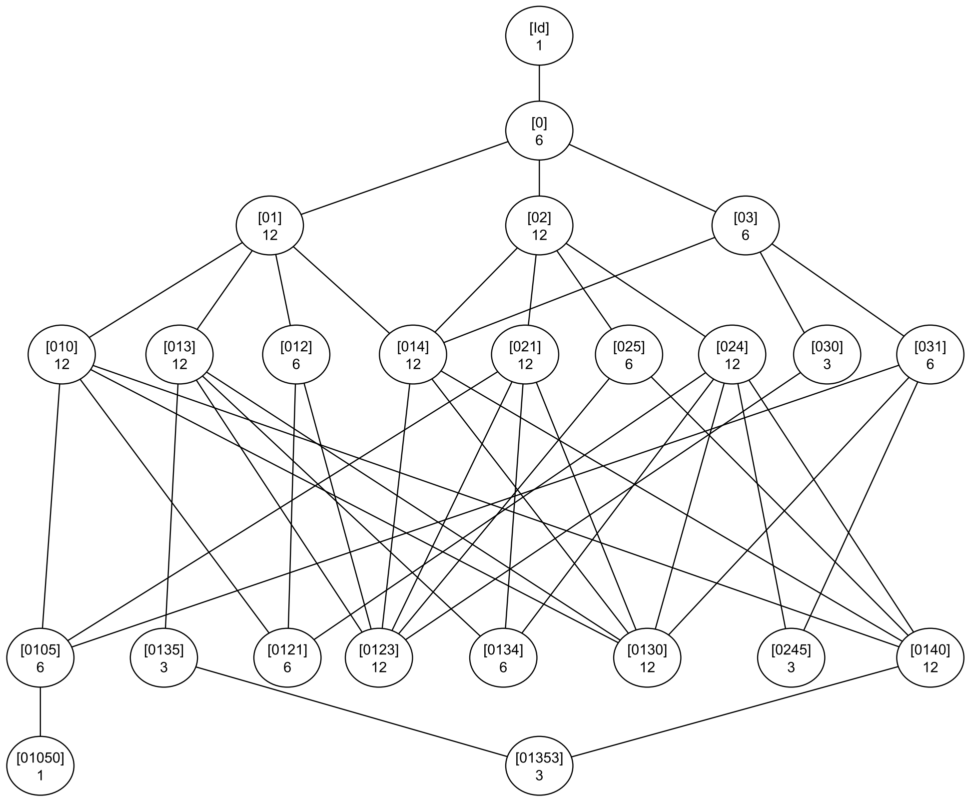
$t_0t_1t_0t_5t_0t_4 = (e)[t_0t_1t_0t_5]^{(0,4,2)(1,5,3)}$

$t_0t_1t_0t_5t_0t_5 = (e)[t_0t_1t_0t_5]^{(0,5,4,3,2,1)}$.

Hence the representatives from $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, and $\{5\}$ all advance back to [0105]. There are no new double cosets coming from the orbits of $N^{(01050)}$ on X.

Because there are no new words, we have completed our double coset enumeration of G over N. Our group is closed under right multiplication of t_i's. The index of N in G is 182. The Cayley graph for G is given below.

Figure 2.3: $PGL(2,13)$ Cayley Graph



2.3.2 Proof of $G \cong PGL(2,13)$

We will now prove that the group of order 2184 is $PGL(2,13)$ utilizing linear fractional mappings.

Let us first define our mapping given by α, β, γ, and δ. We are not able to use the presentation formula as we used earlier since $13 \not\equiv 3 \pmod 4$. We will begin by denoting our permutations on 14 letters by $0, 1, 2, \ldots, 12, \infty$. We find the following:

$\alpha : x \mapsto x{+}1 = (0,1,2,3,4,5,6,7,8,9,10,11,12)(\infty) = (0,1,2,3,4,5,6,7,8,9,10,11,12).$

In $\pmod{13}$, we calculate the nonzero, finite squares of $\{0,1,2,\ldots,12,\infty\}$:

$$\begin{aligned}\{1^2,2^2,3^2,4^2,5^2,6^2,7^2,8^2,9^2,10^2,11^2,12^2,13^2\} &= \{1,4,9,3,12,10,10,12,3,9,4,1\}\\ &= \{1,3,4,9,10,12\}.\end{aligned}$$

We find the following:

$4^1 \equiv 4 \pmod{13}$

$4^2 \equiv 16 \pmod{13} \equiv 3 \pmod{13}$
$4^3 \equiv 64 \pmod{13} \equiv 12 \pmod{13}$
$4^4 \equiv 256 \pmod{13} \equiv 9 \pmod{13}$
$4^5 \equiv 1024 \pmod{13} \equiv 10 \pmod{13}$
$4^6 \equiv 4096 \pmod{13} \equiv 1 \pmod{13}$.

Therefore the powers of 4 generate the set $\{1, 3, 4, 9, 10, 12\}$ in (mod 13). So $k = 4$ and we obtain:

$\beta : x \mapsto 4x = (0)(\infty)(1, 4, 3, 12, 9, 10)(2, 8, 6, 11, 5, 7) = (1, 4, 3, 12, 9, 10)(2, 8, 6, 11, 5, 7)$.

As before, to find $\gamma : x \mapsto -x^{-1}$, we will first find x^{-1}, or the multiplicative inverse of each letter, and then multiply that value by -1 and determine its representative value modulo 13.

For instance, the multiplicative inverse of 1 is itself since $1 \times 1 = 1 \equiv 1 \pmod{13}$, therefore $1 \mapsto -(1) \equiv 12 \pmod{13}$. The multiplicative inverse of 2 is 7, since $2 \times 7 = 14 \equiv 1 \pmod{13}$. We will again define the multiplicative inverse of 0 as ∞ and claim that $-\infty$ corresponds to the letter ∞. Let us first find solve $x = 1$. $1 \mapsto -(1)^{-1} = -1 \equiv 12 \pmod{13}$. So we find that 1 should advance to 12 in permutation γ. Continuing this pattern with the other 13 letters, we obtain the following permutation for γ:

$\gamma : x \mapsto -x^{-1} = (0, \infty)(1, 12)(2, 6)(3, 4)(5)(7, 11)(8)(9, 10)$.

We must also now calculate δ. We must find a mapping of the form: $\dfrac{ax + b}{cx + d}$, where $ad - bc$ is a nonzero, nonsquare in (mod 13).

We will let our mapping be $\dfrac{0x + 7}{1x + 0}$, since $ad - bc = -7 \equiv 6 \pmod{13}$, where 6 is not a square in (mod 13). So we find $\delta \mapsto \dfrac{0x + 7}{1x + 0} = \dfrac{7}{x} = 7x^{-1}$. Following the same method of finding permutations as we did in γ, we will first calculate x^{-1}, then multiply by 7 and determine the value in (mod 13).

For $x = 1$, $1 \mapsto 7x^{-1} = 7(1) \equiv 7 \pmod{13}$. So we find that 1 should advance to 7 in permutation δ. Continuing this pattern with the other 13 letters, we obtain the following permutation for δ:

$\delta : x \mapsto 7x^{-1} = (0, \infty)(1, 7)(2, 10)(3, 11)(4, 5)(6, 12)(8, 9)$.

We let $H =< \alpha, \beta, \gamma, \delta >= PGL(2, 13)$.

After a quick computerized check, we find that our group G is isomorphic to our constructed group H.

Chapter 3

Double Coset Enumeration of Sporadic Groups

3.1 M_{11} over $N = 2^{\bullet}S_4$

3.1.1 Double Coset Enumeration of G

We factor the progenitor $2^{*8} : (2^{\bullet}S_4)$, with $t \sim t_8$, by the two relations $[zt]^3$ and $[w^{-1}vt]^5$ where $v = (1,2)(3,6)(4,5)$, $w = (3,6,8)(4,7,5)$, $x = (1,3,2,5)(4,8,6,7)$, $y = (1,4,2,6)(3,7,5,8)$, and $z = (1,2)(3,5)(4,6)(7,8)$.

We compute the two relations:

$$(zt)^3 = e$$
$$(zt_8)^3 = e$$
$$z^3(t_8)^{z^2}(t_8)^z t_8 = e$$
$$[(1,2)(3,5)(4,6)(7,8)]^3 t_8 t_7 t_8 = e$$
$$(1,2)(3,5)(4,6)(7,8)t_8 = t_8 t_7$$

$$(w^{-1}vt)^5 = e$$
$$(w^{-1}vt_8)^5 = e$$
$$[w^{-1}v]^5 t_8^{[w^{-1}v]^4} t_8^{[w^{-1}v]^3} t_8^{[w^{-1}v]^2} t_8^{[w^{-1}v]} t_8 = e$$
$$[(1,2)(3,8)(5,7)]^5 t_8 t_3 t_8 t_3 t_8 = e$$
$$(1,2)(3,8)(5,7) t_8 t_3 t_8 = t_8 t_3.$$

We let G be $2^{*8} : (2^{\bullet}S_4)/(1,2)(3,5)(4,6)(7,8)t_8t_7t_8$, $(1,2)(3,8)(5,7)t_8t_3t_8t_3t_8$, where $N =< (1,2)(3,6)(4,5), (3,6,8)(4,7,5)$, $(1,3,2,5)(4,8,6,7), (1,4,2,6)(3,7,5,8), (1,2)(3,5)(4,6)(7,8) >$ and $t \sim t_8$.

We will find the index of N in G by manual double coset enumeration of G over N. We take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

We will complete a double coset enumeration of G over N to find the index of N in G. We must find all distinct double cosets $[w]$, where $[w] = \{Nw^n | n\epsilon N\}$, and how many single cosets are contained in each double coset. The manual double coset enumeration is finished when all potentially new double cosets have already been accounted for and when the set of right cosets we find is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n \; in \; N : Nw^n = Nw\}$, where w is a word of t_i's on $X = \{0,1,2,3,4,5,6,7\}$.

We begin with the double coset NeN, which we denote $[*]$. This double coset consists of the single coset N. For convenience, we will let 8 be 0. The single orbit of N on X is $\{0,1,2,3,4,5,6,7\}$. We choose $t_8 = t_0$ as our symmetric generator from the orbit $\{0,1,2,3,4,5,6,7\}$ and find Nt_0 belongs to Nt_0N which is a new double coset. We denote Nt_0N by $[0]$.

To find the number of single cosets contained in $[0]$ we must find the set of coset stabilizers of 0, denoted $N^{(0)}$. This is relevant to us because the number of single cosets in [0] is equal to $\frac{|N|}{|N^{(0)}|}$. We have:

$$\begin{aligned} N^{(0)} &\geq < (1,2)(3,6)(4,5),(1,4)(2,6)(3,5) > \\ &\geq 6. \end{aligned}$$

So the number of single cosets in $Nt_0N = \frac{|N|}{|N^{(0)}|} = \frac{48}{6} = 8$. When we conjugate t_0 by the transversals of [0], we find 6 single cosets are distinct. The index of N is the sum of distinct single cosets in the distinct double cosets, such as [*] and [0]. As of now, we have $1 + 8 = 9$. The orbits of [0] are $\{0\}$ $\{1,2,3,4,5,6\}$, and $\{7\}$.

We continue to the next level of potential double cosets by working with the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$ $\{1,2,3,4,5,6\}$, and $\{7\}$ and we take t_0, t_1, and t_7 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1,2,3,4,5,6\}$ we find a potentially new double coset Nt_0t_1, which we will denote [01]. From the orbit $\{7\}$ we get Nt_0t_7 which belongs to [07].

Consider the double coset [07]. We have the relation: $(1,2)(3,5)(4,6)(7,8)t_8 = t_8t_7$. This implies that any representative from the orbit $\{7\}$ will actually loop back to [0].

We will now determine how many distinct single cosets are contained in [01].

Computing $N^{(01)}$ in N, we obtain:

$$\begin{aligned} |N^{(01)}| &\geq |N^{01}| \\ &\geq | < Id(G) > | \\ &\geq 1. \end{aligned}$$

The number of single cosets in $Nt_0t_1N = \frac{|N|}{|N^{(01)}|} = \frac{48}{1} = 48$.

Hence, our index is increased to $9 + 48 = 57$.

We now explore the potentially new double cosets coming from representatives from the orbits of $N^{(01)}$ on X. We find [01] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ and $\{7\}$. The representative from the orbit $\{1\}$ will advance to [0]. The other orbit representatives will bring the potentially new double cosets [010], [012], [013], [014], [015], [016] and [017]. However, consider the following relations:

$t_0t_1t_0 = (1,8)(2,7)(3,5)[t_0t_2]^e$
$t_0t_1t_2 = (1,2)(3,5)(4,6)(7,8)[t_0t_1]^{(1,2)(3,4)(5,6)(8,7)}$
$t_0t_1t_4 = (1,8,4)(2,7,6)[t_0t_2]^{(1,8)(2,7)(3,5)}$.

Hence in [01], the single representative {0} goes to [01], the single representative {2} goes to [01], and the single representative {4} goes to [01]. So the only new, distinct double cosets are [013], [015], [016], and [017].

Since there are no more possible three letter words, we must now investigate the double cosets: [013], [015], [016], and [017].

Consider the following relations:
$t_0t_1t_3 = t_5t_1t_7$, so $[t_5t_1t_7]^{(3,7)(4,6)(5,0)} = t_0t_1t_3 \Rightarrow [(3,7)(4,6)(5,0)]\epsilon N^{013}$.
$t_0t_1t_6 = t_2t_7t_4$, so $[t_2t_7t_4]^{(1,7)(2,8)(4,6)} = t_0t_1t_6 \Rightarrow [(1,7)(2,8)(4,6))]\epsilon N^{016}$.
$t_0t_1t_7 = t_1t_8t_2$, so $[t_1t_8t_2]^{(1,8)(2,7)(3,5)} = t_0t_1t_7 \Rightarrow [(1,8)(2,7)(3,5)]\epsilon N^{017}$.
$t_0t_1t_7 = t_2t_7t_1$, so $[t_2t_7t_1]^{(1,7)(2,8)(4,6)} = t_0t_1t_7 \Rightarrow [(1,7)(2,8)(4,6)]\epsilon N^{017}$.
Computing $N^{(013)}$ in N, we obtain:

$$\begin{aligned}|N^{(013)}| &\geq |N^{013}| \\ &\geq |<Id(G),(3,7)(4,6)(5,0)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(016)}$ in N, we obtain:

$$\begin{aligned}|N^{(016)}| &\geq |N^{016}| \\ &\geq |<Id(G),(1,7)(2,8)(4,6)>| \\ &\geq 2.\end{aligned}$$

Computing $N^{(017)}$ in N, we obtain:

$$\begin{aligned}|N^{(017)}| &\geq |N^{017}| \\ &\geq |<Id(G),(1,8)(2,7)(3,5),(1,7)(2,8)(4,6)>| \\ &\geq 4.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_3N = \frac{|N|}{|N^{(013)}|} = \frac{48}{2} = 24$. The number of single cosets in $Nt_0t_1t_6N = \frac{|N|}{|N^{(015)}|} = \frac{48}{2} = 24$. The number of single cosets in $Nt_0t_1t_7N = \frac{|N|}{|N^{(010)}|} = \frac{48}{4} = 12$.

Hence our index is increased to $57 + 24 + 24 + 24 + 12 = 141$.

We now explore any potentially new double cosets coming from representatives from the orbits of $N^{(013)}$ on X, $N^{(016)}$ on X, $N^{(017)}$ on X.

The orbits of $N^{(013)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ and $\{7\}$. The representative from the orbit $\{3\}$ advances to [01]. The other representatives will be the potentially new double cosets [0130], [0131], [0132], [0134], [0135], [0136], [0137]. However, consider the following relations:

$t_0t_1t_3t_0 = (1,7,3)(2,8,5)[t_0t_1]^e$

$t_0t_1t_3t_1 = (1,2)(3,8)(5,7)[t_0t_1t_3]^{(3,7)(4,6)(5,8)}$

$t_0t_1t_3t_2 = (1,8)(2,7)(3,5)[t_0t_1t_7]^{(1,5,6)(2,3,4)}$

$t_0t_1t_3t_5 = (1,2)(3,5)(4,6)(7,8)[t_0t_1t_5]^{(1,2)(3,5)(4,6)(7,8)}$

$t_0t_1t_3t_6 = (1,8,3,2,7,5)(4,6)[t_0t_1t_3t_4]^{(3,7)(4,6)(5,8)}$

$t_0t_1t_3t_7 = (1,8,3,2,7,5)(4,6)[t_0t_1]^{(3,7)(4,6)(5,8)}$.

Hence the representative from the $\{0\}$ will advance to [01], the representative from $\{1\}$ will advance to [013], the representative from $\{2\}$ will advance to [017], the representative from $\{5\}$ will advance to [015], the representative from $\{6\}$ will advance to [0134], and the representative from $\{7\}$ will advance to [01]. So the only new, distinct double coset coming from the orbits of $N^{(013)}$ on X is [0134].

The orbits of $N^{(015)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ and $\{7\}$. The representative from the orbit $\{5\}$ will advance to [01]. The other representatives will bring the potentially new double cosets [0150], [0151], [0152], [0153], [0154], [0156], [0157]. Consider the following relations:

$t_0t_1t_5t_0 = (1,3,7)(2,5,8)[t_0t_1t_3]^e$

$t_0t_1t_5t_1 = (3,7)(4,6)(5,8)[t_0t_1t_6]^{(1,8,2,7)(3,4,5,6)}$

$t_0t_1t_5t_2 = (1,8)(2,7)(3,5)[t_0t_1t_3]^{(1,2)(3,5)(4,6)(7,8)}$

$t_0t_1t_5t_3 = (1,2)(3,5)(4,6)(7,8)[t_0t_1t_3]^{(1,2)(3,5)(4,6)(7,8)}$

$t_0t_1t_5t_4 = (1,5,8,6,2,3,7,4)[t_0t_1t_5]^{(1,3,4,7,2,5,6,8)}$

$t_0t_1t_5t_6 = (1,4,8)(2,6,7)[t_0t_1t_6]^{(1,7,5,6,2,8,3,4)}$

$t_0t_1t_5t_7 = e[t_0t_1]^{(1,2)(3,8)(5,7)}$.

Hence the representative from the $\{0\}$ will advance to [013], the representative from {1} will advance to [016], the representative from {2} will advance to [013], the representative from {3} will advance to [013], the representative from {4} will advance to [015], the representative from {6} will advance to [016], and the representative from {7} will advance to [01]. So no new double cosets come from the orbits of $N^{(015)}$ on X.

The orbits of $N^{(016)}$ on X are {0}, {1}, {2}, {3}, {4}, {5}, {6} and {7}. The representative from {6} will advance to [01]. The other representatives will bring the potentially new double cosets [0160], [0161], [0162], [0163], [0164], [0165], [0167]. Consider the following relations:

$t_0t_1t_6t_0 = (1,4,8)(2,6,7)[t_0t_1t_6]^{(1,7,4,2,8,6)}$

$t_0t_1t_6t_1 = (1,6)(2,4)(7,8)[t_0t_1t_5]^{(1,2)(3,7,6,5,8,4)}$

$t_0t_1t_6t_2 = (1,8,4)(2,7,6)[t_0t_1t_6]^{(1,6,8,2,4,7)(3,5))}$

$t_0t_1t_6t_3 = (1,6,7,5,2,4,8,3)[t_0t_1t_3t_4]^{(3,5)(4,8)(6,7)}$

$t_0t_1t_6t_4 = (1,7,4,2,8,6)(3,5)[t_0t_1]^{(1,7)(2,8)(4,6)}$

$t_0t_1t_6t_5 = (1,5,6)(2,3,4)[t_0t_1t_5]^{(1,6,5)(2,4,3)}$

$t_0t_1t_6t_7 = (1,4)(2,6)(3,5)[t_0t_1t_5]^{(1,7,2,8)(3,6,5,4)}$.

Hence the representative from the {0} will advance to [016], the representative from {1} will advance to [015], the representative from {2} will advance to [016], the representative from {3} will advance to [0134], the representative from {4} will advance to [01], the representative from {5} will advance to [015], and the representative from {7} will advance to [015]. So no new double cosets come from the orbits of $N^{(016)}$ on X.

The orbits of $N^{(017)}$ on X are {0}, {1}, {2}, {3}, {4}, {5}, {6} and {7}. The representative from the orbit {7} will advance to [01]. The other representatives will bring the potentially new double cosets [0170], [0171], [0172], [0173], [0174], [0175], [0176]. Consider the following relations:

$t_0t_1t_7t_0 = (1,7)(2,8)(4,6)[t_0t_1]^{(1,2)(3,5)(4,6)(7,8)}$

$t_0t_1t_7t_1 = (1,8)(2,7)(3,5)[t_0t_1]^{(1,7)(2,8)(4,6)}$

$t_0t_1t_7t_2 = (1,2)(3,5)(4,6)(7,8)[t_0t_1]^{(1,8)(2,7)(3,5)}$

$t_0t_1t_7t_3 = (1,3,7)(2,5,8)[t_0t_1t_7]^{(1,7)(2,8)(4,6)}$

$t_0t_1t_7t_4 = (1,2)(4,7)(6,8)[t_0t_1t_3]^{(1,6,5)(2,4,3)}$

$t_0t_1t_7t_5 = (1,8,3,2,7,5)(4,6)[t_0t_1t_7]^{(1,2)(3,5)(4,6)(7,8)}$

$t_0t_1t_7t_6 = (1,7,4,2,8,6)(3,5)[t_0t_1t_3]^{(1,4,3,8,2,6,5,7)}$.

Hence the representative from the $\{0\}$ will advance to [01], the representative from $\{1\}$ will advance to [01], the representative from $\{2\}$ will advance to [01], the representative from $\{3\}$ will advance to [017], the representative from $\{4\}$ will advance to [013], the representative from $\{5\}$ will advance to [017], and the representative from $\{6\}$ will advance to [013]. So no new double cosets come from the orbits of $N^{(017)}$ on X.

Since there are no other possible four letter words, we must now investigate the double cosets: [0134].

Consider the following relations:

$t_0t_1t_3t_4 = t_7t_6t_3t_2$, so $[t_7t_6t_3t_2]^{(1,6)(2,4)(7,8)} = t_0t_1t_3 \Rightarrow [(1,6)(2,4)(7,8)]\epsilon N^{0134}$.

Computing $N^{(0134)}$ in N, we obtain:

$$\begin{aligned}|N^{(0134)}| &\geq |N^{0134}| \\ &\geq |<Id(G),(1,6)(2,4)(7,8)>| \\ &\geq 2.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_3t_4N = \frac{|N|}{|N^{(0134)}|} = \frac{48}{2} = 24$.

Hence our index is increased to $141 + 24 = 165$.

We now explore any potentially new double cosets coming from representatives from the orbits of $N^{(013)}$ on X, $N^{(016)}$ on X, $N^{(017)}$ on X.

The orbits of $N^{(0134)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ and $\{7\}$. The representative from the orbit $\{4\}$ advances to [013]. The other representatives will be the potentially new double cosets [0137], [01341], [01342], [01343], [01345], [0146], and [0147]. However, consider the following relations:

$t_0t_1t_3t_4t_1 = (1,4,2,6)(3,7,5,8)[t_0t_1t_3t_4]^{(3,7)(4,6)(5,8)}$

$t_0t_1t_3t_4t_2 = (1,4,5,2,6,3)(7,8)[t_0t_1t_3]^{(1,6)(2,4)(7,8)}$

$t_0t_1t_3t_4t_3 = (1,7,5,6,2,8,3,4)[t_0t_1t_5]^{(1,7,2,8)(3,6,5,4)}$

$t_0t_1t_3t_4t_5 = (1,5,4,8,2,3,6,7)[t_0t_1t_6]^{(3,5)(4,8)(6,7)}$

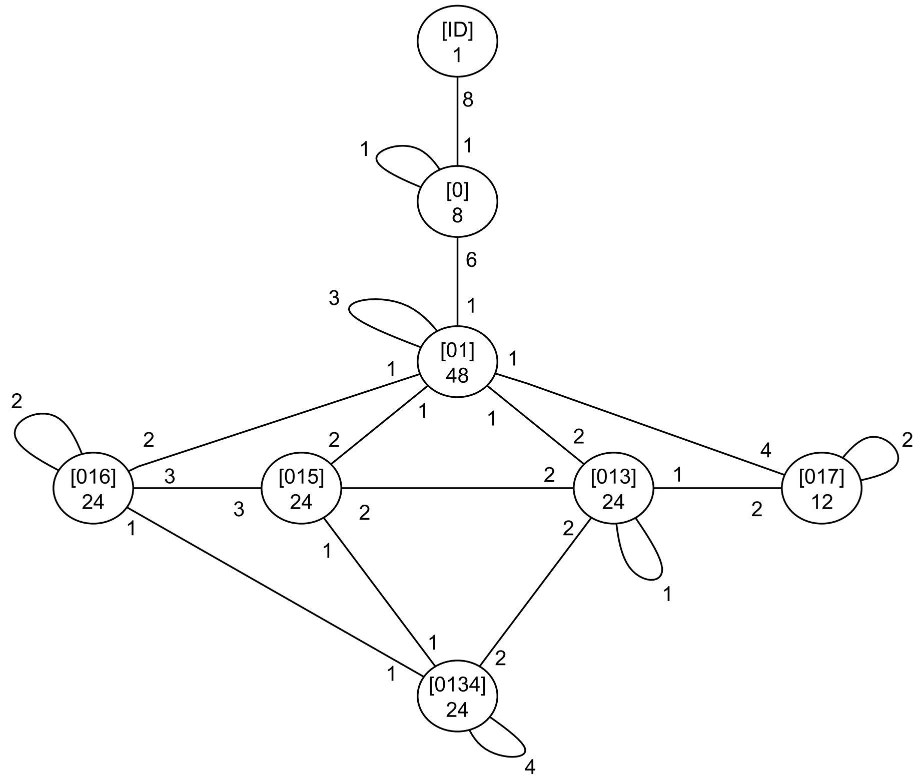
$t_0t_1t_3t_4t_6 = (3,6,8)(4,7,5)[t_0t_1t_3t_4]^{(1,6,2,4)(3,8,5,7)}$

$t_0t_1t_3t_4t_7 = (3,8,6)(4,5,7)[t_0t_1t_3t_4]^{(1,2)(3,6)(4,5)}$.

Because there are no new words of length six, we have completed our double coset enumeration of G over N. Our group is closed under right multiplication of t_i's.

The index of N in G is 165. The Cayley graph for G is given below.

Figure 3.1: M_{11} Cayley Graph



3.1.2 G is a Simple Group Using Iwasawa's Lemma

We let $X = \{N\omega\}$ be the set of single cosets of G over N. We will use Iwasawa's Lemma and the transitive action of G on X to prove G is a simple group. If we can show that G is faithful, G acts primitively on X, $G = G'$, and that there exists a normal, abelian subgroup of G such that $< K^G >= G$, we will have shown that G is a non-abelian simple group of order 7920.

(i) G acts faithfully on X

Since X is a transitive G-set of degree 165, we have:

$$|G| = 165|G_1|,$$

where G_1 is the one point stabiliser of the single coset N. However, N is stabilised by only elements of N. Therefore $G_1 = N$ and $|G_1| = |N| = 48$. It is then evident that $|G| = 7920$. If $|G| > 7920$, X would not be faithful.

(ii) The group G acts primitively on X

Every group constructed by a Cayley graph is transitive. Since G is transitive, we can assume $N \in B$. However, $|B|$ must divide $|X| = 165 = 3 \times 5 \times 11$. By observation of our Cayley graph, there are no possibilities for a nontrivial block. Thus G acts primitively on X.

(iii) The group G is perfect

Let us first begin by showing that G is generated by involutions, or
$G =< N, t_0, t_1, \ldots, t_7 >=< t_0, t_1, \ldots, t_7 >$.

Since N is generated by a, b, c, d, and e, we will show each is generated by t_i's.

- Consider our original relation $t_8t_7 = (1,2)(3,5)(4,6)(7,0)t_8$, where $e = (1,2)(3,5)(4,6)(7,0)$. So we see:

$$t_8t_7 = et_8$$
$$t_8t_7t_8 = e.$$

- Consider the relation $t_8t_1t_3t_4t_6 = (3,6,0)(4,7,5)t_5t_6t_8t_1$, where $b = (3,6,0)(4,7,5)$. So we see:

$$t_8t_1t_3t_4t_6 = bt_5t_6t_8t_1$$
$$t_8t_1t_3t_4t_6t_1t_8t_6t_5 = b.$$

- Consider the relation $t_8t_1t_3t_4t_1 = (1,4,2,6)(3,7,5,0)t_5t_1t_7t_6$, where $d = (1,4,2,6)(3,7,5,0)$. So we see:

$$t_8t_1t_3t_4t_1 = dt_5t_1t_7t_6$$
$$t_8t_1t_3t_4t_1t_6t_7t_1t_5 = d.$$

- Consider the relation $t_8t_3t_8 = (1,2)(3,8)(5,7)t_8t_3$, where $ab = (1,2)(3,8)(5,7)$. But we know $b = t_8t_1t_3t_4t_6t_1t_8t_6t_5$, so we see:

$$t_8t_3t_8 = abt_8t_3$$
$$t_8t_3t_8t_3t_8 = ab$$
$$t_8t_3t_8t_3t_8 = a(t_8t_1t_3t_4t_6t_1t_8t_6t_5)$$
$$t_8t_3t_8t_3t_8t_5t_6t_8t_1t_6t_4t_3t_1t_8 = a.$$

- Consider the relation $t_8t_1t_3t_7 = (1,8,3,2,7,5,1)(4,6)t_5t_1$, where $cb^{-1} = (1,8,3,2,7,5,1)(4,6)$. But $|b| = 3$, so $b^{-1} = t_5t_6t_8t_1t_6t_4t_3t_1t_8$. So we see:

$$t_8t_1t_3t_7 = cb^{-1}t_5t_1$$
$$t_8t_1t_3t_7t_1t_5 = cb^{-1}$$
$$t_8t_1t_3t_7t_1t_5 = c(t_5t_6t_8t_1t_6t_4t_3t_1t_8)$$
$$t_8t_1t_3t_7t_1t_5t_8t_1t_3t_4t_6t_1t_8t_6t_5 = c.$$

So we see that G is generated by the t_i's. Now $G =< N, t >$, where $N = (2^{\bullet}S_4)$, we know $(2^{\bullet}S_4)' \leq G'$. Hence we have
$(2^{\bullet}S_4)' =< (3,6,8)(4,7,5), (1,8,2,7)(3,4,5,6) >=\leq G'$. More importantly,
$(1,8,2,7)(3,4,5,6)(1,8,2,7)(3,4,5,6) = (1,2)(3,5)(4,6)(7,8) \in G'$.
Now consider the following relation:

$$(1,2)(3,5)(4,6)(7,8)t_8 = t_8t_7$$
$$(1,2)(3,5)(4,6)(7,8) = t_8t_7t_8,$$

So we see $t_8t_7t_8 \in G'$. Now we conjugate by the element $t_8 \in G$ and find:

$$[t_8t_7t_8]^{t_8} \in G'$$
$$[t_8]^{-1}[t_8t_7t_8][t_8] \in G'$$
$$t_7 \in G'.$$

Thus $G' \geq< (3,6,8)(4,7,5), (1,8,2,7)(3,4,5,6), t_7 >= G$. But we have already shown that $G =< t_0, t_1, \ldots, t_7 >$. Hence G is perfect.

(iv) The point stabiliser of N of G contains a subgroup K whose conjugates generate G

Since $N = 2^{\bullet} \times S_4$, the center $z = (1,2)(3,5)(4,6)(7,8)$ is a normal abelian subgroup. Utilizing the same relation as before, we also obtain the following:

$$[t_8t_7t_8]^{t_8} \in K^G$$
$$[t_8]^{-1}[t_8t_7t_8][t_8] \in K^G$$
$$t_7 \in K^G,$$

It is then easy to see that $K^G \geq < t_0, t_1, \ldots, t_7 >= G$. But $K^G \leq G$, hence we see that $K^G = G$.

(v) The group G is simple. Furthermore, $G \cong M_{11}$.

We have shown that the group G acts faithfully on X, is primitive, is perfect, and contains a normal abelian subgroup whose conjugates generate G. Therefore by Iwasawa's Lemma, G is a non-abelian simple group. Refering to [WB99], M_{11} is only simple group of order 7920.

3.2 M_{22} over $M = 2^3 : L(3,2)$

We factor the progenitor $2^{*7} : (7 : 3)$ by the two relations $[x^{-1}y^{-1}t]^5$ and $[yxt^{x^2}]^1 1$, where $x = (1,2,3,5,4,6,7)$ and $y = (2,3,4)(5,7,6)$. Letting t be represented by t_7, we compute the two relations:

Letting $\pi = x^{-1}y^{-1} = (1,7,3)(2,5,4)$, we obtain:

$$[x^{-1}y^{-1}t]^5 = e$$
$$[\pi t_7]^5 = e$$
$$\pi t_7 \pi t_7 \pi t_7 \pi t_7 \pi t_7 = e$$
$$\pi^5 t_7^{\pi^4} t_7^{\pi^3} t_7^{\pi^2} t_7^{\pi} t_7 = e$$
$$\pi^2 t_3 t_7 t_1 t_3 t_7 = e$$
$$(1,3,7)(2,4,5) t_3 t_7 t_1 = t_7 t_3.$$

Letting $\pi = yx = (1,3,7)(2,4,5)$, we obtain:

$$[yxt^{x^2}]^{11} = e$$
$$[\pi t_5]^{11} = e$$
$$\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5\pi t_5 = e$$
$$\pi^{11} t_5^{\pi^{10}} t_5^{\pi^9} t_5^{\pi^8} t_5^{\pi^7} t_5^{\pi^6} t_5^{\pi^5} t_5^{\pi^4} t_5^{\pi^3} t_5^{\pi^2} t_5^{\pi} t_5 = e$$
$$(1,7,3)(2,5,4)t_2t_5t_4t_2t_5 = t_5t_2t_4t_5t_2t_4.$$

However, consider the subgroup $(7:3) \in M_{22}$. $(7:3) \leq 2^3 : L(3,2) \leq_{max} M_{22}$. So when $M = 2^3 : L(3,2)$, $N \leq M \leq M$. If we can find a subgroup of order $|2^3 : L(3,2)| = 1344$ generated by the elements x, y, t_1, t_2, $\ldots t_7$, we can construct a double coset enumeration of G over M. We find $M =< x, y, t_1t_2t_1t_2t_7t_1t_2t_1t_2t_1t_7t_2 >$.

We will find the index of M in G by manual double coset enumeration of G over M. We take G and express it as a union of double cosets MgN, where g is an element of G. So $G = MeN \cup Mg_1N \cup Mg_2N \cup ...$, where g_i's are words in the t_i's.

We must find all distinct double cosets $[w]$, where $[w] = \{Mw^n | n \epsilon N\}$, and the number of single cosets contained in each double coset. Our manual double coset enumeration is completed when all potentially new double cosets have previously been accounted for and when the set of right cosets is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Mwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $M^{(w)} = \{n \; in \; N : Mw^n = Mw\}$, where w is a word of t_i's on $\{0,1,2,3,4,5\} = X$.

We begin with the double coset MeN, which we denote $[*]$. This double coset consists of the single coset M. Allowing 7 to be 0, the single orbit of M on X is $\{0,1,2,3,4,5,6\}$. We will choose $t_7 = t_0$ as our symmetric generator from the orbit $\{0,1,2,3,4,5,6\}$ and find Mt_0 belongs to Mt_0N which is a new double coset. We denote Mt_0N by $[0]$.

To find out how many single cosets $[0]$ contains, we find the set of coset stabilizers of $[0]$, denoted $N^{(0)}$. The number of single cosets in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|}$. We have:

$$\begin{aligned}|N^{(0)}| &\geq |<Id(G),(1,2,5)(3,6,4)>| \\ &\geq 3.\end{aligned}$$

The number of single cosets in $Mt_0N = \frac{|N|}{|N^{(0)}|} = \frac{21}{3} = 7$. Our index is the sum of distinct single cosets in each distinct double coset, such as [*] and [0]. As of now, we have $1 + 7 = 8$ single cosets. We note that the orbits of [0] are $\{0\}$, $\{1,2,5\}$, and $\{3,4,6\}$.

We will continue to the next level of potential double cosets by investigating the orbits of $N^{(0)}$ on X. The orbits of $N^{(0)}$ on X are $\{0\}$, $\{1,2,5\}$, and $\{3,4,6\}$ and we take t_0, t_1, and t_3 from each orbit respectively. From the orbit $\{0\}$ we get Nt_0t_0, which belongs to the double coset $[*]$. From the orbit $\{1,2,5\}$ we find a potentially new double coset Nt_0t_1, which we denote [01]. From the orbit $\{3,4,6\}$ we get Nt_0t_3 we find a potentially new double coset Nt_0t_3, which we denote [03]. We must now find the number of distinct single cosets in [01] and [03].

Computing $N^{(01)}$ in M, we obtain:

$$\begin{aligned}|N^{(01)}| &\geq |N^{01}| \\ &\geq |<Id(G)>| \\ &\geq 1.\end{aligned}$$

Computing $N^{(03)}$ in M, we obtain:

$$\begin{aligned}|N^{(03)}| &\geq |N^{03}| \\ &\geq |<Id(G)>| \\ &\geq 1.\end{aligned}$$

So the number of single cosets in $Mt_0t_1N = \frac{|N|}{|N^{(01)}|} = \frac{21}{1} = 21$. The number of single cosets in $Mt_0t_3N = \frac{|N|}{|N^{(03)}|} = \frac{21}{1} = 21$. Hence, our index is now $1+7+21+21 = 50$.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(01)}$ on X. We find [01] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and

$\{6\}$. The representative from the orbit $\{1\}$ advances back to [0]. The other orbit representatives bring the potentially new double cosets [012], [013], [014], [015],[016], and [010]. However, consider the following relations:
$t_0t_1t_3 = (0,3,1)(5,4,2)[t_0t_3]^{(0,1,3)(5,2,4)}$. But $(0,3,1)(5,4,2) \in M$. So $Mt_0t_1t_3 \in Mt_0t_3N$.
$t_0t_1t_0 = [(1,4,7)(3,6,5)t_2t_6t_2t_6t_4t_2t_6t_2t_6t_2t_4t_6t_3t_5t_3t_5t_2t_3t_5t_3t_5t_3t_2t_5](t_7t_1)^e$. But
$(1,4,7)(3,6,5)t_2t_6t_2t_6t_4t_2t_6t_2t_6t_2t_4t_6t_3t_5t_3t_5t_2t_3t_5t_3t_5t_3t_2t_5 \in M$. So $Mt_0t_1t_0 \in Mt_0t_3M$.

Hence in [01], the representative $\{2\}$ advances to [01] and is already being accounted for by the double coset [01]. So the only new double cosets coming from the orbit representatives of $N^{(01)}$ on X are [012], [014], [015] and [016].

The orbits of $N^{(03)}$ on X are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{3\}$ advances back to [0]. The other representatives bring the potentially new double cosets [031], [032], [034], [035], [036] and [030]. Consider the following relations:

$t_0t_3t_1 = (0,1,3)(5,2,4)[t_0t_1]^{(0,3,1)(5,4,2)}$

$t_0t_3t_0 = m[t_0t_3]^n$, for some $m \in M$ and for some $n \in N$.

Hence in [03], the representative $\{1\}$ advances to [01] and the representative $\{0\}$ advances to [03]. However, [032], [034], [035] and [036] are new, distinct double cosets.

The double cosets we must now investigate are [012], [014], [015], [016],[032], [034], [035] and [036].

Computing $N^{(012)}$ in M, we obtain:

$$\begin{aligned}|N^{(012)}| &\geq |N^{012}|\\ &\geq |<Id(G)>|\\ &\geq 1.\end{aligned}$$

Computing $N^{(014)}$ in M, we obtain:

$$\begin{aligned}|N^{(014)}| &\geq |N^{014}|\\ &\geq |<Id(G)>|\\ &\geq 1.\end{aligned}$$

Computing $N^{(015)}$ in M, we obtain:

$$\begin{aligned}|N^{(015)}| &\geq |N^{015}| \\ &\geq | < Id(G) > | \\ &\geq 1.\end{aligned}$$

Computing $N^{(016)}$ in M, we obtain:

$$\begin{aligned}|N^{(016)}| &\geq |N^{016}| \\ &\geq | < Id(G) > | \\ &\geq 1.\end{aligned}$$

The number of single cosets in $Mt_0t_1t_3N = \frac{|N|}{|N^{(012)}|} = \frac{21}{1} = 21$. The number of single cosets in $Mt_0t_1t_4N = \frac{|N|}{|N^{(014)}|} = \frac{21}{1} = 21$. The number of single cosets in $Mt_0t_1t_5N = \frac{|N|}{|N^{(015)}|} = \frac{21}{1} = 21$. And the number of single cosets in $Mt_0t_1t_6N = \frac{|N|}{|N^{(016)}|} = \frac{21}{1} = 21$.

Hence our index is increased to $50 + 21 + 21 + 21 + 21 = 134$.

Computing $N^{(032)}$ in M, we obtain:

$$\begin{aligned}|N^{(032)}| &\geq |N^{032}| \\ &\geq | < Id(G) > | \\ &\geq 1.\end{aligned}$$

Computing $N^{(034)}$ in M, we obtain:

$$\begin{aligned}|N^{(034)}| &\geq |N^{034}| \\ &\geq | < Id(G) > | \\ &\geq 1.\end{aligned}$$

Computing $N^{(035)}$ in M, we obtain:

$$\begin{aligned}|N^{(035)}| &\geq |N^{035}| \\ &\geq |<Id(G)>| \\ &\geq 1.\end{aligned}$$

Computing $N^{(036)}$ in M, we obtain:

$$\begin{aligned}|N^{(036)}| &\geq |N^{036}| \\ &\geq |<Id(G)>| \\ &\geq 1.\end{aligned}$$

The number of single cosets in $Nt_0t_3t_2N = \frac{|N|}{|N^{(032)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_3t_4N = \frac{|N|}{|N^{(034)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_3t_5N = \frac{|N|}{|N^{(035)}|} = \frac{21}{1} = 21$. And he number of single cosets in $Nt_0t_3t_6N = \frac{|N|}{|N^{(036)}|} = \frac{21}{1} = 21$.

Hence our index is increased to $134 + 21 + 21 + 21 + 21 = 218$.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(012)}$ on X. We find [012] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {2} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0121], [0123], [0124], [0125],[0126], and [0120]. However, consider the following relations:

$t_0t_1t_2t_1 = m[t_0t_1t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_5 = (6,3,4)(1,5,2)[t_0t_1t_5]^{(6,4,3)(1,2,5)}$

$t_0t_1t_2t_0 = m[t_0t_1t_2t_3]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [012], the representative {1} advances back to [012], the representative {5} advances to [015], and the representative {0} advances to [0123]. However, [0123], [0124], and [0126] are new, distinct double cosets from the orbits of $N^{(012)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(014)}$ on X. We find [014] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {4} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0141], [0142], [0143], [0145],[0146], and [0140]. However, consider the following relations:

$t_0t_1t_4t_1 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_3 = (0,2,5,6,1,3,4)[t_0t_1t_2t_6]^e$

$t_0t_1t_4t_6 = (0,2,3)(6,4,1)[t_0t_3t_6]^{(0,2,5,6,1,3,4)}$

$t_0t_1t_4t_0 = (0,1,5,4,6,3,2)[t_0t_1t_2t_3]^{(6,3,4)(1,5,2)}$.

Hence in [014], the representative {1} advances to [034], the representative {3} advances to [0126], the representative {6} advances to [0136], and the representative {0} advances to [0123]. However, [0142] and [0145] are new, distinct double cosets from the orbits of $N^{(014)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(015)}$ on X. We find [015] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {5} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0151], [0152], [0153], [0154],[0156], and [0150]. However, consider the following relations:

$t_0t_1t_5t_1 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_2 = (0,4,3)(1,2,5)[t_0t_1t_2]^{(6,3,4)(1,5,2)}$

$t_0t_1t_5t_3 = m[t_0t_1t_2t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_4 = m[t_0t_1t_4t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_0 = m[t_0t_1t_4t_2]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [015], the representative {1} advances to [015], the representative {2} advances to [012], the representative {3} advances to [0124], the representative {4} advances to [0145], and the representative {0} advances to [0142]. However, [0156] is the new, distinct double coset from the orbits of $N^{(015)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(016)}$ on X. We find [016] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {6} advances back to [01]. The other orbit representatives bring the potentially new double cosets [0161], [0162], [0163], [0164],[0165], and [0160]. However, consider the following relations:

$t_0t_1t_6t_1 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_2 = m[t_0t_1t_2t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_3 = m[t_0t_1t_2t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_4 = (0,3,2)(6,1,4)[t_0t_3t_4]^{(0,3,6,2,4,1,5)}$

$t_0t_1t_6t_0 = m[t_0t_1t_2t_6]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [016], the representative {1} advances to [016], the representative {2} advances to [0126], the representative {3} advances to [0124], the representative {4} advances to [034], and the representative {0} advances to [0126]. Hence, [0165] is the only new, distinct double coset from the orbits of $N^{(016)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(032)}$ on X. We find [032] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {2} advances back to [03]. The other orbit representatives bring the potentially new double cosets [0321], [0323], [0324], [0325],[0326], and [0320]. However, consider the following relations:

$t_0t_3t_2t_1 = m[t_0t_1t_2t_3]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_2t_3 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_2t_4 = (0,6,5)(4,2,3)[t_0t_3t_5]^{(0,6,4,5,3,2,1)}$

$t_0t_3t_2t_5 = (0,5,1,4,2,6,3)[t_0t_1t_2t_4]^{(0,4,1)(6,3,5)}$

$t_0t_3t_2t_6 = m[t_0t_1t_2t_3]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_2t_0 = m[t_0t_1t_2t_6]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [032], the representative {1} advances to [0123], the representative {3} advances to [036], the representative {4} advances to [035], the representative {5} advances to [0124], the representative {6} advances to [0123], and the representative {0} advances to [0126]. Hence there are no new double cosets coming from the orbits of $N^{(032)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(034)}$ on X. We find [034] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {4} advances back to [03]. The other orbit representatives bring the potentially new double cosets [0341], [0342], [0343], [0345],[0346], and [0340]. However, consider the following relations:

$t_0t_3t_4t_1 = m[t_0t_1t_4t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_2 = (0,5,6)(4,3,2)[t_0t_1t_6]^{(0,5,1,4,2,6,3)}$

$t_0t_3t_4t_3 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_6 = m[t_0t_1t_6t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_0 = m[t_0t_1t_2t_4]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [034], the representative {1} advances to [0142], the representative

$\{2\}$ advances to [016], the representative $\{3\}$ advances to [014], the representative $\{6\}$ advances to [0165], and the representative $\{0\}$ advances to [0124]. Hence, [0345] is the only new, distinct double coset coming from the orbits of $N^{(034)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(035)}$ on X. We find [035] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{5\}$ advances back to [03]. The other orbit representatives bring the potentially new double cosets [0341], [0342], [0343], [0345],[0346], and [0340]. However, consider the following relations:

$t_0t_3t_5t_1 = m[t_0t_1t_2t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_5t_2 = m[t_0t_1t_4t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_5t_3 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_5t_4 = m[t_0t_3t_4t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_5t_6 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_5t_0 = m[t_0t_1t_2t_4]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [035], the representative $\{1\}$ advances to [0126], the representative $\{2\}$ advances to [0142], the representative $\{3\}$ advances to [035], the representative $\{4\}$ advances to [0345], the representative $\{6\}$ advances to [032] and the representative $\{0\}$ advances to [0124]. Hence there are no new double cosets coming from the orbits of $N^{(035)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(036)}$ on X. We find [036] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{6\}$ advances back to [03]. The other orbit representatives bring the potentially new double cosets [0361], [0362], [0363], [0364],[0365], and [0360]. However, consider the following relations:

$t_0t_3t_6t_1 = m[t_0t_1t_4t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_6t_2 = m[t_0t_1t_2t_3]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_6t_3 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_6t_4 = m[t_0t_1t_5t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_6t_5 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_6t_0 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [035], the representative $\{1\}$ advances to [0142], the representative $\{2\}$ advances to [0123], the representative $\{3\}$ advances to [032], the representative $\{4\}$

advances to [0156], the representative $\{5\}$ advances to [014] and the representative $\{0\}$ advances to [036]. Hence there are no new double cosets coming from the orbits of $N^{(036)}$ on X.

The double cosets we must now investigate are [0123], [0124], [0126], [0142], [0145], [0156], [0165], and [0345].

Consider the following relations:

$t_0t_1t_4t_5 = t_6t_3t_4t_1$, which implies $[t_0t_1t_4t_5]^{(0,6,2)(1,3,5)} = t_6t_3t_4t_1 \Rightarrow [(0,6,2)(1,3,5)] \in N^{(0145)}$.

$t_0t_1t_5t_6 = t_3t_4t_5t_1$, which implies $[t_0t_1t_5t_6]^{(0,3,2)(1,4,6)} = t_3t_4t_5t_1 \Rightarrow [(0,3,2)(1,4,6)] \in N^{(0156)}$.

$t_0t_1t_6t_5 = t_1t_3t_6t_2$, which implies $[t_0t_1t_6t_5]^{(0,1,3)(2,4,5)} = t_1t_3t_6t_2 \Rightarrow [(0,1,3)(2,4,5)] \in N^{(0165)}$.

$t_0t_3t_4t_5 = t_6t_5t_4t_1$, which implies $[t_0t_3t_4t_5]^{(0,6,2)(1,3,5)} = t_1t_3t_6t_2 \Rightarrow [(0,6,2)(1,3,5)] \in N^{(0345)}$.

We find the following:

Computing $N^{(0123)}$ in M, we obtain:

$$\begin{aligned} |N^{(0123)}| &\geq |N^{0123}| \\ &\geq |<Id(G)>| \\ &\geq 1. \end{aligned}$$

Computing $N^{(0124)}$ in M, we obtain:

$$\begin{aligned} |N^{(0124)}| &\geq |N^{0124}| \\ &\geq |<Id(G)>| \\ &\geq 1. \end{aligned}$$

Computing $|N^{(0126)}|$ in M, we obtain:

$$\begin{aligned} |N^{(0126)}| &\geq |N^{0126}| \\ &\geq |<Id(G)>| \\ &\geq 1. \end{aligned}$$

Computing $|N^{(0142)}|$ in M, we obtain:

$$\begin{aligned}|N^{(0142)}| &\geq |N^{0142}|\\ &\geq |<Id(G)>|\\ &\geq 1.\end{aligned}$$

Computing $|N^{(0145)}|$ in M, we obtain:

$$\begin{aligned}|N^{(0145)}| &\geq |N^{0145}|\\ &\geq |<Id(G),(0,6,2)(1,3,5)>|\\ &\geq 3.\end{aligned}$$

Computing $|N^{(0156)}|$ in M, we obtain:

$$\begin{aligned}|N^{(0156)}| &\geq |N^{0156}|\\ &\geq |<Id(G),(0,3,2)(1,4,6)>|\\ &\geq 3.\end{aligned}$$

Computing $|N^{(0165)}|$ in M, we obtain:

$$\begin{aligned}|N^{(0165)}| &\geq |N^{0165}|\\ &\geq |<Id(G),(0,1,3)(2,4,5)>|\\ &\geq 3.\end{aligned}$$

Computing $|N^{(0345)}|$ in M, we obtain:

$$\begin{aligned}|N^{(0345)}| &\geq |N^{0345}|\\ &\geq |<Id(G),(0,6,2)(1,3,5)>|\\ &\geq 3.\end{aligned}$$

The number of single cosets in $Nt_0t_1t_2t_3N = \frac{|N|}{|N^{(0123)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_1t_2t_4N = \frac{|N|}{|N^{(0124)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_1t_2t_6N = \frac{|N|}{|N^{(0126)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_1t_4t_2N = \frac{|N|}{|N^{(0142)}|} = \frac{21}{1} = 21$. The number of single cosets in $Nt_0t_1t_4t_5N = \frac{|N|}{|N^{(0145)}|} = \frac{21}{3} = 7$. The number of single cosets in $Nt_0t_1t_5t_6N = \frac{|N|}{|N^{(0156)}|} = \frac{21}{3} = 7$. The number of single cosets in $Nt_0t_1t_6t_5N = \frac{|N|}{|N^{(0165)}|} = \frac{21}{3} = 7$. The number of single cosets in $Nt_0t_3t_4t_5N = \frac{|N|}{|N^{(0345)}|} = \frac{21}{3} = 7$.

Hence our index is increased to $218 + 21 + 21 + 21 + 21 + 7 + 7 + 7 + 7 = 330$.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0123)}$ on X. We find [0123] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{3\}$ advances back to [012]. The other orbit representatives bring the potentially new double cosets [01231], [01232], [01234], [01235],[01236], and [01230]. However, consider the following relations:

$t_0t_1t_2t_3t_1 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_3t_2 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_3t_4 = m[t_0t_1t_2t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_3t_5 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_3t_6 = m[t_0t_1t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_3t_0 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0123], the representative $\{1\}$ advances back to [036], the representative $\{2\}$ advances to [032], the representative $\{4\}$ advances to [0126], the representative $\{5\}$ advances to [036], the representative $\{6\}$ advances to [012] and the representative $\{0\}$ advances to [032]. Hence there are no new, distinct double cosets from the orbits of $N^{(0123)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0124)}$ on X. We find [0124] has the orbits $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{4\}$ advances back to [012]. The other orbit representatives bring the potentially new double cosets [01241], [01242], [01243], [01245],[01246], and [01240]. However, consider the following relations:

$t_0t_1t_2t_4t_1 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_4t_2 = m[t_0t_1t_2t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_4t_3 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_4t_5 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_4t_6 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_4t_0 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0124], the representative {1} advances back to [016], the representative {2} advances to [0124], the representative {3} advances to [032], the representative {5} advances to [015], the representative {6} advances to [035] and the representative {0} advances to [034]. Hence there are no new, distinct double cosets from the orbits of $N^{(0124)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0126)}$ on X. We find [0126] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {6} advances back to [012]. The other orbit representatives bring the potentially new double cosets [01261], [01262], [01263], [01264],[01265], and [01260]. However, consider the following relations:

$t_0t_1t_2t_6t_1 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_6t_2 = m[t_0t_1t_2t_3]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_6t_3 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_6t_4 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_6t_5 = m[t_0t_3t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_2t_6t_0 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0126], the representative {1} advances back to [016], the representative {2} advances to [0123], the representative {3} advances to [014], the representative {4} advances to [035], the representative {5} advances to [032] and the representative {0} advances to [016]. Hence there are no new, distinct double cosets from the orbits of $N^{(0126)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0142)}$ on X. We find [0142] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {2} advances back to [014]. The other orbit representatives bring the potentially new double cosets [01421], [01423], [01424], [01425],[01426], and [01420]. However, consider the following relations:

$t_0t_1t_4t_2t_1 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_2t_3 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_2t_4 = m[t_0t_1t_4t_2]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_2t_5 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_2t_6 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_2t_0 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0142], the representative {1} advances back to [036], the representative {3} advances to [0135], the representative {4} advances to [0142], the representative {5} advances to [014], the representative {6} advances to [034] and the representative {0} advances to [015]. Hence there are no new, distinct double cosets from the orbits of $N^{(0142)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0145)}$ on X. We find [0142] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {5} advances back to [014]. The other orbit representatives bring the potentially new double cosets [01451], [01452], [01453], [01454],[01456], and [01450]. However, consider the following relations:

$t_0t_1t_4t_5t_1 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_5t_2 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_5t_3 = m[t_0t_1t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_5t_4 = m[t_0t_1t_4t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_5t_6 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_4t_5t_0 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0145], the representative {1} advances to [014], the representative {2} advances to [015], the representative {3} advances to [014], the representative {4} advances to [0145], the representative {6} advances to [015] and the representative {0} advances to [015]. Hence there are no new, distinct double cosets from the orbits of $N^{(0145)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0156)}$ on X. We find [0156] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {6} advances back to [015]. The other orbit representatives bring the potentially new double cosets [01561], [01562], [01563], [01564],[01565], and [01560]. However, consider the following relations:

$t_0t_1t_5t_6t_1 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_6t_2 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_6t_3 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_6t_4 = m[t_0t_1t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_6t_5 = m[t_0t_1t_6t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_5t_6t_0 = m[t_0t_3t_6]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0156], the representative {1} advances to [015], the representative {2} advances to [036], the representative {3} advances to [036], the representative {4} advances to [015], the representative {5} advances to [0165] and the representative {0} advances to [036]. Hence there are no new, distinct double cosets from the orbits of $N^{(0156)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0165)}$ on X. We find [0165] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {5} advances back to [016]. The other orbit representatives bring the potentially new double cosets [01651], [01652], [01653], [01654],[01656], and [01650]. However, consider the following relations:

$t_0t_1t_6t_5t_1 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_5t_2 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_5t_3 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_5t_4 = m[t_0t_1t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_5t_6 = m[t_0t_1t_5t_6]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_1t_6t_5t_0 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0165], the representative {1} advances to [034], the representative {2} advances to [016], the representative {3} advances to [034], the representative {4} advances to [016], the representative {6} advances to [0156] and the representative {0} advances to [034]. Hence there are no new, distinct double cosets from the orbits of $N^{(0165)}$ on X.

We explore the potentially new double cosets coming from representatives from the orbits of $N^{(0345)}$ on X. We find [0345] has the orbits {0}, {1}, {2}, {3}, {4}, {5}, and {6}. The representative from the orbit {5} advances back to [034]. The other orbit representatives bring the potentially new double cosets [03451], [03452], [03453], [03454],[03456], and [03450]. However, consider the following relations:

$t_0t_3t_4t_5t_1 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_5t_2 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_5t_3 = m[t_0t_3t_4]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_5t_4 = m[t_0t_3t_4t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_5t_6 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$

$t_0t_3t_4t_5t_0 = m[t_0t_3t_5]^n$ for some $m \in M$ and for some $n \in N$.

Hence in [0345], the representative {1} advances to [034], the representative {2} advances to [035], the representative {3} advances to [034], the representative {4} advances to [0345], the representative {6} advances to [035] and the representative {0} advances to [035]. Hence there are no new, distinct double cosets from the orbits of $N^{(0345)}$ on X.

Because there are no new words of length five, we have completed our double coset enumeration of G over M. Our group is closed under right multiplication of t_i's. The index of M in G is 330. However, the double cosets of M over N are given by:

$M = N \cup Nt_1t_2t_1t_2t_0t_1t_2t_1t_2t_1t_0t_2N \cup$

$Nt_3t_4t_3t_4t_1t_3t_4t_3t_4t_3t_1t_4t_1t_2t_1t_2t_0t_1t_2t_1t_2t_1t_0t_2N \cup$

$Nt_0t_2t_0t_2t_4t_0t_2t_0t_2t_0t_4t_2t_1t_2t_1t_2t_0t_1t_2t_1t_2t_1t_0t_2N \cup$

$Nt_2t_3t_2t_3t_1t_2t_3t_2t_3t_2t_1t_3t_1t_2t_1t_2t_0t_1t_2t_1t_2t_1t_0t_2N \cup$

$Nt_2t_5t_2t_5t_0t_2t_5t_2t_5t_2t_0t_5t_1t_2t_1t_2t_0t_1t_2t_1t_2t_1t_0t_2N.$

Since the double coset enumeration of G over M gave us $X = \{M, Mt_0, M_t1, \dots\}$, we can perform a double coset decomposition of M over N to find all single cosets of G over N. Since there are 64 single cosets in M and there are 330 singles cosets in X, There are 21120 single cosets in the double coset enumeration of G over N.

The Cayley graph for G over M is given below. Since the orbits on the Cayley graph are difficult to follow, there is a table which illustrates the orbit destinations of the double coset enumeration of G over M.

Figure 3.2: M_{22} Cayley Graph

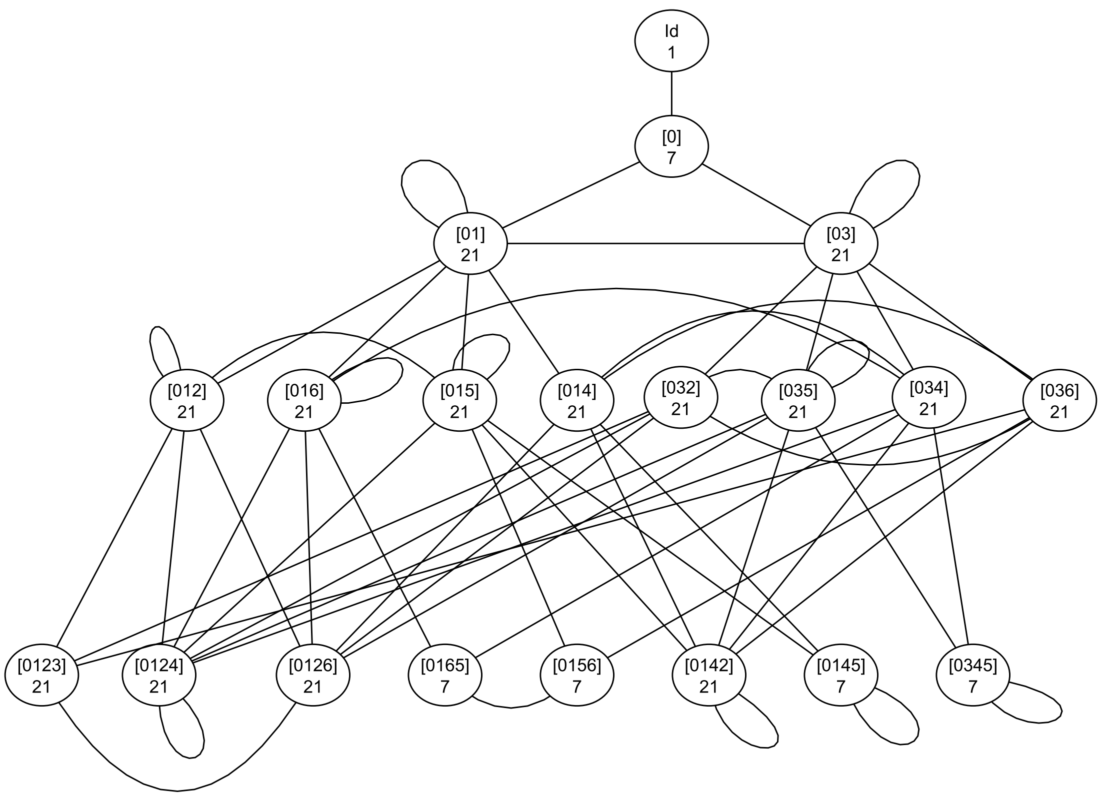


Table 3.1: Orbits of M_{22}(a)

NwN	Orbits	Potentially New DCs	Destination
NeN	$\{0,1,2,3,4,5,6\}$	$Mt_0 \rightarrow$	Mt_0N
Mt_0N	$\{0\},\{1,2,5\},$	$Mt_0t_0 \rightarrow$	N
	$\{3,4,6\}$	$Mt_0t_1 \rightarrow$	Mt_0t_1N
		$Mt_0t_3 \rightarrow$	Mt_3N
Mt_0t_1N	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_0 \rightarrow$	Mt_0t_1N
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_1 \rightarrow$	Mt_0N
		$Mt_0t_1t_2 \rightarrow$	$Mt_0t_1t_2N$
		$Mt_0t_1t_3 \rightarrow$	Mt_0t_3N
		$Mt_0t_1t_4 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_5 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_6 \rightarrow$	$Mt_0t_1t_6N$
Mt_0t_3N	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_3t_0 \rightarrow$	Nt_0t_3N
	$\{4\},\{5\},\{6\}$	$Mt_0t_3t_1 \rightarrow$	Mt_0t_1N
		$Mt_0t_3t_2 \rightarrow$	$Mt_0t_3t_2N$
		$Mt_0t_3t_3 \rightarrow$	$Mt_0t_3t_3N$
		$Mt_0t_3t_4 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_3t_5 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_3t_6 \rightarrow$	$Mt_0t_3t_6N$
$Mt_0t_1t_2N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_2t_0 \rightarrow$	Mt_0t_3N
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_2t_1 \rightarrow$	$Mt_0t_1t_2N$
		$Mt_0t_1t_2t_2 \rightarrow$	Mt_0t_1N
		$Mt_0t_1t_2t_3 \rightarrow$	$Mt_0t_1t_2t_3N$
		$Mt_0t_1t_2t_4 \rightarrow$	$Mt_0t_1t_2t_4N$
		$Mt_0t_1t_2t_5 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_2t_6 \rightarrow$	$Mt_0t_1t_2t_6N$

Table 3.2: Orbits of M_{22}(b)

NwN	Orbits	Potentially New DCs	Destination
$Mt_0t_1t_4N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_4t_0 \rightarrow$	$Mt_0t_1t_4t_2N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_4t_1 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_4t_2 \rightarrow$	$Mt_0t_1t_4t_2N$
		$Mt_0t_1t_4t_3 \rightarrow$	$Mt_0t_1t_2t_6N$
		$Mt_0t_1t_4t_4 \rightarrow$	Mt_0t_1N
		$Mt_0t_1t_4t_5 \rightarrow$	$Mt_0t_1t_4t_5N$
		$Mt_0t_1t_4t_6 \rightarrow$	$Mt_0t_3t_6N$
$Mt_0t_1t_5N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_5t_0 \rightarrow$	$Mt_0t_1t_4t_2N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_5t_1 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_5t_2 \rightarrow$	$Mt_0t_1t_2N$
		$Mt_0t_1t_5t_3 \rightarrow$	$Mt_0t_1t_2t_4N$
		$Mt_0t_1t_5t_4 \rightarrow$	$Mt_0t_1t_4t_5N$
		$Mt_0t_1t_5t_5 \rightarrow$	Mt_0t_1N
		$Mt_0t_1t_5t_6 \rightarrow$	$Mt_0t_1t_5t_6N$
$Mt_0t_1t_6N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_6t_0 \rightarrow$	$Mt_0t_1t_2t_6N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_6t_1 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_6t_2 \rightarrow$	$Mt_0t_1t_2t_6N$
		$Mt_0t_1t_6t_3 \rightarrow$	$Mt_0t_1t_2t_4N$
		$Mt_0t_1t_6t_4 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_6t_5 \rightarrow$	$Mt_0t_1t_6t_5N$
		$Mt_0t_1t_6t_6 \rightarrow$	Mt_0t_1N
$Mt_0t_3t_2N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_3t_2t_0 \rightarrow$	$Nt_0t_1t_2t_6N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_3t_2t_1 \rightarrow$	$Mt_0t_1t_2t_3N$
		$Mt_0t_3t_2t_2 \rightarrow$	Mt_0t_3N
		$Mt_0t_3t_2t_3 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_3t_2t_4 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_3t_2t_5 \rightarrow$	$Mt_0t_1t_2t_4N$
		$Mt_0t_3t_2t_6 \rightarrow$	$Mt_0t_1t_2t_3N$
$Mt_0t_3t_4N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_3t_4t_0 \rightarrow$	$Mt_0t_1t_2t_4N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_3t_4t_1 \rightarrow$	$Mt_0t_1t_4t_2N$
		$Mt_0t_3t_4t_2 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_3t_4t_3 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_3t_4t_4 \rightarrow$	Mt_0t_3N
		$Mt_0t_3t_4t_5 \rightarrow$	$Mt_0t_1t_6t_5N$
		$Mt_0t_3t_4t_6 \rightarrow$	$Mt_0t_1t_2t_3N$
$Mt_0t_3t_5N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_3t_5t_0 \rightarrow$	$Mt_0t_1t_2t_4N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_3t_5t_1 \rightarrow$	$Mt_0t_1t_2t_6N$
		$Mt_0t_3t_5t_2 \rightarrow$	$Mt_0t_1t_4t_2N$
		$Mt_0t_3t_5t_3 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_3t_5t_4 \rightarrow$	$Mt_0t_3t_4t_5N$
		$Mt_0t_3t_5t_5 \rightarrow$	Mt_0t_3N
		$Mt_0t_3t_5t_6 \rightarrow$	$Mt_0t_3t_2N$

Table 3.3: Orbits of M_{22}(c)

MwN	Orbits	Potentially New DCs	Destination
$Mt_0t_3t_6N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_3t_6t_0 \rightarrow$	$Mt_0t_1t_2t_3N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_3t_6t_1 \rightarrow$	$Mt_0t_1t_4t_2N$
		$Mt_0t_3t_6t_2 \rightarrow$	$Mt_0t_1t_2t_3N$
		$Mt_0t_3t_6t_3 \rightarrow$	$Mt_0t_3t_2N$
		$Mt_0t_3t_6t_4 \rightarrow$	$Mt_0t_1t_5t_6N$
		$Mt_0t_3t_6t_5 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_3t_6t_6 \rightarrow$	Mt_0t_3N
$Mt_0t_1t_2t_3N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_2t_3t_0 \rightarrow$	$Mt_0t_3t_2N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_2t_3t_1 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_1t_2t_3t_2 \rightarrow$	$Mt_0t_3t_2N$
		$Mt_0t_1t_2t_3t_3 \rightarrow$	$Mt_0t_1t_2N$
		$Mt_0t_1t_2t_3t_4 \rightarrow$	$Mt_0t_1t_2t_6N$
		$Mt_0t_1t_2t_3t_5 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_1t_2t_3t_6 \rightarrow$	$Mt_0t_1t_2N$
$Mt_0t_1t_2t_4N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_2t_4t_0 \rightarrow$	$Mt_0t_3t_4N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_2t_4t_1 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_2t_4t_2 \rightarrow$	$Mt_0t_1t_2t_4N$
		$Mt_0t_1t_2t_4t_3 \rightarrow$	$Mt_0t_3t_2N$
		$Mt_0t_1t_2t_4t_4 \rightarrow$	$Mt_0t_1t_2N$
		$Mt_0t_1t_2t_4t_5 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_2t_4t_6 \rightarrow$	$Mt_0t_3t_5N$
$Mt_0t_1t_2t_6N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_2t_6t_0 \rightarrow$	$Mt_0t_1t_6N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_2t_6t_1 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_2t_6t_2 \rightarrow$	$Mt_0t_1t_2t_3N$
		$Mt_0t_1t_2t_6t_3 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_2t_6t_4 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_1t_2t_6t_5 \rightarrow$	$Mt_0t_3t_2N$
		$Mt_0t_1t_2t_6t_6 \rightarrow$	$Mt_0t_1t_2N$
$Mt_0t_1t_4t_2N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_4t_2t_0 \rightarrow$	$Mt_0t_1t_5N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_4t_2t_1 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_1t_4t_2t_2 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_4t_2t_3 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_1t_4t_2t_4 \rightarrow$	$Mt_0t_1t_4t_2N$
		$Mt_0t_1t_4t_2t_5 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_4t_2t_6 \rightarrow$	$Mt_0t_3t_4N$
$Mt_0t_1t_4t_5N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_4t_5t_0 \rightarrow$	$Mt_0t_1t_5N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_4t_5t_1 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_4t_5t_2 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_4t_5t_3 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_4t_5t_4 \rightarrow$	$Mt_0t_1t_4t_5N$
		$Mt_0t_1t_4t_5t_5 \rightarrow$	$Mt_0t_1t_4N$
		$Mt_0t_1t_4t_5t_6 \rightarrow$	$Mt_0t_1t_5N$

Table 3.4: Orbits of M_{22}(d)

NwN	Orbits	Potentially New DCs	Destination
$Mt_0t_1t_5t_6N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_5t_6t_0 \rightarrow$	$Mt_0t_3t_6N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_5t_6t_1 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_5t_6t_2 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_1t_5t_6t_3 \rightarrow$	$Mt_0t_3t_6N$
		$Mt_0t_1t_5t_6t_4 \rightarrow$	$Mt_0t_1t_5N$
		$Mt_0t_1t_5t_6t_5 \rightarrow$	$Mt_0t_1t_6t_5N$
		$Mt_0t_1t_5t_6t_6 \rightarrow$	$Mt_0t_1t_5N$
$Mt_0t_1t_6t_5N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_6t_5t_0 \rightarrow$	$Mt_0t_3t_4N$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_6t_5t_1 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_6t_5t_2 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_6t_5t_3 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_6t_5t_4 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_6t_5t_5 \rightarrow$	$Mt_0t_1t_6N$
		$Mt_0t_1t_6t_5t_6 \rightarrow$	$Mt_0t_1t_5t_6N$
$Mt_0t_3t_4t_5N$	$\{0\},\{1\},\{2\},\{3\},$	$Mt_0t_1t_2t_4t_0 \rightarrow$	$Mt_0t_3t_5$
	$\{4\},\{5\},\{6\}$	$Mt_0t_1t_2t_4t_1 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_2t_4t_2 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_1t_2t_4t_3 \rightarrow$	$Mt_0t_3t_4N$
		$Mt_0t_1t_2t_4t_4 \rightarrow$	$Mt_0t_3t_4t_5N$
		$Mt_0t_1t_2t_4t_5 \rightarrow$	$Mt_0t_3t_5N$
		$Mt_0t_1t_2t_4t_6 \rightarrow$	$Mt_0t_3t_5N$

3.2.1 Partial Proof of M_{22} by Iwasawa's Lemma

We let $X = \{N\omega\}$ be the set of single cosets of G over N. We will use Iwasawa's Lemma and the transitive action of G on X to prove G is a simple group. If we can show that G is faithful, G acts primitively on X, $G = G'$, and that there exists a normal, abelian subgroup of G such that $< K^G >= G$, we will have shown that G is a non-abelian simple group of order 443520.

(i) G acts faithfully on X

Since X is a transitive G-set of degree 21120, we have:

$$|G| = 21120|G_1|,$$

where G_1 is the one point stabiliser of the single coset N. However, N is stabilised by only elements of N. Therefore $G_1 = N$ and $|G_1| = |N| = 21$. So we have $|G| > 21 \times 21120 = 443520$. It is then evident that $|G| = 443520$. If $|G| > 443520$, X would not be faithful.

(ii) The group G acts primitively on X

Every group constructed by a Cayley graph is transitive. Since G is transitive, we can assume $N \in B$. However, $|B|$ must divide $|X| = 330 = 2 \times 3 \times 5 \times 11$. So the order of any nontrivial block must be of order 2, 3, 5, 6, 10, 11, 15, 22, 30, 33, 55, 66, or 110.

Let us first exclude a few of these choices. By speculation of our Cayley graph, there is one double coset with 1 double coset, five double cosets with 7 single cosets, and the rest of the double cosets have 21 single cosets. This eliminates the possibilities of obtaining a nontrivial block of size 2, 3, 5, 6, 10, 11, 30, 33, 55, 66, and 110.

We must then determine if there are any nontrivial blocks of size 15 or 22.

Since a block of size 15 must have the double coset M and two double cosets with 7 single cosets, we will look at the various cases. The double cosets with 7 single cosets are [0], [0145], [0156], [0165], and [0345].

First consider a block containing N and Nt_0. Then $B > \{M, Mt_0, Mt_1, \ldots Mt_6\}$. But $Bt_0 = \{Mt_0, Mt_0t_0, Mt_1t_0, \ldots Mt_6t_0\} = \{Mt_0, M, Mt_1t_0, \ldots Mt_6t_0\}$. But this would imply $Mt_0t_1N \in B$, since $Mt_0t_1 \in B$ and double cosets are either disjoint or contained in B. But Mt_0t_1N has 21 single cosets, and we already said there are no potential blocks that contain a double coset with 21 single cosets. Therefore there Mt_0N cannot be contained in a nontrivial block.

Next we consider a block with $Mt_0t_1t_4t_5N$. Then $B > \{M, Mt_0t_1t_4t_5, \ldots\}$. But $Bt_4 > \{Mt_4, Mt_0t_1t_4t_5t_4, \ldots\}$. But we have $0145 \sim 01454$, therefore $Mt_0t_1t_4t_5t_4 = Mt_0t_1t_4t_5$. So $Mt_0N \in B$ since $Mt_4 \in Mt_0N$. Therefore a block with $Mt_0t_1t_4t_5N$ is trivial.

Next we consider a block with $Mt_0t_3t_4t_5N$. Then $B > \{M, Nt_0t_3t_4t_5, \ldots\}$. But $Bt_4 > \{Mt_4, Mt_0t_3t_4t_5t_4, \ldots\}$. But we have $0345 \sim 03454$, therefore $Mt_0t_3t_4t_5t_4 = Mt_0t_3t_4t_5$. So $Mt_0N \in B$ since $Mt_4 \in Mt_0N$. Therefore a block with $Mt_0t_3t_4t_5N$ is trivial.

So a block of size 15 must have [$*$], [0156], and [0165]. So we have
$B = \{M, Mt_0t_1t_5t_6, \ldots, Mt_0t_1t_6t_5, \ldots\}$.

Consider $Bt_5 = \{Mt_5, Mt_0t_1t_5t_6t_5, \ldots, Mt_0t_1t_6t_5t_5, \ldots\}$. But we have $01565 \sim 0165$. So we see $Mt_0N \in B$, since $Mt_5 \in Mt_0N$ and $Mt_5 \in B$. So this block is trivial.

Therefore there are no nontrivial blocks of size 15.

We will now determine if there are any blocks of size 22.

As we seen before, there are no nontrivial blocks that include any double cosets with 7 single cosets. So the only nontrivial blocks of size 22 must be formed utilizing only one double coset with 21 single cosets and N. We will check [01], [03], [012], [014], [015], [016], [032], [034], [035], [036], [0123], [0124], [0126], and [0142] individually joined with [$*$]. If we find that there are any extra cosets in B, the block will be trivial.

If $B = \{M, Mt_0t_1, \ldots\}$, consider $Bt_0 = \{Mt_0, Mt_0t_1t_0, \ldots\}$. But $010 \sim 01$. Then $Bt_0 = \{Mt_0, Mt_0t_1, \ldots\}$. So $Mt_0N \in B$, since $Mt_0 \in Mt_0N$ and $Mt_0 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_3, \ldots\}$, consider $Bt_0 = \{Mt_0, Mt_0t_3t_0, \ldots\}$. But $030 \sim 03$. Then $Bt_0 = \{Mt_0, Mt_0t_3, \ldots\}$. So $Mt_0N \in B$, since $Mt_0 \in Mt_0N$ and $Mt_0 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_2, \ldots\}$, consider $Bt_1 = \{Mt_1, Mt_0t_1t_2t_1, \ldots\}$. But $0121 \sim 012$. Then $Bt_1 = \{Mt_1, Mt_0t_1t_2, \ldots\}$. So $Mt_0N \in B$, since $Mt_1 \in Mt_0N$ and $Mt_1 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_4, \ldots\}$, consider $Bt_1t_3 = \{Mt_1t_3, Mt_0t_1t_4t_1t_3, \ldots\}$. But $0141 \sim 034$. So we have $Bt_1t_3 = \{Mt_1t_3, Mt_0t_3t_4t_3, \ldots\}$. But $0343 \sim 014$. So we have $Bt_1t_3 = \{Mt_1t_3, Mt_0t_1t_4, \ldots\}$. So $Mt_0t_1N \in B$, since $Mt_1t_3 \in Mt_0t_1N$ and $Mt_1t_3 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_5, \ldots\}$, consider $Bt_1 = \{Mt_1, Mt_0t_1t_5t_1, \ldots\}$. But $0151 \sim 015$. So we have $Bt_1 = \{Mt_1, Mt_0t_1t_5, \ldots\}$. So $Mt_0N \in B$, since $Mt_1 \in Mt_0N$ and $Mt_1 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_6, \ldots\}$, consider $Bt_1 = \{Mt_1, Mt_0t_1t_6t_1, \ldots\}$. But $0161 \sim 016$. So we have $Bt_1 = \{Mt_1, Mt_0t_1t_6, \ldots\}$. So $Mt_0N \in B$, since $Mt_1 \in Mt_0N$ and $Mt_1 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_3t_2, \ldots\}$, consider $Bt_4t_6 = \{Mt_4t_6, Nt_0t_3t_2t_4t_6, \ldots\}$. But $0324 \sim 035$. So we have $Bt_4t_6 = \{Mt_4t_6, Nt_0t_3t_5t_6, \ldots\}$. But $0356 \sim 032$. So we have $Bt_4t_6 = \{Mt_4t_6, Mt_0t_3t_2, \ldots\}$. So $Mt_0t_1N \in B$, since $Mt_4t_6 \in Mt_0t_1N$ and $Mt_4t_6 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_3t_4, \ldots\}$, consider $Bt_3t_1 = \{Mt_3t_1, Nt_0t_3t_4t_3t_1, \ldots\}$. But $0343 \sim 014$. So we have $Bt_3t_1 = \{Mt_3t_1, Nt_0t_1t_4t_6, \ldots\}$. But $0146 \sim 034$. So we have $Bt_3t_1 = \{Mt_3t_1, Mt_0t_3t_4, \ldots\}$. So $Mt_0t_3N \in B$, since $Mt_3t_1 \in Mt_0t_3N$ and $Mt_3t_1 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_3t_5, \ldots\}$, consider $Bt_3 = \{Mt_3, Mt_0t_3t_5t_3, \ldots\}$. But $0353 \sim 035$. So we have $Bt_3 = \{Mt_3, Mt_0t_3t_5, \ldots\}$. So $Mt_0N \in B$, since $Nt_3 \in Mt_0N$ and $Mt_3 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_3t_6, \ldots\}$, consider $Bt_5t_6 = \{Mt_5t_6, Mt_0t_3t_6t_5t_6, \ldots\}$. But $0365 \sim 014$. So we have $Bt_5t_6 = \{Mt_5t_6, Mt_0t_1t_4t_6, \ldots\}$. But $0146 \sim 034$. So we have $Bt_5t_6 = \{Mt_5t_6, Mt_0t_3t_4, \ldots\}$. So $Mt_0t_1N \in B$, since $Mt_5t_6 \in Mt_0t_1N$ and

$Mt_5t_6 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_2t_3, \ldots\}$, consider $Bt_4t_2 = \{Mt_4t_2, Mt_0t_1t_2t_3t_4t_2, \ldots\}$. But $01234 \sim 0126$. So we have $Bt_5t_6 = \{Mt_4t_2, Mt_0t_1t_2t_6t_2, \ldots\}$. But $01262 \sim 0123$. So we have $Bt_4t_2 = \{Mt_4t_2, Mt_0t_1t_2t_3, \ldots\}$. So $Mt_0t_1N \in B$, since $Mt_5t_6 \in Mt_0t_1N$ and $Mt_4t_2 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_2t_4, \ldots\}$, consider $Bt_2 = \{Mt_2, Mt_0t_1t_2t_4t_2, \ldots\}$. But $01242 \sim 0124$. So we have $Bt_2 = \{Mt_2, Mt_0t_1t_2t_4, \ldots\}$. So $Mt_0N \in B$, since $Mt_2 \in Mt_0N$ and $Mt_2 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_2t_6, \ldots\}$, consider $Bt_2t_4 = \{Mt_2t_4, Mt_0t_1t_2t_6t_2t_4, \ldots\}$. But $01262 \sim 0123$. So we have $Bt_2t_4 = \{Mt_2t_4, Mt_0t_1t_2t_3t_4, \ldots\}$. But $01234 \sim 0126$. So we have $Bt_2t_4 = \{Mt_2t_4, Mt_0t_1t_2t_6, \ldots\}$. So $Mt_0t_3N \in B$, since $Mt_2t_4 \in Mt_0t_3N$ and $Mt_2t_4 \in B$. So this block is trivial.

If $B = \{M, Mt_0t_1t_4t_2, \ldots\}$, consider $Bt_4 = \{Mt_2, Mt_0t_1t_4t_2t_4, \ldots\}$. But $01424 \sim 0142$. So we have $Bt_4 = \{Mt_2, Mt_0t_1t_4t_2, \ldots\}$. So $Mt_0N \in B$, since $Mt_4 \in Mt_0N$ and $Mt_4 \in B$. So this block is trivial.

Thus G acts primitively on X since there are no nontrivial blocks.

(iii) The group G is perfect

We should begin by showing that $G =< x, y, t_0, t_1, \ldots, t_6 >=< t_0, t_1, \ldots, t_6 >$.

Consider the relations:

$$t_0t_3t_2t_4 = (0,3,2)(1,4,6)t_3t_6t_1, \text{ where } a = (0,3,2)(1,4,6)$$

$$t_0t_3t_2t_4t_1t_6t_3 = a$$

$$t_0t_1t_3 = (0,3,1)(2,5,4)t_1t_0, \text{ where } a^{-1}b^{-1} = (0,3,1)(2,5,4) \; t_0t_1t_3t_0t_1 = a^{-1}b^{-1}.$$

But as we seen before, a is generated by t_i's, so b is also generated by t_i's.

Hence $G =< N, t_0, t_1, \ldots, t_6 >=< t_0, t_1, \ldots, t_6 >$.

We must now show that $G = G'$. Since $N \leq M \leq G$, the derived group of $N \in G'$. The derived group of $N =< b >$.

As we have done before, we find a relation that has only b and some t_i's in it. We then isolate b by putting all the t_i's on the other side of the equation. This helps us see that this product of t_i's is in N'. From here, we try to show that this product of t_i's is actually a product of commutators and one single t_i. If we can accomplish this, we have that a single $t_i \in N' \leq G'$. From here, it is easy to see that $G =< t_0, t_1, \ldots, t_6 >= G'$.

For instance, we have the relation $t_0t_3t_2t_5 = b^4t_4t_0t_2t_1$. So $b^4 = t_0t_3t_2t_5t_1t_2t_0t_4$. But $b^4 \in N'$, so $t_0t_3t_2t_5t_1t_2t_0t_4 \in N' \leq G'$.

Due to time constraints, there is no proof to show $G = G'$.

(iv) The point stabiliser of N of G contains a normal abelian subgroup K whose conjugates generate G

We know a normal abelian subgroup of M is also a normal abelian subgroup of G. In this case, $K = 2^3 \leq [2^3 : L(3,2)] = M$, where the elements that generate $K = 2^3 =< K_a, K_b, K_c >$ are given by:

$$K_a = (2,3,4)(5,7,6)t_4t_2t_4t_2t_1t_4t_2t_4t_2t_4t_1t_3t_2t_3t_1t_2t_3t_2t_1t_3$$
$$K_b = (2,3,4)(5,7,6)t_2t_3t_2t_3t_1t_2t_3t_2t_3t_2t_1t_3t_4t_3t_4t_1t_3t_4t_3t_4t_3t_1t_4$$
$$K_c = (1,7,3)(2,5,4)t_3t_0t_3t_0t_6t_3t_0t_3t_0t_3t_6t_0t_1t_3t_1t_3t_6t_1t_3t_1t_3t_1t_6t_3.$$

To solve this, we want to show that an element in K can be conjugated by elements from G to generate a single t_i.

Due to time constraints, there is no proof to show $K^G = G$.

Chapter 4

Isomorphism Types of Some Groups

4.1 $M_{11} \times S_4$

We begin with the infinite progenitor $2^{*8} : (2^{\bullet}S_4)$ and factor by suitable relations to give us G:

$G =< v, w, x, y, z, t | v^2, w^3, x^2 = z, y^2 = z, z^2, w^v = w^2, x^v = y, x^w = yz, y^v = x, y^w = xy, y^x = yz, z^v = z, z^w = z, z^x = z, z^y = z, t^2, (t, v), (t, wy^{-1}) >$.

To make this group progenitor finite, we factor it by suitable relations. The finite group we will investigate is $G \cong < v, w, x, y, z, t | v^2, w^3, x^2 = z, y^2 = z, z^2, w^v = w^2, x^v = y, x^w = yz, y^v = x, y^w = xy, y^x = yz, z^v = z, z^w = z, z^x = z, z^y = z, t^2, (t, v), (t, wy^{-1}), (xt)^6, (xw^{-1}t)^6, (zt)^3 >$. By the Jordan-Hölder Theorem, we know that every finite group can be factored into simple groups. We can then examine the isomorphism type of our group G.

When determining the isomorphism type, there are different extension types of the groups which have simple factors. If N is normal in G and H is isomorphic to G/N, we say that G is an extension of N by H.

When examining the composition factors of G, we find G consists of one M_{11} group, followed by one C_2 group, a C_3 group, and two C_2 groups. We must now determine the extension problems associated with these simple groups. We first determine that our group has no central element. When observing the minimal normal subgroups of G, we find that there is a normal subgroup of order 24 and another of order 7920. Furthermore, the order of $G = 190080 = 7920 \times 24$. We should then check if our group is a direct product of both these normal subgroups.

The normal subgroup of order 24 has an abelian subgroup of order 4. After a quick computerized check, we find that this subgroup of order 24 is S_4. The normal subgroup of order 7920 is the sporadic Mathieu Group 11 simple group M_{11}. After observing the normal lattice of G, S_4 is denoted by $NL[4]$ and M_{11} is denoted by $NL[5]$.

After a computerized check, we find that G is a direct product between $NL[4]$ and $NL[5]$. Hence $G \cong M_{11} \times S_4$.

A presentation for S_4 is $< a, b | a^4, b^2, (ab)^3 >$.

A presentation for M_{11} is $< c^2, d^4, d^{-1}cd^{-2}cd^{-2}cd^2cd^2cd^2cd^{-1}$, $cdcdcd^{-1}cd^{-1}cdcd^{-1}cd^2cdcd^{-1}, cd^{-2}cd^{-1}cdcd^{-}2cd^{-1}cdcd^2cd^{-1}cd, (cd^{-1})^{11} >$.

In a direct product, the generators of S_4 and M_{11} will commute with one another, hence the presentation for this group can be given by the following:

$G = M_{11} \times S_4 =< a, b, c, d | a^4, b^2, (ab)^3, c^2, d^4, d^{-1}cd^{-2}cd^{-2}cd^2cd^2cd^2cd^{-1}$, $cdcdcd^{-1}cd^{-1}cdcd^{-1}cd^2cdcd^{-1}, cd^{-2}cd^{-1}cdcd^{-}2cd^{-1}cdcd^2cd^{-1}cd, (cd^{-1})^{11}$, $(a, c), (a, d), (b, c), (b, d) >$.

4.2 $Z_8^{\bullet} : [D_{12} : (Z_4 \times Z_4)]$

We begin with the infinite progenitor $G = 2^{*6} : D_{12}$, given by the following presentation: $G \cong< x, y | x^3, y^2, (xy)^2 >$. To make this infinite progenitor finite, we can factor it by suitable relations. The finite group we will investigate is $G \cong< x, y, t | x^3, y^2, (xy)^2, t^2, (t, y), (xt)^8, (xtt^x)^2, (xyt^xt)^8, (ttxt)^8 >$. By the Jordan-Hölder Theorem, we know that every finite group can be factored into simple groups. We can then examine the isomorphism type of our group G.

We will use the normal lattice, denoted NL, to help work with the composition factors of G. We find G consists of ten simple groups; nine C_2 groups and one C_3 group. Furthermore, we have the composition series $G \supset G_1 \supset G_2 \supset G_3 \supset G_4 \supset G_5 \supset G_6 \supset G_7 \supset G_8 \supset G_9 \supset 1$, where $G = (G/G_1)(G_1/G_2) \cdots (G_9/1) = C_2C_3C_2C_2C_2C_2C_2C_2C_2C_2$. We must determine the extension problems associated with G. We first find that $|Z(G)| = 2$, and this centre is equal to $G_9/1$. So we have a central extension of order 2 and can factor G by this centre. After factoring by the centre of G, we obtain a remainder which we denote as A. So we have $G = 2^{\bullet}?A$, where a question mark represents the unknown extension. However, there is no normal subgroup of G that is a direct product with G_9 to equal G. Hence, our central extension is a semi-direct product with the quotient A and we obtain $G = 2^{\bullet} : A$.

In a similar fashion to before, we have a composition series of A, which is $A \supset A_1 \supset \cdots \supset A_8 \supset 1$, where $A = (A/A_1)(A_1/A_2)\cdots(A_8/1) = C_2C_3C_2C_2C_2C_2C_2C_2C_2$. We then find $|Z(A)| = 2$ and this centre is equal to $A_8/1$. After factoring A by the centre, we obtain a remainder which we denote as B. As of now, we have $G = 2^{\bullet}?2^{\bullet} : B$, where a question mark represents the unknown extension. There is also is no normal subgroup of A which is a direct product with A_8 to equal A. Hence we obtain $G = 2^{\bullet} : 2^{\bullet} : B$.

The composition series of B is $B \supset B_1 \supset \cdots \supset B_7$ superset 1, where $B = (B/B_1)(B_1/B_2)\cdots(B_7/1) = C_2C_3C_2C_2C_2C_2C_2C_2$. We find that $|Z(B)| = 2$ and this centre is equal to $B_7/1$. After factoring B by the centre, we obtain a remainder which we denote as Q. As of now, we have $G = 2^{\bullet}?2^{\bullet} : 2^{\bullet} : Q$, , where a question mark represents the unknown extension. There is no normal subgroup of B which is a direct product with B_7 which gives us B. So we now have $G = 2^{\bullet} \times 2^{\bullet} : 2^{\bullet} : Q$.

However, we should note that $(G_7/G8)(G_8/G_9)(G_9/1)$ is abelian. This implies that our three centres can be written as Z_8. So our extension is really $G = Z_8{}^{\bullet}Q$.

We will first determine the isomorphism type of Q before we investigate the central element Z_8. The composition series of Q is $Q \supset Q_1 \supset \cdots \supset Q_6 \supset 1$, where $Q = (Q/Q_1)(Q_1/Q_2)\cdots(Q_6/1) = C_2C_3C_2C_2C_2C_2C_2$. We find that $|Z(Q)| = 1$, so Q does not have a central extension. Since the minimal normal subgroup of Q is of order 4 and there is a normal subgroup of order 16, we should check if
$(Q_3/Q_4)(Q_4/Q_5)(Q_5/Q_6)(Q_6/1) = C_2C_2C_2C_2$ is a direct product of two groups of order 4. This is indeed the case, so we obtain $Q_3 = Z_4 \times Z_4$.

We continue to the next level of our composition series and find $C_2 = (Q_2/Q_3) = Q_2/(Z_4 \times Z_4)$, which implies $Q_2 = C_2?(4 \times 4)$, where a question mark represents the unknown extension. However, we know that there are no normal subgroups of Q of order 2, so this extension must be a semi-direct product. So we now have $Q_2 = C_2 : (Z_4 \times Z_4)$.

We continue to $C_3 = Q_1/Q_2 = Q_1/[C_2 : (Z_4 \times Z_4)]$. So $Q_1 = C_3?[C_2 :$

$(Z_4 \times Z_4)]$, where a question mark represents the unknown extension. Since Q does not have a normal subgroup of order 3, we know this extension must be a semi-direct product. So $Q_1 = C_3 : [C_2 : (Z_4 \times Z_4)]$.

Finally, we arrive at $C_2 = Q/Q_1 = Q/[C_3 : C_2 : Z_4 \times Z_4)]$. So $Q = C_2?[C_3 : C_2 : (Z_4 \times Z_4)]$, where a question mark represents the unknown extension. But, as before, Q has no normal subgroup of order 2, so this extension is a semi-direct product. So we obtain $Q = [C_2 : C_3 : C_2 : (Z_4 \times Z_4)]$.

The presentation for Q is given by $< i, j, k, l, m | i^4, j^4, k^2, l^3, m^2, \ (i, j), i^k = i^{-1}, j^k = j^{-1}, i^l = j, j^l = i^{-1}j^{-1}, k^l = k, i^m = i^{-1}, j^m = ij, k^m = k, l^m = l^{-1} >$.

Now consider $C_2 : C_3 : C_2$. Our presentation of this semi direct product is given by $G < a, b, c >:= Group < a, b, c | a^2, b^3, c^2, a^b = a, a^c = a, b^c = b^{-1} >$. However, this presentation is that of the Dihedral Group 12, D_{12}. Thus, we can rewrite the presentation of $D_{12} \cong C_2 : C_3 : C_2$ as $G < a, b >:= Group < a, b | a^6, b^2, (ab)^2 >$. So we obtain $Q \cong D_{12} : (Z_4 \times Z_4) \cong < j, k, l >:= Group < j, k, l | j^6, k^2, l^2, (jk)^2, (jl)^8, (kl)^2, (jkl)^3 >$.

Now G is a mixed extension of the cyclic group Z_8 by Q. The usual treatment of this is as follows. Our group G has a normal subgroup Z_8 and a quotient group Q. Regard Q as the group of the cosets of G. Pick a "factor set": a representative from each coset, with the one from the identity of the quotient group being the identity of G. Then find a map from $Q \times Q$ to Z_8 such that the factor set is "compatible", i.e. everything fits together correctly. We accomplish this by inserting the MAGMA code below.

```
Let T:=Transversal(G1,NL[6]);
T2:=T[2];
T3:=T[3];
T4:=T[4];
D:=T2*T3*T4;
```

We note that $|D| = 6$ and $|ND| = 3$. Thus $ND^3 = N$. So $D^3 \in N$ and we readily found that $D^3 = a^3$, where a is a generator of Z_8. Also, the action of the generators of Q (as automorphisms of Z_8) on a needs to be determined. We insert an element i of order 8 and determine how j,k, and l act on i . We then determine that $i^j = i^{-1}$, $i^k = i$, and $i^l = i^{-1}$. The presentation of $G = Z_8{}^{\bullet} : [D_{12} : (Z_4 \times Z_4)]$ is $G < i, j, k, l >:= Group < i, j, k, l | i^8, j^6, k^2, l^2, (jk)^2, (jl)^8, (kl)^2, (jkl)^3 = i^3, i^j = i^{-1}, i^k = i, i^l = i^{-1} >$.

Chapter 5

Methods of Finding Progenitors

5.1 Common Finite Groups

Consider the group $N = D_{12}$, or Dihedral Group 12 of the form D_n with $n = 12$. The permutation representation of D_{12} on the minimal number of generators is given by $X = (1, 2, 3, 4, 5, 6)$ and $Y = (1, 5)(2, 4)$. The presentation for any dihedral group is of the form $D_n =< x, y|x^{\frac{n}{2}}, y^2, (xy)^2 >$. We then wish to introduce an element t to N to create our infinite progenitor.

The elements of D_{12} are given by:
$D_{12} = \{Id(N), (1, 5, 3)(2, 6, 4), (1, 2)(3, 6)(4, 5), (1, 2, 3, 4, 5, 6), (1, 5)(2, 4), (1, 3)(4, 6), (1, 6, 5, 4, 3, 2), (1, 4)(2, 3)(5, 6), (1, 6)(2, 5)(3, 4), (1, 3, 5)(2, 4, 6), (1, 4)(2, 5)(3, 6), (2, 6)(3, 5)\}$.

We let $t \sim t_6$. It is then clear that only $Y = (1, 5)(2, 4)$ fixes t pointwise. When constructing a progenitor, you must label which elements of N commute with your t. By saying t and y commute with one another, we are saying that $ty = yt$. We can use a shorthand notation in our presentation and insert (t, y) to imply t commutes with y. We will allow our progenitor to have t_i's over order 2. Hence our infinite progenitor is given by the following:

$$2^{*6} : D_{12} =< x, y, t|x^6, y^2, (xy)^2, t^2, (t, y) >.$$

Elements of this progenitor are simply a product of a, b, t_1, t_2, $\ldots$, t_6. As of now, we know the order of each t_i is 2, but we are unable to collapse multiple t_i's multiplied together. If we can find suitable relations that are able to breakdown the product of multiple t_i's, we can obtain finite homomorphic images of our progenitor $2^{*6} : D_{12}$.

One should first start by determining all possible first ordered relations. These relations appear as relations with an element of N multiplied by a single t_i. Adjusting the order of this first ordered relation will yield different groups. Let us first start by listing some of these first ordered relations:

$(xt)^i, (x^2t)^j, (x^3t)^k, (x^4t)^l, (x^5t)^m, (yt)^n, (xyt)^o, (x^2yt)^p, (x^3yt)^q, (x^4yt)^r, (x^5yt)^s$.

We will omit multiplying t_i's by the identity because there will be a trivial solution each time. Notice above, we have only calculated first ordered relations that have $t = t_6$ as our t_i. Therefore, we should also have 11×5 more relations to list utilizing the other t_i's. Most of these relations will actually be repeats of one another because they are in the same class or have t_i's in the same orbit of one another.

By entering the $Classes(N)$; command, MAGMA determines the different conjugacy classes of D_{12}. For instance, [2] has length 1, which implies there is only one representative in class [2]. In [3], there are 3 representatives of order two that fall under the same conjugacy class. In every level, a representative is given.

```
> Classes(N);
Conjugacy Classes of group N
----------------------------
[1]     Order 1       Length 1
        Rep Id(N)

[2]     Order 2       Length 1
        Rep (1, 4)(2, 5)(3, 6)

[3]     Order 2       Length 3
        Rep (1, 5)(2, 4)

[4]     Order 2       Length 3
        Rep (1, 6)(2, 5)(3, 4)

[5]     Order 3       Length 2
        Rep (1, 3, 5)(2, 4, 6)

[6]     Order 6       Length 2
        Rep (1, 2, 3, 4, 5, 6)
```

So this implies that of the 12 elements of D_{12}, half of the elements are repeats. Because this group is so small in size, it is very easy to determine which element of D_{12} is terms of our generators x and y. In larger progenitors, we will use the Schreier system cide in MAGMA to determine what these permutations are in terms of our generators of N. We then determine the centralizers of each class representative to find which t_i's are in the same orbit. In doing this, we are narrowing down the number of relations even further. For instance, in [2] we can take the representative $(1,4)(2,5)(3,6)$ and multiply it to any of our 6 t_i's. We would like to write our relations as $(\omega t_i)^k$, for some $\omega \in N$ and for some $k \in \mathbb{N}$. However, we will find that if we take the centralizer of a class representative, t_i's that are in the same orbits will actually be repeats of one another.

```
> C2:=Centraliser(N,N!(1, 4)(2, 5)(3, 6));
> Orbits(C2);
[
    GSet{@ 1, 2, 5, 3, 4, 6 @}
]
>
> C3:=Centraliser(N,N!(1, 5)(2, 4));
> Orbits(C3);
[
    GSet{@ 3, 6 @},
    GSet{@ 1, 5, 4, 2 @}
]
>
> C4:=Centraliser(N,N!(1, 6)(2, 5)(3, 4));
> Orbits(C4);
[
    GSet{@ 2, 5 @},
    GSet{@ 1, 6, 4, 3 @}
]
>
> C5:=Centraliser(N,N!(1, 3, 5)(2, 4, 6));
> Orbits(C5);
[
    GSet{@ 1, 3, 4, 5, 6, 2 @}
]
>
> C6:=Centraliser(N,N!(1, 2, 3, 4, 5, 6));
> Orbits(C6);
```

```
[
    GSet{@ 1, 2, 3, 4, 5, 6 @}
```

Table 5.1: Conjugacy Classes of D_{12}

Class	Class Representative	# of Elements	Orbits
C_1	e	1	$\{1,2,3,4,5,6\}$
C_2	$x^3 = (1,4)(2,5)(3,6)$	1	$\{1,2,3,4,5,6\}$
C_3	$y = (1,5)(2,4)$	3	$\{1,2,4,5\}$, $\{3,6\}$
C_4	$yx = (1,6)(2,5)(3,4)$	3	$\{1,3,4,6\}$, $\{2,5\}$
C_4	$x^2 = (1,3,5)(2,4,6)$	2	$\{1,2,3,4,5,6\}$
C_4	$x = (1,2,3,4,5,6)$	2	$\{1,2,3,4,5,6\}$

Since $x^3 = (1,4)(2,5)(3,6)$, the first ordered relation $(x^3t_0)^k$ will be the one distinct relation we should use for C_2. Using $(x^3t_0)^k$ and $(x^3t_1)^k$ will be redundant.

So we continue this pattern by taking a representative from a class and multiplying it by each by a t_i in each orbit, and we obtain all first ordered relations D_{12}.

$(x^3t)^i$, $(yt)^j$, $(yt^a)^k$, $(yxt)^l$, $(yxt^{a^2})^m$, $(x^2t)^n$, $(xt)^o$.

So we factor $2^{*6} : D_{12}$ by the first ordered relations and a few other relations and obtain:

$G < x, y, t | x^6, y^2, (xy)^2, t^2, (t, y), (x^3t)^i, (yxt)^j, (yxt^{x^2})^k, (x^2t)^l, (xt)^m, (xtt^x)^n, (xyt^xt)^o >$.

A table is provided below to show some of the homomorphic images found.

Table 5.2: D_{12} Progenitor Table

D_{12} Progenitor Table								
i	j	k	l	m	n	o	Order of G	Shape of G
3	5	0	0	7	0	0	5040	S_7
0	0	0	0	5	3	0	1320	$L(2,11) \times 2$
3	0	6	10	5	0	0	249600	$[2 : U(3,4)] \times 2$
3	0	0	0	0	3	0	6552	$L(2,13) \times S_3$
3	0	0	4	0	0	0	720	$S_5 \times S_3$
0	0	0	0	0	7	3	2184	$PGL(2,13)$
3	6	0	0	7	0	0	241920	$L(3,4) : (2 \times S_3)$

This is the conventional way of finding progenitors. Most of the small groups and simple groups have been investigated thoroughly already. We must then find progenitors that have not been worked on by other means. Fortunately, as long as we can create presentation of a group, we are able to use MAGMA to find a permutation representation of that group on a minimal number of involutions.

5.2 Group Extension Progenitors

Consider the group $S_3 \times S_3$. With our knowledge of group presentations and direct products, a presentation for this group should be two images of S_3 such that their generators commute with one another. Our presentation is given by:

$$G = < a, b, c, d | a^3, b^2, (ab)^2, c^3, d^2, (cd)^2, (a, c), (a, d), (b, c), (b, d) >.$$

Now we must find a permutation representation of this group so we can form a progenitor. We can use a few MAGMA commands to form this permutation representation.

f,G1,k:=CosetAction(G,sub$< G|Id(G) >$); Creates an image of G.

SL:=Subgroups(G1); Finds all subgroups of G.

T:=X‘subgroup:X in SL; Gathers all subgroups found in SL.

TrivCore:= H:H in T|#Core(G1,H) eq 1; Determines faithful permutation representations of G.

mdeg:=Min(Index(G1,H):H in TrivCore); Gives permutation representations with the least number of letters.

Good:=H:H in TrivCore— Index(G1,H) eq mdeg; Determines how many faithful permutation representations have a minimal number of letters.

H := Rep(Good); Picks a representative from Good.

f,G1,K := CosetAction(G1,H); Creates a permutation representation of the chosen representative from Good.

```
>G<a,b,c,d>:=Group<a,b,c,d|a^3,b^2,(ab)^2,c^3,d^2,(cd)^2,(a,c),

(a,d),(b,c),(b,d)>;

>f,G1,k:=CosetAction(G,sub<G| Id( G )>));

>SL:=Subgroups(G1);

>T:={X`subgroup: X in SL};
```

```
>TrivCore:={ H : H in T | \#Core(G1,H) eq 1};

>mdeg:=Min({Index(G1,H):H in TrivCore});

>Good:={H: H in TrivCore $|$ Index(G1,H) eq mdeg};

>H:=Rep(Good);

>f,G1,K:=CosetAction(G1,H);

>G1;

Permutation group G1 acting on a set of cardinality 6

Order = 36 = 2^2 * 3^2

    (1, 2, 4)(3, 6, 5)

    (1, 3)(2, 5)(4, 6)

    (1, 4, 2)(3, 6, 5)

    (1, 3)(2, 6)(4, 5)
```

So we label $a = (1,2,4)(3,6,5)$, $b = (1,3)(2,5)(4,6)$, $c = (1,4,2)(3,6,5)$, and $d = (1,3)(2,6)(4,5)$.

Since we have a permutation representation of our G, we must now introduce a new element, t, to form our progenitor. Letting our $t \sim t_6$, we can use the MAGMA command Stabiliser(N,6) to find which elements of N fix t.

We find N0:=Stabiliser(N,6) $=< (1,2,4),(2,4)(3,5) >$. Since it is not obvious which elements $(1,2,4)$ and $(2,4)(3,5)$ are, we can utilize the Schreier System in MAGMA. The Schreier System allows us to insert generators of a group and receieve what that permutation is terms of the generators you inserted. Since we have a, b, c, and d as our generators, we give the MAGMA output of what each permutation is in terms of these generators.

```
>S:=Sym(6);
>A:=S!(1, 2, 4)(3, 6, 5);
```

```
>B:=S!(1, 3)(2, 5)(4, 6);
>C:=S!(1, 4, 2)(3, 6, 5);
>D:=S!(1, 3)(2, 6)(4, 5);
>N:=sub<S|A,B,C,D>;
>NN:=G<a,b,c,d>:=Group<a,b,c,d|a^3,b^2,(a*b)^2,c^3,d^2,(c*d)^2,
>(a,c),(a,d),(b,c),(b,d)>;
>
>Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
>ArrayP:=[Id(N): i in [1..36]];
>for i in [2..36] do
>P:=[Id(N): l in [1..#Sch[i]]];
>for j in [1..#Sch[i]] do
>if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
>if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^{-1}; end if;
>if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
>if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
>if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^{-1}; end if;
>if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
>end for;
>PP:=Id(N);
>for k in [1..#P] do
>PP:=PP*P[k]; end for;
>ArrayP[i]:=PP;
>end for;
>
>for i in [1..36] do if ArrayP[i] eq N!(1, 2, 4)
> then print Sch[i]; end if; end for;
>c*a^-1
>for i in [1..36] do if ArrayP[i] eq N!(2, 4)(3, 5)
> then print Sch[i];  end if; end for;
>b*a*d*c
```

So we obtain that $(1,2,4) = ca^{-1}$ and $(2,4)(3,5) = badc$. Our infinite progenitor is given by the following:

$G := < a,b,c,d,t | a^3, b^2, (ab)^2, c^3, d^2, (cd)^2, (a,c), (a,d), (b,c), (b,d), t^2, (t, ca^{-1}), (t, badc) >$.

We must now factor our group by relations to create homomorphic images of $2^{*6} : (S_3 \times S_3)$. Utilizing some of the first ordered relations and a few seperate relations, we will let our group be factored by the following:

$G := < a,b,c,d,t | a^3, b^2, (ab)^2, c^3, d^2, (cd)^2, (a,c), (a,d), (b,c), (b,d), t^2, (t, ca^{-1}), (t, badc),$
$(a^2c^2t)^i, (bct)^j, (tt^{ba})^k = badc, (t(t^c)^a)^l = abc^{-1}d^{-1}, (bt)^m, (cd^2t^a)^n, (at)^o, (adt)^p >$.

A table is provided below to show some of the homomorphic images found.

Table 5.3: $S_3 \times S_3$ Progenitor Table

$S_3 \times S_3$ Progenitor Table									
i	j	k	l	m	n	o	p	Order of G	Shape of G
3	11	0	0	0	0	0	0	190080	$2 : M_{12}$
3	0	4	0	16	0	0	0	3753792	$2 : L(3, 7)$
0	0	0	0	0	0	0	0	241920	$L(3, 4) : D_{12}$
0	0	0	0	0	0	0	0	483840	$S_3 : L(3, 4) \times 2^2$
0	0	2	0	0	0	3	0	5040	S_7

5.3 MAGMA Database Progenitors

5.3.1 Some MAGMA Databases

There are databases that are accessible and stored inside MAGMA. Utilizing these databases, we can work with groups of specific orders and have the option to pick groups with special properties such as transitivity and primitivity.

The **SmallGroupDatabase** is a collection of small groups of order less than or equal to 2000, excluding a few. In order to access groups in this database, we simply insert **D:=SmallGroupDatabase();**. We can then choose a group of a specific order to work with.

There are **TransitiveGroup()** and **PrimitiveGroup()** commands which allow us to choose groups with each respective property. We can ask MAGMA **NumberOfTransitiveGroups(n)** or **NumberOfPrimitiveGroups(n)**, where n represents the number of involutions you wish to have.

For example, we ask MAGMA the following:

```
> NumberOfTransitiveGroups(8);
```

and learn that there are 50 different transitive groups generated by permutations on 8 letters. Once we determine which transitive group we wish to work with in the database, we must label it and determine which group MAGMA has stored it as in the SmallGroupDatabase.

We will use N = TransitiveGroup(8,23) as an example. We input the code below in MAGMA and receieve the following output:

```
> D:=SmallGroupDatabase();
> N:=TransitiveGroup(8,23);
> IdentifyGroup(N);
<48, 29>.
```

This tells us that MAGMA stores TransitiveGroup(8,23) as SmallGroup(D,48,29) in the SmallGroupDatabase. 48 represents the number of elements in N and 29 represents the 29th group of order 48.

As of now, we have neither a permutation representation or even a presentation of our group which we will label G. However, we can use the command **FPGroup(G);** to form a presentation for G.

We use the FPGroup command in MAGMA below to determine a presentation for G.

```
> FPGroup(G);
Finitely presented group on 5 generators
Relations
    $.1^2 = Id($)
    $.2^3 = Id($)
    $.3^2 = $.5
    $.4^2 = $.5
    $.5^2 = Id($)
    $.2^$.1 = $.2^2
    $.3^$.1 = $.4
    $.3^$.2 = $.4 * $.5
    $.4^$.1 = $.3
    $.4^$.2 = $.3 * $.4
    $.4^$.3 = $.4 * $.5
    $.5^$.1 = $.5
    $.5^$.2 = $.5
    $.5^$.3 = $.5
    $.5^$.4 = $.5
Mapping from: GrpFP to GrpPC: G
```

Translating this into a presentation, we obtain the following:
$G =< a, b, c, d, e | a^2, b^3, c^2 = e, d^2 = e, e^2, b^a = b^2, c^a = d, c^b = de, d^a = c, d^b = cd, d^c = de, e^a = e, e^b = e, e^c = e, e^d = e >$.

Now that we have a presentation for G, we are able to use the **TrivCore, mdeg,** and **Good** commands as we had before to create a permutation representation of G. We are then able to form a progenitor by choosing a t_i as t in our progenitor, and showing what elements generate the stabilising group of t.

In this progenitor, we let $t \sim t_8$. Furthermore, the elements that fix t are a and bd^{-1}.

Utilizing some of the first ordered relations of G and a few seperate relations, we let our group be factored by the following:
$G =< a, b, c, d, e, t | a^2, b^3, c^2 = e, d^2 = e, e^2, b^a = b^2, c^a = d, c^b = de, d^a = c,$
$d^b = cd, d^c = de, e^a = e, e^b = e, e^c = e, e^d = e, t^2, (t, a), (t, bd^{-1}),$
$(et)^i, (act)^j, (b^{-1}at)^k, (bct)^l, (ac^{-1}t)^m, (at^c)^n, (ct)^o, (cb^{-1}t)^p >.$

A table is provided below to show some of the homomorphic images found.

Table 5.4: SmallGroup(D,48,29) $\cong 2^{\bullet}S_4$ Progenitor Table

SmallGroup(D,48,29) $\cong 2^{\bullet}S_4$ Progenitor Table									
i	j	k	l	m	n	o	p	Order of G	Shape of G
3	6	0	0	0	0	0	0	33696	$L(3,3) : S_3$
3	0	0	0	0	0	6	6	190080	$M_{11} \times S_4$
0	6	0	0	0	8	0	0	2016	$PGL(2,7) \times S_3$
0	6	0	0	0	0	7	0	120960	$L(3,4) : S_3$
0	0	0	4	0	0	6	0	240	$2^{\bullet}S_5$
3	0	0	8	6	0	0	0	11232	$L(3,3) : 2$
3	0	5	0	0	0	0	0	7920	M_{11}

5.3.2 A Few Tables of Database Progenitors

We let our N = SmallGroup(16,8) $= 2^{\bullet}D_4$, $A = (1,2,4,7)(3,5,8,6)$,
$B = (2,5)(3,8)(6,7)$, $C = (1,3,4,8)(2,6,7,5)$ and $D = (1,4)(2,7)(3,8)(5,6)$ where $N =< A, B, C, D >$. Letting $t \sim t_8$, we factor N by the following relations:
$G =< a, b, c, d, t | a^2 = d, b^2, c^2 = d, d^2, b^a = bc, c^a = cd, c^b = cd, (d, a), (d, b), (d, c), t^2,$
$(t, bd), (bc^{-1}t)^i, (ct)^j, (at)^k, (dt)^l, (a^{-1}bt)^m, (bact)^n >.$

Table 5.5: SmallGroup(16,8) $\cong 2^{\bullet}D_4$ Progenitor Table

SmallGroup(16,8) $\cong 2^{\bullet}D_4$ Progenitor Table							
i	j	k	l	m	n	Order of G	Shape of G
3	3	8	0	0	0	720	S_6
3	3	12	0	0	0	15600	$PGL(2,25)$
3	3	13	0	0	0	5616	$L(3,3)$
0	4	7	3	0	6	40320	$L(3,4):2$
3	3	14	0	0	0	56448	$(L(2,7))^2:2$
3	0	7	0	8	0	336	$PGL(2,7)$
3	0	10	0	8	0	1440	$S_6:2$

We let our $N =$ TransitiveGroup(8,27) $= (2\times8)^{\bullet}:4$, $A = (1,2)(3,7)(4,5,8,6)$, $B = (1,3)(2,5)(4,8)(6,7)$, $C = (1,4)(2,6)(3,8)(5,7)$, $D = (4,8)(5,6)$, $E = (2,7)(5,6)$, and $F = (1,3)(2,7)(4,8)(5,6)$ where $N =< A,B,C,D,E,F >$. Letting $t \sim t_8$, we factor N by the following relations:
$G :=< a,b,c,d,e,f,t|a^2 = d, b^2, c^2, d^2, e^2, f^2, b^a = bc, c^a = ce, c^b = c, d^a = d, d^b = de,$
$d^c = df, e^a = ef, e^b = e, e^c = e, e^d = e, f^a = f, f^b = f, f^c = f, f^d = f, f^e = f, t^2,$
$(t,e),(t,bf),(t,df),(cft)^i,(dct)^j,(abt)^k,(cta)^l,(bat)^m,(tab)^n,(bcet)^o,(aca^{-1}t)^p >.$

A table is provided below to show some of the homomorphic images found.

5.4 Progenitors of Sporadic Subgroups

Progenitors are created by introducing an element to an already existing N to form a new group. Furthermore, that new group must have a subgroup N inside it. We should then consider observing subgroups of sporadic groups in hopes of finding new homomorphic images.

We first will investigate M_{11}'s subgroups. M_{11} has the following maximal subgroups: M_{10}, $L(2,11)$, $M_9:2$, S_5, and $2^{\bullet}S_4$. Since symmetric and linear group progenitors are typically studied a lot, we will work examine the maximal subgroup $M_9:2$. We will first analyze the subgroup $M_9 \subset M_9:2$.

Table 5.6: TransitiveGroup(8,27) $\cong (2 \times 8)^{\bullet} : 4)$ Progenitor Table

TransitiveGroup(8,27) $\cong (2 \times 8)^{\bullet} : 4)$									
i	j	k	l	m	n	o	p	Order of G	Shape of G
3	3	9	0	0	0	0	0	4896	$PGL(2,17)$
3	0	0	9	0	0	0	0	6840	$PGL(2,19)$
3	0	0	11	11	0	0	0	6072	$L(2,23)$
3	0	0	13	10	0	0	0	15600	$PGL(2,25)$
3	0	0	17	10	0	0	0	8160	$PGL(2,16)$
0	0	0	13	0	6	5	0	124800	$U(3,4):2$
3	0	0	12	13	0	0	3	11232	$L(3,3):2$
3	3	6	0	0	0	0	0	240	$2^{\bullet}S_5$
3	3	7	0	0	0	0	0	336	$PGL(2,7)$
0	0	5	0	0	0	0	5	720	S_6

M_9 is saved in the MAGMA database as SmallGroup(D,72,41). Following the methods we used before, we can find a permutation representation and a presentation using MAGMA commands. Since we are pursuing M_{11}, it would be very useful to determine if it is even possible to find it as a homomorphic image of $2^{*9} : M_9$. A presentation for M_9 is given by:

$N =< a, b, c, d, e | a^2 = c, b^2 = c, c^2, d^3, e^3, b^a = bc, c^a = c, c^b = c, d^a = de^2,$
$d^b = e, d^c = d^2, e^a = d^2e^2, e^b = d^2, e^c = e^2, e^d = e >.$

Now that we have a presentation, we can find a permutation representation of M_9 by asking MAGMA. Afterwards, we can form our progenitor. Letting $t \sim t_9$, we obtain the following infinite progenitor:

$P =< a, b, c, d, e, t | a^2 = c, b^2 = c, c^2, d^3, e^3, b^a = bc, c^a = c, c^b = c, d^a = de^2,$
$d^b = e, d^c = d^2, e^a = d^2e^2, e^b = d^2, e^c = e^2, e^d = e, t^2, (t, e^{-1}a), (t, abd^{-1}) >$

Although we can begin by adding relations to this progenitor, there exists a MAGMA program (See [Why06]) that computes all homomorphic images of all almost simple groups. We run the code below in MAGMA.

```
> P<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^2=c,b^2=c,c^2,d^3,e^3,
b^a=b*c,c^a=c,c^b=c, d^a=d*e^2,d^b=e,d^c=d^2,e^a=d^2*e^2,
e^b=d^2,e^c=e^2,e^d=e,t^2,(t,e^-1*a),(t,a*b*d^-1)>;
>
> D:=AlmostSimpleGroupDatabase();
> for i in [1..#D] do
for> G1:=GroupData(D,i)`permrep;
for>  sg:=GroupData(D,i)`subgens;
```

```
for>  if #sg eq 0 then
for|if>  G:=sub<G1|G1.1,G1.2>;
for|if>  else
for|if>  F:=Parent(sg[1]);
for|if> t:=Ngens(G1)-2;
for|if>  phi:= hom<F -> G1 |
for|if> [G1.(i+2) : i in [1..t]] cat [Id(G1) : i in [t+1..\
Ngens(F)]]>;
for|if>  G:= sub <G1 | G1.1, G1.2, [phi(s): s in sg]>;
for|if> end if;
for> if #Homomorphisms(P,G: Limit:=1) gt 0 then GroupData(\
D,i)`name; end if;
for> end for;
```

The AlmostSimpleGroupDatabase contains groups G where $S \leq G \leq Aut(S)$ where S is simple. Groups of this database are those of order less than 16000000, as well as M_{24}, HS, J_3, McL, $Sz(32)$, and $L(6,2)$. The only almost simple group that we can obtain on this progenitor is $PSL(3,4)$. The $L(3,4)$ progenitor is given below after being factored by a few relations.

```
> G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^2=c,b^2=c,c^2,d^3,e^3,
b^a=b*c,c^a=c,c^b=c,d^a=d*e^2,d^b=e,d^c=d^2,e^a=d^2*e^2,
e^b=d^2,e^c=e^2,e^d=e,t^2,(t,e^-1*a),(t,a*b*d^-1),
(c*e^-1*t)^5,  (b*t)^7, (b*d*a*t)^7>;
>
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> CompositionFactors(G1);
    G
    |  A(2, 4)                     = L(3, 4)
    1
```

Since we do not see M_{11} as a possible homomorphic image of M_9, we should suspect $M_9 : 2$ should not have M_{11} as a homomorphic image either. This is the case. So our next aim is to find progenitors of special subgroups of simple groups that have the capabilities of generating those same sporadic groups.

5.5 Progenitors of Specific Sporadic Subgroups

MAGMA stores many sporadic groups which are accessible to any user. Consider the Mathieu sporadic group, M_{22}. To load this group in MAGMA, we type:

load m22;

and MAGMA labels our group as G. By asking MAGMA for G, it gives a permutation representation of M_{22}.

We wish to find an element c of order 2 and a subgroup $H \leq G$, such that $< c, H >= G$, which implies $< c^H >= G$. We can then find a faithful permutation representation of H on n letters, where $n = |c^H|$. Equivalently, n is the quotient of the number elements in H and the number of elements in the centraliser of c in H.

For example, let aa,bb,cc be the permutation representation of $G = M_{22}$. We then take an element $c \in G$ and a subgroup $H =< dd, ee, ff, hh >$ and find $c^H = G$. The MAGMA code below expresses this.

```
> S:=Sym(22);
> aa:=S!(1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)
(5, 10, 20, 17, 11,22, 21, 19, 15, 7, 14);
> bb:=S!(1, 18, 4, 2, 6)(5, 21, 20, 10, 7)(8, 16, 13, 9, 12)
(11,19, 22, 14, 17);
> cc:=S!(1, 18, 2, 4)(3, 15)(5, 9)(7, 16, 21, 8)
(10, 12, 20, 13)(11, 17, 22, 14);
> m22:=sub<S|aa,bb,cc>;
> G:=m22;
> c:=G!(1, 16)(3, 8)(5, 10)(6, 11)(7, 17)(9, 21)
(13, 22)(18, 20);
>
> dd:=G!(2, 17, 15, 11)(3, 19, 6, 12)(4, 21)(5, 13, 16, 8)
(7, 9, 22, 20)(14, 18);
> ee:=G!(2, 11, 20, 22)(3, 5, 13, 12)(4, 18)(6, 16, 8, 19)
(7, 15,17, 9)(14, 21);
> ff:=G!(2, 20)(3, 13)(5, 12)(6, 8)(7, 17)(9, 15)
(11, 22)(16, 19);
> hh:=G!(2, 15)(3, 6)(5, 16)(7, 22)(8, 13)(9, 20)
(11, 17)(12, 19);
> HH:=sub<G|dd,ee,ff,hh>;
21
> #Centraliser(HH,c);
3
> #(c^HH);
7
> #Conjugates(HH,c);
7
> G eq sub<G|c^HH>;
```

```
true
```

So we find that M_{22} is a homomorphic image of $2^{*7} : N$, where N is a transitive subgroup of S_7 with order 21. This result lead us to the discovery of the M_{22} simple group on the progenitor $2^{*7} : [7 : 3]$.

Chapter 6

Other Notable Progenitors Discovered

Some of the progenitors investigated yielded very few interesting homomorphic images. However, many Mathieu Group M_{12} automorphism groups and Symplectic groups were found on these progenitors. Rather than making a table for one group found on a specific progenitor, we will list the progenitor with relations used.

6.1 Non-Simple Mathieu Group M_{12} Groups

6.1.1 $M_{12}:2$

Letting our progenitor be $N = \text{TransitiveGroup(8,30)} = (2^3 : 2) :^{\bullet} 4$ and $t \sim t_8$, we obtain the group below.

```
> G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2=d,b^2,
> c^2=f,d^2,e^2,f^2,b^a=b*c,c^a=c*e,c^b=c*f,d^a=d,
> d^b=d*e*f,d^c=d*f,e^a=e*f,e^b=e,e^c=e,e^d=e,
> f^a=f,f^b=f,f^c=f,f^d=f,f^e=f, t^2,(t,d*f),(t,b*d),
> (t*t^c)^3=b*d, (c*t*a)^6, (t*a*b)^11=d>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d,e,f>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
```

```
    |  M12
    1
> Center(G1);
Permutation group acting on a set of cardinality 23760
Order = 1
```

6.1.2 $M_{12}:2$

Letting our progenitor be $N = 3^{\bullet}A_4$ and our $t \sim t_{12}$, we obtain the group below.

```
> G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^2,b^2,c^2,d^2,e^3,
> (a*b)^2,(a*c)^2,(b*c)^2,(b*d)^2,(c*d)^2,d*e^-1*b*e,
> e^-1*b*a*e*a,e^-1*d*c*e*c,t^2,(t,c),(t,a*b),
> (b*c*t)^0, (e*t)^5, (a*c*d*t)^3>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d,e>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  M12
    1
> Center(G1);
Permutation group acting on a set of cardinality 3960
Order = 1
```

6.1.3 $2^{\bullet}(M_{12}:2)$

Letting our N = TransitiveGroup(8,35); $= 2^{\bullet}(2^4 : 2^2)$ and our $t \sim t_8$, we obtain the group below.

```
> G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^3,
> e^2,f^2,b^a=b*c,c^a=c,c^b=c,d^a=d^2,d^b=d,d^c=d,e^a=f,
> e^b=e,e^c=e,e^d=f,f^a=e,f^b=f,f^c=f,f^c=f,f^d=e*f,f^e=f,
> t^2,(t,a*d),(t,a*b*d*e),(a*c*d^-1*t)^5, (b*a*t)^6>;
>
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  M12
    *
```

```
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 3960
Order = 2
```

6.1.4 $2^{\bullet}(M_{12}:2)$

Letting our N = TransitiveGroup(12,52) $= 2^4 : S_3$ and our $t \sim t_{12}$, we obtain the group below.

```
> G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^3,e^2,
> f^2, b^a=b*c,c^a=c,c^b=c,d^a=d^2,d^b=d,d^c=d,e^a=f,e^b=e,
> e^c=e,e^d=f,f^a=e,f^b=f,f^c=f,f^c=f,f^d=e*f,f^e=f,t^2,
> (t,a*d),(t,a*b*d*e),(a*c*d^-1*t)^5, (b*a*t)^6>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d,e,f>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  M12
    *
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 3960
Order = 2
```

6.1.5 $(2^{\bullet}M_{12}):A_4$

Letting our N =TransitiveGroup(8,32) $= 2^{\bullet}(4^2 : 3)$ and our $t \sim t_8$, we obtain the group below.

```
> G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^3,b^2,c^2,d^2,
> e^2,f^2, b^a=c, c^a=b*c,d^a=e,d^b=d,d^c=d*f, e^a=d*e,
> e^b=e*f, (a,f),(b,f),(c,f),(d,f),(e,f), t^2,(t,a^-1*e*b),
> (t,e*f), (t,d*e), (b*d*c*e*t)^5, (c*t*a)^6, (c*b*d*t)^6>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d,e>);
> CompositionFactors(G1);
    G
    |  M12
```

```
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 23760
Order = 2
```

6.2 Sporadic Simple Groups

6.2.1 M_{12}

Letting our $N = C_{11}$ and our $t \sim t_{11}$, we obtain the group below.

```
> G<a,t>:=Group<a,t|a^11,t^2,
> (a*t)^6, (a^4*t)^3, (a^4*t*a)^6>;
>
> f,G1,k:=CosetAction(G,sub<G|a>);
> CompositionFactors(G1);
    G
    |  M12
    1
> Center(G1);
Permutation group acting on a set of cardinality 8640
Order = 1
```

6.2.2 J_2

Letting our $N = A_5 \times 2$ and our $t \sim t_{10}$, we obtain the group below.

```
> G<a,b,c,t>:=Group<a,b,c,t|a^2,b^3,(a*b)^5,c^2,(a,c),(b,c),
> t^2,(t,a),(t,b^-1*a*b^-1*a*b*a),(c*t)^3,(a*b*t)^5,
> (a*b*c*t)^12>;
> f,G1,k:=CosetAction(G,sub<G|a,b,c>);
> CompositionFactors(G1);
    G
    |  J2
```

```
    1
```

6.3 Non-Sporadic Findings

6.3.1 $8^{\bullet}L(3,4)$

Letting our $N = 2 \times 4$ and $t \sim t_8$, we obtain the group below.

```
> G<a,b,t>:=Group<a,b,t|a^4,b^2,(a,b),t^2,
> (a*t)^5, (b*a*t)^7, (a*b*a*t)^3>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b>);
> CompositionFactors(G1);
    G
    |  A(2, 4)                    = L(3, 4)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 20160
Order = 8 = 2^3
```

6.3.2 $4^{\bullet}S(4,3)$

Letting our $N = 2 \times 3 \times 2$ and our $t \sim t_{12}$, we obtain the group below.

```
> G<a,b,c,t>:=Group<a,b,c,t|a^2,b^3,c^2,(a,b),(a,c),(b,c),
> t^2,(b*a*t)^4, (c*a*t)^4, (b*t)^3, (a*t)^6, (c*t)^4>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c>);
> CompositionFactors(G1);
    G
    |  C(2, 3)                    = S(4, 3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

```
> Center(G1);
Permutation group acting on a set of cardinality 8640
Order = 4 = 2^2
```

6.3.3 $S(4,5)$

Letting our $N = 6^{\bullet}2^2$ and our $t \sim t_{12}$, we obtain the group below.

```
> G<a,b,c,d,t>:=Group<a,b,c,d,t|a^2,b^2,c^3,d^2,b^a=b*d,
> c^a=c,c^b=c,(d,a),(d,b),(d,c),t^2,(t,b*d),
> (d*t)^5, (d*b*c*a*t)^5, (c*a*t)^13>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d>);
> CompositionFactors(G1);
    G
    |  C(2, 5)                       = S(4, 5)
    1
> Center(G1);
Permutation group acting on a set of cardinality 195000
Order = 1
```

6.3.4 $U(3,4):2$

Letting our $N = D_8$ and our $t \sim t_4$, we obtain the group below.

```
> G<a,b,c,t>:=Group<a,b,c,t|a^2,b^2,c^2,b^a=b*c,
> c^a=c,c^b=c,t^2,(t,a*c),(a*t)^5, (b*a*t)^6, (b*c*t)^13>;
>
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  2A(2, 4)                      = U(3, 4)
    1
> Center(G1);
Permutation group acting on a set of cardinality 124800
Order = 1
```

6.3.5 $2^{\bullet}(S(4,3):2)$

Letting our N = TransitiveGroup(12,14) $= 6^{\bullet}2^2$ and our $t \sim t_{12}$, we obtain the group below.

```
> G<a,b,c,d,t>:=Group<a,b,c,d,t|a^2,b^2,c^3,d^2,b^a=b*d,
> c^a=c,c^b=c, d^a=d,d^b=d,d^c=d,t^2,(t,b*d),
> (c*t)^4, (b*c*a*b*t)^5, (d*a*b*t)^4>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c,d>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  C(2, 3)                      = S(4, 3)
    *
    |  Cyclic(2)
    1
>
> Center(G1);
Permutation group acting on a set of cardinality 4320
Order = 2
```

6.3.6 $2^{\bullet}Sz(8)$

Letting our N = PrimitiveGroup(5,3) $= D_{10} : 2$ and our $t \sim t_5$, we obtain the group below.

```
> G<a,b,c,t>:=Group<a,b,c,t|a^2=b,b^2,c^5,b^a=b,
> c^a=c^2,c^b=c^4,t^2,(t,c*a),
> (a*c^-1*a*t)^7, (c*a*b*t)^7>;
>
> f,G1,k:=CosetAction(G,sub<G|a,b,c>);
>
> CompositionFactors(G1);
    G
    |  2B(2, 8)                     = Sz(8)
    *
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 2912
Order = 2
```

Appendix A

MAGMA Code for $L(2,11) \times 2$ DCE

```
/* This code guides a double coset enumeration of G over N.

%-------------------------------------------------

S:=Sym(6);
xx:=S!(1,2,3,4,5,6);
yy:=S!(1,5)(2,4);

G<x,y,t>:=Group<x,y,t|x^6, y^2,(x*y)^2, t^2,(t,y),
(x*t*t^x)^3, (t*t*x*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
sub<N|yy> eq Stabiliser(N,6);

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;
```

```
ts := [Id(G1): i in [1 .. 6] ];
ts[6]:=f(t); ts[1]:=f(t^x); ts[2]:=f(t^(x^2));
ts[3]:=f(t^(x^3)); ts[4]:=f(t^(x^4)); ts[5]:=f(t^(x^5));
cst:=[null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is [Integers() | ];

 for i := 1 to 6 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
m:=0;
for i in [1..110] do if cst[i] ne [] then m:=m+1;
end if; end for; m;


%--------------------------------------------------

N0:=Stabiliser (N,6);

N0s:=N0;
T0:=Transversal(N,N0s);
T0;
for i in [1..#T0] do
ss:=[6]^T0[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N0);

%--------------------------------------------------

N01:=Stabiliser(N0,1);

SSS:={[6,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[6]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01s:=N01;
```

```
T01:=Transversal(N,N01s);
T01;
for i in [1..#T01] do
ss:=[6,1]^T01[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N01);

%--------------------------------------------------

N02:=Stabiliser(N0,2);

SSS:={[6,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[6]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N02s:=N02;

T02:=Transversal(N,N02s);
T02;
for i in [1..#T02] do
ss:=[6,2]^T02[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N02);

%--------------------------------------------------

N03:=Stabiliser(N0,3);

SSS:={[6,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
```

```
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[6]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N03s:=N03;

T03:=Transversal(N,N03s);
T03;
for i in [1..#T03] do
ss:=[6,3]^T03[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N03);

%--------------------------------------------------

/* After inserting the chunk of code for a double
coset,if m increases by a value, the double coset
is new. Your single coset count, m, is increased by
the number of single cosets in the double coset
checked.

/* One should follow this pattern until m = 110, since
the index of our group is |G| / |N| = 1320/12 = 110.


/* Below is an example of what to add for if the loop:

for n in N do
if ts[a]*ts[b]*ts[c] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]] then print Rep(Seqq[i]);
end if; end for; end for;

actually gives equal coset names. In this example, the
double coset [713] = [431]. In this case, we want
all elements in N that send [713] to [431].  If there
were more equal names of [713], we would have to make
N013s include all of those elements in N that send [713]
```

```
to equal names.

%------------------------------------------------

N013:=Stabiliser(N01,3);

SSS:={[6,1,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[6]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]] then print Rep(Seqq[i]);
end if; end for; end for;

for g in N do if 6^g eq 4 and 1^g eq 3 and 3^g eq 1
then N013s:=sub<N|N013s,g>; end if; end for;
#N013s;

T013:=Transversal(N,N013s);
T013;

for i in [1..#T013] do
ss:=[6,1,3]^T013[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1; end if; end for;  m;


... and so on

%------------------------------------------------

/* Once all single cosets have been accounted for,
we must determine which double cosets were equal
to one another.


/* Save equal double cosets up here with
K=ts[7]*ts[1]*ts[7].
```

```
for g in IN do for h in IN do
if ts[6]*ts[4] eq g*(ts[6]*ts[2])^h
then g,h; end if; end for; end for;

for g in IN do for h in IN do
if K eq g*(ts[6]*ts[1])^h
then g,h; end if; end for; end for;



%--------------------------------------------------

/* Change K to a double coset which is a repeat of one
which has already been accounted for.



K:=ts[7]*ts[1]*ts[7];

for g in IN do for h in IN do
if K eq g*(ts[6])^h
then g,h; end if; end for; end for;

for g in IN do for h in IN do
if K eq g*(ts[6]*ts[1])^h
then g,h; end if; end for; end for;

for g in IN do for h in IN do
if K eq g*(ts[6]*ts[2])^h
then g,h; end if; end for; end for;



...

for g in IN do for h in IN do
if K eq g*(ts[6]*ts[1]*ts[2]*ts[3])^h
then g,h; end if; end for; end for;

%--------------------------------------------------

/* If a double coset does not increase m, we should check
```

which double coset that double coset is equal to. It is simple to run as many loops as you have double cosets to check which two are equal. We can label our potentially new double coset as a variable, say K, and check every possible double coset it could be equal to. Note, it can only be equal to one of them. All the other loops ran should give no values. Once all orbits have been accounted for, our group is closed under right multiplication. One should then verify that the Cayley graph works correctly.

Appendix B

MAGMA Code for M_{22} over M DCE

```
/* This code guides a double coset enumeration of
G over M. The process is similar, but we must change one
loop to have M instead of N.

%-------------------------------------------------------

s:=Sym(7);
A:=s!(2,3,4)(5,7,6);
B:=s!(1,2,3,5,4,6,7);
N:=sub<s|A,B>;
G<a,b,t>:=Group<a,b,t|a^3,b^7,b^a=b^2,t^2,(t,a*b),
(a^-1*b^-1*t)^5,(b*a*t^(a^2))^11>;

H:=sub<G|a,b,t^b*t^(b^2)*t^b*t^(b^2)*t*t^b*
t^(b^2)*t^b*t^(b^2)*t^b*t*t^b^2>;
f,G1,k:=CosetAction(G,H);

M:=sub<G1|f(a),f(b),
f(t^b*t^(b^2)*t^b*t^(b^2)*t*t^b*t^(b^2)*t^b*
t^(b^2)*t^b*t*t^b^2)>;

IN:=sub<G1|f(a),f(b)>;

#DoubleCosets(G,sub<G|a,b,t^b*t^(b^2)*t^b*t^(b^2)*
t*t^b*t^(b^2)*t^b*t^(b^2)*t^b*t*t^(b^2)>,sub<G|a,b>);
```

```
Index(G,sub<G|a,b,
t^b*t^(b^2)*t^b*t^(b^2)*t*t^b*t^(b^2)*t^b*
t^(b^2)*t^b*t*t^b^2>);


prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
   end for;
return v;
end function;

ts := [Id(G1): i in [1 .. 7] ];
ts[7]:=f(t); ts[1]:=f(t^b); ts[2]:=f(t^(b^2));
ts[3]:=f(t^(b^3)); ts[4]:=f(t^(b^5));
ts[5]:=f(t^(b^4)); ts[6]:=f(t^(b^6));

cst:=[null : i in [1 .. 330]] where null is [Integers()|];
 for i := 1 to 7 do
       cst[prodim(1, ts, [i])] := [i];
    end for;
m:=0;
for i in [1..15] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

for i in [1..12] do i, cst[i]; end for;

%-------------------------
N0:=Stabiliser(N,7);
Orbits(N0);

N0s:=N0;
T0:=Transversal(N,N0s);
T0;
for i in [1..#T0] do
ss:=[7]^T0[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

%-------------------------
```

```
N01:=Stabiliser(N0,1);
SSS:={[7,1]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[7]*ts[1] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01s:=N01;

T01:=Transversal(N,N01s);
T01;
for i in [1..#T01] do
ss:=[7,1]^T01[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N01);

%--------------------------------------------------

N03:=Stabiliser(N0,3);
SSS:={[7,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[7]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N03s:=N03;

T03:=Transversal(N,N03s);
T03;
for i in [1..#T03] do
ss:=[7,3]^T03[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
```

```
Orbits(N03);

%-------------------------------------------------

N012:=Stabiliser(N01,2);
SSS:={[7,1,2]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[7]*ts[1]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]] then print Rep(Seqq[i]);
end if; end for; end for;
N012s:=N012;

T012:=Transversal(N,N012s);
T012;
for i in [1..#T012] do
ss:=[7,1,2]^T012[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N012);

%-------------------------------------------------

N013:=Stabiliser(N01,3);
SSS:={[7,1,3]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[7]*ts[1]*ts[3] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]] then print Rep(Seqq[i]);
end if; end for; end for;
N013s:=N013;

T013:=Transversal(N,N013s);
T013;
for i in [1..#T013] do
ss:=[7,1,3]^T013[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;


... and so on


%--------------------------------------------------

/* After inserting the chunk of code for a double
coset, if m increases by a value, the double coset
is new. Your single coset count, m, is increased by
the number of single cosets in the double coset checked.

/* Below is an example of what to add for if the loop:

for n in M do
if ts[a]*ts[b]*ts[c] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]] then print Rep(Seqq[i]);
end if; end for; end for;

actually gives equal coset names. In this example, the
double coset [7145] = [6341]. In this case, we want
all elements in N that send [7145] to [6341].  If there
were more equal names of [7145], we would have to make
N0145s include all of those elements in N that send [7145]
to equal names.

%--------------------------------------------------

N0145:=Stabiliser(N014,5);
SSS:={[7,1,4,5]}; SSS:=SSS^N;
#(SSS);
Seqq:=Setseq(SSS);
for i in [1..#SSS] do
for n in M do
if ts[7]*ts[1]*ts[4]*ts[5] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
```

```
for g in N do if 7^g eq 6 and 1^g eq 3 and 4^g eq 4 and
5^g eq 1 then N0145s:=sub<N|N0145s,g>; end if; end for;
#N0145s;

N0145s:=N0145;
T0145:=Transversal(N,N0145s);
T0145;
for i in [1..#T0145] do
ss:=[7,1,4,5]^T0145[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

%---------------------------------------------------

\* Once all single cosets have been accounted for, we must
determine which double cosets were equal to one another.
This loop is also similar to the loop before, except we
must find our element g in M instead of N.


for g in M do for h in IN do
if ts[7]*ts[1]*ts[3] eq g*(ts[7]*ts[3])^h then g,h;
end if; end for; end for;


%---------------------------------------------------

K:=ts[7]*ts[1]*ts[7];

for g in M do for h in IN do
if K eq g*(ts[7])^h then g,h;
end if; end for; end for;

for g in M do for h in IN do
if K eq g*(ts[7]*ts[1])^h then g,h;
end if; end for; end for;

for g in M do for h in IN do
if K eq g*(ts[7]*ts[3])^h then g,h;
end if; end for; end for;
```

```
for g in M do for h in IN do
if K eq g*(ts[7]*ts[1]*ts[2])^h then g,h;
end if; end for; end for;

for g in M do for h in IN do
if K eq g*(ts[7]*ts[1]*ts[4])^h then g,h;
end if; end for; end for;

...

for g in M do for h in IN do
if K eq g*(ts[7]*ts[3]*ts[4]*ts[5])^h then g,h;
end if; end for; end for;

%--------------------------------------------------

/* If a double coset does not increase m, one should check
which double coset that double coset is equal to. It is
simple to label your potentially new double coset as a
variable, say K, and check every possible double coset
it could be equal to. It can only be equal to one of them.
All the other loops ran should give no values.
```

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