2020

Geometry Across the Curriculum

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Investigating Generalizations of Distance Through Arc Length

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May 31, 2020

This project is designed to introduce you to a generalization of arc length. We will start with what you might recall about arc length, and generalize it from there.

1 Arc Length

Suppose $C(t)$ is a curve in space, let's say for concreteness that C is a curve in $\mathbb{R}^2$. Recall how we compute the arc length of the segment of the curve from $t = a$ to $t = b$:

- We partition the interval $[a, b]$ as $a = t_0 < t_1 < \cdots < t_n = b$.
- Approximate the length of the curve from $C(t_i)$ to $C(t_{i+1})$ as the length of the straight line segment (vector)

$$\Delta C_i = ||C(t_{i+1}) - C(t_i)||.$$

- The vector $C(t_{i+1}) - C(t_i) \approx C'(t_i)\Delta t_i$, there C' is the derivative of the curve, and $\Delta t_i = t_{i+1} - t_i$.
- Adding these together provides an approximation for the length of the entire curve:

$$\text{Approximate length} = \sum \Delta C_i \approx \sum ||C'(t_i)||\Delta t_i.$$

- Notice that this is a Riemann Sum for the following integral:

$$\sum ||C'(t_i)||\Delta t_i \to \int_a^b ||C'(t)||dt.$$

Thus, we define the *arc length* of a curve C between $t = a$ and $t = b$ to be

$$\text{Arc length} = \int_a^b ||C'(t)||dt.$$

2 Example

Let's try to find the arc length of the curve $C(t) = (t, t)$, where $t = 0$ to 1. We have $C'(t) = (1, 1)$, $||C'(t)|| = \sqrt{2}$, and so

$$\text{Arc length} = \int_0^1 \sqrt{2}dt = \sqrt{2}.$$

This is not surprising, this curve segment is the hypotenuse of a $1 - 1 - \sqrt{2}$ triangle.

3 Generalize this concept

One idea we have taken for granted is that we measure the lengths of vectors the same way, no matter where those vectors are actually located in space. Consider the possibility that at different points in space we measure lengths of the same vector differently. For example, if $\vec{v} = (x, y)$ is a vector with initial point (a, b), then let's define the square of the length of $\vec{v}$ at the initial point (a, b) to be

$$||\vec{v}||^2_{(a,b)} = (a^2 + b^2 + 1)(x^2 + 2y^2).$$

There's no particular reason why I've suggested this particular formula. But let's get to know this concept a little better.

3.1 Exercises

For the given formula above, compute the square of the length of the vector $(1, 1)$ when ...

1. it's initial point is the origin.
2. it's initial point is $(2, 5)$.

Solution:

1. When the initial point is $(a, b) = (0, 0)$, the origin, then our formula tells us that

$$||(1, 1)||^2_{(0,0)} = (0^2 + 0^2 + 1)(1^2 + 2(1)^2) = 3.$$

2. When the initial point is $(a, b) = (2, 5)$, then our formula tells us that

$$||(1, 1)||^2_{(0,0)} = (2^2 + 5^2 + 1)(1^2 + 2(1)^2) = 90.$$

3.2 Why might we generalize length in this way?

Imagine that you walk along flat ground, one step at a time, and each step is the same length. If you started to walk up a steep hill, the actual distance you travel per step (when viewed from above) might be smaller, and if you started to walk down a steep hill, the actual distance you travel per step might be larger. In the above example, we recall that the regular length of the vector $(1,1)$ is $\sqrt{2}$ (that is, the way we have always computed length). With this modified version of distance, the same vector has a length of $\sqrt{3}$ when the vector is positioned at the origin, and $\sqrt{90}$ when positioned at the point $(2,5)$. Thus, one might imagine this vector $(1,1)$ as one step in a particular direction, and the point $(2,5)$ being a particularly downhill slope.

4 Application to arc length

Since the formula for arc length involves lengths of vectors (in fact, it involves the length of the vectors tangent to a curve), it is reasonable to suspect that changing the notion of how we compute distance wold necessarily change some lengths of some of the curves we might get.

4.1 Exercise

Compute the length of the same curve given above, $C(t) = (t,t)$, but using this new version of the length of a vector in Section 3, rather than our usual notion of length.

4.2 Solution

Our formula for arc length is $\int_{t=a}^{b} ||C'(t)||_{C(t)} dt$, now with the difference that our length will be computed differently depending on what point we are at. The bounds on the integral, 0 and 1, will remain the same. Our derivative $C'(t) = (1,1)$ as before as well. But now, we will be measuring the length of the tangent vector $(1,1)$ at the point $C(t) = (t,t)$ using our new definition of distance. At the point $(a,b) = (t,t)$, we find that

$$||C'(t)||_{(t,t)}^2 = (t^2 + t^2 + 1)(1^2 + 2(1)^2) = 3(2t^2 + 1).$$

Now we find our arc length to be (with some steps omitted)

$$\int_0^1 \sqrt{3(2t^2+1)}dt \approx 2.202.$$

5 Geometric Perspective

Earlier, when our notion of length was the same at each point, we found the arc length to be $\sqrt{2} \approx 1.414$, while with this new notion of distance, we found the length to be *more* (about 2.202). Here are a couple of geometric questions to ponder:

1. If this distance measured how you were walking along a curved surface in $\mathbb{R}^3$ (as we described earlier), would you be walking downhill or uphill?
2. This surface: could it possibly be flat? What other information would you need before you could tell for sure.
3. The existence of a surface to walk along might induce some sort of change in measurement of lengths of vectors. Conversely, if you simply define a different way of measuring lengths of vectors, is there some corresponding surface that matches your formula (with regards to this notion of arc length)?
4. Given that the existence of a curved surface seems to correspond to a varying way of measuring distance, in what way could you sense a surface curving just by examining this varying distance measurement? This is an extremely deep question.

6 The Geometry of an Empty Space-Time

An empty "space-time" is just $\mathbb{R}^4$, where one agrees that in each choice of coordinates there is one exceptional coordinate that we denote as "time", while the other three we refer to as "space". Say the coordinates we choose are x, y, z, and t, where t represents time. Instead of thinking as points as "locations" in space-time, we tend to think of them as "events"–they are, after all, a place in space and a time as well.

For example, if we were to choose one particular spot as our origin, then the coordinate point $(1, 0, 0, 0)$ could be imagined as a flash of light emanating from the point one unit in front of you. The coordinate point $(1, 0, 0, 5)$ would again be a flash of light one unit in front of you, but it would occur 5 seconds in the future.

Generally, the units are chosen so that the spatial coordinates are "light seconds" (the distance light travels in one second), while the time unit is "seconds". If there is no mass in your space-time, then one measures the lengths of vectors everywhere the same way, except the formula for the square of the length of the vector (x, y, z, t) at any point is

$$||(x, y, z, t)||^2 = x^2 + y^2 + z^2 - t^2.$$

Any mass in your space-time creates gravity, and depending on the location and quantity of that mass, it alters the formula above to what might be something else (but still similar).

6.1 An interesting example

Let us consider a 4-dimensional version of the curve we've seen twice already:

$$C(t) = (t, 0, 0, t), \text{ for } 0 \leq t \leq 1.$$

What is the average velocity of this curve over the interval $[0, 1]$? Well, velocity is the (spatial) distance traveled, divided by the time it took to travel that distance. In this example, $C(0) = (0, 0, 0, 0)$, and $C(1) = (1, 0, 0, 1)$. This means that in one second, this path travels from the origin of our universe, to the place one unit from there. Our spatial units are "light-seconds", which means that the spatial distance traveled is one light second. And, it took one second to travel that. Thus, the average velocity of this curve is the speed of light: that is the speed one must travel at to go one light second in one second. This path must be a beam of light!

6.2 Exercises

1. For the curve $C(t) = (t, 0, 0, t)$, compute its arc length for $0 \leq t \leq 1$.
2. Suppose a particle's path has a tangent vector $(a, b, c, 1)$. The length of a path's tangent vector is its speed. If this particle is moving at less than the speed of light, then what can you say about $||(a, b, c, 1)||^2$? (Is it positive or negative?) (Here, we choose $d = 1$ to simply talk about what is happening after one second.)
3. What if the same particle with tangent vector $(a, b, c, 1)$ is traveling *super-luminally*, i.e., traveling faster than the speed of light. What then could you say about $||(a, b, c, 1)||^2$?

6.3 Solutions

1. We start by computing

$$||C'(t)||^2 = ||(1, 0, 0, 1)||^2 = 1^2 - 1^2 = 0!$$

   More on this below, but this means that the arc length is

$$\int_0^1 ||C'(t)||dt = 0.$$

2. If one computes the spatial distance the particle moves, it would be $\sqrt{a^2 + b^2 + c^2}$. Let us compare the square of this distance with how much time is elapsed.

   If the particle is moving *less* than the speed of light, then in one second, $a^2 + b^2 + c^2$ must be less than the square of one light second (traveling exactly the speed of light would mean we would have traveled *exactly* one light second, so this quantity must be less. But this means that $a^2 + b^2 + c^2 < 1$, and so

$$||(a, b, c, 1)||^2 = a^2 + b^2 + c^2 - 1^2 < 0!$$

   Thus, the square of the length of this vector is negative. Thus, the square length of all vectors coming from particles traveling less than the speed of light must be negative.

3. Similarly, a particle traveling super-luminally must travel *more* than one light-second of spatial distance after one second. In this case, $a^2 + b^2 + c^2 > 1$, and so

$$||(a, b, c, 1)||^2 = a^2 + b^2 + c^2 - 1^2 > 0.$$

Thus, super-luminal particles have tangent vectors which have a positive square length.

7 A Partial Explanation

There is a big difference between our definition of distance given above as

$$||(x, y)||^2_{(a,b)} = (a^2 + b^2 + 1)(x^2 + 2y^2)$$

and

$$||(x, y, z, t)||^2 = x^2 + y^2 + z^2 - t^2.$$

In the latter, there are nonzero vectors $\vec{v}$ (such as $\vec{v} = (1, 0, 0, 1)$) with $||\vec{v}||^2 = 0$, whereas in the former, there is no such nonzero vector. This leaves open the possibility that there could be a nonconstant path with an arc length of zero, as we saw in the example above (these are called “lightlike” or “null” vectors, in reference to the fact that something traveling at the speed of light must satisfy that condition). One reason we might want to study such strange notions of distance is that we live in a space-time that works in this way. Indeed, in the example above we considered a beam of light and saw its arc length be zero.

One might then wonder, as we did above: does this notion of measuring distance make space-time curved? It turns out that empty space-times (with this notion of distance) are flat (i.e., not curved), and the existence of mass curves space-time, and the larger the mass, the more sharply space-time is curved. Although this discussion is a very long and advanced one for another time.