6-2015

Discovering and Applying Geometric Transformations: Transformations to Show Congruence and Similarity

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DISCOVERING AND APPLYING GEOMETRIC TRANSFORMATIONS:
TRANSFORMATIONS TO SHOW CONGRUENCE AND SIMILARITY

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Teaching: Mathematics

by
Tamara Lee Voorhies Bonn
June 2015
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Approved by:

Dr. John Sarli, Committee Chair, Mathematics
Dr. Davida Fischman, Committee Member
Dr. Catherine Spencer, Committee Member
ABSTRACT

The use and application of geometric transformations is a fundamental standard for the Common Core State Standards. This study was developed to determine current high school teachers’ prior mathematical content knowledge and develop their content knowledge of transformations and their applications. The design of this study was guided by the questions: “Why is there a level of reluctance amongst secondary teachers when it comes to teaching geometric transformations?” and “How can their content knowledge become deepened to apply geometric transformations to prove that two figures are congruent?” The study provided teachers a chance to gain experience with transformations and use transformations to develop an understanding of congruence and similarity. The teachers’ work with transformations also enhanced their understanding of how transformations are the foundation for Euclidean geometry and begin to lay a foundation for the basics of rigid motion in the plane, with or without the use of coordinates. The results supported the claim that teachers’ transformation content knowledge needs to be deepened overall and in particular with respect to the application of transformations to prove that two figures are congruent. The results also showed that, with an increase of understanding of the mathematical properties of transformations, teachers are better prepared to teach them in their classrooms.
ACKNOWLEDGMENTS

I first would like to thank all of the teachers who participated in this study. Their willingness to be transparent about their content knowledge made this study possible. I have learned and been challenged by their questions and their ability to persevere through this process.

I am extremely grateful to Dr. Davida Fischman, Dr. Catherine Spencer, and Dr. Madeline Jetter for challenging me each step of the way through the MAT program. I greatly appreciate all the hours Dr. Fischman spent reading through my thesis, making informative comments, and challenging what I thought about the mathematical content.

Words cannot express my thanks to my advisor, Dr. John Sarli, for all of the content knowledge he helped me absorb and explain. His hours spent making sure the ‘math’ was correct were invaluable to this process.

I would also like to express my sincere thanks for all the sacrifices my husband, Chris Bonn, made during this process. His love, patience, and encouragement allowed me the time I needed to complete this program.
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CHAPTER ONE

INTRODUCTION

Transformations, both linear and non-linear, are a core concept in mathematics; they appear in various areas of mathematics from algebra to calculus and beyond. In this thesis, I focused on the application of linear transformations in geometry; in particular, I investigated how teachers understand and apply transformations to show congruency.

The recently developed U.S. Common Core State Standards (CCCS; 2010) prominently feature geometric transformations at the middle school level as a transition from informal to more formal reasoning in geometry. (Seago, Nikula, Matassa, & Jacobs, 2012, p. Abcde+3).

With this shift, the need for professional development for teachers and leaders has greatly increased. In the Common Core State Standards (CCSS, 2013), geometric transformations link functions to geometry. Seago (2013) made note of recommended priority areas from the McCallum (2011) report as follows:

Geometry in the Common Core State Standards is based on transformations, an approach that is significantly different from previous state standards. This is a change for students, teachers, and teachers of teachers. Challenges include attention to precision and language about transformations . . . The transformational approach to congruence and similarity is likely unfamiliar to many middle grades teachers (McCallum, 2011. p. 10)
Prior to the implementation of the Common Core State Standards (CCSS, 2013), in the California 1997 standards (CDE, 1997), Geometric Transformations were relegated to a small subset of the overall geometry curriculum. In fact, “of the total geometry and measurement GLE (Grade-level learning expectations) in all 42 state documents, a relatively small portion, 6%, were coded as ‘transformation GLEs’ based on the presence of one or more terms directly related to transformations” (Smith, 2011, p. 45). Based on observations, interviews with teachers, and a review of the current curriculum in Southern California, it appears that geometric transformations are one of the least desirable topics teachers want to teach. As stated by Smith (2011) in *Variability is the Rule*,

> When confronted with views of geometric content structured by transformations (e.g., in mathematics textbooks), they (the teachers) could sensibly ask: ‘What’s so important about transformations (since I did not learn about them), and how do they relate to the geometry that I know?’ (p. 41)

A look at the 1997 California Standards for Mathematics (California Department of Education, CDE, 1997) we see that one standard was used to identify transformations: “22.0: Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections” (p. 43). The 1997 Standards limited transformations to the coordinate plane and made no connections to other areas of geometry or algebra. As further indication of the lack of emphasis placed on transformations,
we note the paucity of items assessing this content area in both district and state benchmark tests. All of these limit transformations to those that occur on a coordinate plane and are treated in isolation of any application of transformations to functions, congruence, or similar figures.

The approach to geometric instruction, as outlined in the CCSS (2013), is likely to pose a significant challenge for teachers as they face teaching new content in substantially different ways (Seago et al., 2012, p. 2). It is evident from the CCSS and research that a deeper understanding of transformations and how they are the basis for more than just their application to the coordinate plane will be needed in order to be fully prepared for the Common Core State Standards.

As we can see, the 1997 California Mathematic Standards (CDE, 1997) limited transformations to one standard: “22.0: Students know the effect of rigid motions on figures in the coordinate plane and space including rotations, reflections, and translations” (p. 43).

However, the Common Core State Standards (2003) give transformations ‘center stage,’ as well as develop their application to other mathematical concepts.

2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. (p. 56)
4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. (p. 56)

6: Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. (p. 72)

7: Use the definition of congruence in terms of rigid motions to show that two triangles are congruent, if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. (p. 72)

8: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. (p. 72)

As shown above, the CCSS (2013) have indeed made a shift to considering transformations foundational for not only congruence and similarity but for proofs, angle relationship, and algebraic functions. According to Dick and Childrey (2012), the Common Core State Standards Initiative (2013) followed Klein’s (1872) lead and centered geometry on transformations stating, “The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometry transformation” (p. 622).
teachers will need opportunities to gain ‘mathematical knowledge for teaching’ (Ball, Thames, & Phelps, 2008) in the areas of geometric transformations and similarity, including a deep understanding of the mathematical content and the fluency to make instructional decisions that support students’ learning of this content. (Seago et al., 2012, p. 2Abcde+3)

Statement of the Problem

During the pre-interviews in this project, the secondary math teachers who were interviewed as to how geometric transformations are being used to make connections to congruence, similarity, and higher order functions, the most frequent response was that they had not been exposed to the connection or the use of transformations in their middle or high school math classes. Teachers also frequently stated that transformations are much too difficult to teach to high school students. Teachers often consider transformations ‘too difficult’ to teach, either because of the difficulty students have in comprehending them, or because teachers themselves have an insufficient understanding of the role transformations play in mathematics, in particular with regard to proving that two objects are congruent. As stated by Mashingaidze (2012),

Some [teachers] said they never attempt to teach it because they don’t quite understand it, whilst others said they usually teach it when there is very little time left to finish the syllabus resulting in a crush programme by teachers. (p. 197)
Dick and Childrey (2012) made the assertion that current high school math teachers treat transformations lightly or not at all:

The study of transformations usually begins—and, unfortunately ends—with these four elements: three simple reflections (across the x- and y-axes and across the line y=x); three simple rotations (of 90, 180, and 270 degrees counterclockwise about the origin); dilations centered at the origin, and finally, translations. (p. 622)

While geometric transformations, also called isometries, can be limited to reflections, rotations, and translations, they actually are an application of functions, congruence, and similarity.

Transformations have always been an important area of mathematics. The use of transformations to show that two triangles are congruent is not an idea created by the Common Core State Standards (2013); it is an essential part of the understanding of why two triangles are congruent. In Postulate 4, Euclid (Heath, 2012) stated: “things that coincide with one another are equal to one another” (p. 247). While Euclid did not formalize in the language that coincidence can be obtained by transformations, we now see that if one object can be mapped onto another by a series of transformations such that their angles and sides coincide, then those objects are congruent.

All these discussions show that a deeper understanding of transformations, their applications to higher order functions and their use in the geometry classroom is essential to the Common Core State Standards (2013).
Transformations must not be treated as the unwanted “black-sheep” of the geometry class.

Based on all the research, there does not seem to be a single method of instruction that stands out that will consistently produce the greatest results. Nevertheless, there is an agreement that teachers need to deepen their understanding of geometric transformations in order to be fully prepared to teach the Common Core State Standards (2013). While working with teachers during the discovery portion of my study, I asked the question, “What is it about transformations that make you feel unprepared to teach it in your classroom?” (see Appendix D, question f). The consensus among those teachers interviewed was that it isn’t clear that transformations need to be taught, since they are not relevant to most other areas of mathematics; consequently they should take a back seat to other aspects of geometry.

Common Core State Standards (2013) include transformations not just in the geometry course but are introduced in the 7th grade. Dick and Childrey (2012) reminded us that NCTM has always held that transformations are a critical part of the whole curriculum: “students should ‘understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices (p. 308).’” (p. 622).

Mathematics is no longer conceived of as a set of disjoint branches, each evolving in its own way. The present conception is that of a unified whole in which all the branches contribute to the development of each other. (Fehr, 2006, p. 166)
As we look over the K-8 CCSS (2013), the argument can be made that transformations are integral to the understanding of congruence, similarity, and much more.

It is apparent from the research, Mashingaidze (2012) stated that math teachers, especially those at the secondary level, need to make the connection of transformations with congruence, similarity, and proportion (p. 198). The use of transformations to identify congruent and similar figures “explain the traditional Euclidean properties” (Smith, 2011, p. 42), thus relying less on the memorization of properties, definitions, and axioms. Many articles, such as those by Shockey and Snyder (2007), indicated that there is a great need for multiple approaches to transformations with a greater emphasis on making connections with algebra, functions, and discovery. “School children are afforded too few opportunities to locate general results via investigation; rather, the norm tends to be that students are expected to learn and apply results obtained by others” (Driscoll et al., 2007, p. 11).

Significance

The greatest significance of this project will be to the teachers and their students. I believe that if we create an atmosphere where transformations are explored, explained, and applied then teachers will teach geometric transformations more effectively and they will be able to apply them across the curriculum.
The Common Core State Standards (2013) rely on transformations from 7th grade and on to explain congruence and similarity in polygons. By studying this unit, teachers will be able to have a deeper understanding of transformations, which will enable them to integrate geometric transformations into their existing scope and sequence.

The ultimate goal is that teachers and students will deepen their content knowledge of geometric transformations and be able to apply this knowledge to the application of transformations to multiple aspects of mathematics. Teachers will “begin to re-think their practice based on this deeper mathematical understanding” (Seago et al., 2012, p. abcde+2) of transformations. Transformations are woven throughout the common core standards beginning in the 7th grade, thus study of transformations is no longer an isolated event in geometry.

Purpose of the Study

My aim was to create a unit of study (see Appendix C) for teachers that place geometric transformations at the heart of understanding congruence and similarity. *This unit will make connections between geometric transformations and the concepts of congruence and similarity. The unit’s key aim will be to support teachers in developing a deep understanding of geometric transformations as they apply to congruence and similarity.*
Math Content

The unit made use of isometries as the foundation for the application to congruent polygons. The pre-test asked teachers to state a transformation or series of transformations that would map one figure onto another. Only one question was represented on a coordinate grid. Teachers had little or no trouble performing a translation in the coordinate plane; however, the majority of the teachers had significant trouble performing a transformation without the coordinate plane. Thus, it was apparent from questions 3, 4, and 5 from the pre-test (see Appendix D) that time would have to be taken to deepen the teachers understanding of transformations without the aid of a coordinate plane. The majority of the teachers could recognize from problem 3 that the triangles were congruent; however, in subsequent problems they were unable to perform a transformation that would map one figure onto another. So often, as the research showed, we isolate transformations to the coordinate plane so that we can derive a set of rules that can be applied to obtain the desired result.

The first transformation that was explored was a translation of a square along a vector (Figure 1). This activity familiarized the participants with vectors and the use of patty paper as a tool of mathematics. This simple activity gave the teachers a better understanding of transformations.

In order to continue the development of the basics of the isometries, the participants also worked with rotations and reflections using patty paper (Figures 2 and 3).
3. Translate Square NEAR 3 cm up line RE. Correctly label the image.

Figure 1. Translation

5. Use tracing paper to reflect triangle MOP across line f.

Figure 2. Reflection
Objectives

The following were the objectives for questions 3, 4, and 5:

- Develop a unit of study (see Appendix C) for teachers regarding the teaching of transformations. (see additional details below)
- Implement the unit with in-service and pre-service teachers in one large urban high school.
- Assess the teachers’ mathematical content knowledge and pedagogy before and after implementation of the unit.
- Increase teacher’s level of Geometric Thinking and evaluate in what ways and to what extent their level of understanding has been enhanced.

The ultimate desire was that teachers would find the unit helpful in developing their own understanding, and thus will be inspired to implement similar units in their classrooms to the benefit of their students.
This unit was given so that teachers had a chance to gain experience with transformations, using transformations to develop the basis for congruence and similarity so that they may use them as the foundation for Euclidean geometry. The unit will lay a foundation in the basics of rigid motion with and without the use of the real number plane. As teachers, it is necessary to have a foundation with which to explain why a rigid motion can be used to show two objects are congruent. This unit will lay that foundation and not just give teachers another set of procedures to explain congruence.

**Pedagogy**

As educators, we must understand that students need to experience the math rather than just absorb what they are being taught in class. Teachers need to move from the Information Phase to the Integration Phase as described by the Van Hiele’s (n.d.) levels of geometric thinking. “Van Hiele warned that these types of ‘tricks’ (e.g., using a ‘z’ to find alternate interior angles) might actually prevent students from moving to the subsequent level of reasoning” (Smith, 2011, p. 77). Driscoll et al. (2007) made note of this idea in their book *Fostering Geometric Thinking*:

Mathematical power is best described by a set of *habits of mind*. People with mathematical power perform thought experiments; tinker with real and imagined machines; invent things; look for invariants (patterns); make reasonable conjectures; describe things both casually and formally (and play other language games); think about methods, strategies, algorithms, and processes; visualize
things (even when the “things” are not inherently visual); seek to explain why things are as they see them; and argue passionately about intellectual phenomena. (p. 9)

In order to accomplish these goals, teachers will work to teach from a problem-solving perspective. It is one thing to teach a teacher about transformations; it is quite another to give teachers the tools they need to implement transformations in their classrooms. This unit of study will make use of the Standards for Mathematical Practice (SMPs) so that teachers, and eventually students, will engage in “doing mathematics” as described in the Common Core State Standards (2013). Some of the SMP’s teachers will use in this unit are: SMP 1: Make sense of problems and persevere in solving them (p. 6); SMP 2: Reason abstractly and quantitatively (p. 6), and SMP 5: Use appropriate math tools strategically (p. 7). In order to accomplish these goals the unit will make use of Van Hiele’s (n.d.) levels of geometric thinking making sure to build the cognitive thinking necessary to progress from visualization (rules) to being able to prove a theorem is true through the application of transformations.
CHAPTER TWO
REVIEW OF LITERATURE

The purpose of the literature review is to demonstrate the findings that provide the foundation of this study. The review of the literature demonstrated that there is a need to increase teachers’ overall understanding of transformations and their application to multiple areas of mathematics. The review is divided into three major sections. The first section presents an overview of the understanding of transformations. The second section reviews findings regarding the application of transformations to prove congruence and similarity in polygons. The third section presents findings about mathematical content knowledge, misconceptions, and experience with teaching transformations, which helped determine the areas of focus of the study.

Understanding Transformations: Overview

This section of the literature review presents an overall discussion of the general understanding of transformations by both the teacher and the student. My literature took a broad look at what content knowledge teachers possess, how they teach transformations, and what limitations there may be in using transformations as a foundation for other topics in mathematics.

Teachers have continually placed geometric transformations on the back burner when it comes to classroom content (Seago et al., 2013). There appears to be a definite lack of deep understanding of the richness of geometric
transformations and how they can be applied to multiple concepts both in and out of the geometry classroom (Mashingaidze, 2012). Teachers have many ‘shortcomings’ when it comes to the application of transformations to show that two polygons are congruent. For example, teachers limit their application of transformations to the coordinate plane where lengths of sides can be determined using the distance formula, instead of seeing the use of transformations as a much more accurate method. This lack of understanding leads teachers and students to believe that transformations are not integral to the teaching of geometry.

Typical high school curriculum present transformations as an isolated topic, much as an afterthought, involving the four basic transformations—reflection, rotation, translation, and dilation—independently of other topics in mathematics, and in particular geometry (Dick & Childrey, 2012). This is evident in that most high school textbooks, as well as district pacing guides, place transformations after state testing. As teachers, we tend to use the curriculum as a guide to the importance of a topic, thus the placement of transformations as the last chapter in a high school textbook and as the last standard in the 1997 California math standards (CDE, 1997) has reinforced the idea that transformations are indeed an isolated topic.

This neglect in the high school geometry classroom leaves a gap in the students’ understanding of how the postulates of SAS, ASA, and SSS are formed. Since congruence can be concretely shown by a sequence of transformations, then transformations are a natural progression to congruence
and “The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation” (Dick & Childrey, 2012, p. 622).

When students lack the knowledge that the figure has actually moved, yet it is still the same figure, the concepts of transformations to show congruence is lost. Students need to understand that any transformation is a transformation of all points in the plane which can be accomplished by having students translate points rather than figures (Hollebrands, 2004). Rigid motions (translation, reflection, and rotation) preserve the distance between vertices of any polygon. Thus, rigid motions are transformations that create congruent figures. In contrast, dilations do not preserve distance and thus it creates similar figures. The idea that all points in the plane move can be shown using technology; thus, students are able to see that there is a direct relationship between transformations and the distance from the segments and the points that make up the line of reflection point of rotation or direction of transformation of figure being transformed (Hollebrands, 2004).

Using Transformations to Understand and Prove Congruence

This section of the literature review highlights the discussion that transformations are not an isolated topic in the geometry classroom but are the basis for a deeper understanding of why two objects are congruent. The literature took a Common Core State Standards (2013) approach to the application of transformations to help students develop the ideas of congruent
figures. As previously stated, teachers and textbooks have traditionally placed little emphasis on transformations as the formal proof for congruence (Bucher & Edwards, 2011). As teachers, we traditionally isolate proofs to those that can be written in the much familiar two-column format. While this type of proof is beneficial to the high school student, we must also embrace the idea of a proof as a series of steps that lead to showing that two figures are indeed congruent.

Most teachers and curriculum lack “... rigorous proof of congruent triangles” (Bucher & Edwards, 2011). This means that as teachers we miss opportunities to connect the rigid motions to congruence as well as other geometric topics. In order for teachers to make that connection both for ourselves, as well as for our students, it means we must take the time to develop transformations as a tool for proofs rather than an isolated topic.

The traditional geometry class misses out on this opportunity. One way to increase the understanding of transformations is with the use of dynamic software as a visual tool in showing transformations lead to congruent polygons as well as similar polygons. Technology is not foreign to the classroom and should be embraced as a tool to aid in deepening the understanding of all aspects of mathematics. GeoGebra, Geometers Sketchpad (Hollebrands, 2004) and other online applications will aid the students to see how transformations are an integral key to students’ understanding of congruence!

A convincing argument can be made that the emphasis on geometric transformations in the Common Core State Standards (2013) will create “some difficulties for teachers, mathematics supervisors, and coordinators” (Smith,
With any new idea, method, or set of rules, the teachers bear the responsibility of becoming “experts” on the content. Any teacher knows that while we can consider ourselves experts in a certain discipline of mathematics, the classes our administrators assign to us may have little to do with our expertise and more to do with the needs of the school. As such, the responsibility to understand and communicate the curriculum is on the teacher. The Common Core State Standards (CCSS, 2013) place transformations as the “unifying” theme, which is a great contrast to the previous standards. Teachers will need professional development in order to be prepared to teach transformations as a tool in order to understand the detail of the mathematical difference in the treatment of transformations from the previous standards to the new CCSS (Smith, 2011). As educators, we know that the vocabulary for transformations will occupy a greater space in the CCSS; this will mean that teachers will need to attend to the correct vocabulary such as reflection rather than flip. In order to apply this deeper understanding of transformations, teachers and students will have to make a shift in how they instruct and learn about transformations (Smith, 2011).

Not only will teachers have to learn about transformations they will also have to continue to refine instructional and assessment models to make use of Van Hiele’s levels. The CCSS (2013) writers may not have designated “Van Hiele’s levels in creating consistent, supportive geometry learning trajectories . . .” (Smith, 2011, p. 93); however the rigor of the performance tasks can be directly linked to Van Hiele’s five levels of geometric thinking:
Level 1: Visual level
Level 2: Descriptive/Analytic
Level 3: Abstract/Relational
Level 4: Formal Deduction
Level 5: Mathematical Rigor (van Hiele, n.d.)

Each level brings with it a unique opportunity to increase student understanding while building upon prior knowledge. These levels are not just for the student but for the teacher as well. We cannot just insert more standards and expect teachers to increase their instruction; we must instead equip these educators to not only understand the standards but how to lead their students to a higher level of thinking, problem solving, and discovery (Ball et al., 2008).

Mathematical Content Knowledge

Content knowledge is an imperative in any classroom, which on the surface appears quite a logical statement; it is one of the hardest to define. In this third section of the review, the foundation is laid for the need for teachers to increase their content knowledge of transformations, which leads directly to the application of the unit with teachers. In *Content Knowledge for Teaching—What Makes It Special?*, Ball et al. (2008) made the argument that “subject matter was little more than context” (p. 390). Content knowledge can be defined as a ‘special’ kind of knowledge that is imperative to the teaching of a subject (Ball et al., 2008). Some questions that help us to define content knowledge are:
What do teachers need to know and be able to do in order to teach effectively? Is content knowledge more than just basic mathematical knowledge? How can we ascertain a teacher’s content knowledge? While it seems obvious that teachers need to know the topics and procedures that they teach—primes, equivalent fractions, functions, translations and rotations, factoring, and so on—we decided to focus on how teachers need to know the content. (Ball, 2008, p. 395)

This last comment instilled in me the need to look further at what teachers “need” to know about transformations in order to be effective in the classroom. This meant that my unit, while applicable to the high school classroom, needed to be administered to teachers in order to better determine their content knowledge of transformations and their application to show congruence.

At each stage of the research, the overarching theme was the immediate need to educate both the teachers and educators of teachers in the richness of transformations. “Teachers who persevere in the profession and ‘change the world of the classroom’ must add ‘knowledge-in-practice’ and ‘knowledge-of-practice’ (Cochran-Smith & Lytle, 1999) to formal knowledge and theory gained in their professional preparation programs” (Bonner, 2006, pp. 28-29). All of the research pointed towards the idea that teachers need more exposure to transformations, their application to other areas of mathematics, and how to apply them to their classroom on a daily basis.
CHAPTER THREE
METHODOLOGY

The purpose of this study was to deepen teachers’ mathematical content knowledge of transformational geometry and the application of transformations to show congruence in polygons. The study took place at a high school in San Bernardino. This school site had approximately 10 math teachers, eight of which participated in the full study, teaching Integrated Math 1, CAHSEE math prep, Geometry, Algebra 2, Pre-Calculus, Calculus, and Statistics. The years of experience for the teachers ranged from first-year teachers to teachers with over 20 years’ experience. The average teaching time of the teachers was 9.5 years.

Over the course of eight weeks, teachers were asked to work through several problems that involved basic transformations in the plane, as well as, the application of those transformations to show congruence in polygons. Bonner (2006) stated “A ‘culture’ of inquiry, implies the social context in which individuals inquire together and thus the component of collaboration that is so desperately needed in the teaching profession” (p. 31).

I developed a unit of study focused on transformations and the use of transformations as a proof method to show that two polygons are congruent to each other.
The project followed the following components:

- **Review of the literature** to determine the area of focus and explore what has been done in this area
- Design a **unit of study** on the meaning of transformations and their application to congruence
- Design and implement a **pre-test** to determine teachers’ familiarity with transformation and their application to congruence
- **Implement the unit** with in-service teachers
- Design and implement a **post-test** to assess teachers’ growth in understanding of these areas
- **Analyze responses** to unit items and reflection questions

**Review of Literature**

In order to make the decision on where to focus my unit, I first began to read articles related to both the application of transformations, as well as, the teaching of transformations in the secondary classroom. Upon review of the articles, I began to develop an area of focus for my thesis and unit. I decided to focus on working with teachers to increase their overall understanding of geometric transformations, their application to show congruence in polygons, and how transformations apply to multiple levels of mathematics.
Unit Design

Criteria for the unit:

1. Closely related to classroom instruction and curriculum

2. Common Core State Standards
   a. Standards for Mathematical Practice (SMPs)
      i. Make sense of problems and persevere in solving them.
      ii. Reason abstractly and quantitatively.
      iii. Construct viable arguments and critique the reasoning of others.
      iv. Model with mathematics.
      v. Use appropriate tools strategically.
      vi. Attend to precision.
      vii. Look for and make sure of structure.
      viii. Look for and express regularity in repeated reasoning.

   (Common Core State Standards (2013, pp. 6-8)

   b. Content standards

3. Concept focus
   a. Introduction to mathematical vocabulary for transformations
   b. Practice with transformations to show congruence in triangles.
   c. Direct application of transformations to congruent triangle theorems such as SSS, SAS, ASA, etc.
   d. Support for teachers to use these topics in their classroom.
In order to make the unit not only meaningful to the teacher, but also applicable in the math classroom, I chose to use the CCSS (2013) as my guide. I searched many resources both digital and non-digital to find exemplary problems that would guide the participants toward a deeper understanding of both transformations and their application to show congruence in polygons. It was also necessary to find problems that were in line with the CCSS for 8th grade and above. It was also essential to find questions that aligned with the Standards for Mathematical Practices and Van Hiele’s (n.d.) levels of geometric thinking.

Pre- and Post-Tests

The pre- and post-tests were designed using Common Core questioning strategies. The CCSS (2013) questioning strategies asked the participants to not only state the correct answer to the problem but be able to justify their answer using correct mathematical language, as well as, be able to apply appropriate problem solving skills to show understanding of the content. The questions in the pre- and post-test made use of these skills; for example, when a participant was asked to state whether two triangles were congruent they had to also state the properties of transformations used to determine if the two triangles were congruent. In addition, they must also state how the properties of transformations connected to the postulate used to determine congruence. I began by researching multiple digital resources such as NCTM’s Illuminations (n.d.), Smarter Balanced (Smarter Balanced Assessment Consortium, n.d.), MARS (n.d.) tasks, and many others.
The pre-test was designed to determine the participant’s level of understanding of transformations using the coordinate plane and without the coordinate plane. The questions were chosen to determine how familiar teachers were with not only the four isometries but also how well they could determine a transformation given both the pre-image and the image. I was careful to choose questions that elicited understanding of isometries rather than questions that may have been vague or easy to guess the answer. A question that I did not choose, because it did not fit my design parameters, is shown in Figure 4. This question was excluded due to the fact that it was a route problem and did not elicit a deep understanding of transformations.

Figure 4. Excluded Question
A question that asked for the same information (a sequence of rigid motions) whose responses would allow me to gauge the participants’ understanding is shown in Figure 5.

In order to design both the pre- and post-test, I began by gathering together approximately 50 questions that dealt with transformations and application of transformations to show congruence. The questions were then classified as to type (multiple-choice or free response), Depth of Knowledge (DOK) level, and which CCSS (2013) standard was being addressed. Once I had created a spreadsheet that reflected each question and each category, I began to choose the questions that would allow me to determine a participant’s level of understanding both on the pre- and post-test.

It was imperative to the unit that the pre-test was designed to access prior knowledge with regards to transformations, gauge the knowledge of the participants with regards to the application of transformations, and also indicate any areas of concern with respect to mathematical vocabulary.

The post-test was designed to use similar problems from the pre-test so that the effectiveness of the unit could be ascertained. The post-test also included more problems than the pre-test in order to not only test growth from pre-test but how much content knowledge had increased over the application of the unit.
Using the graph, below describe the transformation or series of transformations that takes \( \triangle ABC \) to \( \triangle A'B'C' \).

a. If \( \triangle ABC \) is reflected over the \( x \)-axis to yield \( \triangle A'B'C' \), what are the coordinates of the vertices of \( \triangle A'B'C' \)?

b. Using this reflection, write a general rule that will map \( \triangle ABC \) onto \( \triangle A'B'C' \).

c. If \( \triangle ABC \) is translated 4 units to the left and 3 units down to yield \( \triangle RST \), draw \( \triangle RST \) on the coordinate plane below.

d. Using your translation, write a general rule that will map \( \triangle ABC \) onto \( \triangle RST \). Use words, numbers, and/or pictures to show your work.

Figure 5. Included Question
Analysis

This was the critical part of the methodology for this unit. In order to show an increased understanding of transformations and their application to congruence, there must be some method of evaluation. Since the major portion of my unit was based on the teachers’ responses to either the problems themselves or to specific reflection questions, my method of evaluation was qualitative in nature. I also needed to code the questions that I asked of the participants into categories (see Table 1) so that I could focus my analysis based on their responses.

Table 1. Analysis Code

<table>
<thead>
<tr>
<th>CODE</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>Vocabulary</td>
</tr>
<tr>
<td>U</td>
<td>Understand Transformations</td>
</tr>
<tr>
<td>TA</td>
<td>Application of Transformations</td>
</tr>
<tr>
<td>T</td>
<td>Use Tools Strategically and Constructions</td>
</tr>
<tr>
<td>L</td>
<td>Progression from Learning to Teaching</td>
</tr>
<tr>
<td>P</td>
<td>Improvement in Understanding</td>
</tr>
</tbody>
</table>

In the first part, the study participants were asked to perform simple transformations, such as a translation of a figure in the plane by a given distance, to show that two figures are congruent. Teachers then defined a series of transformations that can be applied to show that two polygons are congruent or
similar. As the unit progressed, the participants were asked to demonstrate their understanding of transformations and their applications to geometry through a series of thought provoking questions.

Once the activity had been given, I reviewed the responses and pre-test to find if there were any areas of misunderstandings with regards to basic transformations. Upon reflection, I made notes of any concepts that needed to be reviewed before the participants began the unit. The unit was designed to not only “review” the basics of transformations but to develop transformations as a set of isometries that have a much richer application than reflection, translation, and rotation.

In order to maintain accurate records, the participants were asked to maintain folders of their work, observations, and reflections. As the participants progressed through the unit, some adjustments and additions were made in order to aid in the overall understanding of transformations as they apply to prove that two figures are congruent. Once the unit had been completed, a final assessment was given. This assessment took the form of a Common Core assessment in that the participants were asked to consider multiple applications of transformations from the 8th grade to high school standards in order to see how well the participants have become familiar with all applications of transformations.
CHAPTER FOUR

RESULTS

Introduction

The unit of study that was designed allowed teachers to experience transformations, its application to geometric proofs, and the reflections from the teachers as they worked through the major components of the unit. The unit was preceded by a pre-test that was administered to all of the participants. The purpose of the pre-test was to determine if there was a need to review and thus deepen the understanding of transformations by the teachers.

Presentation of Findings

The questions and their responses are shown in Table 2.
Table 2. Unit Pre-Test

<table>
<thead>
<tr>
<th>Pre-Test Question</th>
<th>Responses</th>
</tr>
</thead>
</table>
| #3) Triangle ABC and triangle LMN are shown in the coordinate plane below.       | • “It tells me they are congruent so what is the point of this question—why do I need a transformation.” (Participant 4)  
• “I do not know where to start.” (Participant 1, Participant 3)  
• Many left blank—when asked they said “they had no idea what to do since there were no markings.” (Participant 2)  
• “Explain? I have no idea how to accomplish this task.” (Participant 6)  
• “Triangles have different names so how can I know what to do?”  
• No Response (Participant 8, Participant 7, Participant 5)  
• “There is no grid so how can I count to find line of reflection?” (Participant 3)  
• “How can I know I found the right line of reflection?” (Participant 7)  
• “Since each point is a different distance apart there is no line of reflection.” (Participant 2)  
• “Wow!” (Participant 4)  
• “Have no idea how to start.” (Participant 1, Participant 5, Participant 6)  
• All participants in the study left this question blank.  
• Verbal questioning led to the following responses.  
  o “I knew how to identify how the triangles were congruent but not what was the ‘criteria?’” (Participant 2, Participant 4)  
  o “Still uncertain how to use a transformation to show the triangles are congruent.” (Participant 1)  
  o “Without a coordinate plane, I do not know where to begin.” (Participant 6) |
| #4) In the figure below, there is a reflection that transforms triangle A to triangle B. Use a straightedge and compass to construct the line of reflection and list the steps of the construction. |                                                                                                                                                                                                           |
| #5) Use the triangles and to answer the questions. Given information:  
  a. What criteria for triangle congruence (ASA, SAS, SSS) implies that  
  b. Describe a sequence of rigid transformations that shows  |                                                                                                                                                                                                           |

Source: Item #3) PACC Geometry Assessments 10.1, no. 9 (see Permission, p. 184); Item #4) EngageNY (2013, p. 161); Item #5) EngageNY, (2013, p. 261);
Based upon the responses to the pre-test, the following topics were focused upon: deepening the understanding of how to construct a transformation, how to notate a function that represents the transformation, and how to apply a transformation in the plane. The “Transformation Review Sheet for Teachers” (see Appendix D, Section III) was created in order to guide the teachers towards proper function notations. A review of transformations without the aid of a coordinate plane was conducted with the participants (see Figure 6, Tables 3 and 4).

When we perform transformations without using a coordinate grid it is commonly called Synthetic Geometry.

Notation: The most “standard” notation for transformations is as follows:

- Translation of $(x, y)$ maps onto $T_{(a, b)} \rightarrow (x \pm a, y \pm b)$
- Reflection over line $P$
- Rotation about point $A$, $\theta$ degrees $R_{A, \theta}$
- Dilation by a scale factor of “$k$” $D_k$

Figure 6. Transformation Cheat Sheet for Teachers
Table 3. Review of Transformations without the Coordinate Plane

<table>
<thead>
<tr>
<th>Pre-Test Question</th>
<th>Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Some practice not on a coordinate grid.</td>
<td>- “The use of patty paper gave me a better understanding of a transformation and how every point moves.” (Participant 8)</td>
</tr>
<tr>
<td>1. Translate square NEAR 3 cm up the line BE.</td>
<td>- “The use of patty paper will definitely be useful to show students how every point moves up the line.” (Participant 2, Participant 4, Participant 5)</td>
</tr>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td>- “Transformations without a grid were interesting for me to learn!” (Participant 1)</td>
</tr>
<tr>
<td></td>
<td>- “I am missing more knowledge of transformations and rotations. I have never worked with them.” (Participant 3, Participant 6, Participant 7)</td>
</tr>
<tr>
<td><img src="image2" alt="Diagram" /></td>
<td>- “The practice of folding to find line of reflection was interesting.” (Participant 3, Participant 6)</td>
</tr>
<tr>
<td></td>
<td>- “I learned that constructions could be applied to find line of reflection.” (Participant 8)</td>
</tr>
<tr>
<td></td>
<td>- “I learned how to construct reflections and transformations.” (Participant 2)</td>
</tr>
<tr>
<td></td>
<td>- “Used to apply midpoint and perpendicular bisector to find line of reflection.” (Participant 1)</td>
</tr>
<tr>
<td><img src="image3" alt="Diagram" /></td>
<td>- “Orientation is reversed on reflection.” (Participant 6)</td>
</tr>
<tr>
<td></td>
<td>- “How to construct perpendicular bisector of like vertices.” (Participant 4)</td>
</tr>
<tr>
<td></td>
<td>- “Labeling but no work.” (Participant 5, Participant 7)</td>
</tr>
</tbody>
</table>
Table 4. Mapping Application Problems

<table>
<thead>
<tr>
<th>Pre-Test Question</th>
<th>Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the line of reflection for the following:</td>
<td>• “I learned how to reflect without a grid by measuring distance.” (Participant 1)</td>
</tr>
<tr>
<td></td>
<td>• “Reflection along a non-horizontal or non-vertical line using patty paper was new to me.” (Participant 4)</td>
</tr>
<tr>
<td></td>
<td>• “I learned how to label pre-image and image.” (Participant 8)</td>
</tr>
<tr>
<td></td>
<td>• “Midpoint or perpendicular bisector?” (Participant 3, Participant 7)</td>
</tr>
<tr>
<td></td>
<td>• “Connect any two corresponding vertices or find midpoint by measuring distances.” (Participant 5, Participant 2)</td>
</tr>
<tr>
<td></td>
<td>• “Construct the perpendicular bisector among the common vertices.” (Participant 1, Participant 6)</td>
</tr>
<tr>
<td>Rotations-rotations are generally performed by finding an angle from each vertex to the point of reflection. While this process works it can be quite &quot;messy”. We will instead perform a rotation using a reflection in two non-parallel lines.</td>
<td>• “It would be nice to continue with more rotations.” (Participant 2)</td>
</tr>
<tr>
<td>Rotate triangle TOP 60 degrees counter clockwise about point M.</td>
<td>• “I want to know why the rotation method works.” (Participant 3)</td>
</tr>
<tr>
<td></td>
<td>• “Even number of reflections = rotations.” (Participant 5, Participant 4)</td>
</tr>
<tr>
<td></td>
<td>• “I totally forgot to look for center of rotation to answer this question.” (Participant 7)</td>
</tr>
<tr>
<td></td>
<td>• “Why does this work?” (Participant 1, Participant 8)</td>
</tr>
<tr>
<td></td>
<td>• “Two reflections through non parallel lines is a rotation.” (Participant 6)</td>
</tr>
</tbody>
</table>
Once the review of transformations and the applications to show congruence had been completed, a quiz (see Table 5) was administered to check for an increase in understanding of transformations.
Table 5. Quiz

<table>
<thead>
<tr>
<th>QUIZ QUESTIONS</th>
<th>RESPONSES</th>
</tr>
</thead>
</table>
| Use the following diagram to answer the questions below. You may use a protractor and a ruler as needed. | • “Transformations include rotations and reflections.” (Participant 3)  
• “Having a hard time finding each line.” (Participant 7, Participant 8)  
• “To find point of rotation from reflections, find intersection of reflection lines.” (Participant 3)  
• No response (Participant 4, Participant 6)  
• “Transformation is a function.” (Participant 2) |

1. Describe the transformation from the pre-image $T_0$ to the image $T_1$.

   • “Image $T_0$ is reflection over line a, where each point is the same distance from line a.” (Participant 7, Participant 3)  
   • “Transformation is a function.” (Participant 5)  
   • “Reflection over line a.” (Participant 4, Participant 2)  
   • “Pre-image is over line which is a reflection.” (Participant 6)  
   • No Response (Participant 8, Participant 1) |

2. Describe the transformation from the image $T_1$ to the image $T_2$.

   • “Image $T_1$ is reflective over line b, where each point of $T_1$ is the same distance from line b.” (Participant 7, Participant 2, Participant 1)  
   • “Reflection over line b.” (Participant 6, Participant 4, Participant 3)  
   • “Reflect over line a and then over line b, rotation about point P.” (Participant 5) |
Table 5. Quiz (cont’d.)

3. Describe a single transformation of the plane that takes the pre-image to the image $T_2$.

- “$T_o$ is rotated 90 degrees and translated.” (Participant 7, Participant 1)
- “Extend lines of rotation, where both lines meet that is the point of rotation to get the final image.” (Participant 6, Participant 5)
- “Rotation along the line a Participant 4.” (Participant 2)
- “Rotation about 87 degrees about Pt. B.” (Participant 3)

Upon the completion of the review of the basic isometries, it was now time to begin applying the transformations to show congruence in polygons (see Table 6; and Appendix D).
Table 6. Application of Congruent Transformations

Rigid motion can be used to show that angle XYM is congruent to angle XZM. We all know that students struggle to understand what seems so simple to us. Transformations allow students to see the congruent parts using rigid motions-line up the congruent parts and thus develop a proof rather than memorize the steps.

The following questions are taken from Geometry by Brannan, Esplen, and Gray (2007, p. 47).

<table>
<thead>
<tr>
<th>APPLICATION OF TRANSFORMATIONS</th>
<th>RESPONSES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prove that if $\triangle ABC$ and $\triangle DEF$ are two triangles such that if $AB \cong DE$, $AC \cong DF$, and $\angle BAC \cong \angle EDF$, then $BC \cong EF$, $\angle ABC \cong \angle DEF$, and $\angle ACB \cong \angle DFE$.</td>
<td></td>
</tr>
</tbody>
</table>

Given that translations and reflections preserve length it is sufficient to show that there is an Isometry, which maps $\triangle ABC$ onto $\triangle DEF$. We construct the Isometry in stages as follows:

1. Start with a translation that maps A to D.

   • “I constructed a vector that maps A to D.” (Participant 1, Participant 3, Participant 6)
   • “Horizontal Translation.” (Participant 5)
   • No Response (Participant 2, Participant 4, Participant 7, Participant 8)

2. Reflect $\triangle DB'C'$ such that point C' coincides with point F. This rotation maps $\triangle DB'C'$ onto $\triangle DB''F''$.

   • “Can we prove SAS using a transformation?” (Participant 2, Participant 7)
   • “How will this look in proof form?” (Participant 1, Participant 4, Participant 8)
   • “Drop A' because A'=D.” (Participant 3)
   • No Response (Participant 5, Participant 6)
Next, the actual unit, which was written to be used in a high school classroom, was given to the teachers to work through. The focus of this unit was Lesson Four (see Appendix C), where teachers began to map one triangle onto another to show that the triangles were congruent. Upon establishing the figures were congruent, teachers compared the SAS (side-angle-side) congruence postulate to the use of rigid motions to establish that the two triangles were indeed congruent (Lesson Six, Appendix C). In order to determine teachers’ responses to the use of transformations rather than reliance on just the postulate, they were asked to reflect on the activity and answer the following questions.
### Table 7. Responses to Reflection on Activities

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>RESPONSEs</th>
</tr>
</thead>
</table>
| 1. Does the use of rigid motion to show congruence give a deeper understanding of the triangle congruence criteria such as SSS, SAS, and ASA? | • “Yes.” (Participant 1, Participant 6)  
• “Yes, because when we use a rigid motion the triangles coincide which show the students the triangles are congruent by SSS, SAS, and ASA.” (Participant 4)  
• “I think it is a discovery of a sort.” (Participant 2, Participant 5)  
• “Yes we are able to use the tools and see that the triangles do coincide with the criteria.” (Participant 3)  
• “Yes, because they know rigid motion and can see why SSS, SAS, and ASA work.” (Participant 4)  
• “Yes, it allows students to prove without using SSS, SAS, and ASA and to understand where the justification comes from.” (Participant 7, Participant 8) |
| 2. Has this unit increased your understanding of transformations? If so, how? Be specific. | • “Yes it has because I did not know that you could do transformations off the grid.” (Participant 1)  
• “Yes, I have a better understanding of how to connect rigid motion into proving SSS, SAS, and ASA. This helps the visual learner.” (Participant 2)  
• “Yes, increased factual and historical knowledge.” (Participant 3)  
• “Yes, before I just memorized the (SSS, SAS, and ASA) criteria and now I have a visual and a more complete understanding of why we have this criteria.” (Participant 4)  
• “Yes, this has increased my understanding of transformations because before I didn’t know why we could only use three criteria to move congruence.” (Participant 5) |
Table 7. Responses to Reflection on Activities (cont’d.)

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>RESPONSES</th>
</tr>
</thead>
<tbody>
<tr>
<td>• “It has increased my understanding, but I would need a lot more exposure. I have never been introduced to it (transformations) before as a student or as a teacher.” (Participant 6)</td>
<td></td>
</tr>
<tr>
<td>• “The unit has given me a refresher in geometry and reminded me what I need to know when assisting students.” (Participant 7, Participant 8)</td>
<td></td>
</tr>
<tr>
<td>3. Has your understanding of the application of rigid motion to show congruence increased? If so, how? Be specific.</td>
<td>• “Yes, I did not know that there was any connection between the congruence postulates and transformations.” (Participant 1)</td>
</tr>
<tr>
<td>• “Yes, the application of rigid motion has showed me visually a proof without using the proof statements. Rigid motion has given me a different view of understanding the postulates.” (Participant 2)</td>
<td></td>
</tr>
<tr>
<td>• “Yes, as I have applied rigid motions in the classroom I have been able to make sense of congruent triangles.” (Participant 3, Participant 4)</td>
<td></td>
</tr>
<tr>
<td>• “Yes, my understanding of application of rigid motion to show congruence increased because before I thought we needed six criteria of congruence but now I see we don’t and can actually visualize it.” (Participant 5)</td>
<td></td>
</tr>
<tr>
<td>• “Yes it has. Not by a lot but more than I had prior to today.” (Participant 6)</td>
<td></td>
</tr>
<tr>
<td>• “I feel like I have relearned a lot more.” (Participant 7, Participant 8)</td>
<td></td>
</tr>
</tbody>
</table>

At the conclusion of the unit teachers, were asked to take a brief assessment to show their mastery of the use of rigid motions and their
application to show congruence in polygons (see Figure 7; Appendix D, Section V, Final Assessment, Problem #2).

2. Rigid motions can be used to prove that the three triangle congruence criteria SSS, ASA, and SAS are sufficient to guarantee that triangles are ALWAYS congruent.

Given: \( \angle A \equiv \angle D, \overline{AB} \equiv \overline{DE}, \overline{B} \equiv \overline{E} \). Mark the diagrams below to show the given information.

Your task: Use rigid motions to prove that \( \triangle ABC \equiv \triangle DEF \).

Describe a sequence of rigid motions that will move \( \triangle ABC \) onto \( \triangle DEF \). Since these triangles represent the general case, describe the motions as precisely as possible, even though we don’t have specific measurements.

Figure 7. Brief Assessment Question

Upon completion of the application of the review, unit, and assessments, the participants were asked to complete an online survey. The questions from the survey are located in Appendix D, Using Transformations to Identify Congruent Triangles section; and the results follow in Figure 8 as well as in Appendix D, Summary section.
Summary

Before the unit I felt knowledgable about transformations

<table>
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<th>Count</th>
<th>Percentage</th>
</tr>
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<tbody>
<tr>
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<td>1</td>
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<td>2</td>
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<td>4</td>
<td>1</td>
<td>20%</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0%</td>
</tr>
</tbody>
</table>

The fact that transformations are functions give me a foundation for their application in other areas of mathematics.

<table>
<thead>
<tr>
<th>Score</th>
<th>Count</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
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<td>60%</td>
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<tr>
<td>5</td>
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<td>0%</td>
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</table>

I will take the concept of end design back to my classroom

<table>
<thead>
<tr>
<th>Option</th>
<th>Count</th>
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<tbody>
<tr>
<td>Yes [4]</td>
<td></td>
</tr>
<tr>
<td>I am not sure [1]</td>
<td></td>
</tr>
<tr>
<td>No [0]</td>
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</tbody>
</table>

Figure 8. Online Survey Results
Figure 8. Online Survey Results (cont’d.)

<table>
<thead>
<tr>
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</tr>
<tr>
<td>I am not sure</td>
<td>1</td>
<td>20%</td>
</tr>
</tbody>
</table>

**My understanding of transformations has increased by a level of**

- 100%: 2 (40%)
- 50%: 2 (40%)
- 25%: 1 (20%)

**The use of transformations as a definition for congruent triangles is clear after the unit.**

- Yes: 3 (60%)
- No: 2 (40%)

**Comments:**

- Methodology is creative and involving.
- Straight forward, aligned with current common core standards.
- Teaching transformations was a positive learning experience as a teacher. There were many concepts in transformations that I had to remind myself. Proving that two triangles are congruent by discovery was one of the assignments I found beneficial to continue on to proofs. Instead of memorizing the conjectures, I was able to discover why they are the conjectures true.
- I thought the unit helped with the understanding of transformations.
- The unit is great just that I have no previous background knowledge of transformations.

**Overall thoughts**

- I will suggest to start from the beginning of the unit with function notation. I think that this will benefit students when they get to two sided proofs.
- Appears to be well organized and easy to follow. Written to allow for student success.
- Thank you for making your work available to us.
- The unit is very helpful. It is easy to follow and understand some of it, even if the person has no knowledge of transformations.
just really did not see how transformations proved congruency

I teach

<table>
<thead>
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<th>Subject</th>
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</thead>
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<tr>
<td>Geometry</td>
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<td>Algebra 2</td>
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<tr>
<td>Statistics</td>
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<tr>
<td>Pre-Calculus</td>
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<td>0%</td>
</tr>
<tr>
<td>Other</td>
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<td>60%</td>
</tr>
</tbody>
</table>

Figure 8. Online Survey Results (cont’d.)
An analysis of the teachers’ responses to the items in the unit revealed that they could be grouped into four themes:

(a) teachers’ content knowledge regarding transformations
(b) teachers’ attitude towards transformations
(c) teachers’ knowledge regarding the teaching and application of transformations to their current subject area
(d) areas of further study for teachers and instruction.

Bonner (2006) stated that inquiry can be one of the five models of teacher professional development. Inquiry can be based on three assumptions noted by Susan Loucks-Horsley and associates in 1987:

- Teachers are intelligent, inquiring individuals with legitimate expertise and important experience.
- Teachers are inclined to search for data to answer pressing questions and to reflect on the data to formulate solutions.
- Teachers will develop new understandings as they formulate their own questions and collect their own data to answer them.

(Sparks and Loucks-Horsley (1989, Underlying Assumptions)
Teacher Content Knowledge Regarding Transformations

As the teachers worked through the problems that were presented to them, discussed them with their partners, and were asked questions during the current study, they began to gain new content knowledge, ideas, and mathematical vocabulary about transformations. These questions, in the form of a pre-test, showed that some time would have to be spent deepening the participants’ mathematical content knowledge of transformations.

As stated by Hollebrands (2004), “students’ initial understandings . . . [of] transformations [is limited to] as actions or motions performed on a figure” (p. 213). Students’ lack the knowledge that the figure has actually moved, yet it is still the same figure. The same “lack of knowledge” of transformations was shown to be true for the teachers in this study. One participant wrote “moving graphs using equations showed the fact that all points move” (Participant 5). Another wrote, “transformations are like functions” (Participant 8). Both of these statements show that teachers also lack the knowledge that a transformation is a function—not just a set of rules applied to a figure in geometry.

Once the initial pre-test was administered, time was taken to review isometries, why a transformation is a function that applies to all points in the plane, and why transformations are essential components in the understanding of functions not just the movement of geometric figures.

When the teachers had a better understanding of transformations in the coordinate plane, the unit moved them toward the application of transformations without the aid of the coordinate plane. This application was difficult for many of
the teachers as most curriculums start and end at transformations with the aid of the coordinate plane.

In order to allow teachers to be comfortable and successful, it was important to give them time to explore options using patty paper, rulers, and compasses. Not only did this help to reduce the immediate stress of “not knowing” what to do, it also allowed them to see it from a student perspective. After time for exploration, before instruction, teachers were asked to reflect on any insights or questions they might have. One teacher wrote, “Transformations without a grid is interesting for me to learn and teach my students” (Participant 7). This one statement guided our further enquiry into transformations not on the coordinate plane.

Increasing teacher knowledge of transformations is the corner stone to increasing instruction of transformations in the classroom.

In 2010, the Common Core State Standards Initiative followed Klein’s lead and centered geometry on transformations, stating, ‘The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations rotations, reflections, and combinations of these . . . (CCSSI 2010, p. 74). (Dick & Childrey, 2012, p. 622)

Beyond the basic ability to apply a single transformation without the aid of the coordinate plane, the participants began to develop an understanding of how to identify the transformation given an image and pre-image. This additional
content knowledge gave them a more complete picture of the importance of transformations and their application to prove two figures are congruent.

**Teacher Attitude towards Transformations**

Dick and Childrey (2012) made the assertion that current high school math teachers are treating transformations lightly or not at all.

The study of transformations usually begins—and, unfortunately ends—with these four elements: three simple reflections (across the x- and y-axes and across the line y=x); three simple rotations (of 90, 180, and 270 degrees counterclockwise about the origin); dilations centered at the origin, and finally, translations. (p. 622)

This claim was also supported by the study. Many teachers reflected on the question, “Why do we as teachers avoid teaching transformations?” The following are the responses from the participants.

- “I think the lack of clear understanding of what is given and the order in which it’s presented. It is not a logical progression” (Participant 7).
- “Transformations are complex and difficult to understand. If I do not truly understand them, then I am not confident enough to teach them” (Participant 5).
- “Many times it is difficult to explain to students the process of what we are thinking” (Participant 2).
- “I first need to understand it myself and see myself how they work in order to teach it to the students” (Participant 4).
• “I don’t know the answer without having come across them while teaching 5, 6, and 7th grade in math. I can see why a math teacher would struggle if they didn’t know how to do them themselves” (Participant 6).
• “I think we avoid them (transformations) because we are uncomfortable with them. We have not seen their importance so we don’t think they are” (Participant 1).

This question was asked during the first week we were together and it is evident from many articles that teachers treat transformations lightly or choose not to teach them at all. It is important to again note that this was a group of experienced teachers who still have a “fear” of transformations. This first question guided our next meeting. The goal was to not only remove the fear element but also to deepen the teachers understanding of transformations as a common thread throughout all of their disciplines at the high school level.

The CCSS (2013) set transformations as the foundation that teachers must understand—that transformations of geometric figures are the building block for applying transformations to algebraic functions. In order to make this connection for the teachers in my study, time was spent developing the function notation of isometries as applied to geometric figures. Teachers were asked to complete the following problem shown in Figure 9.
Once the participants had completed the problem, they were asked to make any connections to other high school mathematics courses. One teacher stated, “The *aha-moment* was the refresher on translations are functions” (Participant 7) which is a reflection on the disconnection between geometric functions and algebraic functions.

From this problem it appears if we are to improve teacher attitude toward transformations then we must help them to make the connections across the disciplines. We must show them that geometry and its applications are not to be treated in “isolation” but that they are an essential building block in the students’ understanding of all functions. As shown in Quiz 2 of Appendix C, problem 2, a deeper understanding of transformations as functions is needed in order to be successful on this assessment.
Teacher Knowledge Regarding the Teaching and Application of Transformations

In order to facilitate deepening of the understanding of transformations, the participants worked through a series of transformations in which they were not given a coordinate grid to work with. Discussion followed as to possible tools that would be needed, along with strategies for success. As a group, the participants decided to make use of patty paper, straight edge, and geometric constructions to aid in their understanding. Once the activity had been completed, the participants were asked, “State two things you learned today.” The responses were as follows:

- “How it is useful to use patty paper and how easy it is to show students how to translate using this method” (Participant 2).
- “Transformations are like functions, when we use them in calculus” (Participant 4).
- “I learned to construct reflections and translations” (Participant 6).
- “I learned to reflect without a grid by measuring distance. I learned constructions could be applied to transformations” (Participant 1).
- “I learned how to use patty paper to translate a figure. You use transformations from algebra to calculus whenever you are translating or transforming an equation” (Participant 8).

Seago et al. (2012) stated, . . . teachers will need opportunities to gain ‘mathematical knowledge for teaching’ (Ball, Thames, & Phelps, 2008) in the areas of geometric
transformations and similarity, including a deep understanding of the mathematical content and the fluency to make instructional decisions that support students' learning of this content. (p. A b c d e + 3) The time spent “learning” and becoming more comfortable with transformations confirms Seago’s idea that teachers need to deepen their knowledge before they can instruct students. As shown by the responses from the participants, their knowledge has begun to increase but still may not be deep enough to make those instructional decisions.

Often as teachers we “glance” over the unit on transformations without making any connections to their application to other areas of geometry. The heart of this study was to show teachers how transformations can be applied to show that two polygons are congruent. Once teachers had an understanding of transformations, it was time to “apply” these transformations to show congruence in polygons.

In Euclid’s Book I, Proposition 4 he stated:

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. (Heath, 2012, p. 247)

Euclid (Heath, 2012) left much for us to think about both as students and as teachers. The teachers in this study were asked to perform the
transformations necessary to show evidence of the Side-Angle-Side (SAS) postulate using two arbitrary triangles. The purpose of this problem was to show that we only need to show congruence using three criteria not six (see Figure 10).

2. Rigid motions can be used to prove that the three triangle congruence criteria SSS, ASA, and SAS are sufficient to guarantee that triangles are ALWAYS congruent.

Given: \( \angle A \equiv \angle D, AB \equiv DE, \angle B \equiv \angle E \). Mark the diagrams below to show the given information.

Your task: Use rigid motions to prove that \( \triangle ABC \equiv \triangle DEF \)

Describe a sequence of rigid motions that will move \( \triangle ABC \) onto \( \triangle DEF \). Since these triangles represent the general case, describe the motions as precisely as possible, even though we don’t have specific measurements.

![Figure 10. Sample Problem from Unit](image)

The teachers in the study worked together to perform the rigid motions necessary to map one triangle onto the other triangle. Working through this “arbitrary” case gave teachers a deeper understanding of why SAS works for proving two triangles congruent. Teachers were then asked to reflect on this problem and state what they learned, what they still needed to understand, and how this applies to their classrooms. Their responses to the question, “Does the
use of rigid motion to show congruence give a deeper understanding of the triangle congruence criteria such as SSS, SAS and ASA?” are as follows:

- “Yes, it allows students to prove without using SSS, SAS, and ASA and to understand where the justification comes from” (Participant 7).
- “Yes, because when we use a rigid motion the triangles coincide which shows the students the triangles are congruent” (Participant 2).
- “Yes, because they know rigid motion and can see why SSS, SAS, and ASA work” (Participant 5).
- “Yes, we are able to use the tools and see that the triangles do coincide with the criteria for SSS, SAS, and ASA” (Participant 4).

It is important to note here that one teacher had a difficult time with this proof and asked “How do I know it ALWAYS works” (Participant 3)? This is a valid question and one students will ask of a teacher. The teacher was asked to create two triangles such that two parts were congruent that could not be mapped on to one another by a series of rigid transformations. The teacher spent quite a bit of time trying to find the exception and is, in fact, still trying to find that exception.

Teachers were then given a series of problems where they were asked to show that two triangles are congruent, given two congruent parts, using a series of rigid motions, and then identify the congruence postulate that was discovered from the rigid motion. The conversation around the groups indicated that teachers’ understanding of congruence through rigid motions had deepened over the course of this study.
Areas of Improvement for Teachers and Instruction

In order to improve instruction, teachers must spend time deepening their understanding of any topic they will be teaching in the classroom. This seems like an obvious idea, yet the time needed to deepen understanding is not easily accessible for most classroom teachers.

Each of the participant’s responses to guiding questions was coded for these categories in order to analyze the effectiveness of the application of the Unit. It is important to note that the unit was designed to be used in a high school geometry class and thus the teachers were asked to analyze their content knowledge based on the unit’s design. The goal was that when the teachers completed this study, they would leave with a knowledge of what a transformation is, how transformations are used to show congruence and similarity in polygons, and how transformations are a common thread from basic mathematics to calculus.

Upon final reflection to the overall unit, the teachers’ responses to both the survey and the problems showed that there was an overall increase in their content knowledge.

While there is no absolute way to measure how much the participants’ content knowledge had increased, it is evident from the results that there is a definite need for both teachers and educators of teachers to spend a significant amount of time deepening their knowledge of transformations and their application to show congruence in polygons as well as their application to other areas of mathematics.
CHAPTER 6
CONCLUSION

Introduction

At the beginning of the Master of Arts in Teaching Mathematics courses, I gained a greater understanding of math and its application to our world. I also learned how to teach transformations across the curriculum instead of treating them in isolation. When I entered this program, I was like most teachers in that I thought transformations were isolated to a small, unimportant unit in geometry. I was unaware of their applications and struggled as a teacher on how to use the transformations in the classroom.

However, as I worked through my thesis, my unit, and through this program, I gained a deeper knowledge of how critical transformations are to our understanding of congruence and similarity theorems as well as, congruence as a concept. Each day the math I thought I knew was challenged, stretched, and sometimes crushed.

While researching this topic, the sources agreed that educators would need a significant amount of professional development to fully understand transformations and their importance to the Common Core State Standards. This unit of study is in no way a complete course in transformations; however, the results of the study show that there was an increase in mathematical content knowledge for the participants.
A significant limitation of this study was that it focused on how the teachers understood the mathematical content rather than on their classroom instruction. Since time was not spent on how the teachers develop their lessons, what guided their instruction, or how they developed the lesson to reflect the standards, this study is in no way a comprehensive view of how transformations are taught in the high school classroom.

Teachers, as well as students, need time to understand transformations and their application to other areas of mathematics. As was evident from this study, transformations are indeed an area where a deeper mathematical content knowledge is needed in order to be able to apply them to show that two figures are congruent. As teachers begin to fully implement the Common Core State Standards (2013), educators, administrators, and professors will need to be mindful of the importance of transformations. Many hours will need to be spent whether it is at the school site, district, or college campus to change how educators view transformations. Without the time spent on gaining the mathematical content knowledge, the Common Core State Standard thread of transformations will be lost and there will be no change in mathematics classes.

Future Research and Recommendations

No single study can fully answer all the questions or areas of concerns. Some of the questions not addressed by this study are:

1. How does direct instruction versus student exploration impact how students learn transformations?
2. How much professional development will teachers require in order to be fully prepared to teach Common Core State Standards?

3. Will giving teachers a greater foundation in transformations impact their students in a significant enough manner to improve standardized test scores?

I do know that I set out with an ambitious goal of changing how one group of teachers view and apply transformations. While this study may not have changed all of the teachers’ understanding of transformations, I do know that this unit challenged them to think much more deeply about transformations and how they can be used in their classrooms. This is demonstrated by responses such as “my understanding of application of rigid motion to show congruence increased because before I thought we needed six criteria of congruence but now I see we don’t” (Participant 5) and “I have applied rigid motions in the classroom and I also have been able to make sense of congruent triangles using rigid motion” (Participant 4). I also know that working with the teachers taught me that even though I may think I know what I want to teach I still need to be mindful of who I am teaching, why I am teaching, and what I can learn from the experience.
APPENDIX A

GLOSSARY
When we communicate in any language we must be careful to use the correct words or our meanings may be misinterpreted. The same is true for the language of transformations. Each of the following terms was used throughout this text; therefore, each term will be defined as well as represented visually.

**Isometry**

An *isometry* is a “*length-preserving transformation’* (Coxeter & Greitzer, 1967, p. 80). Each of the following is an isometry: a translation along a line, a reflection in a line, a rotation about a point, and a glide reflection. A composition of two isometries is also an isometry. A more familiar term for isometry is rigid motion.

**Collineation**

A *collineation* is a transformation that maps lines to lines. Every isometry is a collineation. We will see examples of others. A collineation takes all the points on any line and maps them to another line. The use of collineation has its roots in the ideas of collinear points in a plane.

**Orientation**

Orientation is the way in which we view an object. A collineation either preserves or reverses orientation. A good way to understand orientation in the plane is to picture an angle as two rays with a common vertex. Imagine measuring an angle by sweeping the first ray to the second ray in a given direction, say counterclockwise. This is now an *oriented* angle. Now apply the collineation and keep track of whether the orientation is still counterclockwise or if it has changed to clockwise.
Types of Isometries

A translation is an isometry that moves each point through a given distance in a given direction, more commonly known as a vector. This means that if two points A and B are a specific distance apart on the pre-image then their image points $A'$ and $B'$ are that same distance apart after the transformation. Therefore it maps any line segment to a parallel line segment. A translation not only preserves distance it also preserves orientation. A composition of two translations is a translation.

![Figure A1. Horizontal Translation](image)

A rotation is an isometry that moves each point about a fixed point through a given angle. A rotation also preserves orientation as well as distance. A
composition of two or more rotations about the same point is a rotation about that point.

As teachers we must be careful to not state that a rotation is a “turn” as when we turn something it brings to mind the idea of a carousel or wheel that turns about a fixed point usually described as an axis. A rotation is an Isometry that takes a point or a set of points and rotates all of these points about a center of rotation. That center can be inside, outside or a point on the object. These rotations preserve distance yet do not preserve orientation.

Figure A2. A rotation 90° counter-clockwise about point D

A reflection fixes every point on a given line (axis) and maps every other point to that point such that the segment between these points is perpendicular to
the axis and bisected by it. A reflection preserves distance but not orientation. A reflection reverses orientation of an object. A composition of two reflections is a translation if the two axes are parallel; otherwise it is a rotation about the point where the axes intersect. Any composition of an even number of reflections is either a translation or a rotation.

Figure A3. A reflection over the line $x = -4$

A *glide* is a combination of a reflection with a translation parallel to the axis of the reflection. A glide preserves distance but not orientation.
Figure A4. A reflection over the y-axis and vertical translation -6 units

A rigid motion is another name for an isometry. Every rigid motion of the plane is one of the four types above: translation, rotation, reflection, or glide. The first two types preserve orientation, whereas the second two types reverse orientation.

Non-rigid motion does not preserve distance. For example, any dilation is a non-rigid motion. A dilation will transform any figure into a similar figure. When we create a similar figure using dilation, angles are preserved; however, distances are increased or decreased by some given ratio (Coxeter & Greitzer, 1967, 94).
Another term we will use when discussing transformations is **invariant**. *Invariant* is used to describe quantities that remain unchanged under certain classes of transformations. For example, length is invariant under rigid motion.

While all of the examples used in this section are being performed on the coordinate plane, geometric transformations are limited to the coordinate plane. The importance of this unit is to develop the application of transformations without the use of the coordinate plane.

Figure A5. Dilation about point D by a scale factor of 1.5
APPENDIX B

INFORMED CONSENT
April 15, 2014

Ms. Tamar V. Bonn
cc: Prof. John Sarli
Department of Mathematics
California State University, San Bernardino
5500 University Parkway
San Bernardino, California 92407

Dear Ms. Bonn:

Your application to use human subjects, titled “Discovering and Applying Geometric Transformations to Congruence and Similarity” has been reviewed and approved by the Institutional Review Board (IRB). The attached informed consent document has been stamped and signed by the IRB chairperson. All subsequent copies used must be this officially approved version. A change in your informed consent (no matter how minor the change) requires resubmission of your protocol as amended. Your application is approved for one year from April 15, 2014 through April 14, 2015. One month prior to the approval end date you need to file for a renewal if you have not completed your research. See additional requirements (items 1 – 4) of your approval below.

Your responsibilities as the researcher/investigator reporting to the IRB Committee include the following 4 requirements as mandated by the Code of Federal Regulations 45 CFR 46 listed below. Please note that the protocol change form and renewal form are located on the IRB website under the forms menu. Failure to notify the IRB of the above may result in disciplinary action. You are required to keep copies of the informed consent forms and data for at least three years. Please notify the IRB Research Compliance Officer for any of the following:

1) Submit a protocol change form if any changes (no matter how minor) are proposed in your research protocol for review and approval of the IRB before implemented in your research,
2) If any unanticipated/adverse events are experienced by subjects during your research,
3) To apply for renewal and continuing review of your protocol one month prior to the protocols end date,
4) When your project has ended by notifying the IRB Research Compliance Officer.

The CSUSB IRB has not evaluated your proposal for scientific merit, except to weigh the risk to the human participants and the aspects of the proposal related to potential risk and benefit. This approval notice does not replace any departmental or additional approvals which may be required.

If you have any questions regarding the IRB decision, please contact Michael Gillespie, the IRB Compliance Officer. Mr. Michael Gillespie can be reached by phone at (909) 537-7588, by fax at (909) 537-7028, or by email at mgillespie@csusb.edu. Please include your application approval identification number (listed at the top) in all correspondence.

Best of luck with your research.

Sincerely,

Sharen Ward, Ph.D.
Chair
Institutional Review Board

cc: Prof. John Sarli, Department of Mathematics
INFORMED CONSENT

The study in which you are being asked to participate is designed to investigate transformations and their use with regards to congruence and similarity in the geometry classroom. This study is being conducted by Tamara Voorhies Bonn under the supervision of Dr. John Sarli, Professor of Mathematics, California State University, San Bernardino. This study has been approved by the Institutional Review Board, California State University, San Bernardino.

PURPOSE: To address your understandings as a teacher of transformations and how it applies to congruence and similarity. To test a new unit I created to see if it helps clarify the meaning of transformations, congruence and similarity.

DESCRIPTION: You (the participant) will be given a pretest and then categorized into groups based on your responses. Those that show that they have a limited view of transformations will participate in the unit and will be divided into two groups. One group will learn how transformations are an integral tool to show congruence and similarity of polygons using the unit that I created the other group will only take the pre- and post-test. After you have completed the unit, you will be tested again to see if the unit was a success or not by comparing the scores of those who participated in the unit vs. those who only received the pre- and post-test.

PARTICIPATION: Your participation is voluntary, refusal to participate will involve no penalty or loss of benefits to which you are otherwise entitled and you may discontinue participation at any time without penalty or loss of benefits, to which you are otherwise entitled.

ANONYMITY: You will remain anonymous in my paper. You will be given personal identification codes to be used throughout the study. The information will be stored on my personal hard drive that will be protected by a password.

DURATION: The total duration for your participation will be approximately five 30 minute sessions over the course of 4 weeks.

RISKS: There may be a risk of social discomfort because of the teachers being placed in a group, and they may think that they were placed in the group due to low achievement on the pre-test.

BENEFITS: Teachers will benefit from participation because they will receive additional tools with which to use in the teaching of congruence and similarity.

VIDEO/AUDIO/PHOTOGRAPH: No video/audio/photographs will be used.

CONTACT: Professor John Sarli, Professor of Mathematics, California State University.

mjetter@csusb.edu, (909) 537-5374

RESULTS: Results to this project can be found at Indian Springs High School in room J-10, or at California State University, San Bernardino in the department of Mathematics.
SIGNATURE:

Signature: ____________________________  Date: ______
### Unit 1: Transformations and Congruence

In this unit, students will explore translations, reflections, and rotations. This unit should begin by giving a pre-assessment to determine the teacher/students' understanding of transformations. Once students/teachers have developed an understanding of each transformation and some of their properties, begin verifying the properties through experiments. Design tasks with the initial and transformed geometric item provided such that the student/teacher can determine a transformation (or series of transformations) that takes the pre-image to the image. The criteria for figure congruence will be established through rigid motion and applied to help students/teachers create a sequence of transformations that would carry one figure onto another. You may consider beginning transformations of figures on blank paper and then extend to the coordinate plane. Students/Teachers will begin to formalize the effects of translations, reflections, and rotations in terms of congruence and similarity.

### Major Cluster Standards

<table>
<thead>
<tr>
<th>Understand congruence and similarity using physical models, transparencies, or geometry software. 8.G.1</th>
<th>Verify experimentally the properties of rotations, reflections, and translations: a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines. 8.G.2 Understand congruence and similarity using physical models, transparencies, or geometry software. Understand a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations. Given two congruent figures, describe a sequence that exhibits the congruence between them. Congruence HS.G.CO.5 Experiment with transformations in the plane 8.G.2 Understanding congruence in terms of rigid motion Use geometric descriptions of rigid motions to transform figures and to predict the effect of given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. HS.G.CO.6 Understand congruence in terms of rigid motion Use the definition of congruence in terms of rigid motions to show that two triangles are congruent. HS.G.CO.7 Understand congruence in terms of rigid motions</th>
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<td>8.G.1 The skill of transforming geometric items as well as the properties of transforming these items will extend to develop and establish the criteria for figure similarity. 8.G.2 This same skill is used in 8.G.4 with the addition of dilations. HS.G.CO.5 The skill of being able to transform a figure and be able to specify the sequence that takes one figure to another. HS.G.CO.6 The skill of using a rigid motion to define congruent figures is first applied. HS.G.CO.7 The application of congruent triangles as defined by</td>
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congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

**HS.G.CO.8** Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions.

**HS.G.CO.8** The use of transformations to prove congruence under a rigid motion

<table>
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<th>Focus Standards of Mathematical Practice</th>
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<tr>
<td><strong>MP.3</strong> Construct viable arguments and critique the reasoning of others.</td>
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<tr>
<td><strong>MP.5</strong> Use appropriate tools strategically.</td>
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<td><strong>MP.6</strong> Attend to precision.</td>
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<tr>
<td><strong>MP.7</strong> Look for and make use of structure</td>
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Source: Prince George's County Public School. (n.d.) *Geometry transformation. (Common Core State Standards, G.Co.5; Understand congruence in terms of rigid motions, G.Co.6, G.Co.7, and G.CO.8)*
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<th>Standards</th>
<th>Description</th>
<th>Links/Resources</th>
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<tr>
<td>Lesson 1 of 10 : Unit Pre-Assessment</td>
<td>HS.G.CO.5 <strong>Experiment with transformations in the plane</strong></td>
<td>This lesson reviews transformations in the plane. Teacher will use results in order to know what topics need to be reviewed before new material is presented. Print the following document: Unit Pre-Assessment</td>
<td>Unit Pre-Assessment (pp. 10-11)  Review of Vocabulary for transformations (Word Search PDF-Pg. 12)</td>
</tr>
<tr>
<td>Lesson 2 of 10: Construct and Apply a Sequence of Rigid Motions</td>
<td>HS.G.CO.6 <strong>Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.</strong></td>
<td>This lesson will introduce students to the concept of congruent figures. Teacher will have student's use their knowledge of transformations to map one figure onto another and identify the congruent sides and angles.</td>
<td>EngageNY:Common Core Geometry Lesson 20 <a href="https://www.engageny.org/resource/geometry-module-1-topic-c-lesson-20">https://www.engageny.org/resource/geometry-module-1-topic-c-lesson-20</a>  Exit Ticket included in Engage NY Lesson 20  Prince George's County Public School. (n.d.) Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.6)</td>
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| Lesson 3 of 10: Use function notation to identify the transformation(s) | HS.G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. | This lesson will continue to use transformations to show that two figures are congruent. Teacher will use student’s prior knowledge of transformation notation to describe the transformations applied to the figure. | EngageNY-Common Core Geometry Lesson 21 [https://www.engageny.org/resource/geometry-module-1-topic-c-lesson-21](https://www.engageny.org/resource/geometry-module-1-topic-c-lesson-21)  
Prince George’s County Public School. (n.d.) Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.6) |
| Lesson 4 of 10 : Quiz | HS.G.CO.5 & HS.G.CO.6 | Students will show their understanding of the concepts taught in day 2-3 of the unit. This will be a formative assessment to check for understanding. Note: Teacher may choose to include some problems from the unit pre-assessment to make sure student understanding has increased. | Unit Quiz 1 (pg. 13-14)  
Partner Activity-Class Practice A (pp. 15-16) |
<table>
<thead>
<tr>
<th>Topic</th>
<th>Standards</th>
<th>Description</th>
<th>Links/Resources</th>
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</thead>
</table>
| Lesson 5 of 10: Criteria for showing two triangles are congruent | **HS.G.CO.7**: Understand congruence in terms of rigid motions. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.  
**HS.G.CO.8**: Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions. | Teacher will begin to develop the criteria for proving two triangles are congruent. Students will be introduced to the fact that only three criteria are needed to show congruence such as SSS, SAS and ASA. | Engage“”-Common Core Geometry-Lesson 22  
https://www.engageny.org/resource/geometry-module-1-topic-d-lesson-22  
Prince George’s County Public School. (n.d.) Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.7, and G.CO.8) |
| Lesson 6 of 10: Criteria for showing two triangles are congruent | **HS.G.CO.7**: Understand congruence in terms of rigid motions. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.  
**HS.G.CO.8**: Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions. | Teacher will continue to build the criteria for showing two triangles are congruent. | Engage“”-Common Core Geometry-Lesson 22  
https://www.engageny.org/resource/geometry-module-1-topic-d-lesson-22  
Exit Ticket included in lesson  
Prince George’s County Public School. (n.d.) Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.7, and G.CO.8) |
<table>
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<tr>
<th>Topic</th>
<th>Standards</th>
<th>Description</th>
<th>Links/Resources</th>
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</thead>
</table>
| Lesson 7 of 10 : Unit Quiz 2 | HS.G.CO.7: Understand congruence in terms of rigid motions. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. **HS.G.CO.8** Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions. | Students will demonstrate their understanding of the concepts covered on Days 5-7 by successfully completing Quiz #2. | Quiz 2 (pp. 17-18)  
Prince George’s County Public School. (n.d.)  
*Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.7, and G.CO.8)* |
| Lesson 8 of 10: Application of Transformations to show Congruence (Two Day Activity) | HS.G.CO.7: Understand congruence in terms of rigid motions. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. **HS.G.CO.8** Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. | Students will work in pairs to work on the Performance Task. Teacher will facilitate student understanding by answering questions, guiding students and checking their work as they progress through the activity. | Performance Task: Downton Abbey Gardens (pp. 19-20)  
Note: Extension activities are included and may be assigned at the discretion of the teacher.  
Prince George’s County Public School. (n.d.)  
*Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.7, and G.CO.8)* |
<table>
<thead>
<tr>
<th>Topic</th>
<th>Standards</th>
<th>Description</th>
<th>Links/Resources</th>
</tr>
</thead>
</table>
| Lesson 9 of 10: Unit Review | **HS.G.CO.5** Experiment with transformations in the plane  
**HS.G.CO.6** Understand congruence in terms of rigid motion  
Use geometric descriptions of rigid motions to transform figures and to predict the effect of given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent  
**HS.G.CO.7** Understand congruence in terms of rigid motions  
Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding | Students will review the standards covered in this lesson and prepare for the unit assessment | Teacher should create student review using the item bank provided.  
Prince George’s County Public School. (n.d.)  
Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.6, G.Co.7, and G.CO.8) |
<table>
<thead>
<tr>
<th>Topic</th>
<th>Standards</th>
<th>Description</th>
<th>Links/Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pairs of angles are congruent.</td>
<td><strong>HS.G.CO.8</strong> Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions.</td>
<td></td>
</tr>
<tr>
<td>Lesson 10 of 10: Assessment</td>
<td><strong>HS.G.CO.5</strong> Experiment with transformations in the plane <strong>HS.G.CO.6</strong> Understand congruence in terms of rigid motion  Use geometric descriptions of rigid motions to transform figures and to predict the effect of given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent <strong>HS.G.CO.7</strong> Understand congruence in terms of rigid motions</td>
<td>Students will demonstrate their understanding of the unit by successfully completing the unit assessment.</td>
<td>Unit Assessment and Key (pp. 21-32) Prince George's County Public School. (n.d.) Geometry transformation. (Understand congruence in terms of rigid motions, G.Co.6, G.Co.7, and G.CO.8)</td>
</tr>
<tr>
<td>Topic</td>
<td>Standards</td>
<td>Description</td>
<td>Links/Resources</td>
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<td>-------------------------------------------------------</td>
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<tr>
<td>Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. <strong>HS.G.CO.8</strong> Explain how the criteria for triangle congruence (ASA, SAS and SSS) follow from the definition of congruence in terms of rigid motions.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Appendix</td>
<td>The appendix contains lesson to teach transformations.</td>
<td></td>
<td>pp. 33-39 Lesson 1-6, Appendix C</td>
</tr>
</tbody>
</table>
What will students know and be able to do by the end of this unit?
Students will demonstrate an understanding of the unit focus and meet the expectations of the Common Core State Standards on the unit assessments.

**Standards**

*The major clusters for this unit include:*
- Understand congruent and similarity between geometric figures using physical models, transparencies, or geometry software.
- Understand how to apply transformations to show that two figures are congruent or similar.

**Unit Assessment**

*Students will demonstrate mastery of the content through assessment items and tasks requiring:*
- Conceptual Understanding
- Procedural Skill and Fluency
- Application
- Math Practices

**Objectives and Formative Tasks**

*Objectives and tasks aligned to the CCSS prepare students to meet the expectations of the unit assessments.*

**Concepts and Skills**

*Each objective is broken down into the key concepts and skills students should learn in order to master objectives.*
Identify Desired Results

Performance Task Overview

Unit Pre-Test

Word Search: Transformations

Quiz 1

Class Practice A

Unit Quiz 2

Determining Acceptance Evidence

End-of-Unit Assessment

End-of-Unit Assessment Item Responses

Key to Quiz 2

Lesson One: The Rigid Motions

Lesson Two: Line, Ray, and Point Congruence

Lesson Three: Construct and Apply a Sequence of Rigid Motions

Lesson Four: Applications of Congruence in Terms of Rigid Motions

Lesson Five: Correspondence and Transformations

Lesson Six: Congruence Criteria for Triangles–SAS

Unit References
**Congruence and Similarity through Rigid Motion**

**Unit Title:** Triangle Congruence  
**Grade Levels:** 9th and 10th grade

**Subject/Topic Areas:** Congruence  
**DOK Level:** 1-3

**Key Words:** triangle, congruence, proof, rigid motion, congruence shortcuts

**Designed by:** Tamara Bonn

**Time Frame:**

**School:**

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**Brief Summary of Unit (including curriculum context and unit goals):**

This unit on congruence is designed as an application of a previous unit on transformations and is designed to show how to apply rigid motion to show congruence in triangles. Students will learn about rigid motion transformations in terms of congruence, corresponding parts of congruent triangles are congruent and the shortcuts for proving triangle congruence. Students will be able to describe a series of rigid motions to prove triangle congruence.

This unit is organized so that the performance task applies all of the components of the unit. The skills needed to complete the performance task will be listed in the students “Need to know” category after reading the entry document and will be introduced daily through inquiry investigations. Students will have multiple opportunities, in their group and individually, to reflect on prior knowledge, develop new understandings, and apply their newly discovered knowledge to the performance task.

Students will demonstrate acceptable evidence of understanding through two quizzes and the culminating performance task. The performance task requires students to analyze four new *Downton Abbey* garden designs and act as consultants to the present castle owner. Students will apply their knowledge of triangle congruence to help them justify their evaluation of the track designs. Students will defend their findings through both a formal explanation using appropriate vocabulary and a series of rigid motions.
IDENTIFY DESIRED RESULTS

Established Goals:
Common Core State Standards HS Geometry:
G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
G.CO.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.
G.CO.10 Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

What essential questions will be considered?
- Is there a widely accepted way to construct viable arguments and critique the arguments of others?
- Can we use mathematics to model real life?
- How can rigid motion help us understand congruence?
- How do the criteria for triangle congruence fit with the definition of congruence?
- How do you identify corresponding parts of congruent triangles?

What enduring understandings are desired?

Students will understand that...
- Mathematical proofs are based on deductive reasoning and are a sound way to construct viable arguments.
- Rigid motion transformations move a figure, but don’t change the angle or side measurements.
- Corresponding parts of congruent triangles are congruent.
What key knowledge and skills will students acquire as a result of this unit?

<table>
<thead>
<tr>
<th>Students will know...</th>
<th>Students will be able to...</th>
</tr>
</thead>
<tbody>
<tr>
<td>• How rigid motion relates to congruence</td>
<td>• Prove congruence based on rigid motion</td>
</tr>
<tr>
<td>• Corresponding parts of congruent triangles are congruent</td>
<td>• Find corresponding parts of congruent triangles</td>
</tr>
<tr>
<td>• Side-side-side, angle-side-angle and side-angle-side correspondence between two</td>
<td>• Correctly use and identify SSS, ASA, SAS, AAS, and HL congruence</td>
</tr>
<tr>
<td>triangles determines rigid motions of the plane.</td>
<td>shortcuts</td>
</tr>
<tr>
<td>• Angle-angle-angle and side-side-angle do not prove congruence</td>
<td></td>
</tr>
</tbody>
</table>
PERFORMANCE TASK OVERVIEW

1. Play English “Gardens” Power point to show symmetry.
2. Distribute performance task and have students come up with a “Know” and “Need to know” list.
3. Reflect on what we know about rigid transformations and the definition of congruence.
4. As a group analyze garden designs creating multiple series of rigid transformations to prove corresponding parts of congruent triangles are congruent.
5. Reflect on “Know” and “Need to know” list to evaluate progress.
6. As a group identify given information in each garden design and discuss appropriate shortcuts to use within each proof.
7. Reserve computer lab for students to write their formal report rough draft. Introduce the structure of formal reports and including sources.
8. Peer review of formal report rough draft. Students question each other’s justifications and critique their reasoning.
9. Formal report is submitted and students grade themselves based on the rubric.
<table>
<thead>
<tr>
<th></th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Thinking:</strong></td>
<td>Makes and carries out a plan to identify congruent triangles using appropriate strategies</td>
<td>Makes and carries out a plan to pose and solve problems with transformations, using a limited range of appropriate strategies; rarely results in an accurate solution</td>
<td>Makes and carries out a plan to pose and solve problems with transformations, using some appropriate strategies; frequently results in accurate solution</td>
<td>Makes and carries out a plan to pose and solve problems with transformations, using appropriate strategies that result in an accurate solution; may be innovative</td>
</tr>
<tr>
<td><strong>Knowledge and</strong></td>
<td>May be unable to apply, create, and explain:</td>
<td>May only be partially able to apply, create, and explain:</td>
<td>Appropriately applies, creates and explains:</td>
<td>In various contexts, consistently and appropriately applies, creates, and explains:</td>
</tr>
<tr>
<td><strong>Understanding</strong></td>
<td>-transformations - triangle congruence postulates</td>
<td>-transformations - triangle congruence postulates</td>
<td>-transformations - triangle congruence postulates</td>
<td>-transformations - triangle congruence postulates</td>
</tr>
<tr>
<td><strong>Application</strong></td>
<td>With limited effectiveness:</td>
<td>With some effectiveness:</td>
<td>With considerable effectiveness:</td>
<td>With a high degree of effectiveness:</td>
</tr>
<tr>
<td>Uses appropriate procedures to accurately:</td>
<td>-performs and describes transformations, -identifies congruent figures</td>
<td>-performs and describes transformations, -identifies congruent figures</td>
<td>-performs transformations, -identifies congruent figures</td>
<td>-accurately performs transformations; -identifies congruent figures</td>
</tr>
<tr>
<td><strong>Communication</strong></td>
<td>Presents work with little clarity or precision; needs assistance to describe procedures and express mathematical thinking</td>
<td>Presents work, describes procedures, and expresses mathematical thinking with some clarity and precision, using some appropriate language</td>
<td>Presents work, describes procedures, and expresses mathematical thinking with general clarity and precision, using appropriate terminology</td>
<td>Presents work, describes procedures, and expresses mathematical thinking clearly, precisely, and confidently, using a range of appropriate terminology</td>
</tr>
</tbody>
</table>
Geometry

Name:______________________

Unit Pre-Test

Answer each question to the best of your ability. Make sure to show work!

1. Circle all the transformations that lead to congruent images.

   Rotation       Reflection       Translation       Dilation

2. Which transformation will lead to a similar figure but not a congruent one?

Apply the given transformation to the given pre-image.

3. Translation by vector \( \left( \frac{2}{3} \right) \)

4. Rotation by 90°
Apply the given series of transformations to the given pre-image.

5) Dilation by scale factor 2 and reflect across y-axis

6) Find the measure of each angle.

7) Find the measure of each missing angle.
Word Search: Transformations

center
enlargement
glide reflection
line symmetry
reduction
symmetry
transformation

composition
Escher
image
point symmetry
reflection
tessellation
translation
dilation
glide
isometry
preimage
rotations
tilling
vector
Quiz #1

1. (5 points)
Which rule would you use to find the image of any point \((x, y)\) under a vertical translation of \(b\) units?

   A. \((x, y) \rightarrow (x + b, y + b)\)

   B. \((x, y) \rightarrow (x + b, y)\)

   C. \((x, y) \rightarrow (x, y + b)\)

   D. \((x, y) \rightarrow (3x, y)\)

2. (5 points)

The triangles above are congruent. Which of the following could be false?

   A. Angle C is congruent to angle M

   B. Side CB is congruent to side MN

   C. Side AC is half the length of side NM

   D. Angle A is congruent to Angle O
3. (5 points)

Use the figure below to answer the question:

Which term best describes the transformation shown below?

A. dilation
B. rotation
C. reflection

4. (5 points)

Use the figure below to answer the question:

What rigid motion(s) maps \( \overrightarrow{AB} \) onto \( \overrightarrow{CD} \)?

Describe the rigid motion(s) using function notation.
5. Identify pairs of congruent triangles. For each pair:

- Describe the transformation that proves the triangles are congruent.
- Label the congruent sides of the triangles.
- Write statements of congruence for the triangle pairs and the congruent sides.

<table>
<thead>
<tr>
<th>Pair #1:</th>
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<tbody>
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</table>

<table>
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<tr>
<th>Pair #2:</th>
</tr>
</thead>
<tbody>
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<td></td>
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</table>

<table>
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<tr>
<th>Pair #3:</th>
</tr>
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<td></td>
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</tbody>
</table>
Work with a partner to complete each of the following problems. Make sure to use appropriate vocabulary!

Given: \( \triangle ABC \), shown on the coordinate plane below

![Coordinate Plane with Triangle ABC](image)

a. If \( \triangle ABC \) is reflected over the x-axis to yield \( \triangle A'B'C' \), what are the coordinates of the vertices of \( \triangle A'B'C' \)?

b. Using this reflection, write a general rule that will map \( \triangle ABC \) onto \( \triangle A'B'C' \).
c. If $\triangle ABC$ is translated 4 units to the left and 3 units down to yield $\triangle RST$, draw $\triangle RST$ on the coordinate plane below.

\[\text{Coordinate Plane with grid lines.} \]

\[\text{Label points A, B, C, R, S, and T on the grid.} \]

d. Using your translation, write a general rule that will map $\triangle ABC$ onto $\triangle RST$. Use words, numbers, and/or pictures to show your work.
Unit Quiz 2

1) A rigid motion $J$ of the plane takes point $A$ as input and gives $C$ as output, i.e., $J(A) = C$. Similarly, $J(B) = D$ for input point $B$ and output point $D$.

Jerry claims that knowing nothing else about $J$, we can be sure that $\overline{AC} \cong \overline{BD}$ because rigid motions preserve distance.

   a. Show that Jerry’s claim is incorrect by giving a counterexample (hint: a counterexample would be a specific rigid motion and four points $A$, $B$, $C$, and $D$ in the plane such that the motion takes $A$ to $C$ and $B$ to $D$, yet $\overline{AC} \not\cong \overline{BD}$).

   b. There is a type of rigid motion for which Jerry’s claim is always true. Which type below is it?

      Rotation  Reflection  Translation

   c. Suppose Jerry claimed that $\overline{AB} \cong \overline{CD}$. Would this be true for any rigid motion that satisfies the conditions described in the first paragraph? Why or why not?
2) For the triangles \(\triangle ABC\) and \(\triangle DEF\) in the figure below, 

a. Using the given information, what criterion for triangle congruence (ASA, SAS, SSS) implies that \(\triangle ABC \cong \triangle DEF\)?

b. Describe a sequence of rigid transformations that shows \(\triangle ABC \cong \triangle DEF\).
DETERMINING ACCEPTABLE EVIDENCE

Performance Task:
Downton Abbey Gardens

Background: In England the gardens are a show of wealth and standing. These gardens are designed to have a great deal of symmetry along with strong lines and to be calming along with being quite colorful.

Objective: Mrs. Bonn loves Downton Abbey and has decided to plant her own English garden. She has recruited all of the geometry students at Indian Springs High School to consult with her to design the perfect garden. She wants to build two triangular gardens so that both her dogs can have their own identical garden. In order for this to be fair between the dogs, the gardens need to be congruent. Four landscape designers have submitted the plans below.

Your task is to determine if each garden design meets Mrs. Bonn’s requirements. You will need to submit a formal report with justifications supporting your findings that the two gardens are indeed congruent. You must submit two types of justifications for each garden design in your report. The first type of justification must describe a series of rigid motions with a justification of how you chose each step in your rigid motion and the second must use the appropriate congruence postulate.

1) \( \overline{EH} \cong \overline{GH} \)

2) \( \overline{WU} \cong \overline{VO} \)

Source of Picture: https://encrypted-tbn3.gstatic.com/images?q=tbn:ANd9GcR_2cNHqeDKlM6XYccoGMlc2if2x_zocXEe8NM2fp22EPxdc4DQg
3) \[ \angle GFH \cong \angle EFH \]

4) \[ AD \cong BC \]
EXTENSION

A) Consider either garden #1 or #2 from the previous page. Describe what happens to a fountain (the circle) positioned as indicated on the landscape as we are performing the rigid motions: reflection, translation and rotation.

B) Design your own garden and perform the rigid motion that does not move the fountain for Extension A. Make sure to describe your rigid motion and explain why the fountain does not move.

C) Expand Garden number one by a scale factor of your choosing and describe what has to be done to the other garden so they are still congruent.

D) After much careful consideration and some modification, one of the designers submitted the following garden. However, the owners of Downton Abbey require that the gardens stay symmetrical. Therefore, your task is to draw in the modification so that both gardens A and B are symmetrical.
END-OF-UNIT ASSESSMENT

1) What happens to a vertical line if you rotate it 90 degrees clockwise?
   A. The image line is 90 times as long.
   B. The image is a vertical line.
   C. The image is a line 90 inches long.
   D. The image is a horizontal line.

2) Use the coordinate grid and triangles below to identify the transformation performed and the relationship between the original and the image.

   Part A: Explain why triangle ABC is congruent to triangle LMN using one or more reflections, rotations, and/or translations.

   Part B: Explain how you can use the transformation described in Part A to prove triangle ABC is congruent to triangle LMN by any of the criteria for triangle congruence (ASA, SAS, or SSS.)

A. Translation, not congruent
B. Rotation, congruent
C. Reflection, not congruent
D. Reflection, congruent
3) Triangle ABC and triangle LMN are shown in the coordinate plane below. 

![Coordinate Plane with Triangles](image)

**Part A:** Explain why triangle ABC is congruent to triangle LMN using one or more reflections, rotations, and/or translations.

**Part B:** Explain how you can use the transformation described in Part A to prove triangle ABC is congruent to triangle LMN by any of the criteria for triangle congruence (ASA, SAS, or SSS.)

4) In the figure below, there is a reflection that transforms to triangle.

Use a straightedge and compass to construct the line of reflection and list the steps of the construction.

![Reflection of Triangles](image)
5) The triangles and in the figure below such that , , and

a. What criteria for triangle congruence (ASA, SAS, SSS) implies that ?

b. Describe a sequence of rigid transformations that shows .

6) Complete all parts. You may use a ruler and/or protractor to complete the task.

a. Rotate $\angle XYZ$ 90° counter clockwise about the origin. Draw and label the image, $\angle X'Y'Z'$. 
b. Name the ordered pairs for:  
X' ____________  
Y' ____________  
Z' ____________

c. Reflect \( \angle X'Y'Z' \) over the y-axis. Draw and label \( \angle X''Y''Z'' \).

d. What is the \( m_{\angle XYZ} \) ? _______________ What is the \( m_{\angle X''Y''Z''} \) ? _______________

How do the measures of these angles compare? Explain why this is reasonable for an angle that is rotated and reflected.

e. Sarah thinks that she can produce the \( \angle X''Y''Z'' \) from \( \angle XYZ \) with one transformation, an 180˚ rotation about the origin. Is she correct? Explain why or why not.

7) The graph of a figure and its image are shown below. Identify the type of transformation that maps the pre-image to the image. (Note the image is marked using prime notation.)

Check the correct choice.

___reflection  ___translation  ___reflection ___translation  ___rotation  ___dilation  ___rotation  ___dilation
c. reflection translation
rotation dilation
d. reflection translation
rotation dilation
8) Triangles ABC and PQR are shown below in the coordinate plane.

a. Show that \( \triangle ABC \) is congruent to \( \triangle PQR \) with a reflection followed by a translation.

b. If you reverse the order of your reflection and translation in part (a) does it still map \( \triangle ABC \) to \( \triangle PQR \)?

c. Find a second way, different from your work in parts (a) or (b), to map \( \triangle ABC \) to \( \triangle PQR \) using translations, rotations, and/or reflections.

END-OF-UNIT ASSESSMENT ITEM RESPONSES:

1) **Standard 8.G.A.1a/ HS.G.CO.5**
   Answer choice D
   DOK 1

2) **Standard 8.G.A.2/ HS.G.CO.5**
   Answer choice D. Reflection, congruent
   DOK 1

3) **Standard HS.G.CO.7**
   Part A: $\triangle ABC$ maps to $\triangle A'B'C'$ by a rotation counter clockwise about the point (-5, 0). Translation 10 units to the right maps $\triangle A'B'C'$ onto $\triangle L MN$.
   Part B: Rotation and translations preserve distance such that AC maps to LN, CB maps to NM and BA maps to ML. Thus triangle ABC is congruent to triangle LMN by SSS
   DOK 2, 3

4) **Standard HS.G.CO.7**
   Construct AA' and BB'. Construct the perpendicular bisector of AA' then extend to BB'.
   DOK 2

5) **Standard HS.G.CO.7**
   Part A: SAS
   Part B:
   DOK 2
6) **Standard HS.G.CO.5**

Complete all parts. You may use a ruler and/or protractor to complete the task.

![Graph showing geometric transformations]

a. Rotate \( \angle XYZ \) 90˚ counter clockwise about the origin. Draw and label the image, \( \angle X'Y'Z' \). *(1 point) DOK 1*

b. Name the ordered pairs for:  
   - \( X' \) \((-4, -2)\)  
   - \( Y' \) \((-1, -5)\)  
   - \( Z' \) \((-1, -3)\)  
   *(1 point for all correct) DOK 1*

c. Reflect \( \angle X'Y'Z' \) over the y-axis. Draw and label \( \angle X''Y''Z'' \). *(1 point) DOK 1*

d. What is the \( m\angle XYZ \)? \(45^\circ\)  
   What is the \( m\angle X''Y''Z'' \)? \(45^\circ\)  
   *(1 point for both)*

How do the measures of these angles compare? *The angles are congruent.* *(1 pt.)*

Explain why this is reasonable for an angle that is rotated and reflected. *Rotations and reflections produce congruent transformations, so the angles should be congruent after transformation.* *(2 points) DOK 2*
e. Sarah thinks that she can produce the \( \angle X''Y''Z'' \) from \( \angle XYZ \) with one transformation, a 180˚ rotation about the origin. Is she correct? Explain why or why not.

No, the coordinates for \( \angle X''Y''Z'' \) are \( X''(4, -2), Y''(1, -5), \) and \( Z''(1, -3) \), but the coordinates of the angle if it is rotated 180˚ are \( X(2, -4), Y(5, -1), \) and \( Z(3, -1) \).

(2 points)

Or

In order to produce 180˚ rotation, two reflections are necessary.

Or

Any other student responses that are accurate and demonstrate strong mathematical reasoning.

DOK 2, 3

7) Standard HS.G.CO.
c. _____reflection  _____translation
     _____translation
     ____X____rotation  _____dilation

d. ____X____reflection
     _______rotation  _______dilation
8) 

a. Show that \( \triangle ABC \) is congruent to \( \triangle PQR \) with a reflection followed by a translation. Reflect across the y-axis, then translate down 6 units.

There are other possible combinations of reflections and translations, but this is the one the students will most likely choose. However, other solutions will need to be checked for accuracy.

DOK 1
b. If you reverse the order of your reflection and translation in part (a) does it still map \( \triangle ABC \) to \( \triangle PQR \)?

This answer is dependent on the answer in part (a). Using the reflection and translation chosen above, the answer would be yes. \( \triangle ABC \) translated down 6 units and then reflected over the y-axis, would result in \( \triangle PQR \). See below.
c. Find a second way different from your work in part (a), to map \( \Delta ABC \) to \( \Delta PQR \) using translations, rotations, and/or reflections. 

There are several possible answers. Below are a few samples.
### KEY TO QUIZ 2

<table>
<thead>
<tr>
<th>1</th>
<th>a – c</th>
<th>G-CO.2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>“Translation” is circled for part (b), but no further response is correct or shows clear understanding of the application of rigid motions.</td>
<td>The response includes a counterexample provided in part (a) OR an idea is presented to prove that ( AB \cong CD ) in part (c), whichever is presented is less than perfectly clear in stating the solutions; “translation” is circled for part (b).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
<th>a – b</th>
<th>G-CO.7</th>
<th>G-CO.8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A response shows little or no evidence of understanding for part (a) or (b).</td>
<td>A response shows the correct triangle congruence criteria listed in part (a) and lists only one rigid transformation for part (b).</td>
<td>A response shows the correct triangle congruence criteria listed in part (a) and lists only two rigid transformations for part (b).</td>
</tr>
</tbody>
</table>

**Assessment Task Item**

- **STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.
- **STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.
- **STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.
- **STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
Sample Student Solution

2. A rigid motion, $J$, of the plane takes a point, $A$, as input and gives $C$ as output, i.e., $JA = C$. Similarly, $JB = D$ for input point $B$ and output point $D$.

Jerry claims that knowing nothing else about $J$, we can be sure that $AC = BD$ because rigid motions preserve distance.

a. Show that Jerry's claim is incorrect by giving a counterexample (hint: a counterexample would be a specific rigid motion and four points $A$, $B$, $C$, and $D$ in the plane such that the motion takes $A$ to $C$ and $B$ to $D$, yet $AC \neq BD$).

Here, $J$ is a reflection across a vertical line. The distance from $A$ to the line is different from the distance from $B$ to the line. Therefore, the distance from $A$ to its image ($C$) is different from the distance from $B$ to its image ($D$).

b. There is a type of rigid motion for which Jerry's claim is always true. Which type below is it?

Rotation  Reflection  Translation

c. Suppose Jerry claimed that $AB \equiv CD$. Would this be true for any rigid motion that satisfies the conditions described in the first paragraph? Why or why not?

Yes, because rigid motions always preserve distance.
Lesson One: The Rigid Motions

I. Concept

1. Describe the sequence of rigid motions that begins with triangle ABC and ends with triangle A'''B'''C'''. (Recall that we’re very particular about how we describe a rigid motion. For a translation, we must name the vector and say that we have moved along it. For a reflection, we must name the mirror and say that we have reflected over it. For a rotation, we must mention the center and say whether we’ve rotated clockwise or counterclockwise around it.)

2. The final motion in the diagram above was a rotation about point H. How many degrees was that rotation? _____ How do you know?

3. Benito says that Motion Three is a clockwise rotation about point H. Amber says that it is a counterclockwise rotation. Both agree that it is a rotation of 180°. Who’s right? Why?
4. Triangle ABC below was rotated about a point. Deduce and mark the location of that point. (Don’t guess, and don’t measure. You don’t need to do either.) Explain why you’re right.

5. Triangle ABC below was reflected over a line. Deduce the position of the line and draw it. (Unlike before, I give you permission to measure.) Explain why you’re right.
II. Translations

A translation needs a vector. The translation vector tells us how far and in what direction we translated the object. In the coordinate plane, the translation vector can be given as a pair of numbers, the first of which tells us how far right or left to travel, the second of which tells us how far up or down to travel. (As usual, right and up are positive, left and down are negative.) Let us agree that if we so specify a vector, we’ll use the angle brackets. Thus \( v = (1, 3) \), for instance, is the translation vector that tells us to go 1 right and 3 up, while \( v = (-3, -1) \) tells to go left 3 and down 1.

You can assume throughout that to rigidly transform a segment, it’s sufficient to rigidly transform its endpoints and then connect. (This assumption will later become part of our Rigid Motion Postulate.)

1. In each diagram below, translate the triangle ABC by the given vector. Label the vertices of the image with an A', a B' and a C'. Give the coordinates of A', B' and C'.

\[ \begin{align*}
\text{Coordinates of A': } & \quad \text{Coordinates of A': } \\
\text{Coordinates of B': } & \quad \text{Coordinates of B': } \\
\text{Coordinates of C': } & \quad \text{Coordinates of C': }
\end{align*} \]
2. Let’s say that the vector for a certain translation was \((a, b)\). What if we wished to reverse the translation? What if, that is, we wished to make the pre-image the image and the image the pre-image? What would the vector for that reversed translation be?
III. Reflections

Recall that a reflection of an object over a line, we carry each point on the object straight to the line and then carry it that far again straight away from the line. The line over which we reflect is thus the perpendicular bisector of each segment that connects a point on the pre-image with the point on the image to which it corresponds.

1. In each diagram below, reflect the triangle over the given line. Label the vertices of the image with an A', a B', and a C'. Give the coordinates of A', B', and C'.

![Diagram of triangle ABC with reflections](image1)

Coordinates of A': ________  
Coordinates of B': ________  
Coordinates of C': ________  

![Diagram of triangle with reflections](image2)

Coordinates of A': ________  
Coordinates of B': ________  
Coordinates of C': ________  

2. What happens if we take the image of a reflection and reflect it over the same line again? Where does it end up relative to the original pre-image?

3. The vertices of triangle ABC have coordinates A(2, 3), B(2, 5) and C(-1, 1). If we reflect ABC over the y-axis, what are the coordinates of A', B', and C'?

A': ____________, B': ____________, C': ____________.

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4. If we reflect the triangle of #3 over the x-axis, what are the coordinates of A', B' and C'?

A': ___________. B': ___________. C': ___________.
IV. Rotations

Recall that when we rotate an object, every point on that object is rotated the same degree, either clockwise or counterclockwise, about the same point.

1. In each diagram below, rotate the triangle about the given center by the given amount. Assume that the reflections are counter-clockwise. (That’s the mathematical default. Assume counter-clockwise unless told otherwise.) Label the vertices of the image with an A’, a B’, and a C’. Give the coordinates of A’, B’, and C’.

Reflection of 90° about the Origin

Coordinates of A’: ________
Coordinates of B’: ________
Coordinates of C’: ________

Reflection of 180° about A

Coordinates of A’: ________
Coordinates of B’: ________
Coordinates of C’: ________
2. Compare a rotation $120^\circ$ clockwise with a rotation of $480^\circ$ clockwise. What’s the relation of their images? Why?
V. The Null Case

We never said that a rotation or a translation must move an object. Of course they can. But they don’t have to. We could translate a distance of 0 units or rotation with an angle of 0°. (I can’t make sense of a 0
reflection. When an object is reflected, it has to move.) In such cases – 0 unit translations and 0 degree rotations – though we do have a rotation or a translation, we leave the object just where it was to begin. We’ll use the term ‘null’ to describe 0 degree rotations and 0 unit translations. Thus a null translation is a translation of distance 0 units and a null rotation is a rotation through 0°.

It’s curious that there are sequences of reflections, rotations, and translations equivalent to the null translation and null rotation.

1. Describe a rotation other than the null rotation that is equivalent to the null rotation.

2. Describe another rotation that’s equivalent to the null rotation.

3. Describe a sequence of reflections equivalent to the null rotation.

4. Describe a sequence of non-null translations equivalent to the null translation.
VI. Equivalent Motions

We say that two sequences of rigid motions are equivalent when each carries a given pre-image to exactly the same location in the plane. A trivial example: a sequence of three translations each of which carries the object upwards 1m is equivalent to one translation that takes it upwards 3 m. Another trivial example: a sequence of two reflections about the same center, the first 90° and the second 90°, is equivalent to one reflection of 180° degree about that same center.

The diagram below shows a pre-image and the image which results after a sequence of rigid motions. Give three equivalent sequences that will carry the pre-image to the image. Make sure that each motion is fully specified. If it’s a translation, give the vector in angle bracket notation. If it’s a reflection, give the equation of the line over which we reflect. If it’s a rotation, give the coordinates of the center of the rotation and the number of degrees rotated.

Sequence One:
Sequence Two:

Sequence Three:
Lesson Two: Line, Ray, and Point Congruence

A quick review of rigid motions: (1) We defined reflection, rotation, and translation. (2) We defined rigid motion as a sequence of reflections, rotations, and translations. (3) We postulated that rigid motion leaves length and angle measure unchanged. (Actually, we postulate more, but this is the heart of the Rigid Motion Postulate.) (4) We defined coincidence. Objects coincide when they occupy precisely the same region of space. (5) We defined congruence. Objects are congruent when they can be made to coincide by a sequence of rigid motions.

Recall too that we distinguished equality from congruence. When objects are equal, they’re really one and the same object. When objects are congruent, they’re two objects. Thus, we can’t use the equal sign for congruence. We need another symbol, and the choice we made was ‘‘.’’

I. Lines, Rays, and Points

I proved for you that line segments with the same measure are congruent and that angles with the same measure are congruent. Now you will prove that lines are congruent, that rays are congruent and that points are congruent. In each, you must describe a sequence of rigid motions that will make the two given objects coincide and you must explain why those motions give you coincident objects.

1. Prove that points A and B below are congruent. Prove, that is, that . (It’s really as easy as it seems.)

2. Prove that lines and below are congruent. Prove, that is, that . (Here’s how you should begin: choose points A and B on lines and respectively and then slide until A and B coincide. You take it from there.)
3. Prove that rays AB and CD below are congruent. Prove, that is, that
II. Consequences

Our definitions have a number of important consequences. Let us take a moment to consider a few.

1. Is an object congruent to itself? Why? (Perhaps it will help if I rephrase: Can an object be made to coincide with itself by a sequence of rigid motions?)

2. If one object is congruent to a second and that second congruent to a third, are first and third congruent? Why?

3. If one object is congruent to a second, is the second congruent to the first? Why?

4. If you think back to our work with the Algebraic Postulates, you should recall that we had terms to describe the properties of congruence described in #s 1, 2, and 3 above. The property described in #1 is the ____________ of congruence. The property described in #2 is the ____________ of congruence. The property described in #3 is the ____________ of congruence.
Answers: II.  1. Of course. Figures are self-congruent. For a 0-slide (by which I mean a slide of distance 0) leaves an object coincident with itself. 2. Yes. Congruence is transitive. Let $S$ be the sequence of transformations that takes the first figure onto the second, and let $T$ be the sequence of transformations that takes the second figure into the third. Then if $S$ is applied to $T$ and then to the first, it will coincide with the third. This means that the first and third are congruent. 3. Yes. Any transformation can be reversed. Thus if the first can be rigidly transformed so that it coincides with the second, the reverse transformations applied to the second will leave it coincident with the first. 4. Reflexivity, Transitivity, Symmetry.
Lesson Three: Construct and Apply a Sequence of Rigid Motions

We have been using the idea of congruence already (but in a casual and unsystematic way). In Grade 8, we introduced and experimented with concepts around congruence through physical models, transparencies, or geometry software. Specifically, we had to:

1. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations . . . . [and (2)] describe a sequence that exhibits the congruence between [two congruent figures]. (California State Board of Education, 2013, p. 56)

As with so many other concepts in Grade 10, congruence is familiar, but we now study it with greater precision and focus on the language with which we discuss it.

Let us recall some facts related to congruence that appeared previously in this unit.

1. We observed that rotations, translations, and reflections—and thus all rigid motions—preserve the lengths of segments and the measures of angles. We think of two segments (respectively, angles) as the same or equal in an important respect if they have the same length (respectively, degree measure), and thus, sameness of these objects relating to measure is well characterized by the existence of a rigid motion mapping one thing to another. Defining congruence by means of rigid motions extends this notion of sameness to arbitrary figures, while clarifying the meaning in an articulate way.

2. We noted that symmetry is a rigid motion that carries a figure onto itself.

So how do these facts about rigid motions and symmetry relate to congruence? We define two figures in the plane as congruent if there exists a finite composition of basic rigid motions that maps one figure onto the other.

It might seem easy to equate two figures being congruent to having same size same shape. Same size and same shape is a phrase with intuitive meaning only (it helps to paint a mental picture), but is NOT a definition that we can use in a mathematical argument, where we need specificity. As in a court of law, to establish guilt it is not enough to point out that the defendant looks like a sneaky, unsavory type. We need to point to exact pieces of evidence concerning the specific charges. It is also not enough that the defendant did something bad. It must be a violation of a specific law. Same size, same shape is on the level of, “He looks like a sneaky, bad guy who deserves to be in jail.”

It is also not enough to say that they are alike in all respects except position in the plane. We are saying that there is some particular rigid motion that carries one to another. Almost always, when we use congruence in an explanation or proof, we need to refer to the rigid motion. Moreover, congruence by means of one rigid motion and congruence by means of a different rigid motion are two separate things. Specifying one of the many possible rigid motions, when more than one exists, may be important. We see this when discussing the symmetries of a figure. Symmetry is nothing other than a congruence of an object with itself. A figure may have many different rigid motions that map it onto itself. For example, there are six different rigid motions that take one equilateral triangle with side length 1 to another such triangle. Whenever this occurs, it is because of symmetry in the objects being compared.
Lastly, we discuss the relationship between congruence and correspondence. A correspondence between two figures is a function from the parts of one figure to the parts of the other, with no requirements concerning same measure or existence of rigid motions. If we have rigid motion that takes one figure to another, then we have a correspondence between the parts. For example, if the first figure contains segment , then the second includes a corresponding segment . But we do not need to have a congruence to have a correspondence. We might list the parts of one figure and pair them with the parts of another. With two triangles, we might match vertex to vertex. Then the sides and angles in the first have corresponding parts in the second. But being able to set up a correspondence like this does not mean that there is a rigid motion that produces it. The sides of the first might be paired with sides of different length in the second. Correspondence in this sense is important in triangle similarity.

**Classwork**

**Discussion**

We now examine a figure being mapped onto another figure through a composition of rigid motions.

![Diagram of rigid motions with labeled points and transformations](image)

To map to here, we first rotate 120°, (R_{0,120°}) around the point,. Then reflect the image (EF) across the line segment . Finally, translate the second image ( ) along the given vector to obtain . Since each transformation is a rigid motion, . We use function notation to describe the composition of the rotation, reflection, and translation:

\[
(\text{EF} \circ \text{R}_{0,120°} \circ \text{( ) })
\]

Notice that (as with all composite functions) the innermost function/transformation (the rotation) is performed first, and the outermost (the translation) last.
Example 1

i. Draw and label a triangle $\triangle PQR$ in the space below.

ii. Use your construction tools to apply one of each of the rigid motions we have studied to it in a sequence of your choice.

iii. Use function notation to describe your chosen composition here. Label the resulting image as $\triangle XYZ$:

\[ \Delta XYZ : \text{______________________________} \]

iv. Complete the sentences. Some blanks are single words, others are phrases:

Triangle $\triangle PQR$ is _______________________ to $\triangle XYZ$ because ________________________________

map point $P$ to point $X$, point $Q$ to point $Y$, and point $R$ to point $Z$. Rigid motions map segments onto

______________________________ ________________________________ and angles onto angles

______________________________.

Example 2

On a separate piece of paper, trace the series of figures in your composition but do NOT include the center of rotation, the line of reflection, or the vector of the applied translation.

Swap papers with a partner and determine the composition of transformations your partner used. Use function notation to show the composition of transformations that renders $\triangle PQR = \triangle XYZ$. 
Problem Set

1. Use your understanding of congruence to explain why a triangle cannot be congruent to a quadrilateral.
   a. Why can’t a triangle be congruent to a quadrilateral?
   b. Why can’t an isosceles triangle be congruent to a triangle that is not isosceles?

2. Use the figures below to answer each question:
   a. $\triangle ABD \cong \triangle CDB$. What rigid motion(s) maps $CD$ onto $AB$? Find 2 possible solutions.

   ![Diagram 1]

   b. All of the smaller sized triangles are congruent to each other. What rigid motion(s) map $ZB$ onto $AZ$? Find 2 possible solutions.

   ![Diagram 2]
Lesson Four: Applications of Congruence in Terms of Rigid Motions

Every congruence gives rise to a correspondence.

Under the Common Core definition of congruence, when we say that one figure is congruent to another, we mean that there is a rigid motion that maps the first onto the second. That rigid motion is called congruence.

Recall the Grade 7 definition: A correspondence between two triangles is a pairing of each vertex of one triangle with one and only one vertex of the other triangle. When reasoning about figures, it is useful to be able to refer to corresponding parts (e.g., sides and angles) of the two figures. We look at one part of the first figure and compare it to the corresponding part of the other. Where does a correspondence come from? We might be told by someone how to make the vertices correspond, or, we might notice that two triangles look very much alike, and we might match the parts of one with the parts of the other based on appearance—thus making our own correspondence. Or finally, if we have congruence between two figures, the congruence gives rise to a correspondence.

A rigid motion \( F \) always produces a one-to-one correspondence between the points in a figure (the pre-image) and points in its image. If \( P \) is a point in the figure, then the corresponding point in the image is \( F(P) \). A rigid motion also maps each part of the figure to what we call a corresponding part of the image. As a result, corresponding parts of congruent figures are congruent, since the very same rigid motion that makes congruence between the figures also makes a congruent relationship between each part of the figure and the corresponding part of the image.

The phrases corr. \( \angle \) of \( \cong \Delta \) and corr. sides of \( \cong \Delta \) frequently appear in proofs as a reminder of this fact. This is simply a repetition of the definition of congruence. If \( \Delta ABC \) is congruent to \( \Delta D E G \) because there is a rigid motion \( F \) such that \( F(A) = D, F(B) = E \) and \( F(C) = G \), then \( AB \) is congruent to \( DE \) and \( \Delta ABC \) is congruent to \( \Delta D E G \) and so forth—because the rigid motion \( F \) takes \( AB \) to \( DE \) and takes \( \angle BAC \) to \( \angle EDF \). People who share an understanding of the meaning of congruence might want to put a reminder like corr. \( \angle \) of \( \cong \Delta \) and corr. sides of \( \cong \Delta \) into proofs from time to time, because sometimes it is not clear that it is a congruence that justifies a step. Ritual use of the phrases whenever we refer to congruence is unnecessary.

There are correspondences that do not come from congruences.

The sides (and/or angles) of two figures might be compared even when the figures are not congruent. For example, a carpenter might want to know if two windows in an old house are the same, so the screen for one could be interchanged with the screen for the other. He might list the parts of the first window and the analogous parts of the second, thus making a correspondence between the parts of the two windows. Checking part by part, he might find that the angles in the frame of one window were slightly different from the angles in the frame of the other, maybe because the house had tilted slightly as it aged. He has used a correspondence to help describe the differences between the windows, not to describe congruence.

In general, given any two triangles, one could make a table with two columns and three rows, and then list the vertices of the first triangle in the first column and the vertices of the second triangle in the second column in a random way. This would create a correspondence between the triangles—though not
generally a very useful one. No one would expect a random correspondence to be very useful, but it is a correspondence nonetheless.

Later, when we study similarity, we will find that it is very useful to be able to set up correspondences between triangles despite the fact that the triangles are not congruent. Correspondences help us to keep track of which part of one figure we are comparing to that of another. It makes the rules for associating part to part explicit and systematic, so that other people can plainly see what parts go together.

**Discussion**

Let’s review function notation for rigid motions.

a. To name a translation, we use the symbol $T_{AB}$. We use the letter $T$ to signify that we are referring to a translation, and the letters $A$ and $B$ to indicate the translation that moves each point in the direction from $A$ to $B$ along a line parallel to line $AB$ by distance $AB$. The image of a point $P$ is denoted $T_{AB}(P)$. Specifically, $T_{AB}A = B$.

b. To name a reflection, we use the symbol $A_l$. We use the upper case Greek letter $\Lambda$ (lambda) as a reminder that we are speaking of a reflection; $l$ is the line of reflection. The image of a point $P$ is denoted $A_l(P)$. In particular, if $A$ is a point on $l$, $A_l(A) = A$. For any point $P$, line $l$ is the perpendicular bisector of segment $PA_l(P)$.

c. To name a rotation, we use the symbol $R_{C,\alpha}$ to remind us of the word rotation. $C$ is the center point of the rotation, and $\alpha$ represents the degree of the rotation counterclockwise around the center point. Note: a positive degree measure refers to a counterclockwise rotation, while a negative degree measure refers to a clockwise rotation.
**Classwork**

**Example 1**

In each figure below, the triangle on the left has been mapped to the one on the right by a 240° rotation about \( P \). Identify all six pairs of corresponding parts (vertices and sides).

<table>
<thead>
<tr>
<th>Corresponding vertices</th>
<th>Corresponding sides</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What rigid motion mapped \( \) onto \( ? \) Write the transformation in function notation.
Example 2

Given a triangle with vertices $A, B$ and $C$, list all the possible correspondences of the triangle with itself.

Example 3

Give an example of two quadrilaterals and a correspondence between their vertices such that:
a) corresponding sides are congruent but b) corresponding angles are not congruent.
Problem Set

1. Given two triangles, one with vertices $A$, $B$, and $C$, and the other with vertices $X$, $Y$, and $Z$, there are six different correspondences of the first with the second.
   a. One such correspondence is the following:
      \[
      A \rightarrow Z \\
      B \rightarrow X \\
      C \rightarrow Y
      \]
   Write the other five correspondences.
   b. If all six of these correspondences come from congruence, then what can you say about $\triangle ABC$?
   c. If two of the correspondences come from congruence, but the others do not, then what can you say about $\triangle ABC$?
      Why can there be no two triangles where three of the correspondences come from congruence, but the others do not?

2. Give an example of two triangles and a correspondence between them such that: a) all three corresponding angles are congruent but b) corresponding sides are not congruent.

3. Give an example of two triangles and a correspondence between their vertices such that: a) one angle in the first is congruent to the corresponding angle in the second, b) two sides of the first are congruent to the corresponding sides of the second, but 3) the triangles themselves are not congruent.

4. Given two quadrilaterals, one with vertices $A$, $B$, $C$, and $D$ and the other with vertices $W$, $X$, $Y$, and $Z$, there are 24 different correspondences of the first with the second. List them all.

5. Give an example of two quadrilaterals and a correspondence between their vertices such that a) all four corresponding angles are congruent, b) two sides of the first are congruent to two sides of the second, but c) the two quadrilaterals are not congruent.

6. A particular rigid motion, $M$, takes point $P$ as input and gives point $P'$ as output. That is, $M(P) = P'$. The same rigid motion maps point $Q$ to point $Q'$. Since rigid motions preserve distance, is it reasonable to state $PP' = QQ'$? Does it matter which type of rigid motion $M$ is? Justify your response for each of the three types of rigid motion. Be specific. If it is indeed the case, for some class of transformations, that $PP' = QQ'$ is true for all $P$ and $Q$, explain why. If not, offer a counter-example.
Classwork

Opening Exercise

The figure at the right represents a rotation of 80° around vertex . Name the triangle formed by the image of . Write the rotation in function notation and name all corresponding angles and sides.

Discussion

In the Opening Exercise, we explicitly showed a single rigid motion, which mapped every side and every angle of onto . Each corresponding pair of sides and each corresponding pair of angles was congruent. The two triangles are congruent when each side and each angle on the pre-image maps onto the corresponding side or angle on the image. Conversely, if two triangles are congruent, then each side and angle on the pre-image is congruent to its corresponding side or angle on the image.

Example 1

 is a square, and is one diagonal of the square. is a reflection of across segment . Complete the table below identifying the missing corresponding angles and sides.

<table>
<thead>
<tr>
<th>Corresponding angles</th>
<th>Corresponding sides</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Are the corresponding sides and angles congruent? Justify your response.

Exercises 1–3

Each exercise below shows a sequence of rigid motions that map a pre-image onto a final image. Identify each rigid motion in the sequence, writing the composition using function notation. Trace the congruence of each set of corresponding sides and angles through all steps in the sequence, proving that the pre-image is congruent to the final image by showing that every side and every angle in the pre-image maps onto its corresponding side and angle in the image. Finally, make a statement about the congruence of the pre-image and final image.

1.

<table>
<thead>
<tr>
<th>Sequence of rigid motions (2)</th>
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</thead>
<tbody>
<tr>
<td>Composition in function notation</td>
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<tr>
<td>Sequence of corresponding sides</td>
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<tr>
<td>Sequence of corresponding angles</td>
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<tr>
<td>Triangle congruence statement</td>
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2.

<table>
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<tr>
<th>Sequence of rigid motions (3)</th>
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<tbody>
<tr>
<td>Composition in function notation</td>
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<tr>
<td>Sequence of corresponding sides</td>
</tr>
<tr>
<td>Sequence of corresponding angles</td>
</tr>
<tr>
<td>Triangle congruence statement</td>
</tr>
</tbody>
</table>
Problem Set

1. Exercise 3 above mapped onto in three “steps.” Construct a fourth step that would map back onto .

2. Explain triangle congruence in terms of rigid motions. Use the terms corresponding sides and corresponding angles in your explanation.
Lesson Six: Congruence Criteria for Triangles-SAS

Classwork

Opening Exercise

Answer the following question. Then discuss your answer with a partner.

Is it possible to know that there is a rigid motion that takes one triangle to another without actually showing the particular motion?

__________________________________________________________________________________
__________________________________________________________________________________
__________________________________________________________________________________

Discussion

It is true that we will not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e., existence of rigid motion). We start with the Side-Angle-Side (SAS) criteria.

Side-Angle-Side triangle congruence criteria (SAS): Given two triangles \( \triangle ABC \) and \( \triangle A'B'C' \) so that 

- \( AB = A'B' \) (Side),
- \( \angle A = \angle A' \) (Angle),
- \( AC = A'C' \) (Side).

Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. What is important is that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfies the SAS criteria.

Proof: Provided the two distinct triangles below, assume \( AB = A'B' \) (Side), \( \angle A = \angle A' \) (Angle), \( AC = A'C' \) (Side).
By our definition of congruence, we will have to find a composition of rigid motions that will map \( \triangle A'B'C' \) to \( \triangle ABC \). So we must find a congruence \( F \) so that \( F(\triangle A'B'C') = \triangle ABC \). First, use a translation \( T \) to map a common vertex.

Which two points determine the appropriate vector?

Can any other pair of points be used? ________ Why or why not?

State the vector in the picture below that can be used to translate \( \triangle A'B'C' \): ____________

Using a dotted line, draw an intermediate position of \( \triangle A'B'C' \) as it moves along the vector:

After the translation (below), \( T_{AC}(\triangle A'B'C') \) shares one vertex with \( \triangle ABC \), \( A \). In fact, we can say

\( T____(\triangle ________) = \triangle _________. \)

Next, use a clockwise rotation \( R_{AC} \) to bring the sides \( AC'' \) to \( AC \) (or counterclockwise rotation to bring \( AB'' \) to \( AB \)).
A rotation of appropriate measure will map $\overline{AC''}$ to $\overline{AC}$, but how can we be sure that vertex $C''$ maps to $C$? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps $C''$ to $C$. ($AC = AC''$, the translation performed is a rigid motion, and thereby did not alter the length when $AC'$ became $AC''$.)

![Diagram of a rotation and translation]

After the clockwise rotation, $R_{\angle ACO} (\Delta AB'C'')$, a total of two vertices are shared with $\Delta ABC$, $A$ and $C$. Therefore,

$$R_{\text{_____}} (\Delta \text{_______}) = \Delta \text{________}.$$ 

Finally, if $B'''$ and $B$ are on opposite sides of the line that joins $AC$, a reflection $\Lambda_{AC}$ brings $B'''$ to the same side as $B$.

![Diagram of a reflection]

Since a reflection is a rigid motion and it preserves angle measures, we know that $\angle B'''AC = \angle BAC$ and so $\overline{AB'''}$ maps to $\overline{AB}$. If, however, $\overline{AB'''}$ coincides with $\overline{AB}$, can we be certain that $B'''$ actually maps to $B$? We can, because not only are we certain that the rays coincide, but also by our assumption that $\overline{AB} = \overline{AB'''}$. (Our assumption began as $\overline{AB} = A'B'$, but the translation and rotation have preserved this length now as $\overline{AB'''}$.) Taken together, these two pieces of information ensure that the reflection over $AC$ brings $B'''$ to $B$.

Another way to visually confirm this is to draw the marks of the construction for $AC$. 

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Write the transformations used to correctly notate the congruence (the composition of transformations) that takes $\triangle A'B'C' \cong \triangle ABC$:

\begin{align*}
F \quad \quad \quad \\
G \quad \quad \quad \\
H \quad \quad \quad \\
____(\quad (\triangle A'B'C')) = \triangle ABC
\end{align*}

We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two, distinct triangles, we could perform a similar proof. There are other situations, where the triangles are not distinct, where a modified proof would be needed to show that the triangles map onto each other. Examine these below.
Example 1

What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

<table>
<thead>
<tr>
<th>Case</th>
<th>Diagram</th>
<th>Transformations Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shared Side</td>
<td><img src="image" alt="Diagram" /></td>
<td>B''''</td>
</tr>
<tr>
<td>Shared Vertex</td>
<td><img src="image" alt="Diagram" /></td>
<td>B''''</td>
</tr>
</tbody>
</table>

Exercises 1–3

1. Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.
Directions: Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

2. Given: \[ \triangle LNO \] and \[ \triangle LOM \] meet the SAS criteria?

3. Given: \[ \triangle GHI \] and \[ \triangle GCI \] meet the SAS criteria?
**Problem Set**

*Directions:* Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

1. Given: 
   Do and meet the SAS criteria?

2. Given: 
   Do and meet the SAS criteria?

3. Given: and bisect each other. 
   Do and meet the SAS criteria?
4. Given: 
   Do and meet the SAS criteria?

5. Given: bisects angle 
   Do and meet the SAS criteria?

6. Given: and bisect each other.
   Do and meet the SAS criteria?

7. Given: 
   Do and meet the SAS criteria?
8. Given: \( \text{ } \)
   Do \( \text{ } \) and \( \text{ } \) meet the SAS criteria?

9. Given: \( \text{ } \)
   Do \( \text{ } \) and \( \text{ } \) meet the SAS criteria?

10. Given: \( \text{ } \) is isosceles,
    Do \( \text{ } \) and \( \text{ } \) meet the SAS criteria?
UNIT REFERENCES


APPENDIX D

TEACHER FORMS
I. Interview Questions

(a) What do you think of End Design?
(b) What was the AHA moment?
(c) State two things you learned today.
(d) Why do we use transformations from Algebra to Calculus?
(e) Why do we as teachers avoid teaching transformations?
(f) What is it about transformations that make you feel unprepared to teach it in your classroom?
(g) Does the use of rigid motion to show congruence give a deeper understanding of the triangle congruence criteria such as SSS, SAS and ASA?
(h) Has this unit increased your understanding of transformations? If so, how? Be specific.
(i) Has your understanding of rigid motion to show congruence increased?
(j) Any suggestions on the unit plan or implementation into a high school classroom?

II. Pre-Assessment

1) What happens to a vertical line if you rotate it 90 degrees clockwise?
   A. The image line is 90 times as long.
   B. The image is a vertical line.
   C. The image is a line 90 inches long.
   D. The image is a horizontal line.
2) Use the coordinate grid and triangles below to identify the transformation performed and the relationship between the original and the image.

Part A: Explain why triangle ABC is congruent to triangle LMN using one or more reflections, rotations and/or translations.

Part B: Explain how you can use the transformation described in Part A to prove triangle ABC is congruent to triangle LMN by any of the criteria for triangle congruence (ASA, SAS, or SSS.)

3) Triangle ABC and triangle LMN are shown in the coordinate plane below.

Part A: Explain why triangle ABC is congruent to triangle LMN using one or more reflections, rotations and/or translations.

Part B: Explain how you can use the transformation described in Part A to prove triangle ABC is congruent to triangle LMN by any of the criteria for triangle congruence (ASA, SAS, or SSS.)
4) In the figure below, there is a reflection that transforms \( \triangle ABC \) to \( \triangle A'B'C' \). Use a straightedge and compass to construct the line of reflection and list the steps of the construction.

5) The triangles \( \triangle ABC \) and \( \triangle DEF \) in the figure below such that , and .

   a. What criteria for triangle congruence (ASA, SAS, SSS) implies that ?

   b. Describe a sequence of rigid transformations that shows .
III. Transformation Cheat Sheet for Teachers

When we perform transformations without using a coordinate grid, it is commonly called Synthetic Geometry.

Notation: The most "standard" notation for transformations is as follows:

\[ T_{(x,y)} \rightarrow (x \pm a, y \pm b) \]  Translation of \((x, y)\) maps onto

\[ \rho \]  Reflection over line

\[ R_{A,\theta} \]  Rotation about point \(A\), \(\theta\) degrees

\[ D_k \]  Dilation by a scale factor of "k"

Some practice not on a coordinate grid.

1. Translate square NEAR 3 cm up the line BE.
2. How is reflecting across a diagonal line different from reflecting across a horizontal or vertical line? What is different about your image after the reflection?

Use tracing paper to reflect triangle MOP across line f.

3. Find the line of reflection for the following:
Rotations—rotations are generally performed by finding an angle from each vertex to the point of reflection. While this process works it can be quite “messy.” We will instead perform a rotation using a reflection in two non-parallel lines.

4. Rotate triangle TOP 60 degrees counter clockwise about point M.

5. Find the center of rotation.
IV. Quiz

1. A rigid motion of the plane takes point A as input and gives C as output, i.e., for input point and output point . Similarly, for input point and output point .

Jerry claims that knowing nothing else about rigid motions preserve distance.

a. Show that Jerry’s claim is incorrect by giving a counterexample (hint: a counterexample would be a specific rigid motion and four points , , , and in the plane such that the motion takes to and to , yet ).

b. There is a type of rigid motion for which Jerry’s claim is always true. Which type below is it?

   Rotation  Reflection  Translation

   Rotation

   Reflection

   Translation

c. Suppose Jerry claimed that . Would this be true for any rigid motion that satisfies the conditions described in the first paragraph? Why or why not?

2. For the triangles and in the figure below , , , and .

![Diagram of triangles A, B, C, D, E, F]
a. Using the given information, what criteria for triangle congruence (ASA, SAS, SSS) implies that $\triangle ABC \cong \triangle DEF$?

b. Describe a sequence of rigid transformations that shows $\triangle ABC \cong \triangle DEF$. 

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V. Final Assessment

Quick Review

1. Name the transformation or sequence of transformations that map one figure onto the other. Then complete the congruence statement.

![Diagram with labeled points D, E, C, G, F, T, L, O, Y, and H.]

Transformations: (Start with □DEC)
A rotation about the origin at ________□
Followed by
A translation of ____________
□DEC ≅ □_________

Transformations: (Start with □FLT)
A reflection over the ____________
Followed by
A translation of ____________
□FLT ≅ □_________

2. Rigid motions can be used to prove that the three triangle congruence criteria SSS, ASA, and SAS are sufficient to guarantee that triangles are ALWAYS congruent.

Given: \( \angle A \equiv \angle D, \overline{AB} \equiv \overline{DE}, \angle B \equiv \angle E \). Mark the diagrams below to show the given information.
Your task: Use rigid motions to prove that $\triangle ABC \cong \triangle DEF$

Describe a sequence of rigid motions that will move $\triangle ABC$ onto $\triangle DEF$. Since these triangles represent the general case, describe the motions as precisely as possible, even though we don’t have specific measurements.

3. Based on your results from the previous problem you agree that if we know two angles and their included side can we confidently state the two triangles are congruent?

PRACTICE: For each diagram below state the rigid motion that carries one triangle onto another and then state the triangle congruence (i.e. SSS, SAS, or ASA).
1. $\triangle ACE \cong \triangle TOM$

2. $\triangle MUT \cong \triangle STU$

3. $\triangle ABC \cong \triangle \underline{ }$
   Complete the statement to make it true.
   Give your reasoning.
VI. Reflection:

1) Does the use of rigid motion to show congruence give a deeper understanding of the triangle congruence criteria such as SSS, SAS, and ASA?

2) Has this unit increased your understanding of transformations? If so, how—be specific.

3) Has your understanding of the application of rigid motion to show congruence increase? If so how—be specific.

4) Any suggestions on the unit plan or implementation into a high school classroom?
Using Transformations to identify congruent triangles
* Required

1. Before the unit I felt knowledgable about transformations *
   Mark only one oval.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not at all</td>
<td></td>
<td></td>
<td></td>
<td>I got this!</td>
</tr>
</tbody>
</table>

2. The fact that transformations are functions give me a foundation for their application in other areas of mathematics. *
   Mark only one oval.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>I still have trouble with this</td>
<td></td>
<td></td>
<td></td>
<td>I got this!</td>
</tr>
</tbody>
</table>

3. I will take the concept of end design back to my classroom *
   Mark only one oval.
   □ Yes
   □ No
   □ I am not sure

4. My understanding of transformations has increased by a level of *
   Check all that apply.
   □ 100%
   □ 50%
   □ 25%

5. The use of transformations as a definition for congruent triangles is clear after the unit.
   * Check all that apply.
   □ Yes
   □ No
6. **Comments:**
Please take time to give feedback on the mathematics used in this unit.

7. **Overall thoughts**
Please express any thoughts you would like to convey to the Unit Writer.

8. **I teach**
Check all that apply
*Check all that apply.*

- [ ] Math 1
- [ ] Geometry
- [ ] Algebra 2
- [ ] Statistics
- [ ] Calculus
- [ ] Pre-Calculus
- [ ] Other:
Summary

Before the unit I felt knowledgable about transformations

1 1 20%
2 2 40%
3 1 20%
4 1 20%
5 0 0%

The fact that transformations are functions give me a foundation for their application in other areas of mathematics.

1 1 20%
2 1 20%
3 0 0%
4 3 60%
5 0 0%

I will take the concept of end design back to my classroom
Yes 4 80%
No 0 0%
I am not sure 1 20%

My understanding of transformations has increased by a level of

100% 2 40%
50% 2 40%
25% 1 20%

The use of transformations as a definition for congruent triangles is clear after the unit.

Yes 3 60%
No 2 40%

Comments:

methodology is creative and involving
Straight forward, aligned with current common core standards.
Teaching transformations was a positive learning experience as a teacher. There were many concepts in transformations that I had to remind my self. Proving that two triangles are congruent by discovery was one of the assignments I found beneficial to continue on to proofs. Instead of memorizing the conjectures, I was able to discover why are the conjectures true.
I thought the unit helped with the understanding of transformations
The unit is great its just that I have no previous background knowledge of transformations

Overall thoughts

I will suggest to start from the beginning of the unit with function notation. I think that this will benefit students when they get to two sided proofs.
Appears to be well organized and easy to follow. Written to allow for student success.
Thank you for making your work available to us
The unit is very helpful. It is easy to follow and understand some of it, even if the person has no knowledge of transformations
I teach

<table>
<thead>
<tr>
<th>Subject</th>
<th>Count</th>
<th>Percentage</th>
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<tbody>
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<tr>
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<tr>
<td>Algebra 2</td>
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<td>20%</td>
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<td>Statistics</td>
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</tr>
<tr>
<td>Pre-Calculus</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Other</td>
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<td>60%</td>
</tr>
</tbody>
</table>

just reały did not see how transformations proved congruency
APPENDIX E
COMMON CORE STATE STANDARDS INITIATIVE
STANDARDS FOR MATHEMATICAL PRACTICE
The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s (Kilpatrick, Swafford, & Findell, 2001) report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1. **Make sense of problems and persevere in solving them.**

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if
necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.**

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved;
attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.**

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
4. **Model with Mathematics.**

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.**

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient
students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the
elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.**

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$.

8. **Look for and express regularity in repeated reasoning.**

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same
calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation \((y - 2)/(x - 1) = 3\). Noticing the regularity in the way terms cancel when expanding \((x - 1)(x + 1)\), \((x - 1)(x^2 + x + 1)\), and \((x - 1)(x^3 + x^2 + x + 1)\) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

### Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to
Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.
APPENDIX F

PERMISSION TO USE
Tamara Bonn  
129 S. San Mateo  
Redlands, CA 92373  

Dear Ms. Bonn,

In reference to your request of May 2015:

Permission is granted to include the geometry problem on page 32 of the IFL publication *Mathematics: Investigating Congruence in Terms of Rigid Motion* in your forthcoming dissertation, subject to the following conditions:

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3939 O’Hara Street, 310E LRDC Bldg.  
Pittsburgh, PA 15260  

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Tamara Bonn <mathmom81@me.com> writes:

Dear Sir,

I am in the process of doing my masters thesis on congruent and similar figures as a result of transformations. I would like to know if I have your permission to use a few of your problems as examples in my thesis. I will make sure to cite your site and or you personally.

Thanks for asking Tammy.... go ahead!!!

Thanks

Tamara Bonn
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Mike Patterson
2009 Milken Educator Awardee
2009 NV Presidential Award for Excellence in Math Teaching
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REFERENCES

doi:10.1177/0022487108324554


