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A Fundamental Unit of O_K

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A FUNDAMENTAL UNIT OF O_K

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

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in

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by

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Abstract

In classical algebraic number theory quadratic extensions of $\mathbb{Q}$ are studied. Let $F = \mathbb{Q}(\sqrt{d})$ where d is a squarefree integer. The elements of F which are integral over $\mathbb{Z}$ form a ring called O_F and properties of O_F can be examined. In particular, the units of O_F can be determined by using continued fractions to solve Pell's Equation $x^2 - y^2 d = \pm 1$. By constructing a parallel to the classical case we let K be a quadratic extension of $\mathbb{Q}(x)$. We define $K = \mathbb{Q}(x)(\sqrt{f})$ for $f \in \mathbb{Q}[x]$ a monic squarefree polynomial of even degree. The elements of $\mathbb{Q}(x)(\sqrt{f})$ which are integral over $\mathbb{Q}[x]$ form a ring called O_K. Of interest are the units of this ring. Similar to the classical case, units in O_K are solutions to a generalized Pell's equation $X^2 - Y^2 f = r$ where r is a non-zero rational and f is a monic squarefree polynomial of even degree. We compute these units through the use of continued fractions. Finally we give details of interesting results.

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Chapter 1

Introduction

Classically, Algebraic Number Theory is the study of algebraic extensions F of $\mathbb{Q}$. The elements of F which are integral over $\mathbb{Z}$ form a ring called O_F. We define integral as the following.

Definition 1.1. *Let A and B be integral domains with $A \subseteq B$. The element $b \in B$ is said to be integral over A if it satisfies a polynomial equation*

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

where $a_0, a_1, \ldots, a_n \in A$.

Our interest are the properties of O_F. We are particularly interested when F is a quadratic extension of $\mathbb{Q}$. Consider $F = \mathbb{Q}(\sqrt{d})$ where d is a squarefree integer. Then the set O_F of algebraic integers in F is given by:

$$O_F = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{d}, & if\ d \not\equiv 1 \pmod 4) \\ \mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{d}}{2}), & if\ d \equiv 1 \pmod 4. \end{cases}$$

Units of O_F may be calculated through the use of continued fractions since they are also solutions to Pell's equation $x^2 - dy^2 = \pm 1$. We construct a parallel to $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ by replacing them with $\mathbb{Q}[x]$, $\mathbb{Q}(x)$, $\mathbb{Q}(x)^*$ respectively. While $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to ordinary absolute value, $\mathbb{Q}(x)^*$ is the completion of $\mathbb{Q}(x)$ with respect to a Non-Archimedian valuation which is described below.

In this thesis we will examine K as the quadratic extension of $\mathbb{Q}$, where $K = \mathbb{Q}(x)(\sqrt{f}) = \{g + h\sqrt{f}\,|g, h \in \mathbb{Q}[x]\}$ and $f \in \mathbb{Q}[x]$ is a monic polynomial of even degree.

The elements of $\mathbb{Q}(x)(\sqrt{f})$ which are integral over $\mathbb{Q}[x]$ form a ring called O_K. This is similar to the ring discussed above O_F. First we will show O_K is a ring then through the use of continued fractions we will compute the units of O_K. In Chapter 2 we will show why we have no interest in any case where f is not a monic squarefree polynomial of even degree. It turns out that if f has no restrictions then the units of O_K are trivial. As they are in the classical case, the units of O_K are solutions to a generalized Pell equation $X^2 - fY^2 = r$ for $r \in \mathbb{Q} - 0$. By computing a fundamental unit in O_K we will be able to generate others and observe how they relate to one another.

As we have already seen, there exists many parallels between the two cases. Such parallels were shown in a thesis titled *Solutions To A Generalized Pell Equation* ([Cas]) by Kyle Castro. In his thesis Castro duplicated the results found in Strayer's text ([Str01]) for the new setting $\mathbb{Q}(x)$. This leaves us to focus on calculating fundamental units for this setting. However, our first concern will be to explain the algebraic number theory behind the quadratic extension K. Finally, we will compute the units of O_K and later go into detail with certain interesting examples.

We will refer to the extension of $\mathbb{Z}$ to $\mathbb{Q}$ and its completion to $\mathbb{R}$ with respect to the absolute value as the classical case. The new setting which we have already alluded to, is the extension of $\mathbb{Q}[x]$ to $\mathbb{Q}(x)$ and its completion to $\mathbb{Q}(x)^*$ with respect to a non-Archimedean Valuation. The following definition is also found in [Cas]).

Definition 1.2. *Let P be an ordered field and K be a field with $a, b \in K$. Then a mapping $v : K \longrightarrow P$ is called a* <u>*Multiplicative Non-Archimedean valuation*</u> *if*
(1) $v(a) > 0$ for $a \neq 0$; $v(0) = 0$
(2) $v(ab) = v(a)v(b)$
(3) $v(a + b) \leq max\{v(a), v(b)\}$.

Notice that we have chosen to define the valuation as multiplicative. While most older texts tend to use the multiplicative valuations, modern texts tend to use additive valuations. In the case of $\mathbb{Q}(x)$, we define $v(\frac{f}{g}) = e^{deg(f)-deg(g)}$ where e is the transcendental number. In the Archimedean case we see that $\phi(m \cdot 1) = \phi(1+1+1+1+\ldots+1) > 1$. Where in the Non-Archimedean case we see $\phi(m \cdot 1) = \phi(1+1+1+1+\ldots+1) \leq 1$. The most significant change is seen in the multiplicative Archimedean valuation, where the triangle inequality holds versus the multiplicative Non-Archimedean valuation where we make use of what is known as the strong triangle inequality. This stronger version of the triangle inequality is seen in (3) above.

Castro showed that although there were some minor changes, many theorems and definitions from the traditional study of continued fractions hold true for our new rings and fields. We will now describe our new setting. Elements in $\mathbb{Q}[x]$ have the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ where $a_i \in \mathbb{Q}$ for all $i \geq 0$. That is, $\mathbb{Q}[x]$ is the domain consisting of polynomials with rational coefficients. $\mathbb{Q}(x)$ is the field of fractions of polynomials with rational coefficients. Elements belonging to $\mathbb{Q}(x)$ have the form $\frac{f(x)}{g(x)}$ where $f(x), g(x) \in \mathbb{Q}[x]$ and $g(x) \neq 0$. Elements in $\mathbb{Q}(x)^*$ have the form $\sum\limits_{i=k}^{-\infty} c_i x^i$ where $c_k \in \mathbb{Q}[x]$ for $k \in \mathbb{Z}$. If $f \in \mathbb{Q}[x]$ is a monic polynomial of even degree, $f = x^{2n} + a_{2n-1}x^{2n-1} + \ldots + a_0$, then $\sqrt{f} \in \mathbb{Q}(x)^*$. In particular, $\sqrt{f} = \sum\limits_{i=k}^{-\infty} c_i x^i$ where $c_i \in \mathbb{Q}$ and $c_k = 1$. In Chapter 3 we will use this representation to obtain an infinite continued fraction expansion for $\sqrt{f}$.

We are now ready to elaborate on the algebraic number theory behind our new setting.

Chapter 2

Algebraic Number Theory of O_K

2.1 Elements Integral Over A Domain

In the Introduction we defined elements from a domain B which are integral over a domain A. These elements also form a subdomain.

Theorem 2.1. *The set of all elements of $\mathbb{Q}(x)(\sqrt{f})$ that are integral over $\mathbb{Q}[x]$ is a subdomain of $\mathbb{Q}(x)(\sqrt{f})$ that contains $\mathbb{Q}[x]$.*

For proof of Theorem 2.1 see ([AW03]). Next we define algebraic integers and algebraic closure.

Definition 2.2. *Let S be the algebraic closure of $Q(x)^*$. Elements of S which are integral over $Q[x]$ are called algebraic integers.*

Definition 2.3. *An integral domain B is said to be integrally closed if the only elements of its quotient field that are integral over B are those of B itself.*

In fact, the set of all algebraic integers is an integral domain denoted by Ω. This notation is used to characterize the domain $\mathbb{Q}[x]$.

Theorem 2.4. $\mathbb{Q}(x) \cap \Omega = \mathbb{Q}[x]$

Proof. We need to show that $\mathbb{Q}[x]$ is integrally closed in $\mathbb{Q}(x)$. Theorem 2.2.1 ([AW03]) shows that because $\mathbb{Q}$ is a field, then $\mathbb{Q}[x]$ is a Euclidean domain. Also, a Euclidean domain is a Principal Ideal domain (PID). Further, PID implies integrally closed ([AW03]). Thus $\mathbb{Q}[x]$ is a Euclidean domain and a PID then $\mathbb{Q}[x]$ is integrally closed in $\mathbb{Q}(x)$. □

2.2 Minimal Polynomial Of An Element Algebraic Over A Field

Let K be a subfield of the field S where S is the algebraic closure of $\mathbb{Q}(x)^*$. Let $\alpha \in S$ be algebraic over K. That is, there exists a nonzero monic polynomial $g(X) \in K[X]$ such that $g(\alpha) = 0$. Now let $I_K(\alpha)$ denote the set of all polynomials in $K[X]$ for which this is true.

$$I_K(\alpha) = \{g(X) \in K[X] \,|g(\alpha) = 0\}$$

The interested reader may find that it is easy to check that $I_K(\alpha)$ is an ideal in $K[X]$. Since K is a field, $K[X]$ is a PID; then it is easy to see there exists a unique monic polynomial $p(X) \in K[X]$ such that $I_K(\alpha) = \langle p(X)\rangle$.

Definition 2.5. *Let K be a subfield of the field S and let $\alpha \in S$ be algebraic over K. Then the unique monic polynomial $p(X) \in K[X]$ such that*

$$I_K(\alpha) = \langle p(X)\rangle$$

is called the minimal polynomial of α over K denoted $irr_K(\alpha)$.

It turns out that $\mathrm{irr}_K(\alpha)$ is irreducible in $K[X]$, meaning if $\mathrm{irr}_K(\alpha)$ is the product of two polynomials in $K[X]$ then one must be a unit. This implies that $\mathrm{irr}_K(\alpha)$ is not a unit of $K[X]$.

Next we will discuss the conjugates of α over the subfield K.

2.3 Conjugates Of α Over K

First we define the conjugates of an element over a subfield of S.

Definition 2.6. *Let $\alpha \in S$ be algebraic over a subfield K of S. The conjugates of α over K are the roots in S of $irr_K(\alpha)$.*

The reader may not be surprised that as in the classical case, the conjugates of α over K are distinct. The following proof mimics the proof of the classical case found in ([AW03]).

Theorem 2.7. *Let K be a subfield of F. Let $\alpha \in F$ be algebraic over K. Then the conjugates of α over K are distinct.*

Proof. Suppose that α has two conjugates over K that are the same. Then $\mathrm{irr}_K(\alpha)$ has a root of order greater than or equal to 2. Let $\beta \in S$ be such a multiple root. Then

$$\mathrm{irr}_K(\alpha) = (X-\beta)^2 r(X),$$

where $r(X) \in S[X]$. Differentiating the above equation with respect to X, we obtain

$$\mathrm{irr}_K(\alpha)' = (X-\beta)^2 r'(X) + 2(X-\beta)r(X).$$

Thus β is a root of the derivative $\mathrm{irr}_K(\alpha)'$ of $\mathrm{irr}_K(\alpha)$. As $\mathrm{irr}_K(\alpha)' \in K[X]$ we have

$$\mathrm{irr}_K(\alpha)' \in I_K(\alpha) = \langle irr_K(\alpha)\rangle$$

so that

$$\mathrm{irr}_K(\alpha) \mid \mathrm{irr}_K(\alpha)'$$

and thus

$$\deg(\mathrm{irr}_K(\alpha)) \leq \deg(\mathrm{irr}_K(\alpha)').$$

Which is impossible. Hence the conjugates of α over K are distinct. $\square$

Then we may discuss the conjugates of α as the algebraic integer. The following proof is also found ([AW03]).

Theorem 2.8. *If α is an algebraic integer then it's conjugates over $\mathbb{Q}(x)$ are also algebraic integers.*

Proof. As α is an algebraic integer then there exists a monic polynomial $h(X) \in \mathbb{Q}[X]$ that has α as a root. Where $h(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0 \in \mathbb{Q}(x)[X]$. Since $h(X) \in \mathbb{Q}(x)[X]$ and $h(\alpha) = 0$ we have $h(X) \in I_{\mathbb{Q}(x)}(\alpha) =< \mathrm{irr}_{\mathbb{Q}(x)}(\alpha) >$, where $I_{\mathbb{Q}(x)}(\alpha)$ denotes the set of all polynomials in $\mathbb{Q}(x)[X]$ having α as a root. This implies that

$$h(X) = \mathrm{irr}_{\mathbb{Q}(x)}(\alpha) \cdot q(X)$$

for some $q(X) \in \mathbb{Q}(x)[X]$. Let β be a conjugate of α over $\mathbb{Q}(x)$. Then β is also a root of $\mathrm{irr}_{\mathbb{Q}(x)}(\alpha)$. That is, $h(\beta) = 0$ and β, the conjugate of α over $\mathbb{Q}(x)$, is also an algebraic integer. $\square$

Theorem 2.9. *If α is an algebraic integer then,*

$$irr_{\mathbb{Q}(x)}(\alpha) \in \mathbb{Q}[x][X]$$

Proof. Let the conjugates of the algebraic integer α over $\mathbb{Q}(x)$ be $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$. Then

$$\begin{aligned}\mathrm{irr}_{\mathbb{Q}(x)}(\alpha) &= (X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n)\\ &= X^n - (\alpha_1+\alpha_2+\cdots+\alpha_n)X^{n-1} + (\alpha_1\alpha_2+\cdots,\alpha_{n-1}\alpha_n)X^{n-2}\\ &+\cdots+(-1)^n\alpha_1\alpha_2\cdots\alpha_n.\end{aligned}$$

As $\mathrm{irr}_{\mathbb{Q}(x)}(\alpha) \in \mathbb{Q}(x)[X]$, we have

$$\begin{aligned}\alpha_1+\cdots+\alpha_n &\in \mathbb{Q}(x)\\ \alpha_1\alpha_2+\cdots+\alpha_{n-1}\alpha_n &\in \mathbb{Q}(x)\\ \alpha_1\alpha_2\cdots\alpha_n &\in \mathbb{Q}(x).\end{aligned}$$

By Theorem 2.8 if α is an algebraic integer then its conjugates over $\mathbb{Q}(x)$ are also algebraic integers. That is $\alpha_1, \ldots, \alpha_n$ are also algebraic integers. Now the set of all algebraic integers is an integral domain, which implies that $\alpha_1+\cdots+\alpha_n$, $\alpha_1\alpha_2+\cdots+\alpha_{n-1}\alpha_n$, $\alpha_1\alpha_2\cdots\alpha_n$ are all algebraic integers. Further $\mathbb{Q}[x] \subseteq \mathbb{Q}(x)$ where $\mathbb{Q}[x]$ is a unique factorization domain and is integrally closed, by a previous result. Thus $\alpha_1+\cdots+\alpha_n$, $\alpha_1\alpha_2+\cdots+\alpha_{n-1}\alpha_n$, $\alpha_1\alpha_2\cdots\alpha_n \in \mathbb{Q}[x]$. Finally, $\mathrm{irr}_{\mathbb{Q}(x)}(\alpha) \in \mathbb{Q}[x][X]$ $\square$

We continue with the following definition where we consider the norm of an element $\alpha \in K$.

Definition 2.10. *Let K be an algebraic number field and let $\alpha \in K$. Now let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the $K-$conjugates of α. The norm of α is denoted $N(\alpha)$ and is defined by*

$$N(\alpha) = \alpha_1\alpha_2\cdots\alpha_n.$$

If K is a quadratic field then $K = \mathbb{Q}(x)(\sqrt{f})$ for some squarefree polynomial f. Let $\alpha \in K$, then $\alpha = g + h\sqrt{f}$ for some $g, h \in \mathbb{Q}(x)$. The $K-$conjugates of α are $\alpha = g + h\sqrt{f}$ and $\alpha' = g + h\sqrt{f}$. Then the norm is

$$N(\alpha) = \alpha\alpha' = g^2 - fh^2 \quad \in \mathbb{Q}(x) \text{ and } N(0) = 0.$$

The interested reader may find that it is easy to show that the norm is multiplicative. That is, $N(\alpha\beta) = N(\alpha)N(\beta)$ for any $\alpha, \beta \in K$. Next we characterize the norm of an element.

Lemma 2.11. *Let K be an algebraic number field and let $\alpha \in K$. If α is an algebraic integer then $N(\alpha) \in \mathbb{Q}[x]$.*

Proof. By the definition above we have $N(\alpha) = \alpha_1\alpha_2 \cdots \alpha_n$ where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ are the $K - conjugates$ of α. Now α is an algebraic integer then by Theorem 2.9 the $K - conjugates$ of α are also algebraic integers. Also, $\mathbb{Q}[x]$ is integrally closed. Then $N(\alpha) = \alpha_1\alpha_2 \cdots \alpha_n \in \mathbb{Q}[x]$. □

Finally, we discuss the norm of a unit in O_K, where K is an algebraic number field of *degree* 2. The following theorem is a vital part in the main result of this thesis.

Theorem 2.12. *Let K be an algebraic number field of degree* 2.
(a) If α is a unit of O_K then $N(\alpha) \in \mathbb{Q} - \{0\}$.
(b) If $\alpha \in O_K$ and $N(\alpha) \in \mathbb{Q} - \{0\}$ then α is a unit of O_K.

Proof. (a) Let $\alpha \in O_K$ be a unit. Then there exists a $\beta \in O_K$ such that $\alpha\beta = 1$. Taking the norm we have $N(\alpha)N(\beta) = N(\alpha\beta) = N(1) = 1$. Now if $N(\alpha)$ divides 1 in $\mathbb{Q}[x]$, then $N(\alpha) \in \mathbb{Q} - \{0\}$ as needed.
(b) Let $\alpha \in O_K$ and $N(\alpha) \in \mathbb{Q} - \{0\}$. Say that $N(\alpha) = r$ for some $r \in \mathbb{Q} - \{0\}$ then by definition $\alpha \cdot \alpha' = r$ where both α and $\alpha' \in O_K$ by Theorem 2.9. Now $\frac{1}{r}(\alpha \cdot \alpha') = 1$ implies α is a unit. □

2.4 Algebraic Integers in a Quadratic Field K

We will now determine the algebraic integers in any quadratic extension of $\mathbb{Q}(x)$. First, adjoin the root of an irreducible polynomial to $\mathbb{Q}(x)$. This suggests that $\mathbb{Q}(x)(\alpha)$ is the smallest subfield of S which contains both $\mathbb{Q}(x)$ and α. The field $\mathbb{Q}(x)(\alpha)$ is called a quadratic field or a quadratic field extension of $\mathbb{Q}(x)$. The following theorem gives a way to represent this quadratic field.

Theorem 2.13. *Let K be a quadratic field. Then there exists a unique squarefree polynomial f such that $K = \mathbb{Q}(x)(\sqrt{f})$.*

For proof of this representation see ([AW03]). Next we describe the algebraic integers in the quadratic field $K = \mathbb{Q}(x)(\sqrt{f})$. We have already denoted this set as O_K.

Theorem 2.14. *Let $f \in \mathbb{Q}[x]$ be a squarefree polynomial and set $K = \mathbb{Q}(x)(\sqrt{f})$. Then the set O_K of algebraic integers in K is given by*

$$O_K = \mathbb{Q}[x] \; + \; \mathbb{Q}[x](\sqrt{f})$$

Proof. Let $a + b\sqrt{f} \in \mathbb{Q}[x]+\mathbb{Q}[x](\sqrt{f})$ where $a, b \in \mathbb{Q}[x]$, then there exists a monic polynomial of degree 2

$$Y^2 - 2aY + (a^2 - fb^2) \quad \in \mathbb{Q}[x][Y]$$

such that $a + b\sqrt{f}$ is a root of. This implies that $a + b\sqrt{f} \in O_K$. Thus we have shown inclusion in one direction.

Now let $\alpha \in O_K$ where $K = \mathbb{Q}(x)(\sqrt{f})$. Then let $\alpha = g + h(\sqrt{f})$ where $g, h \in \mathbb{Q}(x)$. Then α is the root of a monic polynomial of degree 2

$$X^2 - 2gX + (g^2 - fh^2) \quad \in \mathbb{Q}(x)[X].$$

The discriminant of this monic polynomial is as follows

$$(2g)^2 - 4(g^2 - h^2 f) = 4h^2 f.$$

Then our polynomial is reducible in $\mathbb{Q}(x)[X]$ if $h = 0$ and irreducible in $\mathbb{Q}(x)[X]$ if $h \neq 0$. Hence,

$$\mathrm{irr}_{\mathbb{Q}(x)}(\alpha) = \begin{cases} X - g, & if\ h = 0 \\ X^2 - 2gX + (g^2 - fh^2), & if\ h \neq 0 \end{cases}$$

By previous result, α is an algebraic integer then $\mathrm{irr}_{\mathbb{Q}(x)}\, \alpha \in \mathbb{Q}[x][X]$ so that

$$\begin{cases} g \in \mathbb{Q}[x], & if\ h = 0 \\ 2g, g^2 - fh^2 \in \mathbb{Q}[x], & h \neq 0 \end{cases}$$

Now $2g \in \mathbb{Q}[x]$ then $g \in \mathbb{Q}[x]$ since 2 is a unit in $\mathbb{Q}$. Also $g^2 - h^2 f \in \mathbb{Q}[x]$ implies $h^2 f \in \mathbb{Q}[x]$. Now $h \in \mathbb{Q}(x)$ then $h = \frac{h_1}{h_2}$ for some $h_1, h_2 \in \mathbb{Q}$. We have $(\frac{h_1}{h_2})^2 f \in \mathbb{Q}[x]$ and simplifying we see $\frac{(h_1)^2}{(h_2)^2} f$ which contradicts that f is a squarefree polynomial. Thus we have shown that $h, g \in \mathbb{Q}[x]$.

This completes the proof of Theorem 2.14 $\square$

We have now described algebraic integers in a quadratic field $K = \mathbb{Q}(x)(\sqrt{f})$ where f is any polynomial. Our goal will be to caculate the set of units in O_K. It turns out that most polynomials will result in trivial units. Our interest will be on a particular f. Consider the following:

Assume $f \in \mathbb{Q}[x]$ is squarefree where $deg(f)$ is odd and $\alpha = g + h\sqrt{f}$ is a unit of O_K. Then $N(\alpha) \in \mathbb{Q} - \{0\}$ and $g^2 - fh^2 \in \mathbb{Q} - \{0\}$. But g^2 has even degree and fh^2 has odd degree which forces $h = 0$ and $g^2 \in \mathbb{Q}$. Further forcing $\alpha = g \in \mathbb{Q}$, giving trivial units in O_K.

Next assume $f \in \mathbb{Q}[x]$ is squarefree where the leading coefficient is negative and $\alpha = g + h\sqrt{f}$ is a unit of O_K. Then $N(\alpha) \in \mathbb{Q} - \{0\}$ and $g^2 - fh^2 \in \mathbb{Q} - \{0\}$. Now assume that $deg(g) > 0$ then $deg(g^2 - fh^2) > 0$. Contradiction since $g^2 - fh^2 \in \mathbb{Q} - \{0\}$. Now supppose that $deg(g) = 0$ then $g \in \mathbb{Q} - \{0\}$ forcing $h = 0$ and $\alpha = g$, giving trivial units in O_K.

Finally assume $f \in \mathbb{Q}[x]$ is squarefree where the leading coefficient is not a perfect square and $\alpha = g + h\sqrt{f}$ is a unit of O_K. Then $N(\alpha) \in \mathbb{Q} - \{0\}$ and $g^2 - fh^2 \in \mathbb{Q} - \{0\}$. But the leading coefficient of f is not a perfect square forcing $h = 0$. Further we see $g^2 \in \mathbb{Q}$ and $\alpha = g \in \mathbb{Q}$, giving trivial units.

Thus any case where f is not a monic squarefree polynomial has trivial unity in O_K.

Our interest, as previously mentioned, will be to compute the units of O_K through the use of continued fractions. Traditionally, continued fractions are used to compute solutions to Pell's equation. Subsequently we will show that the units of O_K are also solutions. First we will define some terms and review important results we will need.

Chapter 3

Continued Fractions

The fundamental units in the ring O_K may be calculated through the use of continued fractions. In order to continue we must first summarize some important topics of continued fractions and its application to the new ring O_K.

Continued fractions were first introduced by Leonardo Fibonacci in his work *Liber abaci* published in 1202. There are two types of continued fraction expansions, those which are infinite and those which are finite. Classically, all rational numbers are expressible as finite continued fractions while irrational numbers are expressible as infinite continued fractions. Further, infinite continued fractions may be subdivided into periodic and non-periodic expansions. In order to characterize those real numbers which are expressible as certain types of infinite continued fractions we will need convergents, which we discuss in section 3.2. Convergents are approximations of infinite continued fractions represented by rational numbers. It is known the the limit of the i^{th} convergent exists and is equal to the real number that it represents.

As mentioned previously all irrational numbers are expressible as infinite continued fractions, however not all have the desired periodic expansion. Consider the following definition.

Definition 3.1. *Let $\alpha \in \mathbb{R} - \mathbb{Q}$. Then α is said to be quadratic irrational number if α is a root of a quadratic polynomial $Ax^2 + Bx + C$ with $A, B, C \in \mathbb{Z}$ and $A \neq 0$.*

A very important result of α being a quadratic irrational number is that the expression of α as an infinite continued fraction is always periodic or eventually periodic. These are the types of expansions that we will take most interest in. Further, this result has an impact on the special case to Pell's equation, introduced in Chapter 1, $x^2 - dy^2 = 1$. Now d must be a positive integer that is not a perfect square then $\sqrt{d}$ is a root of the polynomial $x^2 - d$. Then by the definition above we may conclude that d is a quadratic irrational number and $\sqrt{d}$ has an eventually periodic expansion. We will discuss the generalized Pell's equation $X^2 - Y^2 f = r$. Where there exists polynomials X and Y with rational coefficients which make the equation true. These solutions belong to a field $\mathbb{Q}(x)$ described in Chapter 2 and are also units in O_K. Lagrange was the first to prove that the classical equation $x^2 - dy^2 = 1$ has infinitely many solutions for integers x and y which suggests that there exists infinitely many units. We will discuss how the solutions to this special case are found and later turn our attention to the generalized Pell equation and its units.

As suggested there are infinitely many solutions to Pell's equation, but once we have the least positive solution the others may be generated. That is, when there exists polynomials X and Y that make the equation true, they are some variation of the least positive solution. This least positive solution is better known as the Fundamental solution.

3.1 Finite Continued Fractions

Before we can properly introduce finite continued fractions, we will need the Division Algorithm.

Theorem 3.2. *Given $f(x)$, $g(x) \in \mathbb{Q}[x]$ with $g(x) \neq 0$, then there exists a unique $q(x)$, $r(x) \in \mathbb{Q}[x]$ such that $f(x) = q(x)g(x) + r(x)$ where deg$(r(x))<$deg$(g(x))$ or $r(x) = 0$.*

For the proof of the division algorithm in its new setting see ([Cas]). We are now ready to define a finite continued fraction and its expansion.

Definition 3.3. *The expression*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Q}(x)$, $deg(a_i) > 0$ *for* $0 < i \leq n$, *and* $a_1, a_2, \ldots, a_n \neq 0$ *is said to be a* *finite continued expansion* *and is denoted by* $[a_0, a_1, a_2, \ldots, a_n]$. *A* *finite simple continued fraction* *is a continued fraction expansion in which* $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Q}[x]$.

Notice that if the degree of the numerator is smaller than that of the denominator then the expansion may begin with 0. Also, $deg(a_i)$ must be greater than zero for all i to ensure that each finite continued fraction expansion is unique. Our main interest will be the *simple* continued fractions expansion, however in Chapter 4 we will consider an interesting example with an expansion which is not *simple*. We continue with an example of a finite simple continued fraction.

Example 3.4. *Find the finite simple continued fraction expansion of* α

$$\alpha = \tfrac{18x^8-36x^6+15x^5+30x^3-12x^2-24x+16}{6x^5-3x^2+4}$$

By the repeated use of the Division Algorithm we have the Euclidean Algorithm,

$$\begin{aligned} 18x^8 - 36x^6 + 15x^5 + 30x^3 - 12x^2 - 24x + 16 &= (6x^5 - 3x^2 + 4) \cdot (3x^3 - 9x + 4) + (3x) \\ 6x^5 - 3x^2 + 4 &= (3x) \cdot (2x^4 - x) + (4) \\ 3x &= (4) \cdot (\tfrac{3x}{4}) + 0 \end{aligned}$$

Dividing each equation by the first factor of the product to the right of the equal sign yields,

$$\begin{aligned} \tfrac{18x^8-36x^6+15x^5+30x^3-12x^2-24x+16}{6x^5-3x^2+4} &= 3x^3 - 9x + 4 + \tfrac{3x}{6x^5-3x^2+4} \\ \tfrac{6x^5-3x^2+4}{3x} &= 2x^4 - x + \tfrac{4}{3x} \\ \tfrac{3x}{4} &= \tfrac{3x}{4} \end{aligned}$$

and

$$\frac{18x^8-36x^6+15x^5+30x^3-12x^2-24x+16}{6x^5-3x^2+4} = 3x^3 - 9x + 4 + \frac{1}{\frac{6x^5-3x^2+4}{3x}}$$
$$= 3x^3 - 9x + 4 + \frac{1}{(2x^4-x)+\frac{1}{\frac{3x}{4}}}$$

Then, $\frac{18x^8-36x^6+15x^5+30x^3-12x^2-24x+16}{6x^5-3x^2+4} = [3x^3 - 9x + 4, 2x^4 - x, \frac{3x}{4}]$

Through the manipulation of the Euclidean algorithm we were able to see the expression α in a particular expansion, previously defined as a continued fraction. The interested reader may compute the expansion of an α where the degree of the numerator is smaller than the denominator.

From the previous example it should be clear that for all $\alpha \in \mathbb{Q}(x)$, α is expressible as a finite simple continued fraction. It is also true that every finite simple continued fraction represents an element of $\mathbb{Q}(x)$. These two ideas summarize the topic of the following theorem.

Theorem 3.5. *Let $\alpha \in \mathbb{Q}(x)^*$. Then $\alpha \in \mathbb{Q}(x)$ if and only if α is expressible as a finite simple continued fraction.*

For proof see ([Cas]).

3.2 Convergents

Ultimately we would like for the reader to have a grasp on infinite simple continued fractions. Through the use of convergents we will explore how adding terms to the expansion will get us closer and closer to an $\alpha \in \mathbb{Q}(x)$. Let us first consider convergents of a finite continued fraction expansion.

Definition 3.6. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be expressible as a finite continued fraction. The finite continued fraction $C_i = [a_0, a_1, \ldots, a_i]$ for $0 \leq i \leq n$ is said to be the $\underline{i^{th}\text{ convergent}}$ of α. If $i = n$, then $\alpha = C_n = [a_0, a_1, \ldots, a_n]$.*

Note that the n^{th} convergent of $\alpha = [a_0, a_1, a_2 \ldots, a_n]$ is equal to α, as illustrated by the following example.

Example 3.7. *Find the convergents of $\alpha = [x^4, x^3, x^2, x]$.*

The convergents of α are:

$$C_0 = [x^4] = x^4$$

$$C_1 = [x^4, x^3] = x^4 + \tfrac{1}{x^3} = \tfrac{x^7+1}{x^3}$$

$$C_2 = [x^4, x^3, x^2] = x^4 + \tfrac{1}{x^3+\frac{1}{x^2}} = \tfrac{x^9+x^4+x^2}{x^5+1}$$

$$C_3 = [x^4, x^3, x^2, x] = x^4 + \tfrac{1}{x^3+\frac{1}{x^2+\frac{1}{x}}} = \tfrac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}$$

Thus $\alpha = C_3 = \frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}$.

Note that $\alpha = C_3$. The name is promising as the convergents are indeed converging to α. That is C_1 is 'closer' than C_0, C_2 is 'closer' then C_1 and so on. However, this method of computing convergents is tedious. Imagine having an α with an expansion of 20 terms. These convergents may be more easily computed with the use of certain recurrence relations shown in the next proposition. The proof of Proposition 3.8 can also be found in ([Cas]). We repeat it here to illustrate the technique that we will use.

Proposition 3.8. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be expressible as a finite simple continued fraction. Define $h_0, h_1, \ldots, h_n$ and $k_0, k_1, \ldots, k_n$ by the following recurrence relations: For $2 \leq i \leq n$,*

$$h_0 = a_0 \qquad k_0 = 1$$

$$h_1 = a_1 a_0 + 1 \qquad k_1 = a_1$$

$$h_i = a_i h_{i-1} + h_{i-2} \qquad k_i = a_i k_{i-1} + k_{i-2}$$

Then,

$$C_i = \tfrac{h_i}{k_i}, \text{ for } 0 \leq i \leq n.$$

Proof. Notice, if $i = 0$,

$$C_0 = [a_0] = a_0 = \tfrac{a_0}{1} = \tfrac{h_0}{k_0}.$$

If $i = 1$,

$$C_1 = [a_0, a_1] = a_0 + \tfrac{1}{a_1} = \tfrac{a_0 a_1 + 1}{a_1} = \tfrac{h_1}{k_1}.$$

If $i = 2$,

$$C_2 = [a_0, a_1, a_2] = a_0 + \tfrac{1}{a_1+\frac{1}{a_2}} = \tfrac{a_2(a_0 a_1+1)}{a_2 a_1+1} = \tfrac{a_2 h_1 + h_0}{a_2 k_1 + k_0} = \tfrac{h_2}{k_2}.$$

Assume, for the sake of induction, that $C_m = [a_0, a_1, \ldots, a_m] = \frac{h_m}{k_m}$. Consider,

$$
\begin{aligned}
C_{m+1} &= [a_0, a_1, \ldots, a_m, a_{m+1}] = [a_0, a_1, \ldots, a_m + \tfrac{1}{a_{m+1}}] \\
&= \frac{(a_m+\frac{1}{a_{m+1}})h_{m-1}+h_{m-2}}{(a_m+\frac{1}{a_{m+1}})k_{m-1}+k_{m-2}} \quad (\text{Since } \tfrac{h_m}{k_m} = \tfrac{a_m h_{m-1}+h_{m-2}}{a_m k_{m-1}+k_{m-2}}) \\
&= \frac{a_{m+1}(a_m h_{m-1}+h_{m-2})+h_{m-1}}{a_{m+1}(a_m k_{m-1}+k_{m-2})+k_{m-1}} \\
&= \frac{a_{m+1}h_m+h_{m-1}}{a_{m+1}k_m+k_{m-1}} \quad (\text{By the induction hypothesis}) \\
&= \frac{h_{m+1}}{k_{m+1}}.
\end{aligned}
$$

Thus by Mathematical Induction, $C_i = \frac{h_i}{k_i}$, for $0 \le i \le n$. □

We may use Proposition 3.8 to compute the convergents of Example 3.5.

Example 3.9. *Compute the convergents of α using Proposition 3.8.*

$$\alpha = \frac{18x^8-36x^6+15x^5+30x^3-12x^2-24x+16}{6x^5-3x^2+4} = [3x^3 - 9x + 4, 2x^4 - x, \tfrac{3x}{4}].$$

By Proposition 3.8 we have $a_0 = 3x^3 - 9x + 4, a_1 = 2x^4 - x, a_2 = \frac{3x}{4}$ Then

$$
\begin{aligned}
h_0 &= 3x^3 - 9x + 4 \\
h_1 &= (3x^3 - 9x + 4)(2x^4 - x) + 1 = (6x^7 - 18x^5 + 5x^4 + 9x^2 - 4x + 1) \\
h_2 &= (\tfrac{3x}{4})(6x^7 - 18x^5 + 5x^4 + 9x^2 - 4x + 1) + (3x^3 - 9x + 4) \\
&= \tfrac{9x^8}{2} - \tfrac{27x^6}{2} + \tfrac{15x^5}{4} + \tfrac{39x^3}{4} - 3x^2 - 9x + 4
\end{aligned}
$$

$$
\begin{aligned}
k_0 &= 1 \\
k_1 &= 2x^4 - x \\
k_2 &= (\tfrac{3x}{4})(2x^4 - x) + 1 = \tfrac{3x^5}{2} - \tfrac{3x^2}{4} + 1
\end{aligned}
$$

Then the convergents based on the calculations above are:

$$
\begin{aligned}
C_0 &= 3x^3 - 9x + 4 \\
C_1 &= \frac{6x^7-18x^5+5x^4+9x^2-4x+1}{2x^4-x} \\
C_2 &= \frac{\frac{9x^8}{2}-\frac{27x^6}{2}+\frac{15x^5}{4}+\frac{39x^3}{4}-3x^2-9x+4}{\frac{3x^5}{2}-\frac{3x^2}{4}+1}
\end{aligned}
$$

Now let us consider the results of Example 3.7 and compare them to computations of convergents through the recurrence relations found in Proposition 3.8.

Example 3.10. *Compute the convergents of α using Proposition 3.8 and compare the results with Example 3.7.*

From example 3.7 we have $\alpha = \frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x} = [x^4, x^3, x^2, x]$. Then $a_0 = x^4$, $a_1 = x^3$, $a_2 = x^2$, $a_3 = x$. So,

$$\begin{aligned}
&h_0 = x^4 \\
&h_1 = (x^3) \cdot (x^4) + 1 = x^7 + 1 \\
&h_2 = (x^2) \cdot (x^7 + 1) + (x^4) = x^9 + x^4 + x^2 \\
&h_3 = (x) \cdot (x^9 + x^4 + x^2) + (x^7 + 1) = x^{10} + x^7 + x^5 + x^3 + 1 \\
&k_0 = 1 \\
&k_1 = x^3 \\
&k_2 = (x^2) \cdot (x^3) + 1 = x^5 + 1 \\
&k_3 = (x) \cdot (x^5 + 1) + (x^3) = x^6 + x^3 + x.
\end{aligned}$$

Thus,

$$\begin{aligned}
&C_0 = \tfrac{x^4}{1} = x^4 \\
&C_1 = \tfrac{x^7+1}{x^3} \\
&C_2 = \tfrac{x^9+x^4+x^2}{x^5+1} \\
&C_3 = \tfrac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}.
\end{aligned}$$

The reader may have noticed that in the two previous examples the convergents of $\alpha \in \mathbb{Q}(x)$ are in lowest terms. This observation may be better established in the following proposition and its corollary.

Proposition 3.11. *Let $\alpha = [a_1, a_2, \ldots, a_n]$ be expressible as a finite simple continued fraction and let all notation be as in Proposition 3.8 Then, $h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$ for $1 \le i \le n$.*

The proposition is evident in examples 3.9 and 3.10. The following corollary shows that the convergents of a a finite simple continued fraction are in lowest terms.

Corollary 3.12. *Let $\alpha = [a_1, a_2, \ldots, a_n]$ be a finite simple continued fraction and let all notation be as in Proposition 3.8. Then, $gcd(h_i, k_i) = 1$ for $0 \le i \le n$.*

The second corollary of Proposition 3.11 will begin to extend the theory from finite simple continued fractions to infinite simple continued fractions.

Corollary 3.13. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be a finite simple continued fraction with all notation as in Proposition 3.8*
Then,

$$C_i - C_{i-1} = \frac{(-1)^{i-1}}{k_i k_{i-1}} \; for \; 1 \leq i \leq n \quad and \quad C_i - C_{i-2} = \frac{(-1)^i a_i}{k_i k_{i-2}} \; for \; 2 \leq i \leq n.$$

For proofs of Proposition 3.11 and its two corollaries see ([Cas]). We are now ready to consider infinite simple continued fractions.

3.3 Infinite Continued Fractions

Traditionally in number theory all real numbers may be represented as continued fractions. Rational numbers are characterized by having finite representations, while irrational numbers are characterized by having infinite representations. This is also true for elements in $\mathbb{Q}(x)^*$. As previously stated, for $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, α will have an infinite continued fraction expansion. As seen by previous examples, for every $\beta \in \mathbb{Q}(x)$ β may be characterized by having finite continued fraction expansions. Our interest is in certain $\alpha's$ belonging to $\mathbb{Q}(x)^* - \mathbb{Q}(x)$. In particular these elements will be square roots of monic polynomials with even degrees and rational coefficients, which was discussed in Chapter 2. As demonstrated by Castro, not all traditional Number Theory theorems and proofs hold in the new setting $Q(x)^*$. Unlike the real numbers, the new setting $Q(x)^*$ lacks the notion of absolute value, which is a strong tool used in several proofs. In $Q(x)^*$ however, we have the idea of a valuations as defined in the introduction. Another important difference when considering the new case is that we no longer will have a greatest integer functions since we have moved from the integers to the ring of polynomials with rational coefficients. In consideration of this new ring we will make use of the "'integral part"' of an element defined as follows;

Definition 3.14. *Let* $\alpha = \sum_{i=k}^{-\infty} c_i x^i \in \mathbb{Q}(x)^*$. *Then the* <u>*integral part*</u> *of* α, *denoted* $[[\alpha]]$, *is defined as* $\sum_{i=k}^{0} c_i x^i$.

The focus of this section will be to discuss infinite expansions. In an effort to avoid unpredictable expansions we will focus on periodic and eventually periodic expansions. The following proposition defines the uniqueness of such expansions.

Proposition 3.15. *Let* $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. *Then the expression of* α *as an infinite simple continued fraction is unique.*

Thus both finite and infinite simple continued fraction expansions are unique. This leads us to the following proposition.

Proposition 3.16. *$\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ if and only if α is expressible as an infinite simple continued fraction.*

For proofs of Propositions 3.15 and 3.16 see ([Cas]). Now consider the following example of a unique infinite simple continued fraction expansion.

Example 3.17. *Find the infinite simple continued fraction expansion of $\sqrt{9x^2+1} \in \mathbb{Q}(x)^*$ using proposition 3.16.*

$$\alpha = \alpha_0 = \sqrt{9x^2+1} = 3x + \tfrac{1}{6}x^{-1} + \dots$$
$$\begin{aligned} a_0 &= [[\alpha_0]] = 3x & \alpha_1 &= \tfrac{1}{\alpha_0 - a_0} = 6x + \tfrac{1}{6}x^{-1} + \dots \\ a_1 &= [[\alpha_1]] = 6x & \alpha_2 &= \tfrac{1}{\alpha_1 - a_1} = 6x + \tfrac{1}{6}x^{-1} + \dots \\ a_2 &= [[\alpha_2]] = 6x & \alpha_3 &= \tfrac{1}{\alpha_2 - a_2} = 6x + \tfrac{1}{6}x^{-1} + \dots \end{aligned}$$
$$\vdots$$

Thus $a_i = 6x$ for $i > 0$ making the infinite simple continued fraction expansion of $\sqrt{9x^2+1} = [3x, \overline{6x}]$.

The interested reader will find that a slight change of the the polynomial changes the expansion dramatically.

Example 3.18. *Find the infinite simple continued fraction expansion of $\sqrt{x^2+4x+1} \in \mathbb{Q}(x)^*$ using Proposition 3.16.*

$$\alpha = \alpha_0 = \sqrt{x^2+4x+1} = x + 2 - \tfrac{3}{2}x^{-1} + \dots$$
$$\begin{aligned} a_0 &= [[\alpha_0]] = x + 2 & \alpha_1 &= \tfrac{1}{\alpha_0 - a_0} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots \\ a_1 &= [[\alpha_1]] = -\tfrac{2}{3}x - \tfrac{4}{3} & \alpha_2 &= \tfrac{1}{\alpha_1 - a_1} = 2x + 4 - \tfrac{3}{2}x^{-1} + \dots \\ a_2 &= [[\alpha_2]] = 2x + 4 & \alpha_3 &= \tfrac{1}{\alpha_2 - a_2} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots \\ a_3 &= [[\alpha_3]] = -\tfrac{2}{3}x - \tfrac{4}{3} & \alpha_4 &= \tfrac{1}{\alpha_3 - a_3} = 2x + 4 - \tfrac{3}{2}x^{-1} + \dots \\ a_4 &= [[\alpha_4]] = 2x + 4 & \alpha_5 &= \tfrac{1}{\alpha_4 - a_4} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots \end{aligned}$$
$$\vdots$$

Thus, the infinite simple continued fraction expansion of $\sqrt{x^2+4x+1} = [x+2, -\frac{2}{3}x - \frac{4}{3}, 2x+4, -\frac{2}{3}x - \frac{4}{3}, 2x+4, \dots]$.

Although the two previous examples eventually repeat not all expansions are as neat.

Example 3.19. *Find the infinite simple continued fraction expansion of* $\sqrt{x^4+8x^3+16x^2+x+1}$ *using Proposition 3.16.*

$$\alpha = \alpha_0 = \sqrt{x^4+8x^3+16x^2+x+1} = x^2+4x+\tfrac{1}{2}x^{-1}-\tfrac{3}{2}x^{-2}+\ldots$$

$$\begin{array}{ll} a_0 = [[\alpha_0]] = x^2+4x & \alpha_1 = \frac{1}{\alpha_0-a_0} = \frac{2x^2+8x+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\ldots}{x+1} \\ a_1 = [[\alpha_1]] = 2x+6 & \alpha_2 = \frac{1}{\alpha_1-a_1} = \frac{-2x^2-8x-6-\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\ldots}{12x+35} \\ a_2 = [[\alpha_2]] = -\frac{1}{6}x-\frac{13}{72} & \alpha_3 = \frac{1}{\alpha_2-a_2} = \frac{10368x^2+41472x+1656+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\ldots}{276x-133} \\ a_3 = [[\alpha_3]] = \frac{864}{23}x+\frac{89064}{529} & \alpha_4 = \frac{1}{\alpha_3-a_3} = \frac{559682x^2+2238728x-1298166+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\ldots}{x+1} \end{array}$$

$$\vdots$$

There doesn't seem to be a pattern and notice that the coeffecients are growing rapidly. Unfortunately, determining if a $\sqrt{f}$ is periodic or eventually periodic is difficult.

As we did with the finite expansions we will also study the convergents of these infinite cases. Previously we found that $\alpha = [a_0, a_1, \cdots, a_n] = C_n$ for $i = n$. However, in the infinite case C_i is simply an approximation of α for all i. It is important we know the accuracy of this approximation of α, known as the i^{th} convergent. First consider the following proposition.

Proposition 3.20. *Let* $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ *and let* $\frac{h_i}{k_i}, i = 0,1,2,\ldots,$ *be the convergents of the infinite simple continued fraction expansion of* α*. If* $a, b \in \mathbb{Q}[x]$ *and* $0 < v(b) < v(k_{i+1})$*, then* $v(k_i\alpha - h_i) \leq v(b\alpha - a)$*.*

For the proof see ([Cas]). The use of Proposition 3.20 aids in the proof that the i^{th} convergent of α is the most accurate approximation for α.

Corollary 3.21. *Let* $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ *and let* $\frac{h_i}{k_i}, i = 0,1,2,\ldots$ *be the convergents of the infinite simple continued fraction expansion of* α*. If* $a, b \in \mathbb{Q}[x]$ *and* $0 \leq v(b) \leq v(k_i)$*, then* $v(\alpha - \frac{h_i}{k_i}) \leq v(\alpha - \frac{a}{b})$*.*

Proof. Assume, by contradiction, $v(\alpha - \frac{h_i}{k_i}) > v(\alpha - \frac{a}{b})$. Then, $v(k_i\alpha - h_i) = v(k_i)v(\alpha - \frac{h_i}{k_i})1 > v(b)v(\alpha - \frac{a}{b}) = v(b\alpha - a)$ which contradicts Proposition 3.21. □

Next we show that a close approximation of α is $\frac{a}{b}$ if and only if $\frac{a}{b}$ a convergent of the infinite simple continued fraction expansion.

Proposition 3.22. *Let $a, b \in \mathbb{Q}[x]$ with $gcd(a,b) = 1$ and $v(b) > 0$. Let $\alpha \in \mathbb{Q}(x)^*$, then $v(\alpha - \frac{a}{b}) \leq \frac{1}{v(xb^2)}$ if and only if $\frac{a}{b}$ is a convergent of the infinite simple continued fraction expansion of α.*

For proof see ([Cas]). To illustrate Proposition 3.22 consider the following example.

Example 3.23. $\alpha = \sqrt{x^2+1} = [x, 2x, 2x, \ldots]$.

Then,

$\alpha = x + \frac{1}{2}x^{-1} - \frac{1}{8}x^{-3} + \frac{1}{16}x^{-5} - \frac{5}{128}x^{-7} + \frac{7}{256}x^{-9} - \frac{21}{1024}x^{-11} + \frac{33}{2048}x^{-13} - \frac{143}{65536}x^{-19} + \ldots$.

Consider

$$\frac{a}{b} = \frac{64x^7+112x^5+56x^3+7x}{64x^6+80x^4+24x^2+1}$$

$= x + \frac{1}{2}x^{-1} - \frac{1}{8}x^{-3} + \frac{1}{16}x^{-5} - \frac{5}{128}x^{-7} + \frac{7}{256}x^{-9} - \frac{21}{1024}x^{-11} + \frac{1048x^4+476x^2+21}{65536x^{17}+81920x^{15}+24576x^{13}+1024x^{11}}$.

Since $v(\alpha - \frac{a}{b}) = e^{-13} \leq \frac{1}{v(xb^2)} = e^{-13}$, $\frac{64x^7+112x^5+56x^3+7x}{64x^6+80x^4+24x^2+1}$ is a convergent of the infinite simple continued fraction expansion of $\sqrt{x^2+1}$. In fact,

$$C_6 = [x, 2x, 2x, 2x, 2x, 2x, 2x] = \frac{64x^7+112x^5+56x^3+7x}{64x^6+80x^4+24x^2+1} = \frac{a}{b}.$$

On the other hand, consider each of the convergents of the infinite simple continued fraction expansion of $\sqrt{x^2+1}$.

$$\begin{aligned}
C_0 &= x \\
C_1 &= x + \tfrac{1}{2}x^{-1} = \tfrac{2x^2+1}{2x} \\
C_2 &= x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{32x^5+8x^3} = \tfrac{4x^3+3x}{4x^2+1} \\
C_3 &= x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{16}x^{-5} - \tfrac{1}{32x^7++16x^5} = \tfrac{8x^4+8x^2+1}{8x^3+4x} \\
&\vdots
\end{aligned}$$

Then,

$$\begin{aligned}
v(\alpha - C_0) &= v(\alpha - \tfrac{x}{1}) = e^{-1} \leq \tfrac{1}{v(x(1)^2)} = e^{-1} \\
v(\alpha - C_1) &= v(\alpha - \tfrac{2x^2+1}{2x}) = e^{-3} \leq \tfrac{1}{v(x(2x)^2)} = e^{-3} \\
v(\alpha - C_2) &= v(\alpha - \tfrac{4x^3+3x}{4x^2+1}) = e^{-5} \leq \tfrac{1}{v(x(4x^2+1)^2)} = e^{-5} \\
v(\alpha - C_3) &= v(\alpha - \tfrac{8x^4+8x^2+1}{8x^3+4x}) = e^{-7} \leq \tfrac{1}{v(x(8x^3+4x)^2)} = e^{-7} \\
&\vdots
\end{aligned}$$

In this example, $v(\alpha - \frac{h_i}{k_i}) = \frac{1}{v(xk_i^2)}$ for all i because $v(a_0) = v(a_1) = v(a_2) = \ldots$. Choosing a new example where $v(a_i)$ varies will cause $v(\alpha - \frac{h_i}{k_i}) \leq \frac{1}{v(xk_i^2)}$.

3.4 Eventually Periodic Continued Fractions

Up to this point we have seen expansions where there appears to be a pattern. This repeating behavior is defined as being eventually periodic.

Definition 3.24. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ and let $\alpha = [a_0, a_1, \dots]$ be the infinite simple continued fraction expansion of α. Then α is said to be* <u>*eventually periodic*</u> *if there exists nonnegative integers l and N such that $a_n = a_{n+l}$ for all $n \geq N$. We call the sequence $a_N, a_{N+1}, \dots, a_{N+(l-1)}$ the* <u>*period*</u> *of α where l is minimal. The eventually periodic continued fraction expansion is denoted,*

$$\alpha = [a_0, a_1, \dots a_{N-1}, \overline{a_N, a_{N+1}, \dots a_{N+(l-1)}}].$$

If N=0, the infinite simple continued fraction expansion is called <u>*purely periodic*</u>*, or* <u>*periodic*</u> *for short.*

Although the focus of my thesis will be to compute a fundamental unit of $\mathbb{Q}(x)(\sqrt{f})$, where $\sqrt{f}$ has an infinite simple continued fraction expansion that is eventually periodic. In Chapter 4 we will see a special case where the expansion of $\sqrt{f}$ is not simple. That is, if $\sqrt{f} = [a_0, \overline{a_1, a_2, \dots, 2a_0}]$ is the infinite continued fraction expansion of $\sqrt{f}$ where not all $a_i \in \mathbb{Q}[x]$. Specifically we will chose $a_1 \in \mathbb{Q}(x)$ which will force $\sqrt{f}$ to have an expansion of period 2. For now we continue with an example of an expansion which is purely periodic.

Example 3.25. *By Example 3.17 $\sqrt{9x^2+1} = [3x, \overline{6x}]$. Then the expansion is eventually periodic. Also,*

$$\sqrt{9x^2+1} = [3x, \overline{6x}] = 3x + \cfrac{1}{6x + \cfrac{1}{6x + \cfrac{1}{\ddots}}}$$

Adding $3x$ to both sides we see that

$$3x + \sqrt{9x^2+1} = 6x + \cfrac{1}{6x + \cfrac{1}{6x + \cfrac{1}{\ddots}}} = [\overline{6x}]$$

becomes a purely periodic infinite simple continued fraction expansion.

Next characterize some elements of $\mathbb{Q}(x)^* - \mathbb{Q}(x)$ with the following definition.

Definition 3.26. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. Then α is a quadratic surd if α is a root of a quadratic polynomial $Ax^2 + Bx + C$ with $A, B, C \in \mathbb{Q}[x]$ and $A \neq 0$.*

A more elegant characterization is seen in the following propisition the proof of which is found in ([Cas]).

Proposition 3.27. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, then α is a quadratic surd if and only if $\alpha = \frac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Q}[x]$, $v(b) > 0$, b is not a perfect square, and $c \neq 0$.*

Proof. ($\Rightarrow$) Assume that α is a quadratic surd. Then there exists $A, B, C \in \mathbb{Q}[x]$ with $A \neq 0$ such that $A(\alpha)^2 + B(\alpha) + C = 0$. By the quadratic formula, we have

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Note, $B^2 - 4AC$ is not a perfect square and $v(B^2 - 4AC) > 0$ since α is a quadratic surd. Letting $a = -B$, $b = B^2 - 4AC$, and $c = 2A$ and taking the positive sign yields $\alpha = \frac{a+\sqrt{b}}{c}$. Otherwise, letting $a = B$, $b = B^2 - 4AC$, and $c = -2A$ and taking the negative sign also yields the desired result.
($\Leftarrow$) Assume that $\alpha = \frac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Q}[x]$, $v(b) > 0$, b is not a perfect square, and $c \neq 0$. Then $c^2 \neq 0$ implies α is a root of the quadratic polynomial, $c^2x^2 - 2acx + (a^2 - b)$. Let $A = c^2$, $B = -2ac$, $C = a^2 - b$. Then α is a root of the polynomial $Ax^2 + Bx + C$ with $A, B, C \in \mathbb{Q}[x]$ and $A \neq 0$ as desired. $\square$

Now we will see a series of results that further examine quadratic surds. For proof of these results see ([Cas]).

Lemma 3.28. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, then α is a quadratic surd if and only if $\alpha = \frac{P+\sqrt{d}}{Q}$ where $P, Q, d \in \mathbb{Q}[x]$, $v(d) > 0$, d is not a perfect square, $Q \neq 0$, and $Q | d - P^2$.*

The method seen in the first part of Proposition 3.16 turns out to be complicated when computing expansions of a quadratic surd. The following is an alternate easier method of computing continued fraction expansion.

Proposition 3.29. *By Lemma 3.28, let*

$$\alpha = \frac{P_0+\sqrt{d}}{Q_0}$$

be a quadratic surd where $P_0, d, Q_0 \in \mathbb{Q}[x]$, $d \neq 0$, *d is not a perfect square,* $Q_0 \neq 0$, *and* $Q_0 | (d - P_0^2)$. *Define* $\alpha_0, \alpha_1, \alpha_2, \ldots, a_0, a_1, a_2, \ldots, P_1, P_2, \ldots, Q_1, Q_2, \ldots$ *by the following recurrence relations:*

$$\alpha_i = \frac{P_i+\sqrt{d}}{Q_i} \qquad a_i = [[\alpha_i]] \qquad P_{i+1} = a_i Q_i - P_i \qquad Q_{i+1} = \frac{d-P_{i+1}^2}{Q_i}$$

for all $i \geq 0$. *Then* $\alpha = [a_0, a_1, a_2, \ldots]$.

Proposition 3.29 will be a valuable tool in computing units in O_K.

Lemma 3.30. *Let* α *be a quadratic surd and let* $a, b, c, d \in \mathbb{Q}[x]$. *Then,*

$$\alpha = \frac{a\alpha+b}{c\alpha+d}$$

belongs to either $\mathbb{Q}(x)$ *or* $\mathbb{Q}(x)^* - \mathbb{Q}(x)$.

Next recall the definition of conjugates of α over K and some of the properties of conjugates.

Definition 3.31. *Let* $\alpha = \frac{a+\sqrt{b}}{c}$ *be a quadratic surd where* $a, b, c \in \mathbb{Q}[x]$ *and* $c \neq 0$. *The* <u>*conjugate*</u> *of* α, *denoted* α', *is*

$$\alpha' = \frac{a-\sqrt{b}}{c}.$$

Lemma 3.32. *Let* $\alpha_1 = \frac{a_1+\sqrt{b}}{c_1}$ *and* $\alpha_2 = \frac{a_2+\sqrt{b}}{c_2}$ *where* $a_1, a_2, b, c_1, c_2 \in \mathbb{Q}[x]$ *and* $c_1, c_2 \neq 0$. *Then*

a) $(\alpha_1 + \alpha_2)' = \alpha_1' + \alpha_2'$
b) $(\alpha_1 - \alpha_2)' = \alpha_1' - \alpha_2'$
c) $(\alpha_1 \alpha_2)' = \alpha_1' \alpha_2'$
d) $(\frac{\alpha_1}{\alpha_2})' = \frac{\alpha_1'}{\alpha_2'}$.

We have just seen the best possible characterization of a quadratic surd. Now an important result that will be vital in the solution to Pell's equation and further in computing the fundamental unit of O_K.

Theorem 3.33. *Let* $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. *If the expression of* α *as an infinite simple continued fraction is eventually periodic, then* α *is a quadratic surd.*

For proof see ([Cas]). Theorem 3.33 is best summarized in the following example.

Example 3.34. *Find the quadratic surd represented by* $\alpha = [4x^2, \overline{4x^2, 8x^2}]$.

First let us find the quadratic surd represented by the periodic infinite simple continued fraction $\beta = [\overline{4x^2, 8x^2}]$. Notice $\beta = [4x^2, 8x^2, \overline{4x^2, 8x^2}] = [4x^2, 8x^2, \beta]$ so we have,

$$\beta = 4x^2 + \frac{1}{8x^2 + \frac{1}{\beta}}$$

$$\Rightarrow \beta = \frac{32x^4\beta + 4x^2 + \beta}{8x^2\beta + 1}$$

$$\Rightarrow 8x^2\beta^2 - 32x^4\beta - 4x^2 = 0.$$

By the quadratic formula, $\beta = \frac{32x^4 + \sqrt{1024x^8 + 128x^4}}{16x^2}$. Thus

$$\begin{aligned}
\alpha = [4x^2, \overline{4x^2, 8x^2}] &= 4x^2 + \frac{1}{\frac{32x^4 + \sqrt{1024x^8 + 128x^4}}{16x^2}} \\
&= 4x^2 + \frac{16x^2}{32x^4 + \sqrt{1024x^8 + 128x^4}} \\
&= \frac{128x^6 + 4x^2\sqrt{1024x^8 + 128x^4} + 16x^2}{32x^4 + \sqrt{1024x^8 + 128x^4}} \\
&= \frac{\sqrt{1024x^8 + 128x^4}}{128x^2}.
\end{aligned}$$

Therefore $[4x^2, \overline{4x^2, 8x^2}] = \frac{\sqrt{1024x^8 + 128x^4}}{128x^2} = \frac{\sqrt{16x^4 + 2}}{16}$.

Example 3.19 is a case where α is a quadratic surd but it's infinite simple continued fraction expansion was not periodic. This suggests that the converse of Theorem 3.33 does not hold.

As previously suggested, determining if a quadratic surd will have an infinite simple continued fraction expansion that is eventually periodic is difficult. While we are able to characterize some quadratic surds, there are some that we won't be able to characterize. In the following section we will focus on quadratic surds which are eventually periodic.

3.5 Periodic Continued Fractions

We continue with examining quadratic surds that have eventually periodic expansions. We begin with a theorem.

Theorem 3.35. *If α is a quadratic surd with a periodic continued fraction expansion, then the expansion is purely periodic if and only if $v(\alpha) > 1$ and $v(\alpha') < 1$, where α' denotes the conjugate of α. For ease of notation, we will call such quadratic surds 'reduced'.*

For a proof see ([Cas]).

Example 3.36. *Find the infinite simple continued fraction expansion of*

$$\alpha = x^2 + \tfrac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \tfrac{9}{4}x^2 + 3x + 2}.$$

First, notice $v(\alpha) > 1$ and since $\sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2} = x^2 + \frac{1}{2}x + 1 + x^{-1} + \ldots$, $v(\alpha') = v\left(x^2 + \frac{1}{2}x + 1 - \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}\right) = v(x^2 + \frac{1}{2}x + 1 - (x^2 + \frac{1}{2}x + 1 + x^{-1} + \ldots)) = v(-x^{-1} - \ldots) < 1$ so α is reduced. Moreover, $\sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}$ has been chosen due to its eventually periodic continued fraction expansion. By Proposition 3.29,

$$\begin{array}{lll}
P_0 = x^2 + \frac{1}{2}x + 1 & Q_0 = 1 & \alpha = \alpha_0 = x^2 + \frac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2} \\
 & & a_0 = [[\alpha_0]] = 2x^2 + x + 2 \\
P_1 = x^2 + \frac{1}{2}x + 1 & Q_1 = 2x + 1 & \alpha_1 = \frac{x^2 + \frac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}}{2x+1} \\
 & & a_1 = [[\alpha_1]] = x \\
P_2 = x^2 + \frac{1}{2}x - 1 & Q_2 = 2x + 1 & \alpha_2 = \frac{x^2 + \frac{1}{2}x - 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}}{2x+1} \\
 & & a_2 = [[\alpha_2]] = x \\
P_3 = x^2 + \frac{1}{2}x + 1 & Q_3 = 1 & \alpha_3 = x^2 + \frac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2} \\
 & & a_3 = [[\alpha_3]] = 2x^2 + x + 2.
\end{array}$$

Since $\alpha_3 = \alpha_1$, the infinite simple continued fraction expansion of $\alpha = [\overline{2x^2 + x + 2, x, x}]$ is purely periodic.

The results of Example 3.36 are surprising. Looking at α one wouldn't think that α is reduced and that it has an infinite simple continued fraction expansion that is periodic. However, Theorem 3.35 holds because of the specific pattern that the expansion of $\alpha = \sqrt{f}$ has.

Proposition 3.37. *Let $f \in \mathbb{Q}[x]$, where f is not a perfect square, with the infinite simple continued fraction expansion of $\sqrt{f}$ being eventually periodic. Then, the infinite simple* *continued fraction expansion of* $\sqrt{f}$ *takes the form* $[a_0, \overline{a_1, a_2 \dots, a_{l-1}, 2a_0}]$ *where* l *is the period length and* $a_0 = [[\sqrt{f}]]$.

Proof. Consider $\alpha = [[\sqrt{f}]] + \sqrt{f}$. Clearly, $v(\alpha) > 1$ and $v(\overline{\alpha}) < 1$. So by Theorem 3.35, the infinite simple continued fraction expansion of α is purely periodic, say

$$[\overline{2a_0, a_1, a_2 \dots, a_{l-1}}]$$

where l is the period length. Noting that $2a_0 = [[\ [[\sqrt{f}]] + \sqrt{f}\ ]] = 2[[\sqrt{f}]]$, we have that

$$\begin{aligned}
\sqrt{f} &= ([[\sqrt{f}]] + \sqrt{f}) - [[\sqrt{f}]] = \alpha - [[\sqrt{f}]] \\
&= [\overline{2a_0, a_1, a_2 \dots, a_{l-1}}] - [[\sqrt{f}]] \\
&= [2a_0, \overline{a_1, a_2 \dots, a_{l-1}, a_0}] - [[\sqrt{f}]] \\
&= [2a_0, \overline{a_1, a_2 \dots, a_{l-1}, 2a_0}] - [[\sqrt{f}]] \\
&= [\ a_0, \overline{a_1, a_2 \dots, a_{l-1}, 2a_0}\ ] \text{ as desired.}
\end{aligned}$$

□

The following corollary sums up the results found in this section by showing the similarities between the infinite simple continued fraction expansions of $\sqrt{f}$ and $[[\sqrt{f}]] + \sqrt{f}$.

Corollary 3.38. *Let* $f \in \mathbb{Q}[x]$, *where* f *is not a perfect square, with the infinite simple continued fraction expansion of* $\sqrt{f}$ *being periodic. Then, the infinite simple continued fraction expansion of* $\sqrt{f}$ *and* $[[\sqrt{f}]] + \sqrt{f}$ *differ only in the first component (with the first component of the latter being twice the first component of the former), and the period lengths are equal. Furthermore, the values generated by* $\alpha = \alpha_0 = \sqrt{f}$, $P_0 = 0$, $Q_0 = 1$ *differ from those generated by* $\alpha = \alpha_0 = [[\sqrt{f}]] + \sqrt{f}$, $P_0 = [[\sqrt{f}]]$, $Q_0 = 1$ *only at* P_0 *and* a_0.

For proof see ([Cas]). We continue with interesting examples and their infinite continued fraction expansions.

3.6 Examples

Traditionally in Number Theory $\sqrt{d}$ always has an eventually periodic expansion for $d \in \mathbb{N}$ squarefree. This is not the case in the new setting $\sqrt{f} \in \mathbb{Q}[x]$. Although in Chapter 4 we will show a way to classify which $\sqrt{f} \in \mathbb{Q}[x]$ will have an eventually periodic expansion, the method is not very efficient. The following examples will attempt to classify some expansions of $\sqrt{f}$ using Proposition 3.29.

Example 3.39. *Find the infinite simple continued fraction expansion for $\alpha = \sqrt{x^4+2x}$ Then use the expansion to compute the first 4 convergents of α.*

Let $\alpha = \sqrt{x^4+2x}$. Then,

$$
\begin{array}{lll}
P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^4+2x}. = x^2 + x^{-1} + \dots \\
 & & a_0 = [|\alpha_0|] = x^2 \\
P_1 = x^2 & Q_1 = 2x & \alpha_1 = \frac{x^2+\sqrt{x^4+2x}}{2x} \\
 & & a_1 = [|\alpha_1|] = x \\
P_2 = x^2 & Q_2 = 1 & \alpha_2 = x^2 + \sqrt{x^4+2x} \\
 & & a_2 = [[\alpha_2]] = 2x^2 \\
P_3 = x^2 & Q_3 = 2x & \alpha_3 = \frac{x^2+\sqrt{x^4+2x}}{2x} \\
 & & a_3 = [|\alpha_3|] = x \\
 & & \vdots
\end{array}
$$

Since $\alpha_1 = \alpha_3$, the infinite simple continued fraction will be periodic; therefore, $\alpha = \sqrt{x^4+2x} = [x^2, \overline{x, 2x^2}]$ is period 2.

Now we use Proposition 3.8 to compute the convergents of α

$$
\begin{array}{ll}
h_0 = x^2 & k_0 = 1 \\
h_1 = x^3 + 1 & k_1 = x \\
h_2 = 2x^5 + 3x^2 & k_2 = 2x^3 + 1 \\
h_3 = 2x^6 + 4x^3 + 1 & k_3 = 2x^4 + 2x \\
 & \vdots
\end{array}
$$

Then the convergents based on the calculations above are:

$$
\begin{aligned}
&C_0 = x^2 \\
&C_1 = \tfrac{x^3+1}{x} \\
&C_2 = \tfrac{2x^5+3x^2}{2x^3+1} \\
&C_3 = \tfrac{2x^6+4x^3+1}{2x^4+2x} \\
&\quad\vdots
\end{aligned}
$$

Example 3.40. *Find the infinite simple continued fraction expansion for* $\alpha = \sqrt{x^4 + x + \frac{1}{2}}$ *Then use the expansion to compute the first 5 convergents of* α*.*

Let $\alpha = \sqrt{x^4 + x + \frac{1}{2}}$, then

$$
\begin{aligned}
&P_0 = 0 \qquad Q_0 = 1 \qquad \alpha = \alpha_0 = \sqrt{x^4 + x + \tfrac{1}{2}} = x^2 + \tfrac{1}{2}x^{-1} + \dots \\
&\qquad a_0 = [|\alpha_0|] = x^2 \\
&P_1 = x^2 \qquad Q_1 = x + \tfrac{1}{2} \qquad \alpha_1 = \frac{x^2+\sqrt{x^4+x+\frac{1}{2}}}{x+\frac{1}{2}} \\
&\qquad a_1 = [|\alpha_1|] = 2x - 1 \\
&P_2 = x^2 - \tfrac{1}{2} \qquad Q_2 = x + \tfrac{1}{2} \qquad \alpha_2 = \frac{x^2-\frac{1}{2}+\sqrt{x^4+x+\frac{1}{2}}}{x+\frac{1}{2}} \\
&\qquad a_2 = [|\alpha_2|] = 2x - 1 \\
&P_3 = x^2 \qquad Q_3 = 1 \qquad \alpha_3 = x^2 + \sqrt{x^4 + x + \tfrac{1}{2}} \\
&\qquad a_3 = [|\alpha_3|] = 2x^2, \\
&P_4 = x^2 \qquad Q_4 = x + \tfrac{1}{2} \qquad \alpha_4 = \frac{x^2+\sqrt{x^4+x+\frac{1}{2}}}{x+\frac{1}{2}} \\
&\qquad a_4 = [|\alpha_4|] = 2x - 1 \\
&\qquad \vdots
\end{aligned}
$$

Again $\alpha_1 = \alpha_4$, implies that $\alpha = \sqrt{x^4 + x + \frac{1}{2}} = [x^2, \overline{2x-1, 2x-1, 2x^2}]$ is period 3.

Now we use Proposition 3.8 to compute the convergents of α

$$
\begin{aligned}
h_0 &= x^2 \\
h_1 &= 2x^3 - x^2 + 1 \\
h_2 &= 4x^4 - 4x^3 + 2x^2 + 2x - 1 \\
h_3 &= 8x^6 - 8x^5 + 4x^4 + 6x^3 - 3x^2 + 1 \\
h_4 &= 16x^7 - 24x^6 + 16x^5 + 12x^4 - 16x^3 + 5x^2 + 4x - 2 \\
k_0 &= 1 \\
k_1 &= 2x - 1 \\
k_2 &= 4x^2 - 4x + 2 \\
k_3 &= 8x^4 - 8x^3 + 4x^2 + 2x - 1 \\
k_4 &= 16x^5 - 24x^4 + 16x^3 + 4x^2 - 8x + 3 \\
&\vdots
\end{aligned}
$$

Then the convergents based on the calculations above are:

$$C_0 = x^2$$

$$C_1 = \frac{2x^3-x^2+1}{2x-1}$$

$$C_2 = \frac{4x^4-4x^3+2x^2+2x-1}{4x^2-4x+2}$$

$$C_3 = \frac{8x^6-8x^5+4x^4+6x^3-3x^2+1}{8x^4-8x^3+4x^2+2x-1}$$

$$C_4 = \frac{16x^7-24x^6+16x^5+12x^4-16x^3+5x^2+4x-2}{16x^5-24x^4+16x^3+4x^2-8x+3}$$

$$\vdots$$

Example 3.41. *Find the infinite simple continued fraction expansion for* $\alpha = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}$. *Then use the expansion to compute the first 6 convergents of* α.

$$P_0 = 0 \qquad Q_0 = 1$$

$$\alpha = \alpha_0 = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}}$$

$$= x^7 - \tfrac{1}{2}x^2 - x - \tfrac{1}{2}x^{-1} + \tfrac{1}{4}x^{-4} - \tfrac{1}{4}x^{-7} + \dots \qquad a_0 = [[\alpha_0]] = x^7 - \tfrac{1}{2}x^2 - x$$

$$P_1 = x^7 - \tfrac{1}{2}x^2 - x \qquad Q_1 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_1 = \frac{x^7-\frac{1}{2}x^2-x+\sqrt{x^{14}-x^9-2x^8-x^6+\frac{1}{4}x^4+\frac{3}{2}x^3+x^2+\frac{1}{2}x+\frac{1}{2}}}{-x^6+\frac{1}{2}x^3+\frac{1}{2}x+\frac{1}{2}} \qquad a_1 = [[\alpha_1]] = -2x$$

$$P_2 = x^7 - x^4 - \tfrac{1}{2}x^2 \qquad Q_2 = -2x^5 + 2x^2 + 1$$

$$\alpha_2 = \frac{x^7-x^4-\frac{1}{2}x^2+\sqrt{x^{14}-x^9-2x^8-x^6+\frac{1}{4}x^4+\frac{3}{2}x^3+x^2+\frac{1}{2}x+\frac{1}{2}}}{-2x^5+2x^2+1} \qquad a_2 = [[\alpha_2]] = -x^2$$

$$P_3 = x^7 - x^4 - \tfrac{1}{2}x^2 \qquad Q_3 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_3 = \frac{x^7-x^4-\frac{1}{2}x^2+\sqrt{x^{14}-x^9-2x^8-x^6+\frac{1}{4}x^4+\frac{3}{2}x^3+x^2+\frac{1}{2}x+\frac{1}{2}}}{-x^6+\frac{1}{2}x^3+\frac{1}{2}x+\frac{1}{2}} \qquad a_3 = [[\alpha_3]] = -2x$$

$$P_4 = x^7 - \tfrac{1}{2}x^2 - x \qquad Q_4 = 1$$

$$\alpha_4 = x^7 - \tfrac{1}{2}x^2 - x + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}}$$

$$a_4 = [[\alpha_4]] = 2x^7 - x^2 - 2x$$

$$P_5 = x^7 - \tfrac{1}{2}x^2 - x \qquad Q_5 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_5 = \frac{x^7-\frac{1}{2}x^2-x+\sqrt{x^{14}-x^9-2x^8-x^6+\frac{1}{4}x^4+\frac{3}{2}x^3+x^2+\frac{1}{2}x+\frac{1}{2}}}{-x^6+\frac{1}{2}x^3+\frac{1}{2}x+\frac{1}{2}} \qquad a_5 = [[\alpha_5]] = -2x$$

$$\vdots$$

But $\alpha_1 = \alpha_5$, we have that $\alpha = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}} = [x^7 - \frac{1}{2}x^2 - x, \overline{-2x, -x^2, -2x, 2x^7 - x^2 - 2x}]$ is period 4.

Now we use Proposition 3.8 to compute the convergents of α

$h_0 = x^7 - \frac{1}{2}x^2 - x$

$h_1 = -2x^8 + x^3 + 2x^2 + 1$

$h_2 = 2x^{10} - x^5 - 2x^4 - \frac{3}{2}x^2 + x^7 - x$

$h_3 = -4x^{11} - 4x^8 + 2x^6 + 4x^5 + 4x^3 + 4x^2 + 1$

$h_4 = -8x^{18} - 8x^{15} + 8x^{13} + 16x^{12} + 14x^{10} + 16x^9 - 2x^8 - 5x^7 - 8x^6 - 5x^5 - 14x^4 - 8x^3 - \frac{5}{2}x^2 - 3x$

$h_5 = 16x^{19} + 16x^{16} - 16x^{14} - 32x^{13} - 32x^{11} - 32x^{10} + 4x^9 + 6x^8 + 16x^7 + 12x^6 + 32x^5 + 16x^4 + 9x^3 + 10x^2 + 1$

$k_0 = 1$

$k_1 = -2x$

$k_2 = 2x^3 + 1$

$k_3 = -4x^4 - 4x$

$k_4 = -8x^{11} - 8x^8 + 4x^6 + 8x^5 + 6x^3 + 8x^2 + 1$

$k_5 = 16x^{12} + 16x^9 - 8x^7 - 16x^6 - 16x^4 - 16x^3 - 6x$

$\vdots$

Then the convergents based on the calculations above are:

$C_0 = x^7 - \frac{1}{2}x^2 - x$

$C_1 = \frac{-2x^8+x^3+2x^2+1}{-2x}$

$C_2 = \frac{2x^{10}-x^5-2x^4-\frac{3}{2}x^2+x^7-x}{2x^3+1}$

$C_3 = \frac{-4x^{11}-4x^8+2x^6+4x^5+4x^3+4x^2+1}{-4x^4-4x}$

$C_4 = \frac{-8x^{18}-8x^{15}+8x^{13}+16x^{12}+14x^{10}+16x^9-2x^8-5x^7-8x^6-5x^5-14x^4-8x^3-\frac{5}{2}x^2-3x}{-8x^{11}-8x^8+4x^6+8x^5+6x^3+8x^2+1}$

$C_5 = \frac{16x^{19}+16x^{16}-16x^{14}-32x^{13}-32x^{11}-32x^{10}+4x^9+6x^8+16x^7+12x^6+32x^5+16x^4+9x^3+10x^2+1}{16x^{12}+16x^9-8x^7-16x^6-16x^4-16x^3-6x}$

$\vdots$

Castro showed that expansions of period 2, 3 and 4 have certain characteristics. Say $\sqrt{f}$ has an expansion of period 2, then $\sqrt{f} = [a_0, \overline{a_1, 2a_0}]$ where $a_1 | 2a_0$ and $f = a_0^2 + \frac{2a_0}{a_1}$. Now say $\sqrt{f}$ has an expansion of period 3, then $\sqrt{f} = [a_0, \overline{a_1, a_2, 2a_0}]$ where $a_1^2 + 1 | (2a_0 a_1 + 1)$ and $f = a_0^2 + \frac{2a_0 a_1 + 1}{a_1^2 + 1}$. Finally if $\sqrt{f}$ has an expansion of period 4, then $\sqrt{f} = [a_0, \overline{a_1, a_2, a_3, 2a_0}]$ where $a_2 a_1^2 + 2a_1 | (2a_0 a_1 a_2 + 2a_0 + a_2)$ and

$f = a_0^2 + \frac{2a_0a-1a_2+2a_0+a_2}{a_2a_1^2+2a_1}$. Although such generalizations are interesting they make it difficult to build our own polynomial with a desired period. Consider the following general case where a polynomial of *deg* 2 can be period 2.

Example 3.42. *Find the infinite simple continued fraction expansion for $\alpha = \sqrt{x^2+b}$ for $b \in \mathbb{Q} - \{0\}$. Then use the expansion to compute the first 4 convergents of α.*

Let $\alpha = \sqrt{x^2+b}$. Then,

$$\begin{array}{llll}
P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^2+b} = x + \frac{b}{2}x^{-1} + \dots & a_0 = [|\alpha_0|] = x \\
P_1 = x & Q_1 = b & \alpha_1 = \frac{x+\sqrt{x^2+b}}{b} & a_1 = [|\alpha_1|] = \frac{2x}{b} \\
P_2 = x & Q_2 = 1 & \alpha_2 = x + \sqrt{x^2+b} & a_2 = [[\alpha_2]] = 2x \\
P_3 = x & Q_3 = b & \alpha_3 = x + \sqrt{x^2+b} & a_3 = [|\alpha_3|] = \frac{2x}{b} \\
 & & \vdots &
\end{array}$$

Since $\alpha_1 = \alpha_3$, the infinite simple continued fraction will be periodic; therefore, $\alpha = \sqrt{x^2+b} = [x, \overline{\frac{2x}{b}, 2x}]$ is period 2.

Now we use Proposition 3.8 to compute the convergents of α

$$\begin{array}{ll}
h_0 = x & k_0 = 1 \\
h_1 = \frac{2x^2+b}{b} & k_1 = \frac{2x}{b} \\
h_2 = \frac{4x^3+3xb}{b} & k_2 = \frac{4x^2+b}{b} \\
h_3 = \frac{8x^4+8x^2b+b^2}{b^2} & k_3 = \frac{4x^3+(b+2)x}{b} \\
 & \vdots
\end{array}$$

Then the convergents based on the calculations above are:

$$\begin{aligned}
C_0 &= x \\
C_1 &= \frac{\frac{2x^2+b}{b}}{\frac{2x}{b}} \\
C_2 &= \frac{\frac{4x^3+3xb}{b}}{\frac{4x^2+b}{b}} \\
C_3 &= \frac{\frac{8x^4+8x^2b+b^2}{b^2}}{\frac{4x^3+(b+2)x}{b}} \\
&\vdots
\end{aligned}$$

Next consider another polynomial of *deg* 2, let $\alpha = \sqrt{x^2+bx+c}$. It turns out that the infinite continued fraction expansion of α is also period 2. That is, $\alpha =$

$[x, \overline{\frac{2x}{bx+c}, 2x}]$. Notice, however that $a_1 = \frac{2x}{bx+c} \in \mathbb{Q}(x)$ not in the desired $\mathbb{Q}[x]$. We will further investigate this case in the beginning of Chapter 4.

The previous examples had expansions as described in Proposition 3.37 which were of the form $[a_0, \overline{a_1, a_2, \ldots, a_{l-1}, 2a_0}]$. Our next example will have a form described as being *Almost Periodic* defined below.

Definition 3.43. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. Then α is said to be* *Almost Period $n - i$* *for $n \in \mathbb{N}$ if*

$$\alpha = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, r\alpha_i]$$

for $0 \leq i \leq n$ and $r \in \mathbb{Q} - 0$.

Notice if $n - i$ is odd then the expression of α as a continued fraction expansion has period $2(n - i)$. These types of expansions have the form,

$$\alpha = [a_0, \overline{a_1, a_2, \ldots, a_i, \ldots, a_{n-1}, ra_i, \frac{a_{i+1}}{r}, ra_{i+2}, \ldots, \frac{ra_{i+n-i}}{r} = a_n}]$$

for $r \in \mathbb{Q} - 0$.

Example 3.44. *Find the infinite simple continued fraction expansion for $\alpha = \sqrt{x^2 + \frac{1}{2}}$. Then use the expansion to compute the first 4 convergents of α.*

Let $\alpha = \sqrt{x^2 + \frac{1}{2}}$. Then,

$$\begin{array}{llll}
P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^2 + \frac{1}{2}} = x + \frac{1}{4}x^{-1} + \ldots & a_0 = [|\alpha_0|] = x \\
P_1 = x & Q_1 = \frac{1}{2} & \alpha_1 = \frac{x + \sqrt{x^2 + \frac{1}{2}}}{\frac{1}{2}} & a_1 = [|\alpha_1|] = 4x \\
P_2 = x & Q_2 = 1 & \alpha_2 = x + \sqrt{x^2 + \frac{1}{2}} & a_2 = [[\alpha_2]] = 2x \\
P_3 = x & Q_3 = \frac{1}{2} & \alpha_3 = \frac{x + \sqrt{x^2 + \frac{1}{2}}}{\frac{1}{2}} & a_3 = [|\alpha_3|] = 4x \\
 & & \vdots &
\end{array}$$

Then the infinite simple continued fraction is "'almost period 1"' where $\alpha = \sqrt{x^2 + \frac{1}{2}} = [x, \overline{4x, 2x}] = [x, \overline{2(2x), \frac{4x}{2}}]$ thus $r = 2$. As noted above the infinite simple continued fraction expansion of α has period $2(2 - 1) = 2$. Now we use Proposition 3.8 to compute the convergents of α

$$\begin{aligned} h_0 &= x & k_0 &= 1 \\ h_1 &= 4x^2+1 & k_1 &= 4x \\ h_2 &= 8x^3+3x & k_2 &= 8x^2+1 \\ h_3 &= 32x^4+16x^2+1 & k_3 &= 32x^3+8x \\ &\vdots \end{aligned}$$

Then the convergents based on the calculations above are:

$$\begin{aligned} C_0 &= x \\ C_1 &= \tfrac{4x^2+1}{4x} \\ C_2 &= \tfrac{8x^3+3x}{8x^2+1} \\ C_3 &= \tfrac{32x^4+16x^2+1}{32x^3+8x} \\ &\vdots \end{aligned}$$

Example 3.45. *Find the infinite simple continued fraction expansion for $\alpha = \sqrt{x^6 - 4x^2 + 4}$. Then use the expansion to compute the first 6 convergents of α.*

Let $\alpha = \sqrt{x^6 - 4x^2 + 4}$. Then,

$$\begin{array}{lll} P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^6-4x^2+4} = x^3 - 2x^{-1} + \dots \\ & & a_0 = [|\alpha_0|] = x^3 \\ P_1 = x^3 & Q_1 = -4x^2+4 & \alpha_1 = \frac{x^3+\sqrt{x^6-4x^2+4}}{-4x^2+4} \\ & & a_1 = [|\alpha_1|] = -\frac{1}{2}x \\ P_2 = x^3-2x & Q_2 = x^2+1 & \alpha_2 = \frac{x^3-2x+\sqrt{x^6-4x^2+4}}{-x^2+1} \\ & & a_2 = [[\alpha_2]] = -2x \\ P_3 = x^3 & Q_3 = 4 & \alpha_3 = \frac{x^3+\sqrt{x^6-4x^2+4}}{4} \\ & & a_3 = [|\alpha_3|] = \frac{1}{2}x^3 \\ P_4 = x^3 & Q_4 = -x^2+1 & \alpha_4 = \frac{x^3+\sqrt{x^6-4x^2+4}}{-x^2+1} \\ & & a_4 = [|\alpha_4|] = -2x \\ P_5 = x^3-2x & Q_5 = -4x^2+4 & \alpha_5 = \frac{x^3-2x+\sqrt{x^6-4x^2+4}}{-4x^2+4} \\ & & a_5 = [[\alpha_5]] = -\frac{1}{2}x \\ P_6 = x^3 & Q_6 = 1 & \alpha_6 = x^3 + \sqrt{x^6-4x^2+4} \end{array}$$

$$P_7 = x^3 \qquad Q_7 = -4x^2+4 \qquad \begin{array}{l} a_6 = [|\alpha_6|] = 2x^3 \\ \alpha_7 = \frac{x^3+\sqrt{x^6-4x^2+4}}{-4x^2+4} \\ a_7 = [|\alpha_7|] = -\frac{1}{2}x \end{array}$$

$$\vdots$$

Then the infinite simple continued fraction is "'almost period 3"' where $\alpha = \sqrt{x^6-4x^2+4}$
$= [x^3, \overline{-\frac{1}{2}x, -2x, \frac{1}{2}x^3, -2x, -\frac{1}{2}x, 2x^3}] = [x^3, \overline{-\frac{1}{2}x, -2x, \frac{1}{4}(2x^3), \frac{(-\frac{1}{2}x)}{\frac{1}{4}}, \frac{1}{4}(-2x), \frac{1}{4}(\frac{2x^3}{\frac{1}{4}})}]$ thus $r = \frac{1}{4}$. As noted above the infinite simple continued fraction expansion of α has period $2(6-3) = 6$.

Now we use Proposition 3.8 to compute the convergents of α.

$h_0 = x^3$ $\qquad k_0 = 1$

$h_1 = \frac{-1}{2}x^4 + 1$ $\qquad k_1 = \frac{-1}{2}x$

$h_2 = x^5 + x^3 - 2x$ $\qquad k_2 = x^2 + 1$

$h_3 = \frac{1}{2}x^8 + \frac{1}{2}x^6 - \frac{3}{2}x^4 + 1$ $\qquad k_3 = \frac{1}{2}x^5 + \frac{1}{2}x^3 - \frac{1}{2}x$

$h_4 = -x^9 - x^7 + 4x^5 + x^3 - 4x$ $\qquad k_4 = -x^6 - x^4 + 2x^2 + 1$

$h_5 = \frac{1}{2}x^{10} + x^8 - \frac{3}{2}x^6 - 2x^4 + 2x^2 + 1$ $\qquad k_5 = \frac{1}{2}x^7 + x^5 - \frac{1}{2}x^3 - x$

$h_6 = x^{13} + 2x^{11} - 4x^9 - 5x^7 + 8x^5 + 3x^3 - 4x$ $\qquad k_6 = x^{10} + 2x^8 - 2x^6 - 3x^4 + 2x^2 + 1$

$$\vdots$$

Then the convergents based on the calculations above are:

$$C_0 = x^3$$

$$C_1 = \frac{\frac{-1}{2}x^4+1}{\frac{-1}{2}x}$$

$$C_2 = \frac{x^5+x^3-2x}{x^2+1}$$

$$C_3 = \frac{\frac{1}{2}x^8+\frac{1}{2}x^6-\frac{3}{2}x^4+1}{\frac{1}{2}x^5+\frac{1}{2}x^3-\frac{1}{2}x}$$

$$C_4 = \frac{-x^9-x^7+4x^5+x^3-4x}{-x^6-x^4+2x^2+1}$$

$$C_5 = \frac{\frac{1}{2}x^{10}+x^8-\frac{3}{2}x^6-2x^4+2x^2+1}{\frac{1}{2}x^7+x^5-\frac{1}{2}x^3-x}$$

$$C_6 = \frac{x^{13}+2x^{11}-4x^9-5x^7+8x^5+3x^3-4x}{x^{10}+2x^8-2x^6-3x^4+2x^2+1}$$

$$\vdots$$

We will conclude this chapter with and amazing example of almost period 7, periodic 14 where $r = -\frac{1}{12288}$. To save space we will only show certain terms of the expansion.

Example 3.46. *Find the infinite simple continued fraction expansion for* $\alpha = \sqrt{x^4 + 10x^2 - 96x - 71}$.

Let $\alpha = \sqrt{x^4 + 10x^2 - 96x - 71}$. Then,

$P_0 = 0$ $\quad Q_0 = 1$ $\quad \alpha = \alpha_0 = \sqrt{x^4 + 10x^2 - 96x - 71} = x^2 + 5 + \dots$

$a_0 = [|\alpha_0|] = x^2 + 5$

$P_1 = x^2 + 5$ $\quad Q_1 = -96x - 96$ $\quad \alpha_1 = \frac{x^2+5+\sqrt{x^4+10x^2-96x-71}}{-96x-96}$

$a_1 = [|\alpha_1|] = -\frac{1}{48}x + \frac{1}{48}$

$P_2 = x^2 + 7$ $\quad Q_2 = \frac{5}{4} - \frac{1}{4}x$ $\quad \alpha_2 = \frac{x^2+7+\sqrt{x^4+10x^2-96x-71}}{\frac{5}{4}-\frac{1}{4}x}$

$a_2 = [[\alpha_2]] = -8x - 40$

$P_3 = x^2 - 43$ $\quad Q_3 = -384x - 1536$ $\quad \alpha_3 = \frac{x^2-43+\sqrt{x^4+10x^2-96x-71}}{-384x-1536}$

$a_3 = [|\alpha_3|] = -\frac{1}{192}x + \frac{1}{48}$

$\vdots$

$P_7 = x^2 + 5$ $\quad Q_7 = -12288$ $\quad \alpha_7 = \frac{x^2+5+\sqrt{x^4+10x^2-96x-71}}{-12288}$

$a_7 = [|\alpha_7|] = -\frac{1}{6144}x^- + \frac{5}{6144}$

$\vdots$

$P_{11} = x^2 + 11$ $\quad Q_{11} = -384x - 1536$ $\quad \alpha = \alpha_{11} = \frac{x^2+11+\sqrt{x^4+10x^2-96x-71}}{-384x-1536}$

$a_{11} = [|\alpha_{11}|] = -\frac{1}{192}x + \frac{1}{48}$

$P_{12} = x^2 - 43$ $\quad Q_{12} = \frac{5}{4} - \frac{1}{4}x$ $\quad \alpha_{12} = \frac{x^2-43+\sqrt{x^4+10x^2-96x-71}}{\frac{5}{4}-\frac{1}{4x}}$

$a_{12} = [|\alpha_{12}|] = -8x - 40$

$P_{13} = x^2 - 7$ $\quad Q_{13} = -96x - 96$ $\quad \alpha_{13} = \frac{x^2-7+\sqrt{x^4+10x^2-96x-71}}{-96x-96}$

$a_{13} = [[\alpha_{13}]] = -\frac{1}{48}x + \frac{1}{48}$

$P_{14} = x^2+5$ $\quad Q_{14} = 1$ $\quad \alpha_{14} = x^2+5+\sqrt{x^4 + 10x^2 - 96x - 71}$

$a_{14} = [|\alpha_{14}|] = 2x^2 + 10$

$\vdots$

Then the infinite simple continued fraction is ”‘almost period 7”’ where

$\alpha = \sqrt{x^4 + 10x^2 - 96x - 71} = [x^2{+}5, \overline{-\frac{1}{48}x + \frac{1}{48}, -8x - 40, -\frac{1}{192}x + \frac{1}{48}, \ldots, -\frac{1}{6144}x^2 - \frac{5}{6144},}$ $\overline{\ldots, -\frac{1}{192}x + \frac{1}{48}, -8x - 40, -\frac{1}{48}x + \frac{1}{48}, 2x^2 + 10}] = [x^2{+}5, \overline{-\frac{1}{48}x + \frac{1}{48}, -8x - 40, -\frac{1}{192}x + \frac{1}{48},}$ $\overline{\ldots, 2r(x^2 + 5), \ldots, r(64x - 256), \frac{1}{r}(\frac{1}{1536}x + \frac{5}{1536}), r(256x - 256), \frac{1}{r}(-\frac{1}{6144}x^2 - \frac{5}{6144})}]$ where $r = -\frac{1}{12288}$. As noted above the infinite simple continued fraction expansion of α has an amazing period $2(14 - 7) = 14$.

At the end of Chapter 4 we will compute the units of O_K of the examples we have just discussed.

Chapter 4

The Fundamental Unit of O_K

Castro wrote a thesis where his intent was to calculate solutions to Pell's equation. He did this with the use of continued fractions and two results found in Strayer's text. We will use continued fractions and similar results to compute the units of O_K where $K = \mathbb{Q}(x)(\sqrt{f})$.

Up to this point we have established strict restrictions on $f(x) \in \mathbb{Q}[x]$. We have only considered f as a monic squarefree polynomial of even degree. Further, we have only considered the expansion of $\sqrt{f}$ when it is *simple*. We have seen how these expansions can be unpredictable and often frustrating if a pattern doesn't appear to be periodic. By considering an expansion that is not simple we obtain a continued fraction expansion that is periodic. We show this result in the following proposition.

Proposition 4.1. *Let $f \in \mathbb{Q}[x]$ be a monic squarefree polynomial of even degree. Then there exists $a_1 \in \mathbb{Q}(x)$ so that $\sqrt{f} = [a_0, \overline{a_1, 2a_0}]$ where $a_0 = [[\sqrt{f}]]$.*

Proof. $f \in \mathbb{Q}[x]$ is a monic squarefree polynomial of even degree then $\sqrt{f} \in \mathbb{Q}(x)^*$ where $\sqrt{f}$ has an infinite continued fraction expansion. Let $\alpha = \sqrt{f}$, then

$$\alpha = a_0 + \sqrt{f} - a_0, \quad \text{where } a_0 = [[\sqrt{f}]]$$

$$\alpha = a_0 + \frac{1}{\frac{1}{\sqrt{f}-a_0}}$$

Multiplying by the conjugate of the denominator we see,

$$\alpha = a_0 + \frac{1}{\frac{a_0+\sqrt{f}}{f-a_0^2}}$$

That is $\alpha_1 = \frac{a_0+\sqrt{f}}{f-a_0^2}$. Now set $a_1 = \frac{a_0+[[\sqrt{f}]]}{f-a_0^2}$, then $a_1 \in \mathbb{Q}(x)$. Since $a_0 = [[\sqrt{f}]]$ then $a_1 = \frac{2a_0}{f-a_0^2}$

Continuing with the expansion and maintaining equality we have,

$$\begin{aligned}\alpha &= a_0 + \frac{1}{\frac{2a_0}{f-a_0^2}+\frac{a_0+\sqrt{f}}{f-a_0^2}-\frac{2a_0}{f-a_0^2}} \\ &= a_0 + \frac{1}{a_1+\frac{-a_0+\sqrt{f}}{f-a_0^2}}\end{aligned}$$

Continuing in the same fashion

$$\begin{aligned}\alpha &= a_0 + \frac{1}{a_1+\frac{1}{\frac{1}{\frac{-a_0+\sqrt{f}}{f-a_0^2}}}} \\ &= a_0 + \frac{1}{a_1+\frac{1}{\frac{f-a_0^2}{\sqrt{f}-a_0}}}\end{aligned}$$

Rationalizing $\frac{f-a_0^2}{\sqrt{f}-a_0}$ we obtain

$$\alpha = a_0 + \frac{1}{a_1+\frac{1}{\sqrt{f}+a_0}}$$

Then $a_2 = [[\sqrt{f}+a_0]]$ where $a_0 = [[\sqrt{f}]]$, so $a_2 = 2a_0$. Now

$$\begin{aligned}\alpha &= a_0 + \frac{1}{a_1+\frac{1}{2a_0+\sqrt{f}-a_0}} \\ &= a_0 + \frac{1}{a_1+\frac{1}{2a_0+\alpha_1}} \\ &= [a_0, a_1, 2a_0, \alpha_1] \\ &= [a_0, \overline{a_1, 2a_0}].\end{aligned}$$

That is given $a_1 = \frac{a_0+[[\sqrt{f}]]}{f-a_0^2} \in \mathbb{Q}(x)$ the infinite continued fraction expansion of $\alpha = [a_0, \overline{a_1, 2a_0}]$ is period 2. □

Next we show that the convergents of this expansion also belong to $\mathbb{Q}(x)$.

Proposition 4.2. *Let f be as it is in Proposition 4.1 and $a_0 = [[\sqrt{f}]]$, $a_1 = \frac{a_0+[[\sqrt{f}]]}{f-a_0^2}$ and $a_2 = [[\sqrt{f}+a_0]]$. Also let $\frac{h_i}{k_i}$ be the i^{th} convergent of $\sqrt{f}$. Then $h_i, k_i \in \mathbb{Q}(x)$.*

Proof. Let all notation be as it is in Proposition 3.9 then,

$$h_0 = a_0 \qquad\qquad k_0 = 1$$
$$h_1 = a_0a_1 + 1 = a_0(\tfrac{a_0+[[\sqrt{f}]]}{f-a_0^2}) + 1 = \tfrac{f+a_0^2}{f-a_0^2} \qquad k_1 = a_1 = \tfrac{a_0+[[\sqrt{f}]]}{f-a_0^2} = \tfrac{2a_0}{f-a_0^2}$$
$$h_2 = a_2h_1 + h_0 = 2a_0(\tfrac{2a_0^2}{f-a_0^2}) + a_0 = \tfrac{3a_0^3+a_0f}{f-a_0^2} \qquad k_2 = a_2k_1 + k_0 = 2a_0(\tfrac{2a_0}{f-a_0^2}) + 1 = \tfrac{3a_0^2+f}{f-a_0^2}$$
$$\vdots$$

where $h_1, h_2, \ldots, k_1, k_2, \ldots \in \mathbb{Q}(x)$. So if f is as it is in Proposition 4.1 then the convergents of f are elements of $\mathbb{Q}(x)$. $\square$

Let $d \in \mathbb{N}$ where d is not a perfect square. In the classical case Strayer showed that $\sqrt{d}$ has an infinite continued fraction expansion that is periodic where $\frac{h_i}{k_i}$ denotes the *ith* convergent of $\sqrt{d}$. Now if $Q_i \neq -1$ for all i then the solutions to a generalized Pell's equation $X^2 - dY^2 = 1$ are given by $X = h_{nl-1}$ and $Y = k_{nl-1}$ where l is the even period length and $n \in \mathbb{N}$. The following theorem makes use of this result and follows directly from Proposition 4.2.

Theorem 4.3. *Let $f \in \mathbb{Q}[x]$ then the Pell Equation $X^2 - fY^2 = 1$ has solutions in $\mathbb{Q}(x)$ if f is a monic squarefree polynomial of even degree.*

Proof. Let $f \in \mathbb{Q}[x]$ and be as it is in Proposition 4.1. Then the infinite continued fraction expansion of $\sqrt{f} = [a_0, \overline{a_1, 2a_0}]$ where l is the period of the expansion. Then $l = 2$ is even so a solution to the Pell equation $X^2 - fY^2 = 1$ is given by $X = h_{2n-1}$ and $Y = k_{2n-1}$ when $n = 1$ and $Q_i \neq -1$. We use Proposition 3.29 to show that $Q_i \neq -1$ for all i.

$$P_0 = 0 \qquad Q_0 = 1 \qquad \alpha = \alpha_0 = \sqrt{f} \qquad a_0 = [[\alpha_0]] = \sqrt{f}$$
$$P_1 = a_0 \qquad Q_1 = f - a_0^2 \qquad \alpha_1 = \tfrac{a_0+\sqrt{f}}{f-a_0^2} \qquad a_1 = [[\alpha_1]] = \tfrac{2a_0}{f-a_0^2}$$
$$P_2 = a_0 \qquad Q_2 = 1 \qquad \alpha_2 = a_0 + \sqrt{f} \qquad a_2 = [[\alpha_2]] = 2a_0$$
$$\vdots$$

Then h_1 and k_1 are solutions to $X^2 - \sqrt{f}Y^2 = 1$. In the proof of Proposition 4.2 we saw that h_1 and $k_1 \in \mathbb{Q}(x)$. That is, if $f \in \mathbb{Q}[x]$ and is as it is in Proposition 4.1, then the solution to the Pell equation $X^2 - fY^2 = 1$ belongs to $\mathbb{Q}(x)$. $\square$

We have now shown that if we choose $a_1 \in \mathbb{Q}(x)$ we have a continued fraction expansion that is not simple and we have a solution for Pell's equation $X^2 - fY^2 = 1$ where $f \in \mathbb{Q}[x]$ and $X, Y \in \mathbb{Q}(x)$. Consider the following examples.

Example 4.4. *Find the infinite continued fraction expansion of $\alpha = \sqrt{x^4 + 2x + 2}$ by choosing $a_0 = [[\sqrt{f}]]$, $a_1 = \frac{a_0 + [[\sqrt{f}]]}{f - a_0^2}$ and $a_2 = [[\sqrt{f} + a_0]]$. Then use Proposition 3.8 to compute the convergents of α. Finally find an X and Y that satisfy $X^2 - fY^2 = 1$.*

First we will find the infinite simple continued fraction expansion of α. Let $\alpha = \sqrt{x^4 + 2x + 2} = x^2 + x^{-1} + \ldots$.

$$\begin{array}{llll} P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^4 + 2x + 2} = x^2 + x^{-1} + \ldots & a_0 = [[\alpha_0]] = x^2 \\ P_1 = x^2 & Q_1 = 2x+2 & \alpha_1 = \frac{x^2 + \sqrt{x^4+2x+2}}{2x+2} & a_1 = [[\alpha_1]] = \frac{x^2}{x+1} \\ P_2 = x^2 & Q_2 = 1 & \alpha_2 = x^2 + \sqrt{x^4 + 2x + 2} & a_2 = [[\alpha_2]] = 2x^2 \\ & & \vdots & \end{array}$$

The infinite continued fraction expansion of $\alpha = \sqrt{x^4 + x^2 + 2} = [x^2, \overline{\frac{x^2}{x+1}, 2x^2}]$ has period 2. Now we compute the convergents of α.

Let $\alpha = [x^2, \overline{\frac{x^2}{x+1}, 2x^2}]$ by Proposition 3.8 we have $a_0 = x^2, a_1 = \frac{x^2}{x+1}, a_2 = 2x^2$. Then,

$$\begin{array}{ll} h_0 = x^2 & k_0 = 1 \\ h_1 = \frac{x^4+x+1}{x+1} & k_1 = \frac{x^2}{x+1} \\ h_2 = \frac{2x^6+3x^3+3x^2}{x+1} & k_2 = 2x^2 \\ h_3 = \frac{2x^8+7x^5+7x^4+x^2+2x+1}{(x+1)^2} & k_3 = \frac{2x^4+x^2}{x+1} \\ \vdots & \end{array}$$

Now since the period $l = 2$ is even, then a solution is given by $X = h_{2n-1}$ and $Y = k_{2n-1}$ where n is a positive integer.

$$X = h_1 = \frac{x^4+x+1}{x+1} \quad \text{and} \quad Y = k_1 = \frac{x^2}{x+1}.$$

That is,

$$\left(\frac{x^4+x+1}{x+1}\right)^2 - (x^4 + 2x + 2)\left(\frac{x^2}{x+1}\right)^2 = 1.$$

Example 4.5. *Find the infinite continued fraction expansion of* $\alpha = \sqrt{x^6+2x+1}$ *by choosing* $a_0 = [[\sqrt{f}]]$, $a_1 = \frac{a_0+[[\sqrt{f}]]}{f-a_0^2}$ *and* $a_2 = [[\sqrt{f}+a_0]]$. *Then use Proposition 3.8 to compute the convergents of* α. *Finally find an* X *and* Y *that satisfy* $X^2 - fY^2 = 1$.

First we will find the infinite simple continued fraction expansion of α. Let $\alpha = \sqrt{x^6+2x+1} = x^3 + \frac{1}{2}x^{-2} + \dots$

$$
\begin{array}{lll}
P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^6+2x+1} = x^3 + \frac{1}{2}x^{-2} + \dots \\
 & & a_0 = [[\alpha_0]] = x^3 \\
P_1 = x^3 & Q_1 = 2x+1 & \alpha_1 = \frac{x^3+\sqrt{x^6+2x+1}}{2x+1} \\
 & & a_1 = [[\alpha_1]] = \frac{2x^3}{2x+1} \\
P_2 = x^3 & Q_2 = 1 & \alpha_2 = x^3 + \sqrt{x^6+2x+1} \\
 & & a_2 = [[\alpha_2]] = 2x^3 \\
 & & \vdots
\end{array}
$$

The infinite continued fraction expansion of $\alpha = \sqrt{x^6+2x+1} = [x^3, \overline{\frac{2x^3}{2x+1}, 2x^3}]$ has period 2. Now compute the convergents of α.

Let $\alpha = [x^3, \overline{\frac{2x^3}{2x+1}, 2x^3}]$ by Proposition 3.8 we have $a_0 = x^3, a_1 = \frac{2x^3}{2x+1}, a_2 =, 2x^3$. Then,

$$
\begin{array}{ll}
h_0 = x^3 & k_0 = 1 \\
h_1 = \frac{2x^6+2x+1}{2x+1} & k_1 = \frac{2x^3}{2x+1} \\
h_2 = \frac{4x^9+6x^4+3x^3}{2x+1} & k_2 = \frac{4x^6+2x+1}{2x+1} \\
h_3 = \frac{8x^{12}+16x^7+8x^6+4x^2+4x+1}{(2x+1)^2} & k_3 = \frac{8x^9+8x^4+4x^3}{(2x+1)^2} \\
 & \vdots
\end{array}
$$

Now since the period $l = 2$ is even, then a solution is given by $X = h_{2n-1}$ and $Y = k_{2n-1}$ where n is a positive integer.

$$X = h_1 = \frac{2x^6+2x+1}{2x+1} \quad \text{and} \quad Y = k_1 = \frac{2x^3}{2x+1}.$$

That is,

$$(\frac{2x^6+2x+1}{2x+1})^2 - (x^6+2x+1)(\frac{2x^3}{2x+1})^2 = 1.$$

In the classical case strayer showed that if x_1 and y_1 are solutions of Pell's equation $X^2 - dY^2 = 1$ then another solution is given by x_n, y_n where $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ for $n \in \mathbb{N}$. We may apply this result to the two previous examples to determine another solution to Pell's equation.

Example 4.6. *Find another solution for $X, Y \in \mathbb{Q}(x)$ in $X^2 - (x^4 + 2x + 2)Y^2 = 1$.*

By Example 4.4 a solution for X and $Y \in \mathbb{Q}(x)$ in $X^2 - (x^4 + 2x + 2)Y^2 = 1$ is $X = h_1 = \frac{x^4+x+1}{x+1}$ and $Y = k_1 = \frac{x^2}{x+1}$. We obtain the next solution by squaring this solution.

$$\begin{aligned}
&(\tfrac{x^4+x+1}{x+1} + (\sqrt{x^4+2x+2})\tfrac{x^2}{x+1})^2 \\
&= \tfrac{2x^8+7x^5+7x^4+x^2+2x+1}{(x+1)^2} + \tfrac{2x^4+x^2}{x+1}\sqrt{x^4+2x+2} \\
&= h_3 + \sqrt{f}k_3.
\end{aligned}$$

Another solution appears at h_3 and k_3.

Example 4.7. *Find another solution for $X, Y \in \mathbb{Q}(x)$ in $X^2 - (x^6 + 2x + 1)Y^2 = 1$.*

By Example 4.5 a solution for X and $Y \in \mathbb{Q}(x)$ in $X^2 - (x^6 + 2x + 1)Y^2 = 1$ is $X = h_1 = \frac{2x^6+2x+1}{2x+1}$ and $Y = k_1 = \frac{2x^3}{2x+1}$. We obtain the next solution by squaring this solution.

$$\begin{aligned}
&(\tfrac{2x^6+2x+1}{2x+1} + (\sqrt{x^6+2x+1})\tfrac{2x^3}{2x+1})^2 \\
&= \tfrac{8x^{12}+16x^7+8x^6+4x^2+4x+1}{(2x+1)^2} + \tfrac{8x^9+8x^4+4x^3}{(2x+1)^2}\sqrt{x^6+2x+1} \\
&= h_3 + \sqrt{f}k_3
\end{aligned}$$

Another solution appears at h_3 and k_3.

In the previous examples we chose a monic squarefree polynomial of even degree and found that we could solve $X^2 - fY^2 = 1$ for $X, Y \in \mathbb{Q}(x)$. In fact, we can find infinitely many pairs that work. Next we will show that we need not have such a particular $f \in \mathbb{Q}[x]$.

Proposition 4.8. *Let $f \in \mathbb{Q}[x]$ be a polynomial of any degree, then $X^2 - fY^2 = 1$ has a solution for $X, Y \in \mathbb{Q}(x)$.*

Proof. By Theorem 4.3 h_1 and k_1 are solutions to $X^2 - fY^2 = 1$ then we want to show that this is true for any $f \in \mathbb{Q}[x]$. Let $f = x^2 + f - x^2$ and let $f_1 = f - x^2$ then $f = x^2 + f_1$. Taking the square root, $\sqrt{f} = \sqrt{x^2 + f_1}$,
we mimic Proposition 4.2 to obtain values for h_i, k_i:

$$\begin{array}{ll} h_0 = a_0 = x & k_0 = 1 \\ h_1 = \frac{f+a_0^2}{f-a_0^2} = \frac{2x^2+f_1}{f_1} & k_1 = a_1 = \frac{2x}{f_1} \\ h_2 = \frac{3a_0^3+a_0 f}{f-a_0^2} = \frac{4x^3+f_1 x}{f_1} & k_2 = \frac{3a_0^2+f}{f-a_0^2} = \frac{4x^2+f_1}{f_1} \\ & \vdots \end{array}$$

Now we show h_1 and k_1 are solutions to $X^2 - fY^2 = 1$.

$$\begin{array}{c} (h_1)^2 - f(k_1)^2 \\ (\frac{2x^2+f_1}{f_1})^2 - f(\frac{2x}{f_1})^2 \\ \frac{4x^4+4x^2 f_1+f_1^2}{f_1^2} - \frac{4x^4+4x^2 f_1}{f_1^2} \\ \frac{f_1^2}{f_1^2} = 1 \end{array}$$

That is $X^2 - fY^2 = 1$ has solutions for $X, Y \in \mathbb{Q}(x)$ and $f \in \mathbb{Q}[x]$. □

Example 4.9. *Let $f(x) = x$. Use Proposition 4.8 to show that $X^2 - fY^2 = 1$ has a solution $X, Y \in \mathbb{Q}(x)$.*

Let $f = x$ then $f = x^2 + x - x^2$. Now let $f_1 = x - x^2$ then $f = x^2 + f_1$ and $\sqrt{f} = \sqrt{x^2 + f_1}$. Next we mimic Proposition 4.2 to obtain values for h_i, k_i:

$$\begin{array}{ll} h_0 = x & k_0 = 1 \\ h_1 = \frac{2x^2+f_1}{f_1} & k_1 = \frac{2x}{f_1} \\ h_2 = \frac{4x^3+f_1 x}{f_1} & k_2 = \frac{4x^2+f_1}{f_1} \\ & \vdots \end{array}$$

Where h_1 and k_1 are solutions to Pell's equation (by Proposition 4.3). Then,

$$\begin{aligned}(h_1)^2 - f(k_1)^2 &= 1\\ (\frac{2x^2+f_1}{f_1})^2 - (x^2+f_1)(\frac{2x}{f_1})^2 &= 1.\end{aligned}$$

Substitute for f_1 and simplify

$$\begin{aligned}(\frac{x-1}{1-x})^2 - x(\frac{2}{1-x})^2 &= 1\\ \frac{x^2+2x+1}{x^2-2x+1} - \frac{4x}{x^2-2x+1} &= 1\\ \frac{x^2-2x+1}{x^2-2x+1} &= 1.\end{aligned}$$

Then $h_1, k_1 \in \mathbb{Q}(x)$ are solutions for Pell's equation.

Example 4.10. *Let $f(x) = x^3$. Use Proposition 4.8 to show that $X^2 - fY^2 = 1$ has a solution $X, Y \in \mathbb{Q}(x)$.*

Let $f = x^3$ then $f = x^2 + x^3 - x^2$. Now let $f_1 = x^3 - x^2$ then $f = x^2 + f_1$ and $\sqrt{f} = \sqrt{x^2 + f_1}$. Next we mimic Proposition 4.2 to obtain values for h_i, k_i.

$$\begin{array}{ll} h_0 = x & k_0 = 1\\ h_1 = \frac{2x^2+f_1}{f_1} & k_1 = \frac{2x}{f_1}\\ h_2 = \frac{4x^3+f_1x}{f_1} & k_2 = \frac{4x^2+f_1}{f_1}\\ & \vdots \end{array}$$

Where h_1 and k_1 are solutions to Pell's equation (by Proposition 4.3). Then,

$$\begin{aligned}(h_1)^2 - f(k_1)^2 &= 1\\ (\frac{2x^2+f_1}{f_1})^2 - (x^2+f_1)(\frac{2x}{f_1})^2 &= 1.\end{aligned}$$

Substitute for f_1 and simplify

$$\begin{aligned}(\frac{x^3+2x^2-x^2}{x^3-x^2})^2-(x^3+x^2-x^2)(\frac{2x}{x^3-x^2})^2 &= 1\\ (\frac{x^3+x^2}{x^3-x^2})^2-(x^3)(\frac{4x^2}{x^6-2x^5+x^4}) &= 1\\ \frac{x^6+2x^5+x^4}{x^6-2x^5+x^4}-\frac{4x^5}{x^6-2x^5+x^4} &= 1\\ \frac{x^6-2x^5+x^4}{x^6-2x^5+x^4} &= 1.\end{aligned}$$

Then $h_1, k_1 \in \mathbb{Q}(x)$ are solutions for Pell's equation.

We will now turn our attention to the main focus of this Thesis. So far we have used continued fractions to compute the convergents $\sqrt{f}$ where $f \in \mathbb{Q}[x]$. Now we will use those convergents to compute fundamental units in O_K. As mentioned in the Introduction a fundamental unit of O_K, where $K = \mathbb{Q}(x)(\sqrt{f}) = \{X + Y\sqrt{f}\,|X, Y \in \mathbb{Q}[x]\}$, is a unit by which all others may be generated. That is if $U(O_K)$ is a unit group then the fundamental unit is a generator. We will show $U(O_K)$ is *isomorophic* to $\mathbb{Q}^*$ X $\mathbb{Z}$. The following theorem shows the relationship between the set $U(O_K)$ and the solutions to the generalized Pell's equation $X^2 - fY^2 = r$ where $\sqrt{f}$ has an infinite simple continued fraction expansion that is periodic and $r \in \mathbb{Q} - \{0\}$. A variation of this theorem is found in ([AR80]).

Theorem 4.11. *The following are equivalent*

(a) $U(O_K)$ is not trivial.

(b) Generalized Pell's equation has a non-trivial solution.

(c) There exists $i \in \mathbb{N}$ with $Q_i \in \mathbb{Q} - \{0\}$.

(d) The continued fraction expansion of $\sqrt{f}$ is periodic.

(e) $\sqrt{f} = [a_0, \overline{a_1, a_2, \ldots, a_{n-1}, 2a_0}]$.

We will give a detailed proof of Theorem 4.11 at the end of this chapter. Before we can continue with the main result we need the following lemma which is also found in Castro's Thesis ([Cas]).

Lemma 4.12. *Let $f \in \mathbb{Q}[x]$ where f is a monic squarefree polynomial of even degree and let $\frac{h_i}{k_i}$ be the i^{th} convergent of the infinite simple continued fraction expansion of $\sqrt{f}$.*

Then, $h_i^2 - fk_i^2 = (-1)^{i-1}Q_{i+1}$ *for* $i \geq 0$ *where* $Q_1, Q_2, \ldots$ *are defined as in Proposition 3.29.*

Proof. Let all notation be as in Proposition 3.29 with $\alpha_0 = \sqrt{f}$. Since $\sqrt{f} = \alpha_0 = [a_0, a_1, \ldots, a_i, \alpha_{i+1}]$ for $i > 0$, Proposition 3.8 yields $\sqrt{f} = \frac{\alpha_{i+1}h_i + h_{i-1}}{\alpha_{i+1}k_i + k_{i-1}}$. But

$$\alpha_{i+1} = \frac{P_{i+1}+\sqrt{f}}{Q_{i+1}}$$
$$\Rightarrow \sqrt{f} = \frac{(P_{i+1}+\sqrt{f})h_i + Q_{i+1}h_{i-1}}{(P_{i+1}+\sqrt{f})k_i + Q_{i+1}k_{i-1}}$$

and so,

$$\begin{aligned}
&fk_i + (P_{i+1}k_i + Q_{i+1}k_{i-1})\sqrt{f} = (P_{i+1}h_i + Q_{i+1}h_{i-1}) + h_i\sqrt{f} \text{ for } i > 0\\
&\Rightarrow fk_i = P_{i+1}h_i + Q_{i+1}h_{i-1} \text{ and } P_{i+1}k_i + Q_{i+1}h_{i-1} = h_i\\
&\Rightarrow fk_i^2 = P_{i+1}h_ik_i + Q_{i+1}h_{i-1}k_i \text{ and } P_{i+1}h_ik_i + Q_{i+1}h_ik_{i-1} = h_i^2\\
&\Rightarrow h_i^2 - fk_i^2 = (h_ik_{i-1} - h_{i-1}k_i)Q_{i+1} \text{ for } i > 0
\end{aligned}$$

but by Proposition 3.11, $(h_ik_{i-1} - h_{i-1}k_i) = (-1)^{i-1} \Rightarrow h_i^2 - fk_i^2 = (-1)^{i-1}Q_{i+1}$ for $i > 0$. If $i = 0 \Rightarrow h_0^2 - fk_0^2 = a_0^2 - f = (-1)(f - a_0^2) = (-1)(\frac{f-P_1^2}{1}) = (-1)Q_1$. □

We now continue with the main result which gives a method of calculating the fundamental unit of O_K. For the following theorem let $\frac{h_i}{k_i} = \frac{a_i}{b_i}$.

Theorem 4.13. *Assume* $f \in \mathbb{Q}[x]$ *is a monic squarefree polynomial of even degree. Assume* $\sqrt{f}$ *has an infinite simple continued fraction expansion that is periodic.*
Select $i \in \mathbb{N}$ *minimal so that* $Q_i \in \mathbb{Q} - \{0\}$. *Let* $\eta = a_{i-1} + b_{i-1}\sqrt{f}$ *and* $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$ *is a generated by* η. *Then we may call* η *a fundamental unit of* O_K.

Proof. Let $\eta = a_{i-1} + b_{i-1}\sqrt{f}$ and β be units in O_K, then $N(\eta)$ and $N(\beta) \in \mathbb{Q} - \{0\}$. We will show that $\beta = s\eta^n$ for some $s \in \mathbb{Q} - \{0\}$ and $n \in \mathbb{Z}$. Now there exists some $d \in \mathbb{N}$ so that

$$v(\eta^d) \leq v(\beta) < v(\eta^{d+1}).$$

Multiply through by η^{-d}

$$v(\eta^d\eta^{-d}) \leq v(\beta\eta^{-d}) < v(\eta^{d+1}\eta^{-d})$$

$$v(1) \leq v(\beta\eta^{-d}) < v(\eta).$$

Now β and η^{-d} are units then $\beta\eta^{-d}$ is a unit. So let

$$\beta\eta^{-d} = a + b\sqrt{f}$$

for some $a, b \in \mathbb{Q}[x]$. Then

$$1 \leq v(a + b\sqrt{f}) < v(\eta).$$

Now $(a+b\sqrt{f})(a-b\sqrt{f}) = a^2 - b^2 f \in \mathbb{Q} - \{0\}$. Assume $v(a+b\sqrt{f}) = 1$ forcing $v(a - b\sqrt{f}) = 1$ since $v(a^2 - b^2 f) = 1$. Now if a, b have coefficients of the same sign in $a + b\sqrt{f}$ then $b = 0$ and $a \in \mathbb{Q} - \{0\}$. Let $a = s$ for $s \in \mathbb{Q} - \{0\}$, then $\beta\eta^{-n} = s$ and $\beta = s\eta^n$ as desired.

Next assume $v(a - b\sqrt{f}) = 1$ which forces $v(a+b\sqrt{f}) = 1$ since $v(a^2 - b^2 f) = 1$. If the coefficients of a, b have the same sign then $b = 0$ and $a \in \mathbb{Q} - \{0\}$ which yields the same desired result as above.

Finally assume $1 < v(\beta\eta^{-n}) < v(\eta)$ then Castro showed that the ratio $\frac{a}{b} = s(\frac{a_j}{b_j})$ for some $s \in \mathbb{Q}$ and some $j \in \mathbb{Z}$. Then $(a_j + b_j\sqrt{f})(a_j - b_j\sqrt{f}) \in \mathbb{Q} - \{0\}$ and

$$1 < v(\beta\eta^{-n}) = v(a_j + b_j\sqrt{f}) < v(\eta) = v(a_{i-1} + b_{i-1}\sqrt{f})$$

$$1 < v(a_j + b_j\sqrt{f}) < v(a_{i-1} + b_{i-1}\sqrt{f})$$

implies that $j < i - 1$. Further $j < i$ contradiction since we chose i to be minimal. Thus $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. □

The following lemma shows that if η is a fundamental unit in O_K the η has the smallest $v - value$ greater than 1.

Lemma 4.14. *Let f be as it is in Theorem 4.13 and η be a fundamental unit in O_K. If $\eta = h_{i-1} + k_{i-1}\sqrt{f}$ where h and k have leading coefficients of the the same sign, then η has the smallest $v - value$ in $U(O_K)$ greater than 1.*

Proof. Assume to the contrary that η does not have the smallest $v - value$. Then $v(\eta) > 1$ and there exists $\beta \in U(O_K)$ such that $v(1) < v(\beta) < v(\eta)$. Then $\beta = s\eta^n$ since β is a unit. Now

$$\begin{aligned} v(\beta) &= v(s\eta^n) = v(s)v(\eta^n) \\ &= 1 \cdot v(\eta)^n = v(\eta)^n \end{aligned}$$

since the value of the product is the product of the value. Then,

$$1 < v(\eta)^n < v(\eta).$$

If n is negative then $v(\eta)^n < 1$ while if n is positive then $v(\eta)^n > v(\eta)$ both of which lead to contradictions. Thus n must be zero and $\beta = s\eta^n = s$. Then $1 < v(\beta) = v(s) = 1$, which is a contradiction. Thus η has the smallest $v - value$ as desired. $\square$

Next we will make use of the recurrence relations found in Proposition 3.29 to determine the fundamental unit η of O_K where f is a monic squarefree polynomial of even degree. We will take the examples at the end of Chapter 3 and find the set $U(O_K)$.

Example 4.15. *Compute the fundamental unit of O_K where $f = x^4 + 2x$.*

Let $\alpha = \sqrt{f}$. Then by Example 3.39, $\alpha = [x^2, \overline{x, 2x^2}]$ has an infinite simple continued fraction expansion that is periodic. Also $Q_2 = 1 \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_1 + k_1\sqrt{x^4 + 2x}$ then $N(\eta) = 1$. By Example 3.39, h_1 and k_1 are as follows

$$\eta = (x^3 + 1) + (x)\sqrt{x^4 + 2x}.$$
$$N(\eta) = \left((x^3 + 1) + (x)\sqrt{x^4 + 2x}\right)\left((x^3 + 1) - (x)\sqrt{x^4 + 2x}\right) = 1.$$

We may calculate another unit in O_K by squaring η.

$$\eta^2 = \left((x^3 + 1) + (x)\sqrt{x^4 + 2x}\right)^2$$
$$= (2x^6 + 4x^3 + 1) + (2x^4 + 2x)\sqrt{x^4 + 2x} = h_3 + k_3\sqrt{x^4 + 2x}.$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

Example 4.16. *Compute the fundamental unit of O_K where $f = x^4 + x + \frac{1}{2}$.*

Let $\alpha = \sqrt{f}$. Then by Example 3.40, $\alpha = [x^2, \overline{2x - 1, 2x - 1, 2x^2}]$ has an infinite simple continued fraction expansion that is periodic. Also $Q_3 = 1 \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_2 + k_2\sqrt{x^4 + x + \frac{1}{2}}$ then $N(\eta) = 1$. By Example 3.40, h_2 and k_2 are as follows

$$\eta = (4x^4 - 4x^3 + 2x^2 + 2x - 1) + (4x^2 - 4x + 2)\sqrt{x^4 + x + \tfrac{1}{2}}$$
$$N(\eta) = \left(h_2 + k_2\sqrt{x^4 + x + \tfrac{1}{2}}\right)\left(h_2 - k_2\sqrt{x^4 + x + \tfrac{1}{2}}\right) = 1.$$

We may calculate another unit in O_K by squaring η.

$$\begin{aligned}\eta^2 &= \left((4x^4 - 4x^3 + 2x^2 + 2x - 1) + (4x^2 - 4x + 2)\sqrt{x^4 + x + \tfrac{1}{2}}\right)^2 \\ &= (32x^8 - 64x^7 + 64x^6 - 40x^4 + 32x^3 - 8x + 3) \\ &+ (32\text{x}^6 - 64x^5 + 64x^4 - 16x^3 - 16x^2 + 16x - 4)\sqrt{x^4 + x + \tfrac{1}{2}} \\ &= h_5 + k_5\sqrt{x^4 + x + \tfrac{1}{2}}\end{aligned}$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

Example 4.17. *Compute the fundamental unit of O_K where*

$$f = x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}.$$

Let $\alpha = \sqrt{f}$. Then by Example 3.42, $\alpha = [x^7 - \frac{1}{2}x^2 - x, \overline{-2x, -x^2, -2x, 2x^7 - x^2 - 2x}]$ has an infinite simple continued fraction expansion that is periodic. Also $Q_4 = 1 \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_3 + k_3\sqrt{f}$ then $N(\eta) = 1$. By Example 3.42, h_3 and k_3 are as follows

$$\eta = (-4x^{11} - 4x^8 + 2x^6 + 4x^5 + 4x^3 + 4x^2 + 1) + (-4x^4 - 4x)\sqrt{f}$$
$$N(\eta) = \left(h_3 + k_3\sqrt{f}\right)\left(h_3 - k_3\sqrt{f}\right) = 1.$$

We may calculate another unit in O_K by squaring η.

$$\begin{aligned}&\eta^2 = \left((-4x^{11} - 4x^8 + 2x^6 + 4x^5 + 4x^3 + 4x^2 + 1) + (-4x^4 - 4x)\sqrt{f}\right)^2 \\ &= (32x^{22} + 64x^{19} - 32x^{17} - 32x^{16} + \ldots + 16x^2 + 1) + (32x^{15} + 64x^{12} - 16x^{10} + \ldots - 32x^3 - 8x)\sqrt{f} \\ &= h_7 + k_7\sqrt{f}.\end{aligned}$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

In the previous three examples η^2 was precisely $h_3 + k_3\sqrt{f}, h_5 + k_5\sqrt{f}$ and $h_7 + k_7\sqrt{f}$ respectively. The subscripts 3, 5 and 7 depended upon the period of $\sqrt{f}$. If this pattern continued then η^3 is $h_5 + k_5\sqrt{f}, h_8 + k_8\sqrt{f}$ and $h_{11} + k_{11}\sqrt{f}$ respectively. η^4 is $h_7 + k7\sqrt{f}, h_{11} + k_{11}\sqrt{f}$ and $h_{15} + k_{15}\sqrt{f}$ respectively, and so on.

At the end of Chapter 3 we defined "*Almost period $n - i$*" as α having the expansion $\alpha = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, r\alpha_i]$. Notice again that if n is odd then the expansion has period $2(n - i)$ and is of the form

$$\alpha = [a_0, \overline{a_1, a_2, \ldots, a_i, \ldots, a_{n-1}, ra_i, \tfrac{a_{i+1}}{r}, ra_{i+2}, \ldots, \tfrac{ra_{i+n-i}}{r} = a_n}].$$

Where $r \in \mathbb{Q} - \{0\}$. We next compute the fundamental unit of O_K where f has an expansion that is "*almost periodic*". In the "*almost period*" case we don't see such a nice pattern like the one above.

Example 4.18. *Compute the fundamental unit of O_K where $f = x^2 + \frac{1}{2}$.*

Let $\alpha = \sqrt{f}$. By Example 3.44 $\alpha = [x, \overline{4x, 2x}]$ is almost period 1 where $r = 2$. Also $Q_1 = \frac{1}{2} \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_0 + k_0\sqrt{x^2 + \frac{1}{2}}$ then $N(\eta) = \frac{1}{2} = \frac{1}{r}$. h_0 and k_0 are given by Example 3.44.

$$\eta = x + \sqrt{x^2 + \tfrac{1}{2}}$$
$$N(\eta) = \left(x + \sqrt{x^2 + \tfrac{1}{2}}\right)\left(x - \sqrt{x^2 + \tfrac{1}{2}}\right) = \tfrac{1}{2} = \tfrac{1}{r}$$

We may calculate another unit in O_k by $\frac{1}{r}(\eta)^2$.

$$\begin{aligned}
\eta &= x + \sqrt{x^2 - \frac{1}{2}} \\
\eta^2 &= 2\left(x + \sqrt{x^2 + \frac{1}{2}}\right)^2 \\
&= (4x^2 + 1) + (4x^2)\sqrt{x^2 + \frac{1}{2}} \\
&= h_1 + k_1\sqrt{x^2 + \frac{1}{2}}
\end{aligned}$$

In fact,

$$\begin{aligned}
\eta^3 &= 2\left(x + \sqrt{x^2 + \frac{1}{2}}\right)^3 \\
&= (8x^3 + 3x) + (8x^2 + 1)\sqrt{x^2 - \frac{1}{2}} \\
&= h_2 + k_2\sqrt{x^2 - \frac{1}{2}}.
\end{aligned}$$

If we continue in this fashion a pattern will appear.

$$\eta^4 = 4\left(h_3 + k_3\sqrt{x^2 - \frac{1}{2}}\right)$$
$$\eta^5 = 4\left(h_4 + k_4\sqrt{x^2 - \frac{1}{2}}\right)$$
$$\eta^6 = 16\left(h_5 + k_5\sqrt{x^2 - \frac{1}{2}}\right)$$
$$\eta^7 = 16\left(h_6 + k_6\sqrt{x^2 - \frac{1}{2}}\right)$$

That is if η has an even power then,

$$\eta^{2i} = r^i\left(h_{2i-1} + k_{2i-1}\sqrt{x^2 - \tfrac{1}{2}}\right) \text{ for } i \geq 1.$$

If η has an odd power then,

$$\eta^{2i+1} = r^i\left(h_{2i} + k_{2i}\sqrt{x^2 - \tfrac{1}{2}}\right) \text{ for } i \geq 1.$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

Example 4.19. *Compute the fundamental unit of O_K where $f = x^6 - 4x^2 + 4$.*

Let $\alpha = \sqrt{f}$. By Example 3.45 $\alpha = [x^3, \overline{-\frac{1}{2}x, -2x, \frac{1}{2}x^3, -2x, -\frac{1}{2}x, 2x^3}]$ is almost period 3 where $r = \frac{1}{4}$. Also $Q_3 = 4 \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_2 + k_2\sqrt{x^6 - 4x^2 + 4}$ then $N(\eta) = 4 = \frac{1}{r}$. h_2 and k_2 are given by Example 3.45.

$$\eta = (x^5 + x^3 - 2x) + (x^2 + 1)\sqrt{x^6 - 4x^2 + 4}$$
$$N(\eta) = \left(h_2 + k_2\sqrt{x^6 - 4x^2 + 4}\right)\left(h_2 - k_2\sqrt{x^6 - 4x^2 + 4}\right) = 4 = \tfrac{1}{r}.$$

We may calculate another unit in O_K by $\frac{1}{r}(\eta)^2$.

$$\begin{aligned}\eta^2 &= r\left((x^5 + x^3 - 2x) + (x^2 + 1)\sqrt{x^6 - 4x^2 + 4}\right)^2 \\ &= (\frac{1}{2}x^{10} + x^8 - \frac{3}{2}x^6 - 2x^4 + 2x^2 + 1) + (\frac{1}{2}x^7 + x^5 - \frac{1}{2}x^3 - x)\sqrt{x^6 - 4x^2 + 4} \\ &= h_5 + k_5\sqrt{x^6 - 4x^2 + 4}\end{aligned}$$

In fact,

$$\eta^3 = r\left((x^5 + x^3 - 2x) + (x^2+1)\sqrt{x^6 - 4x^2 + 4}\right)^3$$
$$= h_8 + k_8\sqrt{x^6 - 4x^2 + 4}.$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

Example 4.20. *Compute the fundamental unit of O_K where $f = x^4 + 10x^2 - 96x - 71$.*

Let $\alpha = \sqrt{f}$. By Example 3.46 $\alpha = [x^2{+}5, \overline{-\frac{1}{48}x + \frac{1}{48}, -8x - 40, -\frac{1}{92}x + \frac{1}{48}, \ldots, 2x^2 + 10}]$ is almost period 7 where $r = -\frac{1}{12288}$. Also $Q_7 = -12288 \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_6 + k_6\sqrt{x^4 + 10x^2 - 96x - 71}$ then $N(\eta) = -12288 = \frac{1}{r}$. h_6 and k_6 are given by Example 3.46.

$$\eta = (\tfrac{-1}{108}x^8 + \tfrac{32}{27}x^5 - \tfrac{1}{2}x^4 - \tfrac{5}{2}x^6 + \tfrac{352}{27}x^3 - \tfrac{781}{27}x^2 - \tfrac{10001}{108}) +$$
$$(-\tfrac{1}{27648}x^5 - \tfrac{5}{13824}x^3 + \tfrac{23}{13824}x^2 - \tfrac{1}{27648}x^4 + \tfrac{55}{27648}x + \tfrac{127}{27648})\sqrt{x^4 + 10x^2 - 96x - 71}$$
$$\mathrm{N}(\eta) = \left(h_6 + k_6\sqrt{x^4 + 10x^2 - 96x - 71}\right)\left(h_6 - k_6\sqrt{x^4 + 10x^2 - 96x - 71}\right)$$
$$N(\eta) = -12288 = \tfrac{1}{r}$$

We may calculate another unit in O_K by $\frac{1}{r}(\eta)^2$.

$$\eta^2 = r\left(h_6 + k_6\sqrt{x^4 + 10x^2 - 96x - 71}\right)^2$$
$$= h_{13} + k_{13}\sqrt{x^4 + 10x^2 - 96x - 71}.$$

Then $U(O_K) = \{s\eta^n : s \in \mathbb{Q} - \{0\}, n \in \mathbb{Z}\}$. Thus η is a fundamental unit of O_K.

Since ([AR80]) contains only a sketch of the proof, we conclude this chapter with a detailed proof of Theorem 4.11.

Proof. $(a) \Leftrightarrow (b)$

$(\Rightarrow)$

Let $a + b\sqrt{f} \in U(O_K)$ where $b \neq 0$ since O_K is non-trivial. Then $N(a + b\sqrt{f}) \in \mathbb{Q} - \{0\}$. Now $N(a + b\sqrt{f}) = (a + b\sqrt{f})(a - b\sqrt{f}) \in \mathbb{Q} - \{0\}$ implies $a^2 - b^2 f \in \mathbb{Q} - \{0\}$. Thus a and b are solutions to the Generalized Pell's equation $X^2 - Y^2 f = r$ for $r \in \mathbb{Q} - \{0\}$.

($\Leftarrow$)
Let $X^2 - Y^2 f = r$ for $r \in \mathbb{Q} - \{0\}$ be a Generalized Pell's equation. Let a, b be non-trivial solutions for X and Y. Then $r = a^2 - b^2 f = (a + b\sqrt{f})(a - b\sqrt{f}) = N(a + b\sqrt{f})$. Thus $N(a + b\sqrt{f}) \in \mathbb{Q} - \{0\}$ and $a + b\sqrt{f}$ is a unit in O_K. $U(O_K)$ is not trivial as desired. $\square$

Proof. $(b) \Leftrightarrow (c)$
($\Rightarrow$)
Let $X^2 - Y^2 f = r$ be a Generalized Pell's equation and let h, k be solutions for X, Y. Then $h^2 - k^2 f = r$ for some $r \in \mathbb{Q} - \{0\}$. Now $\frac{h}{k} = \frac{h_{i-1}}{k_{i-1}}$ by for some $i \in \mathbb{Z}$ by Lemma 3.1 ([Cas]). Then $h = sh_{i-1}$ and $k = sk_{i-1}$ for some $s \in \mathbb{Q} - \{0\}$. Now by Lemma 4.12 $h_i^2 - k_i^2 f = (-1)^{i-1} Q_{i+1}$ for $i \geq 0$. substituting $i - 1$ we have

$$\frac{r}{s} = h_{i-1}^2 - k_{i-1}^2 f = (-1)^{i-2} Q_i$$
$$h_{i-1}^2 - k_{i-1}^2 f = (-1)^i Q_i.$$

Thus $(-1)^i Q_i \in \mathbb{Q} - \{0\}$ and $Q_i \in \mathbb{Q} - \{0\}$.
($\Leftarrow$)
By Lemma 4.12 $h_{i-1}^2 - k_{i-1}^2 f = (-1)^{i-1} Q_i \in \mathbb{Q} - \{0\}$. Then $Q_i = r = h_i^2 - k_i^2 f = (h_i + k_i\sqrt{f})(h_i - k_i\sqrt{f}) = N(h_i + k_i\sqrt{f})$. Thus $N(h_i + k_i\sqrt{f}) = r$. Finally h_i, k_i are non-trivial solutions for X and Y in the Generalized Pell's equation $X^2 - Y^2 f = r$. $\square$

For the proof of $(c) \Rightarrow (d)$ see Castro's Lemma 3.3 ([Cas]). The proof of $(d) \Rightarrow (e)$ follows directly from Proposition 3.37.

Proof. $(e) \Leftrightarrow (c)$
($\Rightarrow$)
If $\sqrt{f}$ is periodic then the expansion of $\sqrt{f}$ has the form $\sqrt{f} = [a_0, \overline{a_1, \ldots, 2a_0}]$. Then there exists $n \in \mathbb{N}$ so that $a_n = 2a_0$ and $deg(Q_n) = 0$. Thus $Q_n \in \mathbb{Q} - \{0\}$.
($\Leftarrow$)
Say $Q_n \in \mathbb{Q} - \{0\}$ for some $n \in \mathbb{N}$. Proposition 3.29 says that $\alpha_n = \frac{P_n + \sqrt{f}}{Q_n}$, let $Q_n = r$

for some $r \in \mathbb{Q} - \{0\}$. Thus $\alpha_n = r^{-1}(P_n + \sqrt{f})$. Then $\sqrt{f} = [a_0, a_1, \ldots, a_{n-1}, \alpha_n]$ and

$$\begin{aligned}\sqrt{f} &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}}} \\ &= \frac{h_{n-1}\alpha_n + h_{n-2}}{k_{n-1}\alpha_n + k_{n-2}}\end{aligned}$$

As seen in the proof of Proposition 2.14 in ([Cas]). Then,

$$\sqrt{f}k_{n-1}r^{-1}P_n + fk_{n-1}r^{-1} + \sqrt{f}k_{n-2} = h_{n-1}r^{-1}P_n + h_{n-1}r^{-1}\sqrt{f} + h_{n-2}.$$

Equalizing both sides and multiplying by $\frac{r}{k_{n-1}}$ we get;

$$\begin{aligned}k_{n-1}r^{-1}P_n + k_{n-2} &= h_{n-1}r^{-1} \\ P_n + r(\frac{k_{n-2}}{k_{n-1}}) &= \frac{h_{n-1}}{k_{n-1}}.\end{aligned}$$

Now $\frac{h_{n-1}}{k_{n-1}} = [a_0, a_1, \ldots, a_{n-1}]$ and $\frac{k_{n-1}}{k_{n-2}} = [a_{n-1}, a_{n-2}, \ldots, a_0]$ are results found in Strayer's ([Str01]) text for the classical case and these identities are true in our case as well. Then,

$$\begin{aligned}P_n + r\left(\frac{1}{\frac{k_{n-1}}{k_{n-2}}}\right) &= [a_0, a_1, \ldots, a_{n-1}] \\ P_n + r\left(\frac{1}{[a_{n-1}, a_{n-2}, \ldots, a_0]}\right) &= [a_0, a_1, \ldots, a_{n-1}] \\ P_n + r\left(\frac{1}{[a_{n-1}, a_{n-2}, \ldots, a_0]}\right) &= a_0 + [0, a_1, \ldots, a_{n-1}]\end{aligned}$$

Forcing $P_n = a_0$ since finite expansions are unique. Then,

$$(r)\cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots + \cfrac{1}{a_0}}}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1}}}}}$$

multiplying r through the contunied fraction on the left and we have the following:

$$\cfrac{1}{r^{-1}a_{n-1}+\cfrac{1}{ra_{n-2}+\cfrac{1}{\ddots+\cfrac{1}{ra_0}}}}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_{n-1}}}}}$$

Again expansions of infinite simple continued fractions are unique then continue by equalizing terms.

$$\begin{aligned} r^{-1}a_{n-1} &= a_1 \\ ra_{n-2} &= a_2 \\ r^{-1}a_{n-3} &= a_3 \\ &\vdots \end{aligned}$$

A pattern appears $r^{(-1)^i}a_{n-i} = a_i$. Next assume n is even. Then say $n = 2j$ for some $j \in \mathbb{Z}$ then,

$$\begin{aligned} r^{(-1)^j}a_{2j-j} &= a_j \\ r^{(-1)^j}a_j &= a_j \end{aligned}$$

Then $r^{(-1)j} = 1$ and $r = 1$. Thus $\sqrt{f}$ is periodic as desired. Recall the case where n is odd from the definition of "*almost period*" discussed in Chapter 3. If n is odd then the expansion has period length $2(n-i)$. $\square$

This concludes the proof of Theorem 4.11.

Chapter 5

Conclusion

In this thesis we examined the quadratic extension of $\mathbb{Q}(x)$, where $f \in \mathbb{Q}[x]$ a monic squarefree polynomial of even degree. The elements of K which are integral over $\mathbb{Q}[x]$ formed the ring O_K. Our aim was to compute the unit group of O_K. First we considered any general quadratic extension and in Chapter 2 showed that any case where f is not a monic squarefree polynomial, O_K would have trivial units. In the previous chapter we showed that the unit group of O_K depended upon the continued fraction expansion of $\sqrt{f}$, the theory of which was developed in Chapter 3. We have successfully calculated the set of units in O_K and in each example have found a generator for the unit group.

An obstacle that I faced was the uncertainty that $\sqrt{f}$ may not have an expansion that is periodic. Maple was helpful in many cases since I was able to compute the terms of an expansion in minutes. If an expansion seemed to be getting larger as in Example 3.19, I would assume that the expansion would not be periodic. However, since I usually only calculated the first 25 terms I could never be sure.

I have already mentioned that Maple was a useful tool in terms of accuracy and speed. Through Maple I was able to focus on the interesting results which came from an expansion and not on the calculations. For example, say η is a fundamental unit in O_K. Given any nonzero rational, r, could one find a monic squarefree polynomial so that $N(\eta) = \pm r$? Through many calculations we found that the answer is yes. Consider $f = x^2 + r$. We compute one final expansion of $\sqrt{f}$ using Proposition 3.29.
Let $\alpha = \sqrt{x^2 + r}$. Then,

$$\begin{array}{llll} P_0 = 0 & Q_0 = 1 & \alpha = \alpha_0 = \sqrt{x^2+r}. = x + \frac{r}{2}x^{-1} + \dots & a_0 = [|\alpha_0|] = x \\ P_1 = x & Q_1 = r & \alpha_1 = \frac{x+\sqrt{x^2+r}}{r}. & a_1 = [|\alpha_1|] = \frac{2x}{r} \\ P_2 = x & Q_2 = 1 & \alpha_2 = x + \sqrt{x^2+r} & a_2 = [|\alpha_2|] = 2x \\ P_3 = x & Q_3 = r & \alpha_3 = x + \sqrt{x^2+r} & a_3 = [|\alpha_3|] = \frac{2x}{r} \end{array}$$

$$\vdots$$

Therefore, $\alpha = \sqrt{x^2+r} = [x, \overline{\frac{2x}{r}, 2x}]$ is period 2. Now we use Proposition 3.8 to compute h_i, k_i of α.

$$\begin{array}{ll} h_0 = x & k_0 = 1 \\ h_1 = \frac{2x^2+r}{r} & k_1 = \frac{2x}{r} \\ h_2 = \frac{4x^3+3xr}{r} & k_2 = \frac{4x^2+r}{r} \\ h_3 = \frac{8x^4+8x^2r+r^2}{r^2} & k_3 = \frac{4x^3+(r+2)x}{r} \end{array}$$

$$\vdots$$

Now $\alpha = \sqrt{x^2+r}$ has an infinite simple continued fraction expansion that is periodic. Also $Q_1 = r \in \mathbb{Q} - \{0\}$ then by Theorem 4.13 if $\eta = h_0 + k_0\sqrt{f}$ then $N(\eta) \in \mathbb{Q} - \{0\}$. From the calculations above,

$$\begin{aligned} \eta &= h_0 + k_0\sqrt{f} \\ \eta &= x + \sqrt{x^2+r} \\ N(\eta) &= -r. \end{aligned}$$

Thus given $r \in \mathbb{Q} - \{0\}$ we can find $f \in \mathbb{Q}[x]$ so that $N(\eta) = \pm r$. In fact, let $f = x^{2n} + r$ for $n \in \mathbb{Z}$ and any $r \in \mathbb{Q} - \{0\}$ then $N(\eta) = \pm r$. In the classical case such a question may not be posed since if γ is a unit in O_F, where $F = \mathbb{Q}(\sqrt{d}) = \left\{a + b\sqrt{d} | a, b \in \mathbb{Z}\right\}$, then $N(\gamma) = \pm 1$ for all $\gamma \in O_F$.

Although many interesting results came from using Maple not all of my questions were answered. I was able to find examples of expansions that are "*almost period*" 1, 3 and 7. Still needed to be found are expansions that are "*almost period*" 5 or any odd length greater than 7. Not having a complete set of examples of "*almost period*" lengths does not let us get a clear characterization of this quadratic extension. In the classical case we are able to find an expansion of any length simply by working backwards from an expansion to a quadratic surd, similar to what is demonstrated in Example 3.34.

As I previously mentioned, determining if an expansion will be periodic or eventually periodic is very difficult. Adams and Razar took on this task in an Article titled *Multiples of points on elliptic curves and continued fractions.* They showed that

> "*The continued fraction of f is periodic if and only if the image of P in J has finite order.*"

This result from Algebraic Geometry relies on computing the order of rational points on the hyper elliptic curve $y^2 = f$. Although a proof of this statement may be found in ([AR80]), could one find a more effective way of computing the image of P? Knowing a more effective process to determine if an expansion is periodic would allow us to further examine the characteristics of the unit group $U(O_K)$.

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