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Hyperbolic transformations on cubics in $H^2$

Frank S. Marfai

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HYPERBOLIC TRANSFORMATIONS ON CUBICS IN $H^2$

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Frank S. Marfai
December 2003
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ABSTRACT

Let us consider an equilateral triangle in $H^2$ (using the Poincaré disk model), where one vertex is at the origin, and whose base coincides with the line $y = 0$. If we find all the centroids (the intersection of the medians) of each equilateral triangle that satisfies the aforementioned condition, it can be shown that the path traced out by these centroids is part of a cubic. What is even more remarkable is that the formula for such a cubic is

$$\frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2.$$

The purpose of this thesis is to study the effects of hyperbolic transformations on the above cubic; exploration of the mathematical properties of the original and transformed cubic will lead to several surprising results.
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CHAPTER ONE
INTRODUCTION

The purpose of this thesis is to study the effects of hyperbolic transformations on the cubic that is determined by locus of centroids of the equilateral triangles in $H^2$ whose base coincides with the line $y = 0$, and whose common vertex is at the origin. The derivation of the formulas within this work will be based on the Poincaré disk model of $H^2$, where $H^2$ is understood to mean the hyperbolic plane. The exploration of the properties of both the untransformed cubic (the original locus of centroids) and the transformed cubic (the original cubic taken under a linear fractional transformation) will lead to several surprising results.

In Preliminaries, the section following this one, the definitions and the necessary background information are given to the reader in order to allow smooth development of the core topic of this thesis. In Centroid of a Triangle, a general formula is derived which finds the centroid of a triangle in the hyperbolic plane. This is then converted to a parametric representation. In Locus of Centroids, the parametric representation is converted to a Cartesian representation, and the invariant properties of the curve are explored. The curve will be referred to as $\mathcal{E}$.

In Stereographic Projection of $\mathcal{E}$, it will be shown that $\mathcal{E}$ can be viewed as the stereographic projection of the intersection of two quadric surfaces in $R^3$ onto the unit disk. In Elliptic Curves, the properties of the cubic $\mathcal{E}$ will be studied. $\mathcal{E}$ will be written in the form $f(x, y) = 0$, and algebraic curve theory will be used to derive results for the curve $f$, along with the curve $f$ under a rotation of $\phi$, namely $f_\phi$.

In Hyperbolic Transformations, a study of the cubic $f$ under a hyperbolic transformation is done. This will be done via spin action, so a suitable matrix will be developed which will transform the cubic. With a suitable representation of $f$,
the equation for image of $f$ under a rotation of $\phi$ is determined. Additionally, the image of $f$ under a hyperbolic transformation which takes 0 to $m$, called $f_m$, is studied. The conditions under which the transformed curve remains a cubic are also examined. Also, hyperbolic rotations about the point $m$ are researched. In *Admissible Lines*, the results from the examination of the hyperbolic rotations about the point $m$ are explored. The unique line passing through $m$ (which stems from the results) has some very special characteristics and is explained in further detail. In *Classification of Cubics*, the conjectures of birational equivalence and projective equivalence of the two curves $f$ and $f_m$ are discussed, along with their implications. Finding the flex of the transformed cubic will be explored. The last section, *Conclusion*, summarizes the results of this work and poses additional questions which can lead to future research in this topic.
CHAPTER TWO
PRELIMINARIES

In this section, the definitions and background knowledge necessary to understand the core component of the thesis is explained. It is assumed that the reader has the requisite background knowledge in Linear Algebra, Abstract Algebra, Complex Analysis, and Hyperbolic Geometry.

Let $C \subset \mathbb{C}^2$. $C$ is called a plane (or affine) algebraic curve [4] if there exists a polynomial $f \in \mathbb{C}^2$ such that $\deg f \geq 1$ and

$$C = V(f) = \{(x_1, x_2) \in \mathbb{C}^2 : f(x_1, x_2) = 0\}.$$ 

Note that $V(f)$ represents an algebraic variety of $f$, which can be viewed as the zeros of $f$. To note, an ordinary algebraic curve does not have the restriction of having exactly two coordinates, namely $(x_1, x_2)$, so a plane algebraic curve is a special subset of all algebraic curves. A cubic equation in two variables is a plane algebraic curve, since it can be denoted as $f(x, y) = 0$.

In a homogenous equation, all the terms are of the same degree; to homogenize an equation that is not homogenous, a variable is introduced and suitable powers are multiplied to each term so that the resulting equation has all its terms to the same degree. The homogenized form of an algebraic curve $f$ is sometimes referred to as the projective closure of $f$, and will be denoted $F$. For the algebraic curve $f(x, y)$, $z$ is typically introduced and the homogenized equation is denoted $F(x, y, z)$. To recapture the original affine curve, set $z = 1$. For example, the homogenization of $f(x, y) = x^2y + x + y^2 + 4$ is

$$F(x, y, z) = x^2y + xz^2 + y^2z + 4z^2.$$ 

Let $P$ be a point on the curve $F(x, y, z) = 0$. Informally speaking, the curve is singular at $P$ if there is more than one tangent to the curve at $P$. A point $P$ is
nonsingular if there is only one tangent at that point. Algorithms to determine the
equations of these tangents do exist. Singularities are called ordinary if the
tangents are distinct. The order of a point refers to the number of tangents that
pass through it. Two powerful theorems are obtained though the study of
algebraic curves. I will state them without proof. The first one defines the order of
a singularity [8].

**Theorem 1**

*P is an r-fold point of \( F(\vec{x}) = 0 \) if and only if all \((r-1)\)th derivatives of \( F \), but not all
the \( r \)-th derivatives, vanish at \( P \).*

Here \( \vec{x} \) is a vector. The implications mean that the way we count tangents is
slightly different in some situations. For example, the curve \( f(x) = x^2 \) has a
singularity of order 2 (a 2-fold point) at the origin, while \( f(x) = x^5 \) has a 5-fold
singularity.

A point is defined to be singular if it is 2-fold or higher, otherwise it is
nonsingular.

**Corollary 1**

*A point \( P \) on a curve \( F \) is singular if and only if \( \frac{\partial F}{\partial x_i} = 0 \) for all \( i \).*

To prove the above, let \( P \) be a 2-fold point. From Theorem 1, this implies the
(2-1)th derivatives will vanish, namely, all the first derivatives. A 2-fold point is
singular, so with that the corollary is proven.

Another theorem gives an algorithm to find tangents under certain special
conditions [8].
Theorem 2

If \( f(x,y) \) has no terms of degree less than \( r \) and has some terms of degree \( r \), then the origin is an \( r \)-fold point of \( f=0 \) and the curve defined by equating to 0 the terms of \( f \) of degree \( r \) has as its components the tangents to \( f \) at the origin.

In the context of the theorem, the word *components* is used to mean its (irreducible) factors arrived at upon a complete factorization of the resulting polynomial. For example, suppose we are given the curve \( x^3 + x^2 + y^2 = 0 \). From Theorem 2, it follows that the equation of the tangent at 0 is \( x^2 + y^2 = 0 \), and so the equations of the tangents at the origin are \( x + iy = 0 \) and \( x - iy = 0 \).

In general, if \( a \) is a nonsingular point of \( F(a)=0 \), then the equation of the tangent to \( F \) at \( a \) is

\[
\sum F_{x_i}(a) \cdot x_i = 0 \tag{1}
\]

Suppose we are interested in finding the all tangents to a curve \( C \) that pass through a prescribed point \( q \). The equation of the polar curve of \( C \) with respect to the pole \( q \) in \( \mathbb{C}^3 \) is defined to be

\[
P_qC = q_1 \frac{\partial F}{\partial x} + q_2 \frac{\partial F}{\partial y} + q_3 \frac{\partial F}{\partial z} = 0 \tag{2}
\]

where \( q = (q_1, q_2, q_3) \) is the pole in component form while \( C \) is the curve \( F(x,y,z) = 0 \). Taking the intersection of the polar curve \( P_qC = 0 \) and the original curve \( F = 0 \) leads to finding the points on \( C \) from which the tangents can be drawn to \( q \).

In a similar fashion, it will also be of interest to find all the generalized inflection points of \( C \), which are referred to as the *flexes* of the curve. Let \( C \) again be the curve \( F(x,y,z) = 0 \). Define the *Hessian matrix* \([4]\) to be the symmetric
matrix

\[ H_F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{xz} & F_{yz} & F_{zz} \end{pmatrix}. \]  

(3)

Then the Hessian curve, \( H_C \), is defined to be the polynomial that results from taking the determinant of the Hessian matrix. Therefore

\[ H_C = \text{det}(H_F). \]  

(4)

The location of the flexes of the curve \( C \) is determined by taking the intersection of the original curve \( F = 0 \) and \( H_C = 0 \).

An algebraic curve is irreducible if its polynomial representation cannot factored into algebraic expressions of a lower degree. For example,

\[ f(x, y) = x^3 + xy + 2y + 1 \] is irreducible but

\[ f(x, y) = x^3y + y = y(x + 1)(x^2 + x + 1) \] is not irreducible.

To find the equation of a line in \( \mathbb{C}P^2 \) through the points \([d,e,f]\) and \([g,h,i]\), the following strategy from [2] will be used:

1. Write down the equation

\[
\begin{vmatrix} x & y & z \\ d & e & f \\ g & h & i \end{vmatrix} = 0;
\]

2. Expand the determinant in terms of the entries in its first row to obtain the required equation in the form \( ax + by + cz = 0 \).

(5)

Some topological terms will also be used. In topology it is known that any compact orientable surface (a two-dimensional topological manifold) is homeomorphic to a sphere with \( g \) handles, where \( g \in \mathbb{N} \) is referred to as the genus.
of the curve [4]. Recall that homeomorphic refers to a mapping between two surfaces such that the mapping and its inverse are both one to one and continuous [3]. A handle is simply a topological "hole". Therefore a sphere with 0 handles is a sphere, a sphere with 1 handle is a torus, a sphere with 2 handles is a double torus (one representation of a double torus looks like a figure 8 in space), and so on. So a genus of a curve informally can be thought of as the number of "holes" a suitable topological representation of the curve would have. Two formulas will be used in calculating the genus of an algebraic curve. If the curve is nonsingular, the formula simply is

$$g = \frac{(n - 1)(n - 2)}{2}$$

(6)

where $n$ is the degree of the curve. If the curve has ordinary singularities, then the formula is modified to

$$g = \frac{(n - 1)(n - 2)}{2} - \sum_{i=1}^{j} \frac{(r_i - 1)(r_i)}{2}$$

(7)

where $r_i$ is the order of the singularity of the point $x_i \in C$.

For example, let us say we have an algebraic curve of degree 4 with two singular points. The first point has a singularity of order 2, while the second one also has a singularity of order 2. Applying the preceding formula leads to $n = 4$, $r_1 = 2$, and $r_2 = 2$. Then

$$g = \frac{(4 - 1)(4 - 2)}{2} - \frac{(2 - 1)(2)}{2} - \frac{(2 - 1)(2)}{2} = 3 - 1 - 1 = 1$$

Therefore $g = 1$, and so the above curve is topologically equivalent to a torus.

An elliptic curve is defined to be an algebraic curve of genus 1 [8]. Another concept that comes from both algebraic curve theory and topology (in particular the study of Riemann surfaces) is the concept of birational equivalence. Briefly
stated, every algebraic curve can be modeled by a suitable Riemann surface, with
the representation dependent on the genus of the curve. Two algebraic curves are
birationally equivalent if they have the same function fields. A function field can
be viewed as all the pairs \((U, V)\) for which the polynomial \(\Phi(U, V) = 0\) is an
algebraic equation in \(V\) with rational coefficients in \(U\) and degree \(N\) [3]. One way
to view this is that birational equivalence corresponds to a conformal equivalence
relationship between two Riemann surfaces [3]; birational equivalence implies
belonging to the same equivalence class. From an algebraic point of view, it is
analogous to the concept in Linear Algebra when we have the same space
represented by different basis vectors.

From an algebraic curve point of view, two curves \(f\) and \(f'\) are birationally
equivalent if a birational correspondence can be established between them. To
accomplish this, let \(\xi\) and \(\eta\) be basis vectors of \(f(x, y)\), and let \(\xi'\) and \(\eta'\) be basis
vectors of \(f'(x', y')\) as suggested in [8]. Define \(\Phi\) and \(\Psi\) to be rational functions,
where \(\xi = \Phi(\xi, \eta)\) and \(\eta' = \Psi(\xi, \eta)\).

Then consider the transformation defined by \(x' = \Phi(x, y), y' = \Psi(x, y)\). If
the transformation takes \(f(x, y)\) to \(f'(x', y')\), then half the conditions necessary for
birational equivalence are satisfied. We need to show the inverse exists also.
Assuming \(f'(x', y')\) exists, we need to find a transformation that takes it back to
\(f(x, y)\). Defining \(\Phi'\) and \(\Psi'\) to be rational functions, where \(\xi = \Phi'(\xi', \eta')\) and
\(\eta = \Psi'(\xi', \eta')\), we consider the transformation defined by \(x = \Phi'(x', y'), y = \Psi'(x', y')\). If the transformation takes \(f'(x', y')\) back to \(f(x, y)\), then a
birational correspondence is established between the two functions. With the
mapping and its inverse created, the algebraic curves \(f(x, y) = 0\) and \(f(x', y') = 0\)
are birationally equivalent.
Lemma 1

Linear Fractional Transformations are birational transformations.

To prove the case for genus 0, it is a known fact that LFTs are conformal maps which are circle preserving. Let $T$ be an arbitrary LFT, and let $z$ be a point on a unit sphere. $T$ takes all the points $z$ on a Riemann sphere and permutes them to $T(z)$, therefore preserving the Riemann surface itself. Hence the function field (which in this case is the set of all points on the Riemann sphere) is unaltered. All LFTs are invertible; thus $T^{-1}(T(z))$ exists and is equal to $z$. For curves of higher genus, we will use the fact that in essence an LFT is a conformal bijection on the Riemann sphere which induces a transformation on the quotient of the sphere that represents the curve as a Riemann surface [3]. Therefore all LFTs are birational transformations.

Lemma 2

Linear Fraction Transformations preserve the genus of the curve.

Recall LFTs are birational transformations, as proven in Lemma 1. Since it is birational, the underlying function field is unaltered. Therefore the topological space is unaltered, and that implies that genus is preserved.

From algebraic curve theory two theorems are obtained that relate birational equivalence and the genus of a curve [8]. These will be stated without proof.

Theorem 3

A curve of genus zero is birationally equivalent to a line.

Theorem 4

An elliptic curve is birationally equivalent to a nonsingular plane cubic.
Note that in both theorems, complex coefficients are admissible.

The final concept to cover in this section is cross-ratio. Cross-ratio is a geometric invariant that is preserved under transformations. There are two types of cross-ratio, the projective cross-ratio and the Möbius cross-ratio. The projective cross-ratio is preserved under projective transformations, while the Möbius cross-ratio is preserved under Möbius transformations. Projective cross-ratios are always calculated with consideration to four collinear points.

There are two known ways to calculate projective cross-ratio. One way is done using homogenous coordinates in the complex projective plane \( \mathbb{C}P^2 \) [2].

**Definition 1a**

Let \( A, B, C, D \), be four collinear points in \( \mathbb{C}P^2 \) represented by the position vectors \( \vec{a}, \vec{b}, \vec{c}, \vec{d} \), and let

\[
\vec{c} = \alpha \vec{a} + \beta \vec{b} \quad \text{and} \quad \vec{d} = \gamma \vec{a} + \delta \vec{b}.
\]

Then the (projective) cross-ratio of \( A, B, C, D \) is

\[
(ABCD) = \frac{\beta}{\alpha} / \frac{\delta}{\gamma}.
\]

Note that the points in the above definition correspond to projective points, and the line on which \( A, B, C, D \) reside is a line with (possibly) complex coefficients.

Another way to calculate cross-ratio is through direct calculation of points in \( \mathbb{C} \). Assign \( x \) to the real component, and \( y \) to the imaginary component. Let \( A, B, C, D \) have the coordinates \((x_A, y_A), (x_B, y_B), (x_C, y_C), \) and \((x_D, y_D)\), respectively. We then have the following definition for the cross-ratio \((ABCD)\) [8].

**Definition 1b**

\[
(ABCD) = \frac{(x_A y_C - x_C y_A)(x_B y_D - x_D y_B)}{(x_A y_D - x_D y_A)(x_B y_C - x_C y_B)}
\]

Let \( \lambda = (ABCD) \). By a suitable homogenization of the curve (and the associated coordinates) from which the points in definition 1b are considered, the calculated
cross-ratio $\lambda$ in definition 1a will be the same. Generally speaking, definition 1b is used for points on a given plane algebraic curve, while definition 1a is used when points on a projective curve were given in homogeneous coordinates.

When studying Möbius cross-ratios, complex numbers are naturally used. Let $A, B, C, D$ be points in $\mathbb{C}$. Let the points $A, B, C, D$ be associated with $z_A, z_B, z_C, z_D$, respectively. The (Möbius) cross-ratio obeys the following definition [6].

**Definition 2**

$$(ABCD) = \frac{z_A - z_C}{z_A - z_D} \div \frac{z_B - z_C}{z_B - z_D}$$

Since hyperbolic transformations are Möbius transformations, it follows then that $(ABCD)$ is preserved under a hyperbolic transformation. To differentiate between projective and Möbius cross-ratios in what shall come later, let projective cross-ratios be denoted $(AB; CD)$, and let Möbius cross-ratios be denoted $(AB; CD)$.

Lastly, cross-ratios are unique up to $S_3$ symmetry, as they are dependent on how the points are ordered. Supposing that the order $ABCD$ yields a cross-ratio of $\lambda$, we can denote the cross-ratio as the equivalence class $[\lambda]$, where it is the set of complex numbers obtained from $\lambda$ under the action of the group generated by the transpositions

$$\lambda \mapsto \frac{1}{\lambda}, \quad \lambda \mapsto 1 - \lambda.$$

Working out the details, the members of the equivalence class $[\lambda]$ are

$$\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, 1 - \frac{1}{\lambda}, \text{ and } \frac{1}{1 - \frac{1}{\lambda}}.$$

Therefore obtaining either of those six possibilities implies that the cross-ratio is in the equivalence class $[\lambda]$.  

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With the preceding definitions and background material, we are finally at a point where we are ready to tackle the core topics of this thesis.
CHAPTER THREE

CENTROID OF A TRIANGLE

The core topic of this thesis studies the locus of centroids of equilateral triangles in $H^2$. One may wonder why studying such a locus is worthwhile. For example, it is known that for equilateral triangles in $\mathbb{R}^2$, the locus of centroids is a straight line at an angle of $\frac{\pi}{6}$ with respect to the horizon, if we assume the same conditions as stated in the abstract. Diagram 1 on page 83 illustrates such a locus. If, however, we work in $H^2$, the result is no longer trivial. A depiction of an equilateral triangle in $H^2$ having a vertex at 0 and base through the line $y = 0$ is depicted in Diagram 2 on page 84, along with its centroid. If several equilateral triangles are drawn with their centroids satisfying the same conditions, the results tend to suggest that the path the locus of centroids trace out can be given by a general formula. Please refer to Diagram 3 on page 85 for an illustration.

In order to show that the path of the centroids of equilateral triangles with the base through $y = 0$ and vertex at 0 results in a cubic, a parameterization of the path of the centroids of these triangles needs to be established. To do that, however, a general formula for the location of the centroid needs to be derived.

Let $AOB$ denote an arbitrary triangle in $H^2$, where the vertex $O$ is at the origin, and whose base $OB$ coincides with the line $y = 0$, as depicted in Diagram 2. Let $C$ be the non-Euclidean midpoint of $OA$, and let $D$ be the non-Euclidean midpoint of $OB$; a precise formula for these points will be derived. Then let $BC$ and $AD$ be d-lines, and so they pass through the medians of the triangle. (Recall that a $d$-line is part of a generalized circle that is orthogonal to the unit disk [2].) Define $\Delta$ to be the centroid determined by the intersection of $BC$ and $AD$.

Define $h$ and $k$ to be the Euclidean lengths of $OA$ and $OB$, respectively,
and let $H$ and $K$ be their non-Euclidean lengths. Let $t$ be the angle of $AOB$. It follows then that

\[ A = h = \tanh H \]

\[ O = 0 \]

\[ B = ke^{it} = (\tanh K)e^{it} \]

Recall the formula for the non-Euclidean midpoint (for points $p$ and $q$ on the same side of $0$) is $d = \frac{1}{2}(d(0, p) + d(0, q))$. Then the non-Euclidean midpoint is the Euclidean point $n$ on the radius through $0$, $p$, and $q$ at a Euclidean distance $\tanh d$ from $0$. Let us calculate the midpoints $d_C$ and $d_D$, which will enable us to determine the location of $C$ and $D$.

\[ d_C = \frac{1}{2}(d(0, 0) + d(0, h)) \]

\[ = \frac{1}{2}d(0, h) \]

\[ = \frac{1}{2}\tanh^{-1}h \]

\[ = \frac{1}{2}H \]

\[ \tanh d_C = \tanh \frac{1}{2}H \]

\[ \Rightarrow C = \tanh \frac{1}{2}H \]

\[ d_D = \frac{1}{2}(d(0, 0) + d(0, k)) \]

\[ = \frac{1}{2}d(0, k) \]

\[ = \frac{1}{2}\tanh^{-1}k \]

\[ = \frac{1}{2}K \]
\[
\tanh d_D = \tanh \frac{1}{2} K \\
\Rightarrow D = \tanh \frac{1}{2} K e^{it} 
\]

We shall prove that the medians of \( AOB \) are concurrent at the point
\[
\Delta = \frac{\left( \rho - \sqrt{\rho^2 - \|\beta\|^2} \right)}{\beta},
\]
where \( \rho = 2(\cosh^2 H + \cosh^2 K) - 1, \)
\[
\beta = \sinh 2H + e^{it} \sinh 2K. \tag{8}
\]

To begin this proof, we want the equations of the following Hermitian circles:

a) The circle passing through \( A, D \), which corresponds to the points
\[
z = \tanh H \quad \text{and} \quad z = \tanh \frac{1}{2} K e^{it}. \]
b) The circle passing through \( B, C \), which corresponds to the points
\[
z = \tanh K e^{it} \quad \text{and} \quad z = \tanh \frac{1}{2} H. \]

Recall that a Hermitian matrix associated with such a circle can be expressed in the form
\[
\begin{pmatrix}
1 & m \\
\overline{m} & 1
\end{pmatrix},
\]
and is associated with the equation \( z\overline{z} + mz + \overline{m}z + 1 = 0. \)

For the \( d \)-line through \( AD \) we have \( A = \tanh H, \; D = \tanh \frac{1}{2} K e^{it} \). Applying the Hermitian equation to each point yields:
\[
(tanh H)^2 + m(tanh H) + \overline{m}(tanh H) + 1 = 0 \tag{i} \\
(tanh \frac{1}{2} K)^2 + m(tanh \frac{1}{2} K e^{it}) + \overline{m}(tanh \frac{1}{2} K e^{-it}) + 1 = 0 \tag{ii}
\]

For the \( d \)-line through \( BC \) we have \( B = \tanh K e^{it}, \; C = \tanh \frac{1}{2} H. \) Applying the
Hermitian equation to each point yields:

\[(\tanh K)^2 + m(\tanh Ke^{it}) + \overline{m}(\tanh Ke^{-it}) + 1 = 0 \quad \text{iii)}\]

\[(\tanh \frac{1}{2}H)^2 + m(\tanh \frac{1}{2}H) + \overline{m}(\tanh \frac{1}{2}H) + 1 = 0 \quad \text{iv)}\]

Simplifying each, with appropriate use of identities, we have:

For equation i):

\[(\tanh H)^2 + m(\tanh H) + \overline{m}(\tanh H) + 1 = 0\]

\[\left(\frac{\sinh^2 H}{\cosh^2 H}\right) + m\left(\frac{\sinh H}{\cosh H}\right) + \overline{m}\left(\frac{\sinh H}{\cosh H}\right) + 1 = 0\]

\[\sinh^2 H + \cosh^2 H + m(\sinh H \cosh H) + \overline{m}(\sinh H \cosh H) = 0\]

\[2(\sinh^2 H + \cosh^2 H) + m(\sinh 2H) + \overline{m}(\sinh 2H) = 0\]

\[2 \cosh 2H + m(\sinh 2H) + \overline{m}(\sinh 2H) = 0\]

Thus equation i) simplifies to

\[2 \cosh 2H + m(\sinh 2H) + \overline{m}(\sinh 2H) = 0 \quad (9)\]

For equation ii):

\[(\tanh \frac{1}{2}K)^2 + m(\tanh \frac{1}{2}Ke^{it}) + \overline{m}(\tanh \frac{1}{2}Ke^{-it}) + 1 = 0\]

\[\sinh^2 \frac{1}{2}K + \cosh^2 \frac{1}{2}K + m(\sinh \frac{1}{2}K \cosh \frac{1}{2}Ke^{it}) + \overline{m}(\sinh \frac{1}{2}K \cosh \frac{1}{2}Ke^{-it}) = 0\]

\[\frac{\cosh K - 1}{2} + \frac{\cosh K + 1}{2} + \frac{1}{2} \sinh Ke^{it} + \frac{1}{2} \sinh Ke^{-it} = 0\]

\[2 \cosh K + m(\sinh Ke^{it}) + \overline{m}(\sinh Ke^{-it}) = 0\]

Thus equation ii) simplifies to

\[2 \cosh K + m(\sinh Ke^{it}) + \overline{m}(\sinh Ke^{-it}) = 0 \quad (10)\]
Similarly for equation iii):

\[(\tanh K)^2 + m(\tanh Ke^{it}) + \overline{m}(\tanh Ke^{-it}) + 1 = 0\]
\[\sinh^2 K + \cosh^2 K + m(\sinh K \cosh K)e^{it} + \overline{m}(\sinh K \cosh K)e^{-it} = 0\]
\[\frac{\cosh 2K - 1}{2} + \frac{\cosh 2K + 1}{2} + m\left(\frac{1}{2} \sinh 2Ke^{it}\right) + \overline{m}\left(\frac{1}{2} \sinh 2Ke^{-it}\right) = 0\]
\[2 \cosh 2K + m(\sinh 2Ke^{it}) + \overline{m}(\sinh 2Ke^{-it}) = 0\]

Thus equation iii) simplifies to

\[2 \cosh 2K + m(\sinh 2Ke^{it}) + \overline{m}(\sinh 2Ke^{-it}) = 0 \quad (11)\]

and for equation iv):

\[(\tanh \frac{1}{2}H)^2 + m(\tanh \frac{1}{2}H) + \overline{m}(\tanh \frac{1}{2}H) + 1 = 0\]
\[\left(\frac{\sinh^2 \frac{1}{2}H}{\cosh^2 \frac{1}{2}H}\right) + m\left(\frac{\sinh \frac{1}{2}H}{\cosh \frac{1}{2}H}\right) + \overline{m}\left(\frac{\sinh \frac{1}{2}H}{\cosh \frac{1}{2}H}\right) + 1 = 0\]
\[\sinh^2 \frac{1}{2}H + \cosh^2 \frac{1}{2}H + m(\sinh \frac{1}{2}H \cosh \frac{1}{2}H) + \overline{m}(\sinh \frac{1}{2}H \cosh \frac{1}{2}H) = 0\]
\[2(\sinh^2 \frac{1}{2}H + \cosh^2 \frac{1}{2}H) + m(\sinh H) + \overline{m}(\sinh H) = 0\]
\[2 \cosh H + m(\sinh H) + \overline{m}(\sinh H) = 0\]

Thus equation iv) simplifies to

\[2 \cosh H + m(\sinh H) + \overline{m}(\sinh H) = 0 \quad (12)\]

To determine the d-line through AD, we will solve for \(m\), using equations 9 and 10.

\[2 \cosh 2H + m(\sinh 2H) + \overline{m}(\sinh 2H) = 0\]
\[2 \cosh K + m(\sinh Ke^{it}) + \overline{m}(\sinh Ke^{-it}) = 0\]

Our goal is to cancel the \(\overline{m}\) term.

\[\sinh Ke^{-it}(2 \cosh 2H + m(\sinh 2H) + \overline{m}(\sinh 2H)) = 0\]
\[\sinh 2H(2 \cosh K + m(\sinh Ke^{it}) + \overline{m}(\sinh Ke^{-it})) = 0\]
Subtracting the two equations yields

\[ 2 \cosh K \sinh 2H - 2 \cosh 2H \sinh K e^{-it} + m(\sinh 2H \sinh K e^{it} - \sinh 2H \sinh K e^{-it}) = 0 \]

Solving for \( m \), the following is obtained:

\[
m = \frac{2(\cosh 2H \sinh K e^{-it} - \cosh K \sinh 2H)}{\sinh 2H \sinh K(e^{it} - e^{-it})} = \frac{2}{e^{it} - e^{-it}}(\coth 2He^{-it} - \coth K)
\]

Let \( m_{AD} = m \). Therefore the \( m \) associated with this hermitian circle is

\[
m_{AD} = \frac{2}{e^{it} - e^{-it}}(\coth 2He^{-it} - \coth K) \tag{13}
\]

To determine the d-line through \( BC \), we will solve for the other \( m \), using equations 11 and 12.

\[ 2 \cosh 2K + m(\sinh 2Ke^{it}) + \overline{m}(\sinh 2Ke^{-it}) = 0 \]

\[ 2 \cosh H + m(\sinh H) + \overline{m}(\sinh H) = 0 \]

Our goal is to cancel the \( \overline{m} \) term.

\[ \sinh H(2 \cosh 2K + m(\sinh 2Ke^{it}) + \overline{m}(\sinh 2Ke^{-it})) = 0 \]

\[ \sinh 2Ke^{-it}(2 \cosh H + m(\sinh H) + \overline{m}(\sinh H)) = 0 \]

Subtracting the two equations yields

\[ 2 \cosh 2K \sinh H - 2 \sinh 2K \cosh He^{-it} + m(\sinh H \sinh 2Ke^{it} - \sinh H \sinh 2Ke^{-it}) = 0 \]

Solving for \( m \), the following is obtained:

\[
m = \frac{2(\sinh 2K \cosh He^{-it} - \cosh 2K \sinh H)}{\sinh H \sinh 2K(e^{it} - e^{-it})}
\]
Let $m_{BC} = m$. Therefore the $m$ associated with this hermitian circle is

$$m_{BC} = \frac{2}{e^{it} - e^{-it}}(\coth He^{-it} - \coth 2K) \quad (14)$$

We wish to find the point of intersection for the circles afforded by the lines $AD$ and $BC$. This can be accomplished by developing a formula for the point of intersection for two arbitrary circles. Suppose there is an $a$ and $b$ so that the following is true:

$$z\overline{z} + az + \overline{a}\overline{z} + 1 = 0$$

$$z\overline{z} + bz + \overline{b}\overline{z} + 1 = 0$$

Taking the suitable linear combination of the above equations to eliminate $z$, then

$$(\overline{a} - \overline{b})z\overline{z} + (\overline{a}b - a\overline{b})z + \overline{a} - \overline{b} = 0 \quad (15)$$

If the original equations are subtracted, then

$$(a - b)z - (\overline{a} - \overline{b})\overline{z} = 0$$

$$\Rightarrow \overline{z} = \frac{-(a - b)}{\overline{a} - \overline{b}} \quad (16)$$

Substituting equation 16 into equation 15 yields

$$-(a - b)z^2 + (\overline{a}b - a\overline{b})z + \overline{a} - \overline{b} = 0$$

In the context of this problem, let $a = m_{AD}$ and $b = m_{BC}$, as given in equations 13 and 14, respectively. Then

$$-(m_{AD} - m_{BC})z^2 + (\overline{m}_{AD}m_{BC} - m_{AD}\overline{m}_{BC})z + \overline{m}_{AD} - \overline{m}_{BC} = 0 \quad (17)$$

Let us calculate each of the individual terms, using trigonometric identities to aid
in simplification.

\[
-(m_{AD} - m_{BC})
= \frac{2}{e^{it} - e^{-it}} ((\coth H - \coth 2H)e^{-it} + \coth K - \coth 2K)
= \frac{2}{e^{it} - e^{-it}} \left( \frac{e^{-it}}{\sinh 2H} + \frac{1}{\sinh 2K} \right)
\]

\[
\bar{m}_{AD}m_{BC} - m_{AD}\bar{m}_{BC}
= \frac{-4}{(e^{it} - e^{-it})^2} ((\coth 2He^{it} - \coth K)(\coth He^{-it} - \coth 2K)
- (\coth 2He^{-it} - \coth K)(\coth He^{it} - \coth 2K))
= \frac{-4}{(e^{it} - e^{-it})^2} (\coth K \coth H(e^{it} - e^{-it}) + \coth 2K \coth 2H (e^{-it} - e^{it}))
= \frac{-4}{e^{it} - e^{-it}} (\coth K \coth H - \coth 2K \coth 2H)
= \frac{-4}{e^{it} - e^{-it}} \left( \frac{\cosh H \cosh K}{\sinh H \sinh K} - \frac{(2 \cosh^2 H - 1)(2 \cosh^2 K - 1)}{4 \sinh H \cosh H \sinh K \cosh K} \right)
= \frac{-4}{e^{it} - e^{-it}} \left( \frac{2 \cosh^2 K + 2 \cosh^2 H - 1}{\sinh 2H \sinh 2K} \right)
\]

\[
\bar{m}_{AD} - \bar{m}_{BC}
= \frac{2}{e^{it} - e^{-it}} ((\coth H - \coth 2H)e^{it} + \coth K - \coth 2K)
= \frac{2}{e^{it} - e^{-it}} \left( \frac{e^{it}}{\sinh 2H} + \frac{1}{\sinh 2K} \right)
\]

Substituting these results into equation 17 yields

\[
\frac{2}{e^{it} - e^{-it}} \left( \frac{e^{-it}}{\sinh 2H} + \frac{1}{\sinh 2K} \right) z^2 - \frac{4}{e^{it} - e^{-it}} \left( \frac{2 \cosh^2 K + 2 \cosh^2 H - 1}{\sinh 2H \sinh 2K} \right) z
+ \frac{2}{e^{it} - e^{-it}} \left( \frac{e^{it}}{\sinh 2H} + \frac{1}{\sinh 2K} \right) = 0
\]
Multiplying both sides by \( \frac{(e^{it} - e^{-it}) \sinh 2H \sinh 2K}{2} \) results in the equation

\[
\begin{align*}
(sinh 2K e^{-it} + sinh 2H)z^2 - 2(2(cosh^2 K + cos^2 H) - 1)z \\
+ (sinh 2K e^{it} + sinh 2H) = 0
\end{align*}
\] (18)

Recall the values given to \( \rho \) and \( \beta \) in Equation 8 on page 15. Comparing them to the above equation 18, the coefficients are as follows:

\[
\begin{align*}
sinh 2K e^{-it} + sinh 2H &= \bar{\beta} \\
-2(2(cosh^2 K + cos^2 H) - 1) &= -2\rho \\
sinh 2K e^{it} + sinh 2H &= \beta
\end{align*}
\]

Therefore equation 18 can be written more simply as

\[
\bar{\beta}z^2 - 2\rho z + \beta = 0
\]

Using the quadratic formula to solve for \( z \), we get

\[
z = \frac{(\rho \pm \sqrt{\rho^2 - \|\beta\|^2})}{\bar{\beta}}
\]

The positive root can be disregarded since that would land outside the unit disk.

Since \( z \) represents the locus of the intersection of the curves \( AD \) and \( BC \), that is precisely the centroid \( \Delta \) of the triangle. Therefore the centroid of triangle \( AOB \) is

\[
\Delta = \frac{(\rho - \sqrt{\rho^2 - \|\beta\|^2})}{\bar{\beta}},
\]

where \( \rho = 2(cosh^2 H + cos^2 K) - 1, \)

\[
\beta = sinh 2H + e^{it} sinh 2K.
\]

With the case of the general centroid of a triangle determined, let us focus on the centroid of equilateral triangles. If \( AOB \) is equilateral, we shall show that

\[
\cos t = \frac{1}{2}(1 + h^2), \text{ and equivalently, } sech H = 2 \sin \frac{t}{2}, \text{ where } 0 < t < \frac{\pi}{3}.
\] (19)
First it will shown that \( \cos t = \frac{1}{2}(1 + h^2) \). As \( AOB \) is equilateral, then \( k = h \) and \( K = H \), where \( h, k \) and \( H, K \) are Euclidean and non-Euclidean lengths respectively. Remember that the general formula of the non-Euclidean distance [2] is given by

\[
d(z_1, z_2) = \tanh^{-1} \left| \frac{z_2 - z_1}{1 - z_1 \overline{z_2}} \right|
\]

As \( AOB \) is equilateral, all sides are of length \( H \). Writing equations for \( H \), we get

\[
H = d(0, h) = \tanh^{-1} h \\
H = d(he^{it}, h) = \tanh^{-1} \left| \frac{h - he^{it}}{1 - h^2e^{it}} \right|
\]

As the sides are of the same length, we can equate them. Therefore

\[
\tanh^{-1} h = \tanh^{-1} \left| \frac{h - he^{it}}{1 - h^2e^{it}} \right|
\]

Since \( \tanh^{-1} \) is one-to-one, then by cancellation

\[
h = \left| \frac{h - he^{it}}{1 - h^2e^{it}} \right|
\]

Recall \( |z| = \sqrt{zz^*} \). Then

\[
h^2 = \left( \frac{h - he^{it}}{1 - h^2e^{it}} \right) \left( \frac{h - he^{-it}}{1 - h^2e^{-it}} \right)
\]

\[
h^2 = \frac{h^2 - h^2e^{it} - h^2e^{-it} + h^2}{1 - h^2e^{it} - h^2e^{-it} + h^4}
\]

\[
h^2 = \frac{2h^2 - h^2(e^{it} + e^{-it})}{1 - h^2(e^{it} + e^{-it}) + h^4}
\]

\[
1 \cdot h^2 = \frac{h^2(2 - 2 \cos t)}{1 - 2h^2 \cos t + h^4}
\]
Assuming \( h \neq 0 \), we divide both sides by \( h^2 \). So

\[
1 = \frac{2 - 2 \cos t}{1 - 2h^2 \cos t + h^4}
\]

\[
1 - 2h^2 \cos t + h^4 = 2 - 2 \cos t
\]

\[
2 - 2 \cos t = 1 - 2h^2 \cos t + h^4
\]

\[
2(h^2 - 1) \cos t = -1 + h^4
\]

\[
\cos t = \frac{h^4 - 1}{2(h^2 - 1)} = \frac{h^2 + 1}{2}
\]

Therefore \( \cos t = \frac{h^2 + 1}{2} \).

Notice that \( t \) changes as \( h \) changes. Since \( \cos t = \frac{h^2 + 1}{2} \), it follows that 

\( t = \arccos \left( \frac{h^2 + 1}{2} \right) \). As \( 0 < h < 1 \), then the following is true:

\[
0 < h < 1
\]

\[
\frac{1}{2} < \frac{h^2 + 1}{2} < \frac{2}{2}
\]

\[
\arccos \left( \frac{1}{2} \right) > \arccos \left( \frac{h^2 + 1}{2} \right) > \arccos (1)
\]

\[
\frac{\pi}{3} > t > 0
\]

Therefore \( 0 < t < \frac{\pi}{3} \).

To show \( \text{sech} \, H = 2 \sin \frac{1}{2} \), we begin with \( \cos t = \frac{h^2 + 1}{2} \). Solving for \( h^2 \) yields \( h^2 = 2 \cos t - 1 \). But \( h = \tanh H \), as originally defined for \( AOB \). Therefore

\[
\tanh^2 H = 2 \cos t - 1
\]

\[
1 - \text{sech}^2 H = 2 \cos t - 1
\]

\[
\text{sech}^2 H = 2 - 2 \cos t
\]

\[
\text{sech}^2 H = 2 \cdot 2 \left( 1 - \cos t \right)
\]

\[
\text{sech}^2 H = 4 \sin^2 \frac{t}{2}
\]
Since $0 < t < \frac{\pi}{3}$, we take the positive square root.

$$\sqrt{\text{sech}^2 H} = +\sqrt{4\sin^2 \frac{t}{2}}$$

$$\text{sech} H = 2\sin \frac{t}{2}$$

Therefore it has been demonstrated that $\cos t = \frac{1}{2}(1 + h^2)$, $\text{sech} H = 2\sin \frac{t}{2}$, while $0 < t < \frac{\pi}{3}$.

Next we shall show that when $AOB$ is an equilateral triangle, the centroid $\Delta$ of the triangle can be expressed as

$$\Delta = \sqrt{\frac{\cos \left(\frac{t}{2} + \frac{\pi}{3}\right)}{\cos \left(\frac{t}{2} - \frac{\pi}{3}\right)}} e^{\frac{H}{2}}$$

To begin with, using the information from equation 19, the formulas for $\cosh H$, $\sinh H$, $\sinh 2H$, $\rho$, and $\beta$ need to be derived.

$$\text{sech} H = 2\sin \frac{t}{2} \Leftrightarrow \cosh H = \frac{1}{2}\sin \frac{t}{2}$$

Then:

$$2\sin \frac{t}{2} = \text{sech} H$$

$$4\sin^2 \frac{t}{2} = \text{sech}^2 H$$

$$4\sin^2 \frac{t}{2} = 1 - \tanh^2 H$$

$$\tanh^2 H = 1 - 4\sin^2 \frac{t}{2}$$

$$\frac{\sinh H}{\cosh^2 H} = 1 - 4\sin^2 \frac{t}{2}$$

$$\sinh^2 H = \left(1 - 4\sin^2 \frac{t}{2}\right) \cosh^2 H$$
\[
\sinh^2 H = \left(1 - 4 \sin^2 \frac{t}{2}\right) \left(\frac{1}{2 \sin \frac{t}{2}}\right)^2
\]
\[
\sinh^2 H = \frac{1 - 4 \sin^2 \frac{t}{2}}{4 \sin^2 \frac{t}{2}}
\]
\[
\therefore \quad \sinh H = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin \frac{t}{2}},
\]

and \(\sinh 2H = 2 \sinh H \cosh H = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin^2 \frac{t}{2}}\).

To derive \(\rho\) and \(\beta\), remember that \(H = K\), since \(AOB\) is an equilateral triangle. Let us obtain \(\rho\) first.

\[
\rho = 2(\cosh^2 H + \cosh^2 K) - 1
\]
\[
= 4 \cosh^2 H - 1
\]
\[
= 4 \frac{1}{4 \sin^2 \frac{t}{2}} - 1
\]
\[
= \frac{1}{\sin^2 \frac{t}{2}} - 1
\]
\[
= \frac{1 - \sin^2 \frac{t}{2}}{\sin^2 \frac{t}{2}}
\]
\[
\rho = \frac{\cos^2 \frac{t}{2}}{\sin^2 \frac{t}{2}}
\]

Let us obtain \(\beta\) and \(\overline{\beta}\):

\[
\beta = \sinh 2H + \sinh 2Ke^{it}
\]
\[
= \sinh 2H + \sinh 2He^{it}
\]
\[
= \sinh 2H(1 + e^{it})
\]
\[
\beta = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin^2 \frac{t}{2}} (1 + e^{it}), \text{ and also}
\]
\[
\overline{\beta} = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin^2 \frac{t}{2}} (1 + e^{-it})
\]
In order to show \( \Delta = \sqrt{\frac{\cos \left( \frac{t}{2} + \frac{\pi}{3} \right)}{\cos \left( \frac{t}{2} - \frac{\pi}{3} \right)} e^{it}} \), it is easier to show that

\[
\Delta^2 = \frac{\cos \left( \frac{t}{2} + \frac{\pi}{3} \right)}{\cos \left( \frac{t}{2} - \frac{\pi}{3} \right)} e^{it}
\]  

(21)

The statements are equivalent since \( 0 < t < \frac{\pi}{3} \), so the positive square root will always be taken. Therefore by showing that equation 21 is true, then equation 20 is proven. Recall that it has been proved that:

\[
\cosh H = \frac{1}{2 \sin \frac{t}{2}}
\]

\[
\sinh H = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin \frac{t}{2}}
\]

\[
\sinh 2H = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin \frac{t}{2}}
\]

\[
\rho = \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}}
\]

\[
\beta = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin \frac{t}{2}} (1 + e^{it}), \text{ and also}
\]

\[
\overline{\beta} = \frac{\sqrt{1 - 4 \sin^2 \frac{t}{2}}}{2 \sin \frac{t}{2}} (1 + e^{-it})
\]

During the derivation, the following trigonometric relations will be used:

\[
(1 + e^{-it})(1 + e^{it}) = 2 + 2 \cos t = 4 \cos^2 \frac{t}{2}, \text{ and}
\]

\[
\frac{(1 + e^{it})}{(1 + e^{-it})} = \frac{(e^{-\frac{it}{2}} + e^{\frac{it}{2}}) e^{\frac{it}{2}}}{(e^{\frac{it}{2}} + e^{-\frac{it}{2}}) e^{-\frac{it}{2}}} = e^{it}
\]

Using equation 8, it follows that

\[
\Delta^2 = \frac{(\rho - \sqrt{\rho^2 - \|\beta\|^2}) (\rho - \sqrt{\rho^2 - \|\beta\|^2})}{\beta} = \frac{2 \rho^2 - \beta \overline{\beta} - 2 \rho \sqrt{\rho^2 - \beta \overline{\beta}}}{\beta \overline{\beta}}
\]
The individual terms become the following:

\[
2\rho^2 = \frac{2 \cos^4 \frac{t}{2}}{\sin^4 \frac{t}{2}}
\]

\[
\beta \beta = \frac{(1 - 4 \sin^2 \frac{t}{2})(\cos^2 \frac{t}{2})}{\sin^4 \frac{t}{2}}
\]

\[
2\rho \sqrt{\rho^2 - \beta \beta} = \frac{2 \cos^2 \frac{t}{2} \sqrt{3} \cos \frac{t}{2}}{\sin^2 \frac{t}{2} \sin \frac{t}{2}}
\]

\[
\overline{\beta \beta} = \frac{(1 - 4 \sin^2 \frac{t}{2})(1 + e^{-it})^2}{4 \sin^4 \frac{t}{2}}
\]

Then \(\Delta^2\) becomes the following:

\[
\Delta^2 = \frac{2 \cos^4 \frac{t}{2} - (1 - 4 \sin^2 \frac{t}{2})(\cos^2 \frac{t}{2}) - 2 \cos^2 \frac{t}{2} \sqrt{3} \cos \frac{t}{2}}{(1 - 4 \sin^2 \frac{t}{2})(1 + e^{-it})^2}
\]

\[
\Delta^2 = \frac{\cos^2 \frac{t}{2} \left( \frac{2 \cos^2 \frac{t}{2}}{\sin^2 \frac{t}{2}} - \frac{(1 - 4 \sin^2 \frac{t}{2})}{\sin^2 \frac{t}{2}} - \frac{2\sqrt{3} \cos \frac{t}{2}}{\sin \frac{t}{2}} \right)}{4 \sin^4 \frac{t}{2}(1 - 4 \sin^2 \frac{t}{2})(1 + e^{-it})(1 + e^{it})(1 + e^{-it})}
\]

\[
\Delta^2 = \frac{\cos^2 \frac{t}{2} \left( 2 \cos^2 \frac{t}{2} - (1 - 4 \sin^2 \frac{t}{2}) - 2\sqrt{3} \cos \frac{t}{2} \sin \frac{t}{2} \right)}{4 \sin^4 \frac{t}{2}(1 - 4 \sin^2 \frac{t}{2})((1 + e^{-it})(1 + e^{it}))}
\]

Applying further simplifications to \(\Delta^2\) yields:

\[
\Delta^2 = \frac{(2 \cos^2 \frac{t}{2} - 1 + 4 \sin^2 \frac{t}{2} - 2\sqrt{3} \cos \frac{t}{2} \sin \frac{t}{2})e^{it}}{1 - 4 \sin^2 \frac{t}{2}}
\]

\[
= \frac{(1 + 2 \sin^2 \frac{t}{2} - 2\sqrt{3} \cos \frac{t}{2} \sin \frac{t}{2})e^{it}}{1 - 4 \sin^2 \frac{t}{2}}
\]

\[
= \frac{(1 + (1 - \cos t) - \sqrt{3} \sin t)e^{it}}{1 - 2(1 - \cos t)}
\]

\[
\therefore \Delta^2 = \frac{(2 - \cos t - \sqrt{3} \sin t)e^{it}}{2 \cos t - 1} \quad (22)
\]
At this point, we begin to work backwards to cast equation 22 into the form of equation 21. If the following procedure appears unnatural, the author of this thesis did the original calculation by simplifying equation 21 into equation 22. All the steps are reversible.

\[
\Delta^2 = \frac{(2 - \cos t - \sqrt{3} \sin t) e^{it}}{2 \cos t - 1} = \frac{2(2 - \cos t - \sqrt{3} \sin t) e^{it}}{2(2 \cos t - 1)} = \frac{(4 - 2 \cos t - 2 \sqrt{3} \sin t) e^{it}}{2 \cos t - 1}.
\]

Therefore the claim for \( \Delta^2 \) has been proven, and this implies that the formula for \( \Delta \), given in equation 20, is also true. We now move on to finding a Cartesian representation for \( \Delta \).
CHAPTER FOUR
LOCUS OF CENTROIDS

From this point on we will assume \( AOB \) is an equilateral triangle. Consider the curve \( \varepsilon = \Delta(t) \), where \( 0 < t < \frac{\pi}{3} \). \( \varepsilon \) is part of the Cartesian curve

\[
\frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2. \tag{23}
\]

We shall show that the path traced out by the locus of centroids, \( \Delta \), is part of the above cubic.

Let \( z = \Delta \). Clearly, \( z \) is a complex number that lies along the median of \( AOB \). Expressed in component form, \( z = x + iy \). As the median of an equilateral triangle bisects the angle also, since \( \angle AOB = t \) the angle the median forms at \( O \) with respect to the base \( OA \) is \( \frac{t}{2} \). Thus in polar form \( z = re^{it/2} = r(\cos \frac{t}{2} + i \sin \frac{t}{2}) \).

By equating components of the Cartesian representation and the polar form, it follows that \( x = r \cos \frac{t}{2} \) and \( y = r \sin \frac{t}{2} \). Thus:

\[
\cos \frac{t}{2} = \frac{x}{r}, \quad \sin \frac{t}{2} = \frac{y}{r}
\]

\[
z = x + iy \Rightarrow z\bar{z} = x^2 + y^2
\]

But \( z\bar{z} = \Delta\bar{\Delta} = \frac{\cos \left( \frac{t}{2} + \frac{\pi}{3} \right)}{\cos \left( \frac{t}{2} - \frac{\pi}{3} \right)} \) using information from equation 20. Then:

\[
x^2 + y^2 = \frac{\cos \left( \frac{t}{2} + \frac{\pi}{3} \right)}{\cos \left( \frac{t}{2} - \frac{\pi}{3} \right)}
\]

\[
= \frac{\cos \frac{t}{2} \cos \frac{\pi}{3} - \sin \frac{t}{2} \sin \frac{\pi}{3}}{\cos \frac{t}{2} \cos \frac{\pi}{3} + \sin \frac{t}{2} \sin \frac{\pi}{3}}
\]

\[
= \frac{1/2 \cos \frac{t}{2} - \sqrt{3}/2 \sin \frac{t}{2}}{1/2 \cos \frac{t}{2} + \sqrt{3}/2 \sin \frac{t}{2}}
\]

29
\[
x^2 + y^2 = \frac{\cos \frac{t}{2} - \sqrt{3} \sin \frac{t}{2}}{\cos \frac{t}{2} + \sqrt{3} \sin \frac{t}{2}}
\]
\[
= \frac{\frac{x}{r} - \frac{\sqrt{3}y}{r}}{\frac{x}{r} + \frac{\sqrt{3}y}{r}}
\]
\[
= \frac{\frac{1}{r}(x - \sqrt{3}y)}{\frac{1}{r}(x + \sqrt{3}y)}
\]
\[
= \frac{x - \sqrt{3}y}{x + \sqrt{3}y}
\]

Therefore the equation is verified.

Graphing the part of the cubic that is on the unit disk, we get the curve \( \varepsilon \) as depicted in Diagram 4 on page 86. Let us denote \( \varepsilon^* \) as the conjugate curve of \( \varepsilon \).

It will be shown that \( \varepsilon \) is invariant under the symmetries

\[
z \leftrightarrow -z, \quad \text{i)}
\]
\[
z \leftrightarrow \frac{1}{z}, \quad \text{ii)}
\]

where \( z = x + iy \) as follows.

i) Invariance under \( z \leftrightarrow -z \).

\[
z \leftrightarrow -z
\]
\[
\Rightarrow x + iy \leftrightarrow -(x + iy)
\]
\[
\Rightarrow x \leftrightarrow -x, y \leftrightarrow -y
\]

We are given \( \frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2 \). Then under the symmetry, \( \varepsilon \) is transformed into

\[
\frac{-x + \sqrt{3}y}{-x - \sqrt{3}y} = (-x)^2 + (-y)^2
\]
\[
\frac{- (x - \sqrt{3}y)}{-(x + \sqrt{3}y)} = x^2 + y^2
\]
\[
\therefore \frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2
\]
ii) Invariance under $z \mapsto \frac{1}{z}$.

\[
\begin{align*}
  z & \mapsto \frac{1}{z} \\
  \Rightarrow x + iy & \mapsto \frac{1}{x + iy} \\
  \Rightarrow x + iy & \mapsto \frac{x - iy}{x^2 + y^2} \\
  \Rightarrow x + iy & \mapsto \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2} \\
  \Rightarrow x & \mapsto \frac{x}{x^2 + y^2}, y \mapsto \frac{-y}{x^2 + y^2}
\end{align*}
\]

Therefore under the symmetry, $\varepsilon$ is transformed into

\[
\begin{align*}
  \frac{x}{x^2 + y^2} - \frac{-\sqrt{3}y}{x^2 + y^2} & = \frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} \\
  \frac{x}{x^2 + y^2} + \frac{-\sqrt{3}y}{x^2 + y^2} & = \frac{x^2 + y^2}{(x^2 + y^2)^2} \\
  \frac{x + \sqrt{3}y}{x - \sqrt{3}y} & = \frac{1}{x^2 + y^2}
\end{align*}
\]

Taking the reciprocals, we have

\[
\frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2.
\]

With this property of the curve $\varepsilon$ explored, let us look at another way to form the same curve.
A curious result from the studies of the cubic $\varepsilon$ is, as we shall show, that it can be viewed as the stereographic projection of the intersection of the quadric surfaces $x^2 + y^2 - z^2 = -1$ and $x = \sqrt{3}yz$.

We are given that $t = v, b = v$, 

\[ x = r\cos \theta, \quad y = r\sin \theta. \]

The above equation then becomes;

\[ \frac{x - \sqrt{3}y}{x + \sqrt{3}y} = x^2 + y^2. \]

Change the above into polar coordinates by $x = r\cos \theta, y = r\sin \theta$. The above equation then becomes:

\[ r^2 = \frac{r\cos \theta - \sqrt{3}r\sin \theta}{r\cos \theta + \sqrt{3}r\sin \theta} \]

Assuming $r \neq 0$, then

\[ r^2 = \frac{\cos \theta - \sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

In anticipation for what is to come, we shall solve for $1 - r^2$ and $1 + r^2$.

\[ 1 - r^2 = 1 - \frac{\cos \theta - \sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

\[ = \frac{\cos \theta + \sqrt{3}\sin \theta - \cos \theta + \sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

\[ = \frac{2\sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

\[ 1 + r^2 = 1 + \frac{\cos \theta - \sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

\[ = \frac{\cos \theta + \sqrt{3}\sin \theta + \cos \theta - \sqrt{3}\sin \theta}{\cos \theta + \sqrt{3}\sin \theta} \]

\[ = \frac{2\cos \theta}{\cos \theta + \sqrt{3}\sin \theta} \]
Summarizing these results,

\[ 0 = \frac{Z + \gamma - \varepsilon \Lambda}{\theta \cos Z} \gamma \]

\[ 0 = \frac{\gamma Z + \varepsilon \Lambda - \varepsilon \Lambda}{\theta \cos \gamma} \gamma \]

\[ I - I = I - \sqrt{Z} + \varepsilon \Lambda - \theta \varepsilon \sin \frac{\varepsilon \Lambda}{\theta \cos \Lambda} + \theta \varepsilon \cos \frac{\varepsilon \Lambda}{\theta \cos \Lambda} \]

\[ I - I = \varepsilon (I - \gamma) - \varepsilon (\theta \sin \frac{\varepsilon \Lambda}{\theta \cos \Lambda}) + \varepsilon (\theta \cos \frac{\varepsilon \Lambda}{\theta \cos \Lambda}) \]

Satisfying the equation, we have

The particular we are interested in is the one that lies on the hyperboloid.

\[ (I - \gamma, \theta \sin \frac{\varepsilon \Lambda}{\theta \cos \Lambda}, \theta \cos \frac{\varepsilon \Lambda}{\theta \cos \Lambda}) = \]

\[ (0, \theta \sin \frac{\varepsilon \Lambda}{\theta \cos \Lambda}, \theta \cos \frac{\varepsilon \Lambda}{\theta \cos \Lambda}) = \]

\[ \gamma \Lambda + \gamma (\gamma - 1) = 0 \]

Applying the formula for the current problem, we have

\[ \gamma \Lambda + \gamma (\gamma - 1) = 0 \]

Through the line containing \( \gamma \Lambda \) and \( \gamma \Lambda \), as is given by

\[ \gamma \Lambda = \varepsilon \Lambda = \varepsilon \Lambda \]

Parameterization \( \gamma = \gamma \Lambda \) and so let a point on \( \gamma \Lambda \) and \( \gamma \Lambda \), \( \gamma \Lambda \) is the point refer to Diagram 6 on page 88. Let the South Pole, \( (0, 0, 1) \), be \( \gamma \Lambda \), and has the

arrive at a location on the unit hyperboloid. For a graphical illustration, please

So we want to project from the South Pole through a point on the cubic to

\[ \frac{\theta \sin \varepsilon \Lambda + \theta \cos \varepsilon \Lambda}{\theta \cos \varepsilon \Lambda} = \varepsilon \Lambda + I \]

\[ \frac{\theta \sin \varepsilon \Lambda + \theta \cos \varepsilon \Lambda}{\theta \sin \varepsilon \Lambda \cos \varepsilon \Lambda} = \varepsilon \Lambda - I \]

Summarizing these results:
Clearly $\lambda \neq 0$, so

$$\lambda r^2 - \lambda + 2 = 0$$

$$\Rightarrow \lambda(r^2 - 1) + 2 = 0$$

$$\Rightarrow \lambda = \frac{2}{1 - r^2}$$

Therefore, by substituting for $\lambda$ and simplifying, $Q = (\lambda r \cos \theta, \lambda r \sin \theta, \lambda - 1)$ becomes

$$Q = \left(\frac{2r \cos \theta}{1 - r^2}, \frac{2r \sin \theta}{1 - r^2}, \frac{1 + r^2}{1 - r^2}\right)$$

on the upper half of the unit hyperboloid. Substituting equations 24 and 25 into equation 26, we get

$$Q = \left(\frac{2r \cos \theta}{\cos \theta + \sqrt{3} \sin \theta}, \frac{2r \sin \theta}{\cos \theta + \sqrt{3} \sin \theta}, \frac{2 \cos \theta}{\cos \theta + \sqrt{3} \sin \theta}\right)$$

$$= \left(\frac{r \cos \theta (\cos \theta + \sqrt{3} \sin \theta)}{\sqrt{3} \sin \theta}, \frac{r (\cos \theta + \sqrt{3} \sin \theta)}{\sqrt{3}}, \frac{\cos \theta}{\sqrt{3} \sin \theta}\right)$$

Therefore $Q = (\frac{1}{\sqrt{3}} r \cot \theta (\cos \theta + \sqrt{3} \sin \theta), \frac{1}{\sqrt{3}} r (\cos \theta + \sqrt{3} \sin \theta), \frac{1}{\sqrt{3}} \cot \theta)$.

The three components are related in a remarkable way. Notice that

$$\frac{1}{\sqrt{3}} r \cot \theta (\cos \theta + \sqrt{3} \sin \theta) = \sqrt{3} \cdot \frac{1}{\sqrt{3}} r (\cos \theta + \sqrt{3} \sin \theta) \cdot \frac{1}{\sqrt{3}} \cot \theta,$$

in other words the x-coordinate of $Q$ is $\sqrt{3}$ times the product of the y and z coordinates. So $Q$ belongs to the upper unit hyperboloid and the quadric surface $x^2 + y^2 - z^2 = -1$ and $x = \sqrt{3}yz$. Therefore the claim is proven.

An illustration of the intersection of the quadric surfaces is given in Diagram 5 on page 87. The stereographic projection of the curve of intersection of two surfaces onto the unit disk is given in Diagram 6 on page 88. Therefore two seemingly unrelated methods (namely this method and the one given in Diagram 3
on page 85) to produce the curve $\varepsilon$ have been explored. Let us explore further properties of $\varepsilon$ by using the theory of algebraic curves.
CHAPTER SIX  
ELLIPTIC CURVES

Let us begin by writing the curve $\varepsilon$ in the form $f = f(x, y) = 0$. We then get from equation 23

$$\begin{align*}
f(x, y) &= x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - x + \sqrt{3}y
\end{align*}$$

(27)

As an algebraic curve we can write $\varepsilon$ in homogeneous coordinates as

$$F = F(x, y, w) = 0.$$ (The use of $w$ instead of $z$ for the third coordinate shall be done solely to keep the notation clear in what shall eventually follow.) Therefore $f$ homogenizes to the curve

$$F(x, y, w) = x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - xw^2 + \sqrt{3}yw^2$$

(28)

and the homogenized coordinates of this curve will be denoted $[x, y, w]$. Also, we allow the variables to take on complex values, because this will allow us to take the full use of algebraic curve theory. Hence $F \in \mathbb{CP}^2$. Note that by setting $w = 1$, the original affine curve $f = 0$ can be recaptured.

Is $F$ singular or nonsingular? From Corollary 1 on page 4, we get the conditions for which $F$ is singular. The contrapositive of this statement can be stated as a another corollary.

**Corollary 2** A point $P$ on a curve $F$ is nonsingular if and only if $\frac{\partial F}{\partial x_i} \neq 0$ for some $i$.

In the context of our problem, Corollaries 1 and 2 can be interpreted in the following manner.
Corollary 3 Consider the curves

\[
\begin{align*}
F &= 0 \\
F_x &= 0 \\
F_y &= 0 \\
F_w &= 0
\end{align*}
\]

The common solutions \([x, y, w]\) to this system are the singular points of the curve. If there is no common solution, then the curve is nonsingular.

To prove the above, set \(i = 3\) in Corollary 1 to obtain the conditions for singularity, and \(i = 3\) in Corollary 2 to obtain the conditions necessary for nonsingularity. In both cases we also need \(F = 0\) as an additional equation, since if \(F_x = F_y = F_w = 0\), we need to make sure whether or not the proposed singularity lies on the curve.

Applying the conditions of Corollary 3 to \(F\), we shall show that \(F\) is nonsingular. To do this, we calculate the curve and its partial derivatives.

\[
\begin{align*}
F &= x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - xw^2 + \sqrt{3}yw^2 \\
F_x &= 3x^2 + y^2 + 2\sqrt{3}xy - w^2 \\
F_y &= 2xy + \sqrt{3}x^2 + 3\sqrt{3}y^2 + \sqrt{3}w^2 \\
F_w &= -2xw + 2\sqrt{3}yw
\end{align*}
\]

If \(F_w = 0\), then

\[
\begin{align*}
-2xw + 2\sqrt{3}yw &= 0 \\
-2w(x - \sqrt{3}y) &= 0
\end{align*}
\]
Therefore we conclude that either

\[ w = 0, \text{ or } \]

\[ x - \sqrt{3}y = 0. \]

Let’s explore case a).  

Applying \( w = 0 \) into the equations for \( F, F_x, \) and \( F_y, \) we get

\[ x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 = 0 \quad \text{(i)} \]
\[ 3x^2 + y^2 + 2\sqrt{3}xy = 0 \quad \text{(ii)} \]
\[ 2xy + \sqrt{3}x^2 + 3\sqrt{3}y^2 = 0 \quad \text{(iii)} \]

If \( y = 0, \) it implies \( x = 0, \) and vice-versa. However, \([0,0,0]\) is not part of the projective space \( \mathbb{CP}^2, \) so it is not a point on the curve. Therefore we shall assume \( x \neq 0, y \neq 0. \) Concentrating on the equations ii) and iii) of the above system, multiply equation ii) by -1, and equation iii) by \( \sqrt{3}. \) The following results:

\[ -3x^2 - y^2 - 2\sqrt{3}xy = 0 \]
\[ 2\sqrt{3}xy + 3x^2 + 9y^2 = 0 \]
\[ 8y^2 = 0 \Rightarrow y = 0 \Rightarrow x = 0 \]

However, \( x = 0, y = 0, w = 0 \Rightarrow [0,0,0] \in \mathbb{CP}^2, \) which is impossible. Therefore \( w \neq 0; \) there are no singularities on the line at infinity.

Let’s explore case b).  

\[ x - \sqrt{3}y = 0 \Rightarrow x = \sqrt{3}y. \] Substituting for \( x \) into the equation for \( F, \) we get

\[ 0 = F = (\sqrt{3}y)^3 + \sqrt{3}yy^2 + \sqrt{3}(\sqrt{3}y)^2y + \sqrt{3}y^3 - \sqrt{3}yw^2 + \sqrt{3}yw^2 \]
\[ = 3\sqrt{3}y^3 + \sqrt{3}y^3 + 3\sqrt{3}y^3 + \sqrt{3}y^3 \]
\[ = 8\sqrt{3}y^3 \]
\[ \Rightarrow y = 0 \Rightarrow x = 0 \quad (\text{since } x = \sqrt{3}y) \]
But from $F_x = 0$, it follows that $x = 0$ and $y = 0 \Rightarrow w = 0$. However this is impossible since $[0, 0, 0] \not\in \mathbb{CP}^2$.

Therefore with both cases exhausted, $F = 0, F_x = 0, F_y = 0, \text{and } F_w = 0$ have no common solution. We conclude then that $F$ is nonsingular.

As $F$ is a nonsingular cubic, it has genus 1; this fact is derived from equation 6 on page 7. Now suppose the original curve $f$ is rotated by an angle $\phi$ about the origin. Conceptually one thinks that the nonsingularity should be preserved; after all it is a rigid motion. But is there another way to show it, apart from laborious calculations? The answer is yes.

To show the curve is nonsingular, let's first find the image of the original curve $f$ under the transformation $z \mapsto e^{i\phi}z$. Recall that

$$0 = f = x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - x + \sqrt{3}y$$

$$= (x^2 + y^2)(x + \sqrt{3}y) - x + \sqrt{3}y$$

From the mapping we have

$$z \mapsto e^{i\phi}z$$

$$\Rightarrow x + iy \mapsto (\cos \phi + i\sin \phi)(x + iy)$$

$$\Rightarrow x + iy \mapsto (x\cos \phi - y\sin \phi) + i(x\sin \phi + y\cos \phi)$$

$$\Rightarrow x \mapsto x\cos \phi - y\sin \phi, \ y \mapsto x\sin \phi + y\cos \phi$$

Let us determine the image of $f$, and call it $f_\phi$.

$$f_\phi(x, y) = ((x\cos \phi - y\sin \phi)^2 + (x\sin \phi + y\cos \phi)^2)((x\cos \phi - y\sin \phi)$$

$$+ \sqrt{3}(x\sin \phi + y\cos \phi)) - (x\cos \phi - y\sin \phi) + \sqrt{3}(x\sin \phi + y\cos \phi)$$

$$= x^3(\cos \phi + \sqrt{3}\sin \phi) + x^2y(\sqrt{3}\cos \phi - \sin \phi) + xy^2(\cos \phi + \sqrt{3}\sin \phi)$$

$$+ y^3(\sqrt{3}\cos \phi - \sin \phi) + x(-\cos \phi + \sqrt{3}\sin \phi) + y(\sqrt{3}\cos \phi + \sin \phi)$$

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In anticipation of a future need, simplifying \( f_\phi \) via trigonometric identities, we get

\[
(x^2 + y^2)(x \cos (\phi - \frac{\pi}{3}) - y \sin (\phi - \frac{\pi}{3})) - (x \cos (\phi + \frac{\pi}{3}) - y \sin (\phi + \frac{\pi}{3})) = 0,
\]

and therefore the curve can be expressed as:

\[
x^2 + y^2 = \frac{x \cos (\phi + \frac{\pi}{3}) - y \sin (\phi + \frac{\pi}{3})}{x \cos (\phi - \frac{\pi}{3}) - y \sin (\phi - \frac{\pi}{3})}.
\]

If we homogenize the equation \( f_\phi(x, y) = 0 \), we have

\[
F_\phi(x, y, w) = (x^2 + y^2)(x \cos (\phi - \frac{\pi}{3}) - y \sin (\phi - \frac{\pi}{3}))
- w^2(x \cos (\phi + \frac{\pi}{3}) - y \sin (\phi + \frac{\pi}{3})) = 0
\]

Clearly this is a third degree equation. The genus of this curve is either 1 if \( F_\phi \) is nonsingular, or it is genus 0, meaning the curve has one 2-fold singularity. We have no other cases as genus cannot be negative. Rather than determining whether \( F_\phi \) is nonsingular or not by brute force (which will work, but is very laborious), there is a better way.

First note that the transformation \( z \mapsto e^{i\phi}z \) is a linear fractional transformation, since it can be written as

\[
T(z) = \frac{e^{i\phi}z + 0}{0z + 1}
\]

From Lemma 1 on page 9, \( T(z) \) is a birational transformation. By Lemma 2, page 9, birational transformations preserve the genus of a curve. Since the genus of \( F \) is 1, it follows that the genus of \( F_\phi \) is also 1. Therefore we conclude that \( F_\phi \) is a nonsingular cubic. Recapturing the affine curve at \( w = 1 \), it follows that the curve \( f_\phi \) is nonsingular.

Let us now study what happens to the curve \( f \) under more general hyperbolic transformations.
CHAPTER SEVEN
HYPERBOLIC TRANSFORMATIONS

In order to study the general locus of centroids, recall that figures in the hyperbolic plane transform under Möbius transformations. (Recall Möbius transformations [2] are functions $M : \mathbb{C} \mapsto \mathbb{C}$ of the form $M(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.) If a figure $H$ is represented as zeros of a real polynomial in $x, y$ then there is a Hermitian matrix associated with the figure $H$. (Recall a Hermitian matrix is one where the matrix is equal to its conjugate-transpose.) Note that 2 x 2 Hermitian matrices that have a negative determinant can be used to represent real circles via the equation

$$ZH^*Z = 0,$$

where $H$ is the matrix associated with the curve, $Z = (z \ 1)$, and $Z^*$ is the conjugate transpose of $Z$.

If $H$ is a 3 x 3 Hermitian matrix, then we can represent polynomial curves of degree 3 and 4 by setting $Z = (z^2 \ z \ 1)$. For example, $f = 0$ takes the form

$$\begin{pmatrix} z^2 & z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 - \sqrt{3}i & 0 \\ 1 + \sqrt{3}i & 0 & -1 - \sqrt{3}i \\ 0 & -1 + \sqrt{3}i & 0 \end{pmatrix} \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix} = 0$$

To show this, by simplifying the above, we get

$$\begin{pmatrix} (1 + \sqrt{3}i)z^2 & (1 - \sqrt{3}i)z^2 + 1(-1 + \sqrt{3}i) & (-1 - \sqrt{3}iz) \end{pmatrix} \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix} = 0$$
\( 0 = (1 + \sqrt{3}i)zz' + (1 - \sqrt{3}i)z^2z' - (1 - \sqrt{3}i)\bar{z} - (1 + \sqrt{3}i)z \)
\[= z\bar{z}[(1 + \sqrt{3}i)\bar{z} + (1 - \sqrt{3}i)z] - 1[(1 - \sqrt{3}i)\bar{z} + (1 + \sqrt{3}i)z] \]
\[= (x^2 + y^2)(2x - 2\sqrt{3}i^2y) - (2x + 2\sqrt{3}i^2y) \]
\[. \quad (x^2 + y^2)(x + \sqrt{3}) - (x - \sqrt{3}y) = 0, \text{ which is the curve } f = 0. \]

The transformation of \( H \) by the Möbius transformation \( M \) is obtained via spin action, which for higher degree curves requires a compatible matrix representation of \( M \). The new curve is then represented by the Hermitian matrix

\[ \hat{H} = M_{co}HM_{co}^*, \]

where \( M_{co} \) is the cofactor matrix, and \( M_{co}^* \) is the transpose-conjugate.

(Remember that a cofactor matrix can viewed as the transpose of its adjoint; it is more properly seen as the matrix first formed in the process of finding the adjoint.) Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). We can envision the transformation as

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} (\text{using } \cong \text{ to denote projective equivalence})
\]

So \( z \mapsto \frac{az + b}{cz + d} \). It follows then that \( z^2 \mapsto \left(\frac{az + b}{cz + d}\right)^2 \). So if we use this idea to
create a suitable matrix for a higher degree curve, we have

\[
\begin{bmatrix}
(a^2 + b^2) \\
(c^2 + d^2)
\end{bmatrix} = \begin{bmatrix}
(a + b)^2 \\
(a + b)(c + d)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(a + b)^2 \\
(c + d)^2 \\
(a + b)(c + d)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(a + b)^2 + 2abz + b^2 \\
ac^2z^2 + (ad + bc)z + bd \\
c^2z^2 + 2cd + d^2
\end{bmatrix}
\]

\[
\Rightarrow M(z) = \begin{bmatrix}
a^2 & 2ab & b^2 \\
ac & (ad + bc) & bd \\
c^2 & 2cd & d^2
\end{bmatrix}
\]

Therefore the associated compatible 3 x 3 matrix is

\[
M = \begin{bmatrix}
a^2 & 2ab & b^2 \\
ac & (ad + bc) & bd \\
c^2 & 2cd & d^2
\end{bmatrix}
\]

Regarding the cofactor matrix \( M_{co} \), the cofactor matrix for the 2 x 2 matrix

\[
M = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

is

\[
M_{co} = \begin{bmatrix}
d & -c \\
-b & a
\end{bmatrix}
\]

In the 2 x 2 case, each entry is a result of a 1 x 1 determinant. For the 3 x 3 case, the cofactor matrix of

\[
\begin{bmatrix}
a^2 & 2ab & b^2 \\
ac & (ad + bc) & bd \\
c^2 & 2cd & d^2
\end{bmatrix}
\]

is
With these formulas, we can find the image of curves of degree less than or equal to four under an LFT (linear fractional transformation). An equation of the form (30) is used for curves of degree 1 or 2, while one of the form (31) shall be used for curves of degree 3 or 4. (For higher degree curves, the method of constructing a suitable $M$ is done similarly by accounting for even higher powers of $z$.)

Let us find the image of $f = 0$ under the transformation $z \mapsto e^{i\phi}z$. The image will be $Z\bar{H}Z^* = 0$, where $\bar{H} = M_{co}HM_{co}^*$. Here $H$ is the Hermitian representation of the original curve and $M$ is the suitable matrix associated with the LFT. From previous work, the matrix $H$ associated with the curve $f = 0$ is

$$H = \begin{pmatrix}
0 & 1 - \sqrt{3}i & 0 \\
1 + \sqrt{3}i & 0 & -1 - \sqrt{3}i \\
0 & -1 + \sqrt{3}i & 0
\end{pmatrix}$$

$$= \begin{pmatrix}
0 & e^{-\frac{\pi}{3}i} & 0 \\
e^{\frac{\pi}{3}i} & 0 & -e^{\frac{\pi}{3}i} \\
0 & -e^{-\frac{\pi}{3}i} & 0
\end{pmatrix}$$

The LFT in this case is

$$M(z) = \frac{az + b}{cz + d} = \frac{e^{i\phi}z + 0}{0z + 1}$$
Therefore:

\[ a = e^{i\phi} \]
\[ b = 0 \]
\[ c = 0 \]
\[ d = 1 \]

and the associated $3 \times 3$ matrix $M$ is (upon simplification)

\[
M = \begin{bmatrix}
e^{2i\phi} & 0 & 0 \\
0 & e^{i\phi} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Applying equation 31, we obtain

\[
M_{co} = \begin{bmatrix}
e^{i\phi} & 0 & 0 \\
0 & e^{2i\phi} & 0 \\
0 & 0 & e^{3i\phi}
\end{bmatrix}
\]

It follows then that

\[
M_{co}^* = \begin{bmatrix}
e^{-i\phi} & 0 & 0 \\
0 & e^{-2i\phi} & 0 \\
0 & 0 & e^{-3i\phi}
\end{bmatrix}
\]

Since $\tilde{H} = M_{co}HM_{co}^*$, then

\[
\tilde{H} = \begin{bmatrix}
e^{i\phi} & 0 & 0 \\
0 & e^{2i\phi} & 0 \\
0 & 0 & e^{3i\phi}
\end{bmatrix} \begin{bmatrix}
e^{-\frac{2}{3}i} & 0 & 0 \\
e^{\frac{5}{3}i} & 0 & \frac{8}{3}i \\
0 & -e^{-\frac{2}{3}i} & 0
\end{bmatrix} \begin{bmatrix}
e^{-i\phi} & 0 & 0 \\
0 & e^{-2i\phi} & 0 \\
0 & 0 & e^{-3i\phi}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & e^{-\frac{2}{3}i}e^{-i\phi} & 0 \\
e^{\frac{5}{3}i}e^{i\phi} & 0 & -e^{\frac{5}{3}i}e^{-i\phi} \\
0 & -e^{-\frac{2}{3}i}e^{i\phi} & 0
\end{bmatrix}
\]
The image of \( f \), \( f_\phi \), is given by \( \bar{Z}HZ^* = 0 \), and becomes

\[
\begin{bmatrix}
  z^2 \\
  z \\
  1
\end{bmatrix}
\begin{bmatrix}
  0 & e^{-\frac{\pi}{3}i}e^{-i\phi} & 0 \\
  e^{\frac{\pi}{3}i}e^{i\phi} & 0 & -e^{\frac{\pi}{3}i}e^{-i\phi} \\
  0 & -e^{-\frac{\pi}{3}i}e^{i\phi} & 0
\end{bmatrix}
\begin{bmatrix}
  z^2 \\
  z \\
  1
\end{bmatrix} = 0
\]

Simplifying this equation and using the fact that \( z = x + iy \) and 
\( e^{i\phi} = \cos \phi + i \sin \phi \), the above simplifies to

\[
(x^2 + y^2)(x \cos (\phi + \frac{\pi}{3}) + y \sin (\phi + \frac{\pi}{3})) - (x \cos (\phi - \frac{\pi}{3}) + y \sin (\phi - \frac{\pi}{3})) = 0,
\]

and this is comparable to the equation derived for \( f_\phi \) in the previous chapter. The curve

\[
x^2 + y^2 = \frac{x \cos (\phi - \frac{\pi}{3}) + y \sin (\phi - \frac{\pi}{3})}{x \cos (\phi + \frac{\pi}{3}) + y \sin (\phi + \frac{\pi}{3})}
\]

(32)
is the image of the curve

\[
x^2 + y^2 = \frac{x - \sqrt{3}y}{x + \sqrt{3}y}
\]

under the transformation \( z \mapsto e^{i\phi} z \). Therefore the derivation of the image is complete.

To reconcile the differences between equations 29 and 32, note that equation 29 is actually the image of the inverse of \( f_\phi \), namely \( f_{-\phi} \). Replacing \( \phi \) with \(-\phi\) in equation 32 and simplifying it with trigonometric identities will result in the two equations being the same.

We wish to know what is the image of \( f \) under an arbitrary direct hyperbolic transformation. From our previous work, this can now be accomplished. Any direct hyperbolic transformation \( T \) can be factored uniquely as \( T = RT_m \), where \( R \) is a rotation about \( z = 0 \), and \( T_m \) is the transformation which sends the point 0 to \( m \). The calculation for \( R \) has been completed by finding \( f_\phi \); therefore we
only need to determine the image of the curve under the transformation $T_m$. Define $T_m$ to be the original $2 \times 2$ matrix associated with the LFT that takes $0$ to $m$.

Let $T_m = \begin{bmatrix} 1 & m \\ \bar{m} & 1 \end{bmatrix}$, where $m = re^{i\theta}$. It follows then that $a = 1$, $b = m$, $c = \bar{m}$, and $d = 1$. Then extending $T_m$ to a suitable matrix $M$, we get

$$M = \begin{bmatrix} 1 & 2m & m^2 \\ \bar{m} & 1 + m\bar{m} & m \\ \bar{m}^2 & 2\bar{m} & 1 \end{bmatrix}$$

$M_{co}$ and $M_{co}^*$ are calculated as follows:

$$M_{co} = (1 - m\bar{m}) \begin{bmatrix} 1 & \bar{m} & \bar{m}^2 \\ \bar{m} & 1 + m\bar{m} & -2\bar{m} \\ m^2 & -m & 1 \end{bmatrix}$$

$$M_{co}^* = (1 - m\bar{m}) \begin{bmatrix} 1 & -2\bar{m} & \bar{m}^2 \\ -m & 1 + m\bar{m} & -\bar{m} \\ m^2 & -2m & 1 \end{bmatrix}$$

We need to calculate $\tilde{H} = M_{co}H M_{co}^*$.

$$\tilde{H} = (1 - m\bar{m})^2 \begin{bmatrix} 1 & \bar{m} & \bar{m}^2 \\ -2m & 1 + m\bar{m} & -2\bar{m} \\ m^2 & -m & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-\frac{\pi}{3}i} & 0 \\ e^{\frac{\pi}{3}i} & 0 & -e^{\frac{\pi}{3}i} \\ 0 & -e^{-\frac{\pi}{3}i} & 0 \end{bmatrix} \begin{bmatrix} 1 & -2\bar{m} & \bar{m}^2 \\ -m & 1 + m\bar{m} & -\bar{m} \\ m^2 & -2m & 1 \end{bmatrix}$$

As eventually we will work with $Z\tilde{H}Z^* = 0$, we can divide the $(1 - m\bar{m})^2$ factor out. (This is allowed since $|m| < 1$.) Then $\tilde{H}$ evaluates to

$$\tilde{H} = \begin{bmatrix} A & B & C \\ D & E & G \\ I & J & K \end{bmatrix}$$

, where
\[ A = -\overline{m}e^{\frac{5}{3}i} - me^{-\frac{5}{3}i}(1 - \overline{m}^2) + m^2\overline{m}e^{\frac{5}{3}i} \]
\[ B = 2\overline{m}^2e^{\frac{5}{3}i} + (1 + \overline{m}\overline{m})(1 - \overline{m}^2)e^{-\frac{5}{3}i} - 2m\overline{m}e^{\frac{5}{3}i} \]
\[ C = -\overline{m}^2e^{\frac{5}{3}i} - \overline{m}(1 - m\overline{m}^2)e^{-\frac{5}{3}i} + \overline{m}e^{\frac{5}{3}i} \]
\[ D = (1 + m\overline{m})e^{\frac{5}{3}i} - 2me^{-\frac{5}{3}i}(\overline{m} - m) - m^2(1 + m\overline{m})e^{\frac{5}{3}i} \]
\[ E = -2\overline{m}(1 + m\overline{m})e^{\frac{5}{3}i} + 2(1 + m\overline{m})e^{-\frac{5}{3}i}(\overline{m} - m) + 2m(1 + m\overline{m})e^{\frac{5}{3}i} \]
\[ G = (1 + m\overline{m})e^{\frac{5}{3}i}m^2 - 2me^{-\frac{5}{3}i}(\overline{m} - m) - (1 - m\overline{m})e^{\frac{5}{3}i} \]
\[ I = -me^{\frac{5}{3}i} - me^{-\frac{5}{3}i}(m^2 - 1) + m^3e^{\frac{5}{3}i} \]
\[ J = 2m\overline{m}e^{\frac{5}{3}i} + (1 + m\overline{m})(m^2 - 1)e^{-\frac{5}{3}i} - 2m^2e^{\frac{5}{3}i} \]
\[ K = -m\overline{m}^2e^{\frac{5}{3}i} - \overline{m}e^{-\frac{5}{3}i}(m^2 - 1) + me^{\frac{5}{3}i} \]

Evaluating \( ZHZ^* = 0 \) we get

\[
0 = \begin{bmatrix} z^2 & z \\ I & J & K \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & G \\ I & J & K \end{bmatrix} \begin{bmatrix} \overline{z}^2 \\ \overline{z} \\ 1 \end{bmatrix} 
\]
\[
= (Az^2 + Dz + I)\overline{z}^2 + (Bz^2 + Ez + J)\overline{z} + (Cz^2 + Gz + K) 
\]
\[
= A(x^2 + y^2) + (B + D)(x^3 + xy^2) + i(B - D)(x^2y + y^3) + (C + E + I)x^2 
\]
\[
+ 2i(C - I)xy + (-C + E - I)y^2 + (G + J)x + i(G - J)y + K 
\]

In the above equation, the fact that \( z = x + iy \) was used. Therefore,

\[
f_m(x, y) = A(x^2 + y^2) + (B + D)(x^3 + xy^2) + i(B - D)(x^2y + y^3) + (C + E + I)x^2 
\]
\[
+ 2i(C - I)xy + (-C + E - I)y^2 + (G + J)x + i(G - J)y + K = 0 
\]

At this point, the definition that \( m = re^{i\theta} \) will be used, and the complex exponents will be evaluated in trigonometric form. Simplifying each coefficient of the
preceding equation yields the following:

\[ A = r[(r^2 - 1) \cos \theta - \sqrt{3}(r^2 + 1) \sin \theta] \]

\[ B + D = (1 - r^2) + r^2[(1 - r^2) \cos 2\theta + \sqrt{3}(3 + r^2) \sin 2\theta] \]

\[ i(B - D) = \sqrt{3}(1 + 6r^2 + r^4) - 2\sqrt{3}r^2(3 + r^2) \cos^2 \theta + r^2(1 - r^2) \sin 2\theta \]

\[ C + E + I = -2\sqrt{3}r[\sin \theta(2r^2 \cos 2\theta + 3r^2 + 1)] \]

\[ = -2\sqrt{3}r[\sin \theta(1 + r^2 + 4r^2 \cos^2 \theta)] \]

\[ 2i(C - I) = 4\sqrt{3}r[2 \cos \theta(1 + r^2 - 2r^2 \cos 2\theta)] \]

\[ = 4\sqrt{3}r[2 \cos \theta(1 + 3r^2 - 4r^2 \cos^2 \theta)] \]

\[ -C + E - I = -2\sqrt{3}r \sin \theta(3 - r^2 - 2r^2 \cos 2\theta) \]

\[ = -2\sqrt{3}r \sin \theta(3(1 + r^2) - 4r^2 \cos^2 \theta) \]

\[ G + J = (r^2 - 1)(1 + r^2 \cos 2\theta) + \sqrt{3}r^2(r^2 + 3) \sin 2\theta \]

\[ = (r^2 - 1)(1 - r^2 + 2r^2 \cos^2 \theta) + \sqrt{3}r^2(r^2 + 3) \sin 2\theta \]

\[ i(G - J) = \sqrt{3}(1 + 3r^2 - r^2(3 + r^2) \cos 2\theta) - r^2(1 - r^2) \sin 2\theta \]

\[ = \sqrt{3}(1 + 6r^2 + r^4 - 2r^2(3 + r^2) \cos^2 \theta) - r^2(1 - r^2) \sin 2\theta \]

\[ K = r[(1 - r^2) \cos \theta - \sqrt{3}(1 + r^2) \sin \theta] \]

From the above it follows that the image of \( f = 0 \) under \( T_m \) is the curve \( f_m = 0, \)
where

\[ 0 = f_m(x, y) = r[(r^2 - 1) \cos \theta - \sqrt{3}(r^2 + 1) \sin \theta](x^2 + y^2)^2 
+ ((1 - r^2) + r^2[(1 - r^2) \cos 2\theta + \sqrt{3}(3 + r^2) \sin 2\theta](x^3 + xy^2) 
+ \sqrt{3}[(1 + 6r^2 + r^4) - 2r^2(3 + r^2) \cos^2 \theta + r^2(1 - r^2) \sin 2\theta](x^2y + y^3) 
- 2\sqrt{3}r[\sin \theta(1 + r^2 + 4r^2 \cos^2 \theta)]x^2 
+ 4\sqrt{3}r[2 \cos \theta(1 + 3r^2 - 4r^2 \cos^2 \theta)]xy 
- 2\sqrt{3}r[\sin \theta(3(1 + r^2) - 4r^2 \cos^2 \theta)]y^2 
+ ((r^2 - 1)(1 - r^2 + 2r^2 \cos^2 \theta) + \sqrt{3}r^2(r^2 + 3) \sin 2\theta)x 
+ (\sqrt{3}(1 + 6r^2 + r^4 - 2r^2(3 + r^2) \cos^2 \theta) - r^2(1 - r^2) \sin 2\theta)y 
+ (r[(1 - r^2) \cos \theta - \sqrt{3}(1 + r^2) \sin \theta]). \] (33)

It will now be shown that \( f_m \) is a cubic if and only if \( m \in \varepsilon^* \), and that this will induce the relations

\[ \sin^2 \theta = \frac{1}{4R}(1 - r^2)^2, \] (34)

\[ \cos^2 \theta = \frac{3}{4R}(1 + r^2)^2, \] (35)

where \( R = 1 + r^2 + r^4 \). In this case, \( \theta \in (-\frac{\pi}{6}, 0) \) or \( \theta \in (\frac{5\pi}{6}, \pi) \).

To start, note that \( \varepsilon^* \) is represented by the curve

\[ x^2 + y^2 = \frac{x + \sqrt{3}y}{x - \sqrt{3}y} \] (36)

Since \( m = re^{i\theta} \), it follows that \( x = r \cos \theta, y = r \sin \theta \). Therefore in polar form, equation 36 simplifies to

\[ r^2 = \frac{\cos \theta + \sqrt{3} \sin \theta}{\cos \theta - \sqrt{3} \sin \theta} \]

\[ \Rightarrow 0 = \cos \theta(1 - r^2) + \sqrt{3}(1 + r^2) \sin \theta \]

\[ = \cos \theta(r^2 - 1) - \sqrt{3}(1 + r^2) \sin \theta, \]
which is essentially the coefficient of \((x^2 + y^2)^2\) in equation 33. So the path of the cubic \(e^*\) is also the condition for which the coefficient of the quartic term of \(f_m\) is necessarily zero, when viewed in context of equation 33. To get the trigonometric relations in equations 34 and 35, we start with the path equation

\[
\cos \theta (r^2 - 1) - \sqrt{3}(1 + r^2) \sin \theta = 0.
\]

Then

\[
\cos \theta (r^2 - 1) - \sqrt{3}(1 + r^2) \sin \theta = 0
\]

\[
\Rightarrow \frac{\sqrt{3}(1 + r^2) \sin \theta}{(r^2 - 1) \cos \theta} = 1
\]

From equation 37, we obtain

\[
\frac{\cos \theta}{\sin \theta} = \frac{\sqrt{3}(1 + r^2)}{r^2 - 1}
\]

\[
\cot^2 \theta = \frac{3(1 + r^2)^2}{(r^2 - 1)^2}
\]

\[
1 + \cot^2 \theta = 1 + \frac{3(1 + r^2)^2}{(1 - r^2)^2}
\]

\[
\csc^2 \theta = \frac{4(1 + r^2 + r^4)}{(1 - r^2)^2}
\]

\[
\sin^2 \theta = \frac{(1 - r^2)^2}{4(1 + r^2 + r^4)}
\]

Also, by a similar process, we get \(\cos^2 \theta\). Applying equation 37, we have

\[
\frac{\sin \theta}{\cos \theta} = \frac{r^2 - 1}{\sqrt{3}(1 + r^2)}
\]

\[
\tan^2 \theta = \frac{(r^2 - 1)^2}{3(1 + r^2)^2}
\]

\[
1 + \tan^2 \theta = 1 + \frac{(r^2 - 1)^2}{3(1 + r^2)^2}
\]

\[
\sec^2 \theta = \frac{4(1 + r^2 + r^4)}{3(1 + r^2)^2}
\]

\[
\cos^2 \theta = \frac{3(1 + r^2)^2}{4(1 + r^2 + r^4)}
\]

Letting \(R = 1 + r^2 + r^4\) leads to the desired result obtained in equations 34 and 35.
When \( m \in \mathbb{R}^* \), the angle \( \theta \) is in the quadrants II and IV, and constrained to the angles \((\frac{5\pi}{6}, \pi)\) and \((-\frac{\pi}{6}, 0)\), respectively. (Note that this is a result derived from the condition of the original locus of centroids; the angle bisector can only reach a maximum of \( \frac{\pi}{6} \) in quadrant I.) The formulas for the cosine and sine are

\[
\cos \theta = \sqrt{\frac{3}{R} \left(1 + r^2\right)}, \quad \sin \theta = -\sqrt{\frac{1}{R} \left(1 - r^2\right)}, \quad \text{when } \theta \in \left(-\frac{\pi}{6}, 0\right)
\]
\[
\cos \theta = -\sqrt{\frac{3}{R} \left(1 + r^2\right)}, \quad \sin \theta = \sqrt{\frac{1}{R} \left(1 - r^2\right)}, \quad \text{when } \theta \in \left(\frac{5\pi}{6}, \pi\right).
\]

Then

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{1 + 4r^2 + r^4}{2R}, \quad \text{and}
\]
\[
\sin 2\theta = 2 \sin \theta \cos \theta = -\frac{\sqrt{3}(1 - r^4)}{2R}.
\]

Lastly, if \( m \in \mathbb{R}^* \), then all the preceding trigonometric relations hold and equation 33 simplifies to the following:

\[
r[(r^2 - 1) \cos \theta - \sqrt{3}(r^2 + 1) \sin \theta](x^2 + y^2)^2
\]
\[= 0(x^2 + y^2)^2
\]
\[=((1 - r^2) + r^2[(1 - r^2) \cos 2\theta + \sqrt{3}(3 + r^2) \sin 2\theta]) (x^3 + xy^2)
\]
\[= \frac{r^2 - 1}{R}(r^6 + 3r^4 + 3r^2 - 1)(x^3 + xy^2)
\]
\[= \frac{\sqrt{3}[(1 + 6r^2 + r^4) - 2r^2(3 + r^2) \cos^2 \theta] + r^2(1 - r^2) \sin 2\theta)(x^2y + y^3)
\]
\[= -\frac{r^2 - 1}{R} \sqrt{3}(r^6 + r^4 + 3r^2 + 1)(x^2y + y^3)
\]
\[= -2\sqrt{3}r[\sin \theta(1 + r^2 + 4r^2 \cos^2 \theta)]x^2
\]
\[= \frac{r^2 - 1}{R} \sqrt{3}r(r^2 + 1)(2r^2 + 1)^2x^2, \quad \text{where}
\]
\[\theta \in \left(-\frac{\pi}{6}, 0\right), \text{or } \theta \in \left(\frac{5\pi}{6}, \pi\right), \text{respectively.}
\]

\[52\]
\[
4\sqrt{3}r[2\cos \theta(1 + 3r^2 - 4r^2 \cos^2 \theta)]xy
\]
\[
= \pm \frac{r^2 - 1}{R}\sqrt{\frac{3}{R}}2\sqrt{3}r(1 + r^2)(1 + 2r^2)xy, \text{ where}
\]
\[
\theta \in \left(-\frac{\pi}{6}, 0\right), \text{ or } \theta \in \left(\frac{5\pi}{6}, \pi\right), \text{ respectively.}
\]
\[
-2\sqrt{3}r[\sin \theta(3(1 + r^2) - 4r^2 \cos^2 \theta)]y^2
\]
\[
= \pm \frac{r^2 - 1}{R}\sqrt{\frac{3}{R}}3r(r^2 + 1)y^2, \text{ where}
\]
\[
\theta \in \left(-\frac{\pi}{6}, 0\right), \text{ or } \theta \in \left(\frac{5\pi}{6}, \pi\right), \text{ respectively.}
\]
\[
((r^2 - 1)(1 - r^2 + 2r^2 \cos^2 \theta) + \sqrt{3}r^2(r^2 + 3) \sin 2\theta)x
\]
\[
= \frac{r^2 - 1}{R}(2r^2 + 1)(r^4 + 4r^2 + 1)x
\]
\[
(\sqrt{3}(1 + 6r^2 + r^4 - 2r^2(3 + r^2) \cos^2 \theta) - r^2(1 - r^2) \sin 2\theta)y
\]
\[
= -\frac{r^2 - 1}{R}\sqrt{\frac{3}{R}}(r^4 + 4r^2 + 1)y
\]
\[
(r[(1 - r^2) \cos \theta - \sqrt{3}(1 + r^2) \sin \theta])
\]
\[
= \pm \frac{r^2 - 1}{R}\sqrt{\frac{3}{R}}r(r^2 + 1)R, \text{ where}
\]
\[
\theta \in \left(-\frac{\pi}{6}, 0\right), \text{ or } \theta \in \left(\frac{5\pi}{6}, \pi\right), \text{ respectively.}
\]

Multiplying the equation of the simplified \( f_m(x, y) \) through by \(-\frac{R}{r^2 - 1}\), leads to the following equation for \( f_m(x, y) \), when \( m \in \varepsilon^* \):

\[
f_m(x, y) = (1 - 3r^2 - 3r^4 - r^6)(x^3 + xy^2)
\]
\[
+ \sqrt{3}(1 + 3r^2 + r^4 + r^6)(x^2y + y^3)
\]
\[
- (1 + 4r^2 + r^4)[(1 + 2r^2)x - \sqrt{3}y]
\]
\[
\pm \sqrt{\frac{3}{R}}r(1 + r^2)[(1 + 2r^2)^2x^2 - 2\sqrt{3}(1 + 2r^2)xy + 3y^2 + R],
\]

depending if \( \theta \in \left(-\frac{\pi}{6}, 0\right) \) or \( \theta \in \left(\frac{5\pi}{6}, \pi\right) \), respectively. Examples of cubics generated under various values of \( r \) in equation 38 are given as Diagrams 7 and 8 on pages 89.
and 90. Diagram 7 is rendered through Maple’s *implicitplot* command, and uses the Cartesian form of the graph. Diagram 8 was created by solving for all points that satisfy a given $r$ and $\theta$; a polar form of the graph was used.

By an argument similar to one given for the cubic $f_\phi$ in the previous chapter, the cubic $f_m$ is nonsingular. As an additional note, if $f_m$ is quartic, it necessarily follows that the curve will have two ordinary 2-fold singularities, due to the fact that genus is preserved under LFTs, and because $f_m$ follows the formula given in equation 7.

We now move on to studying the general case of the formation of the locus of centroids of equilateral triangles using a general vertex $m$, since up until now we have been using 0 as the vertex for all the equilateral triangles. This will lead to interesting results.
CHAPTER EIGHT
ADMISSIBLE LINES

Here we shall study the locus of centroids for a general vertex $m$. It will turn out that only under certain conditions will the path of centroids produce a cubic curve. There are many d-lines that pass through a generic point $m$ in $H^2$. It will be shown that there is only one d-line through $m$ that is associated with a locus of centroids that is cubic.

To do this, note that as we follow the orbit of $m$ about the point $z = 0$ (the center of the unit disk), it will cross through the curve $\varepsilon^*$ at exactly two points. Another way to view this is any circle in $H^2$ centered at 0 will intersect the curve $\varepsilon^*$ at exactly two points. Let $q$ be one of these points; it follows then that $F_q = 0$ is a nonsingular cubic, since $q \in \varepsilon^*$. Let $L_q$ be the image of the real axis under a linear fractional transformation $T_q$. Now rotate $L_q$ about 0 so that $q \mapsto m$. Then $L_q$ will become the line such that the locus of centroids with the vertex $m$ is the nonsingular cubic obtained by the rotation of the curve $F_q = 0$ an angle $\phi$ about 0. This can be denoted $R_{0,\phi}T_{me^{-i\phi}}$, since $T_{me^{-i\phi}}$ describes taking the line to $q$, while $R_{0,\phi}$ rotates the line to the correct position at $m$.

To see this line is unique, let us focus on $L_q$. All lines through $q$ can be obtained as hyperbolic rotations of $L_q$ about $q$. The only rotations about $q$ that transform the cubic $F_q = 0$ into a cubic again are the identity and half-turn, since $F_q = 0$ is invariant under the half-turn about $q$. Let us derive a formula for the hyperbolic rotation of $\phi$ about a general point $m$.

We will prove that the hyperbolic rotations about $m$ are the
transformations of the form

\[
R_{m,\phi} = \begin{bmatrix}
  e^{i\phi} - |m|^2 & m(1 - e^{i\phi}) \\
\frac{m(e^{i\phi} - 1)}{1 - |m|^2 e^{-i\phi}} & 1 - |m|^2 e^{-i\phi}
\end{bmatrix}
\] (39)

To achieve a rotation of \( \phi \) about \( m \), it can be viewed in three steps. First, take \( m \) into 0. Second, rotate about 0 by an angle of \( \phi \). Third, take the rotated curve about 0 back to \( m \). As angles are preserved under hyperbolic transformations, the net effect is the rotation of an angle \( \phi \) through the point \( m \).

Building the transformation, we have:

\[
0 \text{ to } m : T(z) = \frac{1z + m}{mz + 1}
\]

\[
m \text{ to } 0 : T^{-1}(z) = \frac{1z - m}{-mz + 1}
\]

\[
\phi \text{ about } 0 : R(z) = \frac{e^{i\phi} + 0}{0z + 1}
\]

Therefore \( R_{m,\phi} = T(z) \circ R(z) \circ T^{-1}(z) = TRT^{-1}(z) \), which has the associated matrix representation

\[
R_{m,\phi} = \begin{bmatrix}
  1 & m \\
\frac{m}{1} & 1
\end{bmatrix}
\begin{bmatrix}
  e^{i\phi} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & -m \\
-m & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  e^{i\phi} & m \\
\frac{m}{e^{i\phi}} & 1
\end{bmatrix}
\begin{bmatrix}
  1 & -m \\
-m & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  e^{i\phi} - |m|^2 & m(1 - e^{i\phi}) \\
\frac{m(e^{i\phi} - 1)}{1 - |m|^2 e^{-i\phi}} & 1 - |m|^2 e^{-i\phi}
\end{bmatrix}
\]

With equation 39 derived, it can shown that \( R_{m,\phi}T_m = R_{0,\phi}T_{me^{-i\phi}} \).

\[
R_{m,\phi}T_m = \begin{bmatrix}
  e^{i\phi} - |m|^2 & m(1 - e^{i\phi}) \\
\frac{m(e^{i\phi} - 1)}{1 - |m|^2 e^{-i\phi}} & 1 - |m|^2 e^{-i\phi}
\end{bmatrix}
\begin{bmatrix}
  1 & m \\
\frac{m}{1} & 1
\end{bmatrix}
= \begin{bmatrix}
  e^{i\phi} & m \\
\frac{m}{e^{i\phi}} & 1
\end{bmatrix}
\]

and

\[
R_{0,\phi}T_{me^{-i\phi}} = \begin{bmatrix}
  e^{i\phi} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & me^{-i\phi} \\
\frac{me^{i\phi}}{1} & 1
\end{bmatrix}
= \begin{bmatrix}
  e^{i\phi} & m \\
\frac{me^{i\phi}}{1} & 1
\end{bmatrix}
\]
Recall that $R_{0,\phi}T_{m e^{-i\phi}}$ mapped the real axis in such a way that the locus of centroids passing through the vertex $m$ is a cubic. $T_{m e^{-i\phi}}$ maps 0 to a point $q \in \varepsilon^*$, and thus the locus of centroids will be a cubic. Then $R_{0,\phi}$ rotates the point $q$ about 0 by $\phi$ so that it maps to $m$; rotation of a cubic about 0 results in another cubic. (It is assumed by the construction that $m$ and $q$ are separated by an angle $\phi$ with respect to 0.) $R_{0,\phi}T_{m e^{-i\phi}}$ is unique since it is a composition of invertible linear fractional transformations. But $R_{m,\phi'}T_m = R_{0,\phi}T_{m e^{-i\phi}}$ only if $\phi' = \phi$.

Therefore there is a unique line through the vertex $m \neq 0$ such that the locus of centroids is a cubic. Diagram 9 on page 91 gives an illustration of this unique line.

If $m \neq 0$, let $L_m$ denote this unique line, and we will refer to it as the axis of the cubic locus. In the case where $m = 0$, every radial line is an axis of a cubic locus, and such a cubic will be referred to as a radial locus; its reflection in the axis will be its radial conjugate.

If $m \in \varepsilon^*$, then it shall be proven that the Hermitian matrix that represents this line is

$$L_m = \begin{pmatrix} 2r\sqrt{R} & \pm(1 + r^2)[\sqrt{3}r^2 + i(2 + r^2)] \\ \pm(1 + r^2)[\sqrt{3}r^2 - i(2 + r^2)] & 2r\sqrt{R} \end{pmatrix}, \quad (40)$$

for $m$ in the second and fourth quadrant, respectively.

To show this, note that the axis of the curve $\varepsilon$, namely the line $y = 0$, can be represented in the complex form as $i(z - \overline{z}) = 0$. The Hermitian matrix associated with this line is

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$
Let $M = T_m$, where $T_m$ is the arbitrary direct transformation as defined on page 47. Then the matrix representation of the image is

$$L_m = \tilde{H} = M_{co}HM_{co}^*$$

where

$$L_m = \begin{pmatrix}
1 -m & 0 & i
-m & 1 & 0
-i & 0 & 1
\end{pmatrix}$$

Since $m = re^{i\theta}$, the entries can be simplified as follows:

$$i(\overline{m} - m) = 2r \sin \theta$$

$$i(1 - \overline{m}^2) = -r^2 \sin 2\theta + i(1 - r^2 \cos 2\theta)$$

$$i(-1 + m^2) = -r^2 \sin 2\theta - i(1 - r^2 \cos 2\theta)$$

Since $m \in \mathbb{C}^*$, these results can simplified using relations derived from equations 34 and 35 on page 50. Therefore we have the relations

$$2r \sin \theta = \pm \frac{r(1 - r^2)}{\sqrt{R}}$$

$$-r^2 \sin 2\theta = \frac{\sqrt{3}r^2(1 + r^2)(1 - r^2)}{2R}$$

$$1 - r^2 \cos 2\theta = \frac{(1 - r^2)(1 + r^2)(2 + r^2)}{2R}$$

Using these results yields the following:

$$L_m = \pm \frac{r(1 - r^2)}{\sqrt{R}} \frac{(1 + r^2)(1 - r^2)(\sqrt{3}r^2 + i(2 + r^2))}{2R} \frac{2R}{r(1 - r^2)}$$
Multiplying through by $\pm 2R$, and factoring out $(1 - r^2)$ results in

\[
L_m = \begin{pmatrix}
2r\sqrt{R} & \pm(1 + r^2)(\sqrt{3}r^2 + i(2 + r^2)) \\
\pm(1 + r^2)(\sqrt{3}r^2 - i(2 + r^2)) & 2r\sqrt{R}
\end{pmatrix}.
\]

Therefore the claim in equation 40 has been verified.

Let $\gamma$ represent the center of the circle afforded by $L_m$, and let $\rho$ be its radius. To begin with, it will be shown that $|\gamma| = r + \frac{1}{r^2} > 2$, and $\rho = \frac{1}{r}\sqrt{R}$.

Recall that Hermitian matrices that represent circles in $H^2$ are of the form

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix},
\]

and the equations for the center and radius [6] are given by

\[
\gamma = \frac{B}{A} \quad \text{and} \quad \rho = \sqrt{\frac{(A^2 - BB)}{-A^2}}.
\]

As $L_m$ is the Hermitian matrix, the radius $\rho$ is

\[
\rho = \sqrt{\frac{4R(r^2 - (1 + r^2)^2)}{-4r^2R}}
= \sqrt{\frac{1 + r^2 + r^4}{r^2}}
= \sqrt{\frac{R}{r^2}}
= \frac{1}{r}\sqrt{R}
\]

The center of the circle that affords $L_m$, $\gamma$, evaluates to

\[
\gamma = \frac{\pm(1 + r^2)[\sqrt{3}r^2 - i(2 + r^2)]}{2r\sqrt{R}}
\]
Then it follows that

\[ |\gamma| = \frac{(1 + r^2)}{2r\sqrt{R}} \sqrt{(\sqrt{3}r^2)^2 + (2 + r^2)^2} = \frac{2(1 + r^2)\sqrt{1 + r^2 + r^4}}{2r\sqrt{R}} = \frac{2(1 + r^2)(\sqrt{R})}{2r(\sqrt{R})} = \frac{1 + r^2}{r} = \frac{1}{r} + r, \text{ as desired.} \]

To show \(|\gamma| > 2\), it needs to determined at what value the function reaches a minimum.

\[ \frac{d|\gamma|}{dr} = -\frac{1}{r^2} + 1 = 0 \]

\[ \Rightarrow r^2 = 1 \]

\[ \Rightarrow r = 1 \text{ is a critical point. (} \because r > 0 \) \]

As \(-\frac{1}{r^2} + 1\) is decreasing on \([0, 1]\) and increasing on \([1, \infty)\), it follows that \(r = 1\) is a minimum. At \(r = 1, |\gamma| = \frac{1}{1} + 1 = 2\). As the domain of valid \(r\) for an admissible line is \(0 < r < 1\), it follows that \(|\gamma| > 2\), as was claimed.

A line in \(H^2\) is admissible if and only if the radius of the circle that affords it is greater than \(\sqrt{3}\). To show this, what needs to be proven is the following:

*Given: \(\rho = \frac{1}{r} \sqrt{R}\)*

\(r > 0, \rho > 0 \text{ (and thus } \sqrt{R} > 0)\)

*Show: \(|\gamma| = \frac{1}{r} + r > 2 \Leftrightarrow \rho > \sqrt{3}\)*
To prove the forward direction of the bijection, note that

\[
r + \frac{1}{r} > 2
\]

\[
\Rightarrow r^2 + 1 > 2r
\]

\[
\Rightarrow r^2 - 2r + 1 > 0
\]

\[
\Rightarrow (r - 1)^2 > 0
\]

\[
\Rightarrow 0 < r < 1 \text{ or } r > 1
\]

Since \( r > 1 \) is outside the unit disk, it is not applicable. Assuming \( r > 0 \) from the given, we have:

\[
r < 1
\]

\[
\Rightarrow \frac{\sqrt{R}}{\rho} < 1
\]

\[
\Rightarrow \frac{\rho}{\sqrt{R}} > 1
\]

\[
\Rightarrow \rho > \sqrt{R} \quad \forall r \in (0, 1)
\]

\[
\Rightarrow \rho > \sqrt{1 + r^2 + r^4} \quad \forall r \in (0, 1)
\]

Since \( 0 < r < 1 \), it follows that \( 1 < \sqrt{1 + r^2 + r^4} < \sqrt{3} \) by applying the minimum and maximum values of \( r \). For the inequality to be true for all \( r \), it is necessary that \( \rho \) is at least \( \sqrt{3} \). Therefore \( \rho > \sqrt{3} \), as desired.

To show the reverse direction of the bijection, we start with the conditions
\[ \rho > \sqrt{3} \text{ and } \rho = \frac{1}{r} \sqrt{R} \text{ (from equation 41). Then} \]

\[ \rho > \sqrt{3} \]
\[ \Rightarrow \frac{\sqrt{R}}{r} > \sqrt{3} \]
\[ \Rightarrow R > 3r^2 \]
\[ \Rightarrow r^4 - 2r^2 + 1 > 0 \]
\[ \Rightarrow (r + 1)^2(r - 1)^2 > 0 \]

Since \(0 < r < 1 \iff 1 < r + 1 < 2\), \((r + 1)^2\) can be divided out without any significant consequence. Then

\[ (r + 1)^2(r - 1)^2 > 0 \]
\[ \Rightarrow (r - 1)^2 > 0 \]
\[ \Rightarrow r^2 - 2r + 1 > 0 \]
\[ \Rightarrow r^2 + 1 > 2r \]
\[ \Rightarrow r + \frac{1}{r} > 2 \]

Therefore \(r + \frac{1}{r} > 2\), as desired. With that the proof is complete.

For \(m \neq 0\), it will be shown that the rotation about 0 through \(\phi\) transforms \(L_m\) to

\[ L_m' = \begin{pmatrix} 1 & e^{-i\phi} \beta(r) \\ e^{i\phi} \beta(r) & 1 \end{pmatrix}, \quad (43) \]

where \(\beta(r) = \pm \frac{(1 + r^2)}{2r \sqrt{R}} \left[ \sqrt{3} r^2 + i(2 + r^2) \right] \).
From equation 40, \( L_m \) was given to be
\[
L_m = \begin{pmatrix}
2r\sqrt{R} & \pm(1 + r^2)(\sqrt{3}r^2 + i(2 + r^2)) \\
\pm(1 + r^2)(\sqrt{3}r^2 - i(2 + r^2)) & 2r\sqrt{R}
\end{pmatrix}
\frac{1}{2r\sqrt{R}}
\begin{pmatrix}
\pm(1 + r^2)(\sqrt{3}r^2 - i(2 + r^2)) \\
\pm(1 + r^2)(\sqrt{3}r^2 + i(2 + r^2)) \\
2r\sqrt{R} & 1
\end{pmatrix}
\]
as multiples of a Hermitian matrix are equivalent.

Define \( \beta(r) = \frac{\pm(1 + r^2)}{2r\sqrt{R}} [\sqrt{3}r^2 + i(2 + r^2)] \). Then \( L_m \) can be written as
\[
L_m = \begin{pmatrix}
1 & \beta(r) \\
\beta(r) & 1
\end{pmatrix}
\]

A rotation about \( \phi \) is defined by the mapping \( z \mapsto e^{i\phi} \). Its matrix representation is
\[
M = \begin{pmatrix}
e^{i\phi} & 0 \\
0 & 1
\end{pmatrix}
\]
The cofactor matrix of \( M \) and its transpose-conjugate are
\[
M^* = \begin{pmatrix}
e^{i\phi} & 0 \\
0 & e^{-i\phi}
\end{pmatrix}
\]

\( L_m \) is transformed by \( M \) into \( L'_m \) via spin action. Thus:
\[
L'_m = M^*L_mM
\]
\[
= \begin{pmatrix}
1 & 0 \\
0 & e^{i\phi}
\end{pmatrix}
\begin{pmatrix}
1 & \beta(r) \\
\beta(r) & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & e^{-i\phi}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & \beta(r)e^{-i\phi} \\
\beta(r)e^{i\phi} & 1
\end{pmatrix}
\]

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We conclude then that the claim made in equation 43 is valid. Thus:

\[ L'_m = \begin{pmatrix} 1 & e^{-i\phi}\beta(r) \\ e^{i\phi}\beta(r) & 1 \end{pmatrix}, \text{ where } \beta(r) = \pm (1 + r^2) \left[ \sqrt{3} r^2 + i(2 + r^2) \right]. \]

The collection of \( L_m \) and \( L'_m \), together with the radial lines, will be referred to as the admissible lines. Examples of admissible lines are given in Diagram 10 on page 92; notice that these particular ones are associated with the cubics given in Diagram 7 on page 89.

Excluding the radial lines, we can partition the admissible lines into equivalence classes based on the unique radial conjugate containing the vertex \( m \). For example, the \( L_m \) for \( m \in \varepsilon^* \), where \( m \neq 0 \), comprise an equivalence class. For an illustration, refer to Diagram 11 on page 93. \( \varepsilon^* \) can be rotated by \( \phi \) to produce another equivalency class; \( m \in \varepsilon^* \) is the case \( \phi = 0 \). A way to represent a member of this equivalency class is to say

\[ [L_{m\phi_j}] = \begin{pmatrix} 1 & e^{-i\phi_j}\beta(r) \\ e^{i\phi_j}\beta(r) & 1 \end{pmatrix}, \text{ where } 0 \leq \phi_j < 2\pi. \]

Excluding radial lines, another partitioning of admissible lines can be done with respect to \( r \), where lines sharing the same radius \( r \) comprise an equivalency class. An illustration of this is given in Diagram 12 on page 94. The corresponding representation of a member of this equivalency class is

\[ [L_{m\gamma_k}] = \begin{pmatrix} 1 & e^{-i\phi}\beta(r_k) \\ e^{i\phi}\beta(r_k) & 1 \end{pmatrix}, \text{ where } 0 < r_k < 1. \]

The correspondence between \( m \) and \( L_m \) induces a transformation of the open disk to the region \(|z| > 2\) (since \(|\gamma| > 2\)) by sending a point on \( H^2 \) to the center of the circle that affords \( L_m \). It will be shown that this transformation will
send $m$ to its inversion in the circle of radius $\sqrt{1 + mm}$ centered at $0$, followed by rotation about $0$ through $\frac{5\pi}{3}$ (or equivalently stated, through an angle of $-\frac{\pi}{3}$). An example of such a construction can be seen in Diagram 13 on page 95.

To prove the claim, first some notation needs to be given. Let $O$ be the center of a circle $C$. Let $A$ be a point in the circle, and $A'$ be the inversion of point $A$ in $C$. Define $s$ to be the radius of the circle. Let $C$ have center $(0,0)$. Then let the coordinates of $A$ be $(x, y)$. Then $A' = (kx, ky)$ for some constant $k$. From the formula for inversion [2], then $OA \cdot OA' = s^2$.

The particular point $A$ for the current situation is $m = (r \cos \theta, r \sin \theta)$, or equivalently, $m = re^{i\theta}$. The desired radius of this circle is $s = \sqrt{1 + mm}$. The goal is to show that the inversion of $m$ in this circle followed by a rotation of $-\frac{\pi}{3}$ maps $m$ to the center $\gamma$ of $L_m$, namely the same point as given in equation 42 on page 59. First of all,

$$s = \sqrt{1 + mm}$$

$$\Rightarrow s = \sqrt{1 + r^2(\cos^2 \theta + \sin^2 \theta)}$$

$$\Rightarrow s = \sqrt{1 + r^2}$$

Then:

$$OA \cdot OA' = s^2$$

$$OA \cdot OA' = 1 + r^2$$

$$OA^2 \cdot OA'^2 = (1 + r^2)^2$$

$$(x^2 + y^2)((kx)^2 + (ky)^2) = (1 + r^2)^2$$

$$\Rightarrow k^2 = \frac{(1 + r^2)^2}{(x^2 + y^2)^2}$$
As \( A \) and \( A' \) lie in the same direction, we take the positive square root. Therefore:

\[
k = \frac{(1 + r^2)}{x^2 + y^2}
\]

Hence this inversion sends the point \((x, y)\) to the point \((kx, ky)\), where \((kx, ky)\) evaluates to

\[
(kx, ky) = \left( \frac{x(1 + r^2)}{x^2 + y^2}, \frac{y(1 + r^2)}{x^2 + y^2} \right)
\]

Recall that \( x = r \cos \theta, \ y = r \sin \theta \). By further simplification,

\[
(kx, ky) = \frac{r \cos \theta (1 + r^2)}{r^2 (\cos^2 \theta + \sin^2 \theta)}, \frac{r \sin \theta (1 + r^2)}{r^2 (\cos^2 \theta + \sin^2 \theta)}
\]

\[
= \frac{r \cos \theta (1 + r^2)}{r^2}, \frac{r \sin \theta (1 + r^2)}{r^2}
\]

\[
= \frac{1 + r^2}{r^2} (r \cos \theta, r \sin \theta)
\]

Let \( T_1 \) be the image of a point \((x, y)\) under inversion. From the above it follows that

\[
T_1(m) = \frac{(1 + r^2)}{r^2} m.
\]

is the inversion of \( m \in H^2 \) in a circle of radius \( \sqrt{1 + mm} \), where \( m = re^{i\theta} \).

For the rotation through an angle of \( \frac{5\pi}{3} \), it is equivalent to a rotation of an angle of \( -\frac{\pi}{3} \). Let the associated transformation be \( T_2(m) = e^{-\frac{\pi}{3}} m \). So inversion, followed by rotation, is given by

\[
T_2 \circ T_1 (m) = \frac{(1 + r^2)e^{-\frac{\pi}{3}}}{r^2} m
\]
Then:

\[ T_2 \circ T_1(z) = \frac{(1 + r^2)(1 - \sqrt{3}i)}{2r^2} \]

\[ = \left(1 + r^2\right)\left(1 - \sqrt{3}i\right) \cdot \frac{r}{2r^2} \left(\cos \theta + i \sin \theta\right) \]

\[ = \frac{(1 + r^2)}{2r} \left[\left(\cos \theta + \sqrt{3} \sin \theta\right) + i\left(\sin \theta - \sqrt{3} \cos \theta\right)\right] \]

Now using the fact that admissible lines associated with \( L_m \) have \( m \in \varepsilon^* \), the trigonometric relations in equations 34 and 35 are valid and thus

\[ \cos \theta + \sqrt{3} \sin \theta = \pm \left(\frac{\sqrt{3}}{2\sqrt{R}}(1 + r^2) - \frac{\sqrt{3}}{2\sqrt{R}}(1 - r^2)\right) \]

\[ = \pm \frac{\sqrt{3}(r^2)}{\sqrt{R}}, \text{ and} \]

\[ \sin \theta - \sqrt{3} \cos \theta = -(\sqrt{3} \cos \theta - \sin \theta) \]

\[ = (-1) \pm \left(\frac{\sqrt{3}\sqrt{3}(1 + r^2)}{2\sqrt{R}} - \frac{1(1 - r^2)}{2\sqrt{R}}\right) \]

\[ = \mp \left(\frac{2r^2 + 1}{\sqrt{R}}\right) \]

for \( \theta \) in the fourth and second quadrants, respectively. Putting it all together, \( T_2 \circ T_1(z) \) simplifies to

\[ T_2 \circ T_1(z) = \pm \frac{(1 + r^2)}{2r\sqrt{R}} \left[\sqrt{3}r^2 - i(1 + 2r^2)\right] \]

\[ = \gamma, \text{ as desired.} \]

Therefore \( m \) has been sent to the center of the circle that affords \( L_m \).

In summary, \( L_m \) is the unique axis through \( m \) associated with the cubic whose vertex is \( m \). An illustration is given in Diagram 14 on page 96 for \( r = \frac{1}{2} \) and \( m \in \varepsilon^* \). Studying how the original equilateral triangles whose centroids form \( \varepsilon \) are transformed under \( T_m \), it comes as no surprise that the admissible line
coincides with all the bases of the transformed triangles. An illustration of this is given in Diagram 15 on page 97.

With the properties of the admissible lines investigated, we turn our attention to classifying the cubics themselves.
CHAPTER NINE
CLASSIFICATION OF CUBICS

It has been stated that any elliptic curve is birationally equivalent to some nonsingular cubic, from Theorem 4 on page 9. Let us denote the cubic $C$. Through any point $p$ on $C$ there are precisely five lines that are tangents to $C$, including the tangent at $p$ [8]. If $p$ is a flex then the tangent there is counted with multiplicity 2 and hence there are three more tangent lines; in this case we take the tangent at $p$ to be one of the lines in the calculation [8]. From four tangents we can calculate the cross-ratio; recall that a cross-ratio is unique up to $S_3$ symmetry since it depends on how we order the tangents.

We shall show that the cross-ratio of $\varepsilon$ is $[\omega]$, where $\omega$ is the cube root of unity.

To do this, we will use a theorem from algebraic curve theory which states that the four tangents have a cross-ratio independent of point [8]. So we want to choose a point that makes calculations simple. Let us pick the point $p = (0, 0)$ on $\varepsilon$. In projective coordinates, $p = [0, 0, 1]$. Recall that from equation 28, $F(x, y, w) = F = x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - xw^2 + \sqrt{3}yw^2$. Calculating the first partials of the curve $F$ we get:

$$F_x = 3x^2 + 2y^2 + 2\sqrt{3}xy - w^2$$
$$F_y = 2xy + \sqrt{3}x^2 + 3\sqrt{3}y^2 + \sqrt{3}w^2$$
$$F_w = -2xw + 2\sqrt{3}w$$
From equation 3 on page 6, the Hessian matrix associated with $F$ is

$$
H_F = \begin{pmatrix}
6x + 2\sqrt{3}y & 2\sqrt{3}x + 2y & -2w \\
2\sqrt{3}x + 2y & 2x + 6\sqrt{3}y & 2\sqrt{3}w \\
-2w & 2\sqrt{3}w & -2x + 2\sqrt{3}y
\end{pmatrix}
$$

From equation 2, the equation of the polar curve is

$$(3x^2 + 2y^2 + 2\sqrt{3}xy - w^2)p_1 + (2xy + \sqrt{3}x^2 + 3\sqrt{3}y^2 + \sqrt{3}w)p_2 + (-2xw + 2\sqrt{3}yw)p_3 = 0,$$

where $p = [p_1, p_2, p_3] = [0, 0, 1]$.

By simplification, the equation of the polar curve $P_F$ is

$$P_F = -xw + \sqrt{3}yw = 0$$

From equation 4 it follows that the equation of the Hessian curve $H_C$ associated with $F$ is

$$H_C = \sqrt{3}x^2y - 2xy^2 + 2xw^2 - \sqrt{3}y^3 + \sqrt{3}yw^2$$

Setting $F = 0$ and $H_C = 0$, $[0, 0, 1]$ is a mutual solution and thus is a flex. Since $[0, 0, 1]$ is a flex, the tangent there will be counted with multiplicity 2. To calculate the tangents, set $w = 1$ to find the tangents not at infinity. Then

$$F(x, y, 1) = x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - x + \sqrt{3}y$$

$$P_F(x, y, 1) = -x + \sqrt{3}y$$

Setting $F(x, y, 1) = 0$ and $P_F(x, y, 1) = 0$, it follows from the second equation that
\[ x = \sqrt{3}y. \] Substituting into \( F \), we get

\[
3\sqrt{3}y^2 + \sqrt{3}y^3 + 3\sqrt{3}y^3 + \sqrt{3}y^3 - \sqrt{3}y + \sqrt{3}y = 0
\]

\[
8\sqrt{3}y = 0
\]

\[
y = 0
\]

\[
\Rightarrow x = 0
\]

Since it was assumed \( w = 1 \), the calculated point of tangency is \([0, 0, 1]\). This answer is not surprising (and almost seems redundant); the tangent at \([0, 0, 1]\) crosses through \([0, 0, 1]\).

Now we have three more points to find. Let \( w = 0 \) in order to find the points of tangency at the line at infinity. Then

\[
F(x, y, 0) = x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3
\]

\[
P_F(x, y, 0) = 0
\]

Setting \( F(x, y, 0) = 0 \), and \( P_F(x, y, 0) = 0 \), it is obvious that the second equation yields no information. Factoring the first equation we get:

\[
(x + \sqrt{3}y)(x^2 + y^2) = 0
\]

\[
(x + \sqrt{3}y)(x + iy)(x - iy) = 0
\]

\[
\therefore x = \sqrt{3}y, x = -iy, x = iy
\]

As the equations are homogeneous, since \( w = 0 \), we can assume \( y = 1 \). Then the points of tangency are \([-\sqrt{3}, 1, 0], [-i, 1, 0], [i, 1, 0]\), and the tangents to these points all cross through \([0, 0, 1]\). With the four points determined, let us determine the equations of the lines through \([0,0,1]\) which are tangents to those points. The first point is \([0,0,1]\). To do its tangent, we use the method as described in equation
1 on page 5. Thus $F_x(0, 0, 1) = -1 F_y(0, 0, 1) = \sqrt{3}$, and $F_z = 0$. Hence, the equation of the first tangent line is

$$-1x + \sqrt{3}y + 0z = 0 \quad \text{(i)}$$

As a sidenote, notice that for the equation $F(x, y, 1) = 0$ previously described, the corresponding equation was

$$x^3 + xy^2 + \sqrt{3}x^2y + \sqrt{3}y^3 - x + \sqrt{3}y = 0$$

This equation passes through the origin $(0, 0)$, if we consider it as the unhomogenized equation $f(x, y)$. Then from Theorem 2, the equation of the tangent line through the origin is $-x + \sqrt{3}y = 0$, and that confirms equation (i), which was recently obtained via the other method.

To obtain the other three tangents, we need to find the equations of the lines through the points of contact and $[0, 0, 1]$, using the method given in equation 5 on page 6. Therefore:

$$
\begin{vmatrix}
  x & y & z \\
-\sqrt{3} & 1 & 0 \\
  0 & 0 & 1
\end{vmatrix} = 0 \Rightarrow 1x + \sqrt{3}y + 0z = 0 \quad \text{(ii)}
$$

$$
\begin{vmatrix}
  x & y & z \\
-i & 1 & 0 \\
  0 & 0 & 1
\end{vmatrix} = 0 \Rightarrow 1x + iy + 0z = 0 \quad \text{(iii)}
$$

$$
\begin{vmatrix}
  x & y & z \\
i & 1 & 0 \\
  0 & 0 & 1
\end{vmatrix} = 0 \Rightarrow 1x - iy + 0z = 0 \quad \text{(iv)}
$$

To find the projective points associated with the lines, write equations (i), (ii), (iii), and (iv) as a dot product and choose $[x_i, y_i, z_i]$ such that the equations are satisfied.
So we have the following:

$$[-1, \sqrt{3}, 0] \cdot [x_1, y_1, z_1] = 0.$$  \(\text{Choose } [x_1, y_1, z_1] = [\sqrt{3}, 1, 0].\)

$$[1, \sqrt{3}, 0] \cdot [x_2, y_2, z_2] = 0.$$  \(\text{Choose } [x_2, y_2, z_2] = [\sqrt{3}, -1, 0].\)

$$[1, i, 0] \cdot [x_3, y_3, z_3] = 0.$$  \(\text{Choose } [x_3, y_3, z_3] = [1, i, 0].\)

$$[1, -i, 0] \cdot [x_4, y_4, z_4] = 0.$$  \(\text{Choose } [x_4, y_4, z_4] = [1, -i, 0].\)

So now we will use the projective points \([\sqrt{3}, 1, 0], [\sqrt{3}, -1, 0], [1, i, 0], \text{and } [1, -i, 0]\) to calculate the cross-ratio. Now let

$$a = [1, i, 0]$$
$$b = [1, -i, 0]$$
$$c = [\sqrt{3}, 1, 0]$$
$$d = [\sqrt{3}, -1, 0]$$

Then applying definition 1a on page 10, we obtain the following:

$$\bar{c} = \alpha \bar{a} + \beta \bar{b}$$

$$\Rightarrow \sqrt{3} = \alpha \cdot 1 + \beta \cdot 1$$

$$1 = \alpha \cdot i - \beta \cdot i$$

$$\therefore \alpha = \frac{1}{2}(\sqrt{3} - i), \beta = \frac{1}{2}(\sqrt{3} + i)$$

$$\bar{d} = \gamma \bar{a} + \delta \bar{b}$$

$$\Rightarrow \sqrt{3} = \gamma \cdot 1 + \delta \cdot 1$$

$$-1 = \gamma \cdot i - \delta \cdot i$$

$$\therefore \gamma = \frac{1}{2}(\sqrt{3} + i), \delta = \frac{1}{2}(\sqrt{3} - i)$$

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Note that in particular $\alpha = \delta$, and $\beta = \gamma$, so therefore we have the cross-ratio
\[(AB, CD) = \frac{\beta}{\alpha} / \frac{\delta}{\gamma} = \frac{\beta^2}{\alpha^2}.
\]

Define $\omega$ to be the cube root of unity, namely $e^{\frac{2\pi i}{3}}$. It follows then that $\omega^2 = \overline{\omega} = e^{\frac{4\pi i}{3}}$, $-\omega^2 = -\overline{\omega} = e^{\frac{2\pi i}{3}}$, and $-\omega = e^{\frac{5\pi i}{3}} = e^{\frac{-\pi i}{3}}$. Now
\[
\begin{align*}
\beta &= \frac{1}{2} (\sqrt{3} + i) = e^{\frac{\pi i}{6}}, \\
\alpha &= \frac{1}{2} (\sqrt{3} - i) = e^{-\frac{\pi i}{6}} \\
\therefore \beta^2 &= e^{\frac{\pi i}{3}} = -\omega^2, \quad \alpha^2 = e^{-\frac{\pi i}{3}} = -\omega
\end{align*}
\]
\[
\Rightarrow \frac{\beta^2}{\alpha^2} = \frac{-\omega^2}{-\omega} = \omega
\]

Putting them into the formula for the cross-ratio, we get $(AB, CD) = \frac{\beta^2}{\alpha^2}$, which simplifies to $\omega$. Since $\omega \in [\omega]$ it follows that the cross-ratio of $\varepsilon$ is $[\omega]$. (Note that the members of the equivalency class of $[\omega]$ are $\omega, \overline{\omega}, 1 - \omega, 1 - \overline{\omega}, \frac{1}{1 - \omega}, \frac{1}{1 - \overline{\omega}}$, and would be obtained by a suitable permutation of the four tangents $\overline{a}, \overline{b}, \overline{c}, \overline{d}$; recall that from page 11 cross-ratios are unique up to $S_3$ symmetry.)

A theorem (sometimes referred to as Salmon’s Theorem) from algebraic curve theory states that two nonsingular cubics have the same cross-ratio if and only if they are birationally equivalent [8].

It will be shown that $f$ and $f_m$ (as defined in equations 27 and 38, on pages 36 and 53, respectively) have the same cross-ratio.

Rather than trying to calculate the cross-ratio analytically, which is very difficult due to the complexity of $f_m$, it is wiser to use earlier developed theory. First, we homogenize $f$ and $f_m$ as $F$ and $F_m$, respectively. An LFT transforms $F$ into $F_m$ through spin action, as described in Chapter 7. LFTs are birational transformations. Hence the cubics are birationally equivalent. Therefore from Salmon’s Theorem it follows that $F$ and $F_m$ have the same cross-ratio.
Two cubics are projectively equivalent if there exists a linear transformation that transforms one into the other. Note that the homogenized form should be used when performing this calculation. For example, the curve $\varepsilon$ can be transformed into the cubic $y^2 = x(x^2 + x + 1)$, through a linear transformation as described in the program listing on page 101 in Appendix B. Written as a single linear transformation, the mapping

\[
\begin{align*}
  x &\mapsto \frac{-3\frac{1}{3}}{2}(x + w) \\
y &\mapsto \frac{-3-\frac{1}{3}}{2}(x - w) \\
w &\mapsto 3^\frac{1}{3} y
\end{align*}
\]

transforms $F(x, y, w) = 0$ into the curve $F_t(x, y, w) = -x^3 - x^2 w - xw^2 + y^2 w = 0$. Recapturing the affine curve at $w = 1$ leads to the equation

\[
f_t(x, y) = -x^3 - x^2 - x + y^2 = 0.
\]

This can be expressed as the cubic

\[
y^2 = x(x^2 + x + 1). \tag{44}
\]

Equation 44, however, is in the standard form given for a nonsingular irreducible cubic [1]. As cross-ratio is a projective property, it is preserved; by direct calculation, it can also be confirmed that the cross-ratio is once again $\omega$.

It seems reasonable that $F_m$ and $F$ are projectively equivalent, since it was established earlier that they have the same cross-ratio. They are birationally equivalent, and they are both irreducible and nonsingular. It is not known whether having the same cross-ratio is sufficient enough to show projective equivalence. An alternate approach is to attempt to find a transformation which takes $F_m$ to a standard form and then calculate a cross-ratio, possibly even transforming $F_m$ to the same curve $y^2 = x(x^2 + x + 1)$. So far this has been elusive, and one strategy being used is to create a chain of non-degenerate transformations that, when taken
in succession, transform the given cubic into a standard form. Work is still in progress. An example of the strategy is given in a program listing on page 105 in Appendix B. So the following is an open conjecture.

**Conjecture**

*F and F_m are projectively equivalent.*

One of the outcomes of determining a projective transformation of F_m to a standard form is that it would be easier to calculate the flexes of F_m. LFTs do not preserve the flexes of a cubic, and it can shown that a flex is transformed to a point that is not a flex.

To prove this, let us homogenize f_m (from equation 38) to F_m; we calculate the first partials of the function. The following is obtained:

\[
F_{mx} = (1 - 3r^2 - 3r^4 - r^6)(3x^2 + y^2) + 2\sqrt{3}(1 + 3r^2 + r^4 + r^6)xy - (1 + 4r^2 + r^4)(1 + 2r^2)w^2 \pm \sqrt{\frac{3}{R}} \sqrt{r(1 + 2r^2)(2(1 + 2r^2)^2 xw - 2\sqrt{3} (1 + 2r^2)yw)}
\]

\[
F_{my} = 2(1 - 3r^2 - 3r^4 - r^6)xy + \sqrt{3}(1 + 3r^2 + r^4 + r^6)(x^2 + 3y^2) + (1 + 4r^2 + r^4)\sqrt{3}w^2 \pm \sqrt{\frac{3}{R}} \sqrt{r(1 + 2r^2)(-2\sqrt{3}(1 + 2r^2)xw + 6yw)}
\]

\[
F_{mw} = -(1 + 4r^2 + r^4)(2(1 + 2r^2)xw - 2\sqrt{3}yw) \pm \sqrt{\frac{3}{R}} \sqrt{r(1 + 2r^2)((1 + 2r^2)^2 x^2 - 2\sqrt{3}(1 + 2r^2)xy + 3y^2 + 3Rw^2)}
\]

The Hessian matrix H_{F_m} is calculated to be

\[
H_{F_m} = \begin{pmatrix}
A & B & C \\
D & E & G \\
I & J & K
\end{pmatrix},
\]
where:

\[ A = 6(1 - 3r^2 - 3r^4 - r^6)x + 2\sqrt{3}(1 + 3r^2 + r^4 + r^6)y + 2 \]
\[ \pm \sqrt{\frac{3}{R}}r(1 + r^2)(1 + 2r^2)^2w \]

\[ B = 2(1 - 3r^2 - 3r^4 - r^6)y + 2\sqrt{3}(1 + 3r^2 + r^4 + r^6)x - 6 \]
\[ \pm \frac{1}{\sqrt{R}}r(1 + r^2)(1 + 2r^2)w \]

\[ C = -2(1 + 4r^2 + r^4)(1 + 2r^2)w \]
\[ \pm \sqrt{\frac{3}{R}}r(1 + r^2)(2(1 + 2r^2)^2x - 2\sqrt{3}(1 + 2r^2)y) \]

\[ D = 2(1 - 3r^2 - 3r^4 - r^6)y + 2\sqrt{3}(1 + 3r^2 + r^4 + r^6)x - 6 \]
\[ \pm \frac{1}{\sqrt{R}}r(1 + r^2)(1 + 2r^2)w \]

\[ E = 2(1 - 3r^2 - 3r^4 - r^6)x + 6\sqrt{3}(1 + 3r^2 + r^4 + r^6)y + 6 \pm \sqrt{\frac{3}{R}}r(1 + r^2)w \]

\[ G = 2(1 + 4r^2 + r^4)\sqrt{3}w \pm \sqrt{\frac{3}{R}}r(1 + r^2)(-2\sqrt{3}(1 + 2r^2)x + 6y) \]

\[ I = -2(1 + 4r^2 + r^4)(1 + 2r^2)w \pm \sqrt{\frac{3}{R}}r(1 + r^2)(2(1 + 2r^2)^2x - 2\sqrt{3}(1 + 2r^2)y) \]

\[ J = 2(1 + 4r^2 + r^4)\sqrt{3}w \pm \sqrt{\frac{3}{R}}r(1 + r^2)(-2\sqrt{3}(1 + 2r^2)x + 6y) \]

\[ K = -(1 + 4r^2 + r^4)(2(1 + 2r^2)x - 2\sqrt{3}y) + 6 \pm \sqrt{\frac{3}{R}}r(1 + r^2)Rw \]

It is not a tempting prospect to calculate the determinant of the above Hessian matrix. However with some simplifying assumptions, the evaluation can be made significantly better. Recall the purpose is to show the flex of a cubic is not preserved under a direct hyperbolic transformation that takes 0 to \( m \in \varepsilon^* \).

Recalling that \( m = [r \cos \theta, r \sin \theta, 1] \) and using the trigonometric relations given in equations 34 and 35, the equation for the Hessian curve becomes

\[ H_{C_m} = \pm 32rR^2\sqrt{3R}(r - 1)^5(r + 1)^5 \]
depending if $\theta \in (-\frac{\pi}{6}, 0)$ or $\theta \in (\frac{5\pi}{6}, \pi)$, respectively. Setting $H_{C_m} = 0$, the only real solutions occur at $r = -1, 0, 1$, and of these possibilities, only $r = 0$ is valid in our given domain. Therefore any nonzero $m \in \mathbb{C}^*$ will not be a flex. At the line at infinity, if $m = [r \cos \theta, r \sin \theta, 0]$, there are still no flexes, since the equation for the Hessian curve becomes

$$H_{C_m} = \pm 32r^3R^2\sqrt{3R}(r - 1)^4(r + 1)^4$$

depending if $\theta \in (-\frac{\pi}{6}, 0)$ or $\theta \in (\frac{5\pi}{6}, \pi)$, respectively. Setting $H_{C_m} = 0$, the real solutions for $r$ are the same, but none of them are viable. $r = -1$ and $r = 1$ are outside the given domain of the unit disk in $H^2$, while $r = 0$ would imply $[0, 0, 0] \in \mathbb{C}^2$, which is clearly false. Therefore an LFT does not preserve flexes.

The point 0, which was a flex in the original equation, was mapped to a point other than 0 which was not a flex. So at this juncture the only theoretical way to find the flexes would be to solve for the mutual solutions of $F_m = 0$ and $H_{C_m} = 0$, which so far has been difficult to solve due to its unwieldy nature.

The above difficulty in finding a generalized flex is a motivation in trying to determine a projective transformation that takes $F$ to $F_m$, if there is one. Flexes are preserved under projective transformations, so by taking the inverse of such a transformation, the flex of the cubic in a standard form can be mapped back into the location of the flex.

An additional avenue of exploration may include trying to find a projective transformation which fixes the 4-dimensional cone. The motivation behind such a pursuit lies in the fact that an alternate derivation of $\varepsilon$ was achieved as a stereographic projection of the curve of intersection of the quadric surfaces $x^2 + y^2 - z^2 = -1$ and $x = \sqrt{3}yz$. Homogenizing both of these equations leads to equations in four variables. It seems reasonable that the homogenized form of the
upper hyperboloid (a 4-dimensional cone) remains fixed setwise through the transformations in this space. Perhaps all cubics whose vertices are on $\varepsilon^*$ can be expressed as the stereographic projection of the curve of intersection of the upper hyperboloid with some quadric surface. If that is the case, not only will this aid in finding the flex, but it may be able to answer a nagging question: how is the stereographic projection of the curve of intersection of two quadric surfaces onto the unit disk, which produce $\varepsilon$, related to the path of the centroids of equilateral triangles having common vertices and collinear bases, which also produce $\varepsilon$?
In this body of work, we have formed the cubic $\varepsilon$ in two different ways: one via the locus of centroids of equilateral triangles in $H^2$ (Diagram 3), and the other by a stereographic projection of the curve of intersection of two quadric surfaces onto the unit disk (Diagram 6). We also studied the conditions under which did $F$, the homogeneous equation corresponding to $\varepsilon$, transform into another cubic under an LFT; surprisingly, the condition imposed was that $m$ (the point to which $0$ was mapped) had to be on the conjugate curve $\varepsilon^*$.

Also explored was the question of whether the axis of the cubic which had $m$ as a vertex was unique or not. It turned out every $m$ is associated with a unique axis; any arbitrary line through $m$ was not associated with a locus of centroids that was necessarily cubic. This lead to a natural equivalency class for the admissible lines.

The cross-ratio of $\varepsilon$ was determined to be $\omega$. Through birational equivalence, it was determined that the cross-ratio of $F_m$, the transformed cubic, was also $\omega$. A conjecture about the projective equivalence of the two cubics was made which to this day remains open. With further study, it can be resolved, perhaps as a future thesis topic.

An illustration of the original set of equilateral triangles, the axis, and centroids, along with their image under a hyperbolic transformation is given in Diagram 16 on page 98. Diagram 17 on page 99 illustrates some important results obtained in this thesis, namely $\varepsilon$, $\varepsilon^*$, the admissible line $L_m$, and $f_m$, where $m \in \varepsilon^*$. 
In conclusion, many aspects of the curve $\varepsilon$ have been studied. Although much has been accomplished, this thesis is only the tip of a much larger iceberg, and may be connected to a deeper underlying theory that has not yet been developed.
Diagram 1: Equilateral triangles and their centroids in $\mathbb{R}^2$
Diagram 2: An equilateral triangle in $H^2$
Diagram 3: Equilateral triangles and their centroids in $H^2$
Diagram 4: Path of centroids in $H^2$ and conjugate curve
Diagram 5: Intersection of quadric surfaces
Diagram 6: Stereographic projection of $\varepsilon$ onto a unit hyperboloid
Diagram 7: $F_n$, rendered by *implicitplot* command
Diagram 8: $F_m$, rendered by finding solutions
Diagram 9: Plot of an admissible line
Diagram 10: Plot of admissible lines with vertices on $\epsilon^*$
Diagram 11: Equivalency class of $L_m$, partitioned by $\phi$
Diagram 12: Equivalency class of $L_m$, partitioned by $r$
Diagram 13: Inversion followed by a rotation
Diagram 14: A unique axis with its associated cubic
Diagram 15: The equilateral triangles associated with a unique axis
Diagram 16: Original and transformed triangles, axes, and centroids
Diagram 17: A summarization of some important results
APPENDIX B:

PROGRAM LISTINGS
Transformation taking $F$ into standard form

```plaintext
restart;

Transformation of $F$ orig into $F_t$

```c:=a;d:=b;
```

Fm:=a*Xo^3+b*Xo^2*Yo+c*Xo*Yo^2+d*Yo^3+e*Xo^2*Zo+f*Xo*Yo*Zo
+g*Yo^2*Zo+h*Xo*Zo^2+i*Yo*Zo^2+j*Zo^3;

c := a

d := b

Fm := a Xo^3 + b Xo^2 Yo + a Xo Yo^2 + b Yo^3 + e Xo^2 Zo + f Xo Yo Zo
+ g Yo^2 Zo + h Xo Zo^2 + i Yo Zo^2 + j Zo^3

Original cubic

```a:=1;
b:=sqrt(3);
c;
d;
e:=0;
f:=0;
g:=0;
h:=-1;
i:=sqrt(3);
j:=0;
expand(Fm);
```

```
a := 1
b := sqrt(3)
1
sqrt(3)
e := 0
```

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\[ f := 0 \]
\[ g := 0 \]
\[ h := -1 \]
\[ i := \sqrt{3} \]
\[ j := 0 \]
\[ X_0^3 + \sqrt{3} X_0^2 Y_0 + X_0 Y_0^2 + \sqrt{3} Y_0^3 - X_0 Z_0^2 + \sqrt{3} Y_0 Z_0^2 \]
\[ X_0 := A x + B z; Y_0 := C x + E y + F z; Z_0 := G x + H y; \]
\[ F_n := \text{sort(collect}(F_m, [x, y, z], \text{distributed}), [x, y, z], \text{tdeg}); \]
\[ X_0 := A x + B z \]
\[ Y_0 := C x + E y + F z \]
\[ Z_0 := G x + H y \]
\[ F_n := (A C^2 + \sqrt{3} C^3 + A^3 - A G^2 + \sqrt{3} A^2 C + \sqrt{3} C G^2) x^3 \]
\[ + (2 A E C + 3 \sqrt{3} E C^2 - 2 A H G + \sqrt{3} A^2 E + \sqrt{3} E G^2 + 2 \sqrt{3} C H G) x^2 y \]
\[ + (3 B A^2 + 2 \sqrt{3} B A C + \sqrt{3} A^3 F + 3 \sqrt{3} F C^2 + B C^2 + 2 A F C \]
\[ + \sqrt{3} F G^2 - B G^2) x^2 z \]
\[ + (3 \sqrt{3} E^2 C + A E^2 - A H^2 + 2 \sqrt{3} E H G + \sqrt{3} C H^2) x y^2 \]
\[ + (2 \sqrt{3} B A E + 2 A F E - 2 B H G + 2 B E C + 6 \sqrt{3} F E C + 2 \sqrt{3} F H G) x y z \]
\[ + (3 B^2 A + \sqrt{3} B^2 C + 2 \sqrt{3} B A F + 2 B F C + 3 \sqrt{3} F^2 C + A F^2) x z^2 \]
\[ + (\sqrt{3} E H^2 + \sqrt{3} E^2) y^3 + (\sqrt{3} F H^2 - B H^2 + B E^2 + 3 \sqrt{3} F E^2) y^2 z \]
\[ + (\sqrt{3} B^2 E + 2 B F E + 3 \sqrt{3} F^2 E) y z^2 + (B^3 + \sqrt{3} B^3 F + B F^2) z^3 \]
\[
\begin{align*}
\text{eq} &= \{ A C^2 + \sqrt{3} C^3 + A^3 - AC^2 + \sqrt{3} A^2 C + \sqrt{3} C G^2 \neq 0, \\
& \quad \sqrt{3} F H^2 - B H^2 + B E^2 + 3 \sqrt{3} F E^2 \neq 0, \\
& \quad 2 A E C + 3 \sqrt{3} E C^2 - 2 A H G + \sqrt{3} A^2 E + \sqrt{3} E C^2 + 2 \sqrt{3} C H G = 0, \\
& \quad 3 \sqrt{3} E^2 C + A E^2 - A H^2 + 2 \sqrt{3} E H G + \sqrt{3} C H^2 = 0, \\
& \quad 2 \sqrt{3} B A E + 2 A F E - 2 B H G + 2 B E C + 6 \sqrt{3} F E C + 2 \sqrt{3} F H G = 0, \\
& \quad \sqrt{3} E H^2 + \sqrt{3} E^3 = 0, \sqrt{3} B^2 E + 2 B F E + 3 \sqrt{3} F^2 E = 0, \\
& \quad B^3 + \sqrt{3} F^3 + \sqrt{3} B^2 F + B F^2 = 0\}; \end{align*}
\]

\[
\text{solve}(\text{eq}, \{A, B, C, E, F, G, H\});
\]

\[
\begin{align*}
\{ & C = C, B = -\sqrt{3} F, F = F, H = H, E = 0, G = 0, A = \sqrt{3} C \}, \\
\{ & C = C, B = \text{RootOf}(Z^2 + 1, \text{label} = .L1) F, F = F, H = H, E = 0, \\
& \quad G = 0, A = \sqrt{3} C \} \\
\end{align*}
\]

\[
\begin{align*}
E &= 0; G &= 0; C &= -1; A &= \text{sqrt}(3)*C; F &= \text{sqrt}(3); B &= -\text{sqrt}(3)*F; H &= 2; \end{align*}
\]

\[
\text{Ft} := \text{sort}(\text{collect}((F/8), [x, y, z], \text{distributed}), [x, y, z], \text{tdeg});
\]

\[
\begin{align*}
E &= 0 \\
& \quad G := 0 \\
& \quad C := -1 \\
& \quad A := -\sqrt{3} \\
& \quad F := \sqrt{3}
\end{align*}
\]
\[ B := -3 \]
\[ H := 2 \]
\[ F_t := -\sqrt{3} x^3 - 3 x^2 z - 3 \sqrt{3} x z^2 + 3 y^2 z \]

> restart;
> \( F_{t2} := -3^{-(1/2)} x^3 - 3 x^2 z - 3 \sqrt{3} x z^2 + 3 y^2 z \)
\[ F_{t2} := -\sqrt{3} x^3 - 3 x^2 z - 3 \sqrt{3} x z^2 + 3 y^2 z \]

> \( X := \text{sqrt}(3) x; Y := \text{sqrt}(3) y; Z := z; F_{t3} := F_{t2}/9; \)
\[ X := \sqrt{3} x \]
\[ Y := \sqrt{3} y \]
\[ Z := z \]
\[ F_{t3} := -x^3 - x^2 z - x z^2 + y^2 z \]

> \( z := 1; F_{t3}; \)
\[ z := 1 \]
\[ -x^3 - x^2 - x + y^2 \]

> Therefore we have accomplished to transform \( F \) to a standard form through a linear transformation. \( F \) is projectively equivalent to \( F_t \), where

\[ F_t(x,y,z) = -x^3 - x^2 z - x z^2 + y^2 z = 0 \]

Taking \( z = 1 \), the equation of the cubic is:

\[ f_t(x,y) = -x^3 - x^2 - x + y^2 = 0 \]
Transformation taking \( F_m \) into standard form

\[ \text{restart;} \]
\[ \text{Transformation of } F_m \text{ into } F_w \]
\[ c := a; d := b; \]
\[ F_m := a X_0^3 + b X_0^2 Y_0 + c X_0 Y_0^2 + d Y_0^3 + e X_0^2 Z_0 + f X_0 Y_0 Z_0 + g Y_0^2 Z_0 + h X_0 Z_0^2 + i Y_0 Z_0^2 + j Z_0^3; \]
\[ X_0 := A x + B z; Y_0 := C y + E z; Z_0 := F x + G y; \]
\[ F_n := \text{sort(collect}(F_m, [x, y, z], \text{distributed}), [x, y, z], \text{tdeg}); \]

\[ X_0 := A x + B z \]
\[ Y_0 := C y + E z \]
\[ Z_0 := F x + G y \]

\[ F_n := (h A F^2 + e A^2 F + a A^3 + j F^3) x^3 \]
\[ + (f A C F + e A^2 G + 2 h A G F + 3 j G F^2 + i C F^2 + b A^2 C) x^2 y \]
\[ + (f A E F + 2 e B A F + h B F^2 + 3 a B A^2 + i E F^2 + b A^2 E) x^2 z \]
\[ + (f A C G + h A G^2 + a A C^2 + 3 j G^2 F + 2 i C G F + g C^2 F) x^2 y \]
\[ + (2 a A E C + f A E G + 2 g E C F + 2 e B A G + 2 h B G F \]
\[ + 2 b B A C + 2 i E G F + f B C F) x y z \]
\[ + (a A E^2 + 2 b B A E + f B E F + g E^2 F + e B^2 F + 3 a B^2 A) x z^2 \]
\[ + (b C^3 + i C G^2 + j G^3 + g C^2 G) y^3 \]
\[ + (f B C G + h B G^2 + a B C^2 + 3 b E C^2 + i E G^2 + 2 g E C G) y^2 z \]
\[ + (3 b E^2 C + f B E G + b B^2 C + 2 a B E C + e B^2 G + g E^2 G) y z^2 \]
\[ + (a B^3 + b E^3 + b B^2 E + a B E^2) z^3 \]
Eq0 := 
\{a*A*E^2 + 2*b*B*A*E+f*B*E+F+g*E^2*F+e*B^2*F+3*a*B^2*E=0, \\
3*b*E^2*C+f*B*E*G+b*B^2*C+2*a*B*E*C+e*B^2*G+g*E^2*G=0, \\
a*B^3+b*E^3+b*B^2*E+a*B*E^2=0\} \\
> solve(Eq0,\{A,B,C,E,F,G\});

\{E = 0, A = A, C = C, F = F, G = G, B = 0\}, \{B = -\frac{bE}{a}, \\
C = -\frac{G(-fba+eb^2+a^2g)}{b(a^2+b^2)}, A = -\frac{F(-fba+eb^2+a^2g)}{a(a^2+b^2)}, F = F, G = G, E = E\}, \\
\{B = %1 E, C = \frac{1}{2} G(-f\%1+e-g), A = \frac{1}{2} F(-f\%1+e-g), F = F; \\
G = G, E = E\}\%

> Let's choose the second set of solutions. The choices for E, F, and G are arbitrary, so we'll make convenient choices that enable cancellation of common factors.

> G := b*(b^2+a^2); \\
> F := a*(b^2+a^2); \\
> E := a; \\
> C := -G*(-f*b*a+g*a^2+e*b^2)/b/(b^2+a^2); \\
> B := -b*E/a; \\
> A := -F*(-f*b*a+g*a^2+e*b^2)/a/(b^2+a^2); \\

\begin{align*}
G & := b(a^2 + b^2) \\
F & := a(a^2 + b^2) \\
E & := a \\
C & := f b a - e b^2 - a^2 g
\end{align*}
\[ B := -b \]
\[ A := f b a - e b^2 - a^2 g \]

> Fw:=sort(collect(simplify(Fn),[x,y,z],distributed),[x,y,z],
tdeg);

> Notice that the \(x^2 z^2\), \(y^2 z^2\), and the \(z^3\) terms have been
successfully eliminated, while (theoretically) not eliminating the
other terms. Hence we have a transformation of the cubic that
appears to be non degenerate.
BIBLIOGRAPHY


