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SYMMETRIC PRESENTATIONS AND RELATED TOPICS

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Symmetric Presentations and Related Topics

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Mashael Umar Alharbi

March 2015
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Abstract

In this thesis, we have presented our discovery of symmetric presentations of a number of non-abelian simple groups, including the Mathieu group $M_{12}$. We have given several progenitors, permutation and monomial, including $2^{4} : (2^{2} : 3)$, $2^{5} : D_{10}$, $2^{8} : ((4 \times 2) \cdot D_{4})$, $3^{7} : m L_{2}(7)$, $2^{6} : (\mathbb{Z}_{3} \wr \mathbb{Z}_{2})$, and $2^{24} : (2 \cdot A_{5})$ and their homomorphic images which include $4 \cdot (M_{12} : 2)$, the group of automorphisms of $M_{12}$ and several classical groups. We have given the isomorphism type of each of the group mentioned in the thesis. In each case, a proof of the isomorphism type is provided, either computer-based or by hand. In addition, by hand constructions, using the technique of double coset enumeration, are given for the groups $L_{2}(11) \times 3$, $L_{2}(11)$, $PGL_{2}(11)$, $S_{5}$, $(A_{5} \times A_{5}) : 4$, $A_{7}$, and $3^{7} : L_{2}(7)$.
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Chapter 1

Introduction

Group theory is a fundamental tool in scientific areas. In mathematics, it is the studies of groups. The study of permutation groups, which is the basic class of groups, leads to abstract groups that can be described in a presentation by the group generators and some suitable relations. Thus, every group $G$ is isomorphic to factor group of a free product ($G \cong F/R^F$), where $F$ is a free group with basis $X$, and $R^F$ is a normal subgroup of $F$ generated by $\Delta$ which is a family of words in $X$, so every group $G$ has a presentation $\langle F | R^F \rangle$. Moreover, the classification of finite simple group theorem stated that every finite simple group may be one of the given groups list below.

- Cyclic group $\mathbb{Z}_p$, where $p$ is prime.
- Alternating group $A_n$, where $n \geq 5$.
- Lie type group such as $L_n(F),U(F),\ldots,etc$
- The 26 Sporadic groups.

By considering these groups, we have to know that the solution for the extension problem is not unique for instance, $S_3$ and $\mathbb{Z}_3$ have the same composition series which is the products of two cyclic groups of order 2 and 3 but $S_3 \ncong \mathbb{Z}_3$.

As this area is enormous, we are interested in finite groups. Since every finite group is composed of simple groups and we are able to solve extension problems, we are interested in finite non-abelian simple groups. Now, it has been shown that
progenitors factored by appropriate relations give finite non-abelian simple groups including sporadic simple groups. So, we are interested in finding homomorphic images of progenitors. Moreover, it has been demonstrated, the groups found in this way are constructed in a manner that reveals some of the important properties of these groups. We also interested in writing progenitors, permutation and monomial. We have already written permutation progenitors. In writing monomial progenitors we will obtain new monomial representations of groups.

The objective of this project is to factor the progenitor \( m^n : N \) by suitable relations of the form \( \pi \omega(t_1, \ldots, t_n) \), where \( \pi \in N \) and \( \omega \) is a word in the symmetric generators, in order to find finite homomorphic images of the infinite progenitor \( m^n : N \). One a finite homomorphich image is found, we determine its isomorphism type in two ways:

1. We use the composition factors of the image to construct a computer based proof to determine the type. The solution to the extension problem gives each composition factor as one of the following types of extensions: direct product, semi-direct product, central extensions, and mixed extensions.

To ward this, we give a small example that explain each of the four extensions type.

**Example 1.1** (Direct Product Extension). Consider the following group \( G \)

\[
< x, y, t \mid x^4, y^3, (xy)^5, (x^2 y), t^2, (t, xy), (xt)^8, (yt)^4, (xyt^2)^8 >
\]

Using MAGMA, we obtain the following composition factors.

\[
\begin{array}{c|cccc}
 & 2A(2, 5) & & & \\
\hline
G & 2A(2, 5) & & & \\
* & Cyclic(2) & & & \\
1 & & & & \\
\end{array}
\]

Thus, \( G \) has the following composition series \( G = G_1 \supseteq 1 \), where \( G = (G/G_1)(G_1/1) = U_3(5)C_2 \). The normal lattice of \( G \) is
First, we investigate the center of $G$ and we find it is of order 2. Moreover, by looking at the normal lattice, we find it consists a normal subgroup of order 2. Thus, we might have a direct product of $[2]$ by the Unitary group $[3] = U_3(5)$.

```plaintext
> D:=DirectProduct(CyclicGroup(2),NL[3]);
> s:=IsIsomorphic(D,G1);s;
true
```

The MAGMA loop, confirms that we have a direct product of a cyclic group of order 2 by $U_3(5)$. By using ATLAS, we were able to write a presentation for the Unitary group.

\[ < a, b | a^3, b^5, (ab)^7, (ab^{-1})^7, aba^{-1}b^2aba^{-1}bab^2a^{-1}b > \]

Since $G$ is a direct product extension, element of $[2]$ commutes with the elements of $[3] = U_3(5)$. Thus, a presentation for $G$ becomes

```plaintext
> H<a,b,c>:=Group<a,b,c|a^-3,b^5,(a*b)^7,(a*b^-1)^7,
  a*b*a^-1*b^2*a*b*a^-1*b*a*b^-2*a^-1*b,c^-2,(c,a),(c,b)>;#H;
252000
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(H1,G1);s;
true
```

Hence, $G \cong 2 \times U_3(5)$.

**Example 1.2** (Semi-direct Extension:). Consider the following group

\[ < x, y, z, t | x^3, y^3, z^2, (x,y), x^2 = y, y^2 = x, t^2, (t,x), (yt)^3, (yztt^y)^{10}, (xyt^2t)^5 > , \]
where

\[ x \approx (456), \quad y \approx (123), \quad \text{and} \quad z \approx (15)(26)(34) \]

\( G \) has the following composition factors:

\[
\begin{array}{|c|c|}
\hline
G & \text{Cyclic}(2) \\
\text{Alternating}(7) & \text{Cyclic}(3) \\
\hline
1 & 1
\end{array}
\]

Therefore, the composition series for this group is

\[ G_1 \supseteq G_2 \supseteq 1, \]

where \( G = (G_1/G_2)(G_2/1) = C_2A_7C_3 \). The normal lattice of \( G \) is

\[ \text{Figure 1.2: The Normal Lattice of } G \cong (A_7 \times 3) : 2 \]

The center of \( G \) is of order 1 which indicates that we do not have a central extension. Moreover, by using MAGMA, we find that the two minimal normal subgroups of \( G \) is one of order 3 and the other is of order 2520. So, we can say that we have \( C_3 \times A_7 \) with order 7560. By looking at the normal lattice of \( G \), we
find that [4] is the isomorphic image of order 7560. We obtain that [4] \cong C_3 \times A_7. By using ATLAS, we were able to write the symmetric presentation for \( G_1 \).

\[ < a, b, c | a^3, b^5, (ab)^7, (aa^b)^2, (ab^{-2}ab^2)^2, c^3, (a, c), (b, c) > . \]

Since we have \( G_1 = 3 \times A_7 \), \( G/G_1 \cong C_2 \). By viewing the normal lattice, we find that it does not have a normal subgroup of order 2. Thus, we have a semi-direct product of \( G_1 = 3 \times A_7 \) by \( C_2 \). Next, we are looking for an element \( d \) of order 2 in \( G \) but outside [4].

\begin{verbatim}
for g in G1 do if Order(g) eq 2 and g notin NL[4] and G1 eq sub<G1|NL[4],g> then U:=g; break; end if; end for;
\end{verbatim}

Since we have a semi-direct extension, we want to find the action of \( d \) on the generators of \( G_1 \) \( a \), \( b \), and \( c \). In order to do this, we use the Schreier System for \( G_1 \).

\[
\begin{align*}
a^d &= ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1} \cdot \\
b^d &= ababa^{-1}b^{-1} \cdot \\
c^d &= c^{-1}.
\end{align*}
\]

Finally, we were able to write a presentation for \( G \) as below.

\[
< a, b, c, d | a^3, b^5, (ab)^7, (aa^b)^2, (ab^{-2}ab^2)^2, c^3, (a, c), (b, c), d^2, a^d = ab^{-1}a^{-1}b^{-1}a^{-1}, b^d = ababa^{-1}b^{-1}, c^d = c^{-1} > .
\]

Hence, we have solved the extension type of \( G \).

\[ G \cong (3 \times A_7) : 2 \]

**Example 1.3** (Central Extension:). Consider the group

\[ G = D_4 = < a, b | a^4, b^2, (ab)^2 > \]

The composition factors of \( G \) are as below.
Therefore, the composition series is

\[ G_1 \supseteq G_2 \supseteq 1, \]

where \( G = (G_1/G_2)(G_2/1) = C_2C_2C_2 \). The normal lattice of \( G \) is

![Normal Lattice of 2•2²](image)

Figure 1.3: The Normal Lattice of 2•2²

Since the center of \( G \) is of order 2, indicates that we might have a central extension. By viewing the normal lattice of \( G \), we see that \([2]\) is of order 2. Therefore, it is possible that \([2]\) is the center of \( G \).

\[ > \text{Center(G1)}; \]
\[ \text{Permutation group acting on a set of cardinality 8} \]
\[ \text{Order = 2} \]
\[ (1, 5) (2, 4) (3, 8) (6, 7) \]
\[ > \text{Center(G1) eq NL[2];} \]
\[ \text{true} \]

The above loop confirms that \([2]\) is the center of \( G \). Thus, we have a central extension of \([2]\) by \( C_2C_2 \). Now, we want to factor \( G \) by \([2]\) to determine the isomorphism type of \( q \cong G/[2] \).
We find out that \( q \cong 2^2 \). A presentation for \( q \) is

\[
H = \langle a, b | a^2, b^2, (a, b) \rangle.
\]

Now, we want to write a presentation for \( G \) by writing the generators of \( q \) in terms of the center \( x \).

\[
H = \langle x, a, b | x^2, a^2, b^2, (a, b) = x \rangle.
\]
Hence, \( G \cong 2^2 \).

**Example 1.4** (Mixed Extension:). Consider the group

\[
G = \langle x, y, t | x^5, y^2, (xy)^2, t^2, (t, y), (xty^2)^2, (xt^2x)^3, (ytt^2x^2)^2 \rangle,
\]

where \( x \approx (12345) \) and \( y \approx (14)(23) \). By using MAGMA, we get the following composition factors of \( G \).

\[
G \\
| \text{Alternating}(5) \\
* \\
| \text{Cyclic}(2) \\
* \\
| \text{Cyclic}(2) \\
* \\
| \text{Cyclic}(2) \\
1
\]

Therefore, the composition series is

\[
G_1 \supseteq G_2 \supseteq G_3 \supseteq 1,
\]

where \( G = (G_1/G_2)(G_2/G_3)(G_3/1) = A_5C_2C_2C_2 \). The normal lattice of \( G \) is

![Figure 1.4: The Normal Lattice of 4 \( \cdot (2 \times A_5) \)]


Since the center of $G$ is of order 2, we might be have a central extension. Now, we want to find the maximal abelian group in $G$ by running the following loop

```plaintext
> for i in [1..11] do if IsAbelian(NL[i]) then i; end if; end for;
1
2
3
4
5
```

Thus, the maximal abelian group is $[5]$ which is of order 4.

```plaintext
> X:=AbelianGroup(GrpPerm,[4]);
> s:=IsIsomorphic(X,NL[5]);s;
true
```

We confirm that $G_2 = C_4$ is the isomorphism type of $[5]$. Moreover, $[5]$ is not the center of $G$. Thus, we have a mixed extension of $[5]$ by $q$ where $q$ is the isomorphic image of $G/G_2 = G/[5]$. By looking at the normal lattice of $q$ below, we see that

![Figure 1.5: The Normal Lattice of $2 \times A_5$](image)

it has $C_2$ and $A_5$ as normal subgroups. So, we have a direct product of $C_2$ by $A_5$.

```plaintext
> q,ff:=quo<G1|NL[5]>;
> D:=DirectProduct(nl[2],Alt(5));
> s:=IsIsomorphic(D,q);s;
true
```

A presentation for $q$ found by
\( H := \text{sub}<q|q.1,q.2,q.3>; \)
\( \text{FPGroup}(H); \)
\( H<a,b,c> := \text{Group}<a,b,c|a^5,b^2,c^2,(a^{-1}+b)^2,(b+c)^2, \)
\( (c*a^{-1})^3>; \)
\( > f1,H1,k1:=\text{CosetAction}(H,\text{sub}<H|\text{Id}(H)>); \)
\( > s:=\text{IsIsomorphic}(H1,q); s; \)
\( \text{true} \)

Now, we want to write the generators of \( q \) in terms of the generator of \([5]\) and find its action on the generators of \([5]\).

\( > T:=\text{Transversal}(G1,NL[5]); \)
\( > a:=T[2]; \)
\( > b:=T[3]; \)
\( > c:=T[4]; \)
\( > x:=NL[5].1; \)
\( \text{for} \ i \ \text{in} \ [1..4] \ \text{do if} \ x^a \ \text{eq} \ x^i \ \text{then} \ i; \ \text{end if; end for;} \)
\( \text{for} \ i \ \text{in} \ [1..4] \ \text{do if} \ x^b \ \text{eq} \ x^i \ \text{then} \ i; \ \text{end if; end for;} \)
\( \text{for} \ i \ \text{in} \ [1..4] \ \text{do if} \ x^c \ \text{eq} \ x^i \ \text{then} \ i; \ \text{end if; end for;} \)
\( \text{for} \ i \ \text{in} \ [1..4] \ \text{do if} \ (c*a^{-1})^3 \ \text{eq} \ x^i \ \text{then} \ i; \ \text{end if; end for;} \)

We now want to write a presentation for \( G \) given by \( H \), by inserting \( x \) as a generator of \([5]\).

\( H = \langle x,a,b,c|x^4,a^5,b^2,c^2,(a^{-1}b)^2,(bc)^2,(ca^{-1})^3 = x^3,x^a = x,x^b = x^3,x^c = x \rangle. \)

Finally, we check if it is isomorphic to \( G_1 \).

\( > f1,H1,k1:=\text{CosetAction}(H,\text{sub}<H|\text{Id}(H)>); \)
\( > s:=\text{IsIsomorphic}(H1,G1); s; \)
\( \text{true} \)

Hence, we have proved that \( G \cong 4 \cdot (2 \times A_5). \)

2. We perform a double coset enumeration of the image over \( N \) and obtain a Cayley graph of the image over \( N \). We use this graph to prove by hand that the image is indeed isomorphic to the group given in 1.

To ward the end of this, we give small examples to explain the process of the double coset enumeration when the progenitor is involutory then when the progenitor is monomial.
Example 1.5. Consider the group $G \cong \langle x, y, t | x^3, y^2, (xy)^2, t^2, (t, xy), (t, y^3) \rangle$ factored by $tt^2 = t^2t$, where $G = 2^3 : S_3$, $N = S_3 = \langle x, y \rangle = \langle (123), (12) \rangle$ and $t = t_1$. The main goal of this example is to show that $G = 2^3 : S_3$. If we conjugate the previous relation by all elements in $S_3$, we obtain:

$$12 \approx 21, \ 32 \approx 23, \ \text{and} \ 13 \approx 31.$$ 

First, we start with the double coset $NeN$ denoted by $[\ast]$ which consists of the single coset $N$. Therefore, the number of right cosets in $N$ is equal to $\frac{|N|}{|N_t|} = \frac{6}{6} = 1$. Then, we consider the double coset $NwN$, where $w$ is a word of length one. Since $N$ is transitive on $T = \{1, 2, 3\}$, The orbit of $N$ on $T$ is $\{1,2,3\}$. Thus, we pick a representative $t_1$ and determine its double coset.

Consider $Nt_1N = \{Nt_1, Nt_2, Nt_3\}$. We denote the double coset $Nt_1N$ by $[\ast]$. Now, we determine the orbits of $N^1 = \{n \in N | t^n_1 = t_1 \} = \{e, (23)\} = N(1)$ on $T = \{1, 2, 3\}$ and these are $\{1\}$ and $\{2, 3\}$, and consider the double cosets $Nt_1t_iN$ for one $t_i$ from each orbit of $N^1$ say $t_1$ and $t_2$, respectively, and determine the double cosets that contain $Nt_1t_1$ and $Nt_1t_2$. We see that $Nt_1t_1 \in [\ast]$, so one symmetric generator will go back to $[\ast]$, and $Nt_1t_2 \in [12]$ is a new double coset.

Now, $N^{12} = \langle e \rangle$, but $N(t_1t_2)^{(12)} = Nt_2t_1 \Rightarrow Nt_1t_2 \Rightarrow (12) \in N^{(12)}$. So, $N^{(12)} \geq \langle (12) \rangle$. The number of right cosets in $[12]$ is equal to $\frac{|N|}{|N^{(12)}|} = \frac{6}{2} = 3$. The orbits of $N^{(12)}$ on $\{1,2,3\}$ are $\{1,2\}$ and $\{3\}$. Pick a representative from each orbit, say $t_2$ and $t_3$ respectively, to determine the double cosets that contain $Nt_1t_2t_2$ and $Nt_1t_2t_3$. It is obvious to see that $Nt_1t_2t_2 = Nt_1 \in [1]$, and $Nt_1t_2t_3 \in [123]$ is a new double coset.

Now, $N^{123} = \langle e \rangle$, but $N(t_1t_2t_3)^{(12)} = Nt_3t_2t_1 \Rightarrow Nt_3t_2t_1 \Rightarrow (12) \in N^{(123)}$. Similarly, $(123) \in N^{(123)}$. So, $N^{(123)} \geq \langle (12), (123) \rangle \cong S_3$. The number of right cosets in $[123]$ is equal to $\frac{|N|}{|N^{(123)}|} = \frac{6}{6} = 1$. The orbits of $N^{(123)}$ on $\{1,2,3\}$ is $\{1,2,3\}$. By picking $t_3$ from the orbit, we see that $Nt_1t_2t_3t_3 = Nt_1t_2 \in [12]$. Thus, three symmetric generators will go back to $[12]$.

The set of right cosets is closed under right multiplication by $t_i$'s where $i = 1,2,3$. 
Thus, we can determine the index of $N$ in $G$. We conclude that

$$|G| \leq (|N| + \frac{|N|}{|N(1)|} + \frac{|N|}{|N(12)|} + \frac{|N|}{|N(123)|}) \times |N|$$

$$|G| \leq (1 + 3 + 3 + 1) \times 6$$

$$|G| \leq (8 \times 6) = 48.$$ 

<table>
<thead>
<tr>
<th>Label</th>
<th>Single Cosets</th>
<th>$x \approx (123)$</th>
<th>$y \approx (12)$</th>
<th>$t_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$N$</td>
<td>$N$</td>
<td>$N$</td>
<td>$Nt_1$</td>
</tr>
<tr>
<td>2</td>
<td>$Nt_1$</td>
<td>$Nt_2$</td>
<td>$Nt_2$</td>
<td>$Nt_1t_1$</td>
</tr>
<tr>
<td>3</td>
<td>$Nt_2$</td>
<td>$Nt_3$</td>
<td>$Nt_1$</td>
<td>$Nt_2t_1$</td>
</tr>
<tr>
<td>4</td>
<td>$Nt_3$</td>
<td>$Nt_1$</td>
<td>$Nt_3$</td>
<td>$Nt_3t_1$</td>
</tr>
<tr>
<td>5</td>
<td>$Nt_1t_2$</td>
<td>$Nt_2t_3$</td>
<td>$Nt_2t_1$</td>
<td>$Nt_1t_2t_1 = Nt_2$</td>
</tr>
<tr>
<td>6</td>
<td>$Nt_1t_3$</td>
<td>$Nt_2t_1$</td>
<td>$Nt_2t_3$</td>
<td>$Nt_1t_3t_3 = Nt_1$</td>
</tr>
<tr>
<td>7</td>
<td>$Nt_2t_3$</td>
<td>$Nt_3t_1$</td>
<td>$Nt_1t_3$</td>
<td>$Nt_2t_3t_1 = Nt_1t_2t_3$</td>
</tr>
<tr>
<td>8</td>
<td>$Nt_1t_2t_3$</td>
<td>$Nt_2t_3t_1$</td>
<td>$Nt_2t_1t_3$</td>
<td>$Nt_1t_2t_3t_1 = Nt_2t_3$</td>
</tr>
</tbody>
</table>

Next, we note that $G = 2^3 : S_3 =< t_1, x, y >$ acts on the eight cosets above and the actions of $t, x, y$ on the eight cosets is well-defined. Thus, we have a homomorphism $f : G \rightarrow S_8$. Then $f(G) = < f(x), f(y), f(t) >$; that is, the homomorphic image of $2^3 : S_3$ is of the order $| < f(x), f(y), f(t) > | = 48$. Since $t$ has exactly 3 conjugates, we have $f(G) \cong < f(x), f(y), f(t) >$, where

$$f(x) = (234)(576).$$

$$f(y) = (23)(67).$$

$$f(t) = (12)(35)(46)(78).$$  (see table 1)

Hence,

$$G/Ker f \cong < f(x), f(y), f(t) > = f(G)$$

$$|G/Ker f| \cong |f(G)|$$

$$|G| = |Ker f| \times |f(G)|$$

$$|G| \geq |Ker f| \times 48.$$
But from above we observed that $|G| \leq 48$. This can also be seen from the Cayley diagram below. Hence,

$$|G| = 48 \Rightarrow |Ker f| = 1$$

$$\Rightarrow G \cong \langle f(x), f(y), f(t) \rangle$$

Please see the Cayley diagram of $G = \frac{2^3 \cdot S_3}{tt^x^3 = t^x t}$ over $S_3$ below.

![Figure 1.6: The Cayley Digram of $\frac{2^3 \cdot S_3}{tt^x^3 = t^x t}$](image)

$G = \langle f(x), f(y), f(t) \rangle$.

Now, we want to show that $2^3 : S_3$ is the homomorphic image of $\frac{2^3 \cdot S_3}{tt^x^3 = t^x t}$.

$$G = \langle x, y, t | x^3, y^2, (xy)^2, t^2, (t, xy), (t, y x), t t^x = t^x t \rangle$$

By using MAGMA, we have the composition series

$$G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq 1,$$

where

We want to determine the isomorphism type of this group. We find that the center of $G$ is of order 2. So, we might have a central extension. Now, we want to see if $[4]$ is an abelian group or not.

```plaintext
> X:=AbelianGroup(GrpPerm,[2,2,2]);
> s:=IsIsomorphic(X,NL[4]); s;
true
```

We confirm that $G_2 = 2 \times 2 \times 2$ is the isomorphism type of $[4]$. Since $[4]$ is an abelian group of $G$, $G$ is not a central extension. Now, we want to factor $G$ by $[4]$ to find $H$ such that $G/\langle 4 \rangle \cong H$.

```plaintext
> q,ff:=quo<G1|NL[4]>;
> s:=IsIsomorphic(q,Sym(3)); s;
true
```

We show that $H \cong S_3$ and a presentation for $H$ is $< A, B | A^3, B^2, (AB)^2 >$. Next, we want to find the right cosets of $G/\langle 4 \rangle$ to store and write the two generators of $H$ in terms of the generators of $[4]$.

```plaintext
T:=Transversal(G1,NL[4]);
a:=T[2]; b:=T[3];
x:=NL[4].1; y:=NL[4].2; z:=NL[4].3;
```
The following loops will determine the action of \(a\) and \(b\) on \(x\), \(y\), and \(z\).

for \(i,j,l\) in [1..2] do if \(x^a \equiv x^i*y^j*z^l\) then \(i,j,l\); end if; end for;
for \(i,j,l\) in [1..2] do if \(x^b \equiv x^i*y^j*z^l\) then \(i,j,l\); end if; end for;

Using MAGMA, we highlight the relation between them and write a presentation for \(G\). At the end, we check if this presentation is isomorphic with \(2^3 : S_3\),

```magma
> M<x,y,z,a,b>:=Group<x,y,z,a,b|x^2,y^2,z^2,(x,y),
(x,z),(y,z),a^3,b^2,(a*b)^2,x^a=x*z,x^b=x*z,y^a=z,
y^b=y*z,z^a=y*z,z^b=z>;
> #M;
48
> f1,M1,k1:=CosetAction(M,sub<M|Id(M)>);
> s:=IsIsomorphic(M1,G1);s;
true
```

Since the above presentation have been written with the action of the generators of \(H\) on the generators of \([4]\), \(G\) is a semi-direct product. So, we obtain that \(G \cong 2^3 : S_3\).

**Example 1.6.** Consider the group \(N = S_5\). A presentation of \(S_5\) is

\[
\langle x,y | x^5, y^2, (xy)^4, (x,y)^3 \rangle,
\]

where \(x \approx (12345)\) and \(y \approx (12)\). We want to write a presentation for the monomial progenitor \(3^*5 : \_m S_5\). Thus, we have to induce a linear character of a subgroup \(H\) of \(G\). By looking at the character table of \(S_5\), we find that the largest index is 5. Thus, we will induce from a subgroup of index 5 such that \(|H| = \frac{120}{|S_5:H|} = \frac{120}{5} = 24 = |S_4|\). We will induce a linear character of a subgroup \(S_4\), say \(\chi_2\), up to \(G = S_5\).

Moreover, we have to find a representation \(B\) of \(H\) such that \(B(\chi_2(g))\) is equal to the value of \(\chi_2\) of \(H\) at \(g \in H\). So,

\[
B(1) = 1, B((25)(34)) = 1, B((34)) = -1, B((235)) = 1, \text{ and } B((2453)) = -1.
\]
Table 1.2: Character Table of $G = S_5$

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>6</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1.3: Character Table of $H = S_4$

<table>
<thead>
<tr>
<th>Classes</th>
<th>1</th>
<th>(25)(34)</th>
<th>(34)</th>
<th>(235)</th>
<th>(2453)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

A monomial representation of $G$ is given by

$$A(x) = \begin{pmatrix} B(t_1x^{-1}t_1) & B(t_1xt_2^{-1}) & B(t_1xt_3^{-1}) & B(t_1xt_4^{-1}) & B(t_1xt_5^{-1}) \\ B(t_2x^{-1}t_1) & B(t_2xt_2^{-1}) & B(t_2xt_3^{-1}) & B(t_2xt_4^{-1}) & B(t_2xt_5^{-1}) \\ B(t_3x^{-1}t_1) & B(t_3xt_2^{-1}) & B(t_3xt_3^{-1}) & B(t_3xt_4^{-1}) & B(t_3xt_5^{-1}) \\ B(t_4x^{-1}t_1) & B(t_4xt_2^{-1}) & B(t_4xt_3^{-1}) & B(t_4xt_4^{-1}) & B(t_4xt_5^{-1}) \\ B(t_5x^{-1}t_1) & B(t_5xt_2^{-1}) & B(t_5xt_3^{-1}) & B(t_5xt_4^{-1}) & B(t_5xt_5^{-1}) \end{pmatrix}$$

To get the representation of monomial representations $A(x)$ and $A(y)$, which are represented by $5 \times 5$ matrices, from $B(u)$, we extend $B$ to give a presentation of $G$. Thus,

$$A(x) = \begin{cases} B(x), & \text{if } x \in H \\ 0, & \text{if } x \notin H. \end{cases}$$

Now, we will be able to find the monomial representations $A(x)$ and $A(y)$ which are represented by $5 \times 5$ matrices. Moreover, by using MAGMA, we run the following code to obtain the right cosets of $S_5/H$. 

```plaintext
To get the representation of monomial representations A(x) and A(y), which are represented by 5 x 5 matrices, from B(u), we extend B to give a presentation of G. Thus,

A(x) = \begin{cases} B(x), & \text{if } x \in H \\ 0, & \text{if } x \notin H. \end{cases}

Now, we will be able to find the monomial representations A(x) and A(y) which are represented by 5 x 5 matrices. Moreover, by using MAGMA, we run the following code to obtain the right cosets of S_5/H.
```
Thus,

\[ G = H_{t_1} \cup H_{t_2} \cup H_{t_3} \cup H_{t_4} \cup H_{t_5}. \]

We are now in a position to give the monomial representation of the progenitor

\[ 3^{*5};_{5} S_{6}. \]

\[ A(x) = \begin{pmatrix} B((12345)) & B((2345)) & B((12)) & B((1543)) & B((14)(235)) \\ B((1345)) & B((13452)) & B(1) & B((12543)) & B((15324)) \\ B((14)(235)) & B((14235)) & B((13452)) & B(1) & B((12543)) \\ B((153)(24)) & B((15324)) & B((14235)) & B(1) & B((13452)) \\ B((5432)) & B((12543)) & B((15324)) & B((14235)) & B((13452)) \end{pmatrix} \]

Now,

\[ A(x) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

Similarly, we can find the monomial representation for \( A(y) \).

\[ A(y) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \]
Since the entries of the two monomial matrices are ±1, the entries are in \( \mathbb{Z}_3 \). Hence, \( t_i \)'s are of order 3. Since the number of \( t_i \)'s is equal to \([S_5 : S_4] = 5\), we label them as shown in the table below.

### Table 1.4: Labeling

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_2 )</td>
<td>( t_3 )</td>
<td>( t_4 )</td>
<td>( t_5 )</td>
<td>( t_1^{-1} )</td>
<td>( t_2^{-1} )</td>
<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
<td>( t_5^{-1} )</td>
<td></td>
</tr>
</tbody>
</table>

Now, \( A(x) \) is a monomial automorphism of \( < t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 > \) given by \( a_{ij} = a \Rightarrow t_i \rightarrow t_j^a \). Thus, \( a_{12} = -1 \) or \( t_1 \rightarrow t_2^{-1} \). So, \( A(x) \) takes 1 to 7 by our labeling.

### Table 1.5: Permutaion of \( x = A(x) \)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_2 )</td>
<td>( t_3 )</td>
<td>( t_4 )</td>
<td>( t_5 )</td>
<td>( t_1^{-1} )</td>
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<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
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<td>( t_2^{-1} )</td>
<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
<td>( t_5^{-1} )</td>
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<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
<td>( t_5^{-1} )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

So, \( A(x) = (1, 7, 8, 9, 10)(2, 3, 4, 5, 6) \).

Now, \( A(y) \) is a monomial automorphism (permutation) of \( < t_1 > * < t_2 > * < t_3 > * < t_4 > * < t_5 > \) given by \( a_{ij} = a \Rightarrow t_i \rightarrow t_j^a \). Thus, \( a_{12} = 1 \) or \( t_1 \rightarrow t_2 \). So, \( A(y) \) takes 1 to 2 by our labeling.

### Table 1.6: Permutaion of \( y = A(y) \)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_2 )</td>
<td>( t_3 )</td>
<td>( t_4 )</td>
<td>( t_5 )</td>
<td>( t_1^{-1} )</td>
<td>( t_2^{-1} )</td>
<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
<td>( t_5^{-1} )</td>
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</tr>
<tr>
<td>( t_2^{-1} )</td>
<td>( t_3^{-1} )</td>
<td>( t_4^{-1} )</td>
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<td>( t_2^{-1} )</td>
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<td>( t_2^{-1} )</td>
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</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Then, \( A(y) = (1, 2)(3, 8)(4, 9)(5, 10)(6, 7) \).

So, \( S_5 = \langle A(x), A(y) \rangle = \langle (1, 7, 8, 9, 10)(2, 3, 4, 5, 6), (1, 2)(3, 8)(4, 9)(5, 10)(6, 7) \rangle \).

We are now able to give a monomial presentation of the progenitor \( 3^* \colon m S_5 \).
We fix one the five $t_i$'s, say $t_1$ and call it $t$. Next, compute the normalizer of the subgroup $< t_1 >$ in $S_5$. We therefore need to compute the set stabilizer in $S_5$ of the set $\{ t_1, t_1^2 = t_1^{-1} \} = \{ 1, 6 \}$.

\begin{verbatim}
> S:=Sym(10);
> xx:=S!(1,7,8,9,10)(2,3,4,5,6);
> yy:=S!(1,2)(3,8)(4,9)(5,10)(6,7);
> N:=sub<S|xx,yy>;
> SS:=Stabiliser(N,{1,6});
> SS;
(2, 3, 4)(7, 8, 9),
(3, 5, 4)(8, 10, 9),
(1, 6)(2, 10, 4, 8)(3, 7, 5, 9)
\end{verbatim}

Thus, the normalizer of $< t_1 >$ generated by

\[
(2, 3, 4)(7, 8, 9) = x^2 y x^{-1} y,
(3, 5, 4)(8, 10, 9) = x^{-1} y y x^{-1}, \quad \text{and}
(1, 6)(2, 10, 4, 8)(3, 7, 5, 9) = y x^{-1}.
\]

The monomial progenitor $3^*5^5 :_m S_5$ has

\[
\langle x, y, t | x^5, y^2, (x y)^4, (x, y)^3, t^3, (t, x^2 y x^{-1} y), (t, x^{-1} y x y x^{-1}), t^y x^{-1} t^{-2} \rangle.
\]

as a (symmetric) presentation. This progenitor is infinite, and in order to make it finite we add a supporting relator such as $(x t)^k$ and run the progenitor in MAGMA to obtain a homomorphic image of $3^*5^5 :_m S_5$.

\begin{verbatim}
> for k in [1..5] do
  for G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y)^4,(x,y)^3,t^3,(t,x^2*y*x^-1*y),(t,x^-1*y*x*y*x^-1),
  t^y*x^-1*t^-2,(x*t)^k>;k,#G; end for;
1 2
2 2
3 6
4 2
5 9720
\end{verbatim}

Thus, we have $G = \frac{3^*5^5 :_m S_5}{(x t)^5} \cong 3^4 : S_5$.

Now, we are interesting in constructing the Cayley diagram of $3^*5^5 :_m S_5$ by proceeding the DCE technique.
Consider the progenitor $G = \frac{3^5 \cdot m \cdot S_5}{|x|}$, where $S_5$ is generated by
\[\langle x, y \rangle = \langle (1, 7, 8, 9, 10)(2, 3, 4, 5, 6), (1, 2)(3, 8)(4, 9)(5, 10)(6, 7) \rangle.
\]
Let $t = t_1$. Thus,
\[
[xt]^5 = 1
\]
\[x^5 t^4 t^3 t^2 t^t = 1
\]
\[t_{10} t_9 t_8 t_7 t_1 = 1
\]
\[t_{10} t_9 = t_1^{-1} t_7^{-1} t_8^{-1}
\]
\[t_{10} t_9 = t_6 t_2 t_3.
\]
Therefore, $N t_{10} t_9 = N t_6 t_2 t_3$. By conjugation with all $n \in S_5$, we obtain the following relations.

43 = 10171 ≈ 7110 ≈ 1107, 34 = 7101 ≈ 1017 ≈ 1710,
26 = 9810 ≈ 1098 ≈ 8109, 62 = 8910 ≈ 1089 ≈ 9108,
23 = 1910 ≈ 9101 ≈ 1019, 32 = 1091 ≈ 1109 ≈ 9110,
42 = 1018 ≈ 8101 ≈ 1810, 24 = 1108 ≈ 1081 ≈ 8110,
46 = 1087 ≈ 7108 ≈ 8710, 64 = 8107 ≈ 1078 ≈ 7810,
63 = 9710 ≈ 1097 ≈ 7109, 36 = 1079 ≈ 9107 ≈ 7910,
109 = 623 ≈ 236 ≈ 362, 910 = 263 ≈ 632 ≈ 326,
101 = 324 ≈ 243 ≈ 432, 110 = 234 ≈ 342 ≈ 423,
107 = 634 ≈ 463 ≈ 346, 710 = 364 ≈ 436 ≈ 643,
108 = 642 ≈ 264 ≈ 426, 810 = 462 ≈ 246 ≈ 624,
54 = 718 ≈ 187 ≈ 871, 45 = 178 ≈ 817 ≈ 781,
98 = 562 ≈ 625 ≈ 256, 89 = 625 ≈ 265 ≈ 526,
71 = 345 ≈ 453 ≈ 534, 17 = 354 ≈ 543 ≈ 435,
25 = 189 ≈ 918 ≈ 891, 52 = 819 ≈ 198 ≈ 981,
65 = 879 ≈ 987 ≈ 798, 56 = 789 ≈ 897 ≈ 978,
87 = 456 ≈ 564 ≈ 645, 78 = 465 ≈ 546 ≈ 654,
18 = 245 ≈ 524 ≈ 452, 81 = 425 ≈ 542 ≈ 254,
\[91 = 523 \approx 352 \approx 235, 19 = 253 \approx 325 \approx 532,\]
\[79 = 356 \approx 635 \approx 563, 97 = 536 \approx 653 \approx 365,\]
\[53 = 791 \approx 179 \approx 917, 35 = 197 \approx 971 \approx 719.\]

The main goal of this example is to show that \[G = \frac{3^*S_5}{t_1t_9 = t_6t_2t_3} \cong S_5.\] Elements of \(3^*\) is of the form \(<t_1 > \ast \ldots \ast < t_5 >= \{t_1, t_1^2 = t_1^{-1}\} \ast \ldots \ast \{t_5, t_5^2 = t_5^{-1}\}\).

First, we start with the double coset \(NeN\) denoted by \([\ast]\) which consists of the single coset \(N\). Then, we consider the double coset \(NwN\), where \(w\) is a word of length one. Since the orbit of \(N\) is \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\), pick a representative, say \(t_1\), and determine the double coset that contains \(Nt_1N\).

Consider \(Nt_1N = \{Nt_1, \ldots, Nt_{10}\} = Nt_2N = \ldots = Nt_{10}N\), which is denoted by \([1]\). \(N^1 = \langle (2, 3, 4)(7, 8, 9), (3, 5, 4)(8, 10, 9) \rangle = N^{(1)}\). Thus, the number of right cosets in \([1]\) is equal to \(\frac{\#N}{\#N^{(1)}} = \frac{120}{12} = 10\). The orbits of \(N^{(1)}\) on \(T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) are \(\{1\}, \{6\}, \{2, 3, 4, 5\}, \) and \(\{7, 8, 9, 10\}\). By our labeling, it is obvious to see that
\[Nt_1t_1 = Nt_1^2 = Nt_1^{-1} = Nt_6 \in [1],\] so 1 symmetric generator will stay on \([1]\).
\[Nt_1t_6 = N \in [\ast],\] so 1 symmetric generator will take back on \([\ast]\).
\[Nt_1t_2 \in [12].\]
\[Nt_1t_7 \in [17].\]

Consider \(Nt_1t_2N\), which is denoted by \([12]\). \(N^{12} = N^{(12)}\) is generated by
\[< (3, 5, 4)(8, 10, 9), (1, 2)(3, 8)(4, 9)(5, 10)(6, 7), (1, 2)(3, 10, 4, 8, 5, 9)(6, 7), (1, 2)(3, 9, 5, 8, 4, 10)(6, 7) >.\]

Thus, the number of right cosets in \([12]\) is equal to \(\frac{\#N}{\#N^{(12)}} = \frac{120}{6} = 20\). The orbits of \(N^{(12)}\) on \(T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) are \(\{1, 2\}, \{6, 7\}, \) and \(\{3, 4, 5, 8, 9, 10\}\). Pick \(t_2, t_7,\) and \(t_3\), respectively, from each orbit and determine the double coset that contains each one of them. We see that
\[Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1t_2^{-1} = Nt_1t_7 \in [17],\] so 2 symmetric generators will take to \([17]\).
\[Nt_1t_2t_7 = Nt_1t_2t_2^{-1} = Nt_1 \in [1],\] so 2 symmetric generator will take back to \([1]\).
\[Nt_1t_2t_3 \in [123].\]
Consider $N t_1 t_7 N$, which is denoted by [17]. $N^{17} = N^{(17)}$ is generated by

$$< (3, 5, 4)(8, 10, 9), (1, 7)(2, 6)(3, 4)(8, 9), (1, 7)(2, 6)(3, 5)(8, 10), (1, 7)(2, 6)(4, 5)(9, 10) >= N^{(17)}. $$

Therefore, the number of right cosets in [17] is equal to $\frac{|N|}{|N^{(17)}|} = \frac{120}{6} = 20$. The orbits of $N^{(17)}$ are $\{1, 7\}, \{2, 6\}, \{3, 4, 5\}$, and $\{8, 9, 10\}$. The double coset that contains $t_7, t_2, t_3,$ and $t_8$ is as the following

$N t_1 t_7 t_7 = N t_1 t_2^3 t_2 = N t_1 t_2 t_3^3 = N t_1 t_2 \in [12]$, so 2 symmetric generators will take to $[12]$.

$N t_1 t_7 t_2 = N t_1 t_2^{-1} t_2 = N t_1 \in [1]$, so 2 symmetric generator will take back to $[1]$.

Consider $N t_1 t_7 t_3 N$ which is denoted by [173].

Because the relation

$$N t_1 t_7 t_3 = N(t_1 t_2 t_3)^{(1, 8, 10, 7, 9)(2, 4, 6, 3, 5)} = N t_3 t_4 t_5$$

where $N t_3 t_4 t_5 \in [123]$.

There exist $\{n \in N|N(t_1 t_2 t_3)^n = N t_3 t_4 t_5\}$.

Thus, $N(t_1 t_2 t_3)^{(1, 8, 10, 7, 9)(2, 4, 6, 3, 5)} = N t_3 t_4 t_5$.

$$[173] = [123]$$

Therefore, $N t_1 t_7 t_3 N$ is not a new double coset and collapses $\Rightarrow$ three symmetric generators will take to $[123]$.

Consider $N t_1 t_7 t_8 N$ which is denoted by [178].

Because the relation

$$N t_1 t_7 t_8 = N(t_1 t_7)^{(1, 4, 8, 6, 9, 3)(2, 10)(5, 7)} = N t_4 t_5$$

where $N t_4 t_5 \in [17]$.

There exist $\{n \in N|N(t_1 t_7)^n = N t_4 t_5\}$.

Thus, $N(t_1 t_7)^{(1, 4, 8, 6, 9, 3)(2, 10)(5, 7)} = N t_4 t_5$. 
Therefore, \( N_{t_1 t_7 t_8} \) is not a new double coset and collapses \( \Rightarrow \) three symmetric generators will take to \([17]\).

Consider \( N_{t_1 t_2 t_3} \), which is denoted by \([123]\). \( N_{123} = N^{(123)} \) is generated by

\[
N^{(123)} = < (2,3)(4,5)(7,8)(9,10), (1,6)(2,9,3,10)(4,8,5,7), \\
(1,6)(2,10,3,9)(4,7,5,8) >.
\]

Therefore, the number of right cosets in \([123]\) is equal to \( \frac{|N|}{|N_{123}|} = \frac{120}{4} = 30 \). The orbits of \( N^{(123)} \) are \{1,6\}, \{2,3,9,10\}, and \{4,5,8,7\}. It is obvious to see that \( N_{t_1 t_2 t_3 t_8} = N_{t_1 t_2 t_3 t_3}^{-1} = N_{t_1 t_2} \in [12] \), so 4 symmetric generators will take back to \([12]\).

Consider \( N_{t_1 t_2 t_3 t_3} \) which is denoted by \([1233]\).

Because the relation

\[
N_{t_1 t_2 t_3 t_3} = N(t_1 t_2 t_3)^{(1,7,9,8,10)(2,4,3,5,6)}
= N_{t_7 t_4 t_5},
\]

where \( N_{t_7 t_4 t_5} \in [123] \).

There exist \( \{n \in N \mid N(t_1 t_2 t_3)^n = N_{t_7 t_4 t_5}\} \).

Thus, \( N(t_1 t_2 t_3)^{(1,7,9,8,10)(2,4,3,5,6)} = N_{t_7 t_4 t_5} \).

\([1233] = [123] \)

Therefore, \( N_{t_1 t_2 t_3 t_3} \) is not a new double coset and collapses \( \Rightarrow \) three symmetric generators will take to \([123]\).

Consider \( N_{t_1 t_2 t_3 t_6} \) which is denoted by \([1236]\).

Because the relation

\[
N_{t_1 t_2 t_3 t_6} = N(t_1 t_7)^{(1,3,9,5)(2,7)(4,10,6,8)}
= N_{t_3 t_2},
\]

where \( N_{t_3 t_2} \in [17] \).

There exist \( \{n \in N \mid N(t_1 t_7)^n = N_{t_3 t_2}\} \).

Thus, \( N(t_1 t_7)^{(1,3,9,5)(2,7)(4,10,6,8)} = N_{t_3 t_2} \).
Therefore, \( Nt_1t_2t_3t_6 N \) is not a new double coset and collapses \( \Rightarrow \) two symmetric generators will take to [17]. Please see the Cayley diagram below.

![Cayley Diagram](image)

**Figure 1.8: The Cayley Diagram of \( 3^*mS_5 \)**

The set of right cosets is closed under right multiplication by \( t_i \)'s where \( i = 1, 2, 3, \ldots, 10 \). Thus, we can determine the index of \( N \) in \( G \). We conclude that

\[
|G| \leq \left( |N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(17)}|} + \frac{|N|}{|N^{(123)}|} \right) \times |N|
\]

\[
|G| \leq (1 + 10 + 20 + 20 + 30) \times 120
\]

\[
|G| \leq (81 \times 120) = 9720.
\]

Next, we consider that \( G = 3^* m S_5 = \langle t_1, x, y \rangle \) acts on the 81 cosets above and the actions of \( t, x, y \) on the 81 cosets is well-defined. Thus, we have a homomorphism \( f : G \rightarrow S_{81} \). Then, \( f(G) = \langle f(x), f(y), f(t) \rangle \); that is, the homomorphic image of \( 3^* m S_5 \) is of the order, \( |< f(x), f(t) >| = 120 \). Since \( t \)
has exactly 10 conjugates, we have $f(G) \cong< f(x), f(y), f(t) >$, where

$$f(x) = (2, 4, 8, 15, 6)(3, 5, 11, 18, 7)(9, 22, 42, 29, 14)(10, 23, 44, 35, 17)(12, 26, 49, 33, 16)
(13, 27, 52, 39, 19)(20, 40, 55, 30, 38)(21, 41, 65, 36, 32)(24, 47, 56, 31, 34)
(25, 48, 66, 37, 28)(43, 58, 46, 63, 57)(45, 73, 70, 59, 67)(50, 60, 62, 53, 61)
(51, 79, 74, 64, 54)(68, 75, 71, 72, 80)(69, 78, 77, 81, 76).

(60, 72)(63, 77)(64, 81)(68, 75)(69, 76)(70, 74)(71, 79)(73, 78).

$$f(t) = (1, 2, 3)(4, 9, 10)(5, 12, 13)(6, 14, 16)(7, 17, 19)(9, 20, 21)(11, 24, 25)(15, 30, 31)
(33, 59, 60)(34, 61, 62)(35, 63, 64)(38, 52, 67)(40, 66, 68)(41, 69, 70)(44, 71, 72)

Next, we will show that $3^4 : S_5$ is a homomorphic image of $G = \frac{3^5 : mS_5}{{xI}}$. The composition series of $G$ is given by

$$G = G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq 1,$$

where

Since the center of $G$ is of order 1, $G$ is not a central extension. The minimal normal subgroup is of order 81 confirm that we have an abelian subgroup of order 81.

\begin{verbatim}
> X:=AbelianGroup(GrpPerm, [3,3,3,3]);
> s:=IsIsomorphic(X, NL[2]);
> s;
true
\end{verbatim}

The above loop prove that $[2]=G_2 \cong 3^4$. Now, we factor $G$ by $G_2$ such that $G/G_2 \cong q$ to determine the isomporphic type of $q$.

\begin{verbatim}
> q, ff := quo<G1|NL[2]>;
> s:=IsIsomorphic(q,Sym(5));
> s;
true
\end{verbatim}

We established that $q \cong S_5$ and the presentation for $q$ is

\[ <a, b|a^5, b^2, (ab)^4, (a, b)^3 >. \]

Now, we have to find the transversal of $q$ to store its generators and find the action of its elements on the generators of $[2]=G_2$.

\begin{verbatim}
> T:=Transversal(G1, NL[2]);
> a:=T[2];
> b:=T[3];
\end{verbatim}
Thus, the symmetric presentation for $G$ is

$$H = \langle x, y, z, w, a, b | x^3, y^3, z^3, w^3, (x, y), (x, z), (x, w), (y, z), (y, w), (z, w), a^5, b^2, (ab)^4, (a, b)^3, x^a = y, x^b = x, y^a = z, y^b = x^2y^2, z^a = w, z^b = z^2, w^a = x^2y^2z^2w^2, w^b = w^2 \rangle$$

Next, we want to show that $H \cong G$.

```plaintext
> f1, H1, k1 := CosetAction(H, sub<H | Id(H)>);
> s := IsIsomorphic(H1, G1);
> s;
true
```

Hence, $G \cong 3^4 : S_5$.

For more examples, please read chapter 3.

In chapter 2, we will state some important definitions and theorems that are related to our research. In chapter 3, we will use the double coset enumeration (DCE) technique to construct the Cayley graph, then we will solve the extension problems for the finite group $G$. Chapter 4 will illustrate some methods for obtaining homomorphic images for the progenitor $m^*n : N$, where $m = 2, 3, 5, \ldots$. In chapter 5, we want to write the progenitor $2^*n : N$ when the control subgroup $N$ is a transitive. In chapter 6, we will construct two interesting groups by using the wreath product process. In chapter 7, we will write the monomial progenitor $p^*n : N$, where $p$ is prime, and find its homomorphic images by factoring the progenitors by suitable relations. In chapter 8, we will state some facts about the Schur multiplier and universal covering groups. In chapter 9, we continue working on finding more involutory and non-involutory progenitors.
Chapter 2

Preliminaries

In this chapter, we will list all definitions and theorems that are related to our research on group theory, and we will give a short example to illustrate them.

2.1 Group Theory Preliminaries

Definition 2.1 (Permutation). If $X$ is a nonempty set, a permutation of $X$ is a bijection $\phi : X \rightarrow X$.

Note 2.2. The set of all permutations of $X$ is a symmetric group denoted by $S_X$.

Example 2.3. Let $\alpha \in S_4$, where $\alpha = (1234)$. There is a bijection from $S_4$ onto $S_4$, where $X = \{1, 2, 3, 4\}$ such that

$\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 4$, and $\alpha(4) = 1$.

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Definition 2.4 (Disjoint). Two permutations $\alpha, \beta \in S_X$ are disjoint if every $x$ moved by one is fixed by the other. In symbols,

if $\alpha(a) \neq a$, then $\beta(a) = a$, and if $\alpha(b) = b$, then $\beta(b) \neq b$. 
Example 2.5. Let $\alpha, \beta \in S_6$, where $\alpha = (12)(34)$ and $\beta = (56)$. $\alpha$ and $\beta$ are disjoint because

$$\alpha(5) = 5, \text{ but } \beta(5) = 6.$$ 

Theorem 2.6 (Rotman). Let $\alpha \in S_X$. $\alpha$ is either a cycle or a product of disjoint cycles.

Definition 2.7 (Transposition). A permutation is said to be transposition if it exchanges two elements and fixes the rest.

Example 2.8. $\alpha = (12) \in S_3$ is a transposition since it fixes 3 and send 1 to 2 and 2 to 1.

Definition 2.9 (Symmetric Group $S_n$). $S_n$ is a symmetric group that formed by all bijective mapping $\phi : X \rightarrow X$, where $X$ is a nonempty set.

Definition 2.10 (Alternating Group $A_n$). The alternating group $A_n$ is a subgroup of $S_n$ with order equal to $\frac{n!}{2}$.

Definition 2.11 (Abelian Group). A group $G$ is abelian if every pair $a, b \in G$ commutes such as $a \ast b = b \ast a$.

Example 2.12. Let $G = A_3 = \{e, (123), (132)\}$. $G$ is abelian group since every pairs of its distinct elements commutes such that

$$(123)(132) = e = (132)(123)$$
$$e(123) = (123) = (123)e$$
$$e(132) = (132) = (132)e$$

Definition 2.13 (Order of Permutation). Let $\alpha = (x_1, \ldots, x_i)(x_1, \ldots, x_j) \in S_X$, where $\alpha$ is a multiple of two disjoint cyclic. The order of $\alpha$ is the least common multiple of the $i$-cycle and $j$-cycle.

$$|\alpha| = lcm(i, j).$$

Example 2.14. Let $\alpha = (123)(45)$, and $\beta = (12)(3456)$. $\alpha$ has two cycles in length 3 and 2. Thus, the order of $\alpha$ is given by
$|\alpha| = \text{lcm}(3, 2) = 6.$

$\beta$ has two cycles in length 2 and 4. Thus, the order of $\beta$ is given by

$|\alpha| = \text{lcm}(2, 4) = 4.$

**Definition 2.15 (Homomorphism).** Let $G$ and $H$ be groups. A map $\phi : G \rightarrow H$ is said to be **homomorphism** if

$$\text{for } \alpha, \beta \in G, \phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$$

**Note 2.16.** If a homomorphism $\phi$ from $G$ onto $H$ is a bijection, $\phi$ is an **isomorphism** function. So, $G$ is isomorphic to $H$ ($G \cong H$).

**Definition 2.17 (Subgroup).** A nonempty subset $H$ of a group $G$ is a **subgroup** of $G$ if

1. $h \in H$ implies $h^{-1} \in H$.
2. $h, k \in H$ imply $hk \in H$.

A **subgroup** $H$ of $G$ denoted by $H \leq G$.

**Note 2.18.**

- A **proper** subgroup $H$ is any subgroup other than $G$.
- A **trivial** subgroup $H$ is the subgroup that generated by the identity.

**Example 2.19.** Let $H = A_3$ and $G = S_3$.

$H$ is a proper subgroup of $G$.

- Since $e, (123),$ and $(132)$ are in $A_3$, there inverses $e, (132),$ and $(123)$, respectively, are also in $A_3$.
- The product of any two distinct elements of $A_3$ is in $A_3$.

Thus, $A_3 \leq S_3$.

**Definition 2.20 (Cyclic Subgroup Generated by an element).** If $G$ is a group and $a \in G$, then the **cyclic subgroup generated by** $a$, denoted by $< a >$, is the set of all powers of $a$. 
Definition 2.21 (Order of Group). If $G$ is a group and $a \in G$, then the order of $a$ is $|<a>|$, the number of elements in $<a>$. 

Example 2.22. Let 

$G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $H = A_3 = \{e, (123), (132)\}$. 

$H \leq G$: 

Let $h_1 = (123)$ and $h_2 = (132)$. 

1. $h_1 = (123) \in S_3$ implies $h_1^{-1} = (123)^{-1} = (132) \in G$. 

2. $h_1, h_2 \in S_3$ imply $h_1 h_2 = e \in S_3$.

$G = S_3$ is generated by a cyclic subgroup $(123)$ and $(12)$. So, $S_3 =< (123), (12) >$. $|S_3| = 6$. 

$H = A_3$ is generated by a cyclic subgroup $(123)$. So, $A_3 =< (123) >$. $|A_3| = 3$.

Note 2.23. In general, $|S_n| = n!$ and $|A_n| = \frac{n!}{2}$ for every $n \geq 2$.

Definition 2.24 (Right Coset). Let $H \leq G$ and $k \in G$. The subset of $G$

$$\text{Hk} = \{hk : h \in H\}$$

is the right coset of $H$ in $G$.

Note 2.25. $k$ is called a representative of $Hk$.

Definition 2.26 (Index). The index of a subgroup $H$ in $G$ is given by

$$[G : H] = \frac{|G|}{|H|} = \text{the number of right cosets of } H \text{ in } G.$$ 

Example 2.27. Let $G = S_3$, and $H = \{e, (13)\}$. The right coset of $H$ in $G$ are

- $He = \{e, (13)\}$.

- $H(12) = \{(12), (132)\}$.

- $H(23) = \{(23), (123)\}$.

$G = He \cup H(12) \cup H(23) = \text{union of single cosets}$. The index of $H$ in $G$ is

$$[S_3 : H] = \frac{|S_3|}{|H|} = 3 = \text{the number of right cosets of } H \text{ in } G.$$
**Definition 2.28 (Double Coset).** Let $H \leq G$ and $g \in G$. The double coset of $G$ is

$$HgH = \{Hgh | h \in H\}.$$ 

**Note 2.29.** Double cosets are composed of single cosets.

**Example 2.30 (continued with EX2.25).** The double coset of $H$ in $G$ are

- $HeH = \{Heh | h \in H\} = \{H\}$.
- $H(13)H = \{H(13)h | h \in H\} = \{(13), e\} = \{H\}$.
- $H(12)H = \{H(12)h | h \in H\} = \{(12), (123)\} = H(123)H$.
- $H(23)H = \{H(23)h | h \in H\} = \{(23), (132)\} = H(132)$.

$G = HeH \cup H(12)H \cup H(23)H =$ union of double cosets.

**Theorem 2.31 (Lagrange).** If $G$ is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$.

**Definition 2.32 (Exponent).** A group $G$ has exponent $n$ if $g^n = 1$, for all $g \in G$.

**Definition 2.33 (Normal Subgroup).** A subgroup $H \leq G$ is normal in $G$, denoted by $H \triangleleft G$, if

$$g^{-1}Hg = H, \text{ for every } g \in G$$

**Example 2.34.** Let $H = \{e, (12)\} \leq S_3$.

Let $h = (12)$. Then,

$$h^g = (12)^{(23)}$$

$$g^{-1}hg = (23)^{-1}(12)(23)$$

$$= (13) \notin H.$$ 

$H$ is not normal in $S_3$.

**Definition 2.35 (Conjugate).** If $a \in G$, then a conjugate of $g$ in $G$ is an element of the form $a^{-1}ga$ for some $a \in G$. 
Example 2.36. Let $G = S_3$, and $a = (12)$, $g = (123)$ are in $G$. The conjugate of $(123)$ in $S_3$ is

$$(123)^{(12)} = (12)^{-1}(123)(12)$$

$$(213) = (12)(123)(12)$$

$$(213) = (132).$$

Theorem 2.37 (Quotient Group). If $H \triangleleft G$, then the cosets $H$ in $G$ form a group $G/H$ of order $[G : H]$.

Definition 2.38 (Commutator). If $g, h \in G$, the commutator of $g$ and $h$, denoted by $[g, h]$, is

$$[g, h] = ghg^{-1}h^{-1}$$

Definition 2.39 (Derived Subgroup). The derived subgroup of $G$, denoted by $\hat{G}$, is the subgroup of $G$ generated by all the commutators.

Theorem 2.40 (First Isomorphism Theorem (F.I.T)). Let $\phi : G \to H$ be a homomorphism with $\ker \phi$. Then

- $\ker \phi \triangleleft G$.
- $G/\ker \phi \cong \text{im } \phi$

Theorem 2.41 (Second Isomorphism Theorem (S.I.T)). Let $H, K \leq G$ and $H \triangleleft G$. Then

- $H \cap K \triangleleft K$.
- $K/H \cap K \cong HK/H$.

Theorem 2.42 (Third Isomorphism Theorem (T.I.T)). Let $K \leq H \leq G$, where both $H$ and $K$ are normal subgroups of $G$. Then

$$G/K/H/K \cong G/H.$$ 

Theorem 2.43. Let $H \leq G$. $H$ is a maximal normal subgroup of $G$ if there is no normal subgroup $K$ of $G$ with $H < K < G$. 
**Definition 2.44 (Simple).** Let $G \neq 1$. A group $G$ is **simple** if it has no normal subgroups other than $G$ and 1.

**Example 2.45.** $A_n$, where $n \geq 5$ is simple.

**Example 2.46.** $A_4 = \langle (1234), (12) \rangle$ is not simple since it has a normal subgroup $V_4 = \langle (12)(34), (13)(24) \rangle$.

**Theorem 2.47 (Feit-Thoupson).** Every simple group is generated by involution (element of order 2).

**Definition 2.48 (p-group).** A group $G$ is a **$p$-group** if the order of every element of $G$ is a power of $p$.

**Note 2.49.** If $|G| = p^2$, then $G$ is an abelian group.

If $|G| = p^k$, then the center of $G$ is greater than 1. Thus, $Z(G)$ is non-trivial, so $|Z(G)| = p^m$, where $1 \leq m \leq k$.

**Definition 2.50 (Sylow $p$-subgroup).** Let $|G| = p^k m$ and $\gcd(p, m) = 1$. Then, a **Sylow $p$-subgroup** is of order $p^k$.

**Example 2.51.** $|A_5| = 2^2 \cdot 3 \cdot 5$. $A_5$ have Sylow 2-subgroups since $\gcd(2, 3, 5) = 1$.

**Definition 2.52 (Elementry Abelian Group).** Let $G$ be a finite group. It said to be **elementry abelian** group if it is abelian such that every nontrivial element $a \in G$ has a prime order $p$. The general presentation for this group is given by

$$(Z/pZ)^n \cong < a_1, ..., a_n | a_i^p = 1, a_ia_j = a_ja_i >$$

**Definition 2.53 (Stabiliser).** The **stabiliser** of $w$ in $N$ is given by

$N^w = \{ n \in N | w^n = w \}$, where $w$ is a word of $t_i$'s.

**Definition 2.54 (Coset Stabiliser).** The **coset stabiliser** of $Nw$ in $N$ is given by

$N^{(w)} = \{ n \in N | Nw^n = Nw \}$, where $w$ is a word of $t_i$'s.

**Note 2.55.** $N^w \leq N^{(w)}$.

**Example 2.56.** Let $G = S_3$. Then $N^1 = \{ e, (23) \} = N^{(1)}$. 

Definition 2.57 (G-set). If $X$ is a set and $G$ is a group, then $X$ is a $G$-set if there is a function $\phi : G \times X \to X$, denoted by $\phi : (g, x) \to gx$, such that:

1. $1x = x$ for all $x \in X$.
2. $gh(x) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Definition 2.58 (Direct Product $(H \times K)$). Let $G$ be a group and $H, K \leq G$. $G$ is a direct product of $H$ and $K$ if

1. $H, K \triangleright G$.
2. $G = HK$.
3. $H \cap K = 1$.

Example 2.59. Let $G = V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle = \{e, a, b, ab\}$, $H = \{e, a\}$, and $K = \{e, b\}$.

Note:

\[(ab)^2 = 1 \implies abab = 1\]
\[
aba = b^{-1}
\]
\[
ab = b^{-1}a^{-1}
\]
\[
ab = (ab)^{-1}.
\]

- $H, K \triangleleft G$ because
  - $H^e = \{e^e, a^e\} = \{e, a\} \in V_4$.
  - $H^a = \{e^a, a^a\} = \{e, a\} \in V_4$.
  - $H^b = \{e^b, a^b\} = \{e, b^{-1}ab = bab\} = \{e, a\} \in V_4$.
  - $H^{ab} = \{e^{ab}, a^{ab}\} = \{e, (ab)^{-1}a(ab)\} = \{e, a\} \in V_4$.
  - Similarly for $K$.

- $H \cap K = \{e, a\} \cap \{e, b\} = \{e\}$.

- $HK = \{e, a, b, ab\} = V_4$

Hence, $V_4$ is a direct product of $H$ by $K$, denoted by $H \times K$.

Theorem 2.60. Let $G$ be a group with normal subgroups $H$ and $K$. If
• \( HK = G \)
• \( H \cap K = 1. \)

Then \( G \cong H \times K. \)

**Example 2.61.** Let \( G = \langle a \rangle \) and \( |a| = 4. \)

\( G = HK. \) If \( |G| = |H|, \) then \( |H| = 2 = |K|. \) But, \( G \) has only \( \langle a^2 \rangle = \{e, a^2\} = H \) as a normal subgroup of order 2.

There does not exist \( K \subseteq G \ni |K| = 2; H \cap K = 1. \)

\( G \) is not a direct product.

**Definition 2.62 (Complement).** Let \( K \) be a (not necessarily normal) subgroup of a group \( G. \) Then a subgroup \( Q \leq G \) is a complement of \( K \) in \( G \) if

1. \( KQ = G. \)
2. \( K \cap Q = 1. \)

**Example 2.63.** Let \( G = S_3 \) and \( K = A_3 = \{e, (123), (132)\}. \) \( K \) has 3 complements in \( S_3 \) such as

\( \{e, (12)\}, \{e, (13)\}, \) and \( \{e, (23)\}. \)

\( \Rightarrow S_3 = QK, \) where \( Q = \{e, (12), (13), (23)\} \leq S_3. \)

**Example 2.64.** Let \( G = \langle a \rangle \) and \( |a| = 4. \)

\( H = \{1, a^2\} \) does not have a complement in \( G \) since there is not a normal subgroup of order 2.

This example showed that a complement does not always exist.

**Definition 2.65 (Semi-direct Product).** A group \( G \) is a semi-direct product of \( K \) by \( Q, \) denoted by \( G = K \rtimes Q, \) if \( K \triangleleft G \) and \( K \) has a complement \( Q_1 \cong Q. \)

**Example 2.66.** Since \( H = \{e, (12)\} \) is a complement of \( A_3 \) in \( S_3 \) (shown in Ex 2.55) isomorphic to \( \mathbb{Z}_2, \) \( S_3 \) is a semi-direct product of \( A_3 \) by \( \mathbb{Z}_2, \) denoted by \( S_3 = A_3 : \mathbb{Z}_2. \)

**Definition 2.67.** Let \( G \) be a group, and \( H \) and \( K \) are normal subgroups of \( G. \) Then \( G \cong H \times K \) if
1. $HK = G$

2. $H \cap K = 1$

**Definition 2.68 (Center).** Let $G$ be a group. The **center** of $G$, denoted by $(Z(G))$, is the set of all $a \in G$ that commute with every element of $G$.

**Example 2.69.** Let $G = S_3$. Since the identity is the onley permutation that commutes with all $g \in S_3$, $Z(G) = \{ e \}$.

Let $G = A_3$. $Z(G) = \{ g^a = g | a \in G \} = \{(123)\}$.

**Definition 2.70 (Centralizer).** If $a \in G$, then the **centralizer** of $a$ in $G$, denoted by $C_G(a)$, is the set of all $x \in G$ which commute with $a$.

$$C_G(a) = \{ a^x = a | x \in G \}.$$

**Example 2.71.** Let $G = A_3$. We want to find the centraliser of $a = (123) \in A_3$.

$$(123)e = (123) = e(123)$$

$$(123)(123) = (132) = (123)(123)$$

$$(123)(132) = e = (132)(123)$$

Thus, $C_{A_3}((123)) = \{ e, (123), (132) \} = A_3$

**Theorem 2.72 (Rotman).** If $a \in G$, the number of conjugates of $a$ is equal to the index of its centralizer.

$$|a^G| = [G : C_G(a)].$$

**Example 2.73.** $|(123)^{A_3}| = [A_3 : A_3] = 1$.

**Definition 2.74 (Normaliser).** If $H \leq G$, then the **normalizer** of $H$ in $G$ is defined by

$$N_G(H) = \{ a \in G | aHa^{-1} = H \}.$$

**Definition 2.75 (Conjugacy Class).** Let $G$ be a group and $a \in G$. The **conjugacy class** of $a$ is

$$a^G = \{ a^g | g \in G \} = \{ g^{-1}ag | g \in G \}$$
Example 2.76. Let $G = A_4$.
The conjugacy classes of $(12)(34) \in A_4$ is the set

$$(12)(34)^G = \{(12)(34), (13)(24), (14)(23)\}.$$ 

Definition 2.77 (G-Orbit). Let $x \in X$. The set $X^G = \{x^g \mid g \in G\}$ is the G-Orbit.

Example 2.78. Let $G = A_3$ and $X = \{1, 2, 3\}$ be a $G$-set.

$$1^G = \{1^e, 1^{(123)}, 1^{(132)}\} = \{1, 2, 3\} = 2^G = 3^G$$ since $2, 3 \in 1^G$.

The $G$-Orbit $X^G$ is $\{1,2,3\}$.

Definition 2.79 (Faithful). A $G$-set $X$ with action $\alpha$ is faithful if $\tilde{\alpha} : G \to S_X$ is injective.

Definition 2.80 ($k$-transitive). Let $X$ be a $G$-set of degree $n$ and let $k \leq n$ be a positive integer. Then $X$ is $k$-transitive if there is $g \in G$ with $gx_i = y_i$ for $i = 1, \ldots, k$.

Note 2.81. $S_n$ is $n$-transitive, and $A_n$ is $(n - 2)$-transitive.

Example 2.82. Let $G = S_3$. Then

- $X = \{1, 2, 3\}$.
- The orbit of 2 is $2^G = \{2^g \mid g \in G\} = \{1, 2, 3\}$.
- $G$ is a 3-transitive on $X$.

Example 2.83. Let $G = A_4$, $H = \{1, (123), (132)\}$.

Since $1^H = \{1^h : h \in H\} = \{1, 2, 3\}$ and 4 takes to 4 by the identity, $H$ is not transitive on $\{1,2,3,4\}$.

$H$ is not a 4-transitive.

Definition 2.84 (Sharply $k$-transitive). A $k$-transitive $G$-set $X$ is Sharply $k$-transitive if only the identity fixes $k$ distinct elements of $X$.

Definition 2.85 (Dihedral Group). The dihedral group $D_{2n}$ is a group generated by two elements $a$ and $b$ with presentation

$$< a, b \mid a^n = b^2 = (ab)^2 = 1 >.$$ 

The order of $D_{2n}$ is equal to $2n$, where $2n \geq 4$. 
Example 2.86. $D_{10} = D_{2(5)} = \langle a, b | a^5 = b^2 = (ab)^2 = 1 \rangle$. The order of $D_{10}$ is equal to 10.

Definition 2.87 (Quaternion Group). The quaternion group $Q$ is a group generated by two elements $a$ and $b$ with presentation

$$< a, b | a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} > .$$

Example 2.88. $Q_8 = \langle a, b | a^4, b^4, aba = b, a^2 = b^2 \rangle$. The order of $Q_8$ is equal to 8.

Definition 2.89 (Normal Series). A chain of subgroup of $G$

$$G_0 = G \supseteq G_1 \supseteq \ldots \supseteq G_n = 1 \ni G_i < G \quad \forall 1 \leq i \leq n$$

is called a normal series of $G$.

Example 2.90. Consider $D_4 = \langle a, b | a^4 = b^2 = (ab)^2 = 1 \rangle = \{ e, a, a^2, a^3, b, ab, a^2b, a^3b \}$.

$$G = G_0 = D_4, G_1 = \langle a \rangle, G_2 = \langle a^2 \rangle, G_3 = 1.$$  

$D_4 \supseteq \langle a \rangle \supseteq \langle a^2 \rangle \supseteq \{1\}$

is a normal series of $D_4$.

Note 2.91. A normal series is a chief series (cannot add any more groups to the chain).

Definition 2.92 (Subnormal Series). A chain of subgroup of $G$

$$G_0 = G \supseteq G_1 \supseteq \ldots \supseteq G_n = 1 \ni G_i < G \quad \forall 0 \leq i \leq n - 1$$

is called a subnormal series of $G$.

Example 2.93. $S_4 \supseteq A_4 \supseteq \{1\}$ is a subnormal series of $S_4$.

Note 2.94. We can add another group to the chain without repetition.

Example 2.95. $S_4 \supseteq A_4 \supseteq V_4 \supseteq \{1\}$

Definition 2.96. $SL(n, \mathbb{F})$ is a special linear of $n \times n$ matrices over finite field $\mathbb{F}$ with determinant equal to 1.

Example 2.97.

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \text{ or a square} \right\}$$
Definition 2.98. $GL(n, \mathbb{F})$ is a general linear group of $n \times n$ matrices over finite field $\mathbb{F}$.

Example 2.99.

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

Definition 2.100. $PSL(n, \mathbb{F})$ is a projective special linear group of $n \times n$ matrices over finite field $\mathbb{F}$ that formed by factoring $SL$ by its center.

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = SL(n, \mathbb{F})/Z(SL(n, \mathbb{F})).$$

Definition 2.101. $PGL(n, \mathbb{F})$ is a projective general linear group of $n \times n$ matrices over finite field $\mathbb{F}$ that formed by factoring $GL$ by its center.

$$PGL(n, \mathbb{F}) = GL(n, \mathbb{F})/Z(GL(n, \mathbb{F})).$$

Definition 2.102 (Minimal Normal Subgroup). A Minimal Normal Subgroup $N$ of $G$ is a direct product of simple groups.

Note 2.103. If $N$ is a minimal $p$-subgroup then $N$ is a direct product of elementry $p$-subgroups

Example 2.104. Let $N$ be a minimal normal subgroup of order 4. Then, $N \cong 2 \times 2$.

2.2 Group Extension Preliminaries

Definition 2.105 (Group Extensions). An extension of a group $N$ by a group $H$ is a group $G$ with a normal subgroup $M$ such that $M \cong N$ and $G/M \cong H$.

Example 2.106. Let $G = \mathbb{Z}_6$. $G$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$.

$$\mathbb{Z}_6 = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5\}, \quad |a| = 6.$$

$$\mathbb{Z}_2 = \langle a^3 \rangle = \{e, a^3\}.$$

$$\mathbb{Z}_6/\mathbb{Z}_2 = \langle a \rangle / \langle a^3 \rangle$$

$$= \{\{e, a^3\}, a\{e, a^3\}, a^2\{e, a^3\}\}$$

$$= \langle a\{e, a^3\} \rangle.$$

$$\mathbb{Z}_6/\mathbb{Z}_2 \cong \mathbb{Z}_3.$$
Example 2.107. Let $G = S_3 = \langle a, b | a^3, b^2, (ab)^2 \rangle$. $S_3$ is not an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ because it does not have a normal subgroup of order 2. Thus, $H_1 = \{e, (12)\}$, $H_2 = \{e, (13)\}$, and $H_3 = \{e, (23)\}$ are subgroups of order 2 but none of them is normal.

$$
\langle (123) \rangle = \{e, (123), (132)\}, \\
S_3/\langle (123) \rangle = \{(123), (12)(13)\} \\
S_3/\langle (123) \rangle \cong \mathbb{Z}_2 \\
S_3/\mathbb{Z}_3 \cong \mathbb{Z}_2.
$$

Hence, $S_3$ is an extension of $\mathbb{Z}_3$ by $\mathbb{Z}_2$.

Note 2.108. The previous two examples showed that the composition series of a group $G$ is not unique. Thus, $S_3 \not\cong \mathbb{Z}_6$.

Given groups $H$ and $N$ there are many extensions of $N$ by $H$ such as

1. Direct product extension.
2. Semi-direct product extension.
3. Central extension.

We have defined the first two extensions in page 26-27.

Definition 2.109 (Central Extensions $N \rhd H$). $N$ is the center of $G$ if $G$ is a central extension of $N$ by $H$. It is based on $\Psi : H \times H \rightarrow N$.

All elements of $N$ commute with all the elements of $G$ (N is abelian). The group operation of $G = N \uparrow H$ is

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot n_2 \cdot \psi(h_1, h_2), h_1 h_2).$$

Example 2.110. Let a group $G$ has the following composition series $G = G_1 \supseteq G_2$, where $G = (G_1/G_2) = (G_1/1) = G_1 = C_3A_7 = NH$ and $A_7 = \langle (123), (234567) \rangle$.

Since the center of $G$ is $Z(G) = \{g^a = g | a \in G\} = \{(123)\} = N$, this is a central extension of $C_3$ by $A_7$ denoted by $3^*A_7$. 
Example 2.111. Let a group \( G = D_4 = \{a^4, b^2, (ab)^2\} = \{1, a, a^3, b, ab, a^2b, a^3, b\} \). 

\( D_4 \) is either a central or a direct product extension. The normal lattice of \( D_4 \) is

\[
\begin{array}{cc}
[1] = e & \\
\downarrow & \\
[2] = \{1, a^2\} & \\
\{1, a^2, b, a^2b\} = [3] & \{1, a, a^2, a^3\} = [4] \\\n\{1, a^2, b\} = [5] = D_4 & \\
\end{array}
\]

\( D_4 \) has 2 normal subgroups of order 4. Thus,

\[
D_4 \cong [2] \times [3], \quad \text{but} \quad [2] \cap [3] \neq 1.
\]

\[
D_4 \cong [2] \times [4], \quad \text{but} \quad [2] \cap [4] \neq 1.
\]

So, \( D_4 \) is must be a central extension \( 2^\bullet (C_2 \times C_2) = 2^\bullet C_4 \),

Where

\[
D_4/Z(D_4) = D_4/\{1, a^2\}
\]

\[
= \{\{1, a^2\}, a\{1, a^2\}, b\{1, a^2\}, ab\{1, a^2\}\}
\]

\[
= C_4
\]

Therefore,

\[
D_4/Z(D_4) \cong C_4 \Rightarrow D_4 \cong 2^\bullet C_4
\]

Definition 2.112 (Mixed Extensions \( N \bullet H \)). It is combined the properties of a semi-direct product and a central extension. \( N \) is a normal subgroup, but it is not a central.

\( \exists \Phi : H \rightarrow Aut(N) \) and \( \Psi : H \times H \rightarrow N \). The group operation of \( G \) is

\[
(n_1, h_1) \ast (n_2, h_2) = (n_1 \ast h_1^{h_2}, \psi(h_1, h_2), h_1 h_2).
\]

Example 2.113. Let \( G \) has the following composition series \( G = C_2C_2C_2A_5 = Q_8A_5 \).
• $G \not\cong Q_8 \times A_5$ because $G$ does not have a normal subgroup of order 60.

• $G \not\cong Q_8 : A_5$ because $A_5 \leq Aut(Q_8) = S_4$.

• $G \not\cong Q_8 \cdot A_5$ because $Z(G) \neq Q_8$.

Thus, $G$ is a mixed extension of $Q_8$ by $A_5$ denoted by $G \cong Q_8 \cdot A_5$
Chapter 3

Double Coset Enumeration and Composition Factors

The objective of this chapter is to find the index of \(N\) in \(G = p^n : N\), where \(G\) is a semi-direct product of a free product \(p^n\) of \(n\) cyclic groups of order \(p\) by the control group \(N\). \(G = p^n : N\) is called a progenitor. Since \(G\) is an infinite group, we must add one or more relations of the form \(\pi_i \omega_i\) in order to form a finite group \(G = \frac{p^n : N}{\pi_i \omega_i}\). To construct these finite groups, we will use the double coset enumeration (DCE) technique. Thus, we need to express \(G\) as a union of double cosets \(NgN\), where \(g\) is an element of \(G\). Thus, \(G = NeN \cup Ng_1N \cup Ng_2N \cup \ldots\), where \(g_i\)'s are words in \(t_i\)'s. Now, we need to find out all \(\{w\} = \{Nw^n|n \in N\}\), which are the double cosets, and how many single cosets each of them contains. The desired result of DCE is complete when the set of right cosets is closed under right multiplication by \(t_i\)'s.

In this chapter, we will consider the progenitor \(2^*^4 : A_4\). Then, this progenitor will factored by appropriate relations to give this homomorphic images \(S_5\), \(L_2(11) \times 3\), and \(L_2(11)\). Similarly, we will consider the progenitor \(2^*^5 : D_{10}\), then we will factore it by suitable relations to give the homomorphic image \(PGL_2(11)\). We have discovered several homomorphic images that all listed in a table at the end of the chapter.

**Note 3.1.** \(G = 2^n : N\) is called an involutory progenitor.
3.1 \( S_5 \) as a Homomorphic Image of \( 2^{*4} : A_4 \)

3.1.1 The Construction of \( S_5 \) over \( A_4 \)

Consider the group \( G = \langle x, y, t | x^2, y^3, (xy)^3, t^2, (t, y) \rangle \) factored by \([y^2xt^x] \) and \([ytt^x] \), where \( G = 2^{*4} : A_4 \), \( N = A_4 = \langle x, y, (12)(34), (123) \rangle \), and let \( t = t_4 \). So,

\[
(y^2xt^x)^2 = e
\]

\[
(y^2x(t_4t_3)^yxt_4t_3) = e
\]

\[
((y^2)x^2t_3t_1t_4t_3 = e
\]

\[
(134)t_4t_3t_4t_3t_4 = e
\]

\[
(134)t_3t_1 = t_3t_4 \implies Nt_3t_1 = Nt_3t_4
\]

\[
(ytt^x)^4 = e
\]

\[
(yt_4t_3t_4)^4 = e
\]

\[
y^4(t_4t_3t_4)^y^3(t_4t_3t_4)^y^3(t_4t_3t_4)^y^2(t_4t_3t_4)^y = e
\]

Thus, we have the following equivalent cosets. Let \( t = t_4 \to t_0 \).

\[
31 \approx 30 \approx 32,
\]

\[
23 \approx 20 \approx 21,
\]

\[
12 \approx 10 \approx 13, \text{ and}
\]

\[
03 \approx 01 \approx 02.
\]

The main goal of this example is to show that \( G = \frac{2^{*4}A_4}{(134)t_3t_1 = t_3t_0} \cong S_5 \). First, we start with the double coset \( N \), denoted by \([*] \), which consists of the single coset \( N \). Then,
we consider the double coset $NwN$, where $w$ is a word of length one and so on until the set of right cosets is closed under right multiplication by $t'_i$s where $i = 0, \cdots, 3$.

- Consider $NeN$, denoted by $[\ast]$.
  
  $NeN = \{N\}$.

  The number of right cosets in $[\ast]$ is equal to $\frac{|N|}{|N|} = \frac{12}{12} = 1$.

  Since $N$ is a transitive on $\{0,1,2,3\}$, the orbit of $N$ is $\{0,1,2,3\}$.

  Pick a representative from the orbit $\{0,1,2,3\}$, say $t_0$, and determine the double coset that contain $Nt_0$.

**Word of length 1.**

- Consider $Nt_0N$ denoted by $[0]$.
  
  $N^0 = \langle (123), (132) \rangle = N^{(0)}$.

  The number of right cosets in $[0]$ is equal to $\frac{|N|}{|N^{(0)}|} = \frac{12}{3} = 4$.

  The orbits of $N^{(0)}$ on $\{0,1,2,3\}$ are $\{0\}$ and $\{1,2,3\}$.

  Pick a representative from each orbit, say $t_0$ and $t_1$ respectively, and determine the double cosets that contain $Nt_0t_0$ and $Nt_0t_1$.

  $Nt_0t_0 \in [\ast]$ (1 symmetric generator goes back to the double coset $[\ast]$).

  $Nt_0t_1 \in [01]$ (3 symmetric generators take to the double coset $[01]$).

**Word of length 2.**

- Consider $Nt_0t_1N$ denoted by $[01]$.
  
  $N^{01} = \langle e \rangle$. The right cosets in $[01]$ are not distinct. From above relations, we see that

  $$01 \approx 02 \approx 03 \Rightarrow Nt_0t_1 = Nt_0t_2 = Nt_0t_3$$

  Thus, there exist $\{n \in N | N(t_0t_1)^n = Nt_0t_1\}$ such that

  $$N(t_0t_1)^{(123)} = Nt_0t_2 \xrightarrow{\text{relation}} Nt_0t_1 \Rightarrow (123) \in N^{(01)}$$

  $$N(t_0t_1)^{(132)} = Nt_0t_3 \xrightarrow{\text{relation}} Nt_0t_1 \Rightarrow (132) \in N^{(01)}$$

  So, $N^{(01)} = \langle (123) \rangle$.

  The number of right cosets in $[01]$ is equal to $\frac{|N|}{|N^{(01)}|} = \frac{12}{3} = 4$. 
The orbits of $N^{(01)}$ on \{0,1,2,3\} are \{0\} and \{1,2,3\}.

$N_{t_0}t_1t_0 \in [010]$ (1 symmetric generator go to the double coset [010]).

$N_{t_0}t_1t_1 = N_{t_0} \in [0]$ (3 symmetric generators take back to the double coset [0]).

**Word of length 3.**

- Consider $N_{t_0}t_1t_0N$ denoted by [010].

$N^{010} = \langle e \rangle$. From previous relations, we obtain the following

\[
\begin{align*}
t_0t_1t_0 &= (301)t_0t_3t_0 \Rightarrow N_{t_0}t_1t_0 = N_{t_0}t_3t_0. \\
t_0t_1t_0 &= (120)t_0t_2t_0 \Rightarrow N_{t_0}t_1t_0 = N_{t_0}t_2t_0. \\
t_0t_1t_0 &= t_0(210)t_1t_2 \\
&= (210)t_2t_1t_2 \Rightarrow N_{t_0}t_1t_0 = N_{t_2}t_1t_2. \\
t_0t_1t_0 &= t_0(031)t_1t_3 \\
&= (031)t_3t_1t_3 \Rightarrow N_{t_0}t_1t_0 = N_{t_3}t_1t_3. \\
t_0t_1t_0 &= (301)t_0t_3t_0 = (301)t_0(130)t_3t_1 \\
&= (310)t_1t_3t_1 \Rightarrow N_{t_0}t_1t_0 = N_{t_1}t_3t_1. \\
t_0t_1t_0 &= (301)t_0t_3t_0 = (301)t_0(023)t_3t_2 \\
&= (32)(10)t_2t_3t_2 \Rightarrow N_{t_0}t_1t_0 = N_{t_2}t_3t_2. \\
t_0t_1t_0 &= (120)t_0t_2t_0 = (120)t_0(012)t_2t_1 \\
&= (102)t_1t_2t_1 \Rightarrow N_{t_0}t_1t_0 = N_{t_1}t_2t_1. \\
t_0t_1t_0 &= (120)t_0t_2t_0 = (120)t_0(320)t_2t_3 \\
&= (10)(23)t_3t_2t_3 \Rightarrow N_{t_0}t_1t_0 = N_{t_3}t_2t_3. \\
t_0t_1t_0 &= (210)t_2t_1t_2 = (210)(021)t_2t_0t_2 \\
&= (201)t_2t_0t_2 \Rightarrow N_{t_0}t_1t_0 = N_{t_2}t_0t_2. \\
t_0t_1t_0 &= (031)t_3t_1t_3 = (031)(103)t_3t_0t_3 \\
&= (013)t_3t_0t_3 \Rightarrow N_{t_0}t_1t_0 = N_{t_3}t_0t_3. \\
t_0t_1t_0 &= (201)t_2t_0t_2 = (201)t_2(102)t_0t_1 \\
&= t_1t_0t_1 \Rightarrow N_{t_0}t_1t_0 = N_{t_1}t_0t_1.
\end{align*}
\]

Hence,
Therefore, \( N^{(010)} = \{ n \in \mathbb{N} \mid N(010)^n = N(010) \} \).

Thus, \( N^{(010)} \geq \langle (12)(34), (123) \rangle = A_4 \).

The number of right cosets in \([010]\) is equal to \( \frac{|N|}{|N^{(010)}|} = \frac{12}{12} = 1 \).

The orbits of \( N^{(010)} \) on \{0, 1, 2, 3\} is \{0, 1, 2, 3\}.

\( Nt_0t_1t_0t_0 = Nt_0t_1 \) (4 symmetric generators go back to the double coset \([01]\))

Now, we can construct the Cayley diagram of \( S_5 \) over \( A_4 \).

![Cayley Diagram for \( S_5 \) Over \( A_4 \)](image)

Since the set of right cosets is closed under right multiplication by \( t_i's \) where \( i = 0, \cdots, 3 \), we can determine the index of \( N \) in \( G \). We conclude that

\[
G = \frac{2^{1+4}A_4}{(134)t_3t_1 = t_3t_0} 
\]

\[
|G| \leq \left( |N| + \frac{|N|}{|N(0)|} + \frac{|N|}{|N(01)|} + \frac{|N|}{|N(010)|} \right) \times |N| 
\]

\[
|G| \leq (1 + 4 + 4 + 1) \times 12 
\]

\[
|G| \leq (10 \times 12) 
\]

\[
|G| \leq 120
\]
Table 3.1: Single Coset Action of $S_5$ Over $A_4$

<table>
<thead>
<tr>
<th>Label</th>
<th>Single Cosets</th>
<th>$x$</th>
<th>$y$</th>
<th>$t_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$N$</td>
<td>$N$</td>
<td>$N$</td>
<td>$N_{t0}$</td>
</tr>
<tr>
<td>2</td>
<td>$N_{t0}$</td>
<td>$N_{t3}$</td>
<td>$N_{t0}$</td>
<td>$N$</td>
</tr>
<tr>
<td>3</td>
<td>$N_{t3}$</td>
<td>$N_{t0}$</td>
<td>$N_{t1}$</td>
<td>$N_{t3t0}$</td>
</tr>
<tr>
<td>4</td>
<td>$N_{t1}$</td>
<td>$N_{t2}$</td>
<td>$N_{t2}$</td>
<td>$N_{t1t0}$</td>
</tr>
<tr>
<td>5</td>
<td>$N_{t3t0}$</td>
<td>$N_{t0t3}$</td>
<td>$N_{t1t0}$</td>
<td>$N_{t3}$</td>
</tr>
<tr>
<td>6</td>
<td>$N_{t2}$</td>
<td>$N_{t1}$</td>
<td>$N_{t3}$</td>
<td>$N_{t2t0}$</td>
</tr>
<tr>
<td>7</td>
<td>$N_{t1t0}$</td>
<td>$N_{t2t3}$</td>
<td>$N_{t2t0}$</td>
<td>$N_{t1t0t0} = N_{t1}$</td>
</tr>
<tr>
<td>8</td>
<td>$N_{t0t3}$</td>
<td>$N_{t3t0}$</td>
<td>$N_{t0t1}$</td>
<td>$N_{t0t3t0}$</td>
</tr>
<tr>
<td>9</td>
<td>$N_{t2t0}$</td>
<td>$N_{t1t3}$</td>
<td>$N_{t3t0}$</td>
<td>$N_{t2t0t0} = N_{t2}$</td>
</tr>
<tr>
<td>10</td>
<td>$N_{t0t1t0}$</td>
<td>$N_{t3t2t3}$</td>
<td>$N_{t0t2t0}$</td>
<td>$N_{t0t1t0t0} = N_{t0t1}$</td>
</tr>
</tbody>
</table>

We find

\[ f(x) = (23)(46)(58)(79). \]

\[ f(y) = (346)(579). \]

\[ f(t) = (12)(35)(47)(69)(8, 10). \]

Next, we note that $G = 2^*4 : A_4 = < t_0, x, y >$ acts on the ten cosets above and the actions of $t$, $x$, and $y$ on the ten cosets is well-defined. Thus, we have a homomorphism $f : G \rightarrow S_{10}$. Then $f(G) = < f(x), f(y), f(t) >$; that is, the homomorphic image of $2^*4 : A_4$ is of the order, $| < f(x), f(y), f(t) > | = 120$. Since $t$ has exactly 4 conjugates, we have $f(G) \cong < f(x), f(y), f(t) >$.

Hence,

\[ G/Ker_f \cong < f(x), f(y), f(t) > = f(G) \]

\[ \Rightarrow |G/Ker_f| = |f(G)| \]

\[ \Rightarrow |G| = |f(G)| = 120 \]

\[ \Rightarrow |G| = |Ker_f| \ast 120 \]

\[ \Rightarrow |G| \geq 120. \]

From the Cayley diagram, we observed that $|G| \leq 120$ and $|Ker_f| = 1$. Hence, $Ker_f = 1$ and $|G| = 120$.

\[ G \cong < f(x), f(y), f(t) > \]

In the next section, we will show that $G \cong S_5$. 

We find

\[ f(x) = (23)(46)(58)(79). \]

\[ f(y) = (346)(579). \]

\[ f(t) = (12)(35)(47)(69)(8, 10). \]
3.1.2 Proof of the Isomorphism

Consider the group

\[ G = \langle x, y, t \mid x^2, y^3, (xy)^3, t^2, (t, y), (y^2xt)^2, (ytttx^2)^4 \rangle. \]

By using MAGMA, we have the composition series \( G \supseteq G_1 \supseteq G_2 \), where

\[ G = (G/G_1)(G_1/G_2) = (G/G_1)(G_1/1) = C_2A_5. \]

Now, we want to determine the extension type of this group.

\[ \text{CompositionFactors}(G_1); \]
\[ G \]
\[ | \text{Cyclic}(2) \]
\[ * \]
\[ | \text{Alternating}(5) \]
\[ 1 \]

Since the center of \( G \) is of order 1, this is not a central extension. Moreover, by looking at the normal lattice of \( G \), we see that it does not have a normal subgroup of order 2. Thus, this is not a direct product.

\[ \text{D:=DirectProduct}(NL[2],NL[3]); \]
\[ s,t:=\text{IsIsomorphic}(D,G_1); \]
\[ s; \]
\[ \text{false} \]

We find that it is a semi-direct product. Thus, we have \( A_5 \) extended by \( C_2 \). Then, we must find the action of \( C_2 \) on the generators \( a \) and \( b \) of \( A_5 \).

\[ \text{H<a,b>:=Group<a,b|a^2,b^3,(a*b)^5>}; \]
\[ \#H; \]
\[ 60 \]
\[ f1,H1,k1:=\text{CosetAction}(H,sub<H|\text{Id}(H)>); \]
\[ s,t:=\text{IsIsomorphic}(H1,NL[2]); \]
\[ s; \]
\[ \text{true} \]
\[ s,t:=\text{IsIsomorphic}(H1,\text{Alt}(5)); \]
\[ s; \]
\[ \text{true} \]
\[ a:=NL[2].1; \]
\[ b:=NL[2].2; \]
By using Schreier System, we compute the action of \( c \) on \( a \) and \( b \) and write a presentation for \( A_5 : C_2 \).

\[
A := [\text{Id(NN)} : i \in [1..2]]; \\
\text{for } i \in [1..60] \text{ do} \\
\text{if } a^i U \text{ eq ArrayP}[i] \text{ then } A[1] := \text{Sch}[i]; \text{ Sch}[i]; \text{ end if; end for;}
\]

\[
\text{for } i \in [1..60] \text{ do} \\
\text{if } b^i U \text{ eq ArrayP}[i] \text{ then } A[2] := \text{Sch}[i]; \text{ Sch}[i]; \text{ end if; end for;}
\]

\[
\text{HH} := \langle a, b, c | a^2, b^3, (a b)^5, c^2, a^c a, b^c a^b - 1 a^b a^b - 1 >; \\
\#HH; \\
f2, H2, k := \text{CosetAction(HH, sub<HH|Id(HH)>)};
\]

\[
s := \text{IsIsomorphic(H2,G1)};
\]

\[
> s; \\
\text{true}
\]

At the end, we check if this presentation is isomorphic with \( S_5 \).

\[
> s := \text{IsIsomorphic(H2, Sym(5))}; \\
> s; \\
\text{true}
\]

So, we obtained that \( G = A_5 : C_2 \cong S_5 \).

### 3.2 \( L_2(11) \times 3 \) as a Homomorphic Image of \( 2^* 4 : A_4 \)

#### 3.2.1 The Construction of \( L_2(11) \times 3 \) over \( A_4 \)

Consider the group

\[
G = 2^* 4 : A_4 = \langle x, y, t | x^2, y^3, (xy)^3, (y^{-1} x^{-1} y x)^2, t^2, (t, y) \rangle \text{ factored by } [xt]^5.
\]
\( N = A_4 = \langle x, y \rangle = \langle (12)(34), (123) \rangle \) and let \( t = t_4 \). So,

\[
(xt)^5 = e \\
(xt_4)^5 = e \\
x^5 t_4 x^3 t_4 x^2 t_4 x t_4 = e \\
x t_4 t_3 t_4 t_3 t_4 = e \\
x t_4 t_3 t_4 = t_4 t_3 \\
(12)(34) t_4 t_3 t_4 = t_4 t_3.
\]

\( t_1 = t_4^y, \ t_2 = t_1^y, \ t_3 = t^x \)

By conjugation with all elements in \( A_4 \), we will have the following relations:

\[
(12)(34) t_3 t_4 t_3 = t_3 t_4, \ (12)(34) t_4 t_3 t_4 = t_4 t_3, \\
(12)(34) t_2 t_1 t_2 = t_2 t_1, \ (12)(34) t_1 t_2 t_1 = t_1 t_2, \\
(23)(14) t_4 t_1 t_4 = t_4 t_1, \ (23)(14) t_1 t_4 t_1 = t_1 t_4, \\
(23)(14) t_3 t_2 t_3 = t_3 t_2, \ (23)(14) t_2 t_3 t_2 = t_2 t_3, \\
(13)(24) t_4 t_2 t_4 = t_4 t_2, \ (13)(24) t_1 t_3 t_1 = t_1 t_3, \\
(13)(24) t_3 t_1 t_3 = t_3 t_1, \ (13)(24) t_2 t_4 t_2 = t_2 t_4.
\]

**The double coset enumeration:**

- Consider \( N e N \) which is denoted by \([*]\).
  
  The number of the right cosets in \([*] = \lvert N \rvert / \lvert N \rvert = \frac{12}{12} = 1.\)

  The orbit of \( N \) on \( \{0, 1, 2, 3\} \) is \( \{0, 1, 2, 3\} \).

  Pick a representative from the orbit \( \{0, 1, 2, 3\} \), say 0, and determine the double coset that contain \( Nt_0 \).

**Word of length 1.**

- Consider \( Nt_0 N \) which is denoted by \([0]\).
  
  \( N^0 = \langle (123), (132) \rangle = N^{(0)} \).

  The number of the right cosets in \([0] \) is equal to \( \lvert N \rvert / \lvert N^{(0)} \rvert = \frac{12}{3} = 4.\)

  The orbit of \( N^{(0)} \) on \( \{0, 1, 2, 3\} \) are \( \{0\} \) and \( \{1, 2, 3\} \).

  \( Nt_0 t_0 \in [*] \) (1 symmetric generator goes back to the double coset \([*]\)).

  \( Nt_0 t_1 \in [01] \) (3 symmetric generators take to the double case \([01]\)).
Word of length 2.

- Consider $N_{t_0t_1N}$ which is denoted by $[01]$.
  
  $N_{01}^{01} = \langle e \rangle = N^{(01)}$.

  The number of the right cosets in $[01] = \frac{|N|}{|N^{(01)}|} = \frac{12}{1} = 12$.

  The orbits of $N^{(01)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.

  $N_{t_0t_1t_0} = N(14)(23)t_0t_1 = Nx^yt_0t_1$ (1 symmetric generator goes back to the double coset $[01]$)

  $N_{t_0t_1t_1} = N_{t_0}$ (1 symmetric generator goes back to the double coset $[0]$)

Word of length 3.

- Consider $N_{t_0t_1t_2N}$ which is denoted by $[012]$.
  
  $N_{012}^{012} = \langle e \rangle = N^{(012)}$.

  The number of the right cosets in $[012] = \frac{|N|}{|N^{(012)}|} = \frac{12}{1} = 12$.

  The orbits of $N^{(012)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.

  $N_{t_0t_1t_2t_0} \in [0120]$.

  $N_{t_0t_1t_2t_1} \in [0121]$.

  $N_{t_0t_1t_2t_2} = N_{t_0t_1}$ (1 symmetric generator goes back to the double coset $[01]$).

  $N_{t_0t_1t_2t_3} \in [0123]$.

- Consider $N_{t_0t_1t_3N}$ which is denoted by $[013]$.
  
  $N_{013}^{013} = \langle e \rangle = N^{(013)}$.

  The number of the single cosets in $[013] = \frac{|N|}{|N^{(013)}|} = \frac{12}{1} = 12$.

  The orbits of $N^{(013)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.

  $N_{t_0t_1t_3t_0} \in [0130]$.

  $N_{t_0t_1t_3t_1} \in [0131]$.

  $N_{t_0t_1t_3t_2} \in [0132]$.

  $N_{t_0t_1t_3t_3} = N_{t_0t_1}$ (1 symmetric generator goes back to the double coset $[01]$)

Word of length 4.

- Consider $N_{t_0t_1t_2t_1N}$ which is denoted by $[0121]$. 
Because the relation

\[ N_{t_0t_1t_2t_1} = N(12)(34)[t_0t_1t_3]^{(432)} = N_{xt_3t_1t_2} = N_{t_3t_1t_2}. \]

where \(N_{t_3t_1t_2} \in [013]\).

There exist \(\{n \in N|N(t_0t_1t_3)^n = N_{t_3t_1t_2}\}\).

Thus, \(N(t_0t_1t_3)^{(432)} = N_{t_3t_1t_2}\).

\[[0121] = [013]\]

Therefore, \(N_{t_0t_1t_2t_1}N\) is not a new double coset and collapses \(\Rightarrow 1\) takes to \([013]\).

- Consider \(N_{t_0t_1t_2t_0}N\) which is denoted by \([0120]\).

\(N_{[0120]} = \langle e \rangle\). The right cosets in \([0120]\) are not distinct. From above relations, we see that

\[
t_0t_1t_2t_0 = t_0t_1(13)(24)t_2t_0t_2
= (13)(24)t_2t_3t_2t_0t_2
= (13)(24)(23)(14)t_2t_3t_2t_0t_2
= (12)(34)t_2t_3t_0t_2
= xt_2t_3t_0t_2. \quad \text{Thus} \quad 0120 \approx 2302.
\]

Therefore, \(N_{(0120)} = \{n \in N|N(0120)^n = N(2302)\}\).

\[ N(0120)^{(13)(24)} = N2302 = N0120 \Rightarrow (13)(24) \in N_{(0120)} \]

\(N_{(0120)} \geq \langle (13)(24) \rangle\).

\[ [0120] = \frac{|N|}{|N_{(0120)}|} = \frac{12}{2} = 6. \]

The orbits of \(N_{(0120)}\) on \(\{0, 1, 2, 3\}\) are \(\{1, 3\}\) and \(\{0, 2\}\).

\(N_{t_0t_1t_2t_0t_1} \in [01201]\) (2 symmetric generators take to the double coset \([01201]\)).

\(N_{t_0t_1t_2t_0t_0} = N_{t_0t_1t_2}\) (2 symmetric generators go back to the double coset \([012]\)).
Consider \( N_{t_0t_1t_2t_3}N \) which is denoted by \([0123]\).
\[ N_{0123} = \langle e \rangle = N^{(0123)} \]
The number of right cosets in \([0123]\) = \( \frac{|N|}{|N_{0123}|} = \frac{12}{1} = 12 \).
The orbits of \( N^{(0123)} \) on \( \{0, 1, 2, 3\} \) are \( \{0\} \), \( \{1\} \), \( \{2\} \), and \( \{3\} \).
\( N_{t_0t_1t_2t_3t_0} \in [01230] \).
\( N_{t_0t_1t_2t_3t_1} \in [01231] \).
\( N_{t_0t_1t_2t_3t_2} \in [01232] \).
\( N_{t_0t_1t_2t_3t_3} = N_{t_0t_1t_2} \) (1 symmetric generator goes back to the double coset [012]).

Consider \( N_{t_0t_1t_3t_1}N \) which is denoted by \([0131]\).
Because the relation
\[
N_{t_0t_1t_3t_1} = N(13)(24)[t_0t_1t_2]^{(423)} = N_{t_2t_1t_3}.
\]
where \( N_{t_2t_1t_3} \in [012] \).
There exist \( \{n \in N| N(t_0t_1t_2)^n = N_{t_2t_1t_3}\} \).
Thus, \( N(t_0t_1t_2)^{423} = N_{t_2t_1t_3} \).
\[
[0131] = [012]
\]
Therefore, \( N_{t_0t_1t_3t_1}N \) is not a new double coset and collapses \( \Rightarrow \) one symmetric generator will take to [012].

Consider \( N_{t_0t_1t_3t_2}N \) which is denoted by \([0132]\).
\[ N_{0132} = \langle e \rangle = N^{(0132)} \]
The number of right cosets in \([0132]\) = \( \frac{|N|}{|N_{0132}|} = \frac{12}{1} = 12 \).
The orbits of \( N^{(0132)} \) on \( \{0, 1, 2, 3\} \) are \( \{0\} \), \( \{1\} \), \( \{2\} \), and \( \{3\} \).
\( N_{t_0t_1t_3t_2t_0} \in [01320] \).
\( N_{t_0t_1t_3t_2t_1} \in [01321] \).
\( N_{t_0t_1t_3t_2t_2} = N_{t_0t_1t_3} \) (1 symmetric generator takes back to the double coset [013]).
\( N_{t_0t_1t_3t_2t_3} \in [01323] \).
• Consider \( N_{t_0 t_1 t_3 t_0} N \) denoted by [0130].
\[ N^{[0130]} = \langle e \rangle. \] The right cosets in [0130] are not distinct. But,
\[
t_0 t_1 t_3 t_0 = t_0 t_1 (12)(34) t_3 t_0 t_3 \\
= (12)(34) t_3 t_2 t_3 t_0 t_3 \\
= (12)(34)(23)(14) t_3 t_2 t_3 t_3 t_0 t_3 \\
= (13)(24) t_3 t_2 t_0 t_3.
\]
Thus, 0130 \( \approx \) 3203.

Therefore, \( N^{(0130)} = \{ n \in N | N(0130)^n = N(3203) \} \).

\( N(0130)^{(12)(34)} = N3203 = N0130 \Rightarrow (12)(34) \in N^{(0130)} \)

\( N^{(0130)} \geq \{ e, (12)(34) \} = \langle (12)(34) \rangle. \)

The number of right cosets in [0130] = \( \frac{|N|}{|N^{[0130]}|} = \frac{12}{2} = 6. \)

The orbits of \( N^{(0132)} \) on \{0,1,2,3\} are \{1, 2\} and \{0, 3\}.

\( N_{t_0 t_1 t_3 t_0 t_1} \in [01301] \) (2 symmetric generators take to the double coset [01301]).
\( N_{t_0 t_1 t_3 t_0 t_0} = N_{t_0 t_1 t_3} \) (2 symmetric generators go back to the double coset [013]).

We now continue the DCE by proceeding in this manner until we obtain all of the 20 double cosets \( [w] \). Now, we can determine the index of \( N \) in \( G \).

\[
|G| \leq \left( |N| + \frac{|N|}{|N^{[0]}|} + \frac{|N|}{|N^{[01]}|} + \frac{|N|}{|N^{[012]}|} + \cdots + \frac{|N|}{|N^{[01231243]}|} \right) \times |N|
\]

\[
|G| \leq (1 + 4 + 12 + 12 + 12 + 6 + 12 + 6 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12) \times 12
\]

\[ |G| \leq (165 \times 12) \]

\[ |G| \leq 1980 \]
Figure 3.2: Cayley Diagram for $L_2(11) \times 3$ Over $A_4$
3.2.2 Proof of the Isomorphism

The composition series for

\[ G = \langle x, y, t | x^2, y^3, (xy)^3, (y^{-1}x^{-1}yx)^2, t^2, (t, y), (xt)^5 \rangle. \]

is \( G \supseteq G_1 \supseteq G_2 \), where \( G = (G/G_1)(G_1/G_2) = (G/G_1)(G_1/1) = L_2(11)C_3 \).

Now, we want to determine the extension type of this group.

\[ \text{CompositionFactors}(G_1); \]
\[ \text{G} \]
\[ | A(1, 11) = L(2, 11) \]
\[ * \]
\[ | \text{Cyclic}(3) \]
\[ 1 \]

Since the center of \( G \) is of order 3, we might have a central extension. Moreover, by looking at the normal lattice of \( G \), we see that it has a normal subgroup of order 3. Thus, this is not a semi-direct product. Now, we want to check if it is a direct product of \( L_2(11) \) by \( C_3 \). Since the minimal normal subgroups is of order equal to \( 660 \times 3 = 1980 \), it is clear that we have a direct product of \( L_2(11) \) by \( C_3 \).

\[ \text{D:=DirectProduct(CyclicGroup}(3), \text{PSL}(2, 11)); \]
\[ \text{s:=IsIsomorphic(D, G_1);} \]
\[ \text{s;} \]
true

By using ATLAS, a presentation for \( L_2(11) \) is

\[ \langle a, b| a^2, b^3, (ab)^{11}, (a, babab)^2 \rangle \]

Moreover, since it is a direct product, an element of order 3 commutes with all generators of \( L_2(11) \). Also, the subgroup of order 3 must be a generator of the center of \( G = L_2(11) \times 3 \).

\[ \text{HH<a,b,c>:=Group<a,b,c|a^2,b^3,(a*b)^{11},(a,b*a*b*a*b)^2, c^3,(a,c),(b,c)>;} \]
\[ \text{#HH;} \]
1980
\[ \text{f2,H2,k2:=CosetAction(HH,sub<HH|Id(HH)>);} \]
\[ \text{s:=IsIsomorphic(H2,G1);} \]
\[ \text{s;} \]
true

So, we have proved that \( G \cong L_2(11) \times 3 \).
### 3.3 Factor $L_2(11) \times 3$ by The Center of $G$

By looking at the Cayley diagram of $G = \frac{2^{*4} : A_4}{x_0y_0t_0 = t_0t_3} \cong L_2(11) \times 3$ over $A_4$, we see that the center of $G$ is $\langle m \ast 01231203 \rangle$.

$$t_0t_1t_2t_3t_1t_2t_0t_3 = m^{-1}$$

$$= xyx$$

$$= (102)$$

We want to factor $G$ by the center

$$\Rightarrow mt_0t_1t_2t_3t_1t_2t_0t_3 = e$$

$$t_0t_1t_2t_3t_1t_2t_0t_3 = m^{-1}$$

$$t_0t_1t_2t_3t_1t_2t_0t_3 = xyx$$

$$t_0t_1t_2t_3 = xyt_3t_0t_2t_1$$

Thus, by adding the new relation

$$(tt^x t^y y^t x^y y^t t^x) = xyx$$

to the representation of $G$, we get

$$G = 2^{*4} : A_4$$

$$= \langle x, y, t|x^2, y^3, (xy)^3, (y^{-1}x^{-1}yx)^2, t^2, (t, y), (xt)^5, (tt^x t^y y^t x^y y^t t^x) = xyx \rangle,$$

where $N = A_4 = \langle x, y \rangle = \langle (12)(30), (123) \rangle$.

Let $t = t_4 \rightarrow t_0$.

Conjugate $Nt_i$ where $i = 0, 1, 2, 3$ by $01231203 = xyx$.

$$Nt_0^{01231203} = t_2$$

$$Nt_1^{01231203} = t_0$$

$$Nt_2^{01231203} = t_1$$

$$Nt_3^{01231203} = t_3$$

If we conjugate $t_0t_1t_2t_3 = (142)t_3t_0t_2t_1$ by all elements in $A_4$, we will have the following relations:
0123 ≈ 3021 ≈ 1320
3210 ≈ 0312 ≈ 2013
2301 ≈ 1203 ≈ 3102
1032 ≈ 2130 ≈ 0231

3.4 \( L_2(11) \) as a Homomorphic Image of \( 2^{*4} : A_4 \)

3.4.1 The Construction of \( L_2(11) \) over \( A_4 \)

- Consider \( NeN \) which is denoted by \([s]\).
  The number of right cosets in \([s]\) = \( \frac{|N|}{|N|} = \frac{12}{12} = 1 \).
  The orbit of \( N \) on \( \{0,1,2,3\} \) is \( \{0,1,2,3\} \).
  Pick a representative from the orbit \( \{0,1,2,3\} \), say 0, and determine the double coset that contain \( Nt_0 \).

Word of length 1.

- Consider \( Nt_0N \) which is denoted by \([0]\).
  \( N^0 = \langle (123), (132) \rangle = N^{(0)} \).
  The number of right cosets in \([0]\) is equal to \( \frac{|N|}{|N^{(0)}|} = \frac{12}{3} = 4 \).
  The orbit of \( N^{(0)} \) on \( \{0,1,2,3\} \) are \{0\} and \{1,2,3\}.
  \( Nt_0t_0 = N \in [s] \) (1 symmetric generators goes back to the double coset \([s]\)).
  \( Nt_0t_1 \in [01] \) (3 symmetric generators take to the double coset \([01]\)).

Word of length 2.

- Consider \( Nt_0t_1N \) which is denoted by \([01]\).
  \( N^{01} = \langle e \rangle = N^{(01)} \).
  The number of right cosets in \([01]\) = \( \frac{|N|}{|N^{(01)}|} = \frac{12}{12} = 12 \).
  The orbits of \( N^{(01)} \) on \( \{0,1,2,3\} \) are \{0\}, \{1\}, \{2\}, and \{3\}.
  \( Nt_0t_1t_0 \in [010] \).
  \( Nt_0t_1t_1 = Nt_0 \) (1 symmetric generator goes back to the double coset \([0]\)).
  \( Nt_0t_1t_2 \in [012] \).
  \( Nt_0t_1t_3 \in [013] \).

Word of length 3.
Consider $Nt_0t_1t_0N$ which is denoted by [010].

Because the relation

$$Nt_0t_1t_0 = N(14)(23)t_0t_1$$

$$= Nx^y t_0t_1$$

where $Nt_0t_1 \in [01]$.

There exist $\{n \in N | N(t_0t_1)^n = Nt_0t_1\}$.

Thus, $N(t_0t_1)^e = Nt_0t_1$. $Nt_0t_1t_0 \in [01]$.

Therefore, $Nt_0t_1t_0N$ is not a new double coset and collapses $\Rightarrow$ one symmetric generator will take to [01].

Consider $Nt_0t_1t_2N$ which is denoted by [012].

$N^{012} = \langle e \rangle = N^{(012)}$.

The number of right cosets in [012] is equal to $\frac{|N|}{|N^{(012)}|} = \frac{12}{1} = 12$.

The orbits of $N^{(012)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.

$Nt_0t_1t_2t_0 \in [0120]$.

$Nt_0t_1t_2t_1 \in [0121]$.

$Nt_0t_1t_2t_2 = Nt_0t_1$ (1 symmetric generators goes back to the double coset [01]).

$Nt_0t_1t_2t_3 \in [0123]$.

Consider $Nt_0t_1t_3N$ which is denoted by [013].

$N^{013} = \langle e \rangle = N^{(013)}$.

The number of right cosets in [013] is equal to $\frac{|N|}{|N^{(013)}|} = \frac{12}{1} = 12$.

The orbits of $N^{(013)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.

$Nt_0t_1t_3t_0 \in [0130]$.

$Nt_0t_1t_3t_1 \in [0131]$.

$Nt_0t_1t_3t_2 \in [0132]$.

$Nt_0t_1t_3t_3 = Nt_0t_1$ (1 symmetric generators goes back to the double coset [01]).

We keep working on the DCE by proceeding in this manner until we obtain a word of length 4. By finding out all of the 8 double cosets $[w]$, we can determine the index of $N$ in $G$.

$$G = \frac{2^{+4}A_4}{t_0t_1t_2t_3 = xyt_3t_0t_2t_1}$$
The following computer-based proof gives that $G \cong L_2(11)$.

```plaintext
> G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,(y^(-1)*x^(-1)*y*x)^2,
> t^2,(t,y),(x*t)^5,(t*(t^x)^y*((t^x)^y)^y*t*(x)^2*(t^x)^y*((t^x)^y)^y*t*t*(x))=(x*y*x)>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);

A(1,11) = L(2,11)
```

In the next section, we will describe how such an isomorphism can be proved by hand (manually).
3.5  $PGL_2(11)$ as a Homomorphic Image of $2^5 : D_{10}$

3.5.1  The Construction of $PGL_2(11)$ over $D_{10}$

Consider the group

$$G = 2^5 : D_{10} = \langle x, y, t | x^5, y^2, (xy)^2, t^2, (t, y), (xt^x)^5, (xt^x)^3 \rangle$$

factored by $[xt]^5$.

$N = D_{10} = \langle x, y \rangle = \langle (12345), (14)(23) \rangle$ and let $t = t_5 \to t_0$. So,

$$(xt)^5 = e$$
$$(xt_0)^5 = e$$

$x_5t_0t_0x_3t_0x_2t_0x_0 = e$

$t_4t_3t_2t_1t_0 = e$

$t_4t_3t_2 = t_0t_1$

$t_4t_3t_2 = t_0t_1 \Rightarrow Nt_4t_3t_2 = Nt_0t_1$

$$(xt^x)^3 = e$$
$$(xt_0t_1)^3 = e$$

$x_3t_0t_1x_2t_0t_1x_0t_1 = e$

$(14203)t_2t_3t_1t_2t_0t_1 = e$

$(14203)t_2t_3t_1 = t_1t_0t_2 \Rightarrow Nt_2t_3t_1 = Nt_1t_0t_2$

If we conjugate the previous relation $(Nt_4t_3t_2 = Nt_0t_1)$ by all elements in $D_{10}$, we will have the following relations:

$$Nt_4t_3t_2 = Nt_0t_1, Nt_1t_2t_3 = Nt_0t_4,$$
$$Nt_0t_1t_2 = Nt_4t_3, Nt_2t_3t_4 = Nt_1t_0,$$
$$Nt_2t_1t_0 = Nt_3t_4, Nt_3t_4t_0 = Nt_2t_1,$$
$$Nt_4t_0t_1 = Nt_3t_2, Nt_0t_4t_3 = Nt_1t_2,$$
$$Nt_3t_2t_1 = Nt_4t_0, Nt_1t_0t_4 = Nt_2t_3.$$

If we conjugate the previous relation $(Nt_2t_3t_1 = Nt_1t_0t_2)$ by all elements in $D_{10}$, we will have the following relations:
\[ Nt_2 t_3 t_1 = Nt_1 t_0 t_2, \quad Nt_3 t_2 t_4 = Nt_4 t_0 t_3, \]
\[ Nt_2 t_1 t_3 = Nt_3 t_4 t_2, \quad Nt_4 t_3 t_0 = Nt_5 t_1 t_4, \]
\[ Nt_0 t_4 t_1 = Nt_1 t_2 t_0, \quad Nt_1 t_0 t_2 = Nt_2 t_3 t_1, \]
\[ Nt_3 t_4 t_2 = Nt_2 t_1 t_3, \quad Nt_1 t_2 t_0 = Nt_0 t_4 t_1, \]
\[ Nt_4 t_0 t_3 = Nt_3 t_2 t_4, \quad Nt_0 t_1 t_4 = Nt_4 t_3 t_0. \]

The double coset enumeration:

- Consider \( NeN \) which is denoted by \([\ast]\).
  The number of right cosets in \([\ast]\) is equal to \( \frac{|N|}{|N|} = \frac{10}{10} = 1. \)
  The orbit of \( N \) on \( \{0,1,2,3,4\} \) is \( \{0,1,2,3,4\} \).
  Pick a representative from the orbit \( \{0,1,2,3,4\} \), say 0, and determine the double coset that contain \( Nt_0 \).

Word of length 1.

- Consider \( Nt_0 N \) which is denoted by \([0]\).
  \( N^0 = \langle (14)(23) \rangle = N^{(0)} \).
  The number of right cosets in \([0]\) is equal to \( \frac{|N|}{|N^0|} = \frac{10}{2} = 5. \)
  The orbit of \( N^{(0)} \) on \( \{0,1,2,3,4\} \) are \( \{0\} \), \( \{1,4\} \) and \( \{2,3\} \).
  \( Nt_0 t_0 \in [\ast] \) (1 symmetric generator goes back to the double coset \([\ast]\)).
  \( Nt_0 t_1 \in [01] \).
  \( Nt_0 t_2 \in [02] \).

Word of length 2.

- Consider \( Nt_0 t_1 N \) which is denoted by \([01]\).
  \( N^{01} = \langle e \rangle = N^{(01)} \).
  The number of right cosets in \([01]\) is equal to \( \frac{|N|}{|N^{01}|} = \frac{10}{1} = 10. \)
  The orbits of \( N^{(01)} \) on \( \{0,1,2,3,4\} \) are \( \{0\} \), \( \{1\} \), \( \{2\} \), \( \{3\} \) and \( \{4\} \).
  \( Nt_0 t_1 t_0 \in [010] \).
  \( Nt_0 t_1 t_1 = Nt_0 \) (1 symmetric generator goes back to the double coset \([0]\)).
  \( Nt_0 t_1 t_2 \in [012] \).
  \( Nt_0 t_1 t_3 \in [013] \).
  \( Nt_0 t_1 t_4 \in [014] \).
• Consider \(Nt_0t_2N\) which is denoted by \([01]\).
\[N^{02} = \langle e \rangle = N^{(02)}.
\]
The number of right cosets in \([02]\) is \(\frac{|N|}{|N^{(02)}|} = \frac{10}{1} = 10\).
The orbits of \(N^{(02)}\) on \(\{0,1,2,3,4\}\) are \(\{0\}\), \(\{1\}\), \(\{2\}\), \(\{3\}\) and \(\{4\}\).
\(Nt_0t_2t_0 \in [020].\)
\(Nt_0t_2t_1 \in [021].\)
\(Nt_0t_2t_2 = Nt_0\) (1 symmetric generator goes back to the double coset \([0]\)).
\(Nt_0t_2t_3 \in [023].\)
\(Nt_0t_2t_4 \in [024].\)

**Word of length 3.**

• Consider \(Nt_0t_1t_2N\) which is denoted by \([012]\).

Because the relation
\[Nt_0t_1t_2 = N(t_0t_1)^{(13)(40)} = Nt_4t_3\]
where \(Nt_4t_3 \in [01]\).

There exist \(n \in N|N(t_0t_1t_2)^n = Nt_4t_3\}.

Thus, \(N(t_0t_1t_2)^{(13)(40)} = Nt_4t_3.\)

\([012] = [01]\)

Therefore, \(Nt_0t_1t_2N\) is not a new double coset and collapses \(\Rightarrow\) one symmetric generator will take to the double coset \([01]\).

• Consider \(Nt_0t_1t_3N\) which is denoted by \([013]\).
\[N^{013} = \langle e \rangle = N^{(013)}.
\]
The number of right cosets in \([013]\) is \(\frac{|N|}{|N^{(013)}|} = \frac{10}{1} = 10\).
The orbits of \(N^{(013)}\) on \(\{0,1,2,3,4\}\) are \(\{0\}\), \(\{1\}\), \(\{2\}\), \(\{3\}\) and \(\{4\}\).
\(Nt_0t_1t_3t_0 \in [0130].\)
\(Nt_0t_1t_3t_1 \in [0131].\)
\(Nt_0t_1t_3t_2 \in [0132].\)
\(Nt_0t_1t_3t_3 = Nt_0t_1\) (1 symmetric generator goes back to the double coset \([01]\)).
\(Nt_0t_1t_3t_4 \in [0134].\)
• Consider $N_{t_0 t_1 t_4}N$ which is denoted by $[014]$. 
$N^{[014]} = \langle e \rangle$. The right cosets in $[014]$ are not distinct. From previous relations, we have that

$$014 \approx 430$$

Therefore, $N^{[014]} = \{ n \in N | N(014)^n = N(430) \}$.

$N(014)^{(13)(45)} = N(430) = N(014) \Rightarrow (13)(45) \in N^{[014]}$

$N^{[014]} \geq \{ e, (13)(45) \} = \langle (13)(45) \rangle$.

The number of right cosets in $[014] = \frac{|N|}{|N^{[014]}|} = \frac{10}{2} = 5$.

The orbits of $N^{[014]}$ on $\{0,1,2,3,4\}$ are $\{2\}, \{1,3\}$, and $\{0,4\}$. $N_{t_0 t_1 t_4 t_2} \in [0142]$.

$N_{t_0 t_1 t_4 t_1} \in [0141]$.

$N_{t_0 t_1 t_4} = N_{t_0 t_1}$ (1 symmetric generator goes back to the double coset $[01]$).

• Consider $N_{t_0 t_1 t_0}N$ which is denoted by $[010]$.

$N^{[010]} = \langle e \rangle = N^{[010]}$.

The number of right cosets in $[010] = \frac{|N|}{|N^{[010]}|} = \frac{10}{1} = 10$.

The orbits of $N^{[010]}$ on $\{0,1,2,3,4\}$ are $\{0\}, \{1\}, \{2\}, \{3\}$ and $\{4\}$.

$N_{t_0 t_1 t_0 t_0} = N_{t_0 t_1}$ (1 symmetric generator takes back to the double coset $[01]$).

$N_{t_0 t_1 t_0 t_1} \in [0101]$.

$N_{t_0 t_1 t_0 t_2} \in [0102]$.

$N_{t_0 t_1 t_0 t_3} \in [0103]$.

$N_{t_0 t_1 t_0 t_4} \in [0104]$.

By continuing this process, we find out the all 19 double cosets $[w]$. Now, we can determine the index of $N$ in $G$.

$$G = \frac{2^{s_A} A_4}{(12)(34)t_4t_3t_4}$$

$$|G| \leq \left( |N| + \frac{|N|}{|N(0)|} + \frac{|N|}{|N(01)|} + \frac{|N|}{|N(02)|} + \ldots + \frac{|N|}{|N(01310)|} + \frac{|N|}{|N(01010)|} \right) \times |N|$$
\[ |G| \leq (1 + 5 + 10 + 10 + 10 + 5 + 10 + 5 + 10 + 10 + 5 + 5 + 10 + 5 + 10 + 10 + 1 + 5) \times 10 \]
\[ |G| \leq (132 \times 10) \]
\[ |G| \leq 1320 \]

Figure 3.4: Cayley Diagram for \( PGL_2(11) \) Over \( D_{10} \)

### 3.5.2 Proof of the Isomorphism

Now, we want to show that \( G \) is isomorphic to \( PGL_2(11) \). First, we will construct \( L_2(11) \), then \( PGL_2(11) \). After this, we see that \( G \) is isomorphic to \( PGL_2(11) \) by constructing a homomorphism \( \phi \) from the progenitor \( 2^*5 : D_{10} \) onto \( PGL_2(11) \). Now, we want to write a permutation for \( L_2(11) \) such that \( L_2(11) = \langle \alpha, \beta, \gamma \rangle \), where

\[ \alpha : x \mapsto x + 1. \]
\[ \beta : x \mapsto kx. \]
\[ \gamma : x \mapsto -\frac{1}{x}. \]

'\( k \)' is a non-zero square whose powers give all non-zero squares in \( \mathbb{Z}_{11} = \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \).
Note 3.2. We will replace 11 by 0.

<table>
<thead>
<tr>
<th>Table 3.2: $\mathbb{Z}_{11}$ Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>0^2</td>
</tr>
<tr>
<td>↓</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

Thus, the nonzero squares in $\mathbb{Z}_{11}$ is \{1,3,4,5,9\}. By using MAGMA, we can see what gives the power of the squares.

```
> Q:=(); for i in [1..10] do Q:=Q join {i^2 mod 11}; end for;
> Q;
{1, 3, 4, 5, 9}
> T:={}; for i in Q do T:=T join {i^2 mod 11};
> if T eq Q then i; end if; end for;
9
```

We find that $k = 9$. Now, we want to write down the permutation for $\alpha, \beta,$ and $\gamma$. To know the inverse of $i \forall 1 \leq i \leq 10$, we run the following code

```
F:=GaloisField(11);
F!1ˆ-1; F!2ˆ-1; F!3ˆ-1; F!4ˆ-1; F!5ˆ-1;
F!6ˆ-1; F!7ˆ-1; F!8ˆ-1; F!9ˆ-1; F!10ˆ-1;
```

<table>
<thead>
<tr>
<th>Table 3.3: Permutation of $\alpha: x \mapsto x + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>↓</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3.4: Permutation of $\beta: x \mapsto 9x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>↓</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

We use MAGMA to confirm that $\langle \alpha, \beta, \gamma \rangle = L_2(11)$.

$\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0)(\infty)$.

$\beta = (1, 9, 4, 3, 5)(2, 7, 8, 6, 10)$.

$\gamma = (0, \infty)(1, 10)(2, 5)(3, 7)(4, 8)(6, 9)$. 
Table 3.5: Permutation of $\gamma : x \mapsto -1/x = -x^{-1}$

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td></td>
</tr>
</tbody>
</table>

$S:=\text{Sym}(12);$
$\alpha:=S!(1,2,3,4,5,6,7,8,9,10,11);$  
$\beta:=S!(1,9,4,3,5)(2,7,8,6,10);$  
$\gamma:=S!(12,11)(1,10)(2,5)(3,7)(4,8)(6,9);$  
$N:=\text{sub}<S|\alpha,\beta,\gamma>;$
$> s,u:=\text{IsIsomorphic}(\text{PSL}(2,11),N);$  
$> s;$  
true

Since it says true, the representation of the Linear Fractional maps $LF_2(11) = L_2(11)$ is given by

$$\langle \alpha, \beta, \gamma | \alpha^{11}, \beta^5, \gamma^2, \alpha^\beta, \alpha^{-9}, (\beta \gamma)^2, (\alpha \gamma)^3 \rangle.$$  

Now, we want to write the permutation representation for $PGL(2,11)$ by adding $\delta : x \mapsto -x$ and find its action on $\alpha$, $\beta$, and $\gamma$.

**Note 3.3.** We denote $\delta$ by $\text{aut}$ in MAGMA.

Table 3.6: Permutation of $\delta : x \mapsto -x$

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td></td>
</tr>
</tbody>
</table>

The MAGMA input for $PGL(2,11)$ is as the following:

$\text{aut}: = S!(1,10)(2,9)(3,8)(4,7)(5,6);$
$\text{pgl211}: = \text{sub}<S|N,\text{aut}>;$  
$\#\text{pgl211};$
$s1,u1:=\text{IsIsomorphic}(\text{PGL}(2,11),\text{pgl211});$
$> s1;$  
true
$s2,u2:=\text{IsIsomorphic}(\text{G1},\text{pgl211});$
$> s2;$  
true
Thus, the representation of $PGL(2, 11)$ is given by

$$\langle \alpha, \beta, \gamma, \delta \mid \alpha^{11}, \beta^5, \gamma^2, \alpha^{\beta^{-9}}, (\beta\gamma)^2, (\alpha\gamma)^3, \delta^2, \alpha^\delta = \alpha^{-1}, \beta^\delta = \beta, \gamma^\delta = \gamma \rangle.$$ 

We define a homomorphism $\phi$ from the progenitor $2^{*5} : D_{10}$ to $PGL_2(11)$ by

$$\phi(x) = (1, 5, 8, 0, 4)(6, 7, \infty, 9, 10).$$
$$\phi(y) = (1, 5)(2, 3)(4, 8)(6, 7)(10, \infty).$$
$$\phi(t_{\infty}) = (1, 4)(2, 6)(3, 7)(5, 8)(9, 0)(10, \infty).$$

$$\phi(G) \cong PGL(2, 11)$$

To show that, I have to find a linear fractional maps in $PGL(2, 11)$ for $\phi(x)$, $\phi(y)$, and $\phi(t)$ such that $x \mapsto \frac{ax + b}{cx + d}$, $ad - bc \neq 0$ for $x$ to be in $PGL(2, 11)$.

- **Map for $\phi(x)$**.
  Since $\phi(x) = (1, 5, 8, 0, 4)(6, 7, \infty, 9, 10)$, we have
  $$0 \mapsto \frac{b}{d} = 4 \Rightarrow b = 4d.$$  
  $$8 \mapsto \frac{8a + b}{8c + d} = 0 \Rightarrow 8a + b = 0 \Rightarrow b = -8a \equiv_{11} 3a.$$  
  $$\infty \mapsto \frac{a}{c} = 9 \Rightarrow a = 9c \Rightarrow c = 9^{-1}a = 5a.$$  
  So, let $a = 1$. We have $b = 3$, $d = 4^{-1}b = 3(4^{-1}) = 3(3) = 9$, and $c = 5$.
  Moreover, since $ad - bc = (1)(9) - (3)(5) = 9 - 15 \neq 0$, our map is in $PGL(2, 11)$.
  The map $\eta$ is
  $$\eta \mapsto \frac{\eta + 3}{5\eta + 9}.$$  

- **Map for $\phi(y)$**.
  Since $\phi(x) = (1, 5)(2, 3)(4, 8)(6, 7)(10, \infty)$, we have
  $$1 \mapsto \frac{a + b}{c + d} = 5 \Rightarrow a + b = 5c + 5d.$$
10 \mapsto \frac{10a+b}{10c+d} = \infty \Rightarrow 10c + d = 0 \Rightarrow d = -10c \equiv_{11} c.
\infty \mapsto \frac{a}{c} = 10 \Rightarrow a = 10c.
So, let \(c = 1\). We have \(a = 10\), \(d = 1\), and \(a+b = 5c+5d \Rightarrow 10+b = 5+5 \Rightarrow b = 0\).
Moreover, since \(ad - bc = (10)(1) - (0)(1) = 10 \neq 0\), our map is in \(PGL(2,11)\).
The map \(\eta\) is
\[
\eta \mapsto \frac{10\eta}{\eta+1}.
\]

• Map for \(\phi(t)\).
Since \(\phi(x) = (1,4)(2,6)(3,7)(5,8)(9,0)(10,\infty)\), we have
\[
1 \mapsto \frac{a+b}{c+d} = 4 \Rightarrow a + b = 4c + 4d.
9 \mapsto \frac{9a+b}{9c+d} = 0 \Rightarrow 9a + b = 0 \Rightarrow b = -9a \equiv_{11} 5a.
0 \mapsto \frac{b}{a} = 9 \Rightarrow b = 9d.
So, let \(d = 1\). We have \(b = 9\), \(d = 1\), and \(a = 5^{-1}b = 9(9) = 81 \equiv_{11} 4\), and
\(4 + 9 = 4c + 4 \Rightarrow c = 9(4^{-1}) = 27 \equiv_{11} 5\).
Moreover, since \(ad - bc = (4)(9) - (9)(5) = 2 \neq 0\) and it is not a square, our map is in \(PGL(2,11)\).
The map \(\eta\) is
\[
\eta \mapsto \frac{4\eta+9}{5\eta+9}.
\]
We construct a homomorphism \(\phi\) from the progenitor \(2^*5 : D_{10}\) to \(PGL_2(11)\) by defining
\[
\phi(x) = \left( \frac{\eta + 3}{5\eta + 9} \right) = (1, 5, 8, 0, 4)(6, 7, \infty, 9, 10).
\phi(y) = \left( \frac{10\eta}{\eta + 1} \right) = (1, 5)(2, 3)(4, 8)(6, 7)(10, \infty).
\]
Since the orders of \(\phi(x), \phi(y), \) and \(\phi(x)\phi(y)\) are 5, 2, and 2, respectively,
\[
N = \langle \phi(x), \phi(y) \rangle \cong D_{10}.
\]
Now, let
\[
\phi(t_\infty) = \left( \frac{4\eta + 9}{5\eta + 9} \right) = (1, 4)(2, 6)(3, 7)(5, 8)(9, 0)(10, \infty).
\]
It is readily verified that \(PGL_2(11) \cong \langle \phi(x), \phi(y), \phi(t_\infty) \rangle\). Next, we show that \(\phi\) preserve the operation of \(2^*5 : D_{10}\). We find that \(|\phi(t_\infty)^N| = 5\) and that \(N\) permutes
the five images of \( t_\infty \), by conjugation, as the group \( D_{10} \). Thus,

\[
\begin{align*}
\phi(t_1) &= \phi(t_-^\infty) = (1, 5)(2, 7)(3, \infty)(4, 10)(6, 9)(8, 0), \\
\phi(t_2) &= \phi(t_+^\infty) = (1, 6)(2, \infty)(3, 9)(4, 0)(5, 8)(7, 10), \\
\phi(t_3) &= \phi(t_x^\infty) = (1, 4)(2, 9)(3, 10)(5, 7)(6, \infty)(8, 0), \\
\phi(t_0) &= \phi(t_y^\infty) = (1, 5)(2, 10)(3, 6)(4, 0)(7, 9)(8, \infty), \\
\phi(t_\infty) &= (1, 4)(2, 6)(3, 7)(5, 8)(9, 0)(10, \infty).
\end{align*}
\]

Thus, \( N \) permutes the five images of \( t_\infty \), by conjugation, as the group \( D_{10} \) given by:

\[
\begin{align*}
\phi(x) : (\phi(t_1), \phi(t_2), \phi(t_3), \phi(t_0), \phi(t_\infty)), \\
\phi(y) : (\phi(t_1), \phi(t_0))(\phi(t_2), \phi(t_3)).
\end{align*}
\]

Hence,

\[
\phi \left( 2^*5 : D_{10} \right) = PGL_2(11).
\]

Look at the additional relation given by \([xt_\infty]^5 = e \iff t_0 t_3 t_2 t_1 t_\infty = e\) is satisfied in \( PGL_2(11) \), because \( \phi(t_0) \phi(t_3) \phi(t_2) \phi(t_1) \phi(t_\infty) = e \) acts as identity on \((\phi(t_1))(\phi(t_2))(\phi(t_3))(\phi(t_0))(\phi(t_\infty)) = e\), by conjugation, on the images of the five symmetric generators.

This shows that \( PGL_2(11) \) is an image of \( G \).

Thus, \(|G| \geq |PGL_2(11)|\); but \(|G| \leq 1320 = |PGL_2(11)|\), and so the equality holds and \( G \cong PGL_2(11) \).

\[
G = \frac{2^*5 : D_{10}}{t_0 t_3 t_2 t_1 t_\infty = 1} \cong PGL_2(11).
\]

To have more homomorphic images, we factored the pogenitor \( 2^*5 : D_{10} \) by the following relators

\[
(xt)^a, (xyt x^2)^b, (xtt x^2)^c, (ytt x^2)^d
\]
Table 3.7: Some finite images of the Progenitor $2^5 : D_{10}$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>Index of $N$ in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>60</td>
<td>$A_5$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>12</td>
<td>120</td>
<td>$A_5 \times 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>240</td>
<td>$2^2 \times A_5$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>48</td>
<td>480</td>
<td>$4 \cdot (2 \times A_5)$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>72</td>
<td>720</td>
<td>$A_6 : 2 \simeq PGL_2(9)$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>132</td>
<td>1320</td>
<td>$PGL_2(11)$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>144</td>
<td>1440</td>
<td>$PGL_2(9) \times 2$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>216</td>
<td>2160</td>
<td>$(3 : A_6) : 2$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
<td>6</td>
<td>264</td>
<td>2640</td>
<td>$PGL_2(11) \times 2$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>9</td>
<td>5</td>
<td>342</td>
<td>3420</td>
<td>$L_2(19)$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>684</td>
<td>6840</td>
<td>$PGL_2(19)$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>972</td>
<td>9720</td>
<td>$3^4 \cdot (2 \times A_5)$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>2436</td>
<td>24360</td>
<td>$PGL_2(29)$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>4320</td>
<td>43200</td>
<td>$PGL_2(9) \times A_5$</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>12960</td>
<td>129600</td>
<td>$(A_5 \times 3) \cdot PGL_2(9)$</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>41040</td>
<td>410400</td>
<td>$(A_5 \times L_2(19)) : 2$</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>75000</td>
<td>750000</td>
<td>$(2 \times 5^6) \cdot (2 \times A_5)$</td>
</tr>
</tbody>
</table>
Chapter 4

Methods for Obtaining Homomorphic Images

In this chapter, we will introduce some useful techniques to find some finite images of an involutory progenitors $m^*n : N$, where $m = 2, 3, \ldots$ and the non-involutory (monomial) progenitors $p^*n :m N$, where $p = 3, 5, 7, \ldots$.

- **Factoring $m^*n : N$ and $p^*n :m N$ by all first order relations.**

  All relations of the first order that the progenitor can be factored by are obtained by computing the conjugacy classes of $N$. Thus, we will obtain a list of representative of classes. Then, we compute the centralisers of representatives of each non-identity class and determine its orbits.

**Example 4.1.** Consider the progenitor of $2^*3 : S_3$ that has the following presentation

$$< x, y | x^3, y^2, (xy)^2 >$$

To find all of the first order relations for the progenitor $2^*3 : S_3$, we run the following code. Then, we summarize the result in the table 4.1.

```plaintext
C:=Classes(N); c;
C2:=Centraliser(N,N!(1,2)); C2;
Orbits(C2);
C3:=Centraliser(N,N!(1,2,3)); C3;
Orbits(C3);
```
Table 4.1: Conjugacy Classes of $S_3$

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the class</th>
<th># of elements in the class</th>
<th>Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>identity</td>
<td>1</td>
<td>${1},{2},{3}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$y = (1,2)$</td>
<td>3</td>
<td>${1,2},{3}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$x = (1,2,3)$</td>
<td>2</td>
<td>${1,2,3}$</td>
</tr>
</tbody>
</table>

Now, we pick a representative from each orbit for the non-identity class $C_2$ and $C_3$. Thus, the all possible relations are

$$(yt)^a, (yt^{x^2})^b, \text{ and } (xt)^c,$$

where $t \sim t_1$ and $t^{x^2} = t_3$.

Hence, for finite images, we factor $2^*3 : S_3$ by the relations

$$(yt)^a, (yt^{x^2})^b, \text{ and } (xt)^c.$$

Hence, the following homomorphic images were obtained.

Table 4.2: Some finite images of the Progenitor $2^*3 : S_3$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>12</td>
<td>$2 \times 3 : 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>24</td>
<td>$S_4$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>48</td>
<td>$2^4 : S_3$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>20</td>
<td>120</td>
<td>$2 \times A_5$</td>
</tr>
</tbody>
</table>

- **Factoring $2^*n : N$ by relations found by Lemma.**

**Lemma 4.2 (Curtis).** $N \cap \langle t_i, t_j \rangle \leq C_N(N_{ij})$, where $N_{ij}$ denotes the stabilizer in $N$ of the two points $i$ and $j$.

To apply this method, we find stabiliser of two elements and determining the centraliser of those points. Moreover, the lemma deals with two types of permutations

1. Permutation $\pi$ that fixes $t_i$ and $t_j$.
   - The relation of this type have the following form $(t_it_j)^k = \pi$, where $k$ is even.
2. Permutation $\pi$ that sends $t_i$ to $t_j$ and vice versa.
   - The relation of this type have the following form $(\pi t)^k = 1$, where $\pi \in N$ and $k$ is odd.
Example 4.3. Consider the progenitor of $2^*4 : S_4$ that has the following presentation

$$< x, y | x^4, y^2, (xy)^3 >$$

To make this progenitor finite, we have to factor it by relations that satisfy the lemma. First, we have to insert a new variable $t \approx t_1$ and find all elements that centralise the subgroup $< t_1 >$. So, the point stabiliser of $t_1$ is

$$N^1 = \{(2, 3), (3, 4)\} = \{y^x, x^2yx^2\}$$

Then, the presentation for $P = 2^*4 : S_4$ becomes as follow:

$$< x, y, t | x^4, y^2, (xy)^3, t^2, (t, y^x), (t, x^2yx^2) >$$

Now, we have to apply the lemma by finding the centraliser of two point stabiliser $N^t_1 t_4$ such that

$$N \cap (t_1, t_4) \leq C_N(N_{14})$$

$$\leq C_N((2, 3))$$

We want to find $n \in N$ such that $(2, 3)^n = (2, 3)$. Hence, the centralisers of $N_{14}$ are the following permutations

$$(1, 4) = y^x \text{ and } (2, 3) = x^2yx^2.$$ We noticed that $(1, 4)$ is a permutation that sends 1 to 4 and 4 to 1 and $(2, 3)$ is a permutation that fixes 1 and 4. So, we might choose either one to factor the progenitor by. The form for these relations is given as bellow

$$(2, 3) \leftrightarrow (t_1 t_4)^k = (2, 3) \Rightarrow (tt^{x^3})^k = x^2yx^2, \quad \text{where } k \text{ is even.}$$

$$(1, 4) \leftrightarrow ((1, 4)t_1)^k = 1 \Rightarrow (y^x t)^k = 1, \quad \text{where } k \text{ is odd.}$$

The presentation for the progenitor over $S_4$ is

$$< x, y, t | x^4, y^2, (xy)^3, t^2, (t, y^x), (t, x^2yx^2), (tt^{x^3})^k = x^2yx^2, (y^x t)^l = 1 >.$$ The following homomorphic image were obtained.
\[
\frac{2^4 \cdot S_4}{[x^2] = x^2 y x^2} \cong PGL_2(7).
\]

- **Factoring** \(m^n : N \) and \(p^* : N \) by \((t_1, t_2)\).

**Theorem 4.4.** The progenitor \(G = m^n : N\), where \(m^n = < t_1 > \cdots < t_n >\) and \(|t_i| = m, 1 \leq i \leq n\) and \(N \leq U(m) \wr S_n\) where \(U(m)\) is the group of positive integers less than \(m\) and relative prime to \(m\) under multiplicaion modulo \(m\), factored by the relations \((t_i, t_j), i, j \in \{1, 2, \ldots, n\}\) is isomorphic to \(m^n : N\).

\[
m^n : N \cong m^n : N.
\]

**Example 4.5.** Let \(G = 3^* : S_3\), where \(S_3\) is generated by \(<x, y> = <(123), (12)>\).

The presentation for this progenitor is

\[
<x, y, t | x^3, y^2, (xy)^2, t^3 >
\]

This progenitor is infinite and in order to make it finite, we have to factore it by relations. Moreover, to prove it is isomorphic to \(3^2 : S_3\), we have to find \((t_1, t_2)\), where \(i, j \in \{1, 2, 3\}\). Thus, we want first to find a permutation in \(S_3\) that fix \(t = t_1\), and we find it is \(<2, 3 > = x y > = N^1\). Then, we have to find the orbits of \(N^1\) on \{0,1,2\} which are \{0,2\} and \{1\}. We pick a representitive from \{0,2\}, say 2. Thus, we obtain the desired relation \((t_1, t_2) = (t, t^2)\). By adding this relator to our progenitor, the homomorphic image is \(3^3 : S_3\). By using MAGMA we were able to prove the isomorphic image.

```plaintext
S:=Sym(3);
x:=S!(1,2,3);
y:=S!(1,2);
N:=sub<S|x, y>;
G<x, y, t>:=Group<x, y, t | x^3, y^2, (x*y)^2, t^3, (t, x*y), (t, t^x)>
> f, G1, k:=CompositionFactors(G, sub<G|x, y>);
> CompositionFactors(G1)
```

<table>
<thead>
<tr>
<th></th>
<th>Cyclic(3)</th>
</tr>
</thead>
</table>

*
By looking at the normal lattice of $G$, we find that there is an abelian group $[4]$ of order $27 = 3^3$. The symmetric presentation for $[4]$ is

$$<a, b, c | a^3, b^3, c^3, (a, b), (a, c), (b, c) >.$$  

Now, we want to factor $G$ by $[4]$ to find $q$.

```
q, ff := quo<G1 | NL[4]>; q;
> s := IsIsomorphic(q, Sym(3)); s;
true
```

We obtain that $q \cong S_3$ generated by $<q.1, q.2> = <(123), (23)>$. Now, we want to find the action of $q$ generators on $[4]$ generators after store them. Thus, we have to find the transversal of $G/[4] \cong q$.

```
T := Transversal(G1, NL[4]);
> ff(T[2]) eq q.1; ff(T[3]) eq q.2;
true true
```

The following loops are for finding the action of $q$ elements on $[4]$ elements. So, we want to write the relations between of them in terms of $[4]$ generators.
for i, j, l in [1..3] do if \(a^x \equiv a^i b^j c^l\) then i, j, l; end if; end for;
for i, j, l in [1..3] do if \(a^y \equiv a^i b^j c^l\) then i, j, l; end if; end for;

Similarly, for \(b\) and \(c\). The symmetric presentation for \(H = [4] : q\) is

\[
< a, b, c, x, y | a^3, b^3, c^3, (a, b), (a, c), (b, c), x^3, y^2, (xy)^2, a^x = ab, a^y = ab, \\
b^x = b^2, b^y = b^2, c^x = c, c^y = c >
\]

The following computer-based proof gives that \(G = \frac{3^*S_3}{(t_1, t_2)} \cong 3^3 : S_3\)

\[
> f2, H2, k2 := CosetAction(H, sub<H | Id(H)>); \\
> s := IsIsomorphic(H2, G1); s; \\
true
\]
Chapter 5

Transitive Group

The objective of this chapter is to factor the progenitor $m^*n : N$ which is infinite group by suitable relations of the form $\pi \omega(t_1, \ldots, t_n)$, where $\pi \in N$ and $\omega$ is a word in the symmetric generators, in order to find finite homomorphic images of the infinite progenitor $m^*n : N$. The notation $m^*n$ denotes a free product of $n$ the number of $t_i$'s of order $m$, and $N$ is a transitive subgroup of the symmetric group $S_n$.

5.1 Transitive group on 8 letters

Consider the progenitor $P = 2^*8 : N$. Since there are 50 transitive group on 8 letters, define $N$ as the 27th transitive subgroup on 8 letters of the symmetric group $S_8$ and define $D$ to be the small group database in order to find a presentation for $N$. The input code onto MAGMA is as follow:

```magma
NumberOfTransitiveGroups(8);
N:=TransitiveGroup(8,27);
D:=SmallGroupDatabase();
G:=SmallGroup(D,64,32);
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL:=Subgroups(G1);

SL will find all subgroups $H$ of $G$.

T:= \{X\'s subgroup: X in SL\};
#T;

#T will show that there are 47 subgroups $H$ in $G$.
```
TrivCore:={H:H in T| #Core(G1,H) eq 1};

#TrivCore will show that there are 21 faithful permutations representation of G.

mdeg:=Min({Index(G1,H):H in TrivCore});

mdeg gives the smallest number of letters in a permutation representation of G.

Good:={H: H in TrivCore| Index(G1,H) eq mdeg};

#Good will show that there are 4 subgroups that give the best, in terms of the smallest number of letters. By running this loop FpGroup(G), we obtain a presentation for G on 8 letters with 6 generators that is isomorphic to our transitive subgroup N in the progenitor 2*8 : N.

\[ G = \langle a, b, c, d, e, r | a^2 = d, b^2, c^2, d^2, e^2, r^2, b^a = bc, c^a = ce, b^b = c, d^a = d, d^b = de, d^c = dr, e^a = er, e^b = e, e^c = e, e^d = e, r^a = r, r^b = r, r^c = r, r^d = r, r^e = r \rangle. \]

Thus, we will label its generators in MAGMA as A, B, C, D, E, and R.

A:=G1!(1, 2)(3, 7)(4, 5, 8, 6);
B:=G1!(1, 3)(2, 5)(4, 8)(6, 7);
C:=G1!(1, 4)(2, 6)(3, 8)(5, 7);
D:=G1!(4, 8)(5, 6);
E:=G1!(2, 7)(5, 6);
R:=G1!(1, 3)(2, 7)(4, 8)(5, 6);
N:=sub<G1|A,B,C,D,E,R>;

Now, to search for relations, we must introduce a new variable t of order 2, let \( t = t_8 \rightarrow t_0 \) and find the stabiliser of 8, and we must write all the elements of the stabiliser in terms of a,b,c,d,e, and r by using the Schreier system.

\[(2,7)(5,6) = e.\]
\[(2,5)(6,7) = ber.\]
\[(1,3)(2,5,7,6) = db.\]

Then, since \( t \) commutes with each generator of the one point stabiliser \( N^0 \), we insert \( t \) and the generators of the centraliser of \( < t > \) into our progenitor. Then the presentation
for $P = 2^8$: $N$ becomes as follow:

\[
\langle a, b, c, d, e, r, t | a^2 = d, b^2, c^2, d^2, e^2, r^2, b^2 = bc, c^a = ce, c^b = c, d^a = d, d^b = de, \\
d^c = dr, e^a = er, e^b = e, e^c = e, e^d = e, r^a = r, r^b = r, r^c = r, r^d = r, r^e = r, t^2, (t, e), \\
(t, ber), (t, db) \rangle
\]

Now, we will utilize the lemma by taking the stabiliser of two elements $N^{01}$ and determining the centraliser of those points. The lemma deals with elements that have a transposition $(0,1)$, or do have either 1 or 0, or do not have either of them.

\[
N \cap \langle t_0, t_1 \rangle \leq C_N(N^{01}).
\]

\[
N^{01} = ((2, 7)(5, 6), (2, 5)(6, 7)).
\]

\[
C_N(N^{01}) = ((1, 4)(3, 8), (1, 3)(4, 8), (2, 7)(5, 6), (2, 5)(6, 7)).
\]

By running the Set($C$), we find that there are four elements that have a transposition $(0,1)$ together and by itself such as:

\[
(1, 0)(2, 6)(3, 4)(5, 7) = aca^{-1},
\]

\[
(1, 0)(2, 5)(3, 4)(6, 7) = cr,
\]

\[
(1, 0)(2, 7)(3, 4)(5, 6) = bc,
\]

\[
(1, 0)(3, 4) = bce.
\]

Thus, we have to apply the lemma as the following:

\[
(aca^{-1}t)^k = 1, \\
(crt)^k = 1, \\
(bcet)^k = 1, \\
(bct)^k = 1.
\]

where $k$ is odd.

By choosing one from the above list of relations, the presentation for this progenitor is

\[
\langle a, b, c, d, e, r, t | a^2 = d, b^2, c^2, d^2, e^2, r^2, b^2 = bc, c^a = ce, c^b = c, d^a = d, d^b = de, \\
d^c = dr, e^a = cr, e^b = e, e^c = e, e^d = e, r^a = r, r^b = r, r^c = r, r^d = r, r^e = r, t^2, (t, e), (t, ber), \\
(t, db), (bcet)^k \rangle
\]
To find some finite images of the progenitor $2^8 : N$, we add more relations and run
them in MAGMA to find some interesting images.

\[
< a, b, c, d, e, r, t | a^2 = d, b^2 = c^2, d^2, e^2, r^2, b^a = bc, c^a = ce, e^b = c, \\
d^a = d, d^b = de, d^c = dr, e^a = er, e^b = e, e^c = c, e^d = e, r^a = r, r^b = r, \\
r^c = r, r^d = r, r^e = r, t^2, (t, e), (t, ber), (t, db), (bct)^k, (dbt^c)^l, \\
(ert)^m, (a(t^r)^c)^n, (ct^b)^o > .
\]

Table 5.1: Some finite images of the progenitors $2^8 : N$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$n$</th>
<th>$o$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 2 2 5 3</td>
<td>15</td>
<td>120</td>
<td>$A_5 : 2 \cong S_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 2 2 5 5</td>
<td>165</td>
<td>1320</td>
<td>$PGL(2, 11)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 2 2 5 0</td>
<td>90</td>
<td>720</td>
<td>$A_6 : 2 \cong PGL_2(9)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0 3 4 8</td>
<td>28224</td>
<td>1806336</td>
<td>$(4 \times 2) \bullet [(L_2(7) \times L_2(7)) : (2^2 : 2)]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 0 3 4 0</td>
<td>225</td>
<td>14400</td>
<td>$(A_5 \times A_5) : 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.1.1 Proof of The Isomorphism for The Shape of $N$

Consider the progenitor $2^8 : N$, Where $N$ is given by

\[
S := \text{Sym}(8); \\
aa := S!(1, 2)(3, 7)(4, 5, 8, 6); \\
bb := S!(1, 3)(2, 5)(4, 8)(6, 7); \\
cc := S!(1, 4)(2, 6)(3, 8)(5, 7); \\
dd := S!(4, 8)(5, 6); \\
ee := S!(2, 7)(5, 6); \\
r := S!(1, 3)(2, 7)(4, 8)(5, 6); \\
N := \text{sub}<S|aa, bb, cc, dd, ee, rr>;
\]

We prove below that $N$ is isomorphic to a mixed extension of the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_2$
by the Dihedral group $D_4$ of order 8. Let’s call $N = G$. The transitive group $G$ has the
following composition series

\[
G = G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq 1,
\]

where

\[
\]
This group has the following normal lattice

```maple
> NL:=NormalLattice(N); NL;
Normal subgroup lattice
-----------------------
[13] Order 64 Length 1 Maximal Subgroups: 10 11 12
---
[10] Order 32 Length 1 Maximal Subgroups: 7 8 9
---
[ 8] Order 16 Length 1 Maximal Subgroups: 6
[ 7] Order 16 Length 1 Maximal Subgroups: 4 5 6
---
[ 6] Order 8 Length 1 Maximal Subgroups: 3
[ 5] Order 8 Length 1 Maximal Subgroups: 3
[ 4] Order 8 Length 1 Maximal Subgroups: 3
---
---
[ 2] Order 2 Length 1 Maximal Subgroups: 1
---
[ 1] Order 1 Length 1 Maximal Subgroups:
```

By looking at the center of $G$, we see that we may have a central extension. Let's check if $[4]$ is an abelian group or not.

```maple
> IsAbelian(NL[4]);
true
> X1:=AbelianGroup(GrpPerm,[4,2]);
> IsIsomorphic(X1,NL[4]);
true
```

We confirm that $G_3 = 4 \times 2$ is the isomorphism type of $[4]$. Since $[4] = \mathbb{Z}_4 \times \mathbb{Z}_2$ is an abelian subgroup of $G$, we have the mixed extension $G = (\mathbb{Z}_4 \times \mathbb{Z}_2) \cdot D_4 = K \cdot H$ but it is not a cent of $G$. Thus, we do not have a central extension but we have a mixed extension. Now, we have to find the generators of $[4]$ and store them.

```maple
A:=NL[4].1;A;
B:=NL[4].2;B;
A:=N!(1, 4, 3, 8)(2, 6, 7, 5);
B:=N!(2, 7)(5, 6);
NL4:=sub<N|A,B>;
```
The presentation for $[4]$ is $< A, B | A^4, B^2, (A, B) >$. Now, we have to find the rest of the group by factoring $G = N$ by $K = \mathbb{Z}_4 \times \mathbb{Z}_2$ to find $H$.

```plaintext
> q, ff := quo<N|NL4>;
> IsAbelian(q);
false
> Center(q);
Permutation group acting on a set of cardinality 4
Order = 2
   (1, 3) (2, 4)
> NumberOfGenerators(q);
6
> q eq sub<q|q.1, q.2>;
true
> q1 := q.1; q2 := q.2;
```

We obtain that $q$ is a Dihedral group $D_4$ of order 8 because $|q.1| = 4$, $|q.2| = 2$, and $|q.1 * q.2| = 2$.

Since $G = K \cdot H$ and $G/K \cong H$, there is a map $\Phi : G/K \to H$ given by $g_i \mapsto n_{q_i}$, where $q \in H$ and $Q = K \cup Kg_1 \cup Kg_i$.

$$
\Phi : G/K \to H,
$$

$$
KT[1] \mapsto Id(K),
$$

$$
KT[2] \mapsto q.1,
$$

$$
KT[3] \mapsto q.2.
$$

In another word, let $g \in G/K$, we want to show that if $g \in Kg_1 \Rightarrow gg_1^{-1} \in K$. This process called 'factor sets'.

```plaintext
> T := Transversal(N, NL4);
> T[1];
Id(N)
> T2 := T[2]; T2;
(1, 2) (3, 7) (4, 5, 8, 6)
> T3 := T[3]; T3;
(1, 3) (2, 5) (4, 8) (6, 7)
> T23;
(1, 5, 4, 2, 3, 6, 8, 7)
> q.1 * q.2;
(1, 4) (2, 3)
```
To find the action of \( q_1 \) and \( q_2 \) and their product on \( A \) and \( B \), we use the Schreier system then include the result in our presentation.

\[
I := \{\text{Id}(N): i \in [1..5]\};
\]

for \( i \) in \([1..8]\) do if \( \text{ArrayP}[i] \) eq \( T_2^3 \) then \( \text{Sch}[i] \); end if; end for;

for \( i \) in \([1..8]\) do if \( \text{ArrayP}[i] \) eq \( A^T_2 \) then \( \text{Sch}[i] \); end if; end for;

for \( i \) in \([1..8]\) do if \( \text{ArrayP}[i] \) eq \( B^T_2 \) then \( \text{Sch}[i] \); end if; end for;

for \( i \) in \([1..8]\) do if \( \text{ArrayP}[i] \) eq \( A^T_3 \) then \( \text{Sch}[i] \); end if; end for;

for \( i \) in \([1..8]\) do if \( \text{ArrayP}[i] \) eq \( B^T_3 \) then \( \text{Sch}[i] \); end if; end for;

\( I; \)

The MAGMA input for the presentation of \( G \) is

\[
H := \langle x, y, z, w \rangle := \text{Group}\langle x, y, z, w \mid x^4, y^2, (x, y), z^4, w^2, (z*w)^2 = x, \\
x^z = x*y, y^z = x^2*y, x^w = x*y, y^w = y \rangle;
\]

\( f, h, k := \text{CosetAction}(H, \text{sub}<H|\text{Id}(H)>); \)

\( #h; \)

64

\( s := \text{IsIsomorphic}(N, h); \)

\( s; \)

true

Thus, we see that \( G \) is isomorphic to the mixed extension of \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) by \( D_4 \).

\[
G = N = (\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4.
\]
5.1.2 $2^*PGL_2(11)$ as a homomorphic Image of $2^*8 : ((\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4)$

Consider the progenitor

$$2^8 : N = \langle a, b, c, d, e, r, t| a^2 = d, b^2, c^2 = d, e^2, r^2, b^a = b, c^a = c, d^a = d, d^b = d, e^a = e, e^b = c, c^a = c, d^c = d, r^a = r, r^b = r, r^c = r, r^d = r, r^e = r, t^2, (t, e), (t, ber), (t, db) \rangle,$$

where $N = (\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4$. We factor the progenitor by two relations $(bcet)^3$ and $(ebta)^5$ to obtain a homomorphic image $G_1$ of order 2640. We now prove that $G_1$ is a central extension of $\mathbb{Z}_2$ by $PGL(2,11)$.

The next MAGMA input code is to make sure that the kernal of the homomorphism $f : G \to S_{|H|}$ is equal to 1, so that $G_1$ is a faithful permutation representation of $G$.

```magma
f,G1,k:=CosetAction(G,sub<G|Id(G)>); SL:=Subgroups(G1); T:={X'subgroup:X in SL}; TrivCore:={H:H in T| #Core(G1,H) eq 1}; mdeg:=Min({Index(G1,H):H in TrivCore}); Good:={H:H in TrivCore|Index(G1,H) eq mdeg}; H:=Rep(Good);
```

The composition series for this group is as the following:

$$G = G \supseteq G_1 \supseteq 1,$$

where $G = (G/G_1)(G_1/1) = C_2L_2(11)$.

By looking at the normal lattice and checking its center, we found that we may have a central extension since the center of $G$ is of order 2. But, we see that the center of $G$ belongs to $[4]$ which indicates that we do not have a direct product. Also, it is not a semi direct product since the automorphism group of order 2 is 1. Thus, $G$ is a central extension of $\mathbb{Z}_2$ by $Q$. To find $Q$, we factor $G_1$ by the center and apply the 'factor sets' process.

```magma
> NL[2] eq Center(G1); true > q,ff:=quo<G1|NL[2]>; > H<x,y,z,w>:=Group<x,y,z,w|x^2,y^2,z^2,w^2,(x*y)^4, > (x*z)^2,(x*w)^10,(y*z)^2,(y*w)^2,(z*w)^6,(x,y)^2, > (x,w)^5,(z,w)^3,(x*y*z)^4,(x*y*w)^12,(x*z*w)^10, > (x*y*z)^2,(x*z*w)^10,(x*y,z),(x*z,w)^5,(x*y*z,w)^5,```
> (x*y*z*w) \cdot 3, (x*y, z*w) \cdot 5, (x*z*y*w*x*y) \cdot 2,
> (x*w+z*y*w*x) \cdot 2, (w*x*y*w*z*x*y*w) \cdot 2,
> (x, y*z*w*y*z*w*z) \cdot 2, (x*y*z*w) \cdot 10,
> (y**w*z*y*z*w*z*xy*w) \cdot 5, (y, x*y*z*w*z*x*y*w) \cdot 3,
> (x*y*z*y*x*w*y*z*w*y*z*w*y*x*w) \cdot 5>
> f, h, k := CosetAction(H, sub<H|Id(H)>);
> #h;
> 1320
> s := IsIsomorphic(q, h); s;
true
> T := Transversal(G1, NL[2]);
> ff(T[2]) eq q.1;
true
> ff(T[3]) eq q.2;
true
> ff(T[4]) eq q.3;
true
> ff(T[5]) eq q.7;
true
> Order(T[2]), Order(T[3]), Order(T[4]), Order(T[5]);
4 2 2 2
> Order(q.1);
2
> T[2] \cdot 2;
(1, 4) (2, 7) (3, 9) (5, 11) (6, 14) (8, 15) (10, 17) (12, 20) (13, 21)
(16, 24) (18, 26) (19, 27) (22, 29) (23, 30) (25, 32) (28, 35) (31, 37)
(33, 39) (34, 40) (36, 42) (38, 43) (41, 44)
> NL[2];
Permutation group acting on a set of cardinality 44
Order = 2
Id(G1)
(1, 4) (2, 7) (3, 9) (5, 11) (6, 14) (8, 15) (10, 17) (12, 20)
(13, 21) (16, 24) (18, 26) (19, 27) (22, 29) (23, 30) (25, 32)
(28, 35) (31, 37) (33, 39) (34, 40) (36, 42) (38, 43) (41, 44)
> K<x, y, z, w, c> := Group<x, y, z, w, c | x^2 = c, y^2, z^2, w^2, (x*y)^4,
(x*z)^2 = c, (x*w)^10 = c, (y*z)^2, (y*w)^2, (z*w)^6, (x, y)^2,
(x, w)^5, (z, w)^3, (x*y*z)^4, (x*y*w)^12, (x*z*w)^10 = c,
(x*y*z)^2 = c, (x*z*w)^10 = c, (x*y, z, w)^5, (x*y*z, w)^5,
(x*y*w, z)^3, (x*y, z*w)^5, (x*z, y*w)^6, (x*w, y*z)^2, (w*x, y*z)^2,
(x*w)^2*y*z)^5 = c, (x*y*z*w)^12, (x*y*(z*w)^2)^10 = c,
We prove that
\[
\frac{2^*N}{[bcet]^5, [ert]^3, [a(t^r)^c]^4} \cong 2^* PGL_2(11).
\]

5.1.3 \( (A_5 \times A_5) : 4 \) as a homomorphic Image of \( 2^* \): \( ((\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4) \)

In this section, we will construct the Cayley diagram of \( (A_5 \times A_5) : 4 \) over \( ((\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4) \) and prove that \( G \cong (A_5 \times A_5) : 4 \).

Consider the group
\[
G = 2^* : ((\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4) =< a, b, c, d, e, r, t | a^2 = d, b^2 = e, c^2 = r, b^a = bc, c^a = ce, \\
c^b = d, d^a = de, d^b = er, c^a = e, e^a = d, r^a = r, r^b = r >,
\]
\[
factored by \[bcet]^5, [ert]^3, \text{and} [a(t^r)^c]^4, \]
\[
N = (\mathbb{Z}_4 \times \mathbb{Z}_2) \bullet D_4 =< a, b, c, d, e, r >,
\]
where
\[
a = (12)(37)(4586).
\]
\[
\]
\[
c = (14)(26)(38)(57).
\]
\[
d = (48)(56).
\]
\[
e = (27)(56).
\]
\[
\]
Let $t = t_8$. So,

\[
[bct]^5 = 1 \\
[bcet_8]^5 = 1 \\
(bce)^5 t_8 (bce)^4 t_8 (bce)^3 t_8 (bce)^2 t_8 (bce) t_8 = 1 \\
(18)(34)t_8 t_1 t_8 t_1 t_8 = 1 \\
t_8 t_1 t_8 = (18)(34)t_8 t_1.
\]

We write $t_0$ instead of $t_8$.

**Note 5.1.** If we conjugate $t_0 t_1 t_0 = (10)(34)t_0 t_1$ by all elements in $N$, we will have more relations as the following.

\[
t_1 t_0 t_1 = (01)(43)t_1 t_0, \ t_7 t_5 t_7 = (57)(62)t_7 t_5, \\
t_5 t_7 t_5 = (75)(26)t_5 t_7, \ t_2 t_6 t_2 = (62)(57)t_2 t_6, \\
t_6 t_2 t_6 = (26)(75)t_6 t_2, \ t_3 t_0 t_3 = (03)(41)t_3 t_0, \\
t_0 t_3 t_0 = (30)(14)t_0 t_3, \ t_4 t_1 t_4 = (14)(30)t_4 t_1, \\
t_1 t_4 t_1 = (41)(03)t_1 t_4, \ t_6 t_7 t_6 = (76)(25)t_6 t_7, \\
t_7 t_6 t_7 = (67)(52)t_7 t_6, \ t_2 t_5 t_2 = (52)(67)t_2 t_5, \\
t_5 t_2 t_5 = (25)(76)t_2 t_5, \ t_3 t_4 t_3 = (43)(01)t_3 t_4, \\
t_4 t_3 t_4 = (34)(10)t_4 t_3.
\]

\[
[ert]^3 = 1 \\
[ert_0]^3 = 1 \\
(ert_0)^3 t_0 (ert)^2 t_0 (ert_0) t_0 = 1 \\
(13)(40)t_0 t_4 t_0 = 1 \\
t_0 t_4 = (13)(40)t_0.
\]

**Note 5.2.** If we conjugate $t_0 t_4 = (13)(40)t_0$ by all elements in $N$, we will have the following relations:

\[
t_1 t_3 = (04)(31)t_1, \ t_2 t_7 = (65)(72)t_2, \ t_3 t_1 = (04)(13)t_3, \\
t_4 t_0 = (13)(04)t_4, \ t_5 t_6 = (72)(65)t_5, \ t_6 t_5 = (27)(56)t_6, \\
t_7 t_2 = (56)(27)t_7.
\]
Note 5.3. If we conjugate $t_2t_1 = t_1t_2$ by all elements in $N$, we will have the following relations:

$$
\begin{align*}
& a(t^r)^4 = 1 \\
& [at_1]^4 = 1 \\
& a^4t_1^a^3t_1^a^2t_1^a = 1 \\
& t_2t_1t_2t_1 = 1 \\
& t_2t_1 = t_1t_2.
\end{align*}
$$

The double coset enumeration:

- Consider $NeN$ which is denoted by $[\ast]$. The number of right cosets in $[\ast] = \frac{|N|}{|N|} = \frac{64}{64} = 1$.

The orbit of $N$ on $\{0,1,2,3,4,5,6,7\}$ is $\{0,1,2,3,4,5,6,7\}$.

Pick a representative from the orbit $\{0,1,2,3,4,5,6,7\}$, say 0, and determine the double coset that contain $Nt_0$.

[0]:

- Consider $Nt_0N$ which is denoted by [0].

$N^0 = \langle (27)(56), (13)(2576), (25)(67) \rangle = N^{(0)}$.

The number of right cosets in [0] = $\frac{|N|}{|N^{(0)}|} = \frac{64}{8} = 16$.

The orbits of $N^{(0)}$ on $\{0,1,2,3,4,5,6,7\}$ are $\{0\}$, $\{4\}$, $\{1,3\}$ and $\{2,5,6,7\}$.

$Nt_0t_0 \in [\ast]$ (1 symmetric generator goes back to the double coset $[\ast]$).

$Nt_0t_4 \in [04]$. $Nt_0t_1 \in [01]$. $Nt_0t_2 \in [02]$.

- Consider $Nt_0t_1N$ which is denoted by [01].

$N^{01} = \langle (27)(56), (26)(57) \rangle = N^{(01)}$.

The number of the right cosets in [01] is equal to $\frac{|N|}{|N^{(01)}|} = \frac{64}{4} = 16$.

The orbits of $N^{(01)}$ on $\{0,1,2,3,4,5,6,7\}$ are $\{0\}$, $\{1\}$, $\{3\}$, $\{4\}$, and $\{2,5,6,7\}$. 
\( Nt_0t_1t_0 \in [010] \).

\( Nt_0t_1 = Nt_0 \) (1 symmetric generator goes back to the double coset \([0]\)).
\( Nt_0t_1t_3 \in [013] \).
\( Nt_0t_1t_4 \in [014] \).
\( Nt_0t_1t_2 \in [012] \).

- Consider \( Nt_0t_2N \) which is denoted by \([02]\).

\( N^{02} = \langle (13)(56) \rangle \). The right cosets in \([02]\) are not distinct. From previous relations, we know that \( Nt_0t_2 = Nt_2t_0 \).

Therefore, \( N^{(02)} = \{ n \in N|N(020)^n = N(02) \} \).

\[
N(02)^{(1635)(20)(47)} = N20 = N02.
\]
\[
N(02)^{(1536)(20)(47)} = N20 = N02.
\]

Thus, \((1635)(20)(47), (1536)(20)(47) \in N^{(02)} \) and

\[
N^{(02)} \geq \langle (13)(56), (1635)(20)(47), (1536)(20)(47) \rangle = N^{(02)}.
\]

The number of right cosets in \([02]\) = \( \frac{|N|}{|N^{(02)}|} = \frac{64}{4} = 16 \).

The orbits of \( N^{(02)} \) on \{0,1,2,3,4,5,6,7\} are \{0,2\}, \{4,7\}, and \{1,3,5,6\}.

\( Nt_0t_2t_2 = Nt_0 \) (1 symmetric generator goes back to the double coset \([0]\)).
\( Nt_0t_2t_4 \in [024] \).
\( Nt_0t_2t_1 \in [021] \).

- Consider \( Nt_0t_4N \) which is denoted by \([04]\).

Because the relation
\[
Nt_0t_4 = N(13)(40)t_0
\]

\[
= Nert_0 = Nt_0.
\]

where \( Nt_0 \in [0] \).

There exist \( \{ n \in N|N(t_0)^n = Nt_0 \} \).

Thus, \( N(t_0)^e = Nt_0 \). \( Nt_0t_4 \in [0] \).

Therefore, \( Nt_0t_4N \) is not a new double coset and collapses \( \Rightarrow \) one symmetric generator will take back to \([0]\).
Consider $N_{t_0t_1t_0}$ which is denoted by [01].

Because the relation

\[ N_{t_0t_1t_0} = N(10)(34)t_0t_1 \]

\[ = Nbcet_0t_1 = N_{t_0t_1} \]

where $N_{t_0t_1} \in [01]$.

There exist \{n $\in N | N(t_0t_1)^n = N_{t_0t_1}$\}.

Thus, $N(t_0t_1)^c = N_{t_0t_1}$. $N_{t_0t_1t_0} \in [01]$.

Therefore, $N_{t_0t_1t_0}$ is not a new double coset and collapses $\Rightarrow$ one symmetric generator will go back to [01].

Consider $N_{t_0t_1t_2}$ which is denoted by [012].

$N^{012} = (e) = N^{(012)}$.

The number of right cosets in [012] = $\frac{|N|}{|N^{(012)}|} = \frac{64}{1} = 64$.

The orbits of $N^{(012)}$ on \{0,1,2,3,4,5,6,7\} are \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} and \{7\}.

$N_{t_0t_1t_2t_0} \in [0120]$.

$N_{t_0t_1t_2t_1} \in [0121]$.

$N_{t_0t_1t_2t_2} = N_{t_0t_1}$ (1 symmetric generator goes back to the double coset [01].

$N_{t_0t_1t_2t_3} \in [0123]$, $N_{t_0t_1t_2t_4} \in [0124]$, $N_{t_0t_1t_2t_5} \in [0125]$, $N_{t_0t_1t_2t_6} \in [0126]$, $N_{t_0t_1t_2t_7} \in [0127]$.

Consider $N_{t_0t_1t_3}$ which is denoted by [013].

Because of the relation

\[ N_{t_0t_1t_3} = N_{t_0}(84)(31)t_1 = N(84)(31)t_4t_1 \]

\[ = Nert_0t_1^{(48)(56)} \]

where $N_{t_4t_1} \in [01]$.

There exist \{n $\in N | N(t_0t_1)^n = N_{t_4t_1}$\}.

Thus, $N(t_0t_1)^{(48)(56)} = N_{t_4t_1}$. $N_{t_0t_1t_3} \in [01]$.

Therefore, $N_{t_0t_1t_3}$ is not a new double coset and collapses $\Rightarrow$ one symmetric generator will take back to [01].
Consider $N_{t_0t_1t_4}N$ which is denoted by $[014]$.

$N^{014} = ((27)(56), (25)(67))$.

The right cosets in $[014]$ are not distinct. By using the above relations, we obtain the following

$$N_{t_0t_1t_4} = N_{t_0}(41)(03)t_4t_1 = N(41)(03)t_2t_1t_4 = N(41)(03)(04)(13)t_3t_4t_1 = N(13)(04)t_4t_3t_0$$

$$N_{t_0t_1t_4} = N_{t_0t_1}(13)(04)t_0t_4 = N(13)(04)t_4t_3t_0$$

$$N_{t_0t_1t_4} = N_{t_0t_1}(10)(34)t_0t_4t_1 = N(10)(34)t_0t_3t_0t_4$$

Hence, $N_{t_0t_1t_4} = N_{t_3t_4t_1} = N_{t_4t_3t_0} = N_{t_1t_0t_3}$.

Therefore, $N^{014} = \{n \in N | N(014)^n = N(014) \}$. 

$$N(014)^{(13)(25)(40)(67)} = N^{430} = N^{014}.$$ 

$$N(014)^{(13)(26)(40)(57)} = N^{430} = N^{014}.$$ 

$$N(014)^{(13)(27)(40)(56)} = N^{430} = N^{014}.$$ 

$$N(014)^{(13)(40)} = N^{430} = N^{014}.$$ 

$$N(014)^{(14)(26)(30)(57)} = N^{341} = N^{014}.$$ 

$$N(014)^{(14)(25)(30)(67)} = N^{341} = N^{014}.$$ 

$$N(014)^{(14)(30)} = N^{341} = N^{014}.$$ 

$$N(014)^{(14)(27)(30)(56)} = N^{341} = N^{014}.$$ 

$$N(014)^{(10)(27)(34)(56)} = N^{103} = N^{014}.$$ 

$$N(014)^{(10)(34)} = N^{341} = N^{014}.$$ 

$$N(014)^{(10)(25)(34)(67)} = N^{103} = N^{014}.$$ 

$$N(014)^{(10)(26)(34)(57)} = N^{103} = N^{014}.$$
Thus, all the above permutations are in $N^{(014)}$ and $N^{(014)} = N^{(014)}$. 

The number of right cosets in $[014] = |N|/|N^{(014)}| = 64/16 = 4$. 

The orbits of $N^{(014)}$ on \{0,1,2,3,4,5,6,7\} are \{0,1,3,4\} and \{2,5,6,7\}. 

$Nt_0t_1t_4t_4 = Nt_0t_1$ (4 symmetric generators go back to the double coset [01]). 

$Nt_0t_1t_4t_2 \in [0142]$.

[02]:

- Consider $Nt_0t_2t_1N$ which is denoted by [021]. 
  Because of the relation
  
  \[ Nt_0t_2t_1 = Nt_0t_1t_2. \]

  There exist \( n \in N|N(t_0t_1t_2)^n = Nt_0t_1t_2 \). 
  Thus, \( N(t_0t_1t_2)^c = Nt_0t_1t_2 \). 
  Therefore, $Nt_0t_2t_1N$ is not a new double coset and collapses \( \Rightarrow \) four symmetric generators will take to [012].

- Consider $Nt_0t_2t_4N$ which is denoted by [024]. 
  Because of the relation
  
  \[ Nt_0t_2t_4 = Nt_0t_4t_2 \]
  \[ = N(13)(40)t_0t_2 \]
  \[ = Nbcet_0t_2. \]

  There exist \( n \in N|N(t_0t_2)^n = Nt_0t_2 \). 
  Thus, \( N(t_0t_2)^d = Nt_0t_2 \). 
  Therefore, $Nt_0t_2t_4N$ is not a new double coset and collapses \( \Rightarrow \) four symmetric generators will go back to [02].

[012]:

- Consider $Nt_0t_1t_2t_0N$ which is denoted by [0120]. 
  Because of the relation
  
  \[ Nt_0t_1t_2t_0 = Nt_0t_1t_0t_2 \]
  \[ = N(18)(34)t_0t_1t_2 \]
  \[ = Nbcet_0t_1t_2 = Nt_0t_1t_2. \]
There exist \( \{ n \in N | N(t_0t_1t_2)^n = Nt_0t_1t_2 \} \).

Thus, \( N(t_0t_1t_2)^{id} = Nt_0t_1t_2 \).

Therefore, \( Nt_0t_1t_2t_0N \) is not a new double coset and collapses \( \Rightarrow \) one symmetric generator will go back to \([012]\).

- Consider \( Nt_0t_1t_2t_1N \) which is denoted by \([0121]\).

  Because of the relation

  \[
  Nt_0t_1t_2t_1 = Nt_0t_1t_2
  \]

  \[
  = Nt_0t_2.
  \]

  There exist \( \{ n \in N | N(t_0t_2)^n = Nt_0t_2 \} \).

  Thus, \( N(t_0t_2)^{id} = Nt_0t_2 \).

  Therefore, \( Nt_0t_1t_2t_1N \) is not a new double coset and collapses \( \Rightarrow \) one symmetric generator will go back to \([02]\).

- Consider \( Nt_0t_1t_2t_3N \) which is denoted by \([0123]\).

  Because of the relation

  \[
  Nt_0t_1t_2t_3 = Nt_0t_1t_3t_2
  \]

  \[
  = Nt_0(84)(31)t_1t_2
  \]

  \[
  = N(84)(31)t_4t_1t_2
  \]

  \[
  = Nert_4t_1t_2
  \]

  \[
  = Ner[t_0t_1t_2]^{(48)(56)}
  \]

  where \( Nt_4t_1t_2 \in [012] \).

  There exist \( \{ n \in N | N(t_0t_1t_2)^n = Nt_4t_1t_2 \} \).

  Thus, \( N(t_0t_1t_2)^{(48)(56)} = Nt_4t_1t_2 \).

  Therefore, \( Nt_0t_1t_2t_3N \) is not a new double coset and collapses \( \Rightarrow \) one symmetric generator will go back to \([012]\).

  We now continue the double coset enumeration by proceeding in this manner until the set of right cosets is closed under right multiplication by \( t_i \)'s where \( i = 0, \cdots, 7 \).
Thus, by finding out all of the 11 double cosets \([w]\), we can determine the index of \(N\) in \(G\). We conclude that

\[
|G| \leq (|N| + \frac{|N|}{|N(0)|} + \frac{|N|}{|N(01)|} + \ldots + \frac{|N|}{|N(01246)|} + \frac{|N|}{|N(012467)|}) \times |N|
\]

\[
|G| \leq (1 + 8 + 16 + 16 + 46 + 4 + 16 + 64 + 16 + 16 + 4) \times 64
\]

\[
|G| \leq (225 \times 64)
\]

\[
|G| \leq 14400
\]

Next, we consider that \(G = \langle t_0, a, b, c, d, e, r \rangle\) acts on the 11 cosets and the actions of \(t, a, b, c, d, e, r\) on the 11 cosets is well-defined. Thus, we have a homomorphism \(f : 2^{*8} : N \rightarrow S_{64}\). Then \(f(2^{*8} : N) = \langle f(a), f(b), f(c), f(d), f(e), f(r), f(t) \rangle\); that is, the homomorphic image of \(G\) is of the order, \(|\langle f(a), f(b), f(c), f(d), f(e), f(r), f(t) \rangle| = 14400\). Since \(t\) has exactly 8 conjugates, we have \(f(G) \cong \langle f(a), f(b), f(c), f(d), f(e), f(r), f(t) \rangle\), where

\[
f(a) = (2, 3, 4, 6)(5, 7)(8, 10) \ldots (199, 216, 206, 219)(222, 223, 224, 225).
\]

\[
f(b) = (2, 4)(3, 7)(5, 10)(6, 8) \ldots (208, 220)(210, 221)(211, 212).
\]

\[
f(c) = (2, 5)(3, 8)(4, 10)(6, 7) \ldots (207, 211)(215, 221)(218, 220).
\]

\[
f(d) = (2, 4)(3, 6)(9, 18)(11, 23) \ldots (220, 221)(222, 224)(223, 225).
\]

\[
f(e) = (3, 6)(7, 8)(9, 12)(13, 14) \ldots (203, 206)(205, 211)(207, 212).
\]

\[
f(r) = (2, 4)(3, 6)(5, 10)(7, 8) \ldots (207, 212)(208, 220)(210, 221).
\]

\[
f(t) = (1, 2)(3, 9)(5, 11)(6, 12) \ldots (215, 217)(216, 224)(220, 225).
\]

Hence,

\[
G/Ker_f \cong \langle f(a), f(b), f(c), f(d), f(e), f(r), f(t) \rangle = f(G)
\]

\[
|G/Ker_f| \cong |f(G)|
\]

\[
|G| \geq |f(G)|
\]

\[
|G| \geq 14400.
\]
But from above we observed that $|G| \leq 14400$. Hence,

\[
\Rightarrow |G| = 14400 \\
\Rightarrow \text{Ker} f = 1 \\
\Rightarrow G \cong < f(a), f(b), f(c), f(d), f(e), f(r), f(t) > \\
\Rightarrow G \cong S_{64}.
\]

\[
G = \left( \frac{2^{8} : N}{[bcet]^{5}, [ert]^{3}, [a(t^{r})^{c}]^{4}} \right)
\]

Please see the Cayley diagram of $(A_{5} \times A_{5}) : 4$ over $(\mathbb{Z}_{4} \times \mathbb{Z}_{2}) \cdot D_{4}$ below.

![Cayley Diagram](image)

**Figure 5.1**: Cayley Diagram for $(A_{5} \times A_{5}) : 4$ Over $(\mathbb{Z}_{4} \times \mathbb{Z}_{2}) \cdot D_{4}$

We now show that $G \cong (A_{5} \times A_{5}) : 4$.

\[
G = \langle a, b, c, d, e, r, t | a^{2} = d, b^{2}, c^{2}, d^{2}, e^{2}, r^{2}, b^{a} = bc, e^{a} = ce, c^{b} = c, d^{a} = d, d^{b} = de, \]

\[
d^{c} = dr, e^{a} = er, e^{b} = e, e^{c} = e, e^{d} = e, r^{a} = r, r^{b} = r, r^{c} = r, r^{d} = r, r^{e} = r, t^{2} = \]

\[
(t, e), (t, ber), (t, db), (bcet)^{5}, (ert)^{3}, (a(t^{r})^{c})^{4} > .
\]

By using MAGMA, we have the composition series $G \supseteq G_{1} \supseteq G_{2} \supseteq G_{3} \supseteq 1$, where

\[
G = (G/G_{1})(G_{1}/G_{2})(G_{2}/G_{3})(G_{3}/1) = C_{2}C_{2}A_{5}A_{5}.
\]

The normal lattice of $G$ is
We want to determine the isomorphism type of this group. We find that the center of $G$ is of order 1. So, this is not a central extension. Moreover, by looking at the normal lattice of $G$, we see that it does not have a normal subgroup of order 2. Thus, this is not a direct product. The order of the minimal normal subgroup is 3600 which indicates that we have a direct product of $A_5$ by $A_5$.

```maple
> H<a,b,c,d>:=Group<a,b,c,d|a^3,b^2,(a*b)^5,c^3,d^2,
(c*d)^5,(a,c),(a,d),(b,c),(b,d)>;
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> #H;
3600
> s,t:=IsIsomorphic(H1,NL[2]);
> s;
true
```

Thus, we have shown $G_2 = A_5 \times A_5 = [2]$. Since the normal lattice of [3] and [4] do not contain a normal subgroup of order 4, it is not a direct product with $C_2^2 = C_4$. Now, we want to check if it is a semi-direct product of [2] by $C_4$. We find that it is. Thus, we have $A_5 \times A_5$ extended by $C_4$. Then, we must find the action of $C_4$ on the generators of $A_5 \times A_5$. We compute the action of $e$ on $a$, $b$, $c$, and $d$ by using the Schreier System for $A_5 \times A_5$ and write a presentation for $(A_5 \times A_5) : 4$.

```maple
> HH<a,b,c,d,e>:=Group<a,b,c,d,e|a^3,b^2,(a*b)^5,c^3,d^2,
(c*d)^5,(a,c),(a,d),(b,c),(b,d),
(e^4,a*e=d*c^1*d*c*d*c,
b*e=d*c*d*c^1*d,c*e=b*a^-1*b*a*b*a,d*e=a*b*a*b*a^-1*b*a^-1>;
```
> f2, H2, k2 := CosetAction(HH, sub<HH|Id(HH)>);
> #HH;
14400
> s := IsIsomorphic(H2, G1);
> s;
true

5.1.4 $S_5$ as a Homomorphic Image of $G$ over $N$

$G = \langle a, b, c, d, e, r, t | a^2 = d, b^2, c^2, d^2, e^2, r^2, b^6 = bc, c^a = ce, c^b = c, d^a = d, d^b = de, d^c = dr, e^a = er, e^b = e, e^c = e, e^d = e, r^a = r, r^b = r, r^c = r, r^d = r, r^e = r, t^2, (t, e), (t, be), (t, db), (dbt^c)^2, (ert)^2, (a(t^e)^c)^5, (ct^b)^3 \rangle.$

By using MAGMA, we have the composition series $G \supseteq G_1 \supseteq 1$, where

$$G = (G/G_1)(G_1/1) = C_2 A_5.$$

We want to determine the isomorphism type of this group. This group is not a central extension since the center of $G$ is of order 1. Moreover, by looking at the normal lattice of $G$, we see that it does not have a normal subgroup of order 2. So, this is not a direct product. The order of the minimal normal subgroup is 60 which indicate that we have a semi-direct product of $[2] = A_5$ by $\mathbb{Z}_2$. Using MAGMA, we highlight the relation between them by finding the generators $A$ and $B$ of $A_5$ and write the action of an element $C$ of order 2 on $A$ and $B$ by using Scherier system to write a presentation for $A_5 : C_2 \cong S_5$.

> a := NL[2].1;
> b := NL[2].2;
> HH < a, b, c > := Group < a, b, c | a^2, b^3, (a*b)^5, c^2,
  a*c = b^-1*a*b^-1*a*b*a*b, b*c = a*b+a*b^-1+a*b^-1 >;
> #HH;
120
> f2, H2, k := CosetAction(HH, sub<HH|Id(HH)>);
> s := IsIsomorphic(H2, G1); s;
true
> s := IsIsomorphic(H2, Sym(5)); s;
true

Thus, we have shown that $G \cong S_5$. 
5.2 The Progenitor $2^{*14} : (L_2(7) \times 2)$

A presentation for the progenitor for the transitive group on 14 letters is

$$< a, b, t | a^7, b^6, (a^{-1}b^{-1})^2, (ab^{-2})^2, (ab^{-1}a)^4, t^2, (t, b^2), (t, a^2b^2a) >$$

In order to find finite images, we factored the progenitor at first by all of the first order relations using the conjugacy classes of $N = L_2(7) \times 2$ (see table 5.2) with a relation that have found by Lemma $(t^ah^{-2}b)^k = b^ob^{-1}a^3$ but it did not produce any images.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the class</th>
<th>1st order relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>identity</td>
<td>$b^3t_1$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$b^3$</td>
<td>$abt_1, abt_2, abt_{10}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$ab$</td>
<td>$(ab^{-1}a)^2t_1, (ab^{-1}a)^2t_4, (ab^{-1}a)^2t_{10}$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$(ab^{-1}a)^2$</td>
<td>$b^2t_1, b^2t_2, b^2t_7$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$b^2$</td>
<td>$ba^3b^2a$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$b^2a^2$</td>
<td>$ba^2b^2at_1, ba^2b^2at_4, ba^2b^2at_{10}$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$b^2a^2$</td>
<td>$b^2a^2t_1, b^2a^2t_2, b^2a^2t_{11}$</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$b$</td>
<td>$bt_1, bt_2, bt_7$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$a$</td>
<td>$at_1$</td>
</tr>
<tr>
<td>$C_{10}$</td>
<td>$a^3$</td>
<td>$a^3t_1$</td>
</tr>
<tr>
<td>$C_{11}$</td>
<td>$ab^{-1}$</td>
<td>$ab^{-1}t_1$</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>$aba^{-1}bab^{-1}$</td>
<td>$aba^{-1}bab^{-1}t_1$</td>
</tr>
</tbody>
</table>

Thus, we tried to run the progenitor given above that have been factored just by the following first order relations. Then, we labeled the result in the table below.

$$(b^3t)^i, ((abt)^m, (abt^a)^n, (abt^a)^o, ((ab^{-1}a)^2 t)^p, ((ab^{-1}a)^2 t^a)^q, ((ab^{-1}a)^2 t^a^2)^r, (b^2t)^s,$$

$$(b^2t^a)^u, (b^2t^a^2)^v, (ba^2b^2bat)^w, (ba^2b^2bat^a)^x, (ba^2b^2bat^a^2)^y, (ba^2b^2bat^a^3)^z, (b^2a^2t^a)^c,$$

$$(b^2a^2t^a)^d, (bt)^e, (ba^2)^f, (ba^2)^g, (at)^i, (a^3)^j, (ab^{-1})^k, (aba^{-1}bab^{-1})^l.$$
5.3 The Progenitor $2^4:5$ : \(2^4: A_5\)

A presentation for $2^4: A_5$ is

\[
< a^3, b^4, b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}b^{-1}, (b^{-1}a^{-1})^5, (b^{-1}a)^5, \\
a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}abab^2a^{-1}bab > .
\]

and a presentation for $2^4:5$ : \(2^4: A_5\) is

\[
< a, b, t | a^3, b^4, b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}b^{-1}, (b^{-1}a^{-1})^5, (b^{-1}a)^5, \\
a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}abab^2a^{-1}bab, t^2, (t, a^{-1}b^{-1}a^{-1}bab), \\
(t, ab^{-1}abab^{-1}a), (t, ab^{-1}a^{-1}bab^{-1}a), (t, a^{-1}ab^{-1}a^{-1}bab) > .
\]

We factored the progenitor by all of the first order relations that have been found by the conjugacy classes of $N = 2^4: A_5$ (see table 5.4).

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the class</th>
<th>1st order relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>identity</td>
<td>(b^2t_1)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$b^2$</td>
<td>(aba^{-1}baba^{-1}b^2t_i), where $i = 1, 2, 3, 5$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$bab$</td>
<td>(b)</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$b$</td>
<td>(bt_1)</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$ba^{-1}b^{-1}ab$</td>
<td>(ba^{-1}b^{-1}abt_1)</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$ab^{-1}aba^{-1}b^{-1}ab$</td>
<td>(ab^{-1}aba^{-1}b^{-1}abt_1)</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$ba$</td>
<td>(bat_{13}t_{13})</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$(ba)^2$</td>
<td>((ba)^2t_1, (ba)^2t_2, (ba)^2t_4, (ba)^2t_5)</td>
</tr>
</tbody>
</table>

The all of the first order relations are

\[
(b^2t)^o, (aba^{-1}baba^{-1}b^2t)^f, (aba^{-1}baba^{-1}b^2t^b)^m, (aba^{-1}baba^{-1}b^2tba)^n, \\
(aba^{-1}baba^{-1}b^2(tba)^a)^o, (bat^p)^f, (babt^b)^g, (babab^ab)^r, (bab^2)^s, (babt^b)^r, (babab^ab)^r, \\
((ba)^2t^b)^h, ((ba)^2t^b)^i, ((ba)^2t^b)^o, ((ba)^2t^b)^o.
\]

When we factored the progenitor by all of the first order relations, no interesting homomorphic images were found. Thus, we change the relators to

\[
(aba^{-1}bab^{-1}ab)^{k}, (ab^{-1}a^2t)^{f}, (at^b)^{o}, (b^2a^{-1}t)^{m}, (a^btt)^{a}, (ab^2tt)^{b},
\]
The following images were obtained.

Table 5.5: Some finite images of the Progenitor $2^{*16} : (2^4 : A_5)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$n$</th>
<th>$o$</th>
<th>$p$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>10</td>
<td>64</td>
<td>61440</td>
<td>$2^6 \cdot (2^4 : A_5)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>5</td>
<td>32</td>
<td>30720</td>
<td>$2^5 \cdot (2 \times A_5)$</td>
</tr>
</tbody>
</table>

5.4 The Progenitor $2^{*4} : (2^2 : 3)$

A presentation for the progenitor is

$$<a, b, c, t \mid a^3, b^2, c^2, b^a = c, c^a = bc, c^b = c, t^2, (t, ac)>$$

By applying the lemma, we got the following

$$G = \frac{2^{*4} : (2^2 : 3)}{[bc]^p = 1} \cong 3 \times L_2(11), \text{ where } t \approx t_4.$$  

To have more images as tabulated in Table 5.6, we factored the progenitor by the following relators $(bc)^k$, $(ba^2t)^q$, $(a^{-1}cbtt^c)^u$, and $(at^c)^v$.

Table 5.6: Some finite images of the Progenitor $2^{*4} : (2^2 : 3)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$q$</th>
<th>$u$</th>
<th>$v$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>24</td>
<td>$2^4 : 3$</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>55</td>
<td>660</td>
<td>$L_2(11)$</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>10</td>
<td>120</td>
<td>$A_5 : 2 \cong S_5$</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>54</td>
<td>648</td>
<td>$3^3 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>91</td>
<td>1092</td>
<td>$L_2(13)$</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>250</td>
<td>3000</td>
<td>$5^2 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>686</td>
<td>8232</td>
<td>$7^3 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>22</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>2662</td>
<td>31944</td>
<td>$11^3 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>4</td>
<td>8</td>
<td>640</td>
<td>7680</td>
<td>$4 \cdot (2^4 : S_5)$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>16</td>
<td>192</td>
<td>$2^4 : A_4$</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>2000</td>
<td>24000</td>
<td>$(2^4 \times 3^3) \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>432</td>
<td>5184</td>
<td>$(2^4 \times 3^4) \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>1458</td>
<td>17496</td>
<td>$9^2 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>1024</td>
<td>12288</td>
<td>$8^2 \cdot (A_4 \times 2)$</td>
</tr>
<tr>
<td>24</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>3456</td>
<td>41472</td>
<td>$(4^3 \times 3^3) : (A_4 \times 2)$</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>128</td>
<td>1536</td>
<td>$4^3 \cdot (A_4 \times 2)$</td>
</tr>
</tbody>
</table>
5.5 The Progenitor $2^*6 : (3^2 : 2^2)$

A presentation for the progenitor is

\[ < a, b, c, d, t| a^2, b^2, c^3, d^3, b^a = b, c^a = c, e^b = e^2, d^a = d^2, d^b = d, d^c = d, \]
\[ t^2, (t, dc^{-1}), (t, abc^{-1}d^{-1}) >, \]

where $t \approx t_6$. The tabulated homomorphic images were obtained by factoring the progenitor $2^*6 : (3^2 : 2^2)$ by

\[ (ad^{-1}t)^k, (a^{-1}dt)^w, (cd^2tt^b)^x, (bcct^d)^y \]

<table>
<thead>
<tr>
<th>$k$</th>
<th>$w$</th>
<th>$x$</th>
<th>$y$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>162</td>
<td>5832</td>
<td>(9 × 3$^2$) : [(2$^2$ × 3) : 2]</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>18</td>
<td>648</td>
<td>3$^4$ : [(3 × 2$^2$) : 2]</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>10</td>
<td>40</td>
<td>(2$^2$ × 5) : 2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>9</td>
<td>2</td>
<td>54</td>
<td>1944</td>
<td>(9 × 3$^2$) • [(2$^2$ × 3) : 2]</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>12</td>
<td>2</td>
<td>72</td>
<td>2592</td>
<td>(4 × 3$^4$) • [(2$^2$ × 3) : 2]</td>
</tr>
</tbody>
</table>

5.6 The Progenitor $2^*10 : [(2 × 5) : 4]$  

A presentation for the progenitor is

\[ < a, b, c, d, t| a^2 = c, b^2, c^2, d^5, b^a = b, c^a = c, e^b = e^2, d^a = d^2, d^b = d, d^c = d^4, \]
\[ t^2, (t, cd), (t, ba^{-1}d^{-1}) >, \]

where $t \approx t_{10}$. The tabulated homomorphic images were obtained by factoring the progenitor $2^*10 : [(2 × 5) : 4]$ by the following relators

\[ (ad^3at)^k, (cabt^b)^h, (d^2tt^a)^i, (a^2t^c)^j \]
Table 5.8: Some finite images of the Progenitor $2^{10} : [(2 \times 5) : 4]$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$i$</th>
<th>$j$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>30</td>
<td>2</td>
<td>6</td>
<td>432</td>
<td>17280</td>
<td>$(2^2 \times 3) \cdot (PGL_2(9) : 2)$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>36</td>
<td>1440</td>
<td>$PGL_2(9) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>30</td>
<td>2</td>
<td>60</td>
<td>2400</td>
<td>$(4 \times 3 \times 5^2) : (4 \times 2)$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>2</td>
<td>15</td>
<td>360</td>
<td>14400</td>
<td>$A_5 \times A_5 : 2^4$</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>2</td>
<td>6</td>
<td>108</td>
<td>4320</td>
<td>$3 \cdot [PGL_2(9) : 2]$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>864</td>
<td>34560</td>
<td>$(4 \times 2 \times 3) \cdot (PGL_2(9) : 2)$</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>3</td>
<td>10</td>
<td>750</td>
<td>30000</td>
<td>$5^4 : [(4 \times 2 \times 3) : 2]$</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>2</td>
<td>6</td>
<td>144</td>
<td>5760</td>
<td>$2^4 : (PGL_2(9) : 2)$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>2</td>
<td>30</td>
<td>2880</td>
<td>115200</td>
<td>$2^3 \cdot (A_5 \times A_5 : 2^2)$</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>2</td>
<td>8</td>
<td>288</td>
<td>11520</td>
<td>$(4 \times 2) : (PGL_2(9) : 2)$</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>2</td>
<td>8</td>
<td>4032</td>
<td>161280</td>
<td>$2 \cdot ([L_3(4) : 2] : 2)$</td>
</tr>
</tbody>
</table>
Chapter 6

Wreath Product of Permutation Group

6.1 Preliminaries

Definition 6.1 (Wreath Product). Let $H$ and $K$ be permutation groups on $X$ and $Y$, respectively, and let $Z = X \times Y$. We define below a permutation group on $Z = \{(x,y)|x \in X, y \in Y\}$, called the wreath product of $H$ by $K$ denoted by $H \wr K$.

Note 6.2. A permutation group on $Z$ denoted by $S_Z$.

For the following examples, consider $H = Z_2 = \langle (1, 2) \rangle$ and $K = Z_2 = \langle (3, 4) \rangle$.

Example 6.3. We have

- $H = \{1, (1, 2)\}$ and $K = \{1, (3, 4)\}$.
- $X = \{1, 2\}$ and $Y = \{3, 4\}$.
- $Z = \{(x, y)|x \in X, y \in Y\} = \{(1,3), (1,4), (2,3), (2,4)\}$.

Definition 6.4. Let $\gamma \in H$ and $y \in Y$ be a fixed element of $Y$. Then,

$$
\gamma(y) = \begin{cases} 
(x,y) & \mapsto ((x)\gamma, y), \\
(x, y_1) & \mapsto (x, y_1) \text{ if } y_1 \neq y.
\end{cases}
$$
Example 6.5. Let $\gamma = (1, 2) \in H$ and $y = 3 \in Y$. Then, $\gamma(y) = (1, 3)$ found by the following table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,3)</td>
<td>(1,4)</td>
<td>(2,3)</td>
<td>(2,4)</td>
</tr>
<tr>
<td>2</td>
<td>(2,3)</td>
<td>(1,4)</td>
<td>(1,3)</td>
<td>(2,4)</td>
</tr>
</tbody>
</table>

\begin{align*}
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
3 & 2 & 1 & 4
\end{align*}

Definition 6.6. Let $k \in K$. Define $k^*: (x, y) \mapsto (x, (y)k)$.

Note 6.7. $K$ acting only on the second component $y_i$'s.

Example 6.8. Let $k = (3, 4) \in K$. Then, $k^* = (1, 3)^* = (1, 2)(3, 4)$ found by the following table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,3)</td>
<td>(1,4)</td>
<td>(2,3)</td>
<td>(2,4)</td>
</tr>
<tr>
<td>2</td>
<td>(1,4)</td>
<td>(1,3)</td>
<td>(2,4)</td>
<td>(2,3)</td>
</tr>
</tbody>
</table>

\begin{align*}
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
2 & 1 & 4 & 3
\end{align*}

Note 6.9. If $\gamma(y)$ and $k^*$ are permutations of $S_Z$, then

1. There is a homomorphism map $\phi : H \to S_Z$ which it is also 1-1 given by $\gamma \mapsto \gamma(y)$.

2. $\phi(y) = \{\gamma(y) | \gamma \in H\}$.

Also,

1. There is a homomorphism map $\Psi : K \to S_Z$ which it is also 1-1 given by $k \mapsto k^*$.

2. $\Psi(K) = \{k^* | k \in K\} = K^*$.

The semi-direct product $H^n : K$ is the wreath product of $H$ by $K$, where

- $n$ is the number of letters on which $H$ acts.
\begin{itemize}
  \item $H^n$ is the direct product ($< H_1 > \times \ldots \times < H_n >$) of $n$ copies permutations of $H$.
\end{itemize}

In the semi-direct part, $K$ permutates the $n$ copies of $H$. Thus,

$$H : K = \langle H(y), k^* \rangle = \prod_{y \in Y} H(y) : K^*.$$ 

6.2 The Progenitor $2^*6 : \mathbb{Z}_3 \wr \mathbb{Z}_2$

Consider the wreath product of $H = \mathbb{Z}_3$ by $K = \mathbb{Z}_2$, denoted by $\mathbb{Z}_3 \wr \mathbb{Z}_2$. In the semi-direct $\mathbb{Z}_3^2 : \mathbb{Z}_2$, $\mathbb{Z}_2$ permutes the 2 copies of $\mathbb{Z}_3$. Thus, we have

$$\mathbb{Z}_3 \wr \mathbb{Z}_2 = \mathbb{Z}_3^2 : \mathbb{Z}_2 = (\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_2$$

Now, we want to write a presentation for $2^*6 : \mathbb{Z}_3 \wr \mathbb{Z}_2$.

$$\mathbb{Z}_3 \wr \mathbb{Z}_2 = \mathbb{Z}_3 \times \mathbb{Z}_3 : \mathbb{Z}_2.$$

$$= \langle (1, 2, 3) \rangle \times \langle (4, 5, 6) \rangle : \langle (7, 8) \rangle.$$ 

Also, we have

$$X = \{1, 2, 3\} \text{ and } Y = \{4, 5\}.$$

$$H = \langle (1, 2, 3) \rangle = \langle \gamma \rangle \text{ and } K = \langle (4, 5) \rangle.$$ 

Hence,

$$Z = X \times Y = \{1, 2, 3, 4, 5, 6\}.$$ 

Now, we want to compute $\gamma(4)$ and $\gamma(5)$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
(1,4) & (1,5) & (2,4) & (2,5) & (3,4) & (3,5) \\
\hline
(2,4) & (1,5) & (3,4) & (2,5) & (1,4) & (3,5) \\
\hline
3 & 2 & 5 & 4 & 1 & 6 \\
\hline
\end{tabular}
\caption{\(\gamma(4)\)}
\end{table}

Thus, we get $\gamma(4) = (135) = H(4)$. 

Table 6.2: $\gamma(5)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,4)</td>
<td>(1,5)</td>
<td>(2,4)</td>
<td>(2,5)</td>
<td>(3,4)</td>
<td>(3,5)</td>
<td></td>
</tr>
<tr>
<td>(1,4)</td>
<td>(2,5)</td>
<td>(2,4)</td>
<td>(3,5)</td>
<td>(3,4)</td>
<td>(1,5)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Thus, we get $\gamma(5) = (246) = H(5)$.

Then, we want to compute $k^* = (45)^*$.  

Table 6.3: $k^*$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,4)</td>
<td>(1,5)</td>
<td>(2,4)</td>
<td>(2,5)</td>
<td>(3,4)</td>
<td>(3,5)</td>
<td></td>
</tr>
<tr>
<td>(1,5)</td>
<td>(1,4)</td>
<td>(2,5)</td>
<td>(2,4)</td>
<td>(3,5)</td>
<td>(3,4)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Thus, we get $k^* = (45)^* = (12)(34)(56)$.

Since $\mathbb{Z}_3 \wr \mathbb{Z}_2 = \langle H(4), H(5), k^* \rangle$, we obtain

$$\mathbb{Z}_3 \wr \mathbb{Z}_2 = \langle (135), (246), (12)(34)(56) \rangle = \langle (135) \rangle \times \langle (246) \rangle : \langle (12)(34)(56) \rangle,$$

where $a = (135)$, $b = (246)$, and $c = (12)(34)(56)$.

To write the general presentation for $\mathbb{Z}_3 \wr \mathbb{Z}_2$, we have to find the action of $c$ on $a$ and $b$.

$$a^c = (135)^{(12)(34)(56)} = (246) = b.$$

$$b^c = (246)^{(12)(34)(56)} = (135) = a.$$

Thus, the general presentation for $\mathbb{Z}_3 \wr \mathbb{Z}_2$ is

$$< a, b, c | a^3, b^3, (a, b), c^2, a^c = b, b^c = a >$$

The order of the wreath product is $|\mathbb{Z}_3 \wr \mathbb{Z}_2| = 3 \times 3 \times 2 = 18$. To write a progenitor for the wreath product, we have to find a faithful permutation representation for $\mathbb{Z}_3 \wr \mathbb{Z}_2$.

The next MAGMA code confirm our presentation.
W:=WreathProduct(CyclicGroup(3),CyclicGroup(2));
> S:=Sym(6);
> aa:=S!(4, 5, 6);
> bb:=S!(1, 2, 3);
> cc:=S!(1, 5)(2, 6)(3, 4);
> N:=sub<S|aa,bb,cc>;
> N eq W;
true

Let \( t = t_1 \). Now, by looking at the 18 elements of \( N = \mathbb{Z}_3 \wr \mathbb{Z}_2 \), we have to determine the permutation of \( N \) that fix the coset stabiliser \( Nt_1 \). The desired permutations are the identity and \( a = (4, 5, 6) \). By using MAGMA, we find that \( t \) commute with \( a \). Thus, the presentation for the progenitor \( 2^*6 : \mathbb{Z}_3 \wr \mathbb{Z}_2 \) is given by

\[
< a, b, c, t | a^3, b^3, (a,b), c^2, a^c = b, b^c = a, t^2, (t,a) > .
\]

Moreover, in order to obtain some finite images, we factored the progenitor \( 2^*6 : C_3 \wr C_2 \) by the following relations

\[
(bt)^k, (btt^b)^l, (abtc^t)^m, (cbt)^n
\]

Table 6.4: Some finite images of the Progenitor \( 2^*6 : C_3 \wr C_2 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( l )</th>
<th>( m )</th>
<th>( n )</th>
<th>Index in ( G )</th>
<th>Order of ( G )</th>
<th>Shape of ( G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>400</td>
<td>7200</td>
<td>( A_5 \times A_5 : 2 )</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>11</td>
<td>10</td>
<td>5280</td>
<td>95040</td>
<td>( M_{12} )</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
<td>15</td>
<td>36000</td>
<td>64800</td>
<td>( A_5 \times A_5 \times A_5 : 3 )</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1600</td>
<td>28800</td>
<td>( 2^*[A_5 \times A_5] : 2^2 )</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>4800</td>
<td>86400</td>
<td>( (4 \times 3) : [(A_5 \times A_5) : 2] )</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>5</td>
<td>0</td>
<td>840</td>
<td>15120</td>
<td>( A_7 \times 3 : 2 )</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>1200</td>
<td>21600</td>
<td>( (A_5 \times A_5) : 3 : 2 )</td>
</tr>
</tbody>
</table>

Then, we tried to apply the Lemma to find more images. Let \( t = t_6 \). By using MAGMA, we find that \( t \) commute with \( b \). Thus, the presentation for the progenitor \( 2^*6 : \mathbb{Z}_3 \wr \mathbb{Z}_2 \) is given by

\[
< a, b, c, t | a^3, b^3, (a,b), c^2, a^c = b, b^c = a, t^2, (t,b) >
\]
Now, we will utilize the lemma by taking the stabiliser of two elements and determining the centraliser of those points. The lemma deals with elements that have a permutation $(6,1)$ by itself, do have either 1 or 6, or do not have either of them.

\[ N \cap \langle t_0, t_1 \rangle \leq C_N(N^{01}). \]
\[ N^{(01)} = \langle e \rangle. \]
\[ C_N(N^{01}) = \langle (4, 5, 6), (1, 2, 3), (1, 5)(2, 6)(3, 4) \rangle. \]

So, the choice for writing product of $2t_i$'s in terms of elements of $N$ are unrestricted.

By running the set of the centralizer, we find that there is an element that includes the transposition $(6,1)$ such as.

\[ (1, 6)(2, 4)(3, 5) = bcb^{-1}. \]

Thus, we have to apply the lemma as the following:

\[ (bcb^{-1}t)^k = 1, \text{ where } k \text{ is odd}. \]

To find some finite images of the progenitor $2^6 : C_3 \wr C_2$, we have to add more relations and run them in the background of MAGMA to have some of interesting group.

\[
< a, b, c, t | a^3, b^3, (a, b), c^2, a^c = b, b^c = a, t^2, (t, b), (bcb^{-1}t)^k = 1, (ca^{-1}t)^l, (btt)^m,
\]

\[
(ab^2c)^n, (bacb)\rangle .
\]

Table 6.5: Some finite images of the Progenitor $2^6 : C_3 \wr C_2$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$n$</th>
<th>$o$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>4</td>
<td>3</td>
<td>10</td>
<td>280</td>
<td>5040</td>
<td>$A_7 : 2 \cong S_7$</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>560</td>
<td>10080</td>
<td>$2 \times (A_7 : 2) \cong 2 \times S_7$</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>12</td>
<td>3</td>
<td>10</td>
<td>840</td>
<td>15120</td>
<td>$(3 \times A_7) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>11</td>
<td>3</td>
<td>11</td>
<td>5280</td>
<td>95040</td>
<td>$M_{12}$</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>8</td>
<td>42240</td>
<td>760320</td>
<td>$4 \circ (M_{12} : 2)$</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>9</td>
<td>144000</td>
<td>2592000</td>
<td>$(4 : [A_5 \times A_5 \times A_5]) : 2$</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>5</td>
<td>4</td>
<td>12</td>
<td>403200</td>
<td>7257600</td>
<td>$2^2 \cdot A_{10}$</td>
</tr>
</tbody>
</table>

Moreover, by factoring the progenitor with all of the first order relations that found from the conjugacy classes of $C_3 \wr C_2$, we obtain similar groups as shown in table 6.5 and the following homomorphic image
6.3 The Progenitor $2^6 : C_2 \wr S_3$

Consider the wreath product of $H$ by $K$, denoted by $H \wr K$, where

$$H = \mathbb{Z}_2$$

and

$$K = S_3.$$

By applying the same process as shown in section 6.2, the presentation for $2^6 : \mathbb{Z}_2 \wr S_3$ is

$$<a^2, b^2, c^2, (a, b), (a, c), (b, c), d^3, e^2, a^d = b, b^d = c, c^d = a, a^e = b, b^e = a, c^e = c, t^2, (t, a), (t, c), (t, ed)>,$$

where

$$a = (34), b = (56), c = (12), d = (145)(236), 	ext{ and } e = (36)(45).$$

In order to obtain homomorphic images, we factored the progenitor by suitable relators. Hence,

$$G = \frac{2^6 : C_2 \wr S_3}{[d^{-1}ct]^5, [d^2etb]^2, [cbea]^5} \cong (4 \times 2^5) : A_5$$

and

$$G = \frac{2^6 : C_2 \wr S_3}{(t, t^b), (t, t^c)} \cong 2^6 : (2^3 : S_3).$$

The homomorphic images that are listed in the table below were found by factoring the progenitor $2^6 : C_2 \wr S_3$ by all of the following first order relations.

$$(abct)^k, (ct)^l, (ct^d)^m, (act)^n, (act^d)^o, (et)^p, (et^d)^q, (cet)^r,$$

$$(cet^d)^s, (dt)^u, (cedt)^v, (cedt^d)^w, (acet)^x, (acet^d)^y, (ced)^z$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>120</td>
<td>$2 \times A_5$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>32</td>
<td>1536</td>
<td>$(4^2 \times 2) \bullet (2^4 : S_3)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>18</td>
<td>162</td>
<td>1944</td>
<td>$(9^2 \times 2) : ([2 \times 3] : 2)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>7</td>
<td>210</td>
<td>10080</td>
<td>$2 \times S_7$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>14</td>
<td>98</td>
<td>1176</td>
<td>$(2 \times 7^2) : ([2 \times 3] : 2)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>20</td>
<td>10240</td>
<td>491520</td>
<td>$2^{12} \bullet (2 \times A_5)$</td>
</tr>
</tbody>
</table>
Chapter 7

Monomial Progenitor of $P^*n : m \ N$

7.1 Preliminaries

Definition 7.1 (Representation of $G$). A representation of a finite group $G$ is defined by $\phi : G \xrightarrow{\text{Homo}} GL(n, F)$, where $GL(n, F)$ is the group of $n \times n$ invertible matrices over a finite field $F$.

Definition 7.2 (Characters). The character $\chi$ afforded by the representation $\rho : G \rightarrow GL(n, F)$ is a function

$$\chi : G \rightarrow F,$$

given by $\chi(g) = Tr(gp) \ \forall g \in G$.

Definition 7.3 (Equivalent Representation). Let $\rho : G \xrightarrow{\text{homo}} GL(n, F)$ and $T \in GL(n, F)$. Then, $T^{-1}\rho T$ is also a representation of $G$ and $T^{-1}\rho T$ and $\rho$ are called equivalent.

Definition 7.4 (Faithful Representation). Let $\rho : G \xrightarrow{\text{Rep}} GL(n, F)$ and $\ker \rho = \{g \in G|g\rho = I_n\}$. Then, $\rho$ is faithful if $\ker \rho = 1$.

Definition 7.5 (Trivial Representation). A representation $\rho : G \xrightarrow{\text{Rep}} GL(n, F)$ given by $g\rho = T_n$ is a trivial representation of $G$.

Facts:

1. $\chi(x) = \chi(y)$ if $x$ and $y$ are conjugates.

2. Equivalent representation have the same character.
3. The number of irreducible character of $G$ is equal to the number of the conjugacy classes of $G$.

**Character Table:**

Think of a character table as the matrix $\left( \chi^{(i)}_\alpha \right)$. In a character table

1. Let $X$ be the row vector $\left( h_\alpha \chi^{(i)}_\alpha \right)$ and $\bar{Y}$ be the conjugate of the row vector $\left( \chi^{(i)}_\alpha \right)$.

If $i \neq j$, then the ordinary dot product $X \cdot \bar{Y} = 0$.

2. Let $X$ be the row vector $\left( h_\alpha \chi^{(i)}_\alpha \right)$. Then, the ordinary dot product $X \cdot \left( \chi^{(i)}_\alpha \right) = |G|$

**Note 7.6.** $\chi^{(i)}_\alpha = \chi_\alpha(g)$, where $g$ is an element of the conjugacy class $C_\alpha$.

**Definition 7.7 (Trivial Character).** The trivial character is the character $\chi$ of the trivial representation, where $\chi: G \to \mathbb{F}$ given by $\chi(g) = 1 \; \forall g \in G$.

**Definition 7.8 (Monomial Matrix).** A square matrix that has exactly one non-zero entry in each row and each column.

**Definition 7.9 (Monomial Representations).** Let $G$ be a group. A monomial representations is a map $A: G \xrightarrow{\text{Homo}} \text{GL}(n, \mathbb{F})$ which provided that $A(x)$ and $A(y)$ are monomial matrices.

**Definition 7.10 (Monomial Character).** A character $\phi$ of $G$ is monomial if $\phi$ is induced by a linear character of a subgroup $H$ (not necessarily proper) of $G$.

**Definition 7.11 (Induced Character).** Let $H \leq G$ and $\phi$ be a character of $G$. The formula for induced character is

$$\phi_G^C = \frac{n}{h_\alpha} \sum_{z \in C_\alpha \cap H} \phi(z),$$

where

$\phi_G^C$ is the value of $\phi^G$ at each element of the class $C_\alpha$.


$h_\alpha$ is the number of elements in the class $C_\alpha$ of $G$.

**Example 7.12.** Let $G = S_3$ and $H = A_3 \leq G$. We want to show that $\phi^G$ is a monomial character.
Table 7.1: Character Table of $G = S_3$

<table>
<thead>
<tr>
<th>Classes</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>#Elements</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

| $\chi_1$ | 1     | 1     | 1     |
| $\chi_2$ | 1     | -1    | 1     |
| $\chi_3$ | 2     | 0     | -1    |

Table 7.2: Character Table of $H = A_3$

<table>
<thead>
<tr>
<th>Classes</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>#Elements</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

| $\chi_1$ | 1     | 1     | 1     |
| $\chi_2$ | 1     | $J - 1$ | $-J$ |
| $\chi_3$ | 1     | $-1 - J$ | $J$ |

$J = \text{RootOfUnity}(3)$

$G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$.

$H = A_3 = \{(1), (123), (132)\} \cong \mathbb{Z}_3$.

The index of $H$ in $G$ is equal to $n = \frac{|S_3|}{|A_3|} = \frac{6}{3} = 2$

Since $A_3 \cong \mathbb{Z}_3$, the cube root of unity is $J$. Thus, $1 + J + J^2 = 0$.

Now, we want to induce $G$ by the second linear character $\phi = \chi^{(2)}$ of $H$.

$\phi_1^G = \frac{2}{1} \sum_{z \in C_1 \cap H} \phi(z) = 2 \cdot \phi(1) = 2(1) = 2$.

$\phi_2^G = \frac{2}{3} \sum_{z \in C_2 \cap H} \phi(z) = 2 \cdot \phi(0) = 0$.

$\phi_3^G = \frac{2}{2} \sum_{z \in C_3 \cap H} \phi(z) = \frac{2}{2} [\phi((123)) + \phi((132))] = J + J^2 = -1$.

We see that

$\phi_1^G = 2 = \chi_1^{(3)}$, $\phi_2^G = 0 = \chi_2^{(3)}$, and $\phi_3^G = -1 = \chi_3^{(3)}$.

We show that $\phi^G$ is a monomial Character.

To find a monomial representation of a finite group $G$, we have to know the following:
1. The number of the linear character of $G$ is equal to $|G/\hat{G}|$.

2. $G/\hat{G}$ is an abelian group.

3. Assume $G/\hat{G} \cong \mathbb{Z}_3$. Then, the linear characters have $3rd$ roots of unity as its entries.

4. Character $\phi$ of the group $G$ is irreducible $\iff \langle \phi, \phi \rangle = 1$, where

$$
\langle \phi, \phi \rangle = \frac{1}{|G|} \sum_{\alpha=1}^{k} h_{\alpha} \phi_{\alpha} \overline{\phi}_{\alpha}.
$$

5. All linear representations of a finite group $G$ are monomial.

6. Let $B$ be a linear representation of a proper subgroup $H$ of $G$, and let $G = \bigcup_{i=1}^{n} Ht_{i}$. Then, a monomial representation of $G$ is given by

$$
A(x) = \begin{pmatrix}
B\left(t_{1}x_{1}t_{1}^{-1}\right) & \cdots & B\left(t_{1}x_{1}t_{n}^{-1}\right) \\
B\left(t_{2}x_{1}t_{1}^{-1}\right) & \cdots & B\left(t_{2}x_{1}t_{n}^{-1}\right) \\
\vdots & \cdots & \vdots \\
B\left(t_{n}x_{1}t_{1}^{-1}\right) & \cdots & B\left(t_{n}x_{1}t_{n}^{-1}\right)
\end{pmatrix}
$$

Example 7.13. Let $G = S_{3} = \langle (123), (12) \rangle$ and $H = A_{3} = \langle (123) \rangle$. The character table of $G$ and $H$ are given in Ex 7.11.

The transversal (right cosets) of $G/H$ is $\{e, (12)\}$. Thus,

$$
G = He \cup H(12) = Ht_{1} \cup Ht_{2}.
$$

Since $J$ is the cube root of unity, $J = 2$ and $J^{2} = 4$. The representative of $H$ are:

$$
B(e) = e.
$$

$$
B((123)) = J - 1 = 1.
$$

$$
B((132)) = -J = -2.
$$
The two monomial representations are

\[ A(x) = \begin{pmatrix} B(t_1 x t_1^{-1}) & B(t_1 x t_2^{-1}) \\ B(t_2 x t_1^{-1}) & B(t_2 x t_2^{-1}) \end{pmatrix} = \begin{pmatrix} B(ex) & B(ex(12)) \\ B((12)x) & B((12)x(12)) \end{pmatrix} \]

\[ = \begin{pmatrix} B((123)) & B((123)(12)) \notin A_3 \\ B((12)(123)) \notin A_3 & B((12)(123)(12)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \]

\[ A(y) = \begin{pmatrix} B(t_1 y t_1^{-1}) & B(t_1 y t_2^{-1}) \\ B(t_2 y t_1^{-1}) & B(t_2 y t_2^{-1}) \end{pmatrix} = \begin{pmatrix} B(e ye) & B(e y(12)) \\ B((12)y e) & B((12)y(12)) \end{pmatrix} \]

\[ = \begin{pmatrix} B((12)) \notin A_3 & B(e) \\ B(e) & B((12)) \notin A_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

**Induced Characters:**

Assume the following:

- \( G \) is a group of order \( q \).

- \( H \) is a subgroup of \( G \) with index \( n \) in \( G \).

- \( B \) is a representation of \( H \) of degree \( m \).

By the latter assumption, there exist a map \( B : H \xrightarrow{\text{Homo}} \text{GL}(m, \mathbb{F}) \) that is given by \( B(uv) = B(u)B(v), \forall u, v \in H \). We want to extend \( B \) to give a presentation of \( G \). Thus, define \( A : G \to \text{GL}(\ell, \mathbb{F}) \) by

\[ A(x) = \begin{cases} B(x), & \text{if } x \in H \\ 0, & \text{if } x \notin H \end{cases} \]

Let \( x, y \in G \).

\[ A(xy) = \begin{cases} B(xy), & \text{if } xy \in H \\ 0, & \text{if } xy \notin H \end{cases} \]

In this chapter, we are going to write monomial presentation for the monomial progenitors of the form \( p^n : m N \), where \( p \geq 3 \) and \( N \) is isomorphic to \( 2^\bullet A_5, L_2(7), \) and \( A_7 \), and look for the homomorphic images of the progenitor, where \( N \) acts monomially on the symmetric generators of order \( p \). Thus, we need a subgroup \( H \) of \( N \) whose index
in $G$ is equal to $n$.

Then, we will be able to form a monomial progenitor $3^n :_m N$. The next examples will illustrate what we have written above.

### 7.2 Monomial Progenitor $3^{12} :_m (2 \cdot A_5)$

Consider the group $3^{12} :_m (2 \cdot A_5)$, where $G = 2 \cdot A_5$. By knowing the presentation for $G = 2 \cdot A_5 = \langle a, b | a^4, b^3, (ab)^5, (a^2, b) \rangle$, where

$$a \approx (1, 2, 5, 4)(3, 6, 8, 7)(9, 13, 11, 14)(10, 15, 12, 16)(17, 19, 18, 20)(21, 24, 23, 22)$$

$$b \approx (1, 3, 2)(4, 5, 8)(6, 9, 10)(7, 11, 12)(13, 16, 17)(14, 15, 18)(19, 21, 22)(20, 23, 24)$$

we can write a monomial permutation representation of $G$. To obtain this, we have to choose a subgroup $H$, not necessarily proper, of $2 \cdot A_5$ and induced it up by a linear character of $H$. First, we chose $H$ to be a central extension of cyclic group of order 2 by $D_{10}$, but we figured out that the character table of $2 \cdot D_{10}$ does not have a faithful character. So, by looking at the character table of $G$, we see that $G$ is not an abelian group since its classes does not have a single element. Thus, not every irreducible character of $G$ is faithful. Then, we choose $H$ to be a Dihedral group of order 10 since it has the same index 12 as the order of the irreducible character of $G$ and it has a faithful character (not necessary irreducible).

To find the value of $\omega$ in the table 7.4, we have to determine the smallest finite field that has square roots of unity. Since the non-zero entries of the two monomial matrices are 1 and 2, we conclude that the smallest finite field is $\mathbb{F}_3$. If $|\mathbb{F}_3| = 3$, then $|\mathbb{F}_3^*| = 2$ is a cyclic group of order 2. So, $\mathbb{F}_3 = \{0, 1, 2\}$ consists a cyclic group of order 2. Since $< 1 >= \{1, 1^2\} = \{1, 2\}$, we see that

$$2^2 = 4 \equiv_3 1$$

So, the value for $\omega$ is equal to 2. Therefore, the value for the different powers of $\omega$ are given by

$$\omega^2 = \omega \cdot \omega = 2 \cdot 2 = 4 \equiv_3 1$$

$$\omega^3 = \omega \cdot \omega^2 = 2 \cdot 1 = 2$$

$$\omega^4 = \omega^2 \cdot \omega^2 = 1 \cdot 1 = 1$$
Table 7.3: Character Table of $G = 2^\bullet A_5$

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
<th>$C_8$</th>
<th>$C_9$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
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<td>-2</td>
<td>-1</td>
<td>0</td>
<td>$Z_1$</td>
<td>$Z_1#2$</td>
<td>1</td>
<td>$-Z_1$</td>
<td>$-Z_1#2$</td>
</tr>
<tr>
<td>$\chi_3$</td>
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<td>-2</td>
<td>-1</td>
<td>0</td>
<td>$Z_1#2$</td>
<td>$Z_1$</td>
<td>1</td>
<td>$-Z_1#2$</td>
<td>$-Z_1$</td>
</tr>
<tr>
<td>$\chi_4$</td>
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<td>$-Z_1#2$</td>
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<td>$-Z_1$</td>
<td>$-Z_1#2$</td>
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<td>0</td>
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<tr>
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<td>1</td>
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<td>-1</td>
</tr>
</tbody>
</table>

# denotes algebraic conjugation

$Z_1$ is the square root of unity

Table 7.4: Character Table of $N = D_{10}$

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
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<th>$C_7$</th>
<th>$C_8$</th>
<th>$C_9$</th>
<th>$C_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
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<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
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<td>1</td>
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</tr>
<tr>
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<td>1</td>
<td>-1</td>
<td>$\omega$</td>
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<td>$\omega^3$</td>
<td>$\omega^4$</td>
<td>$-\omega$</td>
<td>$-\omega^3$</td>
<td>$-\omega^2$</td>
<td>$-\omega^4$</td>
</tr>
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<td>$\vdots$</td>
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</tr>
<tr>
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<td>-1</td>
<td>$\omega^4$</td>
<td>$\omega^3$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$-\omega^4$</td>
<td>$-\omega^3$</td>
<td>$-\omega^2$</td>
<td>$-\omega$</td>
</tr>
</tbody>
</table>

Now, we want to induce $H = D_{10}$ up to $G = 2^\bullet A_5$ to find the two induced representations $A(a)$ and $A(b)$ of degree $\frac{|G|}{|N|} = \frac{120}{10} = 12$ of $G$. Thus, we choose to induce the nonprincipal linear character $\chi_2$ of $H$. It gives the following character (12,-12,0,0,2,2,0,-2,-2). Since we are dealing with 12×12 matrices, we run the following loop to find the non-zero entries for the two monomial representations $A(a)$ and $A(b)$ that generated $2^\bullet A_5$.

```
A:=[0:i in [1..144]];  
for i in [1..12] do if a*T[i]^-1 in N then i,  
    chN[2](a*T[i]^-1); end if; end for;  
for i in [1..12] do if T[2]*a*T[i]^-1 in N then i,  
    chN[2](T[2]*a*T[i]^-1); end if; end for;  
```

for i in [1..12] do if T[12]*a*T[i]^-1 in N then i,
Thus, we obtain

\[
A(a) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

and

\[
A(b) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\(A(a)\) and \(A(b)\) are monomial matrices since there is one non-zero entry other than 1 in each row and column. By checking the order of the two matrices, we see that \(|A(a)| = 4 = |a|\), \(|A(b)| = 3 = |b|\), and \(|A(a) * A(b)| = 5 = |a * b|\). \(\langle A(a), A(b) \rangle\) is a faithful representation of \(2^*A_5\). Now, we want to write a permutation representation of
the two monomial matrices. Elements of the free product $3^{*12} \text{ are of the form } < t_1 > \ast < t_2 > \ast \ldots \ast < t_{11} > \ast < t_{12} >$, where $|t_i| = 3$ and $< t_i > = \{e, t_i, t_i^2\}$ for $\forall 1 \leq i \leq 12$.

To determine the permutation representation of the monomial progenitor, we first find

**Table 7.5: Labeling for $t_i$’s**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
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<tr>
<td>13</td>
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<td>17</td>
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<td>24</td>
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<tr>
<td>$t_7$</td>
<td>$t_7^2$</td>
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<td>$t_{11}$</td>
<td>$t_{11}^2$</td>
<td>$t_{12}$</td>
<td>$t_{12}^2$</td>
</tr>
</tbody>
</table>

the permutation representation of $G$. Let $x = A(a)$. We look at the non-zero entry of $A(a) a_{i,j}$ such that

$a_{1,2} = 1 \Rightarrow$ the automorphism takes $t_1 \rightarrow t_2$, so $A(a)$ takes 1 to 3 in our labeling above.

Moreover, when $t_1^2 \rightarrow t_2^2$, $A(a)$ takes 2 to 4

$a_{2,1} = -1 \equiv_3 2 \Rightarrow$ the automorphism takes $t_2 \rightarrow t_1^2$, so $A(a)$ takes 3 to 2.

Moreover, when $t_2^2 \rightarrow t_1$, $A(a)$ takes 4 to 1.

\[ \vdots \]

**Table 7.6: Permutaion of $x = A(a)$**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
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<tbody>
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</table>

The permutation of $x$ is

\[ (1,3,2,4)(5,8,6,7)(9,17,10,18)(11,22,12,21)(13,23,14,24)(15,19,16,20). \]

Similarly for the permutation of $y = A(b)$.

The permutation for $y$ is
Table 7.7: Permutation of $y = A(b)$

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<th>1</th>
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<td>12</td>
<td>11</td>
<td>21</td>
<td>22</td>
</tr>
</tbody>
</table>


To write a presentation for the monomial progenitor $3^{*12} : \langle 2^* A_5 \rangle$, let $t \approx t_1$. Then, we have to determine what element of $H$ normalise $G$. Thus, we are looking for $t_1$ such that

$$g \in G \ni \langle t_1 \rangle = \langle t_1 \rangle$$

$$\{e, t_1, t_1^2\} = \{e, t_1, t_1^2\}$$

$$\{t_1, t_1^2\}^g = \{t_1, t_1^2\}.$$  

Either $t \to t^2$ and $t^2 \to t$ or $t \to t$ and $t^2 \to t^2$. The normaliser of the subgroup $\langle t_1 \rangle$ is generated by

$$(5, 11, 24, 10, 19)(6, 12, 23, 9, 20)(7, 15, 17, 14, 21)(8, 16, 18, 13, 22), \quad \text{and}$$

$$(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$$

Now, we must write these permutations in terms of $x$ and $y$ by using the Schreier System for $G$, to make them commute with $\langle t \rangle$.

$$(5, 11, 24, 10, 19)(6, 12, 23, 9, 20)(7, 15, 17, 14, 21)(8, 16, 18, 13, 22) = xyxy^{-1}$$

$$(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24) = x^2$$

Note that the first permutation fix $t_1$, so it will have the following form $(t, xyxy^{-1})$ or $t^{xyxy^{-1}}t^{-1}$. The second one sends $t_1 \to t_2$, so $t_1^2 \to t_2^2$. Therefore, we will write it in the
following form $t^{x^2}t^{-2}$.

Now, we can write the desired monomial presentation

$$3^{*12}:_m (2^*A_5) = \langle x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^3, (t, xxy^{-1}), t^{x^2}t^{-2} \rangle.$$ 

To find some finite images, we add the following relations to the progenitor of $3^{*12}:_m (2^*A_5)$.

$$(x^2yt)^k, (xyt^l, (xy^2t(x^y))^m, (x^2yt^xy)^n$$

But, we could not find any homomorphic images except

$$G = \frac{3^{*12}:_m (2^*A_5)}{(t_1, t_i)} \cong 3^{12}: (2^*A_5), \text{ where } i = 2, 3, 4, 5, 9, 7, 8.$$ 

The presentation for $G$ is

$$\langle x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^3, (t, xxy^{-1}), t^{x^2}t^{-2},$$

$$(t, t^{x^2}), (t, t^x), (t, t^3), (t, t^3), (t, t^3), (t, t^3), (t, t^3),$$

$$\rangle = \langle x, y, t, d, e, f, g, h, i, j, k, l \rangle.$$ 

Now, we have to prove that $3^{12}: (2^*A_5)$ is isomorphic to $G$. So, we need $t_1, t_2, \ldots, t_{12}$ in terms of $t^{x*}y^3$. We already knew that the presentation for $3^{*12}:_m (2^*A_5)$ is

$$\langle x, y, t, d, e, f, g, h, i, j, k, l \rangle = \langle x, y, t, d, e, f, g, h, i, j, k, l \rangle.$$ 

We have shown that $N = 2^*A_5 = \{x, y | x^4, y^3, (xy)^5, (x^2, y)\}$. Now, by using the generated of $N$ and our labeling for $t_i's$ we need to find the action of $N$ elements on the elements of $3^{12}$. Since the permutation $x$ takes 1 to 3, by our labeling we see that $t_1^x = t_2$. Also, $x$ takes 3 to 2. Thus, $t_2^x = t_1^3$ and so on for the other $t_i's$. Similarly, the permutation $y$ takes 1 to 5 and 5 to 9, thus $t_1^y = t_3$ and $t_3^y = t_5$, respectively. We keep doing the same process for the rest of $t_i's$. Instead of writing $t_1, t_2, \ldots, t_{12}$, we will use the following labeling

$$\begin{align*}
a & \quad b & \quad c & \quad d & \quad e & \quad f & \quad g & \quad h & \quad i & \quad j & \quad k & \quad l \\
l_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}
\end{align*}$$
Thus, the monomial progenitor has the following symmetric presentation

\[ H = \langle a, b, c, d, e, f, g, h, i, j, k, l, a^3, b^3, c^3, d^3, e^3, f^3, g^3, h^3, i^3, j^3, k^3, l^3, (a, b), (a, c), (a, d), (a, e), (a, f), (a, g), (a, h), (a, i), (a, j), (a, k), (a, l), (b, c), (b, d), (b, e), (b, f), (b, g), (b, h), (b, i), (b, j), (b, k), (b, l), (c, d), (c, e), (c, f), (c, g), (c, h), (c, i), (c, j), (c, k), (c, l), (d, e), (d, f), (d, g), (d, h), (d, i), (d, j), (d, k), (d, l), (e, f), (e, g), (e, h), (e, i), (e, j), (e, k), (e, l), (f, g), (f, h), (f, i), (f, j), (f, k), (f, l), (g, h), (g, i), (g, j), (g, k), (g, l), (h, i), (h, j), (h, k), (h, l), (i, j), (i, k), (i, l), (j, k), (j, l), (k, l), x^4, y^3, (xy)^5, (x^2, y), a^x = b, b^x = a^2, c^x = d^2, d^x = c, e^x = i, f^x = k^2, g^x = l, h^x = j, i^x = c^2, j^x = h^2, k^x = f, l^x = g^2, a^y = c, b^y = d, e^y = e, d^y = g, e^y = a, f^y = l^2, g^y = b, h^y = i, i^y = j, j^y = h, k^y = f^2, l^y = k >. \]

To check, we run the following loop

```plaintext
> V:=CosetSpace(H, sub<H|w,z>: CosetLimit:=10000000, Hard:=true, Print:=1);
#--- Run Parameters ---
Wo:280000056; Max:10000000; Mess:10000; Ti:-1; Ho:-1; Loop:0;
As:0; Path:0; Row:1; Mend:0; No:106; Look:0; Com:10;
C:1000; R:1; Fi:37; PMod:3; PSiz:256; DMod:4; DSiz:1000;
#-----------------------------
SG: a=1 r=1 h=1 n=2; l=1 c=+0.00; m=1 t=1
RS: a=91 r=1 h=1 n=92; l=2 c=+0.00; m=91 t=91

CP: a=530091 r=530 h=14362 n=530092; l=1061 c=+1.23; m=530091 t=530091
INDEX = 531441 (a=531441 r=532 h=531442 n=531442; l=1066 c=32.65; m=531441 t=531441)
> H:=CosetImage(V);
> CompositionFactors(H);
G
| Alternating(5)
* | Cyclic(2)
* | Cyclic(3)
* | Cyclic(3)
```
7.3 Monomial Progenitor $3^7 : m L_2(7)$

Consider

$$G = L_2(7).$$

$$\cong ((2,4)(3,5),(1,2,3)(5,6,7)).$$

$$\cong (x,y|x^2,y^3,(xy)^7,(x,y)^4).$$

The MAGMA input for this group is as the following:

```magma
> G<x,y>:=Group<x,y|x^2,y^3,(x+y)^7,(x,y)^4>; > S:=Sym(7); > xx:=S!(2,4)(3,5); > yy:=S!(1,2,3)(5,6,7); > G1:=sub<S|xx,yy>; > #G; 168
```

To find the monomial representations of $L_2(7)$, we need to find $H \leq L_2(7)$ such that $[H : \hat{H}] = 2$ for entries of the monomial representation to be in $\mathbb{Z}_3$. 

So, the $|H| = 3^{12} \times 120 = 531441 \times 120 = 63772920.$
> S:=Subgroups(G1);
> for i in [1..#S] do if Index(S[i]`subgroup, DerivedGroup(S[i]`subgroup)) eq 2 then
  Index(G1,S[i]`subgroup), i; end if; end for;
 84 2
 28 8
 7 13
 7 14
> H;
Permutation group H acting on a set of cardinality 7
Order = 24 = 2^3 * 3
  (2, 7) (3, 6)
  (1, 2, 4) (3, 6, 5)
  (1, 2) (4, 7)
  (1, 4) (2, 7)
> H:=sub<G1|(2, 7)(3, 6), (1, 2, 4)(3, 6, 5), (1, 2)(4, 7), (1, 4)(2, 7)>;
> #H;
24
> dH:=DerivedGroup(H);
> f,g:=CosetAction(H,dH);
> #Generators(H);
4
> f(H.1), f(H.2), f(H.3), f(H.4);
(1, 2)
Id(g)
Id(g)
Id(g)

By running out all the four generators of H, we get three of them are the identities.

> C<u>:=CyclotomicField(2);
> M:=MatrixAlgebra(C,1);
> HM:=GModule(H, [M![u], M![1], M![1], M![1]]);
> I:=Induction(HM,G1);
> Norm(Character(I));
1
> GP:=MatrixGroup(I);
> #GP;
168
Now, we have to write down the monomial matrices $\phi(x)$ and $\phi(y)$. By using MAGMA, we obtain the following two matrices

$$
\phi(x) = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

$$
\phi(y) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

Now, we have to write the permutation representation of the irreducible monomial representation $\phi(x)$ and $\phi(y)$ of $L_2(7)$ by knowing four cases

- If $a_{ij} = 1$, the automorphism takes $t_i \to t_j$.
- If $a_{ij} = -1$, the automorphism takes $t_i \to t_j^{-1}$.
  
  Where $a_{ij}$ is the non-zero entry in row $i$ column $j$.
- If the automorphism takes $t_i \to t_j$, then $t_i^{-1}$ takes to $t_j^{-1}$.
- If the automorphism takes $t_i \to t_j^{-1}$, then $t_i^{-1}$ takes to $t_j$. 
Table 7.8: The Permutation for $A = \phi(x)$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_5$</td>
<td>$t_6$</td>
<td>$t_7$</td>
<td>$t_1^{-1}$</td>
<td>$t_2^{-1}$</td>
<td>$t_3^{-1}$</td>
<td>$t_4^{-1}$</td>
<td>$t_5^{-1}$</td>
<td>$t_6^{-1}$</td>
<td>$t_7^{-1}$</td>
</tr>
<tr>
<td>$t_1^{-1}$</td>
<td>$t_2$</td>
<td>$t_4$</td>
<td>$t_3$</td>
<td>$t_6$</td>
<td>$t_5$</td>
<td>$t_7$</td>
<td>$t_1^{-1}$</td>
<td>$t_2^{-1}$</td>
<td>$t_3^{-1}$</td>
<td>$t_4^{-1}$</td>
<td>$t_5^{-1}$</td>
<td>$t_6^{-1}$</td>
<td>$t_7^{-1}$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>14</td>
<td>1</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>13</td>
<td>12</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 7.9: The Permutation for $B = \phi(y)$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_5$</td>
<td>$t_6$</td>
<td>$t_7$</td>
<td>$t_1^{-1}$</td>
<td>$t_2^{-1}$</td>
<td>$t_3^{-1}$</td>
<td>$t_4^{-1}$</td>
<td>$t_5^{-1}$</td>
<td>$t_6^{-1}$</td>
<td>$t_7^{-1}$</td>
</tr>
<tr>
<td>$t_1^{-1}$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_5$</td>
<td>$t_6$</td>
<td>$t_4$</td>
<td>$t_7$</td>
<td>$t_1^{-1}$</td>
<td>$t_2^{-1}$</td>
<td>$t_3^{-1}$</td>
<td>$t_4^{-1}$</td>
<td>$t_5^{-1}$</td>
<td>$t_6^{-1}$</td>
<td>$t_7^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>9</td>
<td>10</td>
<td>8</td>
<td>12</td>
<td>14</td>
<td>13</td>
<td>11</td>
</tr>
</tbody>
</table>

Thus, we obtain the two monoial representations

$A = \phi(x) = (18)(34)(56)(7,14)(10,11)(12,13)$.  
$B = \phi(y) = (123)(457)(8,9,10)(11,12,14)$.  

To check these permutations, we store the two given matrices in MAGMA as $\mat1$ and $\mat2$.

```
> mat1:=GP.1; mat2:=GP.2;
> A:=perm(7,2,mat1);
> B:=perm(7,2,mat2);
> A; B;
(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13)
(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14)
> G2:=sub<Sym(14)|A,B>;
> s,t:=IsIsomorphic(G1,G2);
> s;
true
```

Thus, we have find the irreducible monomial representations of $L_2(7)$.

To write a presentation for the progenitor of $L_2(7)$, we must write a representation for $L_2(7)$.

```
a:=G2.1;
Order(a);
b:=G2.2;
Order(b);
G2:=sub<G2|a,b>;
Order(a*b);
```
Order((a,b));
H<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(a,b)^4>;
> #H;
168
> f2,h2,k2:=CosetAction(H,sub<H|Id(H)>);
> CompositionFactors(h2);
G
| A(1, 7) = L(2, 7)
1
N:=G2;

Now, we have to find the normaliser of $H$ in $G$ such that $N(H) = \{g \in G | H^g = H\}$. So, we are looking for elements that normalise $L_2(7)$ and takes $\{7,14\}$ to $\{7,14\}$, then store them.

G22:=Stabiliser(G2,{7,14});
G22;
c:=G2!(1, 4, 10)(2, 5, 6)(3, 8, 11)(9, 12, 13);
d:=G2!(1, 8)(2, 9)(3, 13)(4, 12)(5, 11)(6, 10);
e:=G2!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13);
> G22 eq sub<G2|c,d,e>;
true

After writing a presentation for $L_2(7)$, we write its Schreier system to convert the normaliser of $<t>$ which are $c$, $d$, and $e$ in $L_2(7)$ in terms of $A$ and $B$. Moreover, we see that

$$t_7^c = t_7^{(1,4,10)(2,5,6)(3,8,11)(9,12,13)} = t_7.$$  
$$t_7^d = t_7^{(1,8)(2,9)(3,13)(4,12)(5,11)(6,10)} = t_7.$$  
$$t_7^e = t_7^{(1,8)(3,4)(5,6)(7,14)(10,11)(12,13)} = t_{14}.$$  

The normaliser of $<t>$ in $L_2(7)$ are $t^{(xy^{-1}xyxyy^{-1}xey)}t^{-1}$, $t^{(xy^{-1}xyxyy^{-1})t^{-1}}$, and $t^x = t^{-1}$.

Thus, the presentation for the monomial progenitor is given by

$$3^{*7} \cdot m L_2(7) = \langle x, y, t|x^2, y^3, (xy)^7, (x,y)^4, t^3, t^{(xy^{-1}xyxyy^{-1}xey)}t^{-1}, t^{(xy^{-1}xyxyy^{-1})t^{-1}}, t^x = t^{-1} \rangle.$$  

Generated by
\[x \approx (1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13),
\]
\[y \approx (1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14).
\]

To find some homomorphic images, we factored out the progenitor by some relations such as

\[(xyt)^k, (xty^2t^l), (x^mt), (y^2(xy)^n t), (xy^{-1} t)^o\]

Then, we tabulated the results as shown in Table 7.10.

Table 7.10: Some Finite Images for \(3^* : m L_2(7)\)

<table>
<thead>
<tr>
<th>k</th>
<th>l</th>
<th>m</th>
<th>n</th>
<th>o</th>
<th>Index(G, sub &lt; G[x, y])</th>
<th>Order of G</th>
<th>Shape of G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>15</td>
<td>2520</td>
<td>(A_7)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>2187</td>
<td>367416</td>
<td>(3^f : L_2(7))</td>
</tr>
</tbody>
</table>

7.3.1 \(A_7\) as a Homomorphic Image of \(3^* : m L_2(7)\)

Consider the group

\[G = 3^* : m L_2(7)\]
\[= \langle x, y, t | x^2, y^3, (xy)^7, (x, y)^4, t^3, t(yxy^{-1}xxyxy^{-1}xy)t^{-1}, t(yxy^{-1}xyxy^{-1})t^{-1}, t^n = t^{-1} \rangle,\]

factored by \([xy^{-1}t]^4\).

\[N = L_2(7) = \langle x, y \rangle = ((1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13),\]
\[(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14)),\]

and let \(t = t_7 \rightarrow t_0\). So, by expanding the relation, we get

\[(xy^{-1}t)^4 = e\]
\[(xy^{-1}t_7)^4 = e\]
\[(xy^{-1})^4t_7(yx^{-1})^3t_7(xy^{-1})^2t_7^{-1}t_7 = e\]
\[(1, 6, 10, 4, 14, 2, 5)(3, 11, 7, 9, 12, 8, 13)t_{11}t_{13}t_{12}t_7 = e\]
\[(xy^{-1})^{4}t_{11}t_{13} = t_7t_{12}\]
\[t_{11}t_{13} = (xy^{-1})^{-4}t_7t_{12}\]

The MAGMA input for this group is as the following:
\[ S := \text{Sym}(14); \]
\[ x := S!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13); \]
\[ y := S!(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14); \]
\[ N := \text{sub}<S|xx, yy>; \]
\[ \#N; \]
\[ G<x, y, t> := \text{Group}<x, y, t|x^2, y^3, (x*y)^7, (x, y)^4, t^3, \]
\[ t^*(y*x*y^-1*x*y*x*y^-1*x*y)*t^-1, \]
\[ t^*(y*x*y^-1*x*y*x*y^-1)*t^-1, t^*x = t^-1, (x*y^-1*t)^4>; \]
\[ f, G1, k := \text{CosetAction}(G, \text{sub}<G|x, y>); \]
\[ \#G; \#k; \]
\[ t := [\text{Id}(G1): i \text{ in } [1..14]]; \]
\[ t[7] := f(t); \]
\[ t[1] := f(((t*y)^x)*y); \]
\[ t[2] := f(((t*y)^x)*y)*y; \]
\[ t[3] := f((t*y)^x); \]
\[ t[4] := f(t*y); \]
\[ t[5] := f((t*y)^y); \]
\[ t[6] := f(((t*y)^y)*x); \]
\[ t[14] := f(t*x); \]
\[ t[8] := f(((t*x)^y)*x)*y; \]
\[ t[9] := f(((t*x)^y)*x)*y); \]
\[ t[10] := f(((t*x)^y)*x); \]
\[ t[11] := f((t*x)^y); \]
\[ t[12] := f(((t*x)^y)*y); \]
\[ t[13] := f(((t*x)^y)*y)*x); \]

We need to find the index of \( N \) in \( G \). To obtain this, we need to express \( G \) as a union of double cosets which is here \( G = NeN \cup Nt_0N \).

By using MAGMA, we obtain that \( t_0 = t_7 = t_{14}^x \). Thus, \( t_0 = t_{14} = t_0t_0 = t_0^{-1} \).

**The double coset enumeration:**

- Consider \([*]\) which is the notation of \( NeN = N \).

The number of right cosets in \([*]\) is \( \frac{|N|}{|N|} = \frac{168}{168} = 1 \).

The orbits of \( N \) on \{0,1,2,3,4,5,6,8,9,10,11,12,13,14\} are \{0,1,2,3,4,5,6\} and \{8,9,10,11,12,13,14\}.

Pick a representative from each orbit, say \( t_0 \) and \( t_{14} \), and determine the double coset that contains each one.

\( Nt_0 \in [0] \).

\( Nt_{14} \in [14] \).

Since, \( Nt_0^{-1} = Nt_0 \Rightarrow Nt_{14} = Nt_{0} \), thus, we have 14 go to \([0]\).
Consider \([0]\) which is the notation of \(N_{t_0}N\).

\[
N^0 = \langle (1, 5, 13)(2, 4, 3)(6, 8, 12)(9, 11, 10), (1, 11, 6)(2, 12, 10)(3, 9, 5)(4, 13, 8) \rangle
= N^{(0)}.
\]

The number of right cosets in \([0]\) = \(\frac{|N|}{|N^{(0)}|} = \frac{168}{12} = 14\).

The orbit of \(N\) on \(\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14\}\) are \(\{0\}\), \(\{14\}\), and \(\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13\}\).

\(N_{t_0}t_0 = t_0^{-1} \in [0]\) (1 stays in \([0]\)).

\(N_{t_0}t_{14} = N_{t_0}t_0^{-1} = N \in [\ast]\) (1 symmetric generator goes back to the double coset \([\ast]\)).

\(N_{t_0}t_1 \in [0]\) (12 symmetric generators stay in the double coset \([0]\)).

![Figure 7.1: Cayley Diagram for \(A_7\) Over \(L_2(7)\)](image)

The isomorphic image of the monomial progenitor have proved by MAGMA.

```magma
> S:=Sym(14);
> xx:=S!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13);
> yy:=S!(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14);
> N:=sub<S|xx,yy>;
> #N;
168
> G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^7,(x,y)^4,t^3,
  t*(y*x*y^-1*x*y*x*y*x*y^-1)*t^-1,t*x=t^-1,
  t*(x*y^-1*t)^4>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> #G; #k;
2520
1
```
CompositionFactors(G1);

G
  | Alternating(7)
  1

Hence,

\[
G \cong 3^{14}:mL_2(7) \cong A_7
\]

7.3.2 \(3^7 : L_2(7)\) as a Homomorphic Image of \(3^* : mL_2(7)\)

Consider the group

\[
G = 3^7 : mL_2(7) = \langle x, y, t| x^2, y^3, (xy)^7, (x, y)^4, t^3, t(xy^{-1}xyxy^{-1}y)t^{-1}, t(yxy^{-1}xyxy^{-1}y)t^{-1}, t^x = t^{-1}\rangle,
\]

factored by \([y^2(xy)^{5t}]^4\).

\[
N = L_2(7) = \langle x, y \rangle = \langle (1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13), (1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14) \rangle,
\]

and let \(t = t_7 \to t_0\). So, by expanding the relation, we get

\[
(y^2(xy)^5t)^4 = e
\]
\[
(y^2(xy)^5t_7)^4 = e
\]
\[
(y^2(xy)^5t_7^3)^4t_7^3(y^2(xy)^5)^3t_7^3(y^2(xy)^5)^2(y^2(xy)^5)t_7 = e
\]
\[
t_0t_0t_2t_7 = e
\]
\[
t_0t_{14} = t_7t_2.
\]

We need to find the index of \(N\) in \(G\). To obtain this, we need to express \(G\) as a union of double cosets.

By using MAGMA, we obtain that \(t_0 = t_7 = t_{14}^{-1}\). Thus, \(t_0 = t_{14} = t_0t_0 = t_0^{-1}\).

The double coset enumeration:

- Consider \([\ast]\) which is the notation of \(NeN = N\).

The number of right cosets in \([\ast]\) is \(\frac{|N|}{|N|} = \frac{168}{168} = 1\).

The orbits of \(N\) on \(\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14\}\) are \(\{0, 1, 2, 3, 4, 5, 6\}\) and
\{8,9,10,11,12,13,14\}.

Pick a representative from each orbit, say \(t_0\) and \(t_{14}\), and determine the double coset that contains each one.

\(Nt_0 \in [0]\).

\(Nt_{14} \in [14]\).

Since, \(Nt_{14} = Nt_0 \Rightarrow Nt_0^{-1} = Nt_0 \Rightarrow Nt_{14} \in [0]\) (14 symmetric generators take to the double coset \([0]\)).

- Consider \([0]\).

\[N^0 = \langle (1,13,5)(2,3,4)(6,12,8)(9,10,11),(1,11,6)(2,12,10)(3,9,5)(4,13,8) \rangle = N^{(0)}.

The number of right cosets in \([0]\) = \(\frac{|N|}{|N^{(0)}|} = \frac{168}{12} = 14\).

The orbits of \(N\) on \{0,1,2,3,4,5,6,8,9,10,11,12,13,14\} are \{\}, \{14\}, and \{1,2,3,4,5,6,8,9,10,11,12,13\}.

\(Nt_0 t_0 = t_0^{-1} \in [0]\) (1 stays in \([0]\)).

\(Nt_0 t_{14} = Nt_0 t_0^{-1} = N \in [\ast]\) (1 symmetric generator goes back to the double coset \([\ast]\)).

\(Nt_0 t_1 \in [0]\) (12 symmetric generators go to the double coset \([01]\)).

- Consider \([01]\).

\[N^{01} = \langle e \rangle.

\[N^{(0)} = \langle (1,7)(3,11)(4,10)(5,12)(6,13)(8,14) \rangle.

The number of right cosets in \([01]\) = \(\frac{|N|}{|N^{(0)}|} = \frac{168}{2} = 84\).

The orbits of \(N\) on \{0,1,2,3,4,5,6,8,9,10,11,12,13,14\} are \{2\}, \{9\}, \{1,7\}, \{3,11\}, \{4,10\}, \{5,12\}, \{6,13\}, and \{8,14\}.

\(Nt_0 t_1 t_1 = (Nt_0 t_1)^{18(20)(3,13)(4,12)(5,11)(6,10)}\), so \(Nt_0 t_1 t_1 \in [01]\) (2 stay)

\(Nt_0 t_1 t_2 \in [012]\)

\(Nt_0 t_1 t_3 \in [013]\)

\(Nt_0 t_1 t_4 \in [014]\)

\(Nt_0 t_1 t_5 \overset{MAG}{=} (Nt_0 t_1 t_3)^{1,5,4,13,2,3,7}(6,9,10,14,8,12,11)\), so \(Nt_0 t_1 t_5 \in [013]\) (2 symmetric generators go to the double coset \([013]\))

\(Nt_0 t_1 t_6 \overset{MAG}{=} (Nt_0 t_1 t_3)^{1,7,6,4,12,9,3(2,10,8,14,13,11,5)}\), so \(Nt_0 t_1 t_6 \in [013]\) (2 symmetric
generators go to the double coset \([013]\))

\[ N_{t_0 t_1 t_8} = N_{t_0 t_1 t_9}^{-1} = N_{t_0}, \text{ (2 symmetric generators go back to the double coset } [0]) \]

\[ N_{t_0 t_1 t_9} \in [019] \] (1 symmetric generator goes to the double coset \([019]\)).

- Consider \([012]\).

\[ N^{012} = \langle e \rangle \]

\[ N^{(012)} = \langle (2,7)(3,10)(4,6)(5,12)(9,14)(11,13), (1,7)(3,11)(4,10)(5,12) \]

\[ (6,13)(8,14), (1,7,2)(3,4,13)(6,10,11)(8,14,9), (1,2,7)(3,13,4) \]

\[ (6,11,10)(8,9,14), (1,2)(3,6)(4,11)(5,12)(8,9)(10,13) \].

The number of right cosets in \([012]\) = \( |N| / |N^{(012)}| \) = \( \frac{168}{6} = 28 \).

The orbits of \( N \) on \{0,1,2,3,4,5,6,8,9,10,11,12,13,14\} are \{5,12\}, \{1,7,2\}, \{8,14,9\}, and \{3,4,6,10,11,13\}.

\[ N_{t_0 t_1 t_2 t_2} \overset{MAG}{=} N_{t_0 t_1 t_9} \] (3 symmetric generators take to the double coset \([019]\)).

\[ N_{t_0 t_1 t_2 t_5} \in [0125] \]

\[ N_{t_0 t_1 t_2 t_9} = N_{t_0 t_1 t_2 t_2}^{-1} = N_{t_0 t_1} \] (3 symmetric generators go back to the double coset \([01]\)).

\[ N_{t_0 t_1 t_2 t_3} \in [0123] \]

- Consider \([013]\).

\[ N^{013} = \langle e \rangle = N^{(013)} \]

The number of right cosets in \([013]\) = \( |N| / |N^{(013)}| \) = \( \frac{168}{1} = 168 \).

The orbits of \( N \) on \{0,1,2,3,4,5,6,8,9,10,11,12,13,14\} are \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\} and \{14\}.

\[ N_{t_0 t_1 t_3 t_0} \overset{MAG}{=} (N_{t_0 t_1 t_3})^{(13)(26)(4,11)(7,14)(8,10)(93)} \]

(1 symmetric generator stays in the double coset \([013]\)).

\[ N_{t_0 t_1 t_3 t_1} \overset{MAG}{=} (N_{t_0 t_1 t_3})^{(18)(2,12)(3,7)(4,11)(59)(10,14)} \]

(1 symmetric generator stays in the double coset \([013]\)).

\[ N_{t_0 t_1 t_3 t_2} = N_{t_0 t_1 t_2 t_3} \] (1 symmetric generator goes to the double coset \([0123]\)).

\[ N_{t_0 t_1 t_3 t_3} \overset{MAG}{=} (N_{t_0 t_1 t_4})^{(17)(3,1)(4,10)(5,12)(6,13)(8,14)} \]

(1 symmetric generator takes to the double coset \([014]\)).

\[ N_{t_0 t_1 t_3 t_4} \in [0134] \] (1 symmetric generator goes to the double coset \([0134]\)).
\[ N_{0t_1t_3t_5}^{MAG} = (N_{0t_1t_2t_3})^{(1,5,4,13,2,3,7)(6,9,10,14,8,12,11)} \]

(1 symmetric generator goes to the double coset \([013])

\[ N_{0t_1t_3t_6} \in [0136] \] (1 symmetric generator goes to the double coset \([0136])

\[ N_{0t_1t_3t_8}^{MAG} = (N_{0t_1})^{(1,7,3,2,13,4,5)(6,11,12,8,14,10,9)} \]

(1 symmetric generator goes back to the double coset \([01])

\[ N_{0t_1t_3t_9}^{MAG} = (N_{0t_1t_3t_6})^{(1,3,12,4,6,7)(2,5,11,13,14,8,10)} \]

(3 symmetric generators go to the double coset \([019])

\[ N_{0t_1t_3t_10} = N_{0t_1t_3t_3}^{-1} = N_{0t_1} \]

(1 symmetric generator goes back to the double coset \([01])

\[ N_{0t_1t_3t_11} \in [01311] \] (1 symmetric generator goes to the double coset \([01311])

\[ N_{0t_1t_3t_13}^{MAG} = (N_{0t_1t_2t_5})^{(135)(2,7,13)(6,9,14)(8,10,12)} \]

(1 symmetric generator goes to the double coset \([0125])

\[ N_{0t_1t_3t_14}^{MAG} = (N_{0t_1})^{(1,3,9,12,4,6,7)(2,5,11,13,14,8,10)} \]

(1 symmetric generator goes back to the double coset \([01])

\[ \bullet \text{ Consider } [014]. \]

\[ N^{014} = \langle e \rangle. \]

\[ N^{014} = \langle (1, 7, 4)(2, 5, 13)(6, 9, 12)(8, 14, 11), (1, 4, 7)(2, 13, 5)(6, 12, 9)(8, 11, 14) \rangle. \]

The number of right cosets in \([014] = \frac{|N|}{|N^{014}|} = \frac{168}{3} = 56. \]

The orbits of \(N\) on \(\{0,1,2,3,4,5,6,8,9,10,11,12,13,14\}\) are \(\{3\}, \{10\}, \{0,1,4\}, \{2,5,13\}, \{6,9,12\}, \text{ and } \{8,14,11\}. \)

\[ N_{0t_1t_4t_3}^{MAG} = N_{0t_1t_3t_4} \] (1 symmetric generator goes to the double coset \([0134])

\[ N_{0t_1t_4t_4}^{MAG} = (N_{0t_1t_3t_3})^{(17)(3,11)(4,10)(5,12)(6,13)(8,14)} \] (3 symmetric generators go to the double coset \([013])

\[ N_{0t_1t_4t_10} \in [01410] \] (1 symmetric generator takes to the double coset \([01410])

\[ N_{0t_1t_4t_2}^{MAG} = (N_{0t_1t_2t_3})^{(172)(3,4,13)(6,10,11)(8,14,9)} \] (3 symmetric generators go to the double coset \([0142])

\[ N_{0t_1t_4t_6}^{MAG} = (N_{0t_1t_3t_6})^{(1,7,6,4,12,9,3)(2,10,8,14,13,11,5)} \] (3 symmetric generators go to the double coset \([0136])

\[ N_{0t_1t_4t_{11}} = N_{0t_1t_4t_4}^{-1} = N_{0t_1} \] (3 symmetric generators go back to the double coset \([01])

\[ \bullet \text{ Consider } [019]. \]
$N^{019} = \langle e \rangle$.

$N^{(019)} = \langle (2, 14)(3, 5)(4, 11)(6, 13)(7, 9)(10, 12), (1, 7)(3, 11)(4, 10)(5, 12)(6, 13)
\quad (8, 14), (1, 7, 9)(2, 8, 14)(3, 12, 4)(5, 11, 10), (1, 9, 7)(2, 14, 8)(3, 4, 12)
\quad (5, 10, 11), (1, 9)(2, 8)(3, 10)(4, 5)(6, 13)(11, 12) \rangle$

The number of right cosets in $[019] = |N|/|N^{019}| = 168/6 = 28$.

The orbits of $N$ on $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14\}$ are
\[ \{6, 13\}, \{1, 7, 9\}, \{2, 14, 8\}, \{3, 5, 11, 12, 4, 10\} \].

$Nt_0t_1t_9t_2 = Nt_0t_1t_9t_2^{-1} = Nt_0t_1$ (3 symmetric generator go back to the double coset $[01]$).

$Nt_0t_1t_9t_3^MAG = (Nt_0t_1t_3t_6)^{1, 3, 9, 12, 4, 6, 7)(2, 5, 11, 13, 14, 8, 10}$ (6 symmetric generators go to the double coset $[0136]$).

$Nt_0t_1t_9t_6^MAG = (Nt_0t_1t_3t_5)^{1, 7, 6, 4, 12, 9, 3)(2, 10, 8, 14, 13, 11, 5}$ (3 symmetric generators go back to the double coset $[0135]$).

$Nt_0t_1t_9t_9 = Nt_0t_1t_2$ (3 symmetric generators go to the double coset $[012]$).

We now continue the double coset enumeration by proceeding in this manner until the set of right cosets is closed under right multiplication by $t_i'$s where $i = 0, \ldots, 14$. Thus, we can determine the index of $N$ in $G$. We conclude that

$|G| \leq (|N| + |N|/|N^{019}| + |N|/|N^{012}| + \ldots + |N|/|N^{0123456}| + |N|/|N^{01234625}|) \times |N|$

$|G| \leq (1 + 14 + 84 + 28 + 168 + 56 + 28 + 168 + 56 + 56 + 168 + 42 + 56 + 14 +$
\quad 84 + 84 + 168 + 168 + 168 + 168 + 168 + 56 + 56 + 56 + 56 + 8 + 8) \times 168$

$|G| \leq (2187 \times 168) = 367416$.

By considering the action of $G$ on the 2187 cosets of $L_2(7)$ in $G$, we can show that $|G| \geq 367416$. 

We now show that $G \cong 3^7 : L_2(7)$ over $L_2(7)$. The composition series is

$$G \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \supseteq G_6 \supseteq G_7 \supseteq 1,$$

where


We want to determine the isomorphism type of the group $G$.

Normal subgroup lattice

\[ \text{---------------------------} \]

[2] Order 2187  Length 1  Maximal Subgroups: 1
[1] Order 1  Length 1  Maximal Subgroups:
> MinimalNormalSubgroups(G1);
[ Permutation group acting on a set of cardinality 2187
  Order = 2187 = 3^7
]

We note that the center of \( G \) is of order 1. So, this is not a central extension. Since the minimal normal subgroups is an abelian \( p \)-group, we see that it is of order 2187. Thus, \( |[2]| = 3^7 \). By looking at the normal lattice of \( G \), we see that it does not have a normal subgroup of order 3. Thus, this is not a direct product of \([2]\) by \([3]\). Now, we want to check if it is a semi-direct product of \([2]\) by \( L_2(7) \). We find that it is by investigating the derived group of \( G \).

\[
\begin{align*}
> \text{DD}:=\text{DerivedGroup}(G1);
> \text{DD};
\end{align*}
\]

Permutation group G1 acting on a set of cardinality 2187
Order = 367416 = 2^3 \times 3^8 \times 7

Since the order of the derived group is equal to the order of \([3]\) which is the whole group \( G \), \( G \) is a perfect group. So, we can write a presentation for \( G \) by using the perfect group database.

\[
\begin{align*}
> \text{DB}:=\text{PerfectGroupDatabase}();
> \text{PermutationGroup}(\text{DB},367416,1);
\end{align*}
\]

Permutation group acting on a set of cardinality 2187
\[
> \text{s}:=\text{IsIsomorphic}(\text{PermutationGroup}(\text{DB},367416,1),G1);
> \text{s};
\text{false}
\]
\[
> \text{PermutationGroup}(\text{DB},367416,2);
\]
\[
> \text{s}:=\text{IsIsomorphic}(\text{PermutationGroup}(\text{DB},367416,2),G1);
> \text{s};
\text{true}
\]
\[
> \text{Group}(\text{DB},367416,2);
\]

Since the first presentation is not isomorphic to \([3]\), we chose the second one which it is. We highlight the relation between the generators of \( 3^7 \) and \( L_2(7) \). Thus, we were able to write a presentation for \( 3^7 : L_2(7) \). At the end, we check if this presentation is isomorphic with \( G \).

\[
\begin{align*}
> \text{H}:=\text{Group}<\text{a,b,t,u,v,x,y,w,z}|\text{a}^2,\text{b}^3,\
\end{align*}
\]
(a+b)\(^7\), (a,b)\(^4\), t\(^3\), u\(^3\), v\(^3\), x\(^3\), y\(^3\), w\(^3\), z\(^3\), (a,t\(^{-1}\)),
(t,u), (t,v), (t,w), (t,x), (t,y), (t,z), (u,v), (u,w), (u,x),
(u,y), (u,z), (v,w), (v,x), (v,y), (v,z), (w,x), (w,y), (w,z),
(x,y), (x,z), (y,z), (a,t\(^{-1}\)), a\(^{-1}\)*u*a*w\(^{-1}\), a\(^{-1}\)*v*a*v,
a\(^{-1}\)*w*a*u\(^{-1}\), a\(^{-1}\)*z*a*x\(^{-1}\),
b\(^{-1}\)*t*b*u\(^{-1}\), b\(^{-1}\)*u*b*v\(^{-1}\), b\(^{-1}\)*v*b*t\(^{-1}\),
b\(^{-1}\)*w*b*x\(^{-1}\), b\(^{-1}\)*x*b*y\(^{-1}\), b\(^{-1}\)*y*b*w\(^{-1}\), (b,z\(^{-1}\));
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> #H;
367416
> s:=IsIsomorphic(H1,G1);
> s;
true

So, we obtained that \(G \cong 3^7 : L_2(7)\).

\[
G \approx \frac{3^{3+7} : mL_2(7)}{[y^4(xy)^7]^4-1} \cong 3^7 : L_2(7)
\]

### 7.4 Monomial Progenitor \(3^{21} : m A_7\)

Consider

\[
G = A_7.
\]
\[
\cong \langle (123), (34567) \rangle.
\]
\[
\cong \langle x, y|x^3, y^5, (xy)^7, (xy^y)^2, (xy^{-2}xy^2)^2 \rangle.
\]

By applying the same process as in section 3 and by using MAGMA, we were able to find the following monomial progenitor \(3^{21} : m A_7\)

\[
\langle x, y, t|x^3, y^5, (xy)^7, (xy^y)^2, (xy^{-2}xy^2)^2, t^{21}, t^y = t, t(xyx^{-1}uxy^{-2}x^{-1})t^{-1}, t(xy^{-1}y^{-1}x) \rangle
\]

generated by

\[
x \approx (1,2, 4)(5, 7, 10)(8, 11, 15)(12, 16, 13)(17, 21, 41)(20, 38, 42)(22, 23, 25)
(26, 28, 31)(29, 32, 36)(33, 37, 34),
\]
\[
y \approx (1,3, 6, 9, 13)(2, 5, 8, 33, 17)(4, 7, 11, 37, 20)(10, 14, 18, 36, 19)(12, 38, 23, 26, 29)
\]
We factored the progenitor by the following relators

\[(x^2t)^k, (xyt)^l, (x^yt)^m, (x^2yt)^n, (y^2xy^2t)^o, (y^2xt)^p, (x^2y^3t^2)^q\]

and we obtained that

\[G = \frac{3^{21} \cdot m}{[y^{-2}xy^2t]^2, [y^{-2}xt]^6} \cong A_8\]

The following computer-based proof gives that \(G \cong A_8\).

\[
G<x,y,t>:=\text{Group}<x,y,t|x^3,y^5,(x*y)^7,(x*x*y)^2,t^2,t*y=t,t^*(x*y*x^-1*y*x*y^-2*x^-1)*t^-1,t^*(x*y*x^-1*y^-1*x)*t,(y^-2*x*y^2*t)^2,(y^-2*x*t)^6>;
\]

\[
> \text{CompositionFactors}(G1);
\]

\[
G \mid \text{Alternating}(8)
\]

\[
1
\]

\[
> \#\text{sub}<G|x,y>;
2520
\]

\[
> \#\text{DoubleCosets}(G,\text{sub}<G|x,y>,\text{sub}<G|x,y>);
2
\]
Chapter 8

Covering Group

In this chapter, we want to discover the relation between the central extension and covering group, and how it leads to Schur multiplier.

Definition 8.1 (Central Extension). A central extension of a group $G$ is defined to be a group $H$ such that there exist a homomorphism $\rho$ from $H$ onto $G$ with $\ker \rho \leq Z(H)$, the center of $H$.

Definition 8.2 (Universal Central Extension). A central extension of $H$ is said to be a universal extension of $G$ provided that if $K$ is another central extension of $G$ then there exists a homomorphism $\phi$ from $H$ onto $K$. In other words $H$ is called the universal cover group of $G$.

$$H/\ker \phi \cong K$$

The above definitions showed that a finite group $G$ has a universal central extension if and only if $G$ is perfect.

Note 8.3. $G$ is perfect if $\hat{G} = G$.

Example 8.4. $6^*A_6$ is a universal covering group of $A_6$ and $3^*A_6$ is a covering group of $A_6$.

Let $H = 6^*A_6$ and $K = 3^*A_6$. We want to prove that there is a homomorphism

$$\phi : 6^*A_6 \rightarrow 3^*A_6$$

such that $6^*A_6/\ker \phi \cong 3^*A_6$

By using MAGMA, we obtain the following
> S:=Sym(432);
> a:=S!(1,2,8,5)...(427,430,428,429);
> b:=S!(1,3,14,12,8,22,21,6)...(395,421,431,424,396, \\
> 423,432,422);
> G:=sub<S|a,b>;
> C:=Center(G);C;
> Order(C.1);
  2
> Order(C.2);
  6
> c:=C.2^3;
> Order(c^3);
  2
Thus, we chose to factor \( H \) by the center \( <c^3> \) of \( H \) of order 2 where

\[
  c = Z(H) = (1,7,9,8,18,4)\ldots(425,428,432,426,427,431) = C.2.
\]

\[
  c^3 = (1,8)(2,5)\ldots(429,430)(431,432).
\]

So, the composition for the factored group \( q \) is

> q,ff:=quo<G|c^3>;
> CompositionFactors(q);

\[
G
  | Alternating(6)
  | Cyclic(3)
1
\]

Now, we have to show that it is isomorphic to \( 3\cdot A_6 \) which is generated by

\[
(2,6)(4,11)(7,9)(8,13)(10,14)(12,16), \text{ and}
\]

\[
(1,2,7,4)(3,8,6,10)(5,9,13,12)(11,15)(14,17)(16,18)
\]

> S:=Sym(18);
> a:=S!(2,6)(4,11)(7,9)(8,13)(10,14)(12,16);
> b:=S!(1,2,7,4)(3,8,6,10)(5,9,13,12)(11,15)(14,17)(16,18);
> G1:=sub<S|a,b>;
> s:=IsIsomorphic(q,G1);
> s;
  true
Thus, we have shown that there exist a homomorphism $\phi$ such that $6^*A_6/Ker\phi \cong 3^*A_6$.

Next, we want to exist a homomorphism 

$$ \rho : 3^*A_6 \longrightarrow A_6, \text{ such that } 3^*A_6/Ker\rho \cong A_6 $$

So, we have to factor $q$ by the center of $G1$ which is 

$$ D = Z(G1) = (1, 5, 3)(2, 9, 8)(4, 12, 10)(6, 7, 13)(11, 16, 14)(15, 18, 17) $$

and check if it is isomorphic to $A_6$.

```plaintext
> qq,fff:=quo<G1|D>;
> s:=IsIsomorphic(qq,Alt(6));
> s;
true
```

Hence, $6^*A_6$ is a universal covering group of $A_6$.

Similarly, we can prove that $6^*A_6$ is a universal covering group of $A_6$ and $2^*A_6$ is a covering group of $A_6$.

**Definition 8.5 (Schur Multiplier)**. A **Schur Multiplier** of a group $G$, denoted by $M(G)$, is the Kernel of the homomorphism $\rho$ from $\hat{G}$ onto $G$.

**Note 8.6**. Every simple group is Schur multiplier.

**Note 8.7**. The second homology group is known as the Schur multiplier.

Depending to the transposition of $S_n$, we can extend $2^*A_n$ to $2^*S_n^{\pm}$. Thus, $2^*A_n$ extends to $2^*S_n^+$ if the transposition lift to elements of order 2, and $2^*A_n$ extends to $2^*S_n^-$ if the transposition lift to elements of order 4.

Now, we will state some facts about the Schur multiplier:

1. If $G$ is a finite group, then the Schur multiplier $M(G)$ is a finite abelian group.

2. The Schur multiplier $M(G)$ is trivial if all the Sylow $p$-subgroups of $G$ are cyclic. Thus, the $|M(G)|$ is not divisible by $p$, for some prime $p$.

**Example 8.8**. Let $G$ be the nonabelian group, where $|G| = 6 = 3 \cdot 2$. The Sylow 3-subgroup and Sylow 2-subgroup are cyclic. Thus, $M(G)$ is trivial.

3. The Schur multiplier $M(G)$ is trivial if $G$ is the Quaternion group.
4. $|M(G)| = 2$ if $G$ is the Dihedral groups.

5. The order of the Schur multiplier $M(G)$ can be larger than the order of $G$.

**Example 8.9.** Let $G$ be an elementry abelian group of order 16. The $|M(G)| = 64$.

In the next chapter, we will write more permutation and monoial progenitors when $N$ is a central extension.
Chapter 9

More Progenitors

9.1 $2^\bullet A_5$

Consider the group $N = 2^\bullet A_5 = \langle x, y | x^4, y^3, (xy)^5, (x^2, y) \rangle$, where

\[
x \approx (1, 2, 5, 4)(3, 6, 8, 7)(9, 13, 11, 14)(10, 15, 12, 16)(17, 19, 18, 20)(21, 24, 23, 22)
\]
\[
y \approx (1, 3, 2)(4, 5, 8)(6, 9, 10)(7, 11, 12)(13, 16, 17)(14, 15, 18)(19, 21, 22)(20, 23, 24)
\]

9.1.1 The involutory Progenitor $2^{*24} : (2^\bullet A_5)$

To write a presentation for this type of progenitor, we must introduce a new variable $t = t_1$ of order 2, and find all elements of $N$ that stabilise 1. By using MAGMA, we obtain the following

\[
(2, 8, 11, 15, 7)(3, 9, 16, 6, 4)(10, 18, 23, 19, 14)(12, 17, 21, 20, 13) = xy
\]

Since $t$ commutes with each generator of the one point stabiliser, we have to insert $t$ and with which commutes into our progenitor. Then, the presentation for $2^{*24} : (2^\bullet A_5)$ is

\[
\langle x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^2, (t, xy) \rangle.
\]

In order to search for relations, we use the following two methods

- Factor the progenitor by all of the first order relations.

  By using MAGMA, we obtain a list of the conjugacy classes of $N = 2^\bullet A_5$. 

Table 9.1: Conjugacy Classes of $N = 2^\bullet A_5$

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the Class</th>
<th>Elements of form $\pi t_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$x^2$</td>
<td>$x^2t_1$</td>
</tr>
<tr>
<td>3</td>
<td>$y$</td>
<td>$yt_1, yt_9, yt_{17}, yt_{21}$</td>
</tr>
<tr>
<td>4</td>
<td>$x$</td>
<td>$xt_1, xt_3, xt_9, xt_{10}, xt_{17}, xt_{21}$</td>
</tr>
<tr>
<td>5</td>
<td>$yx$</td>
<td>$yxt_1, yxt_2, yxt_9, yxt_{21}$</td>
</tr>
<tr>
<td>6</td>
<td>$(yx)^2$</td>
<td>$(yx)^2t_1, (yx)^2t_2, (yx)^2t_9, (yx)^2t_{21}$</td>
</tr>
<tr>
<td>7</td>
<td>$yxyxy^{-1}$</td>
<td>$yxyxy^{-1}t_i, i = 1, 2, 9, 21$</td>
</tr>
<tr>
<td>8</td>
<td>$yx^{-1}$</td>
<td>$yx^{-1}t_1, yx^{-1}t_2, yx^{-1}t_9, yx^{-1}t_{21}$</td>
</tr>
<tr>
<td>9</td>
<td>$xy^{-1}x^{-1}y^{-1}$</td>
<td>$xy^{-1}x^{-1}y^{-1}t_i, i = 1, 2, 9, 21$</td>
</tr>
</tbody>
</table>

Since $t \approx t_1$, we get

$$t_2 = t^x, \quad t_3 = t^y, \quad t_9 = t^{yx}$$

$$t_{10} = t^{yx^{-1}}, \quad t_{17} = t^{(xyxy^{-1})^2}, \quad t_{21} = t^{(xy^{-1}xy)^2}$$

- Apply the Lemma.

We will utilize the lemma by taking the stabiliser of two elements and determining the centraliser of those points. The lemma deals with elements that have the transposition $(1,2)$, do have either 1 or 2, or do not have either of them.

$$N \cap \langle t_0, t_1 \rangle \leq C_N(N^{01}).$$

$$N^{01} = \langle e \rangle \quad \Rightarrow \quad C_N(N^{01}).$$

So, the choices for writing product of $2t_i$’s in terms of elements of $N$ are unrestricted. By running the Set($C$), we find that it does not have an element that includes the transposition $(1,2)$. 
By adding all the relations from the above table and write some supporting relations, the presentation for this progenitor is

\[
\begin{align*}
&< x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^2, (t, xy), (x^2t)^k, (yt^{xy})^l, (yt^{(xyxy-1)^2})^n, \\
&\quad (yt^{(xyxy-1)^2})^o, (xt)^p, (xt^{xy})^q, (xt^{(xyxy-1)^2})^r, (xt^y)^s, (xt^{(xyxy-1)^2})^d, (xt^{xyxy-1})^u, \\
&\quad (yxt^{(xyxy-1)^2})^v, (yxt^y)^w, (yxt)^a, (yxt^{xyy})^b, ((yx)^2t^x)^c, ((yx)^2t^{(xy-1xy)})^d, \\
&\quad ((yx)^2t^{xyy})^e, (xyxyxyy^{-1}t^x)^z, (xyxyxy-1t^x)^g, (xyxyxyy^{-1}t^{xy-1xy})^h, \\
&\quad (xyxyxyy^{-1}t^{xyxy-1})^i, (xyxyxyy^{-1}t^{xyxy-1}xyy)^j, (xyxyxyy^{-1}t^{xyxy-1}xyy)^k, \\
&\quad (xyxyxyy^{-1}t^{xyxy-1}xyy)^l, (xyxyxyy^{-1}t^{xyxy-1}xyy)^m, (xyxyxyy^{-1}t^{xyxy-1}xyy)^n, \\
&\quad (xy-1y^{-1}t)^{pp}, (xy-1y^{-1}t^{xyy})^{qq}, (xy-1y^{-1}t^{xyxy-1xyy})^{rr}, (xy-1y^{-1}t^{xyxy-1xyy})^{ss}, (xy^yt)^{uu}, (x+y)^{ll} > .
\end{align*}
\]

By running the progenitors \(2^*24 : (2^*A_5)\) in the background of MAGMA, we obtain some interesting group.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(\cdots)</th>
<th>(ii)</th>
<th>(jj)</th>
<th>(kk)</th>
<th>(ll)</th>
<th>Index in (G)</th>
<th>Order of (G)</th>
<th>Shape of (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\cdots)</td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>260</td>
<td>15600</td>
<td>(L_2(25) : 2)</td>
</tr>
<tr>
<td>0</td>
<td>(\cdots)</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>6</td>
<td>1050</td>
<td>126000</td>
<td>(U_3(5))</td>
</tr>
<tr>
<td>0</td>
<td>(\cdots)</td>
<td>0</td>
<td>6</td>
<td>4</td>
<td>10</td>
<td>240</td>
<td>14400</td>
<td>(2 \cdot (A_5 \times A_5) : 2)</td>
</tr>
<tr>
<td>0</td>
<td>(\cdots)</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>4</td>
<td>25920</td>
<td>1555200</td>
<td>(2 \cdot (A_6 \times A_5) : 2)</td>
</tr>
<tr>
<td>0</td>
<td>(\cdots)</td>
<td>0</td>
<td>8</td>
<td>4</td>
<td>8</td>
<td>2100</td>
<td>252000</td>
<td>(2 \times U_3(5))</td>
</tr>
</tbody>
</table>

### 9.1.2 The Progenitor \(3^*24 : (2^*A_5)\)

A presentation for the progenitor of \(3^*24 : (2^*A_5)\), is

\[
(x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^3, (t, xy)).
\]

We were able to find the images given in Table 8.3 by factoring this progenitor by the following relators

\[
(x^2t)^k, (yt^{xy})^l, (xt^{xyxy-1})^m, (yt^{xyxy-1}t^x)^n, ((yx)^2t^x)^o, (yt^{xyxy-1}xyy)^p, (xyxyxyy^{-1}t^x)^q, \\
(xyxyxyy^{-1}t^x)^r, (xyyt)^s, (xt^x)^t, (yt^{xy})^u, (yt^{xy})^v, (x+y)^w, (xt)^i, (yt)^j, (xy)^l.
\]
9.2 $2^*A_6$

Consider the group $N = 2^*A_6 = \langle x, y|x^4, y^8, (xy)^5, (xy^2)^5, (y^4, x), (x^2, y)\rangle$, where

$x \approx (1, 2, 4, 5)(3, 9, 11, 12)(6, 16, 13, 17)(7, 19, 14, 21)(8, 22, 15, 23)(10, 27, 18, 28)$


$(37, 59, 39, 61)(38, 62, 40, 63)(45, 69, 47, 71)(46, 58, 48, 56)(50, 64, 54, 60)$

$(55, 77, 57, 78)(65, 68, 66, 67)(70, 76, 72, 74)(73, 80, 75, 79)$.

$y \approx (1, 3, 10, 13, 4, 11, 18, 6)(2, 7, 20, 15, 5, 14, 24, 8)(9, 25, 19, 30, 12, 29, 21, 26)$

$(16, 31, 50, 34, 17, 33, 54, 32)(22, 37, 60, 40, 23, 39, 64, 38)(27, 45, 70, 48, 28, 47, 72, 46)$


$(49, 73, 61, 76, 51, 75, 59, 74)$.

9.2.1 The involutory Progenitor $2^{*80} : (2^*A_6)$

To write a presentation for the progenitor of $2^*A_6$, we have to introduce a new variable $t = t_1$ of order 2, and find all elements of $N$ that stabilise 1. By using MAGMA, we obtain the following

$(3, 28, 37)(6, 22, 46)(7, 36, 33)(8, 17, 56)(9, 59, 10)(11, 27, 39)(12, 61, 18)(13, 23, 48)$


$(32, 72, 44)(34, 70, 43)(38, 80, 42)(40, 79, 41)(45, 54, 77)(47, 50, 78)(55, 64, 71)$

$(57, 60, 69) = xy^2xy^{-1}$


$(15, 61, 36)(16, 18, 33)(20, 37, 23)(22, 24, 39)(25, 44, 54)(26, 41, 60)(29, 43, 50)(30, 42, 64)$


$(53, 80, 55) = y^2xy^{-1}xy^2x$
Now, we have to insert $t$ and with which commutes into our progenitors. Then the presentation for $2^*80 : (2^*A_6)$ is

$$\langle x, y, t | x^4, y^8, (xy)^5, (xy^2)^5, (y^4, x), (x^2, y), t^2, (t, xy^2xy^{-1}), (t, y^2xy^{-1}xyxy^2x) \rangle$$

In order to make this progenitor finite, we factor it by some relation. To search for relations, we use the following two methods

- All first order relations.

By using MAGMA, we obtain a list of the conjugacy classes of $N = 2^*A_6$.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the Class</th>
<th># Elements of form $\pi t_i$</th>
<th>Some Elements of form $\pi t_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$x^2$</td>
<td>1</td>
<td>$x^2 t_1$</td>
</tr>
<tr>
<td>3</td>
<td>$xyxy^2xy^{-1}$</td>
<td>8</td>
<td>$xyxy^2xy^{-1}t_i$, $i = 1, 2, 3, 7$</td>
</tr>
<tr>
<td>4</td>
<td>$y^{-1}xy^2xyx$</td>
<td>8</td>
<td>$y^{-1}xy^2xyt_i$, $i = 1, 2, 3, 7$</td>
</tr>
<tr>
<td>5</td>
<td>$x$</td>
<td>10</td>
<td>$xt_1, xt_3, xt_7$</td>
</tr>
<tr>
<td>6</td>
<td>$xy$</td>
<td>8</td>
<td>$xyt_1, xyt_2$</td>
</tr>
<tr>
<td>7</td>
<td>$(xy)^2$</td>
<td>8</td>
<td>$(xy)^2t_1, (xy)^2t_2$</td>
</tr>
<tr>
<td>8</td>
<td>$xyxy^2x^{-1}y^{-1}$</td>
<td>8</td>
<td>$xyxy^2x^{-1}y^{-1}t_i$, $i = 1, 2, 3, 7$</td>
</tr>
<tr>
<td>9</td>
<td>$xy^{-1}xy^2x^{-1}y$</td>
<td>8</td>
<td>$xy^{-1}xy^2x^{-1}yt_i$, $i = 1, 2, 3, 7$</td>
</tr>
<tr>
<td>10</td>
<td>$y$</td>
<td>10</td>
<td>$yt_1, yt_2$</td>
</tr>
<tr>
<td>11</td>
<td>$x^2y^{-1}$</td>
<td>10</td>
<td>$x^2y^{-1}t_1, x^2y^{-1}t_2$</td>
</tr>
<tr>
<td>12</td>
<td>$xy^{-1}$</td>
<td>8</td>
<td>$xy^{-1}t_1, xy^{-1}t_2, xy^{-1}t_3, t_7$</td>
</tr>
<tr>
<td>13</td>
<td>$yxxy^{-1}$</td>
<td>8</td>
<td>$yxxy^{-1}t_i$, $i = 1, 2, 3, 7$</td>
</tr>
</tbody>
</table>

Since $t \approx t_1$, we get

$$t_2 = t^x, t_3 = t^y, \text{ and } t_7 = t^xy$$

- Apply the Lemma.

By finding the centralizer of two points stabiliser $t_1$ and $t_2$, we obtain three generators, thus we choose to factor by the relation

$$(3, 28, 37)(6, 22, 46)(7, 36, 33) \ldots (47, 50, 78)(55, 64, 71)(57, 60, 69) = xyxy^2xy^{-1}$$
$$(3, 48, 19)(6, 49, 27)(7, 58, 12) \ldots (47, 63, 70)(52, 79, 57)(53, 80, 55) = y^2xy^{-1}xyxy^2x$$

Thus, the short relations are given by
\[(t_1t_2)^k = (tt^x)^k = xyxy^2xy^{-1}.
(t_1t_2)^l = (tt^x)^l = y^2xy^{-1}xyxy^2x.\]

In order to have some finite images, we factored the progenitor by some of the relations from the conjugacy classes table and the lemma but no interesting homomorphic images were obtained.

### 9.2.2 Monomial Progenitor $17^{*10} : m (2^*A_6)$

Consider the group $17^{*10} : m (2^*A_6)$, where $G = 2^*A_6$. The presentation for $G$ is given by $\langle x, y|x^4, y^8, (xy)^5, (xy^2)^5, (y^4, x), (x^2, y) \rangle$. To write a monomial presentation for $G$, we chose a subgroup $H = 3^2\cdot 8$ of $G$ and induced it up by a faithful irreducible linear character $\chi_5$.

**Table 9.5: Character Table of $G = 2^*A_6$**

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
<th>$C_8$</th>
<th>$C_9$</th>
<th>$C_{10}$</th>
<th>$C_{11}$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{13}$</td>
<td>10</td>
<td>-10</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-Z2</td>
<td>Z2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

# denotes algebraic conjugation

$I = \text{RootOfUnity}(4)$

$Z_1$ is the primitive eigth root of unity

**Table 9.6: Character Table of $H = 3^2\cdot 8$**

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
<th>$C_8$</th>
<th>$C_9$</th>
<th>$C_{10}$</th>
<th>$C_{11}$</th>
<th>$C_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-I</td>
<td>-I</td>
<td>-I</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-I</td>
<td>I</td>
<td>-I</td>
<td>I</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>I</td>
<td>-I</td>
<td>I</td>
<td>-I</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>I</td>
<td>-I</td>
<td>-1</td>
<td>-1</td>
<td>-I</td>
<td>Z1</td>
<td>Z1#3</td>
<td>-Z1</td>
</tr>
</tbody>
</table>

$I = \text{RootOfUnity}(4)$

To find the value of $I$, we have to determine the smallest finite field that has the fourth roots of unity. We find that it is $F_{17}$. Moreover, since the primitive root of
17 is 3, we know that $|3| = 16$. Also, to know the value for different powers of $I$, we apply the following formula

$$|3^k| = \frac{16}{\gcd(k, 16)}$$

Thus, $|3^2| = \frac{16}{\gcd(2, 16)} = \frac{16}{2} = 8$. Now, we have that $(3^2)^8 = 9^8 \equiv_{17} 1$. By taking $I = 9$, the value for the different powers of $I$ is as the following:

$I^2 = 9^2 = 81 \equiv_{17} 13$
$I^3 = 9^3 = 729 \equiv_{17} 15$
$I^4 \equiv_{17} 1$

Now, we can find the two induced representation $A(x)$ and $A(y)$ of degree $\frac{|G|}{|N|} = \frac{720}{72} = 10$ of $G$.

$$A(x) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -13 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 
\end{pmatrix}$$

and

$$A(y) = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -15 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15
\end{pmatrix}$$
$A(x)$ and $A(y)$ are monomial matrices since there is a non-zero entry in each row and column. By checking the order of the two matrices, we see that $|A(x)| = 4 = |x|$, $|A(y)| = 8 = |y|$, $|A(x) \ast A(y)| = 5 = |xy|$, $|A(x) \ast A(y)^2| = 5 = |xy^2|$, $|(A(x)^2, A(y))| = 1 = |(x^2, y)|$, and $|(A(y)^4, A(x))| = 1 = |(y^4, x)|$. $\langle A(x), A(y) \rangle$ is a faithful representation of $2 \bullet A_6$. Now, we want to write permutation representation of the two monomial matrices. Elements of the free product $17^{\ast} 10$ is of the form $<t_1> \ast <t_2> \ast \ldots \ast <t_9> \ast <t_{10}>$, where $|t_i| = 3$ and $<t_i> = \{e, t_i, t_i^2, \ldots, t_i^{16}\}$ for $\forall 1 \leq i \leq 10$.

<table>
<thead>
<tr>
<th>1 2 3 4 5 12 13 14 15 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$ $&lt;t_1&gt;$ $t_1^2$ $t_1^3$ $t_1^4$ $t_1^5$ $t_1^{12}$ $t_1^{13}$ $t_1^{14}$ $t_1^{15}$ $t_1^{16}$</td>
</tr>
<tr>
<td>17 18 19 20 21 28 29 30 31 32</td>
</tr>
<tr>
<td>$t_2$ $&lt;t_2&gt;$ $t_2^2$ $t_2^3$ $t_2^4$ $t_2^5$ $t_2^{12}$ $t_2^{13}$ $t_2^{14}$ $t_2^{15}$ $t_2^{16}$</td>
</tr>
<tr>
<td>$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$</td>
</tr>
<tr>
<td>145 146 147 148 149 156 157 158 159 160</td>
</tr>
<tr>
<td>$t_{10}$ $&lt;t_{10}&gt;$ $t_{10}^2$ $t_{10}^3$ $t_{10}^4$ $t_{10}^5$ $t_{10}^{12}$ $t_{10}^{13}$ $t_{10}^{14}$ $t_{10}^{15}$ $t_{10}^{16}$</td>
</tr>
</tbody>
</table>

To determine the permutation for $x = A(x)$, we look at the nonzero entry $a_{i,j}$ of $A(x)$ such that

$a_{1,2} = 1 \Rightarrow$ the automorphism takes $t_1 \rightarrow t_2$, so $A(x)$ takes 1 to 17 in our labeling above.

Moreover, when $t_1^2 \rightarrow t_2^2$, $A(x)$ takes 2 to 18, and so on with the all powers of $t_1$.

$a_{2,1} = -1 \equiv_{17} 16 \Rightarrow$ the automorphism takes $t_2 \rightarrow t_1^{16}$, so $A(x)$ takes 17 to 16.

Moreover, when $t_2^2 \rightarrow t_1^{16 \times 2} \equiv_{17} t_1^5$, $A(x)$ takes 18 to 15.
The permutation of $A$ is

$$x \approx (1, 17, 16, 32)(2, 18, 15, 31)(3, 19, 14, 30)(4, 20, 13, 29)(5, 21, 12, 28)(6, 22, 11, 27)
(7, 23, 10, 26)(8, 24, 9, 25)(33, 49, 48, 64)(34, 50, 47, 63)(35, 51, 46, 62)(36, 52, 45, 61)
(37, 53, 44, 60)(38, 54, 43, 59)(39, 55, 42, 58)(40, 56, 41, 57)(65, 68, 80, 77)
(66, 72, 79, 73)(67, 76, 78, 69)(70, 71, 75, 74)(81, 128, 96, 113)(82, 127, 95, 114)
(83, 126, 94, 115)(84, 125, 93, 116)(85, 124, 92, 117)(86, 123, 91, 118)
(87, 122, 90, 119)(88, 121, 89, 120)(97, 145, 112, 160)(98, 146, 111, 159)
(99, 147, 110, 158)(100, 148, 109, 157)(101, 149, 108, 156)(102, 150, 107, 155)
(131, 133, 142, 140)(134, 138, 139, 135)

Similarly for the permutation of $y = A(y)$. The permutation for $A(y)$ is
Now, we want to write a presentation for the monomial progenitor $17^{*10} \cdot m (2^*A_6)$. Let $t \approx t_1$. By using MAGMA, the normaliser of the subgroup $< t_1 >$ is generated by

\[(17, 104, 152) \ldots (64, 66, 95) = xyx^{-1} x y x y^2 x y x, \]

\[(17, 125, 49) \ldots (96, 157, 132) = x y x y^2 x y^{-1} x^{-1} y, \quad and \]

\[(1, 9, 13, 15, 16, 8, 4, 2) \ldots (147, 150, 156, 151, 158, 155, 149, 154) = x y x y^2 x y^{-1} x y^2 x y^{-1} x. \]
Thus, a presentation for the monomial progenitor is given by

\[\begin{align*}
17^{*10} :_{m} (2 \bullet A_6) &= \langle x, y, t | x^4, y^8, (xy)^5, (x^2, y), (y^4, x), t^{17}, \\
& (t, yxy^{-1}xyy^2xyx), (t, xyxy^2xy^{-1}x^{-1}y), t^{xyxy^2xy^{-1}x^{-1}t^{-9}} \rangle.
\end{align*}\]

We want to obtain \(17^{10} : (2 \bullet A_6)\) as a homomorphic image of \(17^{*10} :_{m} (2 \bullet A_6)\). Thus, we factor the progenitor \(17^{*10} :_{m} (2 \bullet A_6)\) by the following commutators, but it seemed hard to prove it as an isomorphic image of \(17^{*10} :_{m} (2 \bullet A_6)\).

\[\begin{align*}
& (t, t^x), (t, t^{x^2}y), (t, t^{xy^{-1}}), (t, t^{y^{-2}}), (t, t^{(xy)^{-1}})^2, \\
& (t, t^{y^{-1}}), (t, t^{(xy^2)^{-1}}), (t, t^{y^{-2}x^{-1}}), (t, t^{y^{-1}x}).
\end{align*}\]

### 9.3 \(3^* A_6\)

Consider the group

\[N = 3^* A_6 = \langle x, y | x^2, y^4, (xy)^{15}, (xy^2)^5, ((xy)^5, x), ((xy)^5, y) \rangle,
\]

where

\[\begin{align*}
x &\approx (2, 6)(4, 11)(7, 9)(8, 13)(10, 14)(12, 16). \\
y &\approx (1, 2, 7, 4)(3, 8, 6, 10)(5, 9, 13, 12)(11, 15)(14, 17)(16, 18).
\end{align*}\]

#### 9.3.1 The involutory Progenitor \(2^{*18} : (3^* A_6)\)

To write a presentation for this type of progenitor, we must introduce a new variable \(t = t_1\) of order 2, and find all elements of \(N\) that stabilise 1. By using MAGMA, we obtain the following

\[\begin{align*}
(2, 6)(4, 11)(7, 9)(8, 13)(10, 14)(12, 16) &= x \\
(2, 7, 17, 10, 16)(4, 14, 9, 13, 15)(6, 18, 12, 11, 8) &= (y^2xy^{-1})^3.
\end{align*}\]

Since \(t\) commutes with each generator of the one point stabiliser, we have to insert \(t\) and with which commutes into our progenitors. Then the presentation for \(2^{*18} : (3^* A_6)\) is

\[\langle x, y, t | x^4, y^3, (xy)^5, (x^2, y), t^2, (t, x), (t, (y^2xy^{-1})^3) \rangle.\]

In order to search for relations, we use the following two methods
Factor the progenitor by some of the first order relations.

By using MAGMA, we obtain a list of the conjugacy classes of $N = 3 \cdot A_6$.

Table 9.10: Conjugacy Classes of $N = 3 \cdot A_6$

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the Class</th>
<th>Elements of form $\pi t_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$y^2$</td>
<td>$y^2 t_i, y^2 t_{11}$</td>
</tr>
<tr>
<td>6</td>
<td>$xy^3 xy xy^4$</td>
<td>$xy^3 xy xy^4 t_i, i = 1, 2, 6, 11$</td>
</tr>
<tr>
<td>12</td>
<td>$yxy^{-1} xy$</td>
<td>$yxy^{-1} xy t_i, i = 1, 2$</td>
</tr>
</tbody>
</table>

Since $t \approx t_1$, we get

$$t_2 = t^y, t_6 = t^{yx}, t_{11} = t^{yx^{-1}}$$

Apply the Lemma.

We will utilize the lemma by taking the stabiliser of two elements and determining the centraliser of those points. The lemma deals with elements that have a transposition $(1,2)$, or do have either 1 or 2, or do not have either of them.

$$N \cap \langle t_0, t_1 \rangle \leq C_N(N^{01}).$$

$$N^{(01)} = \langle e \rangle.$$

$$C_N(N^{01}) = \langle y^{-1} xy^{-1} xy^2 xy, y^2 xy^{-1} x y x^2 \rangle.$$

By adding all the relations from the above table and write some supporting relations, the presentation for this progenitor is

$$< x, y, t | x^2, y^4, (xy^2)^5, (xy)^{15}, ((xy)^5, x), ((xy)^5, y), t^2, (t, x), (t, (y^2xy - 1)^3),$$

$$(tt^y)^k = y^{-1} xy^{-1} x y^2 x y x y, (y^2 t)^m, (y^2 t^{yx^{-1}})^n, (x^2 x y x y^{-1} x y x y^{-1})^o,$$

$$(x y^2 x y x y^{-1} t^{yx^{-1}})^p, (x y^2 x y x y^{-1} t^{yx^{-1}})^q, (x y x y^{-1} x y x t)^r, (x y x y^{-1} x y x t)^u > .$$

The progenitor $2^{*18} : (3 \cdot A_6)$ faild to construct any homomorphic images.

9.3.2 Monomial Progenitor $7^{*6} : m (3 \cdot A_6)$

Consider the group $7^{*6} : m (3 \cdot A_6)$, where $G = 3 \cdot A_6$. The presentation for $G$ is given by $< x, y | x^2, y^4, (xy)^{15}, (xy^2)^5, ((xy)^5, x), ((xy)^5, y) >$. To write a monomial presentation for $G$, we chose a subgroup $H = 3 \times A_5$ of $G$ and induced it up by a
Table 9.11: Character Table of $G = 3 \cdot A_6$

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$\cdots$</th>
<th>$C_{14}$</th>
<th>$C_{15}$</th>
<th>$C_{16}$</th>
<th>$C_{17}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_9$</td>
<td>6</td>
<td>2</td>
<td>$6 \ast J$</td>
<td>$-6 - 6 \ast J$</td>
<td>$\cdots$</td>
<td>$-1 - J$</td>
<td>$J$</td>
<td>$-1 - J$</td>
<td>$J$</td>
</tr>
</tbody>
</table>

# denotes algebraic conjugation

$J = \text{RootOfUnity}(3)$

faithful irreducible linear character $\chi_3 = (1, 1, J, -1 - J, -1 - J, J, 1, 1, 1, -1 - J, J, -1 - J, J, -1 - J)$. To find the value of $J$, we have to determine the smallest finite field that has the cube roots of unity. We find that it is $\mathbb{F}_7$. Thus, the value for $J$ and $J^2$ is 2 and 4, respectively. By inducing the theird linear character of $H$ up to $G$, we can find the two induced representation $A(x)$ and $A(y)$ of degree $\frac{|G|}{|H|} = \frac{1080}{180} = 6$ of $G$.

$$A(x) = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A(y) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$A(x)$ and $A(y)$ are monomial matrices since there is a non-zero entry in each row and column. The two monomial matrices are faithful since they satisfied the following $|A(x)| = 2 = |x|$, $|A(y)| = 4 = |y|$, $|A(x) \ast A(y)| = 15 = |xy|$, $|A(x) \ast A(y)^2| = 5 = |xy^2|$, $|(A(x)A(y))^{15}, A(y))| = 1 = |((xy)^5,y)|$, and $|(A(x)A(y))^{15}, A(x))| = 1 = |((xy)^5, x)|$.

$\langle A(x), A(y) \rangle$ is a faithful representation of $3 \cdot A_6$. Now, we want to write permutation representation of the two monomial matrices. Elements of the free product $7 \ast 6$ is of
the form \(< t_1 > * < t_2 > * \ldots * < t_5 > * < t_6 >\), where \(|t_i| = 7\) and \(< t_i > = \{e, t_i, t_i^2, \ldots, t_i^6\}\) for \(\forall 1 \leq i \leq 6\).

Table 9.12: Labeling for \(t_i\)’s

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_1)</td>
<td>(t_1^2)</td>
<td>(t_1^3)</td>
<td>(t_1^4)</td>
<td>(t_1^5)</td>
<td>(t_1^6)</td>
<td>(t_2)</td>
</tr>
<tr>
<td>(t_3)</td>
<td>(t_3^2)</td>
<td>(t_3^3)</td>
<td>(t_3^4)</td>
<td>(t_3^5)</td>
<td>(t_3^6)</td>
<td>(t_4)</td>
</tr>
<tr>
<td>(t_5)</td>
<td>(t_5^2)</td>
<td>(t_5^3)</td>
<td>(t_5^4)</td>
<td>(t_5^5)</td>
<td>(t_5^6)</td>
<td>(t_6)</td>
</tr>
</tbody>
</table>

By looking at the non-zero entry \(a_{i,j}\) of \(A(x)\), we can determine the permutation for \(x = A(x)\).

\(a_{1,2} = 4 \Rightarrow \) the automorphism takes \(t_1 \rightarrow t_2^4\), so \(A(x)\) takes 1 to 10.

Moreover, when \(t_1^2 \rightarrow t_2^4 \times 2 = t_2\), \(A(x)\) takes 2 to 7

and so on with the all powers of \(t_1\). \(a_{2,1} = 2 \Rightarrow \) the automorphism takes \(t_2 \rightarrow t_1^2\), so \(A(x)\) takes 7 to 2.

Moreover, when \(t_2^2 \rightarrow t_1^2 \times 2 = t_1^4\), \(A(x)\) takes 8 to 4

The permutation of \(A(x)\) is


Similarly for the permutation of \(y = A(y)\). The permutation for \(A(y)\) is
Table 9.14: Permutation of $y = A(y)$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>35</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_1^2$</td>
<td>$t_1^3$</td>
<td>$t_1^4$</td>
<td>$t_1^5$</td>
<td>$t_1^6$</td>
<td>...</td>
<td>$t_6$</td>
<td>$t_6^2$</td>
<td>$t_6^3$</td>
<td>$t_6^4$</td>
<td>$t_6^5$</td>
<td>$t_6^6$</td>
</tr>
<tr>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
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<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>$t_3$</td>
<td>$t_3^2$</td>
<td>$t_3^3$</td>
<td>$t_3^4$</td>
<td>$t_3^5$</td>
<td>$t_3^6$</td>
<td>...</td>
<td>$t_5^2$</td>
<td>$t_5^3$</td>
<td>$t_5^4$</td>
<td>$t_5^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>...</td>
<td>26</td>
<td>28</td>
<td>30</td>
<td>25</td>
<td>27</td>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>

$y \approx (1, 13, 19, 7)(2, 14, 20, 8)(3, 15, 21, 9)(4, 16, 22, 10)$

$(5, 17, 23, 11)(6, 18, 24, 12)(25, 34)(26, 31)$

$(27, 35)(28, 32)(29, 36)(30, 33)$

Now, we want to write a presentation for the monomial progenitor $7^*6 :_m (3^*A_6)$. Let $t \approx t_1$. By using MAGMA, the normalisers of the subgroup $< t_1 >$ that fixes $t_1$ and its powers is generated by

$$(13, 31)(14, 32)(15, 33) \ldots (22, 28)(23, 29)(24, 30) = y^2xyxy^2xy^{-1}xy^2.$$  
$$(13, 19)(14, 20)(15, 21) \ldots (28, 34)(29, 35)(30, 36) = y^{-1}xyxy^2x.$$  
$$(7, 10, 8)(9, 11, 12) \ldots (25, 26, 28)(27, 30, 29) = xyxyxy^2xy^{-1}.$$  
$$(7, 14)(8, 16)(9, 18) \ldots (22, 31)(23, 33)(24, 35) = y^2xy^{-1}xyxy^2.$$  
$$(1, 4, 2)(3, 5, 6) \ldots (19, 20, 22)(21, 24, 23) = xy^{-1}xy^{-1}xyxy^2.$$  

By looking to the above permutations, we see that the last one is the only one that takes $t_1$ to $t_1^4$ (from labeling). Thus, it will have the following form

$$t^{xy^{-1}xy^{-1}xyxy^2} = t^4$$

Thus, a presentation for the monomial progenitor is given by

$$7^*6 :_m (3^*A_6) = \langle x, y, t | x^2, y^4, (xy)^{15}, (xy^2)^5, ((xy)^5, y), ((xy)^5, x), t^7, (t, y^{-1}xyxy^{-1}xy), (t, y^2xyxy^2xy^{-1}xy^2), (t, y^{-1}xy^{-1}xyxy^2x), (t, y^2xy^{-1}xyxy^2), \rangle$$

To make this progenitor finite, we factored it by $(t, t^x)$ and $(t, t^{(xy)^{-1}})$. Thus, we were able to obtain the following homomorphic image

$$\frac{7^*6 :_m (3^*A_6)}{(t, t^x), (t, t^{(xy)^{-1}})} \cong 7^6 : (3^*A_6)$$
9.4 $6^\bullet A_6$

Consider the group

$$N = 6^\bullet A_6 = \langle x, y | x^2 = y^4, (xy)^5 = (y^4(xy)^5)^{-2}, (xy^2)^5, (y^4(xy)^5, x), (y^4(xy)^5, y) \rangle,$$

where

$$x \approx (1, 2, 8, 5)(3, 13, 22, 16)(4, 10, 9, 19) \cdots (421, 422, 423, 424)(427, 430, 428, 429).$$

$$y \approx (1, 3, 14, 12, 8, 22, 21, 6)(2, 9, 23, 20, 5, 4, 15, 11) \cdots (395, 421, 431, 424, 396, 423, 432, 422).$$

9.4.1 The involutory Progenitor $2^*432 : (6^\bullet A_6)$

To write a presentation for the progenitor of $6^\bullet A_6$, we have to introduce a new variable $t = t_1$ of order 2, and find all elements of $N$ that stabilise 1. By using MAGMA, the stabiliser of $t_1$ is the identity. The presentation for $2^*432 : (6^\bullet A_6)$ is

$$\langle x, y, t \mid x^2 = y^4, (xy)^5 = (y^4(xy)^5)^{-2}, (xy^2)^5, (y^4(xy)^5, x), (y^4(xy)^5, y), t^2 \rangle.$$

In order to make this progenitor finite, we factor it by some relation. To search for relations, we use the following two methods

- Some of the first order relations.

  By using MAGMA, we obtain a list of the conjugacy classes of $N = 6^\bullet A_6$.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative of the Class</th>
<th>Some Elements of form $\pi t_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>$xyxy^{-1}xy^{-1}xy^{-1}xy^{-1}$</td>
<td>$xyxy^{-1}xy^{-1}xy^{-1}t_i, i = 1, 3, 14$</td>
</tr>
<tr>
<td>20</td>
<td>$xy$</td>
<td>$xyt_1, xyt_2, xyt_{14}$</td>
</tr>
<tr>
<td>26</td>
<td>$yxyx^{-1}y$</td>
<td>$yxyx^{-1}yt_i, i = 1, 3, 14$</td>
</tr>
<tr>
<td>31</td>
<td>$xy^{-1}$</td>
<td>$xy^{-1}t_1, xy^{-1}t_2, xy^{-1}t_3$</td>
</tr>
</tbody>
</table>

Since $t \approx t_1$, we get

$$t_2 = t^x, t_3 = t^y, \text{ and } t_{14} = t^{y^2}$$

- Apply the Lemma.

  By finding the centralizer of two points stabiliser $t_1$ and $t_2$, we find it is the whole group $G$. Thus, we are not going to use the lemma for this progenitor.
Therefore, by adding all of the relations from the conjugacy classes table, the presentation for the progenitor $2^{*432} : (6^*A_6)$ is

$$< x, y, t | x^2 = y^4, (xy)^5 = (y^4(xy)^5)^{-2}, (xy^2)^5, (y^4(xy)^5, x), (y^4(xy)^5, y), t^2, (xyxy^{-1}xy^{-1}xy^{-1}yt)^k, (xyxy^{-1}xy^{-1}xy^{-1}yt)^l, (xyxy^{-1}xy^{-1}xy^{-1}yt)^m, (xyyt)^n, (xyyt^2)^o, (xyxxy^{-1}yt)^p, (yxyxxy^{-1}yt)^q, (yxyxxy^{-1}yt)^r, (xy^4(xy)^5, x), (xy^4(xy)^5, y), (xy^4(xy)^5, t) >.$$}

By running the above presentation, we find some finite images as given in the Table 9.16.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l$</th>
<th>$s$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>Index in $G$</th>
<th>Order of $G$</th>
<th>Shape of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>360</td>
<td>129600</td>
<td>$A_6 : A_6$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>360</td>
<td>129600</td>
<td>$(2 \times 3) \cdot PGL_2(9)$</td>
<td></td>
</tr>
</tbody>
</table>

### 9.4.2 Monomial Progenitor $61^{*36} : m (6^*A_6)$

Consider the group $61^{*36} : m (6^*A_6)$, where $G = 6^*A_6$. The presentation for $G$ is given by $< x, y | x^2 = y^4, (xy)^5 = (y^4(xy)^5)^{-2}, (xy^2)^5, (y^4(xy)^5, x), (y^4(xy)^5, y) >$.

We want to write a presentation for the monomial progenitor of $6^*A_6$, so we want to induce a faithful linear character (not necessarily irreducible) from a subgroup $H$ up to $6^*A_6$. To do, we run the following loop

```maple
> S:=Subgroups(G1);
> #S;
51
```

So, we have investigated all the subgroup $S[i]$ where $25 \leq i \leq 51$ of $6^*A_6$, and we find that the only subgroup that contain a faithful (not irreducible) linear character are those of index $n$ equal to 36, 60, 90 and 72. Since the degree of the monomial matrices is $n$ implies that we will have $n \times n$ matrices, we consider to induce that one with lower index 36.

Now, we will induce the $9^{th}$ linear character of $H = (3 \times 2 \times 5) : 2$ of order 60 up to
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Group & Subgroup, Index, and Linear Representation & Linear Characters \\
 \hline
$6^*A_6$ & $37, 36, (3 \times 2 \times 5) : 2$ & $\chi_1 \to \chi_{12}$ \\
 & & $\chi_2 \to \chi_{8}$ nonfaithful \\
 & & $\chi_9 \to \chi_{12}$ faithful \\
 \hline
\end{tabular}
\end{center}

$6^*A_6$.

$$\chi^{(9)} = (1, -1, J, -1 - J, -I, I, 1, 1 + J, -J, -1, -1, Z2, Z2^5, -Z, 2 - Z2^5, J, -1 - J, J, -1 - J, 1 + J, 1 + J, -J, -J),$$

where

$J$ is the cube root of unity

$I$ is the fourth root of unity

$Z_1$ is the primitive $5^{th}$ root of unity

$Z_2$ is the primitive $12^{th}$ root of unity

$Z_3$ is the primitive $15^{th}$ root of unity

The smallest finite field that divisible by 5, 12, and 15 is $\mathbb{F}_{61}$ and the primitive root of 61 is 2. To find the third and fourth root of unity $J$ and $I$, respectively, we have to find $k_1$ and $k_2$ that satisfied the following

$$|2^{k_1}| = \frac{60}{\gcd(k_1, 60)} = 3. \quad \text{Thus,} \quad k_1 = \frac{60}{3} = 20.$$

$$|2^{k_2}| = \frac{60}{\gcd(k_2, 60)} = 4. \quad \text{Thus,} \quad k_2 = \frac{60}{4} = 15.$$

Hence,

$$J = 2^{20} \equiv_{61} 47 \quad \text{and} \quad J^2 = 47^2 \equiv_{61} 13.$$

$$I = 2^{15} \equiv_{61} 11, \quad I^2 = 11^2 \equiv_{61} 60, \quad \text{and} \quad I^3 = 11^3 \equiv_{61} 50.$$

The elements of order 3 are 47 and 13, and the elements of order 4 are 11, 60, and 50.

To find monomial matrices $A(x)$ and $A(y)$, we have to find the 36 non-zero entries of both matrices by running the following loops

\begin{verbatim}
A:=[0:i in [1..1296]];
for i in [1..36] do if a*T[i]^(-1) in N then i, chN[9](a*T[i]^(-1)); end if; end for;
for i in [1..36] do if T[2]*a*T[i]^(-1) in N then i,
\end{verbatim}
We will mention the non-zero entries of the two matrices $A(x)$ and $A(y)$ (see table 9.17).
Table 9.17: Nonzero Entries of $A(x)$ and $A(y)$

<table>
<thead>
<tr>
<th>Row ( n )</th>
<th>( A(x) ) Entry</th>
<th>( A(x) ) Value</th>
<th>( A(y) ) Entry</th>
<th>( A(y) ) Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>(-J = -47)</td>
<td>3</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(-J - 1 = J^2 = 13)</td>
<td>5</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>1</td>
<td>8</td>
<td>(I + 1 = -I^2 = -13)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>(I = 11)</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>(-J - 1 = J^2 = 13)</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>(-J - 1 = J^2 = 13)</td>
<td>17</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>(-I = -11)</td>
<td>20</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>-1</td>
<td>10</td>
<td>(J = 47)</td>
</tr>
<tr>
<td>9</td>
<td>24</td>
<td>(-J - 1 = J^2 = 13)</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>26</td>
<td>(-J = -47)</td>
<td>1</td>
<td>(J + 1 = -J^2 = -13)</td>
</tr>
<tr>
<td>11</td>
<td>27</td>
<td>1</td>
<td>4</td>
<td>(-I = -11)</td>
</tr>
<tr>
<td>12</td>
<td>28</td>
<td>-1</td>
<td>29</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>32</td>
<td>1</td>
<td>33</td>
<td>-1</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>(-J = -47)</td>
<td>24</td>
<td>(-J - 1 = J^2 = 13)</td>
</tr>
<tr>
<td>15</td>
<td>23</td>
<td>1</td>
<td>34</td>
<td>(-J - 1 = J^2 = 13)</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>(-J = -47)</td>
<td>26</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>30</td>
<td>-1</td>
<td>35</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>20</td>
<td>1</td>
<td>30</td>
<td>(J + 1 = -J^2 = -13)</td>
</tr>
<tr>
<td>19</td>
<td>33</td>
<td>-1</td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>-1</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>25</td>
<td>1</td>
<td>31</td>
<td>(J + 1 = -J^2 = -13)</td>
</tr>
<tr>
<td>22</td>
<td>36</td>
<td>-1</td>
<td>19</td>
<td>(J = 47)</td>
</tr>
<tr>
<td>23</td>
<td>15</td>
<td>-1</td>
<td>16</td>
<td>(-J - 1 = J^2 = 13)</td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>(-J = -47)</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>21</td>
<td>-1</td>
<td>23</td>
<td>1</td>
</tr>
<tr>
<td>26</td>
<td>10</td>
<td>(-J - 1 = J^2 = 13)</td>
<td>25</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>27</td>
<td>11</td>
<td>-1</td>
<td>36</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>12</td>
<td>1</td>
<td>32</td>
<td>1</td>
</tr>
<tr>
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<td>31</td>
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<td>2</td>
<td>(-J - 1 = J^2 = 13)</td>
</tr>
<tr>
<td>30</td>
<td>17</td>
<td>1</td>
<td>22</td>
<td>1</td>
</tr>
<tr>
<td>31</td>
<td>29</td>
<td>-1</td>
<td>21</td>
<td>(-IJ = -29)</td>
</tr>
<tr>
<td>32</td>
<td>13</td>
<td>-1</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>33</td>
<td>19</td>
<td>1</td>
<td>27</td>
<td>1</td>
</tr>
<tr>
<td>34</td>
<td>34</td>
<td>(I = 11)</td>
<td>28</td>
<td>(-J = -47)</td>
</tr>
<tr>
<td>35</td>
<td>35</td>
<td>(-I = -11)</td>
<td>9</td>
<td>(-J - 1 = J^2 = 13)</td>
</tr>
<tr>
<td>36</td>
<td>22</td>
<td>1</td>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>
Appendix A: MAGMA Code for DCE of $S_5$ over $A_4$

/* This is S5 homographic image of the progenitor 2star4:A4(4 double cosets,10 single cosets) */
-----------------------------------------------------
S:=Sym(4);
xx:=S!(1,2)(3,4);
yy:=S!(1,2,3);
N:=sub<S|xx,yy>;
Stabiliser(N,4);
#N;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,t^2,(t,y),
((y^2)*x*t*(t^x))^2, (y*t*(t^x)*t^*(x^2))^4>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
NN<a,b>:=Group<a,b|a^2,b^3,(a+b)^3>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..12]];
for i in [2..12] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^(-1); end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k];
  end for;
  ArrayP[i]:=PP;
for i in [1..12] do if ArrayP[i] eq N!(1,2)(3,4) then Sch[i]; end if; end for;

prodim := function(pt, Q, I)
  /*
  Return the image of pt under permutations Q[I]
  applied sequentially.
  */
  v:=pt;
  for i in I do
    v:=vˆ(Q[i]);
  end for;
  return v;
end function;

ts:= [Id(G1): i in [1 .. 4] ];
ts[4]:=f(t); ts[1]:=f((tˆx)ˆy);
ts[2]:=ts[1]ˆf(y); ts[3]:=f(tˆ(1));

/* This cst function will keep track of all single cosets*/
cst:= [null : i in [1 .. Index(G,sub<G|x,y>)]]
where null is[Integers() |
for i := 1 to 4 do
cst[prodim(1, ts, [i])]:=i;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
then m:=m+1; end if; end for; m;

N0:=Stabiliser(N,4);
Orbits(N0);
#N0;

N01:=Stabiliser(N,[4,1]);
SSS:={[4,1]};
SSS:=SSSˆN;
SSS;
#SSS;
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
  for n in IN do
    if ts[4]*ts[1] eq
    n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
  end for;
end for;
then print Rep(Seqq[i]);
end if; end for; end for;
N01s:=N01;
for n in N do if 4^n eq 4 and 1^n eq 2 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 4^n eq 4 and 1^n eq 3 then
N01s:=sub<N|N01s,n>; end if; end for;
N01s; #N01s;
T01:=Transversal(N,N01);
for i in [1..#T01] do
ss:=[4,1]^T01[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N01s);
#N01s;

N414:=Stabiliser(N,[4,1,4]);
SSS:={[4,1,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N010s:=N414;
for n in N do if 4^n eq 2 then N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 2 and 1^n eq 3 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 1 and 1^n eq 4 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 2 and 1^n eq 4 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 3 and 1^n eq 2 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 1^n eq 3 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 3 and 1^n eq 4 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 1 and 1^n eq 2 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 1^n eq 2 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 3 then
N010s:=sub<N|N010s,n>; end if; end for;
for n in N do if 4^n eq 1 and 1^n eq 3 then
N010s:=sub<N|N010s,n>; end if; end for;
N010s; #N010s;
T010:=Transversal(N,N010s);
for i in [1..#T010] do
ss:=[4,1,4]^T010[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..10] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N010s);
#(N010s);

//* to print all single cosets */
for i in [1..12] do i, cst[i]; end for;
Appendix B: MAGMA Code for Isomorphism Type of $S_5$

```magma
S:=Sym(4);
xx:=S!(1,2)(3,4);
yy:=S!(1,2,3);
N:=sub<S|xx,yy>;
Stabiliser(N,4);
#N;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,t^2,(t,y),
((y^2)*x*t*(t^x))^2, (y*t*(t^x)*t^x^2)^4>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
Center(G1);
NL:=NormalLattice(G1);
MinimalNormalSubgroups(G1);
s,t:=IsIsomorphic(Alt(5),NL[2]);
s;
D:=DirectProduct(NL[2],NL[3]);
s,t:=IsIsomorphic(D,G1);
s;
H<a,b>:=Group<a,b|a^2,b^3,(a*b)^5>;
#H;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H1,NL[2]);
s;
s,t:=IsIsomorphic(H1,Alt(5));
s;
a:=NL[2].1;
b:=NL[2].2;
for g in G1 do if Order(g) eq 2 and g notin NL[2] and G1 eq
```

sub<\text{G1}|\text{NL}[2],g> then U:=g; break; end if; end for;
\text{G1 eq sub}<\text{G1}|\text{NL}[2],U>;
\text{N}:=\text{sub}<\text{G1}|a,b>;
\#\text{N};
\text{NN}<i,j>:=\text{Group}<i,j|i^2,j^3,(i*j)^5>;
\#\text{NN};
\text{Sch}:=\text{SchreierSystem} (\text{NN}, \text{sub}<\text{NN}|\text{Id}(\text{NN}));
\text{ArrayP}:= [\text{Id}(\text{N})\colon i \in [1..60]];
\text{for } i \text{ in } [2..60] \text{ do }
\text{P}:= [\text{Id}(\text{N})\colon l \in [1..\#\text{Sch}[i]]];
\text{for } j \text{ in } [1..\#\text{Sch}[i]] \text{ do }
\text{if } \text{Eltseq} (\text{Sch}[i])[j] \text{ eq } 1 \text{ then } \text{P}[j]:=a; \text{ end if; }
\text{if } \text{Eltseq} (\text{Sch}[i])[j] \text{ eq } 2 \text{ then } \text{P}[j]:=b; \text{ end if; }
\text{if } \text{Eltseq} (\text{Sch}[i])[j] \text{ eq } -2 \text{ then } \text{P}[j]:=b^{-1}; \text{ end if; }
\text{end for; }
\text{PP}:=\text{Id}(\text{N});
\text{for } k \text{ in } [1..\#\text{P}] \text{ do }
\text{PP}:=\text{PP} \ast \text{P}[k]; \text{ end for; }
\text{ArrayP}[i]:=\text{PP};
\text{end for; }
\text{A}:= [\text{Id}(\text{NN})\colon i \in [1..2]];
\text{for } i \text{ in } [1..60] \text{ do }
\text{if } a^U \text{ eq } \text{ArrayP}[i] \text{ then } \text{A}[1]:=\text{Sch}[i]; \text{ Sch}[i]; \text{ end if; }
\text{end for; }
\text{for } i \text{ in } [1..60] \text{ do }
\text{if } b^U \text{ eq } \text{ArrayP}[i] \text{ then } \text{A}[2]:=\text{Sch}[i]; \text{ Sch}[i]; \text{ end if; }
\text{end for; }
\text{HH}<a,b,c>:=\text{Group}<a,b,c|a^2,b^3,(a*b)^5,c^2,a^c=a,
b^c=a*b^{-1}*a*b*a*b^{-1}>;
\#\text{HH};
\text{f2}, \text{H2}, k:=\text{CosetAction}(\text{HH}, \text{sub}<\text{HH}|\text{Id}(\text{HH}));
\text{s}:={\text{IsIsomorphic}(\text{H2}, \text{G1})};
\text{s};
\text{s}:={\text{IsIsomorphic}(\text{H2}, \text{Sym}(5))};
\text{s};
Appendix C: MAGMA Code for DCE of $L_2(11) \times 3$ over $A_4$

/* This is $L(2,11)X3$ homographic image of the progenitor 2star4:A4(20 double cosets) */
-----------------------------------------------------
S:=Sym(4);
xx:=S!(1,2)(3,4);
yy:=S!(1,2,3);
N:=sub<S|xx,yy>;
Stabiliser(N,4);
#N;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,(y^(-1)*x^(-1)*y*x)^2,
   t^2,(t,y),(x*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^3>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..12]];
for i in [2..12] do
  P:=[Id(N): l in [1..#Sch[i]]];
  for j in [1..#Sch[i]] do
    if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
    if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^(-1); end if;
  end for;
  PP:=Id(N);
  for k in [1..#P] do
    PP:=PP*P[k]; end for;
  ArrayP[i]:=PP;
for i in [1..12] do if ArrayP[i] eq N!(1,4)(2,3) then
   Sch[i]; end if; end for;

prodim := function(pt, Q, I)
   /*
   Return the image of pt under permutations Q[I]
   applied sequentially.
   */
   v:=pt;
   for i in I do
      v:=vˆ(Q[i]);
   end for;
   return v;
end function;

for i := 1 to 4 do
   cst[prodim(1, ts, [i])]:=i;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
   then m:=m+1; end if; end for; m;

N4:=Stabiliser(N,4);
Orbits(N4);
#N4;

N01:=Stabiliser(N,[4,1]);
SS:= {[4,1]};
SS:=SSˆN;
SS;
#SS;
Seqq:=Setseq(SS);
Seqq;
for i in [1..#SS] do
   for n in IN do
         n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
      then print Rep(Seqq[i]);
      end if;
   end for;
end for;
N01; #N01;
T01:=Transversal(N,N01);
for i in [1..#T01] do
ss:=[4,1]^T01[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N01);
#N01;

N412:=Stabiliser(N,[4,1,2]);
SSS:=[[4,1,2]]; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]
then print Rep(Seqq[i]);
end if; end for; end for;
N012s:=N412;
T012:=Transversal(N,N012s);
for i in [1..#T012] do
ss:=[4,1,2]^T012[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N012s);
#(N012s);

N413:=Stabiliser(N,[4,1,3]);
SSS:=[[4,1,3]]; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]
then print Rep(Seqq[i]);
end if; end for; end for;
N013s:=N413;
T013:=Transversal(N,N013s);
for i in [1..#T013] do
ss:=[4,1,3]ˆT013[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N013s);
#N013s;

---------------------------------------------
N414:=Stabiliser(N,[4,1,4]);
SSS:=([4,1,4]); SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N010s:=N414;
T010:=Transversal(N,N010s);
for i in [1..#T010] do
ss:=[4,1,4]ˆT010[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if ts[4]*ts[1]*ts[4] eq
g*(ts[4]*ts[1])^h
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,4)(2,3) then Sch[i];
end if; end for;
xx^yy;

---------------------------------------------
N4121:=Stabiliser(N,[4,1,2,1]);
SSSS:=([4,1,2,1]); SSSS:=SSSS^N;
SSSS;
#(SSSS);
Seqq:=Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]*
  ts[Rep(Seqq[i])][4]
then print Rep(Seqq[i]);
end if; end for; end for;
N0121s:=N4121;
T0121:=Transversal(N,N0121s);
for i in [1..#T0121] do
ss:=[4,1,2,1]ˆT0121[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
xx;
N4124:=Stabiliser(N,[4,1,2,4]);
SSSS:={[4,1,2,4]}; SSSS:=SSSSˆN;
SSSS;
print #SSSS;
Seqq:=Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]*
  ts[Rep(Seqq[i])][4]
then print Rep(Seqq[i]);
end if; end for; end for;
N0124s:=N4124;
for n in N do if 4^n eq 2 and 1^n eq 3 and 2^n eq 4 then
N0124s:=sub<N|N0124s,n>; end if; end for;
N0124s; #N0124s;
T0124:=Transversal(N,N0124s);
for i in [1..#T0124] do
ss:=[4,1,2,4]ˆT0124[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N0124s);
#N0124s;

N4123:=Stabiliser(N,[4,1,2,3]);
SSSS:={[4,1,2,3]}; SSSS:=SSSS^N;
SSSS;
#SSSS;
Seqq:=Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N0123s:=N4123;
T0123:=Transversal(N,N0123s);
for i in [1..#T0123] do
ss:=[4,1,2,3]^T0123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N0123s);
#N0123s;

-----------------------------------------------------

N4131:=Stabiliser(N,[4,1,3,1]);
SSSS:={[4,1,3,1]}; SSSS:=SSSS^N;
SSSS;
#SSSS;
Seqq:=Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N0131s := N4131;
T0131 := Transversal(N, N0131s);
for i in [1..#T0131] do
  ss := [4, 1, 3, 1] \^ T0131[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne [ ]
  then m := m + 1; end if; end for;
g \* (ts[4] \* ts[1] \* ts[2]) \^ h then g, h; break; end if; end for;
end for;
for i in [1..12] do if ArrayP[i] eq N !(1, 3) (2, 4) then Sch[i];
  end if; end for;
yy \* xx \* yy^-1;
-----------------------------------------------------
N4134 := Stabiliser(N, [4, 1, 3, 4];
SSSS := { [4, 1, 3, 4] }; SSSS := SSSS \^ N;
SSSS;
#SSSS;
Seqq := Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
  n \* ts[Rep(Seqq[i]) [1]] \* ts[Rep(Seqq[i]) [2]] \* ts[Rep(Seqq[i]) [3]] \* 
  ts[Rep(Seqq[i]) [4]]
  then print Rep(Seqq[i]);
  end if; end for;
N0134s := N4134;
for n in N do if 4 \^ n eq 3 and 1 \^ n eq 2 and 3 \^ n eq 4 then
  N0134s := sub<N|N0134s, n>; end if; end for;
N0134s; #N0134s;
T0134 := Transversal(N, N0134s);
for i in [1..#T0134] do
  ss := [4, 1, 3, 4] \^ T0134[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne [ ]
  then m := m + 1; end if; end for;
  Stabiliser(N, [4, 1, 3, 4]);
Orbits(N0134s);
#N0134s;
-----------------------------------------------------
N4132:=Stabiliser(N,[4,1,3,2]);
SSSS:={[4,1,3,2]}; SSSS:=SSSS^N;
SSSS;
#SSSS;
Seqq:=Setseq(SSSS);
Seqq;
for i in [1..#SSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N0132s:=N4132;
T0132:=Transversal(N,N0132s);
for i in [1..#T0132] do
ss:=[4,1,3,2]^T0132[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Stabiliser(N,[4,1,3,2]);
Orbits(N0132s);
#N0132s;
-----------------------------------------------------
N41241:=Stabiliser(N,[4,1,2,4,1]);
SSSSS:={[4,1,2,4,1]}; SSSSS:=SSSSS^N;
SSSSS;
#SSSSS;
Seqq:=Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01241s:=N41241;
T01241:=Transversal(N,N01241s);
for i in [1..#T01241] do
ss:=[4,1,2,4,1]^T01241[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne [] then m:=m+1; end if; end for;  m; 
Orbits(N01241s);  
#N01241s; 

-----------------------------------------------------

N41232:=Stabiliser(N,[4,1,2,3,2]);  
SSSSS:={[4,1,2,3,2]};  SSSSS:=SSSSS^N;  
SSSSS;  
#SSSSS;  
Seqq:=Setseq(SSSSS);  
Seqq;  
for i in [1..#SSSSS] do  
for n in IN do  
if ts[4]*ts[1]*ts[2]*ts[3]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]] then print Rep(Seqq[i]); end if; end for;  
end for;  
N01232s:=N41232;  
for n in N do if 4^n eq 1 and 1^n eq 3 and 3^n eq 4 then N01232s:=sub<N| N01232s,n>; end if; end for;  
for n in N do if 4^n eq 3 and 1^n eq 4 and 3^n eq 1 then N01232s:=sub<N| N01232s,n>; end if; end for;  
N01232s;  
#N01232s;  
T01232:=Transversal(N,N01232s);  
for i in [1..#T01232] do  
ss:=[4,1,2,3,2]^T01232[i];  
cst[prodim(1, ts, ss)] := ss;  
end for;  

m:=0; for i in [1..165] do if cst[i] ne [] then m:=m+1; end if; end for;  
for g in IN do for h in IN do if ts[4]*ts[1]*ts[2]*ts[3]*ts[2] eq g*(ts[4]*ts[1]*ts[3]*ts[2])^h then g,h; break; end if; end for; end for;  
for i in [1..12] do if ArrayP[i] eq N!(4,1)(2,3)then Sch[i]; end if; end for;  

xx^yy;  

-----------------------------------------------------

N41234:=Stabiliser(N,[4,1,2,3,4]);  
SSSSS:={[4,1,2,3,4]};  SSSSS:=SSSSS^N;  
SSSSS;  
#SSSSS;
Seqq := Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
  for n in IN do
    if ts[4]*ts[1]*ts[2]*ts[3]*ts[4] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
    then print Rep(Seqq[i]);
    end if;
  end for;
end for;
N01234s := N41234;
T01234 := Transversal(N, N01234s);
for i in [1..#T01234] do
  ss := [4, 1, 2, 3, 4]^T01234[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne [] then m := m+1; end if; end for;
#N01234s;

-----------------------------------------------------

N41231 := Stabiliser(N, [4, 1, 2, 3, 1]);
SSSSS := ([4, 1, 2, 3, 1]); SSSSS := SSSSS^N;
SSSS;
#SSSSS;
Seqq := Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
  for n in IN do
    if ts[4]*ts[1]*ts[2]*ts[3]*ts[1] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
    then print Rep(Seqq[i]);
    end if;
  end for;
end for;
N01231s := N41231;
T01231 := Transversal(N, N01231s);
for i in [1..#T01231] do
  ss := [4, 1, 2, 3, 1]^T01231[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne [] then m := m+1; end if; end for;
#N01231s;
N41341 := Stabiliser(N, [4, 1, 3, 4, 1]);
SSSSS := ([4, 1, 3, 4, 1]); SSSSS := SSSSS^N;
SSSSS;
#SSSSS;
Seqq := Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]]*ts[Rep(Seqq[i])][4]*ts[Rep(Seqq[i])][5]
then print Rep(Seqq[i]);
end if; end for; end for;
N01341s := N41341;
T01341 := Transversal(N, N01341s);
for i in [1..#T01341] do
ss := [4, 1, 3, 4, 1]^T01341[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..165] do if cst[i] ne []
then m := m + 1; end if; end for;
m;
for g in IN do for h in IN do if ts[4]*ts[1]*ts[3]*ts[4]*ts[1]
eq g*(ts[4]*ts[1]*ts[2]*ts[3]*ts[1])^h
then g, h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1, 2)(3, 4) then Sch[i];
end if; end for;
xx;
f(x)*ts[3]*ts[4]*ts[2]*ts[1]*ts[4];

-------------------------------------------------------------------
N41321 := Stabiliser(N, [4, 1, 3, 2, 1]);
SSSSS := ([4, 1, 3, 2, 1]); SSSSS := SSSSS^N;
SSSSS;
#SSSSS;
Seqq := Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]]*ts[Rep(Seqq[i])][4]*ts[Rep(Seqq[i])][5]
then print Rep(Seqq[i]);
end if; end for; end for;
N01321s := N41321;
T01321:=Transversal(N,N01321s);
for i in [1..#T01321] do
  ss:=[4,1,3,2,1]ˆT01321[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if ts[4]*ts[1]*ts[3]*ts[2]*ts[1]
eq g*(ts[4]*ts[1]*ts[2]*ts[4]*ts[1])ˆh
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,4)(2,3) then Sch[i];
end if; end for;
xxˆyy;
f(xˆy)*ts[1]*ts[2]*ts[4]*ts[1]*ts[2];
-----------------------------------------------------
N41323:=Stabiliser(N,[4,1,3,2,3]);
SSSSSS:={[4,1,3,2,3]}; SSSSS:=SSSSSSˆN;
SSSS;
#SSSSSS;
Seqq:=Setseq(SSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01323s:=N41323;
for n in N do if 4ˆn eq 2 and 1ˆn eq 4 and 2ˆn eq 1 then
N01323s:=sub<N| N01323s,n>; end if; end for;
for n in N do if 4ˆn eq 1 and 1ˆn eq 2 and 2ˆn eq 4 then
N01323s:=sub<N| N01323s,n>; end if; end for;
N01323s; #N01323s;
T01323:=Transversal(N,N01323s);
for i in [1..#T01323] do
  ss:=[4,1,3,2,3]ˆT01323[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if ts[4]*ts[1]*ts[3]*ts[2]*ts[3]
eq g*(ts[4]*ts[1]*ts[2]*ts[3])ˆh
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,4)(2,3) then Sch[i]; end if; end for;
xx^yy;
f(x^y)*ts[1]*ts[4]*ts[3]*ts[2];

N41324:=Stabiliser(N,[4,1,3,2,4]);
SSSSS:=[{4,1,3,2,4}]; SSSSS:=SSSSS^N;
SSSSS;
#SSSSS;
Seqq:=Setseq(SSSSS);
Seqq;
for i in [1..#SSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
 ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01324s:=N41324;
T01324:=Transversal(N,N01324s);
for i in [1..#T01324] do
ss:=[4,1,3,2,4]^T01324[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N01324s);
#N01324s;

-----------------------------------------------------------------------------------------------------------------------------------------------

N412412:=Stabiliser(N,[4,1,2,4,1,2]);
SSSSSS:=[{4,1,2,4,1,2}]; SSSSSS:=SSSSSS^N;
SSSSS;
#SSSSSS;
Seqq:=Setseq(SSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
 ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N012412s:=N412412;
T012412:=Transversal(N,N012412s);
for i in [1..#T012412] do
ss:=[4,1,2,4,1,2]^T012412[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if
[4,1,2,4,1,3]^[4,1,2,4,1,3] eq
g*(4,1,2,4,1,3)^[4,1,2,4,1,3]*h
then g,h; break; end if; end for; end for;
end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i];
end if; end for;

for i in [1..4] do for g in IN do for h in IN do if
[4,1,2,4,1,3]^[4,1,2,4,1,3] eq
[4,1,2,4,1,3][4,1,2,4,1,3]*h
then print Rep(Seqq[i]);
end if; end for; end for;
end for;
N012413s:=N412413;
T012413:=Transversal(N,N012413s);
for i in [1..#T012413] do
ss:=[4,1,2,4,1,3]^T012413[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N012413s);
#N012413s;

for i in [1..4] do for g in IN do for h in IN do if
[4,1,2,4,1,3]^[4,1,2,4,1,3] eq

\[
g \ast (ts[4] \ast ts[1] \ast ts[3] \ast ts[2]) \ast h
\]
then \(i; \) break; end if; end for; end for; end for;
for \(g \) in \(IN\) do for \(h \) in \(IN\) do if 
\[
\]
eq \(g \ast (ts[4] \ast ts[1] \ast ts[3] \ast ts[2]) \ast h\) then \(g, h;\) break; end if; end for; end for;
for \(i \) in \([1..12]\) do if \(\text{ArrayP}[i]\) eq \(N!(1,3)(2,4)\) then \(\text{Sch}[i];\) end if; end for;
\[
yy \ast xx \ast yy^{-1};
\]
\[
eq \(f(y \ast x \ast y^{-1}) \ast ts[2] \ast ts[4] \ast ts[3] \ast ts[1];\)
-----------------------------------------------------
\(\text{N412341:=Stabiliser(N,}[4,1,2,3,4,1]);\)
\(\text{SSSSSSS:}\{[4,1,2,3,4,1]\}; \text{SSSSSSS:=SSSSSSS}^{-1};\)
\(\text{SSSSSS};\)
\(\#SSSSSS;\)
\(\text{Seqq:=Setseq(SSSSSSS);}\)
\(\text{Seqq;}\)
for \(i \) in \([1..\#SSSSSS]\) do
for \(n \) in \(IN\) do if 
\[
eq \(n \ast ts[\text{Rep(Seqq}[i])]\{1\} \ast ts[\text{Rep(Seqq}[i])]\{2\} \ast ts[\text{Rep(Seqq}[i])]\{3\} \ast 
\[
ts[\text{Rep(Seqq}[i])]\{4\} \ast ts[\text{Rep(Seqq}[i])]\{5\} \ast ts[\text{Rep(Seqq}[i])]\{6\}\]
then print \(\text{Rep(Seqq}[i]);\) end if; end for; end for;
\(\text{N012341s:=N412341};\)
for \(n \) in \(N\) do if \(4 \ast n\) eq \(3\) and \(1 \ast n\) eq \(2\) and \(2 \ast n\) eq \(1\) and \(3 \ast n\) eq \(4\) then \(\text{N012341s:=sub}<N|\text{N012341s},n>;\) end if; end for;
\(\text{N012341s;}\)
\(\#N012341s;\)
\(\text{T012341:=Transversal(N,N012341s);}\)
for \(i \) in \([1..\#T012341\) do
\(ss:=[4,1,2,3,4,1]^\text{T012341}[i];\)
\(\text{cst[prodim}(1,\ ts, ss)]:=ss;\)
end for;
\(m:=0;\) for \(i \) in \([1..165]\) do if \(\text{cst}[i] \neq \[]\) then \(m:=m+1;\) end if; end for; \(m;\)
\(\text{Orbits(N012341s);}\)
\(\#N012341s;\)
-----------------------------------------------------
\(\text{N412342:=Stabiliser(N,}[4,1,2,3,4,2]);\)
\(\text{SSSSSSS:}\{[4,1,2,3,4,2]\}; \text{SSSSSSS:=SSSSSSS}^{-1};\)
\(\text{SSSSSS};\)
\(\#SSSSSS;\)
\(\text{Seqq:=Setseq(SSSSSSS);}\)
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
if ts[4]*ts[1]*ts[2]*ts[3]*ts[4]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N012342s:=N412342;
T012342:=Transversal(N,N012342s);
for i in [1..#T012342] do
ss:=[4,1,2,3,4,2]ˆT012342[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(2,4)(1,3) then Sch[i];
end if; end for;
yy*xx*yy^-1;
-----------------------------------------------------
for i in [1..4] do for g in IN do for h in IN do if
then i; break; end if; end for; end for;end for;
for g in IN do for h in IN do if
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i];
end if; end for;
-----------------------------------------------------
N412312:=Stabiliser(N,[4,1,2,3,1,2]);
SSSSSSS:={[4,1,2,3,1,2]}; SSSSSS:='SSSSSSS'N;
SSSSSS;
#SSSSSS;
Seqq:=Setseq(SSSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
(ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]])
then print Rep(Seqq[i]);
end if; end for; end for;
N012312s:=N412312;
T012312:=Transversal(N,N012312s);
for i in [1..#T012312] do
ss:=[4,1,2,3,1,2]^T012312[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N012312s);
#N012312s;
---------------------------------------------------------------------
for i in [1..4] do for g in IN do for h in IN do if
g*(ts[4]*ts[1]*ts[3]*ts[4])^h
then i; break; end if; end for; end for;end for;
for g in IN do for h in IN do if
eq g*(ts[4]*ts[1]*ts[3]*ts[4])^h
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,3)(2,4) then Sch[i];
end if; end for;

---------------------------------------------------------------------
N412314:=Stabiliser(N, [4,1,2,3,1,4]);
SSSSSS:=[{[4,1,2,3,1,4]}]; SSSSSS:=SSSSSSˆN;
SSSSSS;
#SSSSSs;
Seqq:=Setseq(SSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
(ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]])
then print Rep(Seqq[i]);
end if; end for; end for;
then print Rep(Seqq[i]);
end if; end for; end for;
N012314s:=N412314;
T012314:=Transversal(N,N012314s);
for i in [1..#T012314] do
ss:=[4,1,2,3,1,4]^T012314[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if
eq g*(ts[4]*ts[1]*ts[3]*ts[2]*ts[4])^h
then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,4)(2,3) then Sch[i];
end if; end for;
xx^yy;
f(x^y)*ts[3]*ts[1]*ts[2]*ts[4]*ts[3];
-------------------------------------------------------------------
N413241:=Stabiliser(N,[4,1,3,2,4,1]);
SSSSSS:={[4,1,3,2,4,1]}; SSSSSS:=SSSSSS^N;
SSSSSS;
#SSSSSS;
Seqq:=Setseq(SSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
 ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N013241s:=N413241;
for n in N do if 4^n eq 2 and 1^n eq 3 and 2^n eq 4 and 3^n
eq 1 then N013241s:=sub<N|N013241s,n>; end if; end for;
N013241s; #N013241s;
T013241:=Transversal(N,N013241s);
for i in [1..#T013241] do
ss:=[4,1,3,2,4,1]^T013241[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if 
then g, h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i]; 
end if; end for;

-----------------------------------------------------
for i in [1..4] do for g in IN do for h in IN do if 
then i; break; end if; end for; end for; end for;
for g in IN do for h in IN do if 
then g, h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,2)(3,4) then Sch[i]; 
end if; end for;
xx;

-----------------------------------------------------
N413243:=Stabiliser(N, [4,1,3,2,4,3]);
SSSSSS:= {[4,1,3,2,4,3]} ; SSSSSS:= SSSSSS^N;
SSSSSS;
#SSSSSS;
Seqq:= Setseq(SSSSSS);
Seqq;
for i in [1..#SSSSSS] do
for n in IN do
n * ts[Rep(Seqq[i])][1] * ts[Rep(Seqq[i])][2] * ts[Rep(Seqq[i])][3] * 
    ts[Rep(Seqq[i])][4] * ts[Rep(Seqq[i])][5] * ts[Rep(Seqq[i])][6] 
then print Rep(Seqq[i]); 
    end if; end for; end for;
N013243s:= N413243;
T013243:= Transversal(N, N013243s);
for i in [1..#T013243] do
ss:= [4,1,3,2,4,3]^T013243[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:= 0; for i in [1..165] do if cst[i] ne []
then m:= m+1; end if; end for; m;
for g in IN do for h in IN do if 
then \(g, h\); break; end if; end for; end for;
for i in \([1..12]\) do if ArrayP[i] eq \(N!(2, 4)(1, 3)\) then Sch[i]; end if; end for;
yy \ast xx \ast y^{-1};
-------------------------------------------------------------------
N4124131:=Stabiliser(N, \([4, 1, 2, 4, 1, 3, 1]\));
SSSSSSS:={\([4, 1, 2, 4, 1, 3, 1]\)}; SSSSSSS:=SSSSSSS\^{N};
SSSSSSSS;
#SSSSSSSS;
Seqq:=Setseq(SSSSSSSS);
Seqq;
for i in \([1..\#SSSSSSSS]\) do
for n in IN do
if 
n \ast ts[Rep(Seqq[i])[1]] \ast ts[Rep(Seqq[i])[2]] \ast ts[Rep(Seqq[i])[3]] \ast
\[ts[Rep(Seqq[i])[4]] \ast ts[Rep(Seqq[i])[5]] \ast ts[Rep(Seqq[i])[6]] \ast
\[ts[Rep(Seqq[i])[7]]\]
then print Rep(Seqq[i]); end if; end for; end for;
N0120131s:=N4124131;
T0120131:=Transversal(N,N0120131s);
for i in \([1..\#T0120131]\) do
ss:=[\([4, 1, 2, 4, 1, 3, 1]\)] \^{T0120131[i]};
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in \([1..165]\) do if cst[i] ne []
then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if 
then \(g, h\); break; end if; end for; end for;
for i in \([1..12]\) do if ArrayP[i] eq \(N!(1, 4)(2, 3)\) then Sch[i]; end if; end for;
xx \^{-yy};
-------------------------------------------------------------------
N4124132:=Stabiliser(N, \([4, 1, 2, 4, 1, 3, 2]\));
SSSSSSSS:=\{\([4, 1, 2, 4, 1, 3, 2]\)}; SSSSSSS:=SSSSSSSS\^{N};
Seqq; for i in [1..#SSSSSSS] do
for n in IN do
then print Rep(Seqq[i]);
end if; end for; end for;
end if; end for; end for;
N0120132s := N4124132;
for n in N do if 4^n eq 2 and 1^n eq 1 and 2^n eq 3 and 3^n eq 4 then N0120132s := sub<N|N0120132s, n>; end if; end for;
N0120132s; #N4124132;
N0120132s := N4124132;
for n in N do if 4^n eq 3 and 1^n eq 1 and 2^n eq 4 and 3^n eq 2 then N0120132s := sub<N|N0120132s, n>; end if; end for;
N0120132s; #N4124132;
T0120132 := Transversal(N, N0120132s);
for i in [1..#T0120132] do
ss := [4, 1, 2, 4, 1, 3, 2]^T0120132[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne []
then m := m + 1; end if; end for;
N0120132s; #N0120132s;
Orbits(N0120132s);

N4124134 := Stabiliser(N, [4, 1, 2, 4, 1, 3, 4]);
SSSSSSS := {[4, 1, 2, 4, 1, 3, 4]}; SSSSSSS := SSSSSSS ^ N;
SSSSSSS;
#SSSSSSS;
Seqq := Setseq(SSSSSSS);
Seqq;
for i in [1..#SSSSSSS] do
for n in IN do
then print Rep(Seqq[i]);
end if; end for; end for;
N0124134s:=N4124134;
T0124134:=Transversal(N,N0124134s);
for i in [1..#T0124134] do
  ss:=[4,1,2,4,1,3,4]ˆT0124134[i];
cst[prodim(1, ts, ss)]:=ss;
end for;

m:=0; for i in [1..165] do if cst[i] ne []
  then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if
eq g*(ts[4]*ts[1]*ts[2]*ts[3]*ts[4])ˆh
  then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1,2)(3,4) then Sch[i];
end if; end for;

for i in [1..4] do for g in IN do for h in IN do if
  g*(ts[4]*ts[1]*ts[3]*ts[2]*ts[4])ˆh
  then i; break; end if; end for; end for;end for;
for g in IN do for h in IN do if
eq g*(ts[4]*ts[1]*ts[3]*ts[2]*ts[4])ˆh
  then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i];
end if; end for;
eq ts[1]*ts[3]*ts[4]*ts[2]*ts[1];

for i in [1..4] do for g in IN do for h in IN do if
  g*(ts[4]*ts[1]*ts[3]*ts[2]*ts[4])ˆh
  then i; break; end if; end for; end for;end for;
for g in IN do for h in IN do if
eq g*(ts[4]*ts[1]*ts[3]*ts[2]*ts[4])ˆh
  then i; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i];
end if; end for;
eq ts[2]*ts[4]*ts[3]*ts[1]*ts[2];
for g in IN do for h in IN do if
then g, h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(1, 3)(2, 4) then Sch[i];
end if; end for;
yy * xx * yy^-1;
-------------------------------------------------------------------
N4123123 := Stabiliser(N, [4, 1, 2, 3, 1, 2, 3]);
SSSSSSSS := {{4, 1, 2, 3, 1, 2, 3}}; SSSSSSS := SSSSSSS ^ N;
SSSSSS;
#SSSSSS;
Seqq := Setseq(SSSSSSSS);
Seqq;
for i in [1..#SSSSSSSS] do
for n in IN do
n * ts[Rep(Seqq[i])][1] * ts[Rep(Seqq[i])][2] * ts[Rep(Seqq[i])][3] *
ts[Rep(Seqq[i])][4] * ts[Rep(Seqq[i])][5] * ts[Rep(Seqq[i])][6] *
ts[Rep(Seqq[i])][7]
then print Rep(Seqq[i]);
end if; end for; end for;
N0123123s := N4123123;
T0123123 := Transversal(N, N0123123s);
for i in [1..#T0123123] do
ss := [4, 1, 2, 3, 1, 2, 3]^T0123123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m := 0; for i in [1..165] do if cst[i] ne []
then m := m + 1; end if; end for; m;
for g in IN do for h in IN do if
then g, h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!(2, 4)(1, 3) then Sch[i];
end if; end for;
yy * xx * yy^-1;
-------------------------------------------------------------------
N4123124 := Stabiliser(N, [4, 1, 2, 3, 1, 2, 4]);
SSSSSSSS := {{4, 1, 2, 3, 1, 2, 4}}; SSSSSSS := SSSSSSS ^ N;
SSSSSSS;
#SSSSSSS;
Seqq:=Setseq(SSSSSSS);
Seqq;
for i in [1..#SSSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]*ts[Rep(Seqq[i])[7]]
then print Rep(Seqq[i]);
end if; end for; end for;
N0123124s:=N4123124;
for n in N do if 4^n eq 2 and 1^n eq 4 and 2^n eq 1 and 3^n eq 3 then N0123124s:=sub<N,N0123124s,n>; end if; end for;
N0123124s:=N4123124;
for n in N do if 4^n eq 1 and 1^n eq 2 and 2^n eq 4 and 3^n eq 3 then N0123124s:=sub<N,N0123124s,n>; end if; end for;
N0123124s; #N0123124s;
T0123124:=Transversal(N,N0123124s);
for i in [1..#T0123124] do
ss:=[4,1,2,3,1,2,4]^T0123124[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N0123124s);
#N0123124s;

N41241321:=Stabiliser(N,[4,1,2,4,1,3,2,1]);
SSSSSSSSS:={[4,1,2,4,1,3,2,1]}; SSSSSSSS:=SSSSSSSSS^N;
SSSSSSS;
#SSSSSSS;
Seqq:=Setseq(SSSSSSSSS);
Seqq;
for i in [1..#SSSSSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]*ts[Rep(Seqq[i])[7]]*ts[Rep(Seqq[i])[8]]
then print Rep(Seqq[i]);
end if; end for; end for;
N01241321s:=N41241321;
for n in N do if 4^n eq 2 and 1^n eq 4 and 2^n eq 1 and 3^n eq 3 then N01241321s:=sub<N|N01241321s,n>; end if; end for;
N01241321s; #N01241321s;
for n in N do if 4^n eq 4 and 1^n eq 3 and 2^n eq 1 and 3^n eq 2 then N01241321s:=sub<N|N01241321s,n>; end if; end for;
N01241321s; #N01241321s;
T01241321:=Transversal(N,N01241321s);
for i in [1..#T01241321] do
  ss:=[4,1,2,4,1,3,2,1]ˆT01241321[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne [] then m:=m+1; end if; end for; m;
Orbits(N01241321s);

N41231241:=Stabiliser(N,[4,1,2,3,1,2,4,1]);
SSSSSSSS:={[4,1,2,3,1,2,4,1]}; SSSSSSS:=SSSSSSSSˆN;
SSSSSSSS;
#SSSSSSSS;
Seqq:=Setseq(SSSSSSSS);
Seqq;
for i in [1..#SSSSSSSS] do
  for n in IN do
      ts[Rep(Seqq[i])][4]*ts[Rep(Seqq[i])][5]*ts[Rep(Seqq[i])][6]*
      ts[Rep(Seqq[i])][7]*ts[Rep(Seqq[i])][8] then print Rep(Seqq[i]); end if; end for; end for;
N01231241s:=N41231241;
T01231241:=Transversal(N,N01231241s);
for i in [1..#T01231241] do
  ss:=[4,1,2,3,1,2,4,1]ˆT01231241[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0; for i in [1..165] do if cst[i] ne [] then m:=m+1; end if; end for; m;
for g in IN do for h in IN do if ts[4]*ts[1]*ts[2]*ts[3]*ts[1]*ts[2]*ts[4]*ts[1] eq g*(ts[4]*ts[1]*ts[2]*ts[3]*ts[1]*ts[2])^h then g,h; break; end if; end for; end for;
for i in [1..12] do if ArrayP[i] eq N!Id(N) then Sch[i]; end if; end for;
N41231243:=Stabiliser(N,[4,1,2,3,1,2,4,3]);
SSSSSSSS:=\{[4,1,2,3,1,2,4,3]\};
SSSSSSSS:=SSSSSSSS^N;
SSSSSSSS;
#SSSSSSSS;
Seqq:=Setseq(SSSSSSSS);
Seqq;
for i in [1..#SSSSSSSS] do
for n in IN do
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
 ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]*
 ts[Rep(Seqq[i])[7]]*ts[Rep(Seqq[i])[8]]
then print Rep(Seqq[i]);
end if;
end for;
end for;
N01231243s:=N41231243s;
for n in N do
if 4^n eq 4 and 1^n eq 3 and 2^n eq 1 and 3^n
eq 2 then N01231243s:=sub<N|N01231243s,n>; end if;
end for;
N01231243s;
for n in N do
if 4^n eq 2 and 1^n eq 4 and 2^n eq 1 and 3^n
eq 3 then N01231243s:=sub<N|N01231243s,n>; end if;
end for;
N01231243s;
T01231243:=Transversal(N,N01231243s);
for i in [1..#T01231243] do
ss:=[4,1,2,3,1,2,4,3]^T01231243[i];
cst[pordim(1, ts, ss)]:=ss;
end for;
m:=0;
for i in [1..165] do
if cst[i] ne []
then m:=m+1;
end if;
end for;
m;
Orbits(N01231243s);
Appendix D: MAGMA Code for Isomorphism Type of $L_2(11) \times 3$

```
S:=Sym(4);
xx:=S!(1,2)(3,4);
yy:=S!(1,2,3);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,(y^-1*x^-1*y*x)^2,t^2,
(t,y),(x*t)^5>;
G1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
Center(G1);
NL:=NormalLattice(G1); NL;
MinimalNormalSubgroups(G1);
D:=DirectProduct(CyclicGroup(3),PSL(2,11));
s:=IsIsomorphic(D,G1); s;
a:=NL[3].1;
b:=NL[3].2;
Order(a);
Order(b);
Order(a*b);
H<a,b>:=Group<a,b|a^2,b^3,(a*b)^11,(a,b*a*b*a*b)^2>; #H;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(H1,PSL(2,11));
s;
HH<a,b,c>:=Group<a,b,c|a^2,b^3,(a*b)^11,(a,b*a*b*a*b)^2,
c^3,(a,c),(b,c)>;
HH;
f2,H2,k2:=CosetAction(HH,sub<HH|Id(HH)>);
s:=IsIsomorphic(H2,G1);
s;
```
Appendix E: MAGMA Code for Factoring $L_2(11) \times 3$ by the center of $G$

\begin{verbatim}
S:=Sym(4);
x:=S!(1,2)(3,4);
y:=S!(1,2,3);
N:=sub<S|xx,yy>;
Stabiliser(N,4);
#N;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^3,(y^-1*x^-1*y*x)^2,t^2,
(t,y),(x*t)^5>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
Index(G,sub<G|x,y>);
Sch;
Center(G1);
No:=NormalSubgroups(G1);
No;
Center(G1).1;
f(t*(t^x)^y*((t^x)^y)^y*t^x*(t^x)^y*((t^x)^y)^y*t*x*(t^x)^y*((t^x)^y)^y*t*x);\end{verbatim}
#G;
#DoubleCosets(G, sub<G|x,y>, sub<G|x,y>);
Index(G, sub<G|x,y>);
Appendix F: MAGMA Code for Isomorphism Type of $PGL_2(11)$

```
S := Sym(5);
xx := S!(1,2,3,4,5);
yy := S!(1,4)(2,3);
N := sub<S|xx,yy>;
N0 := Stabiliser(N,5);
N0;
G<x,y,t> := Group<x,y,t|x^5,y^2,(x*y)^2,t^2,(t,y),(x*t)^5,
(x*t*(t*x))^3>;
#G;
f, G1, k := CosetAction(G, sub<G|x,y>);
CompositionFactors(G1);
Q := {};
for i in [1..10] do Q := Q join {i^2 mod 11}; end for;
Q;
T := {};
for i in Q do T := T join {i^2 mod 11}; end if; end for;
/* the following code will give all the 11 elements
inverses by replacing i with all 11 elements each time*/
F := GaloisField(11);
F!1^1; F!2^1; F!3^1; F!4^1; F!5^1;
F!6^1; F!7^1; F!8^1; F!9^1; F!10^1;
/* let k=9 */
S := Sym(12);
alpha := S!(1,2,3,4,5,6,7,8,9,10,11);
beta := S!(1,9,4,3,5)(2,7,8,6,10);
gamma := S!(12,11)(1,10)(2,5)(3,7)(4,8)(6,9);
/* N here is L(2,11) because L(2,11)=<alpha, beta, gamma>
and PGL(2,11)=<alpha, beta, gamma, delta>, delta here
is equal to (aut) which is a map that takes (x) to (-x) */
N := sub<S|alpha,beta,gamma>;
#N;
```
s, u := IsIsomorphic(PSL(2, 11), N);
s;
aut := S!(1, 10) (2, 9) (3, 8) (4, 7) (5, 6);
pgl211 := sub<S|N, aut>;
#pgl211;
s1, u1 := IsIsomorphic(PGL(2, 11), pgl211);
s1;
s2, u2 := IsIsomorphic(G1, pgl211);
s2;
u2;
xx := pgl211!(1, 5, 8, 11, 4)(6, 7, 12, 9, 10);
yy := pgl211!(1, 5) (2, 3) (4, 8) (6, 7)(10, 12);
tinf := pgl211!(1, 4)(2, 6)(3, 7)(5, 8)(9, 11)(10, 12);
MaximalSubgroups(pgl211);
t1 := tinf^xx; t1;
t2 := t1^xx; t2;
t3 := t2^xx; t3;
t0 := t3^xx; t0;
tinf;
t0*t3*t2*t1*tinf;
xx^5;
Appendix G: MAGMA Code for Progenitor $2^8 \times (\mathbb{Z}_4 \times \mathbb{Z}_2) \cdot D_4$

```magma
NumberOfTransitiveGroups(8);
N:=TransitiveGroup(8,27);
D:=SmallGroupDatabase();
IdentifyGroup(N);
G:=SmallGroup(D,64,32);
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL:=Subgroups(G1);
T:= {X'subgroup: X in SL};
#T;
TrivCore:={H:H in T | #Core(G1,H) eq 1};
#TrivCore;
mdeg:=Min({Index(G1,H):H in TrivCore});
Good:={H: H in TrivCore | Index(G1,H) eq mdeg};
#Good;
H:=Rep(Good);
f,G1,K:=CosetAction(G1,H);
G1;
FPGroup(G);
G<a,b,c,d,e,r>:=Group<a,b,c,d,e,r|a^2=d,b^2,c^2,d^2,e^2,r^2,
b^a=b*c,c^a=c*e,c*b=c,d^a=d,d*b=d*e,d*c=d*r,e^a=e*r,e*b=e,
e*c=e,e*d=e,r^a=r,r*b=r,r^c=r,r*d=r,r*e=r>;
#G;
s:=IsIsomorphic(G1,N);
s;
//* Now, we have a permutaion representation that isomorphic to our transitive group. We will label the previous permutation as a, b, c, d, e, and r. */
A:=G1!(1, 2)(3, 7)(4, 5, 8, 6);
B:=G1!(1, 3)(2, 5)(4, 8)(6, 7);
C:=G1!(1, 4)(2, 6)(3, 8)(5, 7);
```
\[ D:=G_1!(4,\ 8)(5,\ 6); \]
\[ E:=G_1!(2,\ 7)(5,\ 6); \]
\[ R:=G_1!(1,\ 3)(2,\ 7)(4,\ 8)(5,\ 6); \]
\[ N:=\text{sub}<G_1|A,B,C,D,E,R>; \]

\[ \text{//} \text{Now, we must stabilise an element t.} \text{//} \]
\[ N_0:=\text{Stabiliser}(N, 8); \]
\[ N_0; \]

\[ \text{//} \text{We must write all the elements of N0 in terms of a, b, c, d, e,} \]
\[ \text{and r by using Schreier system.} \text{//} \]
\[ \text{NN}<a,b,c,d,e,r>:=\text{Group}<a,b,c,d,e,r|a^2=d,b^2,c^2,d^2,e^2,} \]
\[ r^2,b^a=b*c,c^a=c*e,c^b=c,d^a=d,d^b=d*e,d^c=d*r,e^a=e*r,} \]
\[ e^b=e,e^c=e,e^d=e,r^a=r,r^b=r,r^c=r,r^d=r,r^e=r>; \]
\[ \text{Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);} \]
\[ \text{ArrayP:=[Id(N): i in [1..64]];} \]
\[ \text{for i in [2..64] do} \]
\[ \text{P:=[Id(N): l in [1..#Sch[i]]];} \]
\[ \text{for j in [1..#Sch[i]] do} \]
\[ \text{if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq -1 then P[j]:=A\^-1; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;} \]
\[ \text{if Eltseq(Sch[i])[j] eq 6 then P[j]:=R; end if;} \]
\[ \text{end for;} \]
\[ \text{PP:=Id(N);} \]
\[ \text{for k in [1..#P] do} \]
\[ \text{PP:=PP*P[k]; end for;} \]
\[ \text{ArrayP[i]:=PP;} \]
\[ \text{end for;} \]
\[ \text{for i in [1..64] do} \]
\[ \text{if ArrayP[i] eq N!(2, 7)(5, 6) then print Sch[i];} \]
\[ \text{end if;} \]
\[ \text{end for;} \]
\[ \text{for i in [1..64] do} \]
\[ \text{if ArrayP[i] eq N!(2, 5)(6, 7) then print Sch[i];} \]
\[ \text{end if;} \]
\[ \text{end for;} \]
\[ \text{for i in [1..64] do} \]
\[ \text{if ArrayP[i] eq N!(1, 3)(2, 5, 7, 6) then print Sch[i];} \]
\[ \text{end if;} \]
\[ \text{end for;} \]
\[ \text{Set(N0);} \]

\[ \text{//} \text{we will utilize the lemma by taking the stabiliser} \]
\[ \text{of two elements and determining the centraliser of} \]
\[ \text{those points.} \text{//} \]
\[ \text{N01:=Stabiliser(G1,[8,1]);} \]
N01;
C:=Centraliser(G1,N01);
C;
Set(C);

/* The lemma deals with elements that have a permutation (8,1) by itself or do not have either 1 or 8. By running Set(C), we find that there is four elements that have a permutaion (8,1) together and by itself such as (1, 8)(2, 6)(3, 4)(5, 7),
(1, 8)(2, 5)(3, 4)(6, 7), (1, 8)(3, 4), and
(1, 8)(2, 7)(3, 4)(5, 6). */
for i in [1..64] do
  if ArrayP[i] eq N!(1, 8)(2, 6)(3, 4)(5, 7) then print Sch[i];
  end if;
end for;
for i in [1..64] do
  if ArrayP[i] eq N!(1, 8)(2, 5)(3, 4)(6, 7) then print Sch[i];
  end if;
end for;
for i in [1..64] do
  if ArrayP[i] eq N!(1, 8)(3, 4) then print Sch[i];
  end if;
end for;
for i in [1..64] do
  if ArrayP[i] eq N!(1, 8)(2, 7)(3, 4)(5, 6) then print Sch[i];
  end if;
end for;

/* Thus, we have have to apply the lemma as the following:
(a*c*a^-1*t)^k=1, (c*r)^k=1, (b*c*e)^k=1, or (b*c)^k=1,
where k is odd. */
S:=Sym(8);
aa:=G1!(1, 2)(3, 7)(4, 5, 8, 6);
bb:=G1!(1, 3)(2, 5)(4, 8)(6, 7);
cc:=G1!(1, 4)(2, 6)(3, 8)(5, 7);
dd:=G1!(4, 8)(5, 6);
ee:=G1!(2, 7)(5, 6);
rr:=G1!(1, 3)(2, 7)(4, 8)(5, 6);
N:=sub<G1|aa,bb,cc,dd,ee,rr>;

/* Now, we have to insert t and with which commute into our progenitors.
Note: t commutes with each generator of the one point stabiliser. */
for k in [1..5] do
  G<a,b,c,d,e,r,t>:=Group<a,b,c,d,e,r,t|a^2=d,b^2,c^2,d^2,e^2,r^2,b*a=c,e,c*a=c*e,c*b=c,d^a=d,b^d=d*e,d^c=d*r,
e^a*e^r,e*b=e,e^c=e,e^d=e,r^a=r,r^b=r,r^c=r,r^d=r,
r^e=r,r^2,(t,e),(t,b*e*r),(t,d*b),(b*c*e*t)^k>; #G;
end for;
Appendix H: MAGMA Code for Isomorphism Type of 
\(((\mathbb{Z}_4 \times \mathbb{Z}_2) \cdot D_4)\)

```magma
S := Sym(8);
aa := S!(1, 2)(3, 7)(4, 5, 8, 6);
bb := S!(1, 3)(2, 5)(4, 8)(6, 7);
cc := S!(1, 4)(2, 6)(3, 8)(5, 7);
dd := S!(4, 8)(5, 6);
ee := S!(2, 7)(5, 6);
rr := S!(1, 3)(2, 7)(4, 8)(5, 6);
N := sub<S|aa, bb, cc, dd, ee, rr>;
NL := NormalLattice(N);
IsAbelian(NL[4]);
A := N!(1, 4, 3, 8)(2, 6, 7, 5);
B := N!(2, 7)(5, 6);
NL4 := sub<N|A, B>;
T := Transversal(N, NL4);
q, ff := quo<N|NL4>;
IsAbelian(q);
Center(q);
nl := NormalLattice(q);
nl;
X1 := AbelianGroup(GrpPerm, [4, 2]);
IsIsomorphic(X1, NL[4]);
NumberOfGenerators(NL[4]);
A := N!(1, 4, 3, 8)(2, 6, 7, 5);
B := N!(2, 7)(5, 6);
NL4 := sub<N|A, B>;
```
q, ff := quo<N|NL4>;
NumberOfGenerators(q);
q = sub<q|q.1, q.2>;
q1 := q.1; q2 := q.2;
T := Transversal(N, NL4);
T[1];
T2 := T[2]; T2;
T3 := T[3]; T3;
T23;
/* (1, 5, 4, 2, 3, 6, 8, 7) */
q.1 * q.2;
/* (1, 4)(2, 3) */
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
q eq sub<q|q1, q2>;
ff(T[2]) eq q1;
ff(T[3]) eq q2;
NN<a, b> := Group<a, b|a^4, b^2, (a, b)>;
Sch := SchreierSystem(NN, sub<NN|Id(NN)>);
ArrayP := [Id(NL4) : i in [1..8]];
for i in [2..8] do
P := [Id(NL4) : l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j] := A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j] := A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j] := B; end if;
PP := Id(NL4);
for k in [1..#P] do
PP := PP * P[k]; end for;
ArrayP[i] := PP;
end for;
end for;
I := [Id(NN) : i in [1..5]];
for i in [1..8] do if ArrayP[i] eq T23^2 then Sch[i];
I[1] := Sch[i]; end if; end for;
for i in [1..8] do if ArrayP[i] eq A^T2 then Sch[i];
I[2] := Sch[i]; end if; end for;
for i in [1..8] do if ArrayP[i] eq B^T2 then Sch[i];
I[3] := Sch[i]; end if; end for;
for i in [1..8] do if ArrayP[i] eq A^T3 then Sch[i];
I[4] := Sch[i]; end if; end for;
for i in [1..8] do if ArrayP[i] eq B^T3 then Sch[i];
I[5] := Sch[i]; end if; end for;
I;
H<x,y,z,w>:=Group<x,y,z,w|x^4,y^2,(x,y),z^4,w^2,
(z*w)^2=x,x^z=x*y, y^z=x^2*y,x*w=x*y,y*w=y>;
#H;

f,h,k:=CosetAction(H,sub<H|Id(H)>>);
#h;
s:=IsIsomorphic(N,h);
s;

.FullName := "A presentation for the progenitor is \nG<a,b,c,d,e,r,t>:=Group<a,b,c,d,e,r,t|a^2=d,b^2,c^2,
d^2,e^2,r^2,b*a=b*c,c^a=c*e,c^b=c,d^a=d,d^b=d*e,
d^c=d*r,e^a=e*r,e^b=e,e^c=e,e^d=e,r^a=r,r^b=r,r^c=r,
r^d=r,r^e=r,t^2,(t,e),(t,b*e*r),(t,d*b)>;";
Appendix I: MAGMA Code for Isomorphism Type of $2^\cdot PGL_2(11)$

// We now prove that $G_1$ is central extension of $\mathbb{Z}_2$
by $PGL(2,11)$.*//
S:=Sym(8);
aa:=S!(1, 2)(3, 7)(4, 5, 8, 6);
bv:=S!(1, 3)(2, 5)(4, 8)(6, 7);
c:=S!(1, 4)(2, 6)(3, 8)(5, 7);
dd:=S!(4, 8)(5, 6);
e:=S!(2, 7)(5, 6);
rr:=S!(1, 3)(2, 7)(4, 8)(5, 6);
N:=sub<S|aa,bv,cc,dd,ee,rr>;
G<a,b,c,d,e,r,t>:=Group<a,b,c,d,e,r,t|a^2=d,b^2,c^2,d^2,e^2,
r^2,b^a=b*c,c^a=c*e,c^b=c,d^a=d,d^b=d*e,d^c=d*r,e^a=e*r,
e^b=e,e^c=e,e^d=e,r^a=r,r^b=r,r^c=r,r^d=r,r^e=r,t^2,
(t,e), (t,b*e*r), (t,d*b), (b*c*e*t)^3, (e*b*t^a*t)^5>;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL:=Subgroups(G1);
T:={X\subseteq G: X in SL};
TrivCore:={H: H in T | Core(G1,H) eq 1};
mdeg:=Min({Index(G1,H): H in TrivCore});
Good:={H: H in TrivCore | Index(G1,H) eq mdeg};
H:=Rep(Good);
f,G1,k:=CosetAction(G1,H);
#G; #k;
NL:=NormalLattice(G1);
q,ff:=quo<G1|NL[2]>;
H<x,y,z,w>:=Group<x,y,z,w|x^2,y^2,z^2,w^2, (x*y)^4, (x*z)^2,
(x*w)^10, (y*z)^2, (y*w)^2, (z*w)^6, (x,y)^2, (x,w)^5, (z,w)^3,
(\cdot x)^4, (\cdot x)^12, (\cdot y)^10, (\cdot y)^2, (\cdot z)^10, (\cdot x)^5,
(\cdot x)^5, (\cdot y)^5, (\cdot x)^5, (\cdot y)^3, (\cdot z)^5, (\cdot y)^5,
(\cdot x)^6, (\cdot x)^2, (\cdot w)^2, (\cdot x^2 y^2 z^2) (\cdot (x*w)^2 y^2 z^2)^5, (\cdot x)^5,
(x*y*z*w)^12, (x*y*(z*w)^2)^10, (x*z*y*w*x*y)^2,
(x*w*z+y*w*x)^2, (w*x*y*z+w*z+y*y*w)^2, (x,y*z*w*y*z*w*z+x*w)^2,
(x*y*z+y*w)^10, (y*x*w*z+x*y*w*z)^5, (y,x^z+y*z+w*x+z*x*y*w)^3,
(x*y*z+y*x*w*y*w*z*y*w*x)^10, (x*y*z*w,y*z*x*w*y*z*w*y)^2,
(x*y+z, w*y*x)^5, (x*z+y*w,w*x*z+y)^3,
(x*z*y*w*x*w*z*y*w*z+y*x*w)^5, (x*w*y*z+w*z+y*x*w)^5>;

f,h,k:=CosetAction(H,sub<H|Id(H)>);
#h;

s:=IsIsomorphic(q,h);
T:=Transversal(G1,NL[2]);
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
ff(T[4]) eq q.3;
ff(T[5]) eq q.7;
Order(T[2]), Order(T[3]), Order(T[4]), Order(T[5]);
Order(q.1);
T[2]^2;
NL[2];

K<x,y,z,w,c>:=Group<x,y,z,w,c|x^2=c, y^2, z^2, w^2, (x*y)^4,
(x*z)^2=c, (x*w)^10=c, (y*z)^2, (y*w)^2, (z+w)^6, (x,y)^2, (x,w)^5,
(z,w)^3, (x*y*z)^4, (x*y*w)^12, (x+z+w)^10=c, (x^y*z)^2=c,
(x*z*w)^10=c, (x*y, z, (x*z, w)^5, (x*y*z, w)^5, (x*y*w, z)^3,
(x*y, z*w)^5, (x*z, y*w)^6, (x, w, z)^2, (w, z, y)^2,
((x*w)^2*y*z)^5=c, (x*y*z*w)^12, (x*y*(z+w)^2)^10=c,
(x*z*y*w*x*y)^2, (x*w*z*y*w*x)^2, (w*x*y*w*z*x*y*w)^2,
(x, y, z, w, y*z*w*x)^2, (x*y*z*w*y*z^w)^10=c,
(y*x*w*z*x*y*w)^5=c, (x, z*y*z*w*z*x*y*w)^3,
(x*y*z+y*x*w*y*z+y*w*x)^10=c, (x*y*z*w, y*z*x*w*y*z*w*y)^2,
(x*y*z, w*y*x)^5, (x*z+y*w, w*x*z+y)^3,
(x*z+y*w*x*w*z*y*w*z+y*x*w)^5, (x*w*y*z+w*z+y*x*w)^5=c>;
f,kk,k:=CosetAction(K,sub<K|Id(K)>);
#kk;
s:=IsIsomorphic(G1,kk);
Appendix J: MAGMA Code for Isomorphism Type of \((A_5 \times A_5) : 4\)

\begin{verbatim}
S := Sym(8);
aa := S!(1, 2)(3, 7)(4, 5, 8, 6);
bb := S!(1, 3)(2, 5)(4, 8)(6, 7);
c := S!(1, 4)(2, 6)(3, 8)(5, 7);
dd := S!(4, 8)(5, 6);
e := S!(2, 7)(5, 6);
rr := S!(1, 3)(2, 7)(4, 8)(5, 6);
N := sub<S|aa, bb, cc, dd, ee, rr>;
G<a, b, c, d, e, r, t> := Group<a, b, c, d, e, r, t | a^2 = d, b^2 = c^2, d^2, e^2, r^2, b^a = b*c, c^a = c*e, c*b = c, d^a = d*e, d*c = d*r, e^a = e*r, e*b = e*c = e*d = e, r*a = r*c, r*b = r*c = r*d = r*e = r, t^2, (t, e), (t, b*e*r), (t, d*b), (b*c*e*t)^5, (e*r*t)^3, (a*(t^r)^c)^4>;
f, G1, k := CosetAction(G, sub<G|a, b, c, d, e, r>);
#G; #k;
CompositionFactors(G1);
Center(G1);
NL := NormalLattice(G1);
NL;
MinimalNormalSubgroups(G1);
D := DirectProduct(Alt(5), Alt(5));
s := IsIsomorphic(D, NL[2]);
s;
H<a, b, c, d> := Group<a, b, c, d | a^3, b^2, (a*b)^5, c^3, d^2, (c*d)^5, (a, c), (a, d), (b, c), (b, d)>;
f, H1, k := CosetAction(H, sub<H|Id(H)>);
#H;
s, t := IsIsomorphic(H1, NL[2]);
s;
a := t(f(a));
b := t(f(b));
\end{verbatim}
c:=t(f(c));
d:=t(f(d));
for g in G1 do if Order(g) eq 4 and g notin NL[2] and G1 eq sub<G1|NL[2],g then U:=g; break; end if; end for;
U;
G1 eq sub<G1|NL[2],U>;
N:=sub<G1|a,b,c,d>;
#N;
N eq NL[2];
NN<i,j,l,m>:=Group<i,j,l,m|i^3,j^2,(i*j)^5,l^3,m^2,(l*m)^5,
(i,l),(i,m),(j,l),(j,m)>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..3600]];
for i in [2..3600] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=a; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=a^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=b; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=c; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=c^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=d; end if;
end for;
PP:=Id(N); for k in [1..#P] do PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
A:=[Id(NN): i in [1..3]]; for i in [1..3600] do if a^U eq ArrayP[i] then A[1]:=Sch[i]; end if; for i in [1..3600] do if b^U eq ArrayP[i] then A[2]:=Sch[i]; end if; for i in [1..3600] do if c^U eq ArrayP[i] then A[3]:=Sch[i]; end if; for i in [1..3600] do if d^U eq ArrayP[i] then A[4]:=Sch[i]; end if; end for;
HH<a,b,c,d,e>:=Group<a,b,c,d,e|a^3,b^2,(a*b)^5,c^3,d^2,
Appendix K: MAGMA Code for
The Progenitor $2^6 : (\mathbb{Z}_3 \wr \mathbb{Z}_2)$

G<a,b,c>:=Group<a,b,c|a^3,b^3,c^2,(a,b),a^c=b, b^c=a>;  
#G;  
W:=WreathProduct(CyclicGroup(3), CyclicGroup(2));  
#W;  
CC:=Classes(W);  
CC;  
for i in [1..#CC] do if CC[i][1] eq 2 then i; end if; end for;  
for i in [1..#CC] do if CC[i][1] eq 3 then i; end if; end for;  
for C in Class(W, CC[2][3]) do for A,B in Class(W, CC[5][3])  
join Class(W, CC[6][3]) do if Order(A) eq 3 and Order(B) eq 3  
and Order(C) eq 2 and (A,B) eq Id(W) and A^C eq B and B^C eq A  
and W eq sub<W|A,B,C> then A,B,C; break; end if; end for;  
end for;  
Generators(G);  
S:=Sym(6);  
A:=S!(4, 5, 6);  
B:=S!(1, 2, 3);  
C:=S!(1, 5)(2, 6)(3, 4);  
N:=sub<S|A,B,C>;  
N eq W;  
N1:=Stabiliser(N,1);  
N1;  
#N;  
NN:=G;  
Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);  
ArrayP:=[Id(N): i in [1..18]];  
for i in [2..18] do  
P:=[Id(N): l in [1..#Sch[i]]];  
for j in [1..#Sch[i]] do  
if Eltseq(Sch[i][j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A\(^{-1}\); end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B\(^{-1}\); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
  PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..18] do if ArrayP[i] eq N!(4, 5, 6)
  then print Sch[i]; end if; end for;
H<A,B,C,t>:=Group<A,B,C,t|A\(^3\), B\(^3\), C\(^2\), (A,B),
A\(^C\)=B, B\(^C\)=A, t\(^2\), (t,A)>;
#H;
//* with lemma *//
N0:=Stabiliser(N,6);
N0;
N01:=Stabiliser(N,[6,1]);
N01;
Centraliser(N,N01);
Set(Centraliser(N,N01));
#N;
for i in [1..18] do if ArrayP[i] eq N!(1,6)(2,4)(3,5)
  then print Sch[i]; end if; end for;
for k in [0..5] do
  H<A,B,C,t>:=Group<A,B,C,t|A\(^3\), B\(^3\), C\(^2\), (A,B), A\(^C\)=B,
  B\(^C\)=A, t\(^2\), (t,B),(B*C*B\(^{-1}\)*t)\(^k\)>; #H; end for;
#H;
Appendix L: MAGMA Code for Monomial Progenitor

$3^{12} : m \ (2 \cdot A_5)$

```magma
S:=Sym(24);
a:=S!(1,2,5,4)(3,6,8,7)(9,13,11,14)(10,15,12,16)(17,19,18,20)(21,24,23,22);
b:=S!(1,3,2)(4,5,8)(6,9,10)(7,11,12)(13,16,17)(14,15,18)(19,21,22)(20,23,24);
G1:=sub<S|a,b>;
S:=Subgroups(G1);
#S;
for i in [1..#S] do i, Index(G1,S[i]~'subgroup); end for;
CT:=CharacterTable(G1);CT;
N:=S[7]~'subgroup;
#N;
N;
N eq sub<G1|N.1,N.2>;
;
#N;
CN:=Classes(N);
CN;
CharacterTable(N);
chN:=LinearCharacters(N);
chN;
#chN;
#chN[2];
```
chN[2] (N.1);
chN[2] (N.2);
ind:=Induction(chN[2],G1);
ind;
IsFaithful(ind);
Norm(ind);

for i in [1..9] do if ind eq CT[i] then i; end if; end for;

T:=RightTransversal(G1,N);
T; #T;
A:=[0:i in [1..144]];
for i in [1..12] do if a*T[i]^(-1) in N then i,
chN[2] (a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[2]*a*T[i]^(-1) in N then i,
chN[2] (T[2]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[3]*a*T[i]^(-1) in N then i,
chN[2] (T[3]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[4]*a*T[i]^(-1) in N then i,
chN[2] (T[4]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[5]*a*T[i]^(-1) in N then i,
chN[2] (T[5]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[6]*a*T[i]^(-1) in N then i,
chN[2] (T[6]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[7]*a*T[i]^(-1) in N then i,
chN[2] (T[7]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[8]*a*T[i]^(-1) in N then i,
chN[2] (T[8]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[9]*a*T[i]^(-1) in N then i,
chN[2] (T[9]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[10]*a*T[i]^(-1) in N then i,
chN[2] (T[10]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[11]*a*T[i]^(-1) in N then i,
chN[2] (T[11]*a*T[i]^(-1)); end if; end for;
for i in [1..12] do if T[12]*a*T[i]^(-1) in N then i,
chN[2] (T[12]*a*T[i]^(-1)); end if; end for;
A:=[0:i in [1..144]];
A[2]:=1;
A[13]:=-1;
A[28]:=1;
A[39]:=-1;
A[57]:=1;
A[71]:=-1;
A[84]:=1;
A[94]:=1;
A[101]:=-1;
A[116]:=-1;
A[126]:=1;
A[139]:=-1;
A;
B:=[0:i in [1..144]];
for i in [1..12] do if b*T[i]^−1 in N then i,
chN[2](b*T[i]^−1); end if; end for;
for i in [1..12] do if T[2]*b*T[i]^−1 in N then i,
chN[2](T[2]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[3]*b*T[i]^−1 in N then i,
chN[2](T[3]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[4]*b*T[i]^−1 in N then i,
chN[2](T[4]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[5]*b*T[i]^−1 in N then i,
chN[2](T[5]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[6]*b*T[i]^−1 in N then i,
chN[2](T[6]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[7]*b*T[i]^−1 in N then i,
chN[2](T[7]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[8]*b*T[i]^−1 in N then i,
chN[2](T[8]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[9]*b*T[i]^−1 in N then i,
chN[2](T[9]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[10]*b*T[i]^−1 in N then i,
chN[2](T[10]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[11]*b*T[i]^−1 in N then i,
chN[2](T[11]*b*T[i]^−1); end if; end for;
for i in [1..12] do if T[12]*b*T[i]^−1 in N then i,
chN[2](T[12]*b*T[i]^−1); end if; end for;
B[3]:=1;
B[16]:=1;
B[29]:=1;
B[43]:=1;
B[49]:=1;
B[72]:=-1;
B[74]:=1;
B[93]:=1;
B[106]:=1;
B[116]:=1;
B[126]:=-1;
B[143]:=1;
B;
M := GL(12, 3);
A := M! A; A;
B := M! B; B;
Order(A), Order(B), Order(A*B);
H := sub<M|A, B>;
#H;
s := IsIsomorphic(H, GL);
s;
CompositionFactors(H);
S := Sym(24);
xx := S!(1, 3, 2, 4) (5, 8, 6, 7) (9, 17, 10, 18) (11, 22, 12, 21)
(13, 23, 14, 24) (15, 19, 16, 20);
yy := S!(1, 5, 9) (2, 6, 10) (3, 7, 13) (4, 8, 14) (11, 24, 22)
(12, 23, 21) (15, 17, 19) (16, 18, 20);
Order(xx);
Order(yy);
Order(xx*yy);
Order((xx^2, yy));
N := sub<S|xx, yy>;
#N;
GG := sub<N|Id(N)>;
SS := Stabiliser(N, [1, 2]);
SS;
for g in N do if {1, 2}^g eq {1, 2} then GG := sub<N|GG, g>; end if; end for;
GG eq SS;
SS eq sub<N|5, 11, 24, 10, 19)(6, 12, 23, 9, 20)
(7, 15, 17, 14, 21)(8, 16, 18, 13, 22), (1, 2)(3, 4)
(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)
(19, 20)(21, 22)(23, 24)>;
NN, i, j := Group<i, j|i^4, j^3, (i*j)^5, (i^2, j)>;
#NN;
Sch := SchreierSystem(NN, sub<NN|Id(N)>);
ArrayP := [Id(N): i in [1..120]];
for i in [2..120] do
P := [Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j] := xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j] := xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j] := yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j] := yy^-1; end if;
end for;
PP := Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..120] do if ArrayP[i] eq N!(5, 11, 24, 10, 19)
(6, 12, 23, 9, 20)(7, 15, 17, 14, 21)(8, 16, 18, 13, 22)
then Sch[i]; end if; end for;
for i in [1..120] do if ArrayP[i] eq N!(1, 2)(3, 4)(5, 6)
(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21,22)
(23, 24) then Sch[i]; end if; end for;
SS eq sub<N|xx*yy*xx*yy^-1,xx^-2>;
G<x,y,t>:=Group<x,y,t|x^4,y^3,(x*y)^5,(x^2,y),t^3,
(t,x*y*x*y^-1),t^(x^-2)*t^-2>;
#G;
P<x,y,t>:=Group<x,y,t|x^4,y^3,(x*y)^5,(x^2,y),t^3,
(t,x*y*x*y^-1),t^(x^-2)*t^-2>;
D:=AlmostSimpleGroupDatabase();
for i in [1..#D] do G1:=GroupData(D,i)'permrep;
sg:=GroupData(D,i)'subgens; if #sg eq 0 then
G:=sub<G1|G1.1,G1.2>; else
F:=Parent(sg[1]); t:=Ngens(G1)-2; phi:= hom<F -> G1
|[G1.(i+2) : i in [1..t]] cat [Id(G1) : i in [t+1..Ngens(F)]]>;
G:=sub <G1 | G1.1, G1.2, [phi(s): s in sg]>; end if;
if #Homomorphisms(P,G: Limit:=1) gt 0 then
GroupData(D,i)'name; end if; end for;
Appendix M: MAGMA Code for
Monomial Progenitor $3^7 : m L_2(7)$

perm:=function(n,p,mat)
C<u>:=CyclotomicField(p);
Z:=Integers ();
s:=[];
for i in [1..n] do
s[i]:=i;
end for;
z:=Matrix(C,1,n,s)*mat;
w:=[];
for i in [1..n] do
j:=0; done:=0;
repeat
if z[1,i]/u^j in Z then
if Z!(z[1,i]/u^j) ge 0 then
w[i]:=n*j+Z!(z[1,i]/u^j);
done:=1;
end if; end if;
j:=j+1;
until done eq 1 or j eq p;
end for;
for i in [1..(p-1)] do
for a in [1..n] do
w[a+i*n]:=(Z!w[a]+i*n-1) mod (p*n) + 1;
end for; end for;
S:=Sym(n*p);
w:=S!w;
return w^-1;
end function;
G<x,y>:=Group<x,y|x^2,y^3,(x*y)^7,(x,y)^4>;
S:=Sym(7);
\begin{verbatim}
xx := S!(2,4)(3,5);
yy := S!(1,2,3)(5,6,7);
G1 := sub<S|xx,yy>;
#G1;
#G;
S := Subgroups(G1);
for i in [1..#S] do if Index(S[i]`subgroup,DerivedGroup(S[i]`subgroup)) eq 2 then Index(G1,S[i]`subgroup), i; end if; end for;
H;
H := sub<G1|(2, 7)(3, 6), (1, 2, 4)(3, 6, 5), (1, 2)(4, 7), (1, 4)(2, 7)>;
#H;
n := Index(G1,H); n;
dH := DerivedGroup(H);
#H/#dH;
f, g := CosetAction(H,dH);
#Generators(H);
f(H.1), f(H.2), f(H.3), f(H.4);
C<\Phi> := CyclotomicField(2);
M := MatrixAlgebra(C, 1);
HM := GModule(H, [M!(\Phi), M![1], M![1], M![1]]);
I := Induction(HM, G1);
Norm(Character(I));
GP := MatrixGroup(I);
#GP;
GP;
mat1 := GP.1; mat2 := GP.2;
A := perm(7, 2, mat1);
B := perm(7, 2, mat2);
A; B;
G2 := sub<Sym(14)|A,B>;
s, t := IsIsomorphic(G1, G2);
s;
t;
A, B;
G2;
a := G2.1;
Order(a);
b := G2.2;
Order(b);
G2 := sub<G2|a,b>;
Order(a*b);
\end{verbatim}
Order((a,b));
H<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(a,b)^4>;
#H;
f2,h2,k2:=CosetAction(H,sub<H|Id(H)>);
CompositionFactors(h2);
N:=G2;
NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(a,b)^4>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(G2): i in [1..168]];
for i in [2..168] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
end for;
PP:=Id(G2);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
G22:=Stabiliser(G2,{7,14});
G22;
c:=G22.1; c;
d:=G22.2; d;
e:=G22.3; e;
c:=G2!(1, 4, 10)(2, 5, 6)(3, 8, 11)(9, 12, 13);
d:=G2!(1, 8)(2, 9)(3, 13)(4, 12)(5, 11)(6, 10);
e:=G2!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13);
G22 eq sub<G2|c,d,e>;
for i in [1..#G2] do if ArrayP[i] eq c then Sch[i]; end if; end for;
for i in [1..#G2] do if ArrayP[i] eq d then Sch[i]; end if; end for;
for i in [1..#G2] do if ArrayP[i] eq e then Sch[i]; end if; end for;
B*A*B^-1*A*B*A*B^-1, A>;
for k,l,m,n,o,p in [0..16] do
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^7,(x,y)^4, t^3,
t^(y*x*y^1*x*y*x*y^1*x*y)*t^-1,
t^(y*x*y^1*x*y*x*y^-1)*t^-1,t^x=t^-1,(x*y*t)^k,
(x*t*y*t^-2)^1,(x*y*t)^m,(y*x*t^2*t)^n,(y^2*(x*y)^5*t)^o,
end for;
(x*y^{-1}+t)^p; \text{if Index}(G,\text{sub}<G|x,y>) \gt 7 \text{ then } k, l, m, n, o, p, \text{Index}(G,\text{sub}<G|x,y>), \#G; \text{ end if; end for;
Appendix N: MAGMA Code for DCE of $A_7$ over $L_2(7)$

/* This is $A_7$ homographic image of the progenitor 3star7:L2(7) (2 double cosets, 15 single cosets) */
-----------------------------------------------------
S:=Sym(14);
xx:=S!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13);
yy:=S!(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14);
N:=sub<S|xx,yy>;
#N;
k:=0; l:=0; m:=0; n:=0; o:=4;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^7,(x,y)^4, t^3,
t^((y*x*y^(-1)*x*y*x*y^(-1)*x*y)*t^(-1),
t^((y*x*y^(-1)*x*y*x*y^(-1))*t^(-1),t^x=t^(-1), (x*y*t)^k,	(x*t^y*t^2)^l, (x^y*t)^m, (y^2*(x*y)^5)^n, (x*y^(-1)*t)^o>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1; #k;
CompositionFactors(G1);
#sub<G|x,y>;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(a,b)^4>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..168]];
for i in [2..168] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^(-1); end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

ts:=[Id(G1): i in [1..14]];
ts[7]:=f(t); ts[1]:=f(((t^y)^x)^y);
ts[2]:=f(((t^y)^x)^y)^y); ts[3]:=f((t^y)^x);
ts[4]:=f(t^y); ts[5]:=f((t^y)^y);
ts[6]:=f(((t^y)^y)^x); ts[14]:=f(t^x);
ts[8]:=f(((t^x)^y)^x)^y); ts[9]:=f(((t^x)^y)^x)^y)^y); ts[10]:=f((t^x)^y)^x); ts[11]:=f((t^x)^y);
ts[12]:=f(((t^x)^y)^y); ts[13]:=f(((t^x)^y)^y)^x);

f((x*y^-1)^4)*ts[11]*ts[13]*ts[12]*ts[7];
(xx*yy^-1)^4;
f((x+y^-1+t)^4);
for g,h in IN do if ts[7] eq g*(ts[14])^h then g,h; end if; end for;
for i in [1..15] do i, cst[i]; end for;
for i in [1..168] do if ArrayP[i] eq
N!(7, 14)(4, 3)(5, 6)(11, 10)(12, 13)(8, 1) then Sch[i]; end if; end for;

prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I]
applied sequentially.
*/
v:=pt;
for i in I do
v:=v^(Q[i]);
end for;
return v;
end function;
cst:=[null: i in [1..Index(G,sub<G|x,y>)]]
where null is[Integers()];
for i := 1 to 14 do
cst[prodim(1,ts,[i])]:=i;
end for;
m:=0; for i in [1..15] do if cst[i] ne []
then m:=m+1; end if; end for; m;

N0:=Stabiliser(N,7);
Orbits(N0);
#N0;
N0;

-----------------------------------------------------

N0:=Stabiliser(N,7);
Orbits(N0);
#N0;
N0;

-----------------------------------------------------

ts[14] eq ts[7]*ts[7];
ts[7]*ts[14] eq ts[7]*ts[7]*ts[7];

-----------------------------------------------------

N01:=Stabiliser(N,[7,1]);
N01;
#N01;
SSS:={[7,1]};
SSS:=SSS \N;
SSS;
#SSS;
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[7]*ts[1] eq n*ts[\text{Rep(Seqq}[i][1])]*ts[\text{Rep(Seqq}[i][2])]
then print \text{Rep(Seqq}[i]); end if; end for; end for;
N01s:=N01;
for n in N do if 7^n eq 4 and 1^n eq 2 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 5 and 1^n eq 3 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 3 and 1^n eq 8 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 9 and 1^n eq 7 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 1 and 1^n eq 9 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 11 and 1^n eq 12 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 10 and 1^n eq 4 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 12 and 1^n eq 14 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 2 and 1^n eq 10 then
N01s:=sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 8 and 1^n eq 5 then
N01s := sub<N|N01s,n>; end if; end for;
for n in N do if 7^n eq 14 and 1^n eq 11 then
N01s := sub<N|N01s,n>; end if; end for;
N01s; #N01s;
end if; end for;
end if; end for;
end if; end for;
end if; end for;
end if; end for;
for i in [1..15] do i, cst[i]; end for;

/*
n1 = (2, 5, 8) (3, 7, 12) (4, 13, 14) (6, 11, 10),
thus by using i, cst[i], we convert n to be
n = (7,5,3) (14,12,10) (4,1,13) (11,8,6) and so on for the
rest permutations
n2 = (2, 14, 8) (3, 10, 12) (4, 5, 9) (6, 7, 15) =
(7,13,3) (14,6,10) (4,5,2) (11,12,9)
n3 = (2, 11, 12, 6) (3, 13, 8, 4) (5, 9, 7, 15) (10, 14) =
(7,8,10,11) (14,1,3,4) (5,2,12,9) (6,13)
n4 = (2, 4, 7) (3, 6, 5) (8, 10, 13) (11, 12, 14) =
(7,4,12) (14,11,5) (3,6,1) (8,10,13)
n5 = (4, 15, 10) (5, 11, 12) (6, 9, 14) (7, 13, 8) =
(4,9,6) (5,8,10) (11,2,13) (12,1,3)
n6 = (2, 4, 8, 5, 9, 13, 14) (3, 6, 12, 7, 15, 11, 10) =
(7,4,3,5,2,1,13) (14,11,10,12,9,8,6)
n7 = (2, 11, 4, 8) (3, 13, 6, 12) (5, 7) (9, 14, 15, 10) =
for i in [1..168] do if ArrayP[i] eq N!(7,5,3)(14,12,10)(4,1,13)(11,8,6) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,13,3)(14,6,10)(4,5,2)(11,12,9) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,8,10,11)(14,1,3,4)(5,2,12,9)(6,13) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,4,12)(14,11,5)(3,6,1)(8,10,13) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(4,9,6)(5,8,10)(11,2,13)(12,1,3) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,4,3,5,2,1,13)(14,11,10,12,9,8,6) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,8,4,3)(14,1,11,10)(5,12)(2,13,9,6) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(4,3,2)(5,13,1)(11,10,9)(12,6,8) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,9,6,3,4,1,12)(14,2,13,10,11,8,5) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,12,6)(14,5,13)(4,9,3)(11,2,10) then Sch[i]; end if; end for;
for i in [1..168] do if ArrayP[i] eq N!(7,4,1)(14,11,8)(5,13,2)(12,6,9) then Sch[i]; end if; end for;
\begin{align*}
&f(y\times y\times y\times y^{\sim}1\times x\times y^{\sim}1\times x\times y^{\sim}1)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[4]\times ts[2]; \\
&f(y^{\sim}1\times x\times y\times x\times y^{\sim}1\times x\times y)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[5]\times ts[3]; \\
&f(x\times y\times y^{\sim}1\times x\times y^{\sim}1)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[3]\times ts[8]; \\
&f(y\times x\times y\times x\times y^{\sim}1\times x\times y^{\sim}1\times x)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[9]\times ts[7]; \\
&f(x\times y^{\sim}1\times x\times y^{\sim}1\times x\times y\times x)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[1]\times ts[9]; \\
&f(y\times x\times y\times y^{\sim}1\times x\times y^{\sim}1\times x\times y)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[11]\times ts[12]; \\
&f(y\times x\times y^{\sim}1\times x\times y^{\sim}1\times x)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[10]\times ts[4]; \\
&f(x\times y\times x\times y\times y^{\sim}1\times x\times y^{\sim}1)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[12]\times ts[14]; \\
&f(y\times x\times y^{\sim}1\times x\times y^{\sim}1\times x\times y)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[2]\times ts[10]; \\
&f(y\times x\times y\times y^{\sim}1\times x\times y\times x\times y\times x)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[8]\times ts[5]; \\
&f(x\times y\times x\times y\times y^{\sim}1\times x)\times ts[7]\times ts[1] \quad \text{eq} \quad ts[14]\times ts[11];
\end{align*}
Appendix O: MAGMA Code for Isomorphism Type of $3^7 : L_2(7)$

```magma
S:=Sym(14);
xx:=S!(1, 8)(3, 4)(5, 6)(7, 14)(10, 11)(12, 13);
yy:=S!(1, 2, 3)(4, 5, 7)(8, 9, 10)(11, 12, 14);
N:=sub<S|xx,yy>;
#N;
k:=0; l:=0; m:=0; n:=3; o:=4; p:=0;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^7,(x,y)^4, t^3,
             t^(y*x*y^-1*x*y*x*y*x*y^-1*x*y)*t^-1,
             t^(y*x*y^-1*x*y*x*y^-1)*t^-1, t*x=t^-1, t*x=y*t^k,
             (x*y*x*t)^2)^1, (x^y*x*t)^m, (y^x*t^2*t)^n, (y^2*(x*y)^5*t)^o,
             (x*y^-1*t)^p>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G; #G1; #k;
CompositionFactors(G1);
#sub<G|x,y>;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
Center(G1);
NL:=NormalLattice(G1);
NL;
MinimalNormalSubgroups(G1);
X:=AbelianGroup(GrpPerm,[3,3,3,3,3,3,3,3]);
s:=IsIsomorphic(X,NL[2]);
s;
D:=DirectProduct(NL[2],NL[3]);
s:=IsIsomorphic(NL[3],D);
s;
DD:=DerivedGroup(G1);
DD;
IsPerfect(DD);
```
DB:=PerfectGroupDatabase();
PermutationGroup(DB,367416,1);
s:=IsIsomorphic(PermutationGroup(DB,367416,1),G1);
s;
PermutationGroup(DB,367416,2);
s:=IsIsomorphic(PermutationGroup(DB,367416,2),G1);
s;
Group(DB,367416,2);
H<a,b,t,u,v,x,y,w,z>::=Group<a,b,t,u,v,x,y,w,z|a^2,
b^3,(a*b)^7,(a,b)^4,t^3,u^3,v^3,x^3,y^3,w^3,z^3,(a,t^-1),
(t,u), (t,v), (t,w), (t,x), (t,y), (t,z), (u,v), (u,w), (u,x),
(u,y), (u,z), (v,w), (v,x), (v,y), (v,z), (w,x), (w,y), (w,z),
(x,y), (x,z), (y,z), (a,t^-1), a^-1*u*a*w^-1,a^-1*v*a*v,
a^-1*w*a*u^-1,a^-1*x*a*z^-1,a^-1*y*a*y,a^-1*z*a*x^-1,
b^-1*t*b*u^-1,b^-1*u *b*v^-1,b^-1*v*b*t^-1,b^-1*w*b*x^-1,
b^-1*x*b*y^-1,b^-1*y*b*w^-1,(b,z^-1)>;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
#H;
s:=IsIsomorphic(H1,G1);
s;
Appendix P: MAGMA Code for
Universal Cover of $A_6$

\begin{verbatim}
S:=Sym(432);
a:=S!(1,2,8,5)(3,13,22,16)(4,10,9,19)(6,7,12,18)(11,25,20,26)(14,31,21,32)
(15,28,30,35)(17,30,34,38)(24,39,33,40)(27,45,36,48)(29,51,37,52)(41,67,43,69)
(80,122,94,126)(88,133,91,135)(89,105,92,109)(96,143,98,144)(100,149,104,150)
(119,175,124,177)(121,179,125,180)(129,183,131,185)(130,147,132,151)(134,190,137,191)
(333,375,335,378)(337,380,340,373)
\end{verbatim}
b:=S!(1,3,14,12,8,22,21,6) (2,9,23,20,5,4,15,11)
(7,17,33,19,18,34,24,10) (13,27,46,37,16,36,53,29)
(25,41,68,44,26,43,72,42) (28,47,78,60,35,59,87,50)
(30,49,80,62,38,61,94,54) (31,55,96,58,32,57,98,56)
(39,63,100,66,40,65,104,64) (45,73,113,83,48,82,117,75)
(51,88,134,92,52,91,137,89) (67,105,155,110,69,109,158,106)
(70,111,127,81,71,112,114,74) (76,103,154,128,84,102,153,118)
(77,119,149,125,79,124,150,121) (85,129,184,132,86,131,186,130)
(90,136,192,140,93,139,196,138) (95,141,197,146,97,145,200,142)
(99,147,201,152,101,151,202,148) (107,159,126,162,108,161,122,160)
(115,169,226,173,116,172,231,170) (120,176,144,182,123,181,143,178)
(167,221,294,224,168,223,296,222) (171,228,301,234,174,233,306,232)
(175,235,307,240,177,239,308,237) (179,243,313,246,180,245,314,244)
(183,247,266,199,185,250,265,198) (190,256,321,260,191,259,325,257)
(203,269,337,272,204,271,340,270) (205,273,342,276,206,275,344,274)
(209,277,319,251,210,278,320,253) (211,279,345,282,212,281,346,280)
(225,297,351,290,227,302,352,291) (229,304,283,238,230,305,285,236)
(258,322,373,328,261,327,380,326) (263,329,299,332,264,331,300,330)
(298,357,355,393,303,360,356,338) (323,374,408,378,324,377,411,375)
(363,391,419,394,364,393,420,392) (367,397,425,400,368,399,426,398)
(369,401,413,381,370,402,414,382) (371,403,427,406,372,405,428,404)
(385,415,430,409,386,416,429,407) (395,421,431,424,396,423,432,422);

G:=sub<S|a,b>;
C:=Center(G);C;
Order(C.1);
Order(C.2);
Order(C.2^3);
Order(C.1*C.2);
Order(C.1*C.2^2);
c:=C.2;
Order(c);
Order(c^2);
Order(c^3);
q,ff:=quo<G|c^3>;
q;
CompositionFactors(q);
S:=Sym(18);
a:=S!(2,6)(4,11)(7,9)(8,13)(10,14)(12,16);
b:=S!(1,2,7,4)(3,8,6,10)(5,9,13,12)(11,15)(14,17)(16,18);
G1:=sub<S|a,b>;
s:=IsIsomorphic(q,G1);
s;
D:=Center(G1);D;
qq,fff:=quo<G1|D>;
qq;
CompositionFactors(qq);
s:=IsIsomorphic(qq,Alt(6));
s;
Bibliography


