Radio Number for Fourth Power Paths

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Radio Number for Fourth Power Paths

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Linda Victoria Alegria

December 2014
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Abstract

A path on \( n \) vertices, denoted by \( P_n \), is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the order. A fourth power path, \( P^4_n \), is obtained from \( P_n \) by adding edges between any two vertices, \( u \) and \( v \), whose distance in \( P_n \), denoted by \( d_{P_n}(u,v) \), is less than or equal to four. The diameter of a graph \( G \), denoted \( \text{diam}(G) \) is the greatest distance between any two distinct vertices of \( G \). A radio labeling of a graph \( G \) is a function \( f \) that assigns to each vertex a label from the set \( \{0, 1, 2, \ldots\} \) such that \( |f(u) - f(v)| \geq \text{diam}(G) - d(u,v) + 1 \) holds for any two distinct vertices, \( u \) and \( v \) in \( G \) (i.e., \( u,v \in V(G) \)). The greatest value assigned to a vertex by \( f \) is called the span of the radio labeling \( f \), i.e., \( \text{span}f = \max\{f(v) : v \in V(G)\} \). The radio number of \( G \), \( \text{rn}(G) \), is the minimum span of \( f \) over all radio labelings \( f \) of \( G \). In this paper, we provide a lower bound for the radio number of the fourth power path.
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Chapter 1

Introduction

1.1 Channel Assignment Problem

Radio waves are a type of electromagnetic radiation. They have frequencies from as little as three kHz, where one kHz is 1000 cycles per second, up to 300 GHz, where one GHz is 1,000,000,000 cycles per second. Radio waves that are made artificially have many useful applications including radio communication, broadcasting, radar, communications satellites and more. Depending on the frequency, radio waves behave differently in the Earth’s atmosphere. Whether the wave travels to one general location or whether it makes it across state lines depends on the length of the wave. In order to avoid interference, the use of radio waves is controlled and monitored by law by the International Telecommunications Union (ITU). In the United States, the Federal Communications Commission (FCC) controls the use of specific frequencies and their purposes.

The radio spectrum is split up into radio bands based on frequency and is designated for different uses. AM radio stations, for instance, operate in a band of 535 kHz to 1705 kHz. FM radio stations transmit radio waves in a band of frequencies between 88 MHz and 108 MHz. This band is further divided into 100 channels, each having a width of 200 kHz, or 0.2 MHz. This is why your FM radio will be tuned in to a station with a channel of 88.1, 88.3, 88.5, etc. For the most part, FM radio waves or signals travel in a straight line, or are “line of sight”. The farther away from the FM station, the higher your antenna must be in order to receive signal. Also, since FM signals travel in straight
lines, they can easily be interrupted by buildings, or other geographical structures such as mountains. This would clearly explain why you would experience an interruption while driving around mountains. It would make sense that the higher the FM transmitter antennas are, the greater the area they will cover. There are more factors other than antenna height that are involved in avoiding interference. One of the main factors is distance, namely the distance between the stations or transmitters. Since FM radio waves are able to travel over limited distance, stations are able to share the same channel provided that they are at least a certain minimum distance apart. These stations would be called co-channel stations. If restrictions on minimum distances are not met, the stations would experience interference.

Generally speaking, stations are assigned channels based on their proximity to one another. Stations that are closer to each other geographically must be assigned channels that have larger difference in value. Stations that have greater distance between them can be assigned channels that are closer in value. The task of efficiently assigning channels to radio transmitters is known as the Channel Assignment Problem. The Channel Assignment Problem became a problem of particular interest in the field of graph theory since radio transmitters could be easily represented by vertices on a graph, and distances between them could be represented by edges. There are different models that have been inspired by the Channel Assignment Problem. We will later look at some models that were inspired by the Channel Assignment Problem after first covering some basics.

### 1.2 Basics of Graph Theory

In order to understand much of the language of this paper, let us define a few basic terms and concepts concerning graphs.

A **graph** $G$ consists of a finite, non-empty set $V(G)$ of objects called **vertices** and a set $E(G)$ of two-element subsets of $V(G)$ called **edges**. The set $V$ is called the **vertex set** of $G$, denoted $V(G)$, and the set $E$ is called the **edge set** of $G$, denoted $E(G)$. If $u, v \in V(G)$ and $e = \{u, v\}$ is an edge in $E(G)$, then $u$ and $v$ are **adjacent**.
Figure 1.1 below is of a graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{v_1v_6, v_1v_5, v_2v_6, v_2v_5, v_3v_5, v_3v_4, v_4v_5\}$.

![Figure 1.1: A graph $G$](image)

A **simple graph** is a graph having no loops or multiple edges.

![Figure 1.2: Simple graph vs. non-simple graph](image)

The graphs mentioned in this paper will be simple graphs.

A **subgraph** of a graph $G$ is a graph $H$ such that the $V(H) \subseteq V(G)$ and the $E(H) \subseteq E(G)$. The assignment of vertices to edges in $H$ is the same as in $G$.

A **walk** in $G$ is a sequence of vertices in $G$ beginning with a vertex $u$ and ending at some vertex $v$ such that consecutive vertices in the sequence are adjacent. We can also call this walk a **$u-v$ walk**. Refer to the graph of $G$ in Figure 1.3. A $d-f$ walk is $\{d, c, e, f\}$. 
The **length** of a walk is the same as the number of edges traveled. In Figure 1.3 above, the length of the $d - f$ walk, $\{d, c, e, f\}$, is three. Notice that the $d - f$ walk, $\{d, c, e, f\}$, is not the shortest walk, or path in $G$ from vertex $d$ to vertex $f$. A shorter $d - f$ walk would be $\{d, c, f\}$, making the length of that $d - f$ walk equal to two.

In general, the **distance** between two vertices $u$ and $v$ of a graph $G$, denoted by $d_G(u, v)$ or simply $d(u, v)$ when the graph $G$ is clearly understood, is defined to be the minimum length of any $u - v$ walk. Moreover, the **diameter** of a graph $G$, denoted $\text{diam}(G)$, is the greatest distance between any two vertices of $G$. In Figure 1.3, $\text{diam}(G)$ is equal to two.

Observe the graph $T$ above in Figure 1.4 and let us investigate arbitrary walks between
vertices, distances between vertices, as well as the diameter of the graph.

Figure 1.5: Two $a - d$ walks in $T$

Now, in Figure 1.5a, the $a - d$ walk is $\{a, b, h, g, f, d\}$. The length of this $a - d$ walk is five. In Figure 1.5b, the $a - d$ walk is $\{a, b, c, d\}$. The length of this $a - d$ walk is three. Now, with regard to the distance between $a$ and $d$, the question becomes whether either of these walks is the shortest possible walk in $T$. If there exists an $a - d$ walk of length less than three, then three would not be the distance between vertices $a$ and $d$.

Figure 1.6: Distance between $a$ and $d$

Figure 1.6 shows an even shorter $a - d$ walk which is $\{a, b, d\}$. In this case, it is the most direct and thus shortest $a - d$ walk in $T$. The length of this $a - d$ walk is two, therefore the distance between vertices $a$ and $d$ is equal to two. For fixed vertices $u$ and $v$, there may exist more than one walk in a graph whose length is equal to the distance between $u$ and $v$. In Figure 1.6, there happens to be one walk of minimum length from $a$ to $d$. Now consider the diameter of $T$. The diameter of a graph is defined to be the maximum
distance between any two vertices of the graph. From the graph of $T$, it seems as though $a$ and $f$ may have the largest distance between them.

In Figure 1.7, we have an $a-f$ walk. We are seeking the diameter of graph $T$ and so we are looking at the maximum distance between any two vertices of $T$. The length of the $a-f$ walk is five. This, however is not the diameter of $T$. Again, the diameter of $T$ is the maximum distance among all distances between pairs of vertices in $T$, not necessarily lengths of walks in $T$. The distance between $a$ and $f$, in fact, is not five.

Now observe the $a-f$ walk in Figure 1.8. This $a-f$ walk has length equal to three and is the shortest walk possible in $T$ from $a$ to $f$, therefore the distance between $a$ and $f$ is equal to three. The diameter of $T$, $diam(T)$ actually turns out to be three as well. In order to find the diameter of the graph, distances between all pairs of vertices need to be considered, rather than arbitrary walks between vertices since these walks may not be equal to the distance between the vertices.
A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the order. Paths are denoted by $P_n$ where $n$ is an integer and is equal to the number of vertices in the path. Figure 1.9 is an example of $P_8$, that is, a path on eight vertices.

![Figure 1.9: $P_8$](image)

For a path $P_n$, the distance between any two vertices is equal to the number of edges between them. Furthermore, the diameter of $P_n$ is equal to the distance between the first vertex and the last vertex. That is, $\text{diam}(P_n) = n - 1$.

A graph $G$ is said to be connected if there is a path between any pair of vertices of $G$.

### 1.3 Fourth Power Paths

The main graph of interest in this paper is the fourth power path, denoted $P^4_n$. A fourth power path, $P^4_n$, is obtained from $P_n$ by adding edges between vertices $u$ and $v$ whose distance is at most four. Thus $V(P^4_n)$ is equal to $V(P_n)$ and $E(P^4_n)$ is equal to $E(P_n) \cup \{uv : 2 \leq d(u,v) \leq 4\}$. Figure 1.10 is an example of $P_8$ along with $P^4_8$.

![Figure 1.10: $P_8$ vs. $P^4_8$](image)
In this paper, we will denote the distance between vertices \( u \) and \( v \) of \( P_n \) by \( d_{P_n}(u,v) \) and we will denote the distance between vertices \( u \) and \( v \) of \( P_n^4 \) by \( d(u,v) \).

Recall that the distance between two vertices of a path is equal to the number of edges between them. Also, notice that the distance between two adjacent vertices is one, because there is one edge between any two adjacent vertices. For vertices \( u \) and \( v \), if \( d_{P_n}(u,v) \leq 4 \) then the \( d(u,v) = 1 \). Similarly, for vertices \( u \) and \( v \), if \( 4 < d_{P_n}(u,v) \leq 8 \) then the \( d(u,v) = 2 \). Furthermore, for vertices \( u \) and \( v \), if \( 4(m-1) < d_{P_n}(u,v) \leq 4m \), \( m \in \mathbb{N} \), then the \( d(u,v) = m \). For this reason, the ceiling function is utilized in defining the distance between any two vertices of \( P_n^4 \).

**Proposition 1.1.** For any \( u,v \in V(P_n^4) \), we have
\[
d(u,v) = \left\lceil \frac{d_{P_n}(u,v)}{4} \right\rceil
\]

This paper focuses entirely on fourth power paths, and so we will devote the remainder of this section to explaining properties of \( P_n \), which can further be extended to \( P_n^4 \), as well as defining terms that will be used throughout the paper.

We define a **center** of \( P_n \) as a vertex of \( P_n \) that is equidistant from both the first vertex, \( v_1 \), and the last vertex, \( v_n \), that is, the “middle vertex”. An odd path, \( P_n \), where \( n = 2m + 1 \) for some \( m \in \mathbb{N} \) will have only one center, namely \( v_{m+1} \). A path, \( P_n \), where \( n = 2m \) for some \( m \in \mathbb{N} \) will have two centers, namely \( v_m \) and \( v_{m+1} \). For \( n \) even, \( d(v_1,v_m) = d(v_{m+1},v_n) \). Refer to Figure 1.11.

![Figure 1.11: Centers of odd and even paths](image)

**Figure 1.11:** Centers of odd and even paths
For each vertex \( u \in V(P_n) \), the **level** of \( u \), denoted by \( L(u) \) is the smallest distance in \( P_n \) from \( u \) to a center of \( P_n \). The labeling of the levels can be viewed as being analogous to labeling the number line with the exception of negative values. Levels of vertices are greater than or equal to zero, where the only vertex whose level is equal to zero is the center, or centers. Note that for \( P_{2m} \), there are two centers and thus two vertices whose levels are equal to zero. Refer to Figure 1.12.

Figure 1.12: Levels of vertices for \( n \) even and \( n \) odd

For \( P_n \), where \( n = 2m + 1 \), \( L(v_1) = m \) and \( L(v_{m+1}) = 0 \). We will denote the levels of a sequence of vertices \( A \) by \( L(A) \).

If \( n = 2m + 1 \), then

\[
L(v_1, v_2, ..., v_{2m+1}) = (m, m - 1, ..., 3, 2, 1, 0, 1, 2, 3, ..., m - 1, m).
\]

If \( n = 2m \), then

\[
L(v_1, v_2, ..., v_{2m}) = (m - 1, m - 2, ..., 3, 2, 1, 0, 1, 2, 3, ..., m - 2, m - 1).
\]

**Left** and **right** vertices are defined as follows:

If \( n = 2m + 1 \), then the left and right vertices, respectively are

\[
\{v_1, v_2, ..., v_m, v_{m+1}\} \text{ and } \{v_{m+1}, v_{m+2}, ..., v_{2m}, v_{2m+1}\}.
\]
If \( n = 2m \), then the left and right vertices, respectively are

\[ \{v_1, v_2, \ldots, v_m\} \text{ and } \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}. \]

If two vertices are both right vertices or both left vertices, then we say that they are on the same side. Otherwise, they are on opposite sides. Note that when \( n = 2m + 1 \), the center is on both the right and the left side.

Paths have slightly different properties because of the parity of \( n \). Thus, we will look at these two cases separately.

Let us investigate how the levels of vertices and the distances between them are related. For each case, we will compare level sums of vertices to the respective values found on the number line. For \( n = 2m + 1 \), there is only one center, thus only one vertex with level equal to zero. When finding distances between two numbers or points, \( a \) and \( b \), on a number line, we simply compute \(|a - b|\). Now, when two vertices \( u \) and \( v \) are on the same side, we can find the distance between them, in \( P_{2m+1} \) by doing the same operation with the vertices’ respective levels.

\[
P_7:
\]

![Figure 1.13: Number line analogous to levels of \( P_n \) for \( n \) odd](image)

In Figure 1.13, the distance between the points \( a \) and \( b \) is \(|a - b|\) which is \(|(-3) - (-1)|\) and equals two. Looking at \( P_7 \) in Figure 1.13, the distance between vertices \( v_1 \) and \( v_3 \) is also \(|L(v_1) - L(v_3)|\) which is \(|3 - 1|\) and equals 2. This is true whenever both vertices, \( u \) and \( v \) are on the same side, that is \( d_{P_{2m+1}}(u, v) = |L(u) - L(v)| \). Note that the center is on both the left side and the right side for \( n \) odd.
We further investigate what would happen for vertices on opposite sides. On the number line, finding the distance between two points \(a\) and \(b\) remains the same. It is important to note, however, that when the two values have opposite signs, that is \(a\) is positive and \(b\) is negative, when distance is calculated, the values are added. More specifically, let \(a, b \in \mathbb{N}\) and consider \(-b\). The distance between \(a\) and \(-b\) is \(|a - (-b)|\) which is \(|a + b|\) and equals \(|a| + |b|\). Thus the distance between two numbers \(a\) and \(b\) is equal to the sum of their absolute values, only when \(a\) and \(b\) are on opposite sides of the number line with respect to zero. We can apply this concept to the distance between vertices \(u\) and \(v\) of \(P_{2m+1}\) when the vertices are on opposite sides. Refer to the Figure 1.14. The distance between points \(a\) and \(-b\) is \(|-3 - 1|\) which is \(|-4|\) and equals \(|3| + |1|\). In \(P_7\), the distance between \(v_1\) and \(v_5\) is \(|L(v_1) + L(v_5)|\) which is \(|3 + 1|\) and \(|3| + |1|\) equals 4. Absolute value is not necessary when adding the levels of vertices since the levels are always positive. Thus, for vertices \(u\) and \(v\) of \(P_{2m+1}\) that are on opposite sides, \(d_{P_{2m+1}}(u, v) = L(u) + L(v)\).

Figure 1.14: Number line vs. vertices on opposite sides of \(P_n\) for \(n\) odd

Now let us turn our attention to \(P_{2m}\), that is when \(n\) is even. The only difference in the levels of vertices of \(P_{2m}\) is that there is an extra center, thus a second vertex whose level is equal to zero. Finding the distance between vertices that are on the same side remains equivalent to finding the distance between points on a number line. Therefore we have \(d_{P_{2m}}(u, v) = |L(u) - L(v)|\) when \(u\) and \(v\) are vertices on the same side of \(P_{2m}\). Things change slightly when vertices \(u\) and \(v\) of \(P_{2m}\) are on opposite sides. This case becomes analogous to the number line, and to the case when \(n\) is odd, with the exception of the extra center. Having an extra center adds an edge between the centers and thus adds a distance of one to the distance between vertices on opposite sides.

Refer to Figure 1.15. Just as before, the distance between points \(a\) and \(-b\) is...
Figure 1.15: Number line vs. vertices on opposite sides of $P_n$ for $n$ even

$|a| + |b|$. But the distance between $v_1$ and $v_6$ is not $|L(v_1) + L(v_6)|$. The extra edge between the two centers must be taken into account. We fix this by simply adding one to the level sums. That is, for $u$ and $v$ in $P_{2m}$, $d_{P_{2m}}(u, v) = L(u) + L(v) + 1$ when $u$ and $v$ are on opposite sides. Again, absolute value is not needed since the levels of vertices are always positive. Recall that for $u, v \in P_n^4$, $d(u, v) = \left\lceil \frac{d_{P_n}(u,v)}{4} \right\rceil$. Combining this with the observations above, we have the following:

**Lemma 1.2.** If $n$ is odd, then for any $u, v \in V(P_n^4)$, we have:

$$d(u, v) = \begin{cases} \left\lceil \frac{L(u)+L(v)}{4} \right\rceil, & \text{if } u \text{ and } v \text{ are on opposite sides;} \\ \left\lceil \frac{|L(u)-L(v)|}{4} \right\rceil, & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

If $n$ is even, then for any $u, v \in V(P_n^4)$, we have:

$$d(u, v) = \begin{cases} \left\lceil \frac{L(u)+L(v)+1}{4} \right\rceil, & \text{if } u \text{ and } v \text{ are on opposite sides;} \\ \left\lceil \frac{|L(u)-L(v)|}{4} \right\rceil, & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

(See Appendix A.1 for proof of Lemma 1.2)

Due to the nature of the ceiling function, close and special attention needs to be placed when adding or subtracting particular values prior to applying the ceiling function. For example, $\left\lceil \frac{5+9}{4} \right\rceil = \left\lceil \frac{14}{4} \right\rceil = 4$. However, $\left\lceil \frac{5}{4} \right\rceil + \left\lceil \frac{9}{4} \right\rceil = 2 + 3 = 5$, thus $\left\lceil \frac{5+9}{4} \right\rceil \neq \left\lceil \frac{5}{4} \right\rceil + \left\lceil \frac{9}{4} \right\rceil$. 
Similarly, \( \lceil \frac{10-5}{4} \rceil = \lceil \frac{5}{4} \rceil = 2 \). However, \( \lceil \frac{10}{4} \rceil - \lceil \frac{5}{4} \rceil = 3 - 2 = 1 \), thus \( \lceil \frac{10-5}{4} \rceil \neq \lceil \frac{10}{4} \rceil - \lceil \frac{5}{4} \rceil \). This leads us to the following proposition, which we will make much use of in later proofs.

**Proposition 1.3.** For any \( d_1, d_2 \) in \( \mathbb{N} \), we have:

\[
\left\lfloor \frac{d_1 + d_2}{4} \right\rfloor = \begin{cases} \left\lfloor \frac{d_1}{4} \right\rfloor + \left\lfloor \frac{d_2}{4} \right\rfloor - 1, & \text{if } (d_1, d_2) \equiv_4 (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1); \\ \left\lfloor \frac{d_1}{4} \right\rfloor + \left\lfloor \frac{d_2}{4} \right\rfloor, & \text{otherwise}. \end{cases}
\]

\[
\left\lfloor \frac{d_1 - d_2}{4} \right\rfloor = \begin{cases} \left\lceil \frac{d_1}{4} \right\rceil - \left\lceil \frac{d_2}{4} \right\rceil + 1, & \text{if } (d_1, d_2) \equiv_4 (2, 1), (3, 1), (3, 2), (0, 1), (0, 2), (0, 3); \\ \left\lceil \frac{d_1}{4} \right\rceil - \left\lceil \frac{d_2}{4} \right\rceil, & \text{otherwise}. \end{cases}
\]

More will be said about the level sums of vertices. First, let us briefly go over radio labeling.

### 1.4 Radio Labeling

Motivated by the Channel Assignment Problem with only two levels of interference, a **distance-two labeling** (also called a \( \lambda \)-labeling and \( L(2,1) \)-labeling) for a graph \( G \) is a function \( f : V(G) \to \{0, 1, 2, 3, \ldots\} \), where \( V(G) \) is the vertex set of \( G \) such that the following holds for all distinct \( u, v \in V(G) \):

\[
|f(u) - f(v)| \geq \begin{cases} 2, & \text{if } d(u, v) = 1; \\ 1, & \text{if } d(u, v) = 2. \end{cases}
\]

It is very important to note that there is no restriction for the labeling given to vertices whose distance is strictly greater than two. This means that vertices whose distance is greater than two are able to have the same label. Radio labeling extends the number of interference levels considered in distance-two labeling from two, to the greatest distance between any two vertices, that is, the diameter of \( G \). For a connected graph \( G \) of diameter \( d \), a **radio labeling** of \( G \) is a function \( f : V(G) \to \{0, 1, 2, \ldots\} \) such that the following holds for any two distinct vertices \( u \) and \( v \) of \( G \):

\[
d(u, v) + |f(u) - f(v)| \geq 1 + \text{diam}(G).
\]
Note that in a radio labeling, restrictions are placed on every pair of vertices of $G$. If we denote the last vertex labeled as $x_n$, then $f(x_n)$ is the largest value assigned to a vertex and thus is called the span of the radio labeling, i.e., $\text{span} f = \max\{f(v) : v \in V(G)\}$.

The radio number of $G$, $\text{rn}(G)$, is the minimum span of $f$ over all radio labelings $f$ of $G$. A radio labeling $f$ of $G$ with $f(x_n) = \text{rn}(G)$ is called a minimum radio labeling or an optimal radio labeling.

We will now look at a few examples of radio labelings for a path on seven vertices. It is important to understand that there is more than one way to label a graph, that is, there are many radio labelings. There is, however, one radio number, which requires a more specific, sometimes unique, labeling.

Before we attempt to label $P_7$, we determine the graph’s diameter. Since we are dealing with a path, $\text{diam}(P_7) = d(v_1,v_7) = 6$. Now, in order to obtain a radio labeling of $P_7$, it must be true that for any two distinct vertices of $P_7$, $d(u,v) + |f(u) - f(v)| \geq 1 + \text{diam}(P_7)$, which implies $d(u,v) + |f(u) - f(v)| \geq 7$. Hence $|f(u) - f(v)| \geq 7 - d(u,v)$.

This means that the difference in the values or labels of two vertices must be greater than or equal to seven minus the distance between the two vertices. Now, for the radio labeling, we can choose any vertex to begin with. We start with $v_1$. We can begin our labeling with any value, but in this paper we will always begin with zero. Starting with zero makes calculating the span of the labeling much easier. The value assigned to the next vertex in the labeling depends upon its distance from $v_1$. If we choose to simply label the vertices from left to right, and therefore label $v_2$ next, we must consider the distance between $v_1$ and $v_2$. We have $d(v_1,v_2) = 1$. Furthermore, $|f(u) - f(v)| \geq 7 - d(v_1,v_2)$ which implies $|f(u) - f(v)| \geq 7 - 1 = 6$.

Therefore, the difference between the first label given to $v_1$ and the second label given to $v_2$ must be at least six. Thus, $v_2$ must be labeled six or any value larger than six.

If we continue to label vertices from left to right, each new vertex will have a label that is at least six greater than the previous vertex since the distance between each adjacent vertex is one. Thus we have the radio labeling illustrated in Figure 1.17.
Note that the span of the labeling in Figure 1.17 is 36.

It is important to note that for the labeling to be a valid radio labeling, \( d(u, v) + |f(u) - f(v)| \geq 1 + diam(P_7) \) needs to hold for all vertices \( u \) and \( v \) of \( P_7 \). We will take a look at another labeling of \( P_7 \). This time, we begin by labeling \( v_5 \). Again, we will start the labeling with zero. Next, we choose to label \( v_1 \). The distance, \( d(v_5, v_1) = 4 \) so we must obey the rule \( |f(v_5) - f(v_1)| \geq 7 - 4 = 3 \).

Thus the difference between zero and the label given to \( v_5 \) must be at least three. We will label \( v_1 \) with three.

Next, we label the vertex, \( v_7 \). Note that \( d(v_1, v_7) = 6 \). Therefore, just as before, \( |f(u) - f(v)| \geq 7 - 6 = 1 \). That is, the difference in label given to \( v_7 \) must be at least one.
Thus we will label $v_7$ with four.

$$P_7$$

3 6 4

Figure 1.19: $v_5$, $v_1$, and $v_7$ labeled in a radio labeling

Again, it needs to be emphasized that $|f(u) - f(v)| \geq 1 + \text{diam}(G) - d(u, v)$ needs to hold for all vertices $u$ and $v$ of $G$ in order for the labeling to be a valid radio labeling. For $P_7$, $|f(u) - f(v)| \geq 7 - d(u, v)$. We quickly verify whether this holds for vertices $v_5$ and $v_7$. The distance between the vertices, $d(v_7, v_5) = 2$, therefore, $|f(v_7) - f(v_5)| \geq 7 - 2$ and so $|f(v_7) - f(v_5)| \geq 5$.

Thus the difference in labels given to $v_7$ and $v_5$ needs to be at least five. Therefore we must either change the label given to $v_7$ to five, or choose a different vertex to label. For the labeling illustrated in Figure 1.20, we will simply change the label on $v_7$ to five.

It is worth mentioning that what was done was considered a “jump” to the next integer in order to satisfy the conditions of a radio labeling. Next we label $v_4$. The distance between $v_7$ and $v_4$, $d(v_7, v_4)$ is three. As before, $|f(v_7) - f(v_4)| \geq 7 - 3 = 4$ so we will label $v_4$ with nine. Once more, we must check that the appropriate condition holds for $v_4$ and $v_1$, which it does.

$$P_7$$

3 9 6 5

Figure 1.20: $v_5$, $v_1$, $v_7$, and $v_4$ labeled in a radio labeling

We will continue by labeling the following vertices: $v_6$, $v_2$, and $v_3$ respectively, while making sure that the radio labeling condition is satisfied.

$$d(v_4, v_6) = 2 \quad \text{and so } |f(v_4) - f(v_6)| \geq 7 - 2 = 5,$$
\[d(v_6, v_2) = 4 \quad \text{and so} \quad |f(v_6) - f(v_2)| \geq 7 - 4 = 3,\]
\[d(v_2, v_3) = 1 \quad \text{and so} \quad |f(v_2) - f(v_3)| \geq 7 - 1 = 6.\]

Again, we need to verify that the appropriate conditions are met for all vertices of \(P_7\). So, in Figure 1.21 we have a radio labeling of \(P_7\) with a span of 23.

\[\begin{array}{cccccccc}
3 & 17 & 23 & 9 & 0 & 14 & 5 \\
\end{array}\]

Figure 1.21: A radio labeling of \(P_7\) with span equal to 23

As we mentioned before, while there are many ways to give a valid radio labeling of a graph, there is only one radio number and a special, many times unique, way to label. Let us look at an optimal radio labeling of \(P_7\) in Figure 1.22. The span of \(P_7\) is equal to \(rn(P_7)\) which is 21.

\[\begin{array}{cccccccc}
0 & 7 & 14 & 21 & 3 & 10 & 17 \\
\end{array}\]

Figure 1.22: An optimal radio-labeling of \(P_7\)

When looking to minimize the span, and in doing so, find the radio number of a graph \(G\), we want to label vertices in such a way that the distance between consecutively labeled vertices is maximized. As we have seen, labeling vertices from left to right definitely does not achieve this because the distance between consecutively labeled vertices is minimized. Starting on one side of the graph and then labeling vertices in an alternating fashion seems to be a better choice since the distances between consecutively labeled vertices almost stays constant and is not the minimum distance between vertices. Therefore there is a very strong connection between maximizing distances between consecutively
labeled vertices of a graph, and obtaining an optimal radio labeling of a graph.

The main topic of this paper concerns radio labelings of much more interesting graphs. Namely, fourth power paths, denoted $P^4_n$. Everything that we have covered thus far with regard to radio labeling will be dealt with in the same manner for $P^4_n$ with the exception of the distances between vertices. Since $P^4_n$ is obtained by adding edges between vertices of $P_n$ that are distance four or less apart, the definition of distances between vertices of $P^4_n$ is slightly altered. We must also keep in mind all of the properties of fourth power paths covered in the previous section. Thus, for $P^4_n$, the following is true.

\[ d(u,v) = \left\lceil \frac{d_{P_n}(u,v)}{4} \right\rceil \quad u,v \in V(P^4_n) \quad (1) \]

\[ \text{diam}(P^4_n) = \left\lceil \frac{n-1}{4} \right\rceil \quad (2) \]

For example, consider $P^4_8$

\[ d(u,v) = \left\lceil \frac{d_{P_8}(u,v)}{4} \right\rceil \quad u,v \in V(P^4_8) \]

\[ \text{diam}(P^4_8) = \left\lceil \frac{8-1}{4} \right\rceil \]

By (2), the diameter of $P^4_8$, equals $\lceil \frac{8-1}{4} \rceil$ which is 2. Thus we see that $|f(u) - f(v)| \geq 1 + \text{diam}(P^4_8) - d(u,v)$ implies $|f(u) - f(v)| \geq 3 - d(u,v)$.

If we choose to label vertices from left to right, the radio labeling would be as in Figure 1.24.

Notice that all vertices, if labeled consecutively from left to right, are distance one away from each other and so the label assigned to each vertex must have a difference of at least two. In this example, the span of the radio labeling is 14. We will try alternating sides and see if this lowers the span.

The span of this labeling in Figure 1.25 was not reduced. This occurred because, still, the distance between consecutively labeled vertices was equal to one, and thus the
labeling for each consecutively labeled vertex had to have a difference of at least two. In order to reduce the span at all, the distance between at least one pair of consecutively labeled vertices must be greater than one. We want to try to avoid consecutively labeling adjacent vertices as much as possible. Let us identify pairs of vertices $u$ and $v$ such that the distance between them is strictly greater than one. Consider the following pairs of vertices: \{v_1, v_6\}, \{v_1, v_7\}, \{v_1, v_8\}, \{v_2, v_7\}, \{v_2, v_8\}, and \{v_3, v_8\}. For these pairs of vertices, the distance between them is strictly greater than one. The goal now becomes to label these pairs of vertices consecutively in the labeling so that the difference in their label will be at least one, rather than at least two. Observe the labeling of $P_8$ in Figure 1.26.
The span of the labeling in Figure 1.26 was slightly reduced simply by maximizing distances between consecutively labeled vertices. Now, $P^4_8$ was a rather simple fourth power graph. For larger fourth power graphs, say $P^4_{25}$, labeling and obtaining an optimal radio labeling becomes a much more complex task. A key component, however, has been identified and that is that the distance between consecutively labeled vertices must be maximized in order to reduce the span. In this paper, we will provide a lower bound for the radio number of $P^4_n$. Since, as we mentioned before, we approach $P^4_n$ slightly differently depending on the parity of $n$, we will be considering two cases.

1.5 Notation and Labeling

Before we begin, it is important for the reader to understand much of the notation that is used in the labeling of $P^4_n$ so we will define a few terms and notation first.

Let $M, N \in \mathbb{N} \cup \{0\}$, then we define the ordered pair $(M, N)$ to be a block, which indicates a pattern to follow when labeling consecutive vertices. For example, if we begin labeling using a $(M, N)$ block, the first vertex labeled, $x_i$, will have a level that is congruent to $M \pmod{p}$, where $p$ is the power of the path. The next vertex labeled, $x_{i+1}$, will have a level that is congruent to $N \pmod{p}$. The following vertex labeled, $x_{i+2}$, will have level congruent to $M \pmod{p}$, continuing in this fashion until all of the vertices of levels congruent to $M,N \pmod{p}$ have been labeled. Labeling would then continue with a new block with specified values. We may also choose to specify what side the vertex is on by writing $(LM, RN)$. This would mean that the first vertex labeled, $x_i$, would be on the left side and would have a level congruent to $M \pmod{p}$, then $x_{i+1}$ would be on the right side and would have a level congruent to $N \pmod{p}$, so on and so forth. Without loss of generality, in this paper, when labeling, blocks will always begin with vertices on the left side of the graph. Since $P^4_n$ is symmetric, beginning the labeling on the right side would behave in the same way as beginning on the left side.

A disconnection occurs (or we say that there is a disconnect) when $L(x_i) + L(x_{i+1})$ is not congruent to a specified value modulo $p$ that maximizes the distance between two consecutively labeled vertices. This specific value changes depending upon the specific
case under consideration.

Disconnections will be ranked as being the best type, the second best type, the third best type, ..., or the worst type of disconnection. The goal is to maximize distances between consecutively labeled vertices. This is achieved by consecutively labeling vertices of specific levels or belonging to specific blocks until these vertices are exhausted, at which point we have a disconnection. The worst type of disconnection is one which minimizes the distance between the consecutively labeled vertices according to the respective level-sum definition of distance.

A **labeling pattern** is a specific arrangement of blocks that specifies how vertices will be labeled.

For any labeling pattern, $P^4_n$ will be said to have an “even” pairing if, for each block, $(M, N)$, in the labeling pattern, the number of vertices with level congruent to $M \pmod{p}$ on one side equals the number of vertices with level congruent to $N \pmod{p}$ on the other side. If the labeling of $P^4_n$ does not have an “even” pairing, then $P^4_n$ will be said to have extra vertices. These extra vertices, along with their respective levels, will be closely noted, as they may end up altering the blocks in the labeling pattern. We will investigate the lower bound of $rn(P^4_n)$ when $n$ is even in Chapter 2, and when $n$ is odd in Chapter 3.
Chapter 2

Lower Bound of $r_n(P_n^4)$ for $n$ Even

2.1 General Lower Bound of $r_n(P_n^4)$ for $n$ Even

Lemma 2.1 (General Lower Bound). Let $P_n^4$ be a fourth power path on $n$ vertices where $n \geq 6$ and let $k = \lceil \frac{n-1}{4} \rceil$, i.e., $k = \text{diam}(P_n^4)$.

If $n$ is even, then

$$r_n(P_n^4) \geq \begin{cases} 2k^2 + 1, & \text{if } n \equiv 0 \pmod{8}; \\ 2k^2, & \text{if } n \equiv 2 \pmod{8}; \\ 2k^2 + 1, & \text{if } n \equiv 4 \pmod{8}; \\ 2k^2, & \text{if } n \equiv 6 \pmod{8}. \end{cases}$$

Proof of Lemma 2.1. Let $f$ be a radio-labeling for $P_n^4$. Re-arrange $V(P_n^4) = \{x_1, x_2, ..., x_n\}$ with $0 = f(x_1) < f(x_2) < f(x_3) < ... < f(x_n)$. Note that $f(x_n)$ is the span of $f$.

By definition, $f(x_{i+1}) - f(x_i) \geq k + 1 - d(x_i, x_{i+1})$ for $1 \leq i \leq n - 1$. Summing up these $n - 1$ inequalities, we have

$$f(x_n) \geq (n - 1)(k + 1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}).$$

Thus to minimize $f(x_n)$ it suffices to maximize $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$.

Since $n$ is even,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lfloor \frac{L(x_i) + L(x_{i+1}) + 1}{4} \right\rfloor.$$

Observe from the above inequality we have:
1) For each $i$, the equality for $d(x_i, x_{i+1}) \leq \left\lceil \frac{L(x_i) + L(x_{i+1}) + 1}{4} \right\rceil$ holds only when $x_i$ and $x_{i+1}$ are on opposite sides, unless one of the vertices is a center and the other vertex is of level not congruent to 0 (mod 4).

2) In the summation $\sum_{i=1}^{n-1} \left\lceil \frac{L(x_i) + L(x_{i+1}) + 1}{4} \right\rceil$, each vertex of $P_n^4$ occurs exactly twice, except for $x_1$ and $x_n$, for which each occurs only once.

By direct calculation (see Appendix A.2), we have

$$\left\lceil \frac{L(u) + L(v) + 1}{4} \right\rceil = \begin{cases} \frac{L(u) + L(v) + 4}{4}, & \text{if } L(u) + L(v) \equiv 0 \pmod{4}; \\ \frac{L(u) + L(v) + 1}{4} - \frac{1}{4}, & \text{if } L(u) + L(v) \equiv 1 \pmod{4}; \\ \frac{L(u) + L(v) + 4}{4} - \frac{2}{4}, & \text{if } L(u) + L(v) \equiv 2 \pmod{4}; \\ \frac{L(u) + L(v) + 4}{4} - \frac{3}{4}, & \text{if } L(u) + L(v) \equiv 3 \pmod{4}. \end{cases}$$

Therefore,

$$\left\lceil \frac{L(x_i) + L(x_{i+1}) + 1}{4} \right\rceil \leq \frac{L(x_i) + L(x_{i+1}) + 4}{4},$$

and the equality holds only if $L(x_i) + L(x_{i+1}) \equiv 0 \pmod{4}$. Combining this with 1) above, there exist at most $n - 4$ of the $i$’s such that $d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 4)/4$, i.e., there are at least three disconnections in the labeling. Moreover, since among all the vertices only the two centers are of level equal to zero, therefore $L(x_1) + L(x_n) \geq 0 + 0 = 0$, we conclude that

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lceil \frac{L(x_i) + L(x_{i+1}) + 4}{4} \right\rceil - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}$$

$$= \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + (n - 1) - \frac{3}{4}$$

$$\leq \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - 0 - 0 \right] + (n - 1) - \frac{3}{4}$$

$$= \frac{1}{2} \left[ 2 \left( 0 + 1 + 2 + \ldots + \left( \frac{n}{2} - 1 \right) \right) \right] + n - \frac{7}{4}$$

$$= \frac{n^2}{8} + \frac{3}{4} n - \frac{7}{4}.$$  \hspace{1cm} (2.1)

Hence,

$$\text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4} n - \frac{7}{4} \right).$$
There are four cases according to \( n \mod 8 \) when \( n \) is even. By direct calculation (see Appendix B.1) and considering that \( \text{rn}(P^4_n) \) is an integer, we have:

\[
\text{rn}(P^4_n) \geq \begin{cases} 
\left\lceil \frac{2k^2 + \frac{3}{4}}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv_8 0 \quad (\text{i.e., } n = 4k \text{ and } k \text{ is even}); \\
\left\lceil \frac{2k^2 - \frac{1}{4}}{4} \right\rceil = 2k^2, & \text{if } n \equiv_8 2 \quad (\text{i.e., } n = 4k - 1 \text{ and } k \text{ is odd}); \\
\left\lceil \frac{2k^2 + \frac{3}{4}}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv_8 4 \quad (\text{i.e., } n = 4k \text{ and } k \text{ is odd}); \\
\left\lceil \frac{2k^2 - \frac{1}{4}}{4} \right\rceil = 2k^2, & \text{if } n \equiv_8 6 \quad (\text{i.e., } n = 4k - 2 \text{ and } k \text{ is even}).
\end{cases}
\]

Further investigation for a sharper lower bound of \( \text{rn}(P^4_n) \) when \( n \equiv_8 4 \) and when \( n \equiv_8 6 \) is needed.

### 2.2 Sharper Lower Bound of \( \text{rn}(P^4_n) \) for the Case when \( n \equiv 4 \) or 6 (mod 8)

**Lemma 2.2.** Let \( P^4_n \) be a fourth power path on \( n \) vertices where \( n \geq 6 \) and let \( k = \lceil \frac{n-1}{4} \rceil \), i.e., \( k = \text{diam}(P^4_n) \).

\[
\text{rn}(P^4_n) \geq \begin{cases} 
2k^2 + 2, & \text{if } n \equiv_8 4; \\
2k^2 + 1, & \text{if } n \equiv_8 6.
\end{cases}
\]

There are three cases to consider based on the number of disconnections that occur in the labeling pattern. The types of disconnections will be investigated and looked at in more detail in order to view the affects they have on the lower bound of \( \text{rn}(P^4_{8q+4}) \) and \( \text{rn}(P^4_{8q+6}) \).

Before we discuss the different cases, we must discuss the general labeling techniques that will be applied. Recall that when \( n \) is even, in order to maximize the distance between consecutively labeled vertices, the level sums of the consecutively labeled vertices must be maximized as well. Therefore, we wish to label consecutive vertices \( x_i \) and \( x_{i+1} \) so that \( L(x_i) + L(x_{i+1}) \equiv_4 0 \). Also note that for two consecutively labeled vertices \( x_i \) and
Proof of Lemma 2.2. For $P_n^4$ where $n$ is even, without consideration of any extra vertices, blocks in the labeling pattern will be of the following type (without loss of generality, we start with a left vertex):

$$\begin{align*}
(L0, R0) & \quad (L1, R3) \\
(L2, R2) & \quad (L3, R1)
\end{align*}$$

We investigate separate cases based on the number of disconnections that occur in the labeling pattern in the following subsections.

2.2.1 Case 1: At Least 5 Disconnections

There are at most $n - 6$ of the $i$’s such that $d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 4)/4$. That is, there are at least five disconnections in the labeling pattern and $L(x_1) + L(x_n) \geq 0 + 0 = 0$.

Then,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left[ \sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1}) + 4}{4} \right] - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}$$

$$= \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + (n - 1) - \frac{5}{4}$$

$$\leq \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - 0 - 0 \right] + (n - 1) - \frac{5}{4}$$

$$= \frac{1}{2} \left[ \frac{1}{2} \left( 0 + 1 + 2 + ... + \left( \frac{n}{2} - 1 \right) \right) \right] + n - \frac{9}{4}$$

$$= \frac{n^2}{8} + \frac{3}{4} n - \frac{9}{4}. \quad (2.2)$$

Note that Equation (2.2) is equal to Equation (2.1) minus $\frac{3}{4}$.

Hence,

$$\text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4} n - \frac{9}{4} \right).$$

By direct calculation (see Appendix B.2) and considering that $\text{rn}(P_n^4)$ is an integer, we have:
\[
\begin{align*}
\text{rn}(P^4_n) \geq \left\{ \begin{array}{ll}
\lceil (2k^2 + \frac{3}{4}) + \frac{2}{4} \rceil = 2k^2 + 2, & \text{if } n \equiv_8 4 \\
\lceil (2k^2 - \frac{1}{4}) + \frac{2}{4} \rceil = 2k^2 + 1, & \text{if } n \equiv_8 6 \\
& (\text{i.e., } n = 4k \text{ and } k \text{ is odd}); \end{array} \right.
\end{align*}
\]

2.2.2 Case 2: Exactly 4 Disconnections

There are exactly \( n - 5 \) of the \( i \)'s such that \( d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 4)/4 \), that is, there are exactly four disconnections in the labeling pattern. This case will be broken down into the following two sub-cases based on \( L(x_1) + L(x_n) \).

Case 2.1 \( L(x_1) + L(x_n) \geq 0 + 1 = 1 \), therefore

\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left[ \sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1}) + 4}{4} \right] - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4} \left[ \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) \right] + (n - 1) - \frac{4}{4} = \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + (n - 1) - \frac{4}{4} \leq \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - 0 - 1 \right] + (n - 1) - \frac{4}{4} = \frac{2}{4} \left( 0 + 1 + \ldots + \left( \frac{n}{2} - 1 \right) \right) - \frac{1}{4} + (n - 1) - \frac{4}{4} = \frac{1}{2} \left[ 2 \left( \frac{n^2}{8} - \frac{n}{4} \right) \right] + n - \frac{9}{4} = \frac{n^2}{8} + \frac{3}{4} n - \frac{9}{4}.
\]

Note that Equation (2.3) is equal to Equation (2.1) minus \( \frac{2}{4} \).

Hence,

\[
\text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4} n - \frac{9}{4} \right).
\]
Direct calculations (see Appendix B.3) lead to

\[
\text{rn}(P_n^4) \geq \begin{cases} 
    \left\lceil (2k^2 + \frac{3}{4}) + \frac{2}{4} \right\rceil = 2k^2 + 2, & \text{if } n \equiv 4 \pmod{8} \quad \text{(i.e., } n = 4k \text{ and } k \text{ is odd)}; \\
    \left\lceil (2k^2 - \frac{1}{4}) + \frac{2}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv 6 \pmod{8} \quad \text{(i.e., } n = 4k - 2 \text{ and } k \text{ is even)}. 
\end{cases}
\]

**Case 2.2**  \(L(x_1) + L(x_n) = 0 + 0 = 0.\)

**Claim:** In this case, at least two of the disconnections that occur cannot be of the best type.

By direct calculation, our claim, and noting that \(L(x_1) + L(x_n) = 0 + 0 = 0,\) we have,

\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left[ \sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1}) + 4}{4} \right] - \frac{1}{4} - \frac{1}{4} - \frac{2}{4} - \frac{2}{4} \\
= \frac{1}{4} \left[ \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) \right] + (n - 1) - \frac{6}{4} \\
= \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + (n - 1) - \frac{6}{4} \\
\leq \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - 0 - 0 \right] + (n - 1) - \frac{6}{4} \\
= \frac{1}{2} \left[ 2 \left( \frac{n^2}{8} - \frac{n}{4} \right) \right] + n - \frac{10}{4} \\
= \frac{n^2}{8} + \frac{3}{4} n - \frac{10}{4}. \quad (2.4)
\]

Note that Equation (2.4) is equal to Equation (2.1) minus \(\frac{3}{4}.\)

Hence,

\[
\text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4} n - \frac{10}{4} \right). 
\]
By direct calculation (see Appendix B.3),
\[
\mathrm{rn}(P_n^4) \geq \begin{cases} 
\lceil (2k^2 + \frac{3}{4}) + \frac{3}{4} \rceil = 2k^2 + 2, & \text{if } n \equiv 8 \ 4 \\
(\text{i.e., } n = 4k \text{ and } k \text{ is odd}); \\
\lceil (2k^2 - \frac{1}{4}) + \frac{3}{4} \rceil = 2k^2 + 1, & \text{if } n \equiv 8 \ 6 \\
(\text{i.e., } n = 4k - 2 \text{ and } k \text{ is even}).
\end{cases}
\]

**Proof of claim:** For \( n \equiv 8 \ 4 \) or \( 6 \), we have the following blocks as well as extra vertices:

\[(L_0, R_0), (L_1, R_3), (L_2, R_2), (L_3, R_1) \quad L_1, R_1\]

We wish to have exactly four disconnections and we also want \( L(x_1) + L(x_n) = 0 + 0 = 0 \) under this case. Therefore we must break the \((L_0, R_0)\) block in order to begin with a vertex whose level is equal to zero and end with a vertex whose level is equal to zero. Without considering any extra vertices for \( P_{8q+4}^4 \) and \( P_{8q+6}^4 \), we have the following blocks:

\[(L_0, R_0), (L_1, R_3), (L_2, R_2), (L_3, R_1), (L_0, R_0)\]

Note that any permutation of the above blocks will yield exactly four disconnections. Now, \( P_{8q+4}^4 \) and \( P_{8q+6}^4 \) both have an extra set of vertices whose level is congruent to 1 (mod 4). We must arrange these vertices wisely in the labeling as to not increase the number of disconnections that occur. Thus our new blocks become:

\[(L_0, R_0), (L_1, R_3) - L_1, (L_2, R_2), R_1 - (L_3, R_1), (L_0, R_0)\]

Since we want \( L(x_1) + L(x_n) = 0 + 0 = 0 \), we must fix the \((L_0, R_0)\) blocks so that our labeling pattern starts and ends with the \((L_0, R_0)\) blocks. Special attention is given to the vertices whose levels are congruent to 1 (mod 4), that is, either the first or the last vertex of the \((L_1, R_3) - L_1\) or the \( R_1 - (L_3, R_1) \) blocks in the labeling pattern. We call these vertices “end-1” vertices. In total, there are four “end 1”
vertices and so all disconnections in the labeling pattern will occur at these “end 1” vertices. The best type of disconnection would occur if an “end 1” vertex was followed or preceded by a vertex whose level was congruent to 0 (mod 4). However, there are only two such vertices available. Therefore, at least two of the four “end 1” vertices cannot have disconnections of the best type.

Note: We can look at all possible permutations of the blocks for \( n \) even having 4 disconnections and look at all possible labeling patterns and observe that at least two of the disconnections in the labeling patterns are indeed not of the best type.

<table>
<thead>
<tr>
<th>Labeling pattern</th>
<th>Disconnections</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0) \rightarrow (1, 3) -1 \rightarrow (2, 2) \rightarrow (3, 1) \rightarrow (0, 0))</td>
<td>(-\frac{1}{4} - \frac{3}{4} - \frac{3}{4} - \frac{1}{4})</td>
<td>(-\frac{8}{4})</td>
</tr>
<tr>
<td>((0, 0) \rightarrow (1, 3) -1 \rightarrow (3, 1) \rightarrow (2, 2) \rightarrow (0, 0))</td>
<td>(-\frac{1}{4} - \frac{3}{4} - \frac{3}{4} - \frac{1}{4})</td>
<td>(-\frac{8}{4})</td>
</tr>
<tr>
<td>((0, 0) \rightarrow (2, 2) \rightarrow (1, 3) -1 \rightarrow (3, 1) \rightarrow (0, 0))</td>
<td>(-\frac{1}{4} - \frac{3}{4} - \frac{3}{4} - \frac{1}{4})</td>
<td>(-\frac{8}{4})</td>
</tr>
</tbody>
</table>

Table 2.1: All labeling patterns for cases \( n \equiv 4 \) or 6 (mod 8) with four disconnections

### 2.2.3 Case 3: Exactly 3 Disconnections

There are exactly \( n - 4 \) of the \( i' \)s such that \( d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 4)/4 \), that is, there are exactly three disconnections in the labeling pattern. Also note that \( L(x_1) + L(x_n) \geq 0 + 1 = 1 \).

**Claim:** In this case, at least one of the disconnections in the labeling pattern will not be of the best type.

By calculation, our claim, and noting that \( L(x_1) + L(x_n) \geq 0 + 1 = 1 \), we have,
\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left[ \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) + 4 \right] - \frac{1}{4} - \frac{1}{4} - \frac{2}{4} \\
= \frac{1}{4} \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) + (n-1) - \frac{4}{4} \\
= \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + (n-1) - \frac{4}{4} \\
\leq \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - 0 - 1 \right] + (n-1) - \frac{4}{4} \\
= \frac{2}{4} \left[ 2 \left( 0 + 1 + \ldots + \left( \frac{n}{2} - 1 \right) \right) \right] - \frac{1}{4} + n - \frac{8}{4} \\
= \frac{1}{2} \left[ 2 \left( \frac{n^2}{8} - \frac{n}{4} \right) \right] + n - \frac{9}{4} \\
= \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4}, \quad (2.5)
\]

Note that Equation (2.5) is equal to Equation (2.1) minus \( \frac{2}{4} \).

Hence,
\[
\text{rn}(P_{n}^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right).
\]

Direct calculations (see Appendix B.4) lead to,
\[
\text{rn}(P_{n}^4) \geq \begin{cases} 
\left\lceil (2k^2 + \frac{3}{4}) + \frac{2}{4} \right\rceil = 2k^2 + 2, & \text{if } n \equiv 8 \text{ 4} \\
( \text{i.e., } n = 4k \text{ and } k \text{ is odd}) \\
\left\lceil (2k^2 - \frac{1}{4}) + \frac{2}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv 8 \text{ 6} \\
( \text{i.e., } n = 4k - 2 \text{ and } k \text{ is even}) 
\end{cases}
\]

**Proof of Claim:**

For \( n \equiv 8 \text{ 4} \) or 6, we have the following blocks as well as extra vertices:

\[(L0, R0), \quad (L1, R3), \quad (L2, R2), \quad (L3, R1) \quad L1, R1\]

Both \( P_{8q+4}^4 \) and \( P_{8q+6}^4 \) have an extra pair of vertices whose level is congruent to 1 (mod 4). Thus, the arrangements of blocks must be modified to ensure that we are alternating
sides when labeling in order to achieve the best types of disconnects. To ensure that there are only three disconnections, our new blocks must be:

\[(L0, R0), (L1, R3)-L1, (L2, R2), R1-(L3, R1)\]

Observe that we have two blocks whose first and last vertex have level congruent to 1 (mod 4). Therefore there will be at least two disconnections that occur at these “end 1” vertices. The best case scenario consists of following or preceding each (1-(3-1)) or ((1-3)-1) block with a (L0, R0) block in the labeling to achieve disconnects of the best type. This, however, is impossible to do without increasing the number of disconnections that occur in the labeling. This means that at least one of the “end 1” vertices will have to be followed or preceded by a vertex whose level is not congruent to 0 (mod 4) in the labeling. If an “end 1” vertex is not followed or preceded by a vertex whose level is congruent to 0 (mod 4), then it must be followed or preceded by a vertex whose level is either congruent to 1 (mod 4) or congruent to 2 (mod 4). Thus, out of the three disconnections that occur, at least two of them will occur at the “end 1” vertices. Furthermore, out of these two disconnects that occur at the “end 1” vertices, at least one of them will not be of the best type.

By Case 1, Case 2, and Case 3, we obtain the result stated in Lemma 2.2. □

2.3 Summary of Results of Lower Bound of \(r_n(P^4_n)\) for \(n\) Even

We have considered an arbitrary radio labeling of \(P^4_n\) for \(n\) even, having at least 3 disconnections, as well as further investigated special cases for \(n \equiv 4 \mod 8\) and \(n \equiv 6 \mod 8\) in order to obtain a sharper lower bound of \(r_n(P^4_n)\).

Combining Lemma 2.1 and Lemma 2.2, we have the following:

**Lemma 2.3.** Let \(P^4_n\) be a fourth power path on \(n\) vertices where \(n \geq 6\) and let \(k = \left\lceil \frac{n-1}{4} \right\rceil\),
i.e., $k = diam(P^4_n)$.

If $n$ is even, then $rn(P^4_n) \geq \begin{cases} 
2k^2 + 1, & \text{if } n \equiv_8 0; \\
2k^2, & \text{if } n \equiv_8 2; \\
2k^2 + 2, & \text{if } n \equiv_8 4; \\
2k^2 + 1, & \text{if } n \equiv_8 6.
\end{cases}$

Refer to Figures 4.1, 4.3, 4.5, and 4.7 for examples of radio labelings of $P^4_n$ for $n$ even, whose spans are greater than or equal to the lower bounds given in Lemma 2.3.
Chapter 3

Lower Bound of $r_n(P^4_n)$ for $n$ Odd

3.1 Lower Bound of $r_n(P^4_n)$ for $n$ Odd

Lemma 3.1. Let $P^4_n$ be a fourth power path on $n$ vertices where $n \geq 6$ and let $k = \lceil \frac{n-1}{4} \rceil$, i.e., $k = diam(P^4_n)$.

If $n$ is odd, then

$$r_n(P^4_n) \geq \begin{cases} 
2k^2 + 2, & \text{if } n \equiv 1 \pmod{8} \text{ and } n \geq 17; \\
2k^2 + 1, & \text{if } n \equiv 3; \\
2k^2 + 2, & \text{if } n \equiv 5; \\
2k^2 + 1, & \text{if } n \equiv 7; \\
2k^2 + 1, & \text{if } n = 9.
\end{cases}$$

Proof of Lemma 3.1. We retain the same notation and employ the same method used in the proof of Lemma 2.1. Since $n$ is odd,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lceil \frac{L(x_i) + L(x_{i+1})}{4} \right\rceil.$$

Observe from the above inequality we have:

1) For each $i$, the equality for $d(x_i, x_{i+1}) \leq \left\lceil \frac{L(x_i) + L(x_{i+1})}{4} \right\rceil$ holds only when $x_i$ and $x_{i+1}$ are on opposite sides, unless one of them is a center, and

2) In the summation $\sum_{i=1}^{n-1} \left\lceil \frac{L(x_i) + L(x_{i+1})}{4} \right\rceil$, each vertex of $P^4_n$ occurs exactly twice, except $x_1$ and $x_n$, for which each occurs only once.
By direct calculation (see Appendix A.2), we have

\[
\left\lfloor \frac{L(u) + L(v)}{4} \right\rfloor = \begin{cases} 
\frac{L(u) + L(v) + 3}{4} - \frac{3}{4}, & \text{if } L(u) + L(v) \equiv 4 0; \\
\frac{L(u) + L(v) + 3}{4}, & \text{if } L(u) + L(v) \equiv 4 1; \\
\frac{L(u) + L(v) + 3}{4} - \frac{1}{4}, & \text{if } L(u) + L(v) \equiv 4 2; \\
\frac{L(u) + L(v) + 3}{4} - \frac{2}{4}, & \text{if } L(u) + L(v) \equiv 4 3. 
\end{cases}
\]

Therefore

\[
\left\lfloor \frac{L(x_i) + L(x_{i+1})}{4} \right\rfloor \leq \frac{L(x_i) + L(x_{i+1}) + 3}{4},
\]

and the equality holds only if \(L(x_i) + L(x_{i+1}) \equiv 4 1\). Combining this with 1), there are two possible cases to consider, which will be covered in subsections 3.1.1 and 3.1.2.

### 3.1.1 Case 1: At Least 3 Disconnections

There exist at most \(n - 4\) of the \(i's\) such that \(d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 3)/4\). That is, there are at least three disconnections in the labeling. In this case, since \(n\) is odd, there is only one center. Therefore, \(L(x_1) + L(x_n) \geq 0 + 1 = 1\).

Then

\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lfloor \frac{L(x_i) + L(x_{i+1}) + 3}{4} \right\rfloor - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + \frac{3}{4} (n - 1) - \frac{3}{4} \\
\leq \frac{1}{4} \left[ \left( 2 \sum_{i=1}^{n} L(x_i) \right) - 0 - 1 \right] + \frac{3}{4} (n - 1) - \frac{3}{4} \\
= \frac{2}{4} \left[ 2 \left( 1 + 2 + 3 + \ldots + \frac{n-1}{2} \right) \right] - \frac{1}{4} + \frac{3}{4} n - \frac{6}{4} \\
= \frac{n^2}{8} + \frac{3}{4} n - \frac{15}{8}. \tag{3.1}
\]

Thus,

\[
\text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4} n - \frac{15}{8} \right).
\]
There are four cases according to \( n \pmod{8} \) when \( n \) is odd. By direct calculation (see Appendix C.1) and considering that \( \text{rn}(P^4_n) \) is an integer, we have:

\[
\text{rn}(P^4_n) \geq \begin{cases} 
2k^2 + 1, & \text{if } n \equiv_8 1 \quad (\text{i.e., } n = 4k + 1 \text{ and } k \text{ is even}); \\
\lceil 2k^2 + \frac{1}{2} \rceil = 2k^2 + 1, & \text{if } n \equiv_8 3 \quad (\text{i.e., } n = 4k - 1 \text{ and } k \text{ is odd}); \\
2k^2 + 1, & \text{if } n \equiv_8 5 \quad (\text{i.e., } n = 4k + 1 \text{ and } k \text{ is odd}); \\
\lceil 2k^2 + \frac{1}{2} \rceil = 2k^2 + 1, & \text{if } n \equiv_8 7 \quad (\text{i.e., } n = 4k - 1 \text{ and } k \text{ is even}).
\end{cases}
\]

### 3.1.2 Case 2: Exactly 2 Disconnections

Recall that when \( n \) is odd, the distance between consecutively labeled vertices, \( x_i \) and \( x_{i+1} \), is maximized when \( L(x_i) + L(x_{i+1}) \equiv_4 1 \). Also keep in mind that for vertices \( x_i \) and \( x_{i+1} \), there is a disconnection in the labeling pattern when \( L(x_i) + L(x_{i+1}) \not\equiv_4 1 \). So the general blocks that must be used, without considering any extra vertices for a particular case, are the following:

\[(L_0, R_1) \quad (L_1, R_0) \quad (L_2, R_3) \quad (L_3, R_2)\]

We will investigate when there exist exactly \( n - 3 \) of the \( i \)'s such that \( d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 3)/4 \). That is, there are exactly two disconnections in the labeling. We divide this discussion into two further sub-cases for \( n \equiv_8 1 \) and then for \( n \equiv_8 3, 5, \) or 7. It is recommended that the readers try a couple of examples to gain a better understanding of the following claims.

If we wish to achieve a labeling pattern with only two disconnections, extra attention must be placed on the positioning of the center, denoted by \( C \), as to reduce the number of disconnections by one.

**Case 2.1: \( n \equiv 1 \pmod{8} \)** If \( n \equiv_8 1 \), then neither \( x_1 \) nor \( x_n \) is the center. Therefore, \( \{L(x_1), L(x_n)\} \equiv_4 \{0, 2\}, \{0, 3\}, \{2, 2\}, \text{ or } \{3, 3\} \). Note that for \( P^4_{8q+1} \) there is an even pairing of vertices so there are no extra vertices other than the center. Thus,
in order for there to be exactly two disconnections, we have the following blocks:

\[(L_0, R_1) - C - (L_1, R_0) \quad (L_2, R_3) \quad (L_3, R_2)\]

Therefore \(L(x_1) + L(x_n) \geq 2 + 2 = 4\). By similar calculation to Case 1, we have

\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left[ \sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1})}{4} + 3 \right] - \frac{1}{4} \cdot \frac{1}{4} \\
= \frac{1}{4} \left[ \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) \right] + \frac{3}{4}(n-1) - \frac{2}{4} \\
= \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right] + \frac{3}{4}(n-1) - \frac{2}{4} \\
\leq \frac{1}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - 2 - 2 \right] + \frac{3}{4}(n-1) - \frac{2}{4} \\
= \frac{2}{4} \left[ 2 \left( 1 + 2 + \ldots + \left( \frac{n-1}{2} \right) \right) \right] - \frac{4}{4} + \frac{3}{4}(n-1) - \frac{2}{4} \\
= \frac{1}{2} \left[ 2 \left( \frac{n^2}{8} - \frac{1}{8} \right) \right] + \frac{3}{4} \cdot \frac{n}{4} - \frac{9}{4} \\
= \frac{n^2}{8} + \frac{3}{4}n - \frac{19}{8}.
\] (3.2)

Note that Equation (3.2) is equal to Equation (3.1) minus \(\frac{2}{4}\).

Therefore by direct calculations (see Appendix C.2), since \(n = 4k + 1\) and \(k\) is even, we have

\[\text{rn}(P_n^4) \geq \left( 2k^2 + 1 \right) + \frac{3}{4} \cdot \frac{1}{4} = \left( \frac{2k^2}{2} + \frac{3}{2} \right) = 2k^2 + 2.\]

**Case 2.2: \(n \equiv 3, 5, \text{ or } 7 \pmod{8}\)** If \(n \equiv 3, 5, \text{ or } 7 \pmod{8}\), then \(\{L(x_1), L(x_n)\} \equiv_4 \{1, 2\}, \{1, 3\}, \{2, 2\}, \text{ or } \{3, 3\}\). Note that \(P_{8q+3}^4\) and \(P_{8q+7}^4\) both have an extra pair of vertices whose level is congruent to 1 (mod 4). These extra vertices must be placed wisely,
on different sides, in the labeling as to not increase the number of disconnections. Therefore, the blocks become:

\[
\begin{align*}
R1 &- (L0, R1) - C - (L1, R0) - L1 \\
&\quad (L2, R3) \quad (L3, R2)
\end{align*}
\]

If \( n \equiv 8 \), then \( \{L(x_1), L(x_n)\} \equiv \{1, 2\} \) or \( \{2, 2\} \).

Now, \( P_{8q+5}^4 \) has two extra pairs of vertices whose levels are congruent to 1 (mod 4) and 2 (mod 4). Their placement in the labeling must be done in a way that does not increase the number of disconnections that occur in the labeling. Thus we have the following blocks:

\[
\begin{align*}
R1 &- (L0, R1) - C - (L1, R0) - L1 \\
&\quad (L2, R3) \quad L2 \quad R2 - (L3, R2)
\end{align*}
\]

Therefore, for \( n \equiv 3, 5, \) or 7, \( L(x_1) + L(x_n) \geq 1 + 2 = 3 \).

\[
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left\lfloor \sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1}) + 3}{4} \right\rfloor - \frac{1}{4} - \frac{1}{4}
\]

\[
= \frac{1}{4} \left\lfloor \sum_{i=1}^{n-1} L(x_i) + L(x_{i+1}) \right\rfloor + \frac{3}{4}(n-1) - \frac{2}{4}
\]

\[
\leq \frac{1}{4} \left\lfloor 2 \left( \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right\rfloor + \frac{3}{4}(n-1) - \frac{2}{4}
\]

\[
= \frac{2}{4} \left[ 2 \left( \sum_{i=1}^{n} L(x_i) \right) - 1 - 2 \right] + \frac{3}{4}(n-1) - \frac{2}{4}
\]

\[
= \frac{1}{2} \left\lfloor 2 \left( \frac{n^2}{8} - \frac{1}{8} \right) \right\rfloor + \frac{3}{4}n - \frac{8}{4}
\]

\[
= \frac{n^2}{8} + \frac{3}{4}n - \frac{17}{8}.
\]

Note that Equation (3.3) is equal to Equation (3.1) minus \( \frac{1}{4} \).

Thus, by direct calculation (see Appendix C.2) we have,

\[
\text{For } n \equiv 8, \quad \text{rn}(P_{n}^4) \geq \left\lfloor \left( 2k^2 + \frac{1}{2} \right) + \frac{2}{4} - \frac{1}{4} \right\rfloor = \left\lfloor 2k^2 + \frac{3}{4} \right\rfloor.
\]
For \( n \equiv 8 \), \( r_n(P_4^n) \geq \left\lceil \left( 2k^2 + 1 \right) + \frac{2}{4} - \frac{1}{4} \right\rceil = \left\lceil 2k^2 + \frac{5}{4} \right\rceil \).

For \( n \equiv 7 \), \( r_n(P_4^n) \geq \left\lceil \left( 2k^2 + \frac{1}{2} \right) + \frac{2}{4} - \frac{1}{4} \right\rceil = \left\lceil 2k^2 + \frac{3}{4} \right\rceil \).

Thus we have,

\[
 r_n(P_4^n) \geq \begin{cases} 
\left\lceil 2k^2 + \frac{3}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv 3 \\
\left\lceil 2k^2 + \frac{5}{4} \right\rceil = 2k^2 + 2, & \text{if } n \equiv 5 \\
\left\lceil 2k^2 + \frac{3}{4} \right\rceil = 2k^2 + 1, & \text{if } n \equiv 7
\end{cases}
\]

\( (\text{i.e., } n = 4k - 1 \text{ and } k \text{ is odd};) \)

\( (\text{i.e., } n = 4k + 1 \text{ and } k \text{ is odd};) \)

\( (\text{i.e., } n = 4k - 1 \text{ and } k \text{ is even}). \)

Note that the lower bound of \( r_n(P_4^n) \) under Case 2 for \( n \equiv 3 \) or 7 coincides with the corresponding formulas in Case 1.

Now assume \( n \equiv 8 \) and \( n \geq 17 \), that is, \( n = 4k + 1 \), \( k \) is even and \( k \geq 4 \).
Assume to the contrary that \( f(x_n) = 2k^2 + 1 \). Then only Case 1 is possible and all of the following must hold:

1) \( \{x_1, x_n\} = \{v_{2k+1}, v_{2k+2}\} \) or \( \{v_{2k+1}, v_{2k}\} \)

That is, \( \{x_1, x_n\} = \{\text{center, a vertex right next to center}\} \)

2) \( f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1}) \) for all \( i \)

3) For all \( i \geq 1 \), the two vertices \( x_i \) and \( x_{i+1} \) are on opposites sides unless one of them is the center.

4) There exist three \( t' \)'s, \( 1 \leq t \leq n - 1 \) such that \( L(x_t) + L(x_{t+1}) \equiv 2 \) while \( L(x_t) + L(x_{t+1}) \equiv 1 \) for all other \( i \neq t \).
By 1) and by symmetry, we can assume that \( x_1 = v_{2k+1} \), i.e., \( x_1 \) is the center. Note that here, \( n \) is odd and so \( v_{2k+1} \) is the only center. Also, \( n = 4k + 1 \) for some even \( k \) and \( k \geq 4 \). Excluding the center, there are \( 4k \) vertices in total with \( 2k \) vertices on each side. In particular, there are \( \frac{k}{2} \) many vertices whose level is congruent to 0 (mod 4), 1 (mod 4), 2 (mod 4), and 3 (mod 4) on each side. Since \( x_n \) is either \( v_{2k} \) or \( v_{2k+2} \) and both of these vertices have level equal to 1, \( x_n \) is right next to the center. By 2) and 3), the only three \( t' \)'s in 4) must be \( k + 1 \), \( 2k + 1 \), and \( 3k + 1 \). For otherwise, there would be at least four \( t' \)'s with \( L(x_t) + L(x_{t+1}) \equiv 0, 2, \) or \( 3 \). Therefore we have:

5)

\[
\begin{align*}
L(x_i) &\equiv_4 0 & \text{if } i \in \{1, 3, 5, ..., k + 1, 3k + 2, 3k + 4, ..., 4k\} \\
L(x_i) &\equiv_4 1 & \text{if } i \in \{2, 4, 6, ..., k, 3k + 3, 3k + 5, ..., 4k + 1\} \\
L(x_i) &\equiv_4 2 & \text{if } i \in \{k + 2, k + 4, k + 6, ..., 2k, 2k + 3, 2k + 5, ..., 3k + 1\} \\
L(x_i) &\equiv_4 3 & \text{if } i \in \{k + 3, k + 5, k + 7, ..., 2k + 1, 2k + 2, 2k + 4, ..., 3k\}
\end{align*}
\]

That is, the labeling pattern is the following arrangement of blocks of vertices:

\[
\text{C-(1, 0) - (2, 3) - (3, 2) - (0, 1)}
\]

**Claim:** \( \{v_n, v_1\} = \{x_{k+1}, x_{3k+2}\} \) (i.e., \( v_1 \) and \( v_n \) are the last vertex whose level is congruent to 0 (mod 4) in the (1, 0) block and the first vertex whose level is congruent to 0 (mod 4) in the (0, 1) block)

Assuming this claim, we can further assume that \( v_n = x_{k+1} \) and \( v_1 = x_{3k+2} \). The proof for the other case is symmetric and thus equivalent.

By 5), \( L(x_k) = a \equiv_4 1 \) and \( L(x_{k+2}) = b \equiv_4 2 \). By 2), 3), the fact that \( k \) is even, and our assumption that \( L(x_{k+1}) = L(v_n) = L(v_{4k+1}) = 2k \), we have:
\begin{align*}
f(x_{k+1}) - f(x_k) &= k + 1 - \left\lfloor \frac{2k + a}{4} \right\rfloor \\
&= k + 1 - \left( \left\lfloor \frac{2k}{4} \right\rfloor + \left\lfloor \frac{a}{4} \right\rfloor \right) \quad (\because (2k, a) \equiv_4 (0, 1)) \\
&= \frac{k}{2} + 1 - \left\lfloor \frac{a}{4} \right\rfloor ,
\end{align*}

\begin{align*}
f(x_{k+2}) - f(x_{k+1}) &= k + 1 - \left\lfloor \frac{b + 2k}{4} \right\rfloor \\
&= k + 1 - \left( \left\lfloor \frac{b}{4} \right\rfloor + \left\lfloor \frac{2k}{4} \right\rfloor \right) \quad (\because (b, 2k) \equiv_4 (2, 0)) \\
&= \frac{k}{2} + 1 - \left\lfloor \frac{b}{4} \right\rfloor ,
\end{align*}

and so,

\begin{align*}
f(x_{k+2}) - f(x_k) &= k + 2 - \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor .
\end{align*}

By definition and by Lemma 1.2,

\begin{align*}
f(x_{k+2}) - f(x_k) &\geq k + 1 - \left\lfloor \frac{|a - b|}{4} \right\rfloor .
\end{align*}

Therefore,

\begin{align*}
k + 2 - \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor &\geq k + 1 - \left\lfloor \frac{|a - b|}{4} \right\rfloor .
\end{align*}

Thus,

\begin{align*}
\left\lfloor \frac{|a - b|}{4} \right\rfloor &\geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1.
\end{align*}

Case 1: $a \geq b$

Then by Proposition 1.3 and $(a, b) \equiv_4 (1, 2),
\begin{align*}
\left\lfloor \frac{|a - b|}{4} \right\rfloor = \left\lfloor \frac{a - b}{4} \right\rfloor = \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor \geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1,
\end{align*}

\begin{align*}
1 &\geq 2 \left\lfloor \frac{b}{4} \right\rfloor .
\end{align*}

This implies that $b = 0$ which cannot happen because $b \equiv_4 2$. 
Case 2: $b \geq a$
Then by Proposition 1.3 and $(b, a) \equiv_4 (2, 1)$,
\[
\left\lfloor \frac{|a - b|}{4} \right\rfloor = \left\lfloor \frac{b - a}{4} \right\rfloor = \left\lfloor \frac{b}{4} \right\rfloor - \left\lfloor \frac{a}{4} \right\rfloor + 1 \geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1,
\]
\[
2 \geq 2 \left\lfloor \frac{a}{4} \right\rfloor , \text{ and thus } a = 0, 1, 2, 3, \text{ or } 4.
\]
This implies that $a = 1$ since $a \equiv_4 1$.

Thus, since $a = 1$, we have $L(x_k) = 1$ which means that it is the left vertex that is right next to the center since $x_{k+1} = v_n$ is a right vertex. Thus $x_k = v_{2k}$. In a similar fashion, we will show that $x_{3k+3}$ has level equal to one, is also right next to the center on the right side, and thus is equal to $v_{2k+2}$.

Recall that we let $v_1 = x_{3k+2}$ and so $L(x_{3k+2}) = 2k$. By 5) $L(x_{3k+1}) = a \equiv_4 2$ and $L(x_{3k+3}) = b \equiv_4 1$.

\[
f(x_{3k+2}) - f(x_{3k+1}) = k + 1 - \left\lfloor \frac{2k + a}{4} \right\rfloor \\
= k + 1 - \left( \left\lfloor \frac{2k}{4} \right\rfloor + \left\lfloor \frac{a}{4} \right\rfloor \right) (\because (2k, a) \equiv_4 (0, 2)) \\
= k + 1 - \left\lfloor \frac{2k}{4} \right\rfloor - \left\lfloor \frac{a}{4} \right\rfloor ,
\]
\[
f(x_{3k+3}) - f(x_{3k+2}) = k + 1 - \left\lfloor \frac{b + 2k}{4} \right\rfloor \\
= k + 1 - \left( \left\lfloor \frac{b}{4} \right\rfloor + \left\lfloor \frac{2k}{4} \right\rfloor \right) (\because (b, 2k) \equiv_4 (1, 0)) \\
= k + 1 - \left\lfloor \frac{b}{4} \right\rfloor ,
\]
and so,
\[
f(x_{3k+3}) - f(x_{3k+1}) = k + 2 - \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor .
\]
By definition and by Lemma 1.2,
\[
f(x_{3k+3}) - f(x_{3k+1}) \geq k + 1 - \left\lfloor \frac{|a - b|}{4} \right\rfloor .
\]
Therefore,

\[ k + 2 - \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor \geq k + 1 - \left\lfloor \frac{|a - b|}{4} \right\rfloor. \]

Thus,

\[ \left\lfloor \frac{|a - b|}{4} \right\rfloor \geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1. \]

Case 1: \( a \geq b \)

Then by Proposition 1.3 and \((a, b) \equiv_4 (2, 1)\),

\[ \left\lfloor \frac{|a - b|}{4} \right\rfloor = \left\lfloor \frac{a - b}{4} \right\rfloor = \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor + 1 \geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1, \]

\[ 2 \geq 2 \left\lfloor \frac{b}{4} \right\rfloor, \]

and thus \( b = 0, 1, 2, 3, \) or 4.

This implies that \( b = 1 \) since \( b \equiv_4 1 \).

Case 2: \( b \geq a \)

Then by Proposition 1.3 and \((b, a) \equiv_4 (1, 2)\),

\[ \left\lfloor \frac{|a - b|}{4} \right\rfloor = \left\lfloor \frac{b - a}{4} \right\rfloor = \left\lfloor \frac{b}{4} \right\rfloor - \left\lfloor \frac{a}{4} \right\rfloor \geq \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor - 1, \]

\[ 1 \geq 2 \left\lfloor \frac{a}{4} \right\rfloor, \]

and thus \( a = 0 \).

This implies that \( a = 0 \) which cannot happen because \( a \equiv_4 2 \).

Therefore, \( b = 1 = L(x_{3k+3}) \) which implies that \( x_{3k+3} \) is the right vertex that is right next to the center since \( x_{3k+2} = v_1 \) is a left vertex. Therefore, \( x_{3k+3} = v_{2k+2} \).

By \( L(x_1) + L(x_n) = 0 + 1 = 1 \) under Case 1, without loss of generality we assume \( L(x_1) = 0 \) which means \( x_1 \) is the center, and \( L(x_n) = 1 \) which means \( x_n \) is right next to the center. Now, \( x_n \) is a right vertex since \( x_{3k+2} = v_1 \) is a left vertex, and so \( x_n = v_{2k+2} \).

This implies that \( x_n = v_{2k+2} = x_{3k+3} \). But \( n = 4k + 1 \) so \( 4k + 1 = 3k + 3 \) implies that \( k = 2 \).

Thus we have arrived at a contradiction because we started with \( k \geq 4 \). Therefore \( \text{rn}(P_n^4) \geq 2k^2 + 2 \) if \( n \equiv_8 1 \) and \( n \geq 17 \).
Proof of claim: Suppose $v_1 \not\in \{x_{k+1}, x_{3k+2}\}$, that is $v_1$ is not the first vertex of the $(0,1)$ block nor the last vertex of a $(1,0)$ block. We know that $L(v_1) = 2k \equiv 0 \pmod{4}$. This means that $v_1$ is inside one of the $(0,1)$ or $(1,0)$ blocks.

Let $v_1 = x_c$ for some $c$ where $x_{c-1}$ and $x_{c+1}$ are both vertices on the right side. $L(x_{c-1}) \equiv 0 \pmod{4}$ $L(x_{c+1}) \equiv 1 \pmod{4}$. Let $L(x_{c-1}) = y$ and $L(x_{c+1}) = z$. By (2),

$$f(x_c) - f(x_{c-1}) = k + 1 - \left\lfloor \frac{2k + y}{4} \right\rfloor = k + 1 - \left( \left\lfloor \frac{2k}{4} \right\rfloor + \left\lfloor \frac{y}{4} \right\rfloor \right) \quad (\because (2k,y) \equiv (0,1) \pmod{4})$$

$$= \frac{k}{2} + 1 - \left\lfloor \frac{y}{4} \right\rfloor .$$

$$f(x_{c+1}) - f(x_c) = k + 1 - \left\lfloor \frac{z + 2k}{4} \right\rfloor = k + 1 - \left( \left\lfloor \frac{z}{4} \right\rfloor + \left\lfloor \frac{2k}{4} \right\rfloor \right) \quad (\because (z,2k) \equiv (1,0) \pmod{4})$$

$$= \frac{k}{2} + 1 - \left\lfloor \frac{z}{4} \right\rfloor .$$

Therefore,

$$f(x_{c+1}) - f(x_{c-1}) = k + 2 - \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{z}{4} \right\rfloor .$$

By definition and by Lemma 1.2,

$$f(x_{c+1}) - f(x_{c-1}) \geq k + 1 - \left\lfloor \frac{|z - y|}{4} \right\rfloor .$$

Therefore,

$$k + 2 - \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{z}{4} \right\rfloor \geq k + 1 - \left\lfloor \frac{|z - y|}{4} \right\rfloor .$$

Thus,

$$\left\lfloor \frac{|z - y|}{4} \right\rfloor \geq \left\lfloor \frac{y}{4} \right\rfloor + \left\lfloor \frac{z}{4} \right\rfloor - 1 .$$

Case 1: $y \geq z$

Then by Proposition 1.3 and $(y,z) \equiv (1,1)$,

$$\left\lfloor \frac{|y - z|}{4} \right\rfloor = \left\lfloor \frac{y - z}{4} \right\rfloor = \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{z}{4} \right\rfloor \geq \left\lfloor \frac{y}{4} \right\rfloor + \left\lfloor \frac{z}{4} \right\rfloor - 1 .$$

Therefore,

$$1 \geq 2 \left\lfloor \frac{z}{4} \right\rfloor .$$
Thus $z = 0$, which is a contradiction because $z \equiv 4 \pmod{1}$.

Case 2: $z \geq y$

Then by Proposition 1.3 and $(z, y) \equiv (1, 1)$,

$$\left\lceil \frac{z - y}{4} \right\rceil = \left\lceil \frac{z}{4} - \left\lfloor \frac{y}{4} \right\rfloor \right\rceil \geq \left\lceil \frac{y}{4} \right\rceil + \left\lfloor \frac{z}{4} \right\rfloor - 1.$$ 

Therefore,

$$1 \geq 2 \left\lfloor \frac{y}{4} \right\rfloor.$$ 

Thus $y = 0$, which is a contradiction because $y \equiv 4 \pmod{1}$.

Therefore $v_1 \in \{x_{k+1}, x_{3k+2}\}$. Similarly, we can show that $v_n \in \{x_{k+1}, x_{3k+2}\}$.

Similar techniques can be applied for the case $n \equiv 5 \pmod{8}$.

Assume $n \equiv 5 \pmod{8}$ and $n \geq 21$, that is, $n = 4k + 1$, $k$ is odd, and $k \geq 5$. Assume to the contrary that $f(x_n) = 2k^2 + 1$. Then only Case 1 is possible and all of the following must hold:

1) $\{x_1, x_n\} = \{v_{2k+1}, v_{2k+2}\}$ or $\{v_{2k+1}, v_{2k}\}$
   That is, $\{x_1, x_n\} = \{\text{center, vertex right next to center}\}$

2) $f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1})$ for all $i$

3) For all $i \geq 1$, the two vertices $x_i$ and $x_{i+1}$ are on opposite sides unless one of them is the center.

4) There exist three $t$'s, $1 \leq t \leq n - 1$ such that $L(x_t) + L(x_{t+1}) \equiv 2 \pmod{4}$ while $L(x_t) + L(x_{t+1}) \equiv 1 \pmod{4}$ for all other $i \neq t$.

By 1) and by symmetry, we can assume that $x_1 = v_{2k+1}$, i.e., $x_1$ is the center. Note that here, $n$ is odd and so $v_{2k+1}$ is the only center. Also, $n = 4k + 1$ for some odd $k$ and $k \geq 5$. Excluding the center, there are $4k$ vertices in total with $2k$ vertices on each side. In particular, there are $\frac{k-1}{2}$ many vertices whose level is congruent to $0 \pmod{4}$, $\frac{k+1}{2}$ many vertices whose level is congruent to $1 \pmod{4}$, $\frac{k+1}{2}$ many vertices whose level is congruent to $2 \pmod{4}$, and $\frac{k-1}{4}$ many vertices whose level
is congruent to 3 (mod 4) on each side. Since \( x_n \) is either \( v_{2k} \) or \( v_{2k+2} \) and both of these vertices have level equal to 1, \( x_n \) is right next to the center. By 2) and 3), the only three \( t' \)'s in 4) must be \( k + 1, 2k + 1, \) and \( 3k + 1 \). For otherwise, there would be at least four \( t' \)'s with \( L(x_t) + L(x_{t+1}) \equiv_4 0, 2, \) or 3. Therefore we have:

5)

\[
\begin{align*}
L(x_i) &\equiv_4 0 & \text{if } i \in \{1, 3, 5, \ldots, k, 3k + 3, 3k + 5, \ldots, 4k\} \\
L(x_i) &\equiv_4 1 & \text{if } i \in \{2, 4, 6, \ldots, k + 1, 3k + 2, 3k + 4, \ldots, 4k - 1, 4k + 1\} \\
L(x_i) &\equiv_4 2 & \text{if } i \in \{k + 2, k + 4, k + 6, \ldots, 2k + 1, 2k + 2, 2k + 4, \ldots, 3k + 1\} \\
L(x_i) &\equiv_4 3 & \text{if } i \in \{k + 3, k + 5, k + 7, \ldots, 2k, 2k + 3, 2k + 5, \ldots, 3k\}
\end{align*}
\]

By 1) and 5), the labeling pattern must be the following arrangement of blocks of vertices:

\[
C - (1- 0-1) - (2- 3 - 2) - (2 - 3 - 2) - (1- 0 - 1)
\]

However, in this arrangement the three \( t' \)'s for which \( L(x_t) + L(x_{t+1}) \) is not congruent to 1 (mod 4) are not all congruent to 2 (mod 4), which contradicts 4). Therefore, only Case 2 is possible.

\[\square\]

3.2 Summary of Results of Lower Bound of \( r_n(P^4_{n}) \) for \( n \) Odd

We have considered an arbitrary radio labeling of \( P^4_{n} \) for \( n \) odd having at least 3 disconnections in Case 1. We then considered radio labelings for \( n \) odd having exactly 2 disconnections in Case 2 and obtained sharper lower bounds of \( r_n(P^4_{n}) \) for \( n \equiv_8 1 \) and \( n \equiv_8 5 \), as well as the same lower bounds of \( r_n(P^4_{n}) \) as in Case 1 for \( n \equiv_8 3 \) or 7. We have proved that for \( n \equiv_8 1 \) and \( n \geq 17 \) or \( n \equiv_8 5 \), only Case 2 is possible. Therefore we have:
Lemma 3.2. Let $P^4_n$ be a fourth power path on $n$ vertices where $n \geq 6$ and let $k = \lceil \frac{n-1}{4} \rceil$, i.e., $k = \text{diam}(P^4_n)$.

If $n$ is odd, then $rn(P^4_n) \geq \begin{cases} 2k^2 + 2, & \text{if } n \equiv_8 1 \text{ and } n \geq 17; \\ 2k^2 + 1, & \text{if } n \equiv_8 3; \\ 2k^2 + 2, & \text{if } n \equiv_8 5; \\ 2k^2 + 1, & \text{if } n \equiv_8 7; \\ 2k^2 + 1, & \text{if } n = 9. \end{cases}$

Refer to Figures 4.2, 4.4, 4.6, and 4.8 for examples of radio labelings of $P^4_n$ for $n$ odd, whose spans are greater than or equal to the lower bounds given in Lemma 3.2.
Chapter 4

Conclusion

This paper has focused on obtaining a lower bound for \( \text{rn}(P_n^4) \). We have divided this task into two cases based on the parity of \( n \). Combining the two cases and their respective results, we have the following:

**Theorem 4.1.** Let \( P_n^4 \) be a fourth power path on \( n \) vertices where \( n \geq 6 \) and let \( k = \lceil \frac{n-1}{4} \rceil \), i.e., \( k = \text{diam}(P_n^4) \).

\[
\text{rn}(P_n^4) \geq \begin{cases} 
2k^2 + 1, & \text{if } n \equiv_8 0; \\
2k^2 + 2, & \text{if } n \equiv_8 1 \text{ and } n \geq 17; \\
2k^2, & \text{if } n \equiv_8 2; \\
2k^2 + 1, & \text{if } n \equiv_8 3; \\
2k^2 + 2, & \text{if } n \equiv_8 4; \\
2k^2 + 2, & \text{if } n \equiv_8 5; \\
2k^2 + 1, & \text{if } n \equiv_8 6; \\
2k^2 + 1, & \text{if } n \equiv_8 7; \\
2k^2 + 1, & \text{if } n = 9.
\end{cases}
\]

We conjecture that there are no sharper lower bounds of \( \text{rn}(P_n^4) \) than those given in Theorem 4.1 with the possible exception of the case when \( n \equiv 1 \pmod{8} \). By definition, the radio number of \( P_n^4 \) is equal to the minimum span over all radio labelings of \( P_n^4 \). Therefore, the span of a radio labeling of \( P_n^4 \) is an upper bound of \( \text{rn}(P_n^4) \). The examples provided below are radio labelings whose spans match the lower bounds given
in Theorem 4.1 and therefore are optimal radio labelings (i.e., span is equal to radio number) of the respective graphs, except when \( n \) is of the form \( 8q + 1 \). Based on the multiple radio labelings we have tried for all cases and Theorem 4.1, we conjecture that the radio number of \( P_{n}^{4} \) is as follows:

**Conjecture 4.2.** Let \( P_{n}^{4} \) be a fourth power path on \( n \) vertices where \( n \geq 6 \) and let \( k = \lceil \frac{n-1}{4} \rceil \), i.e., \( k = diam(P_{n}^{4}) \).

\[
rn(P_{n}^{4}) = \begin{cases} 
2k^2 + 1, & \text{if } n \equiv_8 0; \\
2k^2 + q, & \text{if } n \equiv_8 1 \quad (\text{where } n \text{ is of the form } 8q + 1); \\
2k^2, & \text{if } n \equiv_8 2; \\
2k^2 + 1, & \text{if } n \equiv_8 3; \\
2k^2 + 2, & \text{if } n \equiv_8 4; \\
2k^2 + 2, & \text{if } n \equiv_8 5; \\
2k^2 + 1, & \text{if } n \equiv_8 6; \\
2k^2 + 1, & \text{if } n \equiv_8 7.
\end{cases}
\]

\( P_{18}^{4} = P_{8(2)+2}^{4} \) with \( k = diam(P_{18}^{4}) = 5 \quad rn(P_{18}^{4}) \geq 2(5)^2 = 50 \)

Figure 4.1: An optimal radio labeling of \( P_{18}^{4} \) with span equal to 50
$P_{19}^4 = P_{8(2)+3}^4$ with $k = diam(P_{19}^4) = 5$ \hspace{1cm} \text{rn}(P_{19}^4) \geq 2(5)^2 + 1 = 51$

Figure 4.2: An optimal radio labeling of $P_{19}^4$ with span equal to 51

$P_{20}^4 = P_{8(3)+4}^4$ with $k = diam(P_{20}^4) = 5$ \hspace{1cm} \text{rn}(P_{20}^4) \geq 2(5)^2 + 2 = 52$

Figure 4.3: An optimal radio labeling of $P_{20}^4$ with span equal to 52

$P_{21}^4 = P_{8(2)+5}^4$ with $k = diam(P_{21}^4) = 5$ \hspace{1cm} \text{rn}(P_{21}^4) \geq 2(5)^2 + 2 = 52$

Figure 4.4: An optimal radio labeling of $P_{21}^4$ with span equal to 52
Figure 4.5: An optimal radio labeling of $P_{22}^4$ with span equal to 73

$P_{22}^4 = P_{n(3)+6}^{l}$ with $k = diam(P_{22}^4) = 6 \quad rn(P_{22}^4) \geq 2(6)^2 + 1 = 73$

Figure 4.6: An optimal radio labeling of $P_{23}^4$ with span equal to 73

$P_{23}^4 = P_{n(3)+7}^{l}$ with $k = diam(P_{23}^4) = 6 \quad rn(P_{23}^4) \geq 2(6)^2 + 1 = 73$

Figure 4.7: An optimal radio labeling of $P_{24}^4$ with span equal to 73

$P_{24}^4 = P_{n(3)}^{l}$ with $k = diam(P_{24}^4) = 6 \quad rn(P_{24}^4) \geq 2(6)^2 + 1 = 73$
Figure 4.8: A radio labeling of $P_{25}^4$ with span $= 75 = 2k^2 + q = 2(6)^2 + 3$
Appendix A

General Calculations

A.1 Proof of Lemma 1.2

Proof. For \( n \) odd, the distance between vertices on opposite sides is equal to the distance between vertices on the same side only when one of the vertices is the center. That is,

\[
\left\lceil \frac{L(u) + L(v)}{4} \right\rceil = \left\lceil \frac{|L(u) - L(v)|}{4} \right\rceil \quad \text{only when } L(v) = 0.
\]

For \( n \) even, the distance between vertices on opposite sides is also equal to the distance between vertices on the same side only when one of the vertices is the center and the other is not congruent to 0 (mod 4). That is,

\[
\left\lceil \frac{L(u) + L(v) + 1}{4} \right\rceil = \left\lceil \frac{|L(u) - L(v)|}{4} \right\rceil \quad \text{only when } L(v) = 0 \text{ and } L(u) \not\equiv 4 \text{ 0}.
\]

Notice what happens when \( L(u) \equiv 4 0 \), that is \( L(u) = 4m \) for some \( m \in \mathbb{N} \) and \( L(v) = 0 \).

\[
\left\lceil \frac{L(u) + L(v) + 1}{4} \right\rceil = \left\lceil \frac{4m + 0 + 1}{4} \right\rceil = \left\lceil \frac{4m + 1}{4} \right\rceil = \left\lceil m + \frac{1}{4} \right\rceil = m + 1
\]

and

\[
\left\lceil \frac{|L(u) - L(v)|}{4} \right\rceil = \left\lceil \frac{|4m - 0|}{4} \right\rceil = \left\lceil \frac{4m}{4} \right\rceil = m.
\]

The value \( \left\lceil \frac{L(u) + L(v) + 1}{4} \right\rceil \) gets pushed up to the next integer by the ceiling function because of the added 1. When \( L(u) \equiv 1, 2, \text{ or } 3 \) this does not happen. Observe what happens for the following cases:
If $L(u) \equiv 4 \cdot 1$, that is $L(u) = 4m + 1$ for some $m \in \mathbb{N}$ and $L(v) = 0$,
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 1) + 0 + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 2}{4} \right\rfloor = \left\lfloor m + \frac{2}{4} \right\rfloor = m + 1,
\]
and
\[
\left\lfloor \frac{|L(u) - L(v)|}{4} \right\rfloor = \left\lfloor \frac{|(4m + 1) - 0|}{4} \right\rfloor = \left\lfloor \frac{4m + 1}{4} \right\rfloor = \left\lfloor m + \frac{1}{4} \right\rfloor = m + 1.
\]

If $L(u) \equiv 4 \cdot 2$, that is $L(u) = 4m + 2$ for some $m \in \mathbb{N}$ and $L(v) = 0$,
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 2) + 0 + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 3}{4} \right\rfloor = \left\lfloor m + \frac{3}{4} \right\rfloor = m + 1,
\]
and
\[
\left\lfloor \frac{|L(u) - L(v)|}{4} \right\rfloor = \left\lfloor \frac{|(4m + 2) - 0|}{4} \right\rfloor = \left\lfloor \frac{4m + 2}{4} \right\rfloor = \left\lfloor m + \frac{2}{4} \right\rfloor = m + 1.
\]

If $L(u) \equiv 4 \cdot 3$, that is $L(u) = 4m + 3$ for some $m \in \mathbb{N}$ and $L(v) = 0$,
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 3) + 0 + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 4}{4} \right\rfloor = \left\lfloor m + \frac{4}{4} \right\rfloor = m + 1,
\]
and
\[
\left\lfloor \frac{|L(u) - L(v)|}{4} \right\rfloor = \left\lfloor \frac{|(4m + 3) - 0|}{4} \right\rfloor = \left\lfloor \frac{4m + 3}{4} \right\rfloor = \left\lfloor m + \frac{3}{4} \right\rfloor = m + 1.
\]

Therefore, the two values are equivalent only when one of the vertices is the center and the other one is not congruent to 0 (mod 4), for vertices $u$ and $v$ of $P_{2m}$. \qed

\section*{A.2 Level Sums}

Before continuing, we will investigate \(\left\lfloor \frac{L(x_i) + L(x_{i+1}) + 1}{4} \right\rfloor\) in order to get a better understanding of the different possible values it can take based on more specific $x_i$ and $x_{i+1}$.
Let \( L(u) + L(v) \equiv 0 \). Thus \( L(u) + L(v) = 4m \) for some \( m \in \mathbb{N} \) and
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 1}{4} \right\rfloor = \frac{4m + 1}{4} + \frac{3}{4} = \frac{L(u) + L(v)}{4} + \frac{4}{4} = \frac{L(u) + L(v) + 4}{4}.
\]

Let \( L(u) + L(v) \equiv 1 \). Thus \( L(u) + L(v) = 4m + 1 \) for some \( m \in \mathbb{N} \) and
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 1) + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 2}{4} \right\rfloor = \frac{4m + 2}{4} + \frac{2}{4} = \frac{L(u) + L(v)}{4} + \frac{3}{4} = \frac{L(u) + L(v) + 4}{4} - \frac{1}{4}.
\]

Let \( L(u) + L(v) \equiv 2 \). Thus \( L(u) + L(v) = 4m + 2 \) for some \( m \in \mathbb{N} \) and
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 2) + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 3}{4} \right\rfloor = \frac{4m + 3}{4} + \frac{1}{4} = \frac{L(u) + L(v)}{4} + \frac{2}{4} = \frac{L(u) + L(v) + 4}{4} - \frac{2}{4}.
\]

Let \( L(u) + L(v) \equiv 3 \). Thus \( L(u) + L(v) = 4m + 3 \) for some \( m \in \mathbb{N} \) and
\[
\left\lfloor \frac{L(u) + L(v) + 1}{4} \right\rfloor = \left\lfloor \frac{(4m + 3) + 1}{4} \right\rfloor = \left\lfloor \frac{4m + 4}{4} \right\rfloor = \frac{4m + 4}{4} = \frac{L(u) + L(v)}{4} + \frac{1}{4} = \frac{L(u) + L(v) + 4}{4} - \frac{3}{4}.
\]

We can also investigate \( \left\lfloor \frac{L(x_i) + L(x_{i+1})}{4} \right\rfloor \) in order to get a better idea of the different possible values it can take based on more specific \( x_i \) and \( x_{i+1} \).

Let \( L(u) + L(v) \equiv 0 \). Thus \( L(u) + L(v) = 4m \) for some \( m \in \mathbb{N} \) and
\[
\left\lfloor \frac{L(u) + L(v)}{4} \right\rfloor = \left\lfloor \frac{4m}{4} \right\rfloor = \frac{4m}{4}.
\]
\[
= \frac{L(u) + L(v)}{4} = \frac{L(u) + L(v) + 3}{4} - \frac{3}{4}.
\]

Let \(L(u) + L(v) \equiv_4 1\). Thus \(L(u) + L(v) = 4m + 1\) for some \(m \in \mathbb{N}\) and
\[
\left\lfloor \frac{L(u) + L(v)}{4} \right\rfloor = \left\lfloor \frac{(4m + 1)}{4} \right\rfloor = \frac{4m + 1}{4} + \frac{3}{4}.
\]
\[
= \frac{L(u) + L(v)}{4} + \frac{3}{4} = \frac{L(u) + L(v) + 3}{4}.
\]

Let \(L(u) + L(v) \equiv_4 2\). Thus \(L(u) + L(v) = 4m + 2\) for some \(m \in \mathbb{N}\) and
\[
\left\lfloor \frac{L(u) + L(v)}{4} \right\rfloor = \left\lfloor \frac{(4m + 2)}{4} \right\rfloor = \frac{4m + 2}{4} + \frac{2}{4}.
\]
\[
= \frac{L(u) + L(v)}{4} + \frac{2}{4} = \frac{L(u) + L(v) + 3}{4} - \frac{1}{4}.
\]

Let \(L(u) + L(v) \equiv_4 3\). Thus \(L(u) + L(v) = 4m + 3\) for some \(m \in \mathbb{N}\) and
\[
\left\lfloor \frac{L(u) + L(v)}{4} \right\rfloor = \left\lfloor \frac{(4m + 3)}{4} \right\rfloor = \frac{4m + 3}{4} + \frac{1}{4}.
\]
\[
= \frac{L(u) + L(v)}{4} + \frac{1}{4} = \frac{L(u) + L(v) + 3}{4} - \frac{2}{4}.
\]
Appendix B

Calculations for \( n \) Even

B.1 General Lower Bound of \( \text{rn}(P^4_n) \): Calculations for \( n \) Even

If \( n \equiv_8 0 \), then \( n = 4k \) and \( k \) is even,

\[
\text{rn}(P^4_n) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{7}{4} \right)
\]

\[
= (4k-1)(k+1) - \left( \frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{7}{4} \right)
\]

\[
= 4k^2 + 3k - 1 - \left( \frac{16k^2}{8} + \frac{12}{4}k - \frac{7}{4} \right)
\]

\[
= 4k^2 + 3k - 1 - \left( 2k^2 + 3k - \frac{7}{4} \right)
\]

\[
= 2k^2 + \frac{3}{4}
\]

and \( \left\lceil 2k^2 + \frac{3}{4} \right\rceil = 2k^2 + 1 \).

Thus for \( n \equiv_8 0 \) \( \text{rn}(P^4_n) \geq 2k^2 + 1 \) since \( \text{rn}(P^4_n) \) is an integer.

If \( n \equiv_8 2 \), then \( n = 4k - 2 \) and \( k \) is odd,
\[ \text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{7}{4} \right) \]
\[ = (4k - 3)(k + 1) - \left( \frac{(4k-2)^2}{8} + \frac{3}{4}(4k - 2) - \frac{7}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( \frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{7}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( 2k^2 - 2k + \frac{1}{2} + 3k - \frac{3}{2} - \frac{7}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( 2k^2 + k - \frac{11}{4} \right) \]
\[ = 2k^2 - \frac{1}{4} \]

and \[ \left\lceil 2k^2 - \frac{1}{4} \right\rceil = 2k^2. \]

Thus for \( n \equiv_8 2 \) \( \text{rn}(P_n^4) \geq 2k^2 \) since \( \text{rn}(P_n^4) \) is an integer.

If \( n \equiv_8 4 \), then \( n = 4k \) and \( k \) is odd,
\[ \text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{7}{4} \right) \]
\[ = (4k - 1)(k + 1) - \left( \frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{7}{4} \right) \]
\[ = 4k^2 + 3k - 1 - \left( \frac{16k^2}{8} + \frac{12k}{4} - \frac{7}{4} \right) \]
\[ = 4k^2 + 3k - 1 - \left( 2k^2 + 3k - \frac{7}{4} \right) \]
\[ = 2k^2 + \frac{3}{4} \]

and \[ \left\lceil 2k^2 + \frac{3}{4} \right\rceil = 2k^2 + 1. \]

Thus for \( n \equiv_8 4 \) \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.
If $n \equiv 8 \pmod{6}$, then $n = 4k - 2$ and $k$ is even,

$$
\begin{align*}
\text{rn}(P^4_n) & \geq (n - 1)(k + 1) - \left(\frac{n^2}{8} + \frac{3}{4}n - \frac{7}{4}\right) \\
&= (4k - 3)(k + 1) - \left(\frac{(4k - 2)^2}{8} + \frac{3}{4}(4k - 2) - \frac{7}{4}\right) \\
&= 4k^2 + k - 3 - \left(\frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{7}{4}\right) \\
&= 4k^2 + k - 3 - \left(2k^2 - 2k + \frac{1}{2} + 3k - \frac{3}{2} - \frac{7}{4}\right) \\
&= 4k^2 + k - 3 - \left(2k^2 + k - \frac{11}{4}\right) \\
&= 2k^2 - \frac{1}{4}
\end{align*}
$$

and $\left\lceil 2k^2 - \frac{1}{4}\right\rceil = 2k^2$.

Thus for $n \equiv 6 \pmod{8}$, $\text{rn}(P^4_n) \geq 2k^2$ since $\text{rn}(P^4_n)$ is an integer.

### B.2 At Least 5 Disconnections

If $n \equiv 4 \pmod{8}$, then $n = 4k$ and $k$ is odd,

$$
\begin{align*}
\text{rn}(P^4_n) & \geq (n - 1)(k + 1) - \left(\frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4}\right) \\
&= (4k - 1)(k + 1) - \left(\frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{9}{4}\right) \\
&= 4k^2 + 3k - 1 - \left(\frac{16k^2}{8} + \frac{12}{4}k - \frac{9}{4}\right) \\
&= 4k^2 + 3k - 1 - \left(2k^2 + 3k - \frac{9}{4}\right) \\
&= 2k^2 + \frac{5}{4}
\end{align*}
$$

and $\left\lceil 2k^2 + \frac{5}{4}\right\rceil = 2k^2 + 2$. 
Thus for \( n \equiv 8 \mod 4 \) \( \text{rn}(P^4_n) \geq 2k^2 + 2 \) since \( \text{rn}(P^4_n) \) is an integer.

If \( n \equiv 8 \mod 6 \), then \( n = 4k - 2 \) and \( k \) is even,

\[
\text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right)
\]

\[
= (4k - 3)(k + 1) - \left( \frac{(4k - 2)^2}{8} + \frac{3}{4}(4k - 2) - \frac{9}{4} \right)
\]

\[
= 4k^2 + k - 3 - \left( \frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{9}{4} \right)
\]

\[
= 4k^2 + k - 3 - \left( \frac{2k^2 - 2k + 1}{2} + 3k - \frac{3}{2} - \frac{9}{4} \right)
\]

\[
= 2k^2 + \frac{1}{4}
\]

and \( \left\lceil 2k^2 + \frac{1}{4} \right\rceil = 2k^2 + 1 \).

Thus for \( n \equiv 8 \mod 6 \) \( \text{rn}(P^4_n) \geq 2k^2 + 1 \) since \( \text{rn}(P^4_n) \) is an integer.

### B.3 Exactly 4 Disconnections

1) \( L(x_1) + L(x_n) \geq 0 + 1 = 1 \)

If \( n \equiv 8 \mod 4 \), then \( n = 4k \) and \( k \) is odd,
\[ \text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right) \]
\[ = (4k - 1)(k + 1) - \left( \frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{9}{4} \right) \]
\[ = 4k^2 + 3k - 1 - \left( \frac{16k^2}{8} + \frac{12}{4}k - \frac{9}{4} \right) \]
\[ = 4k^2 + 3k - 1 - \left( 2k^2 + 3k - \frac{9}{4} \right) \]
\[ = 2k^2 + \frac{5}{4} \]

and \[ \left\lceil 2k^2 + \frac{5}{4} \right\rceil = 2k^2 + 2. \]

Thus for \( n \equiv 8 \text{mod} 4 \) \( \text{rn}(P^4_n) \geq 2k^2 + 2 \) since \( \text{rn}(P^4_n) \) is an integer.

If \( n \equiv 8 \text{mod} 6 \), then \( n = 4k - 2 \) and \( k \) is even,

\[ \text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right) \]
\[ = (4k - 3)(k + 1) - \left( \frac{(4k - 2)^2}{8} + \frac{3}{4}(4k - 2) - \frac{9}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( \frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{9}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( 2k^2 - 2k + \frac{1}{2} + 3k - \frac{3}{2} - \frac{9}{4} \right) \]
\[ = 4k^2 + k - 3 - \left( 2k^2 + k - \frac{13}{4} \right) \]
\[ = 2k^2 + \frac{1}{4} \]

and \[ \left\lceil 2k^2 + \frac{1}{4} \right\rceil = 2k^2 + 1. \]

Thus for \( n \equiv 8 \text{mod} 6 \) \( \text{rn}(P^4_n) \geq 2k^2 + 1 \) since \( \text{rn}(P^4_n) \) is an integer.
2) \( L(x_1) + L(x_n) = 0 + 0 = 0 \)

If \( n \equiv 4 \), then \( n = 4k \) and \( k \) is odd,

\[
\text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{10}{4} \right) \\
= (4k - 1)(k + 1) - \left( \frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{10}{4} \right) \\
= 4k^2 + 3k - 1 - \left( \frac{16k^2}{8} + \frac{12}{4}k - \frac{10}{4} \right) \\
= 4k^2 + 3k - 1 - \left( \frac{2k^2 + 3k - 10}{4} \right) \\
= 2k^2 + \frac{6}{4}
\]

and \( \left\lceil 2k^2 + \frac{6}{4} \right\rceil = 2k^2 + 2 \).

Thus for \( n \equiv 4 \) \( \text{rn}(P^4_n) \geq 2k^2 + 2 \) since \( \text{rn}(P^4_n) \) is an integer.

If \( n \equiv 6 \), then \( n = 4k - 2 \) and \( k \) is even,

\[
\text{rn}(P^4_n) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{10}{4} \right) \\
= (4k - 3)(k + 1) - \left( \frac{(4k - 2)^2}{8} + \frac{3}{4}(4k - 2) - \frac{10}{4} \right) \\
= 4k^2 + k - 3 - \left( \frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{10}{4} \right) \\
= 4k^2 + k - 3 - \left( \frac{2k^2 - 2k + \frac{1}{2} + 3k - \frac{3}{2} - \frac{10}{4}}{} \right) \\
= 4k^2 + k - 3 - \left( \frac{2k^2 + k - \frac{14}{4}}{} \right) \\
= 2k^2 + \frac{2}{4}
\]

and \( \left\lceil 2k^2 + \frac{2}{4} \right\rceil = 2k^2 + 1. \)
Thus for \( n \equiv 6 \mod 8 \) \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.

### B.4 Exactly 3 Disconnections

If \( n \equiv 4 \mod 8 \), then \( n = 4k \) and \( k \) is odd,

\[
\text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right)
\]
\[
= (4k-1)(k+1) - \left( \frac{(4k)^2}{8} + \frac{3}{4}(4k) - \frac{9}{4} \right)
\]
\[
= 4k^2 + 3k - 1 - \left( \frac{16k^2}{8} + \frac{12}{4}k - \frac{9}{4} \right)
\]
\[
= 4k^2 + 3k - 1 - \left( 2k^2 + 3k - \frac{9}{4} \right)
\]
\[
= 2k^2 + \frac{5}{4}
\]

and \( \lceil 2k^2 + \frac{5}{4} \rceil = 2k^2 + 2 \).

Thus for \( n \equiv 4 \mod 8 \) \( \text{rn}(P_n^4) \geq 2k^2 + 2 \) since \( \text{rn}(P_n^4) \) is an integer.

If \( n \equiv 6 \mod 8 \), then \( n = 4k - 2 \) and \( k \) is even,

\[
\text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{9}{4} \right)
\]
\[
= (4k - 3)(k+1) - \left( \frac{(4k-2)^2}{8} + \frac{3}{4}(4k-2) - \frac{9}{4} \right)
\]
\[
= 4k^2 + k - 3 - \left( \frac{16k^2 - 16k + 4}{8} + \frac{12k - 6}{4} - \frac{9}{4} \right)
\]
\[
= 4k^2 + k - 3 - \left( 2k^2 - 2k + \frac{1}{2} + 3k - \frac{3}{2} - \frac{9}{4} \right)
\]
\[
= 4k^2 + k - 3 - \left( 2k^2 + k - \frac{13}{4} \right)
\]
\[
= 2k^2 + \frac{1}{4}
\]
and \[ \left\lfloor 2k^2 + \frac{1}{4} \right\rfloor = 2k^2 + 1. \]

Thus for \( n \equiv 6 \mod 8 \) \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.
Appendix C

Calculations for \( n \) Odd

C.1 General Lower Bound of \( r_n(P_n^4) \): Calculations for \( n \) Odd

If \( n \equiv 1 \mod{8} \), then \( n = 4k + 1 \) and \( k \) is even,

\[
r_n(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{15}{8} \right)
\]

\[
= (4k)(k+1) - \left( \frac{(4k+1)^2}{8} + \frac{3}{4}(4k+1) - \frac{15}{8} \right)
\]

\[
= 4k^2 + 4k - \left( \frac{16k^2 + 8k + 1}{8} + \frac{12k + 3}{4} - \frac{15}{8} \right)
\]

\[
= 4k^2 + 4k - \left( \frac{2k^2 + k + \frac{1}{8}}{8} + \frac{3}{4}k + \frac{3}{4} - \frac{15}{8} \right)
\]

\[
= 4k^2 + 4k - \left( 2k^2 + 4k - \frac{8}{8} \right)
\]

\[
= 2k^2 + 1.
\]

Thus for \( n \equiv 1 \mod{8} \) \( r_n(P_n^4) \geq 2k^2 + 1 \) since \( r_n(P_n^4) \) is an integer.

If \( n \equiv 3 \mod{8} \), then \( n = 4k - 1 \) and \( k \) is odd,
\[ \text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{15}{8} \right) \]

\[ = (4k - 2)(k + 1) - \left( \frac{(4k - 1)^2}{8} + \frac{3}{4}(4k - 1) - \frac{15}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( \frac{16k^2 - 8k + 1}{8} + \frac{12k - 3}{4} - \frac{15}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( \frac{2k^2 - k + \frac{1}{8} + 3k - \frac{3}{4} - \frac{15}{8}}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( \frac{2k^2 + 2k - \frac{20}{8}}{8} \right) \]
\[ = 2k^2 + \frac{1}{2} \]

and \[ \left\lceil 2k^2 + \frac{1}{2} \right\rceil = 2k^2 + 1. \]

Thus for \( n \equiv 3 \mod 8 \), \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.

If \( n \equiv 5 \mod 8 \), then \( n = 4k + 1 \) and \( k \) is odd,

\[ \text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{15}{8} \right) \]
\[ = (4k + 1)(k + 1) - \left( \frac{(4k + 1)^2}{8} + \frac{3}{4}(4k + 1) - \frac{15}{8} \right) \]
\[ = 4k^2 + 4k - \left( \frac{16k^2 + 8k + 1}{8} + \frac{12k + 3}{4} - \frac{15}{8} \right) \]
\[ = 4k^2 + 4k - \left( \frac{2k^2 + k + \frac{1}{8} + 3k + \frac{3}{4} - \frac{15}{8}}{8} \right) \]
\[ = 4k^2 + 4k - \left( \frac{2k^2 + 4k - \frac{8}{8}}{8} \right) \]
\[ = 2k^2 + 1. \]

Thus for \( n \equiv 5 \mod 8 \), \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.
If \( n \equiv 7 \), then \( n = 4k - 1 \) and \( k \) is even,

\[
\begin{align*}
\text{rn}(P_n^4) & \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{15}{8} \right) \\
& = (4k-2)(k+1) - \left( \frac{(4k-1)^2}{8} + \frac{3}{4}(4k-1) - \frac{15}{8} \right) \\
& = 4k^2 + 2k - 2 - \left( \frac{16k^2 - 8k + 1}{8} + \frac{12k - 3}{4} - \frac{15}{8} \right) \\
& = 4k^2 + 2k - 2 - \left( 2k^2 - k + \frac{1}{8} + 3k - \frac{3}{4} - \frac{15}{8} \right) \\
& = 4k^2 + 2k - 2 - \left( 2k^2 + 2k - \frac{20}{8} \right) \\
& = 2k^2 + \frac{1}{2}
\end{align*}
\]

and \( \left\lceil 2k^2 + \frac{1}{2} \right\rceil = 2k^2 + 1 \).

Thus for \( n \equiv 7 \) \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.

### C.2 Exactly 2 Disconnections

If \( n \equiv 1 \), then \( n = 4k + 1 \) and \( k \) is even,

\[
\begin{align*}
\text{rn}(P_n^4) & \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{19}{8} \right) \\
& = (4k)(k+1) - \left( \frac{(4k+1)^2}{8} + \frac{3}{4}(4k+1) - \frac{19}{8} \right) \\
& = 4k^2 + 4k - \left( \frac{16k^2 + 8k + 1}{8} + \frac{12k + 3}{4} - \frac{19}{8} \right) \\
& = 4k^2 + 4k - \left( 2k^2 + k + \frac{1}{8} + 3k + \frac{3}{4} - \frac{19}{8} \right) \\
& = 4k^2 + 4k - \left( 2k^2 + 4k - \frac{12}{8} \right) \\
& = 2k^2 + \frac{3}{2}
\end{align*}
\]
and \[2k^2 + \frac{3}{2}\] = 2k^2 + 2.

Thus for \(n \equiv 8 \pmod{8}\), \(\text{rn}(P_n^4) \geq 2k^2 + 2\) since \(\text{rn}(P_n^4)\) is an integer.

If \(n \equiv 3 \pmod{8}\), then \(n = 4k - 1\) and \(k\) is odd,

\[
\text{rn}(P_n^4) \geq (n - 1)(k + 1) - \left(\frac{n^2}{8} + \frac{3}{4}n - \frac{17}{8}\right)
\]
\[
= (4k - 2)(k + 1) - \left(\frac{(4k - 1)^2}{8} + \frac{3}{4}(4k - 1) - \frac{17}{8}\right)
\]
\[
= 4k^2 + 2k - 2 - \left(\frac{16k^2 - 8k + 1}{8} + \frac{12k - 3}{4} - \frac{17}{8}\right)
\]
\[
= 4k^2 + 2k - 2 - \left(\frac{2k^2 - k + \frac{1}{8}}{8} + \frac{3}{4}k - \frac{3}{4} - \frac{17}{8}\right)
\]
\[
= 4k^2 + 2k - 2 - \left(\frac{2k^2 + 2k - \frac{22}{8}}{8}\right)
\]
\[
= 2k^2 + \frac{6}{8}
\]

and \[2k^2 + \frac{6}{8}\] = 2k^2 + 1.

Thus for \(n \equiv 3 \pmod{8}\), \(\text{rn}(P_n^4) \geq 2k^2 + 1\) since \(\text{rn}(P_n^4)\) is an integer.

If \(n \equiv 5 \pmod{8}\), then \(n = 4k + 1\) and \(k\) is odd,
\[ \text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{17}{8} \right) \]
\[ = (4k)(k+1) - \left( \frac{(4k+1)^2}{8} + \frac{3}{4}(4k+1) - \frac{17}{8} \right) \]
\[ = 4k^2 + 4k - \left( \frac{16k^2 + 8k + 1}{8} + \frac{12k + 3}{4} - \frac{17}{8} \right) \]
\[ = 4k^2 + 4k - \left( 2k^2 + k + \frac{1}{8} + 3k + \frac{3}{4} - \frac{17}{8} \right) \]
\[ = 4k^2 + 4k - \left( 2k^2 + 4k - \frac{10}{8} \right) \]
\[ = 2k^2 + \frac{5}{4} \]

and \[ \left\lceil 2k^2 + \frac{5}{4} \right\rceil = 2k^2 + 2. \]

Thus for \( n \equiv 8 \) \( \text{rn}(P_n^4) \geq 2k^2 + 2 \) since \( \text{rn}(P_n^4) \) is an integer.

If \( n \equiv 8 \) \( 7 \), then \( n = 4k - 1 \) and \( k \) is even,

\[ \text{rn}(P_n^4) \geq (n-1)(k+1) - \left( \frac{n^2}{8} + \frac{3}{4}n - \frac{17}{8} \right) \]
\[ = (4k-2)(k+1) - \left( \frac{(4k-1)^2}{8} + \frac{3}{4}(4k-1) - \frac{17}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( \frac{16k^2 - 8k + 1}{8} + \frac{12k - 3}{4} - \frac{17}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( 2k^2 - k + \frac{1}{8} + 3k - \frac{3}{4} - \frac{17}{8} \right) \]
\[ = 4k^2 + 2k - 2 - \left( 2k^2 + 2k - \frac{22}{8} \right) \]
\[ = 2k^2 + \frac{6}{8} \]

and \[ \left\lceil 2k^2 + \frac{6}{8} \right\rceil = 2k^2 + 1. \]

Thus for \( n \equiv 8 \) \( 7 \) \( \text{rn}(P_n^4) \geq 2k^2 + 1 \) since \( \text{rn}(P_n^4) \) is an integer.
Bibliography


