Homomorphic Images and their Isomorphism Types

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Homomorphic Images and their Isomorphism Types

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Diana Herrera Gil
June 2014
HOMOMORPHIC IMAGES AND THEIR ISOMORPHISM TYPES

A Thesis
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June 2014
Approved by:

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Dr. Belisario Ventura, Committee Member

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Date

Dr. Charles Stanton
Graduate Coordinator,
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Abstract

In this thesis we have presented original homomorphic images of permutations and monomial progenitors. In some cases we have used the double coset enumeration technique to construct the images and for all of the homomorphic images that we have discovered, the isomorphism type of each group is given. The homomorphic images discovered include Linear groups, Alternating groups, and two sporadic simple groups $J_1$ and $J_2 \times 2$ where $J_1$ is the smallest Janko group and $J_2$ is the second Janko sporadic group.
Acknowledgements

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Gracias a mis padres por todo el apoyo que me han brindando ya que ellos son y serán mi más grande motivación para seguir adelante.
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Chapter 1

Background Information

The following terminology and theorems will be used throughout this thesis:

1.1 Groups

Definition 1.1 [Rot95] A group is a nonempty set, $G$, equipped with an associative operation, $*$, such that:

(i) $e * a = a = a * e$ for all $a \in G$.

(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$.

To avoid any confusion, we will write $G$ instead of $(G, *)$ and we’ll keep in mind that $*$ continues to exist.

Definition 1.2 [Rot95] Let $G$ and $H$ be groups. A function $f : G \mapsto H$ is a homomorphism if, for all $a, b \in G$,

$$f(ab) = f(a)f(b)$$

An isomorphism is a homomorphism that is also a bijection. We say that $G$ is isomorphic to $H$, denoted by $G \cong H$, if there exist an isomorphism $f : G \mapsto H$.

1.2 Group Action

A group action is a way to describe elements of a group acting on elements of a set in certain ways. Its a very useful abstraction, and is used in many fields, such as
geometry and algebra itself. The following definitions and theorems are used:

**Definition 2.1** [Rot95] If $\chi$ is a set and $G$ is a group, then $\chi$ is a $G$-set if there is a function $\alpha : G \times \chi \mapsto \chi$ (called an action), denoted by $\alpha : (g, x) \mapsto gx$, such that:

(i) $1a = a$ for all $a \in \chi$; and
(ii) $g(ha) = (gh)a$ for all $g, h \in G$ and $a \in \chi$.

One also says that $G$ acts on $\chi$. And $|\chi| = n$, the order of $\chi$, then $n$ is called the degree of the $G$-set $\chi$.

**Definition 2.2** [Rot95] The set of all permutations of $n$ letters is called the symmetric group on $n$ letters, and is denoted by $S_n$.

**Definition 2.3** [Rot95] If $\chi$ is a $G$-set and $x \in \chi$, then the $G$-orbit of $x$ is

$$\vartheta(x) = \{gx : g \in G\} \subset \chi.$$    

Throughout this thesis we will be referring to the $G$-orbit as orbit.

**Definition 2.4** [Rot95] If $\chi$ is a $G$-set and $x \in \chi$, then the stabiliser of $x$, denoted by $G_x$, is the subgroup

$$G_x = \{g \in G : gx = x\} \leq G.$$    

### 1.3 Normal Series

**Definition 3.1** [Rot95] Anormal series $G = H_0 \geq H_1 \geq \ldots \geq H_m = 1$ is a refinement of a normal series $G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ if $G_0, G_1, \ldots, G_n$ is a subsequence of $H_0, H_1, \ldots, H_m$.

**Definition 3.2** [Rot95] If $G$ has a composition series, then the factor groups of this series are called the composition factors of $G$. 
Chapter 2

Presentation of $2^5 : A_5$

We will show that $2^* : A_5$, where $x \sim (01234)$ and $y \sim (421)$, by performing a double coset enumeration of $G$ over $A_5$. A symmetric presentation of $G$ is given by:

$$G = \langle x, y, t \mid x^5, y^3, (x * y)^2, t^2, (t, y), (t, x^2 * y * x^{-1}), (t * t^x)^2 \rangle.$$ 

2.1 Relations

As mentioned above, we have the progenitor $2^* : A_5$ being factored by the relation $t_0t_1t_0t_1 = 1$. We will utilize this relation to determine the equal cosets with words composed of $t_1$, $t_2$, $t_3$, and $t_4$. Simplifying our relation:

$$t_0t_1t_0t_1 = 1$$
$$t_0t_1t_0 = t_1$$
$$t_0t_1 = t_1t_0$$

Hence, our relation in simplest terms is $t_0t_1 = t_1t_0$. We will use this relation to help us determine the equal cosets within words of length two. That is, we will conjugate $Nt_0t_1 = Nt_1t_0$ with every element contained in our control group $A_5$ to obtain the following:

- $t_0t_2 \sim t_2t_0$
- $t_1t_2 \sim t_2t_1$
- $t_2t_3 \sim t_3t_2$
- $t_0t_3 \sim t_3t_0$
- $t_1t_3 \sim t_3t_1$
- $t_2t_4 \sim t_4t_2$
- $t_0t_4 \sim t_4t_0$
- $t_1t_4 \sim t_4t_1$
- $t_3t_4 \sim t_4t_3$
To obtain the relations for the words of length three we will use the relations found for words of length two and right multiply each relation to obtain longer relations. We’ll look in detail as to how we can obtain a relation of length three using the relation \( t_0t_1 \sim t_1t_0 \), right multiply both sides by \( t_2 \) and use the above relations:

\[
\begin{align*}
t_0t_1t_2 &= t_1t_0t_2 \\
&= t_1t_2t_0 (\because t_0t_2 \sim t_2t_0) \\
&= t_2t_1t_0 (\because t_1t_2 \sim t_2t_1) \\
&= t_2t_0t_1 (\because t_1t_0 \sim t_0t_1) \\
&= t_0t_2t_1 (\because t_2t_0 \sim t_0t_2)
\end{align*}
\]

Applying this process we can figure out all the relations that will be useful for the words of length three. We will use the following notation for the relations found:

\[
\begin{align*}
012 &\sim 102 \sim 120 \sim 210 \sim 201 \sim 021 \\
031 &\sim 301 \sim 310 \sim 130 \sim 103 \sim 013 \\
041 &\sim 401 \sim 410 \sim 140 \sim 104 \sim 014 \\
241 &\sim 421 \sim 412 \sim 142 \sim 124 \sim 214 \\
231 &\sim 321 \sim 312 \sim 132 \sim 123 \sim 213 \\
341 &\sim 431 \sim 413 \sim 143 \sim 134 \sim 314
\end{align*}
\]

Similarly we can find the relations in regards to the words of length four and length five.
2.2 Double Coset Enumeration

\( N_eN \)

We begin with \( N_eN \), the first double coset, which contains all the words of length zero. We have that \( N_eN = \{N\} \) and it will be denoted \([*]\). Also, \( N = \langle x, y \rangle \cong A_5 \) and is of order 60. The number of elements in \([*]\) is \( \frac{|N|}{|N_e|} = \frac{60}{60} = 1 \). Hence \([*]\) consists of the single coset, \( N \). This single coset contains the single orbit \( \{0, 1, 2, 3, 4\} \). We then take an element from the orbit and right multiply it with the representative coset \( N_e \) to obtain \( Nt_0N \). We now have a new double coset, \( Nt_0N \), denoted as \([0]\).

\( Nt_0N \)

In this double coset we have the words of length one and the representative is \( Nt_0 \). We first find the coset stabilizer, \( N^{(0)} \), which consists of all the permutations in \( N \) that fix the element 0 and permute 1, 2, 3, 4. Hence, \( N^{(0)} = \langle (142), (234) \rangle \), is the coset stabilizer in \( N \) which contains 12 elements. Also, the number of single cosets in the double coset \([0]\) are found by \( \frac{|N|}{|N^{(0)}|} = \frac{60}{12} = 5 \). Now, we find the orbits of \( N^{(0)} \) on \( \{0, 1, 2, 3, 4\} \) by taking the representative from the double coset \([0]\), and conjugating it by the coset stabilizer. Since the element 0 is the only fixed, then \( N^{(0)} \) has \( \{0\} \) and \( \{1, 2, 3, 4\} \) as its orbits. Next, we will take a representative from \([0]\) and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

\[ Nt_0 \cdot t_0 = N(t_0)^2 \in N_eN \]

Since the orbit of 0 is of length one, then 1 element will collapse from \([0]\) to \([*]\).

\[ Nt_0 \cdot t_1 = Nt_0t_1 \in Nt_0t_1N \]

Since the orbit of 1 is of length four, then 4 elements will extend to the double coset \( Nt_0t_1N \) denoted as \([01]\).

\( Nt_0t_1N \)

We begin by finding the point stabilizer of 0 and 1, \( N^{01} \). To find the point stabilizer we need to find elements that belong to \( N \) which fix 0 and 1 and permute 2, 3, 4. Hence \( N^{01} = \{(142), (132), e\} \). Also, since we now have words of length two,
our relations will increase the stabiliser, $N^{(01)}$. Now to find the set stabiliser, $N^{(01)}$, we must find a relation in $N$ such that when $Nt_0t_1$ is conjugated by such relation, $Nt_0t_1$ goes back to itself. Say that such relation is $(10)$, then we have the following:

$$Nt_0t_1 = Nt_1t_0 \Rightarrow Nt_0t_1^{(10)} = Nt_1t_0 = Nt_0t_1$$

Hence, we have $(10) \in N^{(01)}$. So, $N^{(01)} > N^{01}$ and contains 6 elements. Also, we have that the number of single cosets in $Nt_0t_1$ is $\frac{|N|}{|N^{(01)}|} = \frac{60}{6} = 10$.

Now, to find the orbits of $N^{(01)}$ on $\{0, 1, 2, 3, 4\}$ we’ll conjugate $Nt_0t_1$, a representative from the double coset $[01]$ and conjugating it by the point stabilizer to obtain the following:

$$0^{N^{(01)}} = \{1\}, 1^{N^{(01)}} = \{0\}, \text{ and } 2^{N^{(01)}} = \{2, 3, 4\}.$$ 

Hence the orbits of $N^{(01)}$ are $\{0\}, \{1\}, \{2, 3, 4\}$. Next, we will take the representative $Nt_0t_1$ of $[01]$ and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0t_1 \cdot t_0 = Nt_0t_1t_0 = Nt_1t_0t_0 = Nt_1 \in N_0N$$

Since the orbit of 0 is of length one, then 1 element will collapse from $[01]$ to $[0]$.

$$Nt_0t_1 \cdot t_1 = Nt_0t_1t_1 = Nt_0 \in N_0N$$

Since the orbit of 1 is of length one, then 1 element will collapse from $[01]$ to $[0]$.

$$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

Since the orbit of 2 is of length three, then 3 elements will expand from $[01]$ to $[012]$. Now $Nt_0t_1t_2N$ is a new double coset which will be represented as $[012]$.

$Nt_0t_1t_2N$

We begin by finding the point stabilizer of 0,1 and 2, $N^{012}$. To find the point stabilizer we need to find elements that belong to $N$ which fix 0,1, and 2 and permute 3, 4. Hence $N^{012} = \{e\}$. Also, since we now have words of length three, our relations will increase the stabiliser, $N^{(012)}$. Now to find the set stabiliser, $N^{(012)}$, we must find a relation in $N$ such that when $Nt_0t_1t_2$ is conjugated by such relation $Nt_0t_1t_2$ goes back to itself. Say that such relation is $(012)$, then we have the following:

$$Nt_0t_1t_2 = Nt_1t_0t_2 \Rightarrow Nt_0t_1t_2^{(012)} = Nt_1t_2t_0 = Nt_0t_1t_2$$
So, we have \((012) \in N^{(012)}\). We also know that \(|N^{(012)}| = 6\) and the number of single cosets in \([012]\) is \(|N| \cdot |N^{(012)}| = 60\). Now, we find the orbits of \(N^{(012)}\) on \([0, 1, 2, 3, 4]\) by taking the representative \(Nt_0t_1t_2\) from the double coset \([012]\) and conjugating it by the point stabilizer to obtain the following:

\[
0^{N^{(012)}} = \{0, 1, 2\}, \quad 3^{N^{(012)}} = \{3, 4\}.
\]

Hence the orbits of \(N^{(01)}\) are \(\{0, 1, 2\}\) and \(\{3, 4\}\). Next, we will take the representative \(Nt_0t_1t_2\) and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

\[
Nt_0t_1t_2 \cdot t_2 = Nt_0t_1t_2t_2 = Nt_0t_1 = Nt_1 \in Nt_0t_1N
\]

Since the orbit of 2 is of length three, then 3 elements will collapse from \([012]\) to \([01]\).

\[
Nt_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N
\]

Since the orbit of 3 is of length two, then 2 elements will extend from \([012]\) to \([0123]\).

Now \(Nt_0t_1t_2t_3N\) is a new double coset which will be represented by \([0123]\).

**\(Nt_0t_1t_2t_3N\)**

We begin by finding the point stabilizer of 0,1,2, and 3, \(N^{0123}\). To find the point stabilizer we need to find elements that belong to \(N = S_5\) which fix 0,1,2,3 and permute 4. Hence \(N^{0123} = \{e\}\). Also, since we now have words of length four, our relations will increase the stabilizer, \(N^{(0123)}\). Now to find the set stabiliser, \(N^{(0123)}\), we must find a realiton in \(N\) such that when \(Nt_0t_1t_2t_3\) is conjugated by such relation \(Nt_0t_1t_2t_3\) goes back to itself. Say that such relation is \((012)\), then we have the following:

\[
Nt_0t_1t_2t_3 = Nt_1t_0t_2t_3 \Rightarrow Nt_0t_1t_2t_3^{(012)} = Nt_1t_2t_0t_3 = Nt_1t_0t_2t_3 = Nt_0t_1t_2t_3
\]

So, we have \((012) \in N^{(0123)}\). Also, notice that if we conjugate \(Nt_0t_1t_2t_3\) by \((13)(02)\) we obtain the following:

\[
Nt_0t_1t_2t_3^{(13)(02)} = Nt_2t_3t_0t_1 = Nt_2t_0t_3t_1 = Nt_0t_2t_3t_1 = Nt_0t_2t_1t_3 = Nt_0t_1t_2t_3
\]

We can conclude that \((13)(02) \in N^{(0123)}\). We also know that \(|N^{(0123)}| = 12\) and the number of single cosets in \([0123]\) is \(|N| \cdot |N^{(0123)}| = 60\).
Now, we find the orbits of $N^{(0123)}$ on $\{0, 1, 2, 3, 4\}$ by taking the representative $Nt_0t_1t_2t_3$ from the double coset [0123] and conjugating it by the point stabilizer to obtain the following:

$$0^{N^{(0123)}} = \{0, 1, 2, 3\}, \ 4^{N^{(0123)}} = \{4\}$$

Hence the orbits of $N^{(0123)}$ are $\{0, 1, 2, 3\}, \{4\}$. Next, we will take the representative $Nt_0t_1t_2t_3$ and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0t_1t_2t_3 \cdot t_3 = Nt_0t_1t_2t_3t_3 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

Since the orbit of $3$ is of length four, then $4$ elements will collapse from [0123] to [012].

$$Nt_0t_1t_2t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4 = Nt_0t_1t_2t_3t_4N$$

Since the orbit of $4$ is of length one, then $1$ element will expand from [0123] to [01234]. Now $Nt_0t_1t_2t_3t_4N$ is a new double coset which will be represented as [01234].

**$Nt_0t_1t_2t_3t_4N$**

We begin by finding the point stabilizer of $0,1,2,3,$ and $4$, $N^{01234}$. To find the point stabilizer we need to find elements that belong to $N = S_5$ which fix $0,1,2,3,4$. Hence $N^{01234} = \{e\}$. Also, since we now have words of length five, our relations will increase the stabiliser, $N^{(01234)}$. Now to find the stabiliser, $N^{(01234)}$, we must find a relation in $N$ such that when $Nt_0t_1t_2t_3t_4$ is conjugated by such relation, $Nt_0t_1t_2t_3t_4$ goes back to itself. Say that such relation is $(0123)$, then we have the following:

$$Nt_0t_1t_2t_3t_4^{(0123)} = Nt_1t_2t_3t_0t_4 = Nt_1t_2t_0t_3t_4 = Nt_1t_0t_2t_3t_4 = Nt_0t_1t_2t_3t_4$$

Hence we have $(0123) \in N^{(01234)}$. Also, notice that if we conjugate $Nt_0t_1t_2t_3t_4$ by $(12)(34)$ we obtain the following:

$$Nt_0t_1t_2t_3^{(12)(34)} = Nt_0t_2t_1t_4t_3 = Nt_0t_1t_2t_4t_3 = Nt_0t_1t_2t_3t_4$$

We also know that $|N^{(01234)}| = 60$ and the number of single cosets in [01234] is $\frac{|N|}{|N^{(01234)}|} = \frac{60}{60} = 1$. 
Now, we find the orbits of $N^{(01234)}$ on $\{0, 1, 2, 3, 4\}$ by taking the representative $N t_0 t_1 t_2 t_3 t_4$ from the double coset $[01234]$ and conjugating it by the point stabilizer to obtain the following:

$$N t_0 t_1 t_2 t_3 t_4 N^{(01234)} = \{0, 1, 2, 3, 4\}$$

Hence the single orbit of $N^{(01234)}$ is $\{0, 1, 2, 3, 4\}$. Next, we will take the representative $N t_0 t_1 t_2 t_3 t_4$ from $[01234]$ and conjugate it with a representative from the orbit found to determine if it’ll extend or collapse.

$$N t_0 t_1 t_2 t_3 t_4 \cdot t_4 = N t_0 t_1 t_2 t_3 t_4 = N t_0 t_1 t_2 t_3 \in N t_0 t_1 t_2 t_3 N$$

Since the orbit of 4 is of length five, then 5 elements will collapse from $[01234]$ to $[0123]$. Since the set of right cosets is closed under right coset multiplication, the double coset enumeration is now complete. Thus, we have summarize all of our work in the Cayley graph.

![Cayley Graph of $2^5 : A_5$ over $A_5$](image)

Figure 2.1: Cayley Graph of $2^5 : A_5$ over $A_5$
Chapter 3

$L_2(49)$ as a Homomorphic Image of $7^8 : m \ L_2(7)$

We’ll begin with the group $L_2(7)$, generated by $x \sim (3,6,7)(4,5,8)$ and $y \sim (1,6,2)(3,8,7)$. We want to induce a linear character from a subgroup $H$ up to $L_2(7)$. We will induce a linear character of $H$ to get an irreducible character of $L_2(7)$ of degree 3. To do so, we find the largest index from the character table of $L_2(7)$. We will induce from a subgroup of index 8. To make the notation easier, let $L_2(7) = G$. Now we must find the character tables for $G$ and $H$, using MAGMA we have the following:

Now, using Magma, the character table of $H$ is as follows and $Z_1 = \frac{-1+i\sqrt{7}}{2}$:

<table>
<thead>
<tr>
<th>Conjugacy Classes</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>$Z_1$</td>
<td>$Z_1 #3$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>$Z_1 #3$</td>
<td>$Z_1$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Also using Magma, the character table of $G$ is as follows, with $Z_1 = \frac{-1+i\sqrt{7}}{2}$.
and \( J \) is a root of unity 3:

\[
\begin{array}{c|ccccc}
\text{Conjugacy Classes} & C_1 & C_2 & C_3 & C_4 & C_5 \\
\hline
\text{Order} & 1 & 3 & 3 & 7 & 7 \\
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & J & -1 - J & 1 & 1 \\
\chi_3 & 1 & -1 - J & J & 1 & 1 \\
\chi_4 & 3 & 0 & 0 & Z_1 & Z_1 \#3 \\
\chi_5 & 3 & 0 & 0 & Z_1 \#3 & Z_1 \\
\end{array}
\]

Now, we will induce the second character of \( H \) up to \( G \). To do so, we first need to find the right transversals of \( G \). The transversals of \( H \) in \( G \) are: \{\( e, (3,6,7)(4,5,8), (3,7,6)(4,8,5), (1,6,3,2)(4,5,7,8), (1,2,6,8,4,5,3), (1,2,7,3)(4,8,5,6), (1,4,5,2,7,8,3), (1,6,4,3,2,8,5) \}. Then we will label the transversals as \( t_1, t_2, \ldots, t_8 \) respectively. Each transversal will represent a \( t_i \), since there are eight transversals, we will have an \( 8 \times 8 \) matrix representation. and we have:

\( G = Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4 \cup Ht_5 \cup Ht_6 \cup Ht_7 \cup Ht_8 \).

Then the matrices \( A(xx) \) and \( A(yy) \) are a representation of \( G \) induced from the representative of \( H \). Hence the general form for the matrices \( A(xx) \) and \( A(yy) \) are as follows:

\[
A(x) = \begin{bmatrix}
B(t_1 x t_1^{-1}) & B(t_1 x t_2^{-1}) & B(t_1 x t_3^{-1}) & \ldots & B(t_1 x t_7^{-1}) & B(t_1 x t_8^{-1}) \\
B(t_2 x t_1^{-1}) & B(t_2 x t_2^{-1}) & B(t_2 x t_3^{-1}) & \ldots & B(t_2 x t_7^{-1}) & B(t_2 x t_8^{-1}) \\
B(t_3 x t_1^{-1}) & B(t_3 x t_2^{-1}) & B(t_3 x t_3^{-1}) & \ldots & B(t_3 x t_7^{-1}) & B(t_3 x t_8^{-1}) \\
B(t_4 x t_1^{-1}) & B(t_4 x t_2^{-1}) & B(t_4 x t_3^{-1}) & \ldots & B(t_4 x t_7^{-1}) & B(t_4 x t_8^{-1}) \\
B(t_5 x t_1^{-1}) & B(t_5 x t_2^{-1}) & B(t_5 x t_3^{-1}) & \ldots & B(t_5 x t_7^{-1}) & B(t_5 x t_8^{-1}) \\
B(t_6 x t_1^{-1}) & B(t_6 x t_2^{-1}) & B(t_6 x t_3^{-1}) & \ldots & B(t_6 x t_7^{-1}) & B(t_6 x t_8^{-1}) \\
B(t_7 x t_1^{-1}) & B(t_7 x t_2^{-1}) & B(t_7 x t_3^{-1}) & \ldots & B(t_7 x t_7^{-1}) & B(t_7 x t_8^{-1}) \\
B(t_8 x t_1^{-1}) & B(t_8 x t_2^{-1}) & B(t_8 x t_3^{-1}) & \ldots & B(t_8 x t_7^{-1}) & B(t_8 x t_8^{-1}) \\
\end{bmatrix}
\]

We begin to compute the first entry, \( B(t_1 x t_1^{-1}) \) by substituting in the values \( t_1 = e \) and \( x \sim (3,6,7)(4,5,8) \). Substituting the corresponding values we obtain:

\( B(e (3,6,7)(4,5,8) e) = B((3,6,7)(4,5,8)) \), now we look for the element \((3,6,7)(4,5,8)\)
in the character table of $H$ along the second row; since $(3,6,7)(4,5,8)$ is not in it $B((3,6,7)(4,5,8)) = 0$. Let’s look at the next entry on the matrix: $B(t_1 x t_2^{-1}) = B(e (3,6,7)(4,5,8) ((3,6,7)(4,5,8))^{-1}) = B(e)$ since $(e)$ is in $H$ under the column corresponding to 1 then $B(e) = 1$. Continuing with the above process we obtain the matrix:

$$A(xx) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}$$

We do a similar process to obtain the matrix $A(yy)$:

$$A(yy) = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0
\end{pmatrix}$$

Now $A(xx)$ and $A(yy)$ are a faithful representation of $L_2(7)$, since $|x| = |A(xx)| = 3$, $|y| = |A(yy)| = 3$ and $|xx * yy| = |A(xx) \cdot A(yy)| = 4$. Now, we can use the matrices to find permutation representations to use for the progenitor along with the following rule: If $a_{ij} = 1$ then $t_i \rightarrow t_j$ and if $a_{ij} = -1$ then $t_i \rightarrow t_j^{-1}$. 
To find the permutation representation for $A(xx)$ and $A(yy)$, we will label the $t_i$’s as follows:

<table>
<thead>
<tr>
<th>Table 3.3: Labeling $t_i$’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $t_1$ 7. $t_2$ 13. $t_3$ 19. $t_4$ 25. $t_5$ 31. $t_6$ 37. $t_7$ 43. $t_8$</td>
</tr>
<tr>
<td>2. $t_1^2$ 8. $t_2^2$ 14. $t_3^2$ 20. $t_4^2$ 26. $t_5^2$ 32. $t_6^2$ 38. $t_7^2$ 44. $t_8^2$</td>
</tr>
<tr>
<td>3. $t_1^3$ 9. $t_2^3$ 15. $t_3^3$ 21. $t_4^3$ 27. $t_5^3$ 33. $t_6^3$ 39. $t_7^3$ 45. $t_8^3$</td>
</tr>
<tr>
<td>4. $t_1^4$ 10. $t_2^4$ 16. $t_3^4$ 22. $t_4^4$ 28. $t_5^4$ 34. $t_6^4$ 40. $t_7^4$ 46. $t_8^4$</td>
</tr>
<tr>
<td>5. $t_1^5$ 11. $t_2^5$ 17. $t_3^5$ 23. $t_4^5$ 29. $t_5^5$ 35. $t_6^5$ 41. $t_7^5$ 47. $t_8^5$</td>
</tr>
<tr>
<td>6. $t_1^6$ 12. $t_2^6$ 18. $t_3^6$ 24. $t_4^6$ 30. $t_5^6$ 36. $t_6^6$ 42. $t_7^6$ 48. $t_8^6$</td>
</tr>
</tbody>
</table>

Let’s begin with matrix $A(xx)$. Say we begin with entry $a_{12} = 1$ using the relation as mentioned, it implies that $t_1 \rightarrow t_2$ using the labeling we have $1 \rightarrow 7$. Having found this relation it implies that all the powers of $t_1$ go to the corresponding powers of $t_2$. The relations for the remaining powers of $t_1$ and $t_2$ using the labeling are: $2 \rightarrow 8$, $3 \rightarrow 9$, $4 \rightarrow 10$, $5 \rightarrow 11$, and $6 \rightarrow 12$. Similarly, for entry $a_{24} = 1$ implies that $t_2 \rightarrow t_4$, with the labeling we obtain $7 \rightarrow 19$. Having found this relation we know that all the powers of $t_2$ go to the powers of $t_4$. Continuing with the process for the remaining powers we have the remaining relations: $8 \rightarrow 20$, $9 \rightarrow 21$, $10 \rightarrow 22$, $11 \rightarrow 23$, $12 \rightarrow 24$. Continuing with the pattern, the results are summarized on the following table:
Table 3.4: Permutation of the $t_i$’s using $A(xx)$

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2^2$</th>
<th>$t_3^3$</th>
<th>$t_4^4$</th>
<th>$t_5^5$</th>
<th>$t_2$</th>
<th>$t_2^2$</th>
<th>$t_2^3$</th>
<th>$t_2^4$</th>
<th>$t_2^5$</th>
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<th>$t_3^2$</th>
<th>$t_3^3$</th>
<th>$t_3^4$</th>
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<tr>
<td>1</td>
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$A(xx)$

<table>
<thead>
<tr>
<th>$t_3^0$</th>
<th>$t_4$</th>
<th>$t_4^2$</th>
<th>$t_4^3$</th>
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<th>$t_4^5$</th>
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<th>$t_6^8$</th>
<th>$t_6^9$</th>
<th>$t_6^{10}$</th>
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</tbody>
</table>

$A(yy)$

| $t_6^0$ | $t_6^1$ | $t_7$ | $t_7^2$ | $t_7^3$ | $t_7^4$ | $t_7^5$ | $t_8$ | $t_8^2$ | $t_8^3$ | $t_8^4$ | $t_8^5$ | $t_8^6$ | $t_8^7$ | $t_8^8$ | $t_6^9$ | $t_6^{10}$ | $t_6^{11}$ | $t_6^{12}$ |
|---------|---------|-------|---------|---------|---------|---------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 35      | 36     | 37    | 38      | 39      | 40      | 41     | 42    | 43      | 44      | 45     | 46      | 47      | 48      | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       |
| ↓       | ↓     | ↓       | ↓       | ↓       | ↓      | ↓     | ↓     | ↓       | ↓       | ↓      | ↓     | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       |
| 17      | 18    | 40     | 37    | 41     | 38     | 42    | 39    | 44     | 46     | 48     | 47      | 45      | 47      | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       | ↓       |


Now, for matrix $A(yy)$, we’ll be using the same labeling of the $t_i$’s used for $A(xx)$. Along with the conditions, if $a_{ij} = 1$ then $t_i \rightarrow t_j$ and if $a_{ij} = -1$ then $t_i \rightarrow t_j^{-1}$.

We’ll begin with the entry $a_{13} = 1$ which implies that $t_1 \rightarrow t_3$, using the labeling we have $1 \rightarrow 13$. Continuing with the same process, with all 48 $t_i$’s, the following table summarizes our findings:
Table 3.5: Permutation of the \( t_i \)'s using \( A(\mathbf{y}) \)

<table>
<thead>
<tr>
<th>( t_1 )</th>
<th>( t_1^2 )</th>
<th>( t_1^3 )</th>
<th>( t_1^4 )</th>
<th>( t_1^5 )</th>
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<th>( t_2^3 )</th>
<th>( t_2^4 )</th>
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</table>


Now the progenitor for our group \( G \) is as follows:

\[ \text{Group} < x, y, t|x^3, y^3, (x*y)^4, (x, y)^3, (x^2 + y)^7, t^7, < t >^N = < t >> \]

where \( < t >^N = < t > \) is the normaliser of \( < t > \) in \( N \). The normalizer of the subgroup \( < t_1 > \) in \( N \) is \( \{ g \in N \mid < t_1 > n = n < t_1 > \} \). The normaliser of \( < t_1 > \) is the stabiliser of all powers of \( t_1 \) in \( N \). The permutations that fix all powers of \( t_1 \) are:


To be able to use the above permutations in our progenitor we need to represent them in terms of \( x \) and \( y \). Using the Schreier System we are able to find out what each of the above look like: \( xy^{-1}x^{-1} \) and \( y^{-1}x^{-1}y^{-1} \). Hence, we have that \( < t_1 > = < xy^{-1}x^{-1}, y^{-1}x^{-1}y^{-1} > \). So far we have \( 7^{*8} :_{m} L_{2}(7), \) where \( 7^{*8} \) denotes the order of the matrices, 7, and the number of our \( t_i \)'s, being 8. Also, to complete the representation of
our progenitor we find relations for which \( t \) conjugates with. Since we’re dealing with a monomial representation, from \( 7^8 \), we are working with eight symmetric generators, denoted by \( \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8\} \). The elements \( x \) and \( y \) act on the generators by conjugation, so a presentation for the progenitor is:

\[
7^8 : m \ L_2(7) = \langle x^3, y^3, (x * y)^4, (x, y)^3, (x^2 * y)^7, t^7, t^{(x*y^{-1}*x^{-1})} = t^2, (t, y^{-1} * x^{-1} * y^{-1}), (x^2 * y * t * t^x)^2 >
\]

Having completed this monomial progenitor, we can now look at its composition factors. MAGMA prints the composition factors of this progenitor as:

\[
A(1, 49) = L(2, 49)
\]

which is a computer-based proof that we will verify by constructing the following group.

### 3.1 Presentation of \( L_2(49) \)

We will be showing that \( L_2(49) \cong \frac{7^8 : m \ L_2(7)}{t_2 t_1^{t_2} = t_2 t_1} \), where \( x \sim (1, 7, 19)(2, 8, 20)(3, 9, 21)(4, 10, 22)(5, 11, 23)(6, 12, 24)(13, 25, 31)(14, 26, 32)(15, 27, 33)(16, 28, 34)(17, 29, 35)(18, 30, 36)(37, 40, 38)(39, 41, 42)(43, 44, 46)(45, 48, 47) \) and \( y \sim (1, 13, 25)(2, 14, 26)(3, 15, 27)(4, 16, 28)(5, 17, 29)(6, 18, 30)(7, 10, 8)(9, 11, 12)(19, 20, 22)(21, 24, 23)(31, 37, 43)(32, 38, 44)(33, 39, 45)(34, 40, 46)(35, 41, 47)(36, 42, 48) \), by performing a double coset enumeration of \( L_2(49) \) over \( L_2(7) \). Hence the progenitor is:

\[
L_2(49) = \langle x^3, y^3, (x * y)^4, (x, y)^3, (x^2 * y)^7, t^7, t^{(x*y^{-1}*x^{-1})} = t^2, (t, y^{-1} * x^{-1} * y^{-1}), (x^2 * y * t * t^x)^2 >
\]

As mentioned above, we have the progenitor \( 7^8 : m \ L_2(7) \) being factored by the relation \([x^2yt_1t_1^x]^2 = e\). Simplify the relation by conjugating \( t_1 \) by \( x \) we have \( t_1 \rightarrow t_2 \). Our relation now becomes: \([x^2yt_1t_2]^2 = e\). Let \( x^2y = \pi \) then \( \pi = (1, 20, 7, 13, 37, 44, 31)(2, 22, 8, 14, 38, 46, 32)(3, 24, 9, 15, 39, 48, 33)(4, 19, 10, 16, 40, 43, 34)\)
\[(5, 21, 11, 17, 41, 45, 35)(6, 23, 12, 18, 42, 47, 36)\]. The relation \((\pi t_1 t_2)^2 = e\) yields:

\[
\begin{align*}
\pi t_1 t_2 \pi t_1 t_2 &= e \\
\pi \pi \pi^{-1} t_1 t_2 \pi t_1 t_2 &= e \\
\pi^2 (t_1 t_2)^\pi t_1 t_2 &= e \\
\pi^2 t_4^2 t_4 t_1 t_2 &= e \\
t_2 t_4^2 t_4 t_1 t_2 &= e,
\end{align*}
\]

if we use the operation of right hand multiplication, we can simplify this relation to

\[
t_2 t_4^2 t_4 t_1 t_2 = t_2 t_1.
\]

### 3.2 Double Coset Enumeration

We will consider the following labeling for our \(t_i\)'s to facilitate the double coset enumeration notation:

\[
\begin{align*}
1 &= t_1, \quad 1^2 = t_1^2, \quad 1^3 = t_1^3, \quad 1^4 = t_1^4, \quad 1^5 = t_1^5, \quad 1^6 = t_1^6, \\
2 &= t_2, \quad 2^2 = t_2^2, \quad 2^3 = t_2^3, \quad 2^4 = t_2^4, \quad 2^5 = t_2^5, \quad 2^6 = t_2^6, \\
3 &= t_3, \quad 3^2 = t_3^2, \quad 3^3 = t_3^3, \quad 3^4 = t_3^4, \quad 3^5 = t_3^5, \quad 3^6 = t_3^6, \\
4 &= t_4, \quad 4^2 = t_4^2, \quad 4^3 = t_4^3, \quad 4^4 = t_4^4, \quad 4^5 = t_4^5, \quad 4^6 = t_4^6, \\
5 &= t_5, \quad 5^2 = t_5^2, \quad 5^3 = t_5^3, \quad 5^4 = t_5^4, \quad 5^5 = t_5^5, \quad 5^6 = t_5^6, \\
6 &= t_6, \quad 6^2 = t_6^2, \quad 6^3 = t_6^3, \quad 6^4 = t_6^4, \quad 6^5 = t_6^5, \quad 6^6 = t_6^6, \\
7 &= t_7, \quad 7^2 = t_7^2, \quad 7^3 = t_7^3, \quad 7^4 = t_7^4, \quad 7^5 = t_7^5, \quad 7^6 = t_7^6, \\
8 &= t_8, \quad 8^2 = t_8^2, \quad 8^3 = t_8^3, \quad 8^4 = t_8^4, \quad 8^5 = t_8^5, \quad 8^6 = t_8^6,
\end{align*}
\]

NeN

We begin with \(NeN\), the first double coset, which contains all the words of length zero. We have that \(NeN = \{N\}\) and it will be denoted \([\ast]\). Also, \(N = \langle x, y \rangle \cong L_2(7)\) and is of order 168. The number of elements in \([\ast]\) is \(|N| = 168\) \(168\), 1, hence \([\ast]\) consists of the single coset, \(N\). The single coset contains two orbits:

\[
\{1, 1^2, 1^4, 2, 2^2, 2^4, 3, 3^2, 3^4, 4, 4^2, 4^4, 5, 5^2, 5^4, 6, 6^2, 6^4, 7, 7^2, 7^4, 8, 8^2, 8^4\}\) and \(\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6, 3^3, 3^5, 3^6, 4^3, 4^5, 4^6, 5^3, 5^5, 5^6, 6^3, 6^5, 6^6, 7^3, 7^5, 7^6, 8^3, 8^5, 8^6\}\). Now take a representative element from each orbit, say \(t_1\) and \(t_1^3\), and right multiply each with the representative coset \(Ne\):
\[ Ne \cdot t_1 = Nt_1 \in [1] \]
\[ Ne \cdot t_1^3 = Nt_1^3 \in [1^3] \]

**\( Nt_1N \)**

The double coset \( Nt_1N \) denoted by [1]. To find the elements in [1], we find the point stabiliser of 1, denoted as \( N^1 \). The point stabiliser is made up of the permutations in \( N = L_2(7) \) that fix 1 and permutes the rest of the \( t_i \)'s. Thus, we have that \( |N^1| = 7 \). We also have that the set stabiliser is the same as the set point stabiliser, \( N^{(1)} = N^1 \). Hence \( |N^{(1)}| = 7 \).

The orbits of \( N^{(1)} \) are \{1\}, \{1^2\}, \{1^3\}, \{1^4\}, \{1^5\}, \{1^6\}, \{2, 3^2, 4, 5^2, 6^4, 7^2, 8^2\}, \{2^2, 3^4, 4^2, 5^4, 6, 7^4, 8^4\}, \{2^3, 3^6, 4^3, 5^6, 6^5, 7^6, 8^6\}, \{2^4, 3, 4^4, 5, 6^2, 7, 8\}, \{2^5, 3^5, 4^5, 5^5, 6^5, 7^3, 8^3\}, and \{2^6, 3^5, 4^6, 5^5, 6^3, 7^5, 8^5\}. The number of single cosets in the double coset [1] is \( \frac{|N|}{|N^{(1)}|} = \frac{168}{7} = 24 \).

Next we must take a representative from each orbit and right multiply it with the representative \( Nt_1 \), to determine if the \( t_i \)'s will expand or collapse:

\[ Nt_1 \cdot t_1 = Nt_1^2 \in [1] \quad Nt_1 \cdot t_1^3 = Nt_1^6 \in [1^3] \quad Nt_1 \cdot t_2^3 = Nt_1 t_2^3 \in [1, 2^3] \]
\[ Nt_1 \cdot t_2^3 = Nt_1^2 \in [1^3] \quad Nt_1 \cdot t_1^6 = Nt_1^6 = Ne \in [s] \quad Nt_1 \cdot t_2^4 = Nt_1 t_2^4 \in [1, 2^4] \]
\[ Nt_1 \cdot t_2^4 = Nt_1^2 \in [1, 2^4] \quad Nt_1 \cdot t_3 = Nt_3 t_2 \in [1, 2] \quad Nt_1 \cdot t_2^5 = Nt_1 t_2^5 \in [1, 2^5] \]
\[ Nt_1 \cdot t_3^5 = Nt_1^2 \in [1^3] \quad Nt_1 \cdot t_2^5 = Nt_1 t_2^5 \in [1, 2^5] \quad Nt_1 \cdot t_2^5 = Nt_1 t_2^5 \in [1, 2^5] \]

**\( Nt_1^3N \)**

The double coset \( Nt_1^3N \) denoted by [1^3]. To find the elements in [1^3], we find the point stabiliser of \( t^3 \), denoted as \( N^{13} \). The point stabiliser is made up of the permutations in \( N = L_2(7) \) that fix \( t^3 \) and permutes the rest of the \( t_i \)'s. Thus, we have that \( |N^{13}| = 7 \). We also have that the set stabiliser is the same as the set point stabiliser, \( N^{(13)} = N^{13} \). Hence \( |N^{(13)}| = 7 \).

The orbits of \( N^{(13)} \) are \{1\}, \{1^2\}, \{1^3\}, \{1^4\}, \{1^5\}, \{1^6\}, \{2, 3^2, 4, 5^2, 6^4, 7^2, 8^2\}, \{2^2, 3^4, 4^2, 5^4, 6, 7^4, 8^4\}, \{2^3, 3^6, 4^3, 5^6, 6^5, 7^6, 8^6\}, \{2^4, 3, 4^4, 5, 6^2, 7, 8\}, \{2^5, 3^3, 4^5, 5^3, 6^6, 7^3, 8^3\}, and \{2^6, 3^5, 4^6, 5^5, 6^3, 7^5, 8^5\}. The number of single cosets in the double coset [1^3] is obtained by the quotient \( \frac{|N|}{|N^{(13)}|} = \frac{168}{7} = 24 \).
To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(13)}$ in $N$. Then we conjugate $Nt_1^3$, a representative of the coset, with the transversals to obtain the 24 single cosets in $[1^{13}]$. Next, take a representative from each orbit of $N^{(13)}$ and right multiply each with $Nt_1^3$, to determine if the elements will expand or collapse:

\[
\begin{align*}
Nt_1^3 \cdot t_1 &= Nt_1^3 t_1 = Nt_1^4 \in [1] & Nt_1^3 \cdot t_2 &= Nt_1^3 t_2 \in [1, 2^3]
Nt_1^3 \cdot t_1^2 &= Nt_1^3 t_1^2 = Nt_1^5 \in [1^3] & Nt_1^3 \cdot t_2^2 &= Nt_1^3 t_2^2 \in [1, 2^6]
Nt_1^3 \cdot t_1^3 &= Nt_1^3 t_1^3 = Nt_1^6 \in [1^3] & Nt_1^3 \cdot t_2^3 &= Nt_1^3 t_2^3 \in [1, 2^2]
Nt_1^3 \cdot t_1^4 &= Nt_1^3 t_1^4 = Nt_1^7 \in [1] & Nt_1^3 \cdot t_2^4 &= Nt_1^3 t_2^4 \in [1, 2^4]
Nt_1^3 \cdot t_1^5 &= Nt_1^3 t_1^5 = Nt_2^4 \in [1] & Nt_1^3 \cdot t_2^5 &= Nt_1^3 t_2^5 \in [1, 2^4]
\end{align*}
\]

$Nt_1 t_2 N$

The double coset $Nt_1 t_2 N$ denoted by $[1, 2]$. To find the elements in $[1, 2]$, we find the point stabiliser of $\{1, 2\}$, denoted as $N^{12}$. The point stabiliser is made up of the permutations in $N = S_{48}$ that fix 1 and 2 and permute the rest of the $t_i$'s. Thus, we have that $|N^{12}| = 3$. We also have that the set stabiliser, $N^{(12)}$ has 3 elements. The number of single cosets in the double coset $[1, 2]$ is obtained by the quotient $\frac{|N|}{|N^{(12)}|} = \frac{1968}{3} = 56$.

The orbits of $N^{(12)}$ are $\{1, 2, 3^4\}$, $\{1^2, 2^4, 3\}$, $\{1^3, 2^6, 3^5\}$, $\{1^4, 2^3, 3^2\}$, $\{1^5, 2^3, 3^6\}$, $\{1^6, 2^5, 3^3\}$, $\{4, 6, 8^4\}$, $\{4^2, 2^8, 8\}$, $\{4^3, 6^3, 8^5\}$, $\{4^4, 6^4, 8^2\}$, $\{4^5, 6^5, 8^6\}$, $\{4^6, 6^6, 8^3\}$, $\{5, 5^2, 5^4\}$, $\{5^3, 5^5, 5^6\}$, $\{7, 7^2, 7^4\}$, and $\{7^3, 7^5, 7^6\}$.

To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(12)}$ in $N$. Then we conjugate $Nt_1 t_2$ , a representative of the coset, with the transversals to obtain the single cosets in $[1, 2]$ which total to 56. Next we take a representative from each orbit of $N^{(12)}$ and right multiply each with $Nt_1 t_2$, to determine if the elements will expand or collapse:

\[
\begin{align*}
Nt_1 t_2 \cdot t_1 &= Nt_1 t_2 t_1 \in [1, 2^3] & Nt_1 t_2 \cdot t_1^6 &= Nt_1 t_2 t_1^6 \in [1, 2^6]
Nt_1 t_2 \cdot t_1^2 &= Nt_1 t_2 t_1^2 \in [1, 2^5] & Nt_1 t_2 \cdot t_4 &= Nt_1 t_2 t_4 \in [1^3]
Nt_1 t_2 \cdot t_1^3 &= Nt_1 t_2 t_1^3 \in [1] & Nt_1 t_2 \cdot t_2^3 &= Nt_1 t_2 t_2^3 \in [1, 2^6]
Nt_1 t_2 \cdot t_1^4 &= Nt_1 t_2 t_1^4 \in [1, 2^2] & Nt_1 t_2 \cdot t_2^4 &= Nt_1 t_2 t_2^4 \in [1, 2^4]
Nt_1 t_2 \cdot t_1^5 &= Nt_1 t_2 t_1^5 \in [1, 2^4] & Nt_1 t_2 \cdot t_4^4 &= Nt_1 t_2 t_4^4 \in [1, 2^4]
\end{align*}
\]
\[ N_t t_2 \cdot t_3 \in [1, 2^3] \]
\[ N_t t_2 \cdot t_4 \in [1, 2^2] \]
\[ N_t t_2 \cdot t_5 \in [1, 2] \]
\[ N_t t_2 \cdot t_6 \in [1, 2^3] \]
\[ N_t t_2 \cdot t_7 \in [1, 2] \]
\[ N_t t_2 \cdot t_8 \in [1, 2] \]

\[ N_t t_2^2 N_t \]

The double coset \( N_t t_2^2 N_t \) denoted by \([1, 2^2]\). To find the elements in \([1, 2^2]\), we find the point stabiliser of \(\{1, 2^2\}\), denoted as \(N^{12^2}\). The point stabiliser is made up of the permutations in \(N = L_2(7)\) that fix 1 and 2\(^2\) and permutes with the remaining \(t_i\)'s. Thus, we have that \(|N^{12^2}| = 4\). We also have that the set stabiliser is \(N^{(12^2)}\) also contains 4 elements. The number of single cosets in the double coset \([1, 2^2]\) is obtained by the quotient \(\frac{|N|}{|N^{(12^2)}|} = \frac{168}{4} = 42\).

The orbits of \(N^{(12^2)}\) are \([1, 4, 7^4, 8^2]\), \([1^2, 4^2, 7, 8^4]\), \([1^3, 4^3, 7^5, 8^6]\), \([1^4, 4^4, 7^2, 8]\), \([1^5, 4^5, 7^6, 8^3]\), \([1^6, 4^6, 7^3, 8^5]\), \([2, 3, 5^2, 6^3]\), \([2^2, 3^2, 5^4, 6^4]\), \([2^3, 3^3, 5^6, 6^6]\), \([2^4, 3^4, 5, 6]\), \([2^5, 3^5, 5^3, 6^3]\), and \([2^6, 3^6, 5^5, 6^5]\). Next we must take a representative from each orbit of \(N^{(12^2)}\) and conjugate each with \(N_t t_2^2\), a representative of the coset, to determine if the elements will expand or collapse:

\[ N_t t_2^2 \cdot t_1 = N_t t_2^2 t_1 \in [1, 2] \]
\[ N_t t_2^2 \cdot t_2 = N_t t_2^2 t_2 \in [1, 2^3] \]
\[ N_t t_2^2 \cdot t_3^2 = N_t t_2^2 t_3^2 \in [1, 2^3] \]
\[ N_t t_2^2 \cdot t_4 = N_t t_2^2 t_4 \in [1, 2] \]
\[ N_t t_2^2 \cdot t_5 = N_t t_2^2 t_5 \in [1, 2^3] \]
\[ N_t t_2^2 \cdot t_6 = N_t t_2^2 t_6 \in [1, 2] \]
\[ N_t t_2^2 \cdot t_7 = N_t t_2^2 t_7 \in [1, 2^3] \]
\[ N_t t_2^2 \cdot t_8 = N_t t_2^2 t_8 \in [1, 2] \]

\[ N_t t_3^2 N_t \]

The double coset \( N_t t_3^2 N_t \) denoted by \([1, 2^3]\). To find the elements in \([1, 2^3]\), we find the point stabiliser of \(\{1, 2^3\}\), denoted as \(N^{12^3}\). The point stabiliser is made up of the permutations in \(N = L_2(7)\) that fix 1 and 2\(^3\) and permutes the rest of the \(t_i\)'s. Thus, we have that \(|N^{12^3}| = 4\). We also have that the set stabiliser is \(N^{(12^3)}\). The number of single cosets in the double coset \([1, 2^3]\) is obtained by the quotient \(\frac{|N|}{|N^{(12^3)}|} = \frac{168}{4} = 42\).

The orbits of \(N^{(12^3)}\) are \([1, 6^4, 7^8]\), \([1^2, 6, 7^2, 8^2]\), \([1^3, 6^5, 7^3, 8^3]\), \([1^4, 6^2, 7^4, 8^4]\), \([1^5, 6^6, 7^5, 8^5]\), \([1^6, 6^3, 7^6, 8^6]\), \([2, 3^4, 4, 5^4]\), \([2^2, 3^2, 4^2, 5]\), \([2^3, 3^5, 4^3, 5^5]\), \([2^4, 3^4, 4^5, 5^2]\), \([2^5, 3^6, 4^5, 5^6]\), and \([2^6, 3^3, 4^6, 5^3]\). Next, we take a representative from
each orbit of $N^{(123)}$ and conjugate each with $Nt_1t_2^3$, a representative of the coset, to determine if the elements will expand or collapse:

$$Nt_1t_2^3 \cdot t_1 = Nt_1t_2^3t_1 \in [1, 2]$$
$$Nt_1t_2^3 \cdot t_2 = Nt_1t_2^3t_2 \in [1, 2^4]$$

$$Nt_1t_2^3 \cdot t_2^4 = Nt_1t_2^3t_2^4 \in [1, 2^6]$$
$$Nt_1t_2^3 \cdot t_2^5 = Nt_1t_2^3t_2^5 \in [1, 2^7]$$

$$Nt_1t_2^3 \cdot t_2^3 = Nt_1t_2^3t_2^3 \in [1, 2^4]$$
$$Nt_1t_2^3 \cdot t_2^6 = Nt_1t_2^3t_2^6 \in [1, 2^7]$$

$Nt_1t_2^3N$

The double coset $Nt_1t_2^3N$ denoted by $[1, 2^4]$. To find the elements in $[1, 2^4]$, we find the point stabiliser of $\{1, 2^4\}$, denoted as $N^{12^4}$. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2 and permutes the rest of the $t_i$'s. Thus, we have that $|N^{12^4}| = 3$. We also have that the order of the set stabiliser is $|N^{(12^4)}| = 6$. The number of single cosets in the double coset $[1, 2^4]$ is obtained by the quotient $\frac{|N|}{|N^{(12^4)}|} = \frac{168}{6} = 56$. The orbits of $N^{(12^4)}$ are $\{1, 2^4, 6^2\}$, $\{1, 2^4, 6^4\}$, $\{1, 2^4, 6^3\}$, $\{1, 2^4, 6^5\}$, $\{2^2, 5^0, 8^1\}$, $\{2^2, 5^1, 8^0\}$, $\{2^3, 5^4, 8^0\}$, $\{2^5, 5^5, 8^3\}$, $\{2^6, 5^6, 8^5\}$, $\{3^2, 3^4\}$, $\{3^3, 3^5, 3^6\}$, $\{7, 7^2, 7^4\}$, and $\{7^3, 7^5, 7^6\}$. Next, we take a representative from each orbit of $N^{(12^4)}$ and conjugate each with $Nt_1t_2^3$, a representative of the coset, to determine if the elements will expand or collapse:

$$Nt_1t_2^3 \cdot t_1 = Nt_1t_2^3t_1 \in [1, 2^3]$$
$$Nt_1t_2^3 \cdot t_2 = Nt_1t_2^3t_2 \in [1, 2^4]$$

$$Nt_1t_2^3 \cdot t_2^4 = Nt_1t_2^3t_2^4 \in [1, 2^6]$$
$$Nt_1t_2^3 \cdot t_2^5 = Nt_1t_2^3t_2^5 \in [1, 2^7]$$

$$Nt_1t_2^3 \cdot t_2^3 = Nt_1t_2^3t_2^3 \in [1, 2^4]$$
$$Nt_1t_2^3 \cdot t_7 = Nt_1t_2^3t_7 \in [1, 2^4]$$

$Nt_1t_2^5N$

The double coset $Nt_1t_2^5N$ denoted by $[1, 2^5]$. To find the elements in $[1, 2^5]$, we find the point stabiliser of $\{1, 2^5\}$, denoted as $N^{12^5}$. The point stabiliser is made up of...
the permutations in \( N = L_2(7) \) that fix 1 and \( 2^5 \) and permutes the rest of the \( t_i \)’s. Thus, we have that \(|N^{12^5}| = 1\). We also have that the set stabiliser is \(|N^{(12^5)}| = 7\). The number of single cosets in the double coset \([1, 2^5]\) is obtained by the quotient \( \frac{|N|}{|N^{(12^5)}|} = \frac{168}{7} = 24\). The orbits of \( N^{(12^5)} \) are \{8\}, \{8^2\}, \{8^3\}, \{8^4\}, \{8^5\}, \{8^6\}, \{1, 2^2, 3, 4^1, 5^2, 6^4, 7^4\}, \{1^2, 2^4, 3^2, 4, 5^4, 6, 7\}, \{1^3, 2^6, 3^3, 4^5, 5^6, 6^5, 7^5\}, \{1^4, 2, 3^4, 4^2, 5, 6^2, 7^2\}, \{1^5, 2^3, 3^5, 4^6, 5^3, 6^6, 7^6\}, \{1^6, 2^5, 3^6, 4^3, 5^5, 6^3, 7^3\}. Next, we take a representative from each orbit of \( N^{(12^5)} \) and conjugate each with \( N t_1 t_2^5 \), to determine if the elements will expand or collapse:

\[
\begin{align*}
N t_1 t_2^5 \cdot t_1 &= N t_1 t_2^2 t_1 \in [1] & N t_1 t_2^5 \cdot t_8 &= N t_1 t_2^4 t_8 \in [1^3, 2^4] \\
N t_1 t_2^2 \cdot t_2^5 &= N t_1 t_2^4 t_2^5 \in [1, 2^2] & N t_1 t_2^5 \cdot t_8^2 &= N t_1 t_2^4 t_8^2 \in [1, 2^5] \\
N t_1 t_2^2 \cdot t_2^3 &= N t_1 t_2^4 t_2^3 \in [1, 2^4] & N t_1 t_2^5 \cdot t_8^3 &= N t_1 t_2^4 t_8^3 \in [1, 2^5] \\
N t_1 t_2^2 \cdot t_2^4 &= N t_1 t_2^4 t_2^4 \in [1, 2^6] & N t_1 t_2^5 \cdot t_8^4 &= N t_1 t_2^4 t_8^4 \in [1, 2^5, 8^4] \\
N t_1 t_2^2 \cdot t_2^5 &= N t_1 t_2^4 t_2^5 \in [1, 2] & N t_1 t_2^5 \cdot t_8^5 &= N t_1 t_2^4 t_8^5 \in [1^3, 2^4] \\
N t_1 t_2^2 \cdot t_2^6 &= N t_1 t_2^4 t_2^6 \in [1, 2^3] & N t_1 t_2^5 \cdot t_8^6 &= N t_1 t_2^4 t_8^6 \in [1^3, 2^4]
\end{align*}
\]

**Nt_1t_2^5N**

The double coset \( N t_1 t_2^5 N \) denoted by \([1, 2^6]\). To find the elements in \([1, 2^6]\), we find the point stabiliser of \([1, 2^6]\), denoted as \( N^{12^6} \). The point stabiliser is made up of the permutations in \( N = L_2(7) \) that fix 1 and \( 2^6 \) and permutes the remaining \( t_i \)’s. Thus, we have that \(|N^{12^6}| = 1\). We also have that the set stabiliser is \(|N^{(12^6)}| = 3\). The number of single cosets in the double coset \([1, 2^6]\) is obtained by the quotient \( \frac{|N|}{|N^{(12^6)}|} = \frac{168}{3} = 56\). The orbits of \( N^{(12^6)} \) are \{1, 1^2, 1^4\}, \{1^3, 1^5, 1^6\}, \{2, 3, 5^4\}, \{2^2, 3^2, 5\}, \{2^3, 3^3, 5^5\}, \{2^4, 3^4, 5^2\}, \{2^5, 3^5, 5^6\}, \{2^6, 3^6, 5^5\}, \{4, 6^2, 7^4\}, \{4^2, 6^4, 7\}, \{4^3, 6^6, 7^5\}, \{4^4, 6, 7^2\}, \{4^5, 6^3, 7^6\}, \{4^6, 6^5, 7^3\}, \{8, 8^2, 8^4\} and \{8^3, 8^5, 8^6\}. Next, we take a representative from each orbit of \( N^{(12^6)} \) and conjugate each with \( N t_1 t_2^6 \), to determine if the elements will expand or collapse:

\[
\begin{align*}
N t_1 t_2^3 \cdot t_1 &= N t_1 t_2^6 t_1 \in [1, 2^6] & N t_1 t_2^3 \cdot t_4 &= N t_1 t_2^6 t_4 \in [1, 2^4] \\
N t_1 t_2^6 \cdot t_2^3 &= N t_1 t_2^4 t_2^3 \in [1, 2^6] & N t_1 t_2^6 \cdot t_2^4 &= N t_1 t_2^4 t_2^4 \in [1, 2^4] \\
N t_1 t_2^6 \cdot t_3 &= N t_1 t_2^4 t_3 \in [1] & N t_1 t_2^6 \cdot t_2 &= N t_1 t_2^4 t_2 \in [1, 2^5] \\
N t_1 t_2^6 \cdot t_5 &= N t_1 t_2^4 t_5 \in [1, 2] & N t_1 t_2^6 \cdot t_4 &= N t_1 t_2^4 t_4 \in [1, 2^4] \\
N t_1 t_2^6 \cdot t_6 &= N t_1 t_2^4 t_6 \in [1, 2^2] & N t_1 t_2^6 \cdot t_4^2 &= N t_1 t_2^4 t_4^2 \in [1, 2^2]
\end{align*}
\]
\[ Nt_1 t_2^2 \cdot t_4^3 = Nt_1 t_2^6 t_4^3 \in [1^3] \]
\[ Nt_1 t_2^3 \cdot t_4^3 = Nt_1 t_2^6 t_4^3 \in [1^3, 2^4] \]
\[ Nt_1 t_2^6 \cdot t_4^3 = Nt_1 t_2^3 t_4^3 \in [1^3, 2^4] \]
\[ Nt_2^7 \cdot t_8^3 = Nt_2^1 t_8^3 \in [1, 2^6] \]
\[ Nt_2^1 t_6^2 \cdot t_8^3 = Nt_1 t_2^6 t_8^3 \in [1, 2^6] \]

**\text{Nt}_1^3 t_2^4 N**

The double coset \( Nt_1^3 t_2^4 N \) denoted by \([1^3, 2^4]\). To find the elements in \([1^3, 2^4]\), we find the point stabiliser of \( \{1^3, 2^4\} \), denoted as \( N^{1^32^4} \). The point stabiliser is made up of the permutations in \( N = L_2(7) \) that fix \( 1^3 \) and \( 2^4 \) and permutes the remaining \( t_i \)'s. Thus, we have that \( |N^{1^32^4}| = 1 \). We also have that the set stabiliser is \( |N(1^32^4)| = 7 \). The number of single cosets in the double coset \([1^3, 2^4]\) is obtained by the quotient \( \frac{|N|}{|N^{1^32^4}|} = \frac{168}{7} = 24 \). The orbits of \( N^{1^32^4} \) are \( \{4\}, \{4^2\}, \{4^3\}, \{4^4\}, \{4^5\}, \{4^6\}, \{1^2, 2^2, 3^4, 5^6, 6^6, 7^5, 8^3\}, \{1^4, 2^4, 3^5, 6^7, 8^3, 8^4\} \), \{1^5, 2^5, 3^3, 5^6, 6^3, 7^6, 8^5\}, \{1^6, 2^6, 3^5, 5^3, 6^5, 7^3, 8^6\}. Next, we take a representative from each orbit of \( N^{1^32^4} \) and conjugate each with \( Nt_1^3 t_2^4 \), a representative of the coset, to determine if the elements will expand or collapse:

\[ Nt_1^3 t_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
\[ Nt_1^3 t_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
\[ Nt_1^3 t_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
\[ Nt_1^3 t_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
\[ Nt_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
\[ Nt_2^4 \cdot t_4 = Nt_1 t_2^4 t_4 \in [1, 2^5] \]
The double coset $Nt_1 t_2^5 t_3^4 N$ denoted by $[1, 2^5, 8^4]$. To find the elements in $[1, 2^5, 8^4]$, we find the point stabiliser of $\{1, 2^5, 8^4\}$, denoted as $N^{12584}$. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix $1, 2^5$, and $8^4$ and permutes with the remaining $t_i$’s. Thus, we have that $|N^{12584}| = 1$. We also have that the set stabiliser is $|N^{(12584)}| = 168$. The number of single cosets in the double coset $[1, 2^5, 8^4]$ is obtained by the quotient $\frac{|N|}{|N^{(12584)}|} = \frac{168}{168} = 1$. The orbits of $N^{(12584)}$ are $\{1, 1^2, 1^4, 2, 2^2, 2^4, 3, 3^2, 3^4, 4, 4^2, 4^4, 5, 5^2, 5^4, 6, 6^2, 6^4, 7, 7^2, 7^4, 8, 8^2, 8^4\}$ and $\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6, 3^3, 3^5, 3^6, 4^3, 4^5, 4^6, 5^3, 5^5, 5^6, 6^3, 6^5, 6^6, 7^3, 7^5, 7^6, 8^3, 8^5, 8^6\}$. Next, we take a representative from each orbit of $N^{(12584)}$ and conjugate each with $Nt_1 t_2^5 t_3^4$, to determine if the elements will expand or collapse:

$$Nt_1 t_2^5 t_3^4 \cdot t_1 = Nt_1 t_2^5 t_3^4 t_1 \in [1^3, 2^4]$$

$$Nt_1 t_2^5 t_3^4 \cdot t_3 = Nt_1 t_2^5 t_3^4 t_3 \in [1, 2^5]$$

Hence, we summarize all of our work on the following Cayley graph:
Figure 3.1: Cayley Graph of $L_2(49)$ over $L_2(7)$
Chapter 4

Composition Factors

4.1 Direct Products

We will be performing a composition series on a given group with the aid of MAGMA, a computer software that aids with the work on group theory. The progenitors that we investigated are:

\[ <x,y,t | x^{15}, y^2, (x \ast y)^2, t^2, (t,y) > \cong 2^{*15} : D_{15}, \]

where \( x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15) \) and \( y \sim (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(13, 15). \)

\[ <x,y,t | x^3, y^3, (x \ast y)^4, (x, y)^3, (x^2 \ast y)^7, t^7 > \cong L_2(7), \]


\[ <x,y,t | x^3, y^2, (xy)^2, t^2, (t,y) > \cong S_3, \]

where \( x \sim (1, 2, 3) \) and \( y \sim (1, 2). \)
We will begin with small simple groups and work our way up to more complex groups. Let’s begin with the group presentation:

\[ G < x, y, t > := \text{Group} < x, y, t | x^{11}, y^{2}, (x * y)^{2}, t^{2}, (t, y), (x * y * t^{x})^{3}, (x^{2} * y * t * t^{x})^{4} > \]

We have the composition factors of our group as well as its normal lattice. Both are used as a map/guide to find the structure of the composition series of our group:

Hence we begin to list the extensions from the composition factors: \( G_{1}/1 \cong C_{2} \) which implies that \( G_{1} \cong C_{2} \), where \( C_{2} \) is normal. Moving up the composition factors, we have \( G/G_{1} \cong PSL_{2}(23) \) which implies \( G \cong G_{1} \? PSL_{2}(23) \). Now, to solve the extension problem we need to figure out whether we have a direct product, or a semi-direct product.

We for a direct product first, that is, do we have \( G \cong C_{2} \times PSL_{2}(23) \). Now, looking at the two minimal normal subgroups of \( G \), one of order 2 and the other of order 6072, we can conclude that we have \( C_{2} \times PSL_{2}(23) \) with order 12144. Looking at the subgroup lattice, note that \( NL[4] \) has order 12144 as well. We can verify that \( C_{2} \times PSL_{2}(23) \) is indeed isomorphic to \( NL[4] \). The progenitor for our group is:

\[ G :< A, B, C, D \mid A^{2}, B^{23}, C^{11}, D^{2}, B^{C \ast B^{-2}}, (C \ast D)^{2}, (B \ast D)^{3}, (A, B), (A, C), (A, D) > \]

**4.1.1 Example**

By applying a similar process, the following can be shown:

1. Given:

\[ G < x, y, t > := \text{Group} < x, y, t | x^{15}, y^{2}, (x * y)^{2}, t^{2}, (t, y), (x^{y^{2}} * t^{y})^{3}, (x^{2} * y * t^{x})^{2} > \]
we can show that it is isomorphic to the progenitor:

\[ G \triangleleft A, B, C | A^2, B^2, C^3, (B \ast C)^5 > \]

and is of the form: \( G \cong (C_2 \times \text{Alt}(5)) \).

### 4.2 Semi-Direct Products

Our approach will begin the same way as we did on Section 4. Consider the following group presentation:

\[ G < x, y, t > : = \text{Group} < x, y, t|x^{15}, y^2, (x \ast y)^2, t^2, (t, y), (x^{y^2} \ast t^y)^5, (x^{yst} \ast t^{x^2})^3, (y \ast t^{x^2})^8 > . \]

We have the composition factors of our group as well as its normal lattice. Both are used as a map/guide to find the structure of the composition series of our group:

\[
\begin{array}{c|c|c}
\text{G} & \text{Cyclic}(2) & [4] \text{ Order 720} \\
\text{G1}\ast & \text{Alternating}(6) & [3] \text{ Order 360} \\
\text{1} & & [1] \text{ Order 1}
\end{array}
\]

Hence we begin to list the extensions from the composition factors: \( G_1/1 \cong A(6) \) which implies that \( G_1 \cong A(6) \), and \( A(6) \) is normal. Moving up the composition factors, we have \( G/G_1 \cong C_2 \) which implies that \( G \cong G_1 \ast C_2 \), hence \( G \cong A(6) \ast C_2 \). To solve the extension problem we will check for a direct product first, that is, do we have \( G \cong A(6) \ast C_2 \). Looking at the minimal normal subgroup of \( G \) of order 360 we check the order of the group \( A(6) \) and see that it is also 360. Looking at the subgroup lattice, note that \( NL[2] \) has order 360 as well. So far this is our progenitor for \( G_1 \):

\[ G_1 :< A, B \ast A^2, A^4, (A \ast B)^5, (A \ast B^2)^5 > \]
Back to our extension problem, in order to check if we have $G \cong A(6) \times C_2$ we look at the normal lattice. Since we are at $NL[2]$, moving up the normal lattice, we can only move to $NL[3]$, since $NL[2]$ is the only maximal subgroup contained in $NL[3]$. Note that $NL[3]$ is isomorphic to $G$. Since the order of $NL[3]$ is twice the order of $NL[2]$ we look at the normal lattice of $G$ to find a normal subgroup of order 2. Since there’s no such normal subgroup then we know that we don’t have a direct extension. Now we need to check if we have a semi-direct product. To do so, we will find an element $C$, in $NL[3]$ but not in $NL[2]$ of order 2. Hence, we have a semi-direct extension: $G \cong A(6) : C_2$. Since we are extending our group by the element $C$, once we determine how $C$ affects $A$ and $B$ we can complete our progenitor:

$$G : < A, B, C | A^2, B^4, (A \ast B)^5, (A \ast B^2)^5, C^2, A^C = A \ast B^{-1} \ast A \ast B \ast A, B^C = A \ast B^{-1} \ast A \ast B^{-1} \ast A \ast B \ast A >$$

Our progenitor is of order 720 and is isomorphic to our group $G$.

### 4.2.1 Example

By applying a similar process, the following can be shown:

1. Given:

   $$G < x, y, t > := Group < x, y, t | x^{11}, y^2, (x \ast y)^2, t^2, (t, y), (x^2 \ast y \ast t \ast t^x)^4, (x \ast t^x)^3 >$$

   we can show that is isomorphic to the progenitor:

   $$G : < A, B, C | A^2 = B^3 = (A \ast B)^{11} = (A, B \ast A \ast B \ast A \ast B)^2 = 1, C^2,$$
   $$A^C = A \ast B \ast A \ast B^{-1} \ast A \ast B^{-1} \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B^{-1} \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} \ast A \ast B \ast A \ast B^{-1} >$$

   and is of the form: $G \cong (PSL(2, 11) : C_2)$.

2. Given:

   $$G < x, y, t > := Group < x, y, t | x^3, y^3, (x \ast y)^4, (x, y)^3, (x^2 \ast y)^7, t^7,$$
   $$t^{x \ast y^{-1} \ast x^{-1}} = t^2, (t, y^{-1} \ast x^{-1} \ast y^{-1}), (y^2 \ast t \ast t^{y \ast x})^2 >$$
we can show that is isomorphic to the progenitor:

\[ G : < A, B, C, D, E | A^7, B^7, C^7, (A, B), (A, C), (B, C), D^2 = E^3 = (D * E)^7 = (D, E)^4 = 1, A^D = B^2 * A^{-2} * C^{-3}, A^E = C^3 * A^{-2} * B^{-3}, B^D = B * A^{-1} * C^{-3}, B^E = A^3 * C * B^{-1}, C^D = A^{-2} * B^{-3}, C^E = A * C^3 * B^{-1} > \]

and is of the form: \( G \cong (7^3 : PSL(2, 7)) \).

3. Given:

\[ G < x, y, t > : = Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (x^{y^t} * t^{x^2})^2, (x^2 * y * t^{x^t})^2 > \]

we can show that is isomorphic to the progenitor:


and is of the form: \( G \cong ((2^3 : C_3) : C_2) \).

4. Given:

\[ G < x, y, t > : = Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (x^{y^t} * t^{x^2})^2, (x^2 * y * t^{x^t})^5 > \]

we can show that is isomorphic to the progenitor:


and is of the form: \( G \cong (((5^4 : C_2) : C_2) : C_3) : C_2) \).
4.3 Semi-Direct and Direct Products

Given the following presentation:

\[ G < x, y, t > = \text{Group} < x, y, t | x^3, y^2, (x*y)^2, t^2, (t, y), (y*t*t) > \]

Listing the composition factors for which the group is composed of:

\[
\begin{array}{c|c}
G & \text{Cyclic(2)} \\
G_1 & \text{Cyclic(2)} \\
G_2 & \text{Cyclic(3)} \\
G_3 & \text{Cyclic(3)} \\
1 & \\
\end{array}
\]

The extensions from the composition factors: \( G_3/1 \cong C_3 \) imply that \( G_3 \cong C_3 \), and \( C_3 \) is normal. Moving up the composition factors, we have \( G_2/G_3 \cong C_3 \) which implies that \( G_2/C_3 \cong C_3 \), hence \( G_2 \cong C_3 \times C_3 \). To solve the extension problem we will check for a direct product first, that is, do we have \( G_2 \cong C_3 \times C_3 \). Now, we begin by looking at the two minimal normal subgroups of \( G \), each of order 3. Hence we have \( C_3 \times C_3 \) of order 9. Looking at the subgroup lattice, note that \( NL[6] \) has order 9 as well. Note that \( C_3 \times C_3 \) is Abelian and is isomorphic to \( NL[6] \). Therefore we have \( G_2 \cong C_3 \times C_3 \) and \( G_2 \cong NL[6] \). The progenitor so far is:

\[ G_2 : < A, B | A^3, B^3, (A, B) > \]

Continuing up the ladder of the composition factors, we have \( G_1/G_2 \cong C_2 \) which implies that \( G_1/C_3 \times C_3 \cong C_2 \), so \( G_1 \cong (C_3 \times C_3) \times C_2 \). Now, to solve the extension problem let’s check for a direct product first. Now, in reference to the normal lattice, we are located at \( NL[6] \), to continue moving up the lattice we can either move to \( NL[7] \) or \( NL[9] \) since \( NL[6] \) is contained in both \( NL[7] \) and \( NL[9] \). Say we choose to go up to \( NL[9] \), since the order of \( NL[9] \) is twice the order of \( NL[6] \), we need to find
a normal subgroup within $NL[9]$ of order 2. In order to look at the normal subgroups within $NL[9]$ we can generate the normal lattice of $NL[9]$. Since there’s no normal subgroup of order 2, then $G_1$ is not a direct product. So, we check for a semi-direct product. To do so, we find an element, $C$, in $NL[9]$ but not in $NL[6]$ of order 2. $C = (1,2)(3,6)(4,5)$ is the required element. Hence, we have a semi-direct extension: $G_1 \cong (C_3 \times C_3) : C_2$. Since we are extending our group by the element $C$, we will determine how $C$ affects $A$ and $B$ to elaborate on our progenitor:

$$G_1 := A, B, C | A^3, B^3, (A, B), C^2, A^C = A^2 \ast B^2, B^C = A^3 \ast B$$

The above progenitor is of order 18 and is isomorphic to $NL[9]$. Continuing with the composition factors we are now at $G/G_1 \cong C_2$ which implies $G/((C_3 \times C_3) : C_2) \cong C_2$. Our last extension problem is $G \cong ((C_3 \times C_3) : C_2) ? C_2$. To solve it, we will repeat the process as we’ve been doing so far. Notice that on the normal lattice of $G$ we are in $NL[9]$, from here the only place left to move up is to get to $NL[10]$ which represents our entire group, $G$. To be able to get there we need to find a normal subgroup of order 2 in $NL[10]$. Since $NL[10] = G$ then we refer back to the normal lattice belonging to $G$. Since there’s no normal subgroup of order 2, then we don’t have a direct product. So, we check for a semi-direct product. We need an element, $D$ of order 2 in $NL[10]$ but not in $NL[9]$. The required element $D$ is found and we conclude that $G \cong ((C_3 \times C_3) : C_2) : C_2$. We then check how $D$ affects $A$, $B$, and $C$ to be able to complete our progenitor. Hence the progenitor has order 36 and is isomorphic to $G$:

$$G := A, B, C, D | A^3, B^3, (A, B), C^2, A^C = A^2 \ast B^2, B^C = A^3 \ast B, D^2;$$

$$A^D = A \ast B \ast C^2, B^D = B^2 \ast C^2, C^D = C$$

### 4.3.1 Examples

By applying a similar process, the following can be shown:

1. Given:

$$G < x, y, t > := Group < x, y, t | x^{15}, y^2, (x \ast y)^2, t^2, (t, y), (x^2 \ast y \ast t^2) >$$
we can show that is isomorphic to the progenitor:

$$G := A, B, C | A^2, B^2, (A, B), C^3, A^C = A \ast B, B^C = A, D^2, A^D = A \ast B;$$

$$B^D = B, C^D = C^{-1} \ast A$$
and is of the form: \( G \cong (((C_2 \times C_2) : C_3) : C_2) \).

2. Given:

\[
G < x, y, t > := Group < x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y), (y * t * t \times)^2 >
\]
we can show that is isomorphic to the progenitor:

\[
\]
and is of the form: \( G \cong (((C_3 \times C_3) : C_2) : C_2) \).

3. Given:

\[
G < x, y, t > := Group < x, y, t | x^{11}, y^2, (x * y)^2, t^2, (t, y), (x^2 * y * t * t^x)^2 >
\]
we can show that is isomorphic to the progenitor:

\[
\]
and is of the form: \( G \cong ((C_{11} \times C_{11}) : C_2) : C_2 \).

4. Given:

\[
G < x, y, t > := Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^2 * y * t^y)^4, (x^{y^t * t^x})^2, (x^2 * y * t^{t^x})^5 >
\]
we can show that is isomorphic to the progenitor:

\[
\]
and is of the form: \( G \cong (((5^4 : C_2) : C_2) : C_3) : C_2 \).
5. Given:

\[ G < x, y, t > : = \text{Group} < x, y, t | x^{15}, y^2, (x \ast y)^2, t^2, (t, y), (x^{y^2} \ast t^y)^4, (y^t \ast t^{yx})^3 > \]

we can show that is isomorphic to the progenitor:

\[ G : \langle A, B, C, D, E, F, J, K \rangle | A^5, B^5, C^5, (A, B), (A, C), (B, C), D^2, A^D = A^{-1}, B^D = B \ast A^{-2}, C^D = B^2 \ast A^{-2} \ast C^{-1}, E^2, A^E = A, B^E = A^2 \ast B^{-1}, C^E = A^2 \ast C^{-1}, D^E = B \ast C^{-1} \ast D, F^3, A^F = A \ast B^{-1}, B^F = A \ast C \ast B^{-2}; \]
\[ C^F = C \ast B^{-2}, D^F = B \ast D \ast E \ast C^2, E^F = A^2 \ast D, J^2, A^J = B \ast A^{-1}, B^J = B, C^J = B^2 \ast C^{-1}, D^J = F \ast D \ast F^{-1}, E^J = B \ast F^{-1} \ast D \ast F \ast E, F^J = F^{-1} > \]

and is of the form: \(((5^3 : C_2) : C_2) : C_3) : C_2).\]

6. Given:

\[ \text{Group} = \langle x, y, t | x^{15}, y^2, (x \ast y)^2, t^2, (t, y), (x^{y^2} \ast t^y)^4, (y^t \ast t^{yx})^6, (x^2 \ast y \ast t^x)^2 > \]

we can show that is isomorphic to the progenitor:

\[ G : \langle A, B, C, D, E, F, J, K \rangle | A^5, B^5, C^5, (A, B), (A, C), (B, C), D^2, A^D = A^{-1}, B^D = B^{-1}, C^D = C^{-1}, E^2, A^E = C^{-1}, B^E = C^2 \ast A^{-2} \ast B^{-1}, C^E = A^{-1}, D^E = B \ast C^2 \ast D \ast A, F^2, A^F = A \ast E \ast B \ast E, B^F = B^{-1}, C^F = C \ast B^{-1}; \]
\[ D^F = D \ast E \ast B \ast E, E^F = B \ast E \ast B^{-1}, A^J = B \ast C^{-1}, B^J = A^{-1} \ast C^{-1}, C^J = A^{-1}, D^J = D \ast A \ast C, E^J = C \ast F \ast E \ast B, F^J = E \ast B^2 \ast C, K^2, A^K = C \ast B^{-1}, B^K = A^{-1} \ast C^{-1}, C^K = C^{-1}, D^K = A \ast B \ast C \ast D, E^K = J \ast E \ast J^{-1}, F^K = B \ast E \ast B^{-1}, J^K = J^{-1} > \]

and is of the form: \(((C_5 \times C_5 \times C_5) : C_2) : C_2) : C_3) : C_2).\]
4.4 Central Extensions

The following examples are of a different type of extensions. The approach is very similar as we’ve done so far, but now we’ll be dealing with central extensions, direct, and semi-direct products. We will begin with a small group and work our way up to a more complex group. Say we begin with the group presentation:

\[ G < x, y, t > := \text{Group} < x, y, t | x^{11}, y^2, (x \ast y)^2, (t, y), (x \ast y \ast t)^3, (x^2 \ast y \ast t \ast t^x)^4 > \]

Listing the normal lattice and the composition factors:

\[
\begin{array}{c|c}
G & A(1, 7) = L(2, 7) \\
G_1 * & A(1, 7) = L(2, 7) \\
G_2 * & \text{Cyclic}(2) \\
1 & \\
\end{array}
\]

We begin to list the extensions from the composition factors: \( G_2/1 \cong C_2 \) which implies that \( G_2 \cong C_2 \), and \( C_2 \) is normal. Moving up the composition factors, we have \( G_1/G_2 \cong PSL_27 \) which implies that \( G_1 \cong G_2 \wr PSL_27 \), hence \( G_1 \cong C_2 \wr PSL_27 \). To solve the extension problem we check for a direct product first, that is, do we have \( G_1 \cong C_2 \times PSL_27 \). Now, looking at the minimal normal subgroup of \( G \) of order 2 we can’t conclude that we have \( C_2 \times PSL_223 \). Looking at the subgroup lattice, note that \( NL[2] \) has order 2 just like \( C_2 \) has order 2. We refer to the normal lattice instead. Our choices are \( NL[3] \) and \( NL[4] \) since \( NL[2] \) is the only maximal subgroup contained in each. Say we choose to move up to \( NL[3] \). We need to look within \( NL[3] \) for a normal subgroup of order 168, since the order of \( NL[3] \) is 168 times the order of \( NL[2] \). Looking within the normal lattice of \( NL[3] \) we’ll notice that there’s no subgroup of order 168. Hence, we can now conclude that \( G_1 \) is not a direct product.
So, we now have to check if we have a semi-direct product. To do so, we will have to find an element, we’ll call it \( B \), in \( NL[3] \) but not in \( NL[2] \) of order 168. So, we are unable to find the required element \( B \). Hence, we don’t have a semi-direct product.

Now, we know that we don’t have a direct or a semi-direct product. So, our next step is to check if we have a central extension. That is, does our group have a center and if it does we will attempt to factor our group \( G \) by the central element.

The center of \( G \) is a permutation group of order 2. Now, our task is to find which element in our lattice is the center of \( G \). Since, the order of \( NL[2] \) the center of \( G \) is indeed \( NL[2] \). Now, we can factor \( G \) by its center. Doing so will generate a smaller group that we’ll work with. \( Q \), will be our new group for which we will apply the composition series approach as we’ve done so far.

So far this is how \( G \) looks like, where “•” indicates that we’ve factored by the center:

\[
G \cong 2\cdot Q
\]

Now, working with \( Q \) we will list the composition factors and the normal lattice of \( Q \).

\[
\begin{array}{c|c}
Q & A(1, 7) = L(2, 7) \\
Q1 & \ast \\
1 & A(1, 7) = L(2, 7)
\end{array}
\]

Listing the extensions from the composition factors: \( Q_1/1 \cong PSL_27 \) which implies that \( Q_1 \cong PSL_27 \), hence \( PSL_27 \) is normal. Moving up the composition factors, we have \( Q/Q_1 \cong PSL_27 \) which implies that \( Q \cong Q_1 \cdot PSL_27 \), hence \( Q \cong PSL_27 \cdot PSL_27 \). Now, to solve the extension problem we check for a direct product first, that is, do we have \( Q \cong PSL_27 \cdot PSL_27 \). Looking at the two minimal normal subgroups of \( Q \) each of order 168 we can conclude that together they’ll have order \((168)^2 = 28224\). Note that \( nl[4] \) has order 28224 just like the order of the two minimal subgroups. Before we can
conclude that we have a direct product. We check with MAGMA if \( nl[2] \) and \( nl[3] \) are a direct product and isomorphic to \( nl[4] \):

\[
D:=\text{DirectProduct}(nl[2],nl[3]);
\]
\[
s:=\text{IsIsomorphic}(D,Q);
\]
\[
s;
\]
\[
> \text{true}
\]

Also, since \( nl[4] \cong Q \), we know that \( Q \) is a direct product extension:

\[
Q \cong PSL_27 \times PSL_27.
\]

Now we can write the progenitor for \( Q \):

\[
Q :< A, B, C, D | A^2 = B^3 = (A+B)^8 = (A,B)^4 = 1, C^2 = D^3 = (C*D)^8 = (C,D)^4 = 1 >
\]

Hence, our complete composition series for \( G \) is:

\[
G \cong 2 \cdot (PSL_27 \times PSL_27).
\]

**4.4.1 Examples**

By applying a similar process, the following can be shown:

1. Given:

\[
\text{Group} = < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^y * t^y)^4, (x^{y*t} * t^x)^2 >
\]

we will factor by the center and obtain a presentation for the group \( Q \):

\[
\]

we can then conclude that our group is of the form \( 2^\bullet(((5^4 : C_2) : C_2) : C_3) : C_2) \).

We have investigated the progenitor \( 2^\bullet 15 : D_{15} \) along with relations added to it, to be given as \(< x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x * y * t^x)^a, (x * y * t^y)^b, (x^t)^c, (x * y^t)^d, (x^y * t^y)^e, (x^{y*t} * t^x)^f, (y^t * t^y * x)^g, (x^2 * y * t^x)^h > \) where values for \( a, b, c, d, e, f, g, h \) where found:
Table 4.1: Homomorphic images of $2^{*15} : D_{15}$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of G</th>
<th>Isomorphic class</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 3 0 0 2</td>
<td>360</td>
<td>$2 \times A_5$</td>
</tr>
<tr>
<td>0 0 0 0 5 3 8 0</td>
<td>2160</td>
<td>$A_6 : 2$</td>
</tr>
<tr>
<td>0 0 0 0 4 2 0 2</td>
<td>24</td>
<td>$(2^3 : 3) : 2$</td>
</tr>
<tr>
<td>0 0 0 0 4 2 0 5</td>
<td>15000</td>
<td>(((((5^4 : 2) : 2) : 3 : 2))2)</td>
</tr>
<tr>
<td>0 0 0 0 4 0 6 2</td>
<td>6000</td>
<td>(((((5^3 : 2) : 2) : 3 : 2))2)</td>
</tr>
<tr>
<td>0 0 0 0 0 0 0 1</td>
<td>24</td>
<td>((2^2 : 3) : 2)</td>
</tr>
<tr>
<td>0 0 0 0 4 0 3 0</td>
<td>3000</td>
<td>(((((5^3 : 2) : 2) : 3 : 2))2)</td>
</tr>
<tr>
<td>0 0 0 0 4 2 0 0</td>
<td>30000</td>
<td>$2^*(((5^4 : 2) : 2) : 3 : 2))$</td>
</tr>
<tr>
<td>0 0 0 0 0 3 5 0</td>
<td>7200</td>
<td>$PGL(2,59)$</td>
</tr>
<tr>
<td>0 0 3 0 10 5 0 5</td>
<td>175560</td>
<td>$J_1$</td>
</tr>
<tr>
<td>0 0 3 0 0 9 7 0</td>
<td>178920</td>
<td>$PGL(2,71)$</td>
</tr>
<tr>
<td>0 0 3 0 7 0 0 5</td>
<td>12180</td>
<td>$PGL(2,29)$</td>
</tr>
</tbody>
</table>
Chapter 5

$J_1$ as a Homomorphic Image of $2^{*15} : D_{15}$

As mentioned in Chapter 4, the progenitor $2^{*15} : D_{15}$ has the Janko group, namely $J_1$, as its homomorphic image. We will investigate our progenitor to verify if it is a new presentation of $J_1$ In Curtis [Cur07], $J_1$ is given as a homomorphic image of $2^{*3} : S_3$ along with the relations: $[(0,1,2)t_0]^{10}$, $[(0,1)t_0]^{19}$, and $[(0,1)t_0t_2]^7$. We expand these relations below:

$[(0,1,2)t_0]^{10} = 1 \Rightarrow (0,1,2)^{10}t_0t_2t_1t_0t_2 = t_0t_1t_2t_0t_1$

$[(0,1,2)t_0]^{10} = 1 \Rightarrow (0,1)^7t_0t_2t_1t_0t_2t_1 = t_2t_0t_2t_1t_0t_2$

$[(0,1)t_0]^{19} = 1 \Rightarrow (0,1)^{19}t_0t_1t_0t_1t_0t_1t_0t_1t_0 = t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0$

Now, expand one of our relations $(x^t)^3 = x^3t_2t_{14} = 1$ which implies that $t_2 \sim t_{14}$.

Having made this discovery, we can obtain the remaining equivalent $t_i$’s:

$t_1 \sim t_4 \sim t_7 \sim t_{10} \sim t_{13}$

$t_2 \sim t_5 \sim t_8 \sim t_{11} \sim t_{14}$

$t_3 \sim t_6 \sim t_9 \sim t_{12} \sim t_{15}$

Having found the above relations we are able to reduce our presentation of $2^{*15} : D_{15}$ as a presentation of $2^{*3} : S_3$, and the progenitor now becomes

$< x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y), (x^y + t^y)^{10}, (x^{yt} * t^2)^5, (x^2 * y * t^r)^5 >$
where $x \sim (1, 2, 0)$ and $y \sim (1, 2)$. We will now proceed to expand our remaining relations $(x^{y^2} * y^1)^{10}, (x^{y^x} * t^{x^2})^5,$ and $(x^2 * y * t^x)^5$:

$$(x^{y^2} * y^1)^{10} = 1 \Rightarrow (1, 2, 0)^{10}t_0t_2t_1t_2t_0 = t_0t_1t_2t_0t_1$$

$$(x^{y^x} * t^{x^2})^5 = 1 \Rightarrow (1, 0, 2)^5t_1t_2t_1t_2t_0t_1 = t_2t_0t_2t_1t_2t_0$$

$$(x^2 * y * t^x)^5 = 1 \Rightarrow (1, 3)^5t_1t_0t_1t_0t_1t_0 = t_1t_0t_1t_0t_1t_0$$

Comparing our findings to the expanded relations of Curtis, only one of our relations coincides. Because the other relations don’t match then we will input our relations into the progenitor:

$$< x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y)$$

where $x \sim (1, 2, 0)$ and $y \sim (1, 2)$:

$$< x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y), (xt)^{10}, (xyt^x)^{10}, (x^2yt^x)^5, (x^2yt^x)^5 > .$$

Once we ran it in MAGMA to verify the progenitor, it does give the homomorphic image of $J_1$. Hence, our relations from the progenitor $2^*_{15} : D_{15}$ imply Curtis’ relations.

The following code may be used to start a double coset enumeration:

```plaintext
S:=Sym(15); xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15); yy:=S!(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(13, 15); N:=sub<S|xx,yy>; G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^5,(x^2*y*t^x)^5>; Index(G,sub<G|x,y>); f,G1,k:=CosetAction(G,sub<G|x,y>); IN:=sub<G1|f(x),f(y)>; /* creating the t_i s */ ts:=[Id(G1):i in [1..15]]; ts[14]:=f(t); ts[2]:=f(t^x); ts[3]:=f(t^x^2); ts[4]:=f(t^x^3); ts[5]:=f(t^x^4); ts[6]:=f(t^x^5);```
ts[6]:=f(t^x7);
.ts[7]:=f(t^x8);
.ts[8]:=f(t^x9);
.ts[9]:=f(t^x10);
.ts[10]:=f(t^x11);
.ts[11]:=f(t^x12);
.ts[12]:=f(t^x13);
.ts[13]:=f(t^x14);
.ts[1]:=f(t^x2);
cst:=[null:i in [1..29260]] where null is [Integers() | ];
prodim:=function(pt,Q,I)
  v:=pt;
  for i in I do
    v:=v^(Q[i]);
  end for;
  return v;
end function;

Dbl:=DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#Dbl;
/* Double coset [14] */
N14:=Stabiliser(N,14);
for g in N14 do g; end for;
S:={[14]};
SS:=S\N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
  for g in IN do if ts[14] eq g*(ts[(Rep(SSS[i]))[1]]) then print Rep(SSS[i]); end if; end for;
  N14s:=N14;
  for g in N do if [14]^g eq [3] then N14s:=sub<N\N14s,g>; end if; end for;
  [14]^N14s;
/* List the elements that 14 is equal to */
T14:=Transversal(N,N14);
for i in [1..#T14] do
  ss:=[14]^T14[i];
  cst[prodim(1,ts,ss)]:=ss;
  end for;
  m:=0;
  for i in [1..29260] do if cst[i] ne [] then m:=m+1; end if; end for; m;
/* List the orbits and the # of single cosets in [14] */
Orbits(N14);
#N/#N14;
Chapter 6

Wreath Product of $\mathbb{Z}_5 \wr S_2$

Let $H$ and $K$ be permutation groups on $X$ and $Y$, respectively. Let $Z = X \times Y$ define a permutation group on $Z$, called the \textit{wreath product} of $H$ by $K$, $H \wr K$.

Consider the following:

$X = \{1, 2, 3, 4, 5\}$, $Y = \{6, 7\}$, $H = \langle (1, 2, 3, 4, 5) \rangle$, and $K = \langle (6, 7) \rangle$.

Let $\gamma \in H$ and $y$ be a fixed element of $Y$. Then, we define:

$$\gamma(y) = \begin{cases} (x, y) & \mapsto (x(\gamma), y) \\ (x, y) & \mapsto (x, y_1) \text{ if } y_1 \neq y \end{cases}$$

As well as $k^* : (x, y) \mapsto (x, (y)k)$ where $k = \{6, 7\}$ and $k \in K$. Also,

$Z = X \times Y = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7)\}$.

Now, we also need to find $\gamma(6)$, $\gamma(7)$, and $\gamma(6, 7)^*$ where $\gamma = (1, 2, 3, 4, 5)$. The process is summarized in the following table:
Table 6.1: $\gamma$ Function

<table>
<thead>
<tr>
<th></th>
<th>$\gamma(6)$</th>
<th>$\gamma(7)$</th>
<th>$(6,7)^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 6)</td>
<td>$\mapsto$ (2, 6)</td>
<td>(1, 6) $\mapsto$ (1, 6)</td>
<td>(1, 6) $\mapsto$ (1, 7)</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>$\mapsto$ (3, 6)</td>
<td>(2, 6) $\mapsto$ (2, 6)</td>
<td>(2, 6) $\mapsto$ (2, 7)</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>$\mapsto$ (4, 6)</td>
<td>(3, 6) $\mapsto$ (3, 6)</td>
<td>(3, 6) $\mapsto$ (3, 7)</td>
</tr>
<tr>
<td>(4, 6)</td>
<td>$\mapsto$ (5, 6)</td>
<td>(4, 6) $\mapsto$ (4, 6)</td>
<td>(4, 6) $\mapsto$ (4, 7)</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>$\mapsto$ (1, 6)</td>
<td>(5, 6) $\mapsto$ (5, 6)</td>
<td>(5, 6) $\mapsto$ (5, 7)</td>
</tr>
<tr>
<td>(1, 7)</td>
<td>$\mapsto$ (1, 7)</td>
<td>(1, 7) $\mapsto$ (2, 7)</td>
<td>(1, 7) $\mapsto$ (1, 6)</td>
</tr>
<tr>
<td>(2, 7)</td>
<td>$\mapsto$ (2, 7)</td>
<td>(2, 7) $\mapsto$ (3, 7)</td>
<td>(2, 7) $\mapsto$ (2, 6)</td>
</tr>
<tr>
<td>(3, 7)</td>
<td>$\mapsto$ (3, 7)</td>
<td>(3, 7) $\mapsto$ (4, 7)</td>
<td>(3, 7) $\mapsto$ (3, 6)</td>
</tr>
<tr>
<td>(4, 7)</td>
<td>$\mapsto$ (4, 7)</td>
<td>(4, 7) $\mapsto$ (5, 7)</td>
<td>(4, 7) $\mapsto$ (4, 6)</td>
</tr>
<tr>
<td>(5, 7)</td>
<td>$\mapsto$ (5, 7)</td>
<td>(5, 7) $\mapsto$ (1, 7)</td>
<td>(5, 7) $\mapsto$ (5, 6)</td>
</tr>
</tbody>
</table>

Hence from the table above we have the relations:

$$\gamma(6) : (1, 2, 3, 4, 5), \gamma(7) : (6, 7, 8, 9, 10), (6, 7)^* : (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$$

Since $\gamma(y)$ and $k^*$ are permutations of $S_Z$, we have $\gamma(6) \times \gamma(7) : Sym(2)$. We verify that our work is correct in MAGMA:

```magma
S := Sym(10);
N := sub<S|S!(1,2,3,4,5),S!(6,7,8,9,10),S!(1,6)(2,7)(3,8)(4,9)(5,10)>;
#N;
W := WreathProduct(CyclicGroup(5),Sym(2));
#W;
s := IsIsomorphic(N,W);
s;
```

Since the above prints true, we can now write the presentation for $Z_5 \rtimes S_2$:

$$G < x, y, z > := Group < x, y, z | x^5, y^5, (x, y), z^2, xz = y, yz = x >$$

From here we can expand the progenitor by adding new relations hoping to discover new groups. After running the code in MAGMA the following group is obtained:
\[ G<x,y,z,t>:=\text{Group}\langle x,y,z,t|x^5,y^5,(x,y),z^2,x^z=y,y^z=x,t^2,(t,y), \\
(x^2t^y)^z6, (t^xzt^xz^y)^3 \rangle; \]
\[ f1,G1,k1:=\text{CosetAction}(G,\text{sub}<G|x,y,z>); \]
\[ \text{CompositionFactors}(G1); \]

\begin{center}
\begin{tabular}{c|c|c|c|}
 & J2 & * & Cyclic(2) \\
\hline
#N & & & 1 \\
\end{tabular}
\end{center}

6.1 \textbf{\( J_2 \times C_2 \) as a Homomorphic Image of \( \mathbb{Z}_5 \wr S_2 \)}

The following code is used to begin the double coset enumeration of \( J_2 \times C_2 \).

At the time or writing, the construction is not complete.

\[ S:=\text{Sym}(10); \]
\[ xx:=S!(1,2,3,4,5); \]
\[ yy:=S!(6,7,8,9,10); \]
\[ zz:=S!(1,6)(2,7)(3,8)(4,9)(5,10); \]
\[ N:=\text{sub}<S|x,y,z>; \]
\[ G<x,y,z,t>:=\text{Group}\langle x,y,z,t|x^5,y^5,(x,y),z^2,x^z=y,y^z=x,t^2,(t,y), \\
(x^2t^y)^z6, (t^xzt^xz^y)^3 \rangle; \]
\[ f,G1,k1:=\text{CosetAction}(G,\text{sub}<G|x,y,z>); \]
\[ \text{Index}(G,\text{sub}<G|x,y,z>); \]
\[ ИN:=\text{sub}<G1|f(x),f(y),f(z)>; \]
\[ /* \text{Creating the} \ t_i \text{ s} */ \]
\[ \text{ts}:=\{\text{Id}(G1) : i \in [1..10]\}; \]
\[ \text{ts}[1]:=f(t); \]
\[ \text{ts}[2]:=f(t^x); \]
\[ \text{ts}[3]:=f(t^{x^2}); \]
\[ \text{ts}[4]:=f(t^{x^3}); \]
\[ \text{ts}[5]:=f(t^{x^4}); \]
\[ \text{ts}[6]:=f(t^z); \]
\[ \text{ts}[7]:=f((t^x)^z); \]
\[ \text{ts}[8]:=f((t^{x^2})^z); \]
\[ \text{ts}[9]:=f((t^{x^3})^z); \]
\[ \text{ts}[10]:=f((t^{x^4})^z); \]
\[ \text{cst}:=\{\text{null} : i \in [1..24192]\} \text{ where null is } \text{[Integers()] | } \];
\[ \text{prodim}:=\text{function}(\text{pt},Q,I) \]
\[ v:=\text{pt}; \]
\[ \text{for } i \text{ in } I \text{ do} \]
v:=v^(Q[i]);
end for;
return v;
end function;
Appendix A: Construction of $2^5 : A_5$

S:=Sym(5);
xx:=S!(5,1,2,3,4);
yy:=S!(4,2,1);
N:=sub<S|xx,yy>;
#N;
G<x,y,t>:=Group<x,y,t|x^5,y^3,(x*y)^2,t^2,(t,y),(t,x^2*y*x^1),
(t*t^x)^2>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1): i in [1..5]];
for i in [1..5] do ts[i]:=f(t^(x^i)); end for;
prodim := function(pt, Q, I) v := pt;
for i in I do v := v^(Q[i]); end for; return v; end function;
cst := [null : i in [1 .. 90]] where null is [Integers() | ];
for i := 1 to 5 do
cst[prodim(1, ts, [i])] := [i]; end for;
m:=0;
for i in [1..<#cst] do if cst[i] ne [] then m:=m+1; end if; end for; m;
\\\\\\\[5\\\\\\\]
N5:=Stabiliser(N,5);
Orbits(N5);
N51:=Stabilizer(N5,1); S:=[5,1];
SS:=S^N;
\\\\\\\[5 1\\\\\\\]
N51:=Stabilizer(N5,1);
S:=[5,1];
SS:=S^N;
#N51;
Orbits(N51);
tr51 := Transversal(N, N51);
for i := 1 to #tr51 do ss := [5, 1]^tr51[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
\[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\] \[5 1 2\]
N512:=Stabiliser(N51,2); S:=[5,1,2];
SS:=S\N;
N512s:=N512;
for g in N do if [5,1,2]^g eq [1,5,2] then N512s:=sub<N|N512s,g>
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [1,2,5] then N512s:=sub<N|N512s,g>
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [2,1,5] then N512s:=sub<N|N512s,g>
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [2,5,1] then N512s:=sub<N|N512s,g>
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [5,2,1] then N512s:=sub<N|N512s,g>
end if;end for;
[5,1,2]^N512s;
#N512s; #N/#N512s;
Orbits(N512s);
\[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\] \[5 1 2 3\]
N5123:=Stabiliser(N512,3);
S:=[5,1,2,3];
SS:=S\N;
N5123s:=N5123;
for g in N do if [5,1,2,3]^g eq [1,5,2,3] then N5123s:=sub<N|N5123s,g>
end if;end for;
[5,1,2,3]^N5123s;
#N5123s;
for g in N do if [5,1,2,3]^g eq [1,2,5,3] then N5123s:=sub<N|N5123s,g>
end if;end for;
[5,1,2,3]^N5123s;
#N5123s;
for g in N do if [5,1,2,3]^g eq [2,3,5,1] then N5123s:=sub<N|N5123s,g>
end if;end for;
[5,1,2,3]^N5123s;
#N5123s;
#N/#N5123s;
Orbits(N5123s);
\\\\\\\\\\\\  [5 1 2 3 4]  \\\\\\\\\\\\\nN51234:=Stabiliser(N5123,4);
S:={[5,1,2,3,4]};
SS:=S^N;
N51234s:=N51234;
for g in N do if [5,1,2,3,4]^g eq [1,5,2,3,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [1,2,5,3,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,1,3,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,1,4,3] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,3,1,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
#N/#N51234s;
Appendix B: Construction of $7^8 : m \ L_2(7)$

SS := Sym(48);
xx := SS!(1,7,19)(2,8,20)(3,9,21)(4,10,22)(5,11,23)(6,12,24)
(37,40,38)(39,41,42)(43,44,46)(45,48,47);
yy := SS!(1,13,25)(2,14,26)(3,15,27)(4,16,28)(5,17,29)(6,18,30)
(7,10,8)(9,11,12)(19,20,22)(21,24,23)(31,37,43)(32,38,44)(33,39,45)
(34,40,46)(35,41,47)(36,42,48);
N := sub<SS|xx,yy>;
#N;
G<x,y,t> := Group<x,y,t|x^3,y^3,(x*y)^4,(x,y)^3,(x^2*y)^7,t^7,
t^(x * y^-1 * x^-1 )=t^2,(t,y^-1 * x^-1 * y^-1),
(x^2 * y * t * t^x)^2>;
Index(G,sub<G|x,y>);
f,G1,k := CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN := sub<G1|f(x),f(y)>;
#G;

/* gives the sequences of the double cosets */
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
prodim := function(pt, Q, I)
  v := pt;
  for i in I do v:=v^(Q[i]);
  end for; return v; end function;

/* Defining the t_i's */
ts := [Id(G1): i in [1..48] ];
\[ ts[5]:=(ts[1])^5; ts[6]:=(ts[1])^6; ts[7]:=f(t^x); ts[8]:=(ts[7])^2; \\
 ts[9]:=(ts[7])^3; ts[10]:=(ts[7])^4; ts[11]:=(ts[7])^5; \\
 ts[12]:=(ts[7])^6; ts[13]:=f(t^y); ts[14]:=(ts[13])^2; \\
 ts[15]:=(ts[13])^3; ts[16]:=(ts[13])^4; \\
 ts[17]:=(ts[13])^5; ts[18]:=(ts[13])^6; ts[19]:=f(t^{x^2}); \\
 ts[20]:=(ts[19])^2; ts[21]:=(ts[19])^3; ts[22]:=(ts[19])^4; \\
 ts[23]:=(ts[19])^5; ts[24]:=(ts[19])^6; ts[25]:=f(t^{y^2}); \\
 ts[26]:=(ts[25])^2; ts[27]:=(ts[25])^3; ts[28]:=(ts[25])^4; \\
 ts[29]:=(ts[25])^5; ts[30]:=(ts[25])^6; ts[31]:=f(t^{x^2+y^2}); \\
 ts[32]:=(ts[31])^2; ts[33]:=(ts[31])^3; ts[34]:=(ts[31])^4; \\
 ts[35]:=(ts[31])^5; ts[36]:=(ts[31])^6; ts[37]:=f(t^{x^2+y^2}); \\
 ts[38]:=(ts[37])^2; ts[39]:=(ts[37])^3; ts[40]:=(ts[37])^4; \\
 ts[41]:=(ts[37])^5; ts[42]:=(ts[37])^6; ts[43]:=f(t^y); \\
 ts[44]:=(ts[43])^2; ts[45]:=(ts[43])^3; ts[46]:=(ts[43])^4; \\
 ts[47]:=(ts[43])^5; ts[48]:=(ts[43])^6; \]

Orbits(N);

cst:= [null : i in [1 .. Index(G, sub<G|x,y>)]) where null is 
[Integers() | ]; for i in {1, 2, 4, 7, 8, 10, 13, 14, 16, 19, 20, 
22, 25, 26, 28, 31, 32, 34, 37, 38, 40, 43, 44, 46 }
do cst[prodim(1, ts, [i])]:= [i]; end for; m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for;
m;
for i in {3, 5, 6, 9, 11, 12, 15, 17, 18, 21, 23, 24, 27, 29, 30, 
33, 35, 36, 39, 41, 42, 45, 47, 48 }
do cst[prodim(1, ts, [i])]:= [i]; end for; m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* Double coset [1] */

N1:= Stabiliser(N, 1);
for g in N1 do g; end for;

SSS:= { [1] };
SSS:= SSS"N;
N1s:= N1;

SSS:= Setseq(SSS);
for i in [1..#SSS] do
for g in IN do if ts[1] eq g*(ts[Rep(SSS[i])][1])
then print Rep(SSS[i]);
end if; end for; end for;

T1:= Transversal(N, N1);
for i in [1..#T1] do
ss:= [1]^T1[i];
cst[prodim(1, ts, ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
T1;
Orbits(N1);
#N/#N1s;

/* Double coset [3] */
N3:=Stabiliser(N,3);
for g in N3 do g;end for;
SSS:={[3]};
SSS:=SSS^N;
N3s:=N3;
SSS:=Setseq(SSS);
for i in [1..#SSS] do
for g in IN do if ts[3] eq g*(ts[(Rep(SSS[i]))[1]]) then print Rep(SSS[i]);
end if;
end for; end for;

T3:=Transversal(N,N3);
for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
T3;
Orbits(N3s);
#N/#N3s;

/* double coset [1,7] */
N17:=Stabiliser(N,[1,7]);
SSS:={[1,7]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1] * ts[7] 
eq g * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;

N17s := N17;
for g in N do if 1^g eq 8 and 7^g eq 14 then N17s := sub<N|N17s, g>;
end if; end for;
#N17s;
[1, 7]^N17s;

T17 := Transversal(N, N17);
for i in [1..#T17] do
ss := [1, 7]^T17[i];
cst [prod1m(1, ts, ss)] := ss;
end for;
m := 0;
for i in [1..350] do if cst[i] ne [] then m := m+1;
end if; end for; m;

#N17; Orbits(N17s);
#N/#N17s;

/* double coset [1, 8] */

N18 := Stabiliser(N, [1, 8]);
SSS := {[1, 8]};
SSS := SSS^N;
SSS;
#(SSS);
Seqq := Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1] * ts[8] 
eq g * ts[Rep(Seqq[i])[1]] * ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;

N18s := N18;
for g in N do if 1^g eq 19 and 8^g eq 34 then N18s := sub<N|N18s, g>;
end if; end for;
#N18s;
[1, 8]^N18s;
T18:=Transversal(N,N18);
for i in [1..#T18] do
  ss:=[1,8]^T18[i];
  cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N18; Orbits(N18);
#N/#N18s;

/* double coset [1,9] */
N19:=Stabiliser(N,[1,9]);
SSS:={{1,9}};
SSS:=SSS\N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
  for g in IN do if ts[1]*ts[9]
    eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
  then print Rep(Seqq[i]);
  end if;
end for;
end for;

N19s:=N19;
for g in N do if 1^g eq 37 and 9^g eq 17 then N19s:=sub<N|N19s,g>;
end if; end for;
#N19s;
[1,9]^N19s;

T19:=Transversal(N,N19);
for i in [1..#T19] do
  ss:=[1,9]^T19[i];
  cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N19s; Orbits(N19s);
/* double coset [1,10] */
N110 := Stabiliser(N,[1,10]);
SSS := {[1,10]};
SSS := SSS \ N;

for i in [1..#SSS] do
    for g in IN do if ts[1]*ts[10] eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        then print Rep(Seqq[i]);
    end if;end for;end for;

N110s := N110;
for g in N do if 1^g eq 22 and 10^g eq 43 then
    N110s := sub<N|N110s,g>;
end if; end for;

T110 := Transversal(N,N110);
for i in [1..#T110] do
    ss := [1,10] \ T110[i];
    cst[prodim(1,ts,ss)] := ss;
end for;

m := 0;
for i in [1..350] do if cst[i] ne [] then m := m+1; end if;
end for; m;

/* double coset [1,11] */
N111 := Stabiliser(N,[1,11]);
SSS := {[1,11]};
SSS := SSS \ N;

for i in [1..#SSS] do
    for g in IN do if ts[1]*ts[11] eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
        then print Rep(Seqq[i]);
    end if;end for;end for;

N111s := N111;
for g in N do if 1^g eq 22 and 10^g eq 43 then
    N111s := sub<N|N111s,g>;
end if; end for;

T111 := Transversal(N,N111);
for i in [1..#T111] do
    ss := [1,11] \ T111[i];
    cst[prodim(1,ts,ss)] := ss;
end for;

m := 0;
for i in [1..350] do if cst[i] ne [] then m := m+1; end if;
end for; m;
for i in [1..#SSS] do
for g in IN do if ts[i]*ts[11]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N111s:=N111;
for g in N do if 1^g eq 8 and 11^g eq 29
then N111s:=sub<N|N111s,g>;
end if; end for;
#N111s;
[1,11]^N111s;

T111:=Transversal(N,N111);
for i in [1..#T111] do
ss:=[1,11]^T111[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N111s; Orbits(N111s);
#N/#N111s;

/* double coset [1,12] */
N112:=Stabiliser(N,[1,12]);
SSS:={(1,12)};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[i]*ts[12]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N112s:=N112;
for g in N do if 1^g eq 2 and 12^g eq 18
then N112s:=sub<N|N112s,g>;
end if; end for;

#N112s;
[1,12]^N112s;

T112:=Transversal(N,N112);
for i in [1..#T112] do
ss:=[1,12]^T112[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N112s; Orbits(N112s);
#N/#N112s;

/* double coset [3,10] */
N310:=Stabiliser(N,[3,10]);
SSS:={[3,10]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[3]*ts[10] eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N310s:=N310;
for g in N do if 3^g eq 9 and 10^g eq 31
then N310s:=sub<N|N310s,g>;
end if; end for;
#N310s;
[3,10]^N310s;

T310:=Transversal(N,N310);
for i in [1..#T310] do
ss:=[3,10]^T310[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N310s; Orbits(N310s);
#N/#N310s;

/* double coset [1,11,46] */
N11146:=Stabiliser(N,[1,11,46]);
SSS:={[1,11,46]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[11]*ts[46]
eq g*ts[Rep(Seqq[i])][1]*ts[Rep(Seqq[i])][2]*ts[Rep(Seqq[i])][3]
then print Rep(Seqq[i]);
end if;display;end for;end for;

N11146s:=N11146;
for g in N do if 1^g eq 37 and 11^g eq 27 and 46^g eq 13 then
N11146s:=sub<N|N11146s,g>;end if; end for;

for g in N do if 1^g eq 40 and 11^g eq 9 and 46^g eq 14 then
N11146s:=sub<N|N11146s,g>;end if; end for;

for g in N do if 1^g eq 4 and 11^g eq 9 and 46^g eq 25 then
N11146s:=sub<N|N11146s,g>;end if; end for;

#N11146s;
[1,11,46]^N11146s;
T11146:=Transversal(N,N11146);
for i in [1..#T11146] do
ss:=[1,11,46]^T11146[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
#N11146s; Orbits(N11146s);
#N/#N11146s;
Appendix C: Images of $2^{*15} : D_{15}$

We have investigated the progenitor $2^{*15} : D_{15}$ along with relations added to it, to be given as $< x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x*y*t^x)^a, (x*y*t^y)^b, (x*t)^c, (x*y^t)^d, (x^y^t*y^t)^e, (x*y^t*t^x)^f, (y^t+t^y*x)^g, (x^2*y*t^x)^h >$ where values for $a, b, c, d, e, f, g, h$ where found as well as their composition factors:

/* 0 0 3 0 10 5 0 5 29260 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^y^t*y^t)^10,(x^y*t^x*t^x)^5,(x^2*y*t^x)^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| J1
1

/* 0 0 3 0 8 8 0 5 322560 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^y^t*y^t)^8,(x*y^t*t^x)(x^2)^8,(x^2*y*t^x)(t^x)^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
* 
| Cyclic(3)
* 
| Cyclic(2)
* 
| Cyclic(2)
/*
| A(2, 4) = L(3, 4)
* 
| Cyclic(2)
* 
| Cyclic(2)
1
*/
0 0 3 0 8 10 0 4 50000 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))#10,(x^2*y*t^(t^x))^4>;
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^5,(x^2*y*t^(t^x))^10>;
CompositionFactors(g1);
G
| Cyclic(2)
* 
| Cyclic(3)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(5)
* 
| Cyclic(5)
* 
| Cyclic(5)
* 
| Cyclic(5)
* 
| Cyclic(5)
1
*/
0 0 5 0 4 5 10 262144 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^5,(x^2*y*t^(t^x))^10>;
CompositionFactors(g1);
G
| Cyclic(2)
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
* 
| Cyclic(3)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Alternating(5)
* 
| Cyclic(2)

/* 0 0 3 0 0 8 0 2 128*/
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)~2,t~2,(t,y), (x~t)~3,
(x~(y*t)*t~(x~2))~8,(x~2*y*t~(t~x))~2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
* 
| Cyclic(3)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
* 
| Cyclic(2)
1

/* 0 0 3 0 10 10 0 41600 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)~2,t~2,(t,y), (x~t)~3,
(x~(y~2)*t~(y)~10),(x~(y*t)*t~(x~2))~10, (y~t*t~(y*x))~10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
 | Cyclic(2)
 *
 | 2A(2, 4) = U(3, 4)
 *
 | Cyclic(2)
 1

/*0 0 3 0 8 6 0 4 3888*/
G<x,y,t>::Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^((y^t)*t^((x^2)))^6,(x^2*y^t*(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
 | Cyclic(2)
 *
 | Cyclic(3)
 *
 | Cyclic(2)
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 | Cyclic(2)
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 | Cyclic(3)
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 | Cyclic(3)
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 | Cyclic(3)
 *
 | Cyclic(3)
 *
 | Cyclic(3)
 *
 | Cyclic(3)
 1

/*0 0 5 0 0 8 4 6 12288*/
G<x,y,t>::Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y^t)*t^((x^2)))^8,(y^t*t^((y*x)))^4,(x^2*y^t*(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
<table>
<thead>
<tr>
<th>Cyclic(2) *</th>
<th>Alternating(5) *</th>
<th>Cyclic(2) *</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyclic(2) *</td>
<td>Cyclic(2) *</td>
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<td>Cyclic(2) *</td>
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<tr>
<td>Cyclic(2)</td>
<td>Cyclic(2)</td>
<td>Cyclic(2)</td>
</tr>
</tbody>
</table>

/* 0 0 3 0 0 4 0 6 78336 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

<table>
<thead>
<tr>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyclic(2) *</td>
</tr>
<tr>
<td>Cyclic(3) *</td>
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<td>Cyclic(2) *</td>
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<td>Cyclic(2) *</td>
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<tr>
<td>Cyclic(2) *</td>
</tr>
<tr>
<td>A(1, 17) = L(2, 17) *</td>
</tr>
<tr>
<td>Cyclic(2) *</td>
</tr>
</tbody>
</table>
Cyclic(2)

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, 
(x^y*t)^2*(x^2))^4,(x^2*y*t^2*(t^x))^5>;

CompositionFactors(g1);

G
  | Cyclic(2)
  |
  | Cyclic(3)
  |
  | Cyclic(2)
  |
  | Cyclic(2)
  |
  | A(1, 29) = L(2, 29)
  |
  1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, 
(x^y*t)^2*(x^2))^9,(y^t*t^2*(y^x))=7>;

CompositionFactors(g1);

G
  | A(1, 71) = L(2, 71)
  |
  1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, 
(x^y*t)^2*(x^2))^10,(y^t*t^2*(y^x))=7>;

CompositionFactors(g1);

G
  | Cyclic(2)
  |
  | A(1, 29) = L(2, 29)
  |
  1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, 
(x^y*t)^2*(x^2))^7,(x^2*y*t^2*(t^x))^5>;

CompositionFactors(g1);
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
  | A(1, 29) = L(2, 29)
  | 1

/* 0 0 3 0 8 5 0 0 4960 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^-2,(t,y), (x^-t)^3,
(x^-y^2)*t^-y^-8,(x^-y^2)*t^-y^-2)^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
  | Cyclic(2)
  | *
  | A(1, 31) = L(2, 31)
  | 1

/* 0 0 3 0 8 8 0 4 8192 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^-2,(t,y), (x^-t)^3,
(x^-y^2)*t^-y^-8,(x^-y^2)*t^-y^-2)^8,(x^-y^2)*t^-y^-4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
  | Cyclic(2)
  | *
  | Cyclic(3)
  | *
  | Cyclic(2)
  | *
  | Cyclic(2)
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  | Cyclic(2)
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  | Cyclic(2)
  | *
  | Cyclic(2)
  | *
/* 0 0 5 0 3 10 0 12960 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x*(y^t)*t^(-2))^(3), (y^t*t^(-y*x))^(10)>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
| *
| Cyclic(2)
| *
| Cyclic(2)
| *
| Cyclic(2)
1

G
| Alternating(6)
| *
| Alternating(5)
| *
| Cyclic(3)
1

/* 0 0 5 0 9 4 5 35244 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x*(y^2)*t^y)^9,(x^((y*t)*t^(-2))^(9), (y^t*t^(-y*x))^(4),
(x^2*y*t^(t*x))^(5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| A(1, 89) = L(2, 89)
1

/* 0 0 5 0 8 4 6 12288 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x*(y^t)*t^(-2))^(8), (y^t*t^(-y*x))^(4),(x^2*y*t^(t*x))^(6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
| *
| Alternating(5)
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x"(y^2)*t^y)"10,(x"(y*t)*t"(x^2))"6,(x"2*y*t"(t*x))"4>; f1,g1,k1:=CosetAction(G,sub<G|x,y>); CompositionFactors(g1);
G
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)

/* 0 0 3 0 10 6 0 4 15120 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x"(y^2)*t^y)"10,(x"(y*t)*t"(x^2))"6,(x"2*y*t"(t*x))"4>; f1,g1,k1:=CosetAction(G,sub<G|x,y>); CompositionFactors(g1);
G
| Cyclic(2)
| Cyclic(2)
| Alternating(7)
| Cyclic(2)
| Cyclic(2)
| Cyclic(3)
| Cyclic(3)
| Cyclic(3)

/* 0 0 3 0 9 10 10 0 1140 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x"(y^2)*t^y)"9,(x"(y*t)*t"(x^2))"10, (y"t*t"(y*x))"10>; f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| A(1, 19) = L(2, 19)
* |
| Cyclic(2)
1

/* 0 0 3 8 7 0 0 5 2030 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,(x*y^t)^8,
(x^(y^2)*t^y)^7,(x^2*y*t^t(x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| A(1, 29) = L(2, 29)
1

/* 0 0 3 2 7 8 0 0 3584 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,(x*y^t)^2,
(x^(y^2)*t^y)^7,(x*(y*t)*t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
* |
| A(1, 7) = L(2, 7)
* |
| Cyclic(2)
* |
| Cyclic(2)
* |
| Cyclic(2)
* |
| Cyclic(2)
* |
| Cyclic(2)
* |
| Cyclic(2)
* |
| Cyclic(2)
1

/* 0 0 3 0 9 0 10 0 3420 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x+y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^9,(y^t*t^((y*x)^10))>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
\[
\text{G} \\
| \text{A}(1, 19) = \text{L}(2, 19) \\
* \\
| \text{Cyclic}(2) \\
* \\
| \text{Cyclic}(3) \\
1
\]

\[
/* 0 0 3 0 9 4 0 0 816 */ \\
G<x,y,t>:=\text{Group}<x,y,t|x^{15},y^2,(x*y)^2,t^2,(t,y), (x^t)^3 \\
,(x^{(y^2)*t^y})^9,(x^{(y*t)*t^y})^4,g1,k1:=\text{CosetAction}(G,\text{sub}<G|x,y>); \\
\text{CompositionFactors}(g1); \\
\text{G} \\
| \text{Cyclic}(2) \\
* \\
| \text{A}(1, 17) = \text{L}(2, 17) \\
1 \\
\]

\[
/* 0 0 3 0 7 8 0 11480 */ \\
G<x,y,t>:=\text{Group}<x,y,t|x^{15},y^2,(x*y)^2,t^2,(t,y),(x^t)^3, \\
x^{(y*t)*t^y(x^2)}^7, (y^t*t^y(x*x))^8,g1,k1:=\text{CosetAction}(G,\text{sub}<G|x,y>); \\
\text{CompositionFactors}(g1); \\
\text{G} \\
| \text{Cyclic}(2) \\
* \\
| \text{A}(1, 41) = \text{L}(2, 41) \\
1 \\
\]

\[
/* 0 0 3 0 4 0 4 1024 */ \\
G<x,y,t>:=\text{Group}<x,y,t|x^{15},y^2,(x*y)^2,t^2,(t,y), (x^t)^3, \\
(x^{(y*t)*t^y(x^2)})^4, (y^t*t^y(x*x))^0,(x^2*y*t^y(t*x))^4; \\
f1,g1,k1:=\text{CosetAction}(G,\text{sub}<G|x,y>); \\
\text{CompositionFactors}(g1); \\
\text{G} \\
| \text{Cyclic}(2) \\
* \\
| \text{Cyclic}(3) \\
* \\
| \text{Cyclic}(2) \\
* \\
| \text{Cyclic}(2) \\
* \\
|
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3, (x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
 | Cyclic(2)
 | Cyclic(3)
 | Cyclic(2)
 | Cyclic(2)
 | A(1, 7) = L(2, 7)

/* 0 0 3 0 0 4 10 10 10 440 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^10,(x^2*y*t^(t^x))^10;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
| A(1, 11) = L(2, 11)
| Cyclic(2)
1

/* 0 0 3 0 0 6 7 7 364 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^7,(x^2*y*t^(t^x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| A(1, 13) = L(2, 13)
| Cyclic(2)
1

/* 0 0 3 0 6 0 0 8 288 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^6,(x^2*y*t^(t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
| Cyclic(3)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(3)
| Cyclic(2)
| Cyclic(3)
Cyclic(2)

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^-(x^2))^5,(x^2*y*t^-(t*x))^3>;

f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| A(1, 19) = L(2, 19)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8, (y^t*t^-(y*x))^10>;

f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
| Alternating(6)
| Cyclic(2)
| Cyclic(3)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^-(t*x))^4,(x^2*y*t^-(t*x))^8>;

f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
| Cyclic(3)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
|
\[
\text{G} := \text{Group}\langle x, y, t | x^{15}, y^2, (xy)^2, t^2, (t, y), (x^t)^3, (x^{2}y^t)^6, (x^{2}y^t)(t^x)^6 \rangle;
\]
\[
f_1, g_1, k_1 := \text{CosetAction}(G, \text{sub}<G|x, y>);
\]
\[
\text{CompositionFactors}(g_1);
\]

/* 0 0 3 0 6 0 6 162 */
\[
\text{G} := \text{Group}\langle x, y, t | x^{15}, y^2, (xy)^2, t^2, (t, y), (x^t)^3, (x^{2}y^t)^6, (x^{2}y^t)(t^x)^6 \rangle;
\]
\[
f_1, g_1, k_1 := \text{CosetAction}(G, \text{sub}<G|x, y>);
\]
\[
\text{CompositionFactors}(g_1);
\]
Cyclic(3)

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x*(y*t)*t^(-2))*(x^2))^5, (y^t*t^(-y*x))^10;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
  | Cyclic(2)
  *
  | A(1, 11) = L(2, 11)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^t*(y*t)*t^(-2))^2, (y^t*t^(-y*x))^8>
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^t*(y*t)*t^(-2))^2, (y^t*t^(-y*x))^8>
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^t*(y*t)*t^(-2))^2, (y^t*t^(-y*x))^8>
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
\begin{verbatim}
(x^(y^2)*t^y)^8,(x*(y*t)*t^*(x^2))^2,(x^2*y*t^((t^x))^8);
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
 | *
| Cyclic(3)
 | *
| Cyclic(2)
 | *
| Cyclic(2)
 | *
| Cyclic(2)
 | *
| Cyclic(2)
 | *
| Cyclic(2)
 | 1

/* 0 0 3 0 6 0 10 0 50 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^6,(y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Cyclic(2)
 | *
| Cyclic(2)
 | *
| Cyclic(3)
 | *
| Cyclic(3)
 | *
| Cyclic(5)
 | *
| Cyclic(5)
 | *
| Cyclic(5)
 | 1

/* 0 0 3 0 6 5 0 20 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^t)*t^*(x^2))^6, (y^t*t^((y*x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Alternating(5)
 | *
| Cyclic(2)
 | 1
\end{verbatim}
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^3,(x^2*y*t^(t^x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Alternating(5)
| Cyclic(2)
| Cyclic(3)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^3,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Alternating(5)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
G
| Cyclic(2)
| Cyclic(3)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
| Cyclic(2)
1
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^3,
   (x^(y*t)*t^(x^2))^8, (y^t*t^((y*x)^-7));
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
        | Cyclic(2)
   *      
        | A(1, 7)= L(2, 7)
   
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^3,
   (x^(y^2)*t^y)^6,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
        | Cyclic(2)
   *      
        | Cyclic(2)
   *      
        | Cyclic(3)
   *      
        | Cyclic(3)
   *      
        | Cyclic(3)
   
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,
   (x^(y^2)*t^y)^6, (y^t*t^((y*x)^-6),(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
        | Alternating(5)
   *      
        | Cyclic(2)
   *      
        | Cyclic(2)
   
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^-t)^5,
   (x^(y^-2)*t^-y)^5,(x^(-y*t)*t^(-x^2))^6, (y^-t*t^((y*x)^-5),
(x^2*y*t^(t-x))^5;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Alternating(5)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 1

/* 0 0 5 0 0 10 6 2 120 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,
 (x^((y^2)*t^-y))^-6,(y^-t^y^-t^x))^-5,(x^-2*y*t^-t^-x))^-2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| Alternating(5)
 *
 | Cyclic(2)
 *
 | Cyclic(5)
 *
 | Cyclic(2)
 1

/* 0 0 5 0 6 0 5 3 132 */
G<x,y,t>:=Group<x,y,t|x^15,y^-2,(x*y)^2,t^-2,(t,y),(x^-t)^5,
 (x^((y^2)*t^-y))^-6,(y^-t^y^-t^x))^-5,(x^-2*y*t^-t^-x))^-3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
| A(1, 11) = L(2, 11)
 *
 | Cyclic(2)
 1

/* 0 0 5 0 4 0 0 8 144 */
G<x,y,t>:=Group<x,y,t|x^15,y^-2,(x*y)^2,t^-2,(t,y),(x^-t)^5,
 (x^((y^2)*t^-y))^-4,(x^-2*y*t^-t^-x))^-8>;
\texttt{f1,g1,k1:=CosetAction(G,sub<G|x,y>);}
\texttt{CompositionFactors(g1);}
\begin{verbatim}
G
 | Cyclic(2)
 *
 | Alternating(6)
 *
 | Cyclic(2)
 1

/* 0 0 5 0 5 6 0 5 192 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^y*(x^2))^6,(x^-2*y*t^y*(t^-x))^5>;
\end{verbatim}
\texttt{f1,g1,k1:=CosetAction(G,sub<G|x,y>);}
\texttt{CompositionFactors(g1);}
\begin{verbatim}
G
 | Alternating(5)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 *
 | Cyclic(2)
 1

/* 0 0 5 2 0 3 10 8 216 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x*y^t)^2,(x^y*t)^2,(x^y*t^y*(x^2))^3,(y^-2*y*t^y*(y^x))^-10,(x^-2*y^t^y*(t^-x))^8>;
\end{verbatim}
\texttt{f1,g1,k1:=CosetAction(G,sub<G|x,y>);}
\texttt{CompositionFactors(g1);}
\begin{verbatim}
G
 | Cyclic(2)
 *
 | Alternating(6)
 *
 | Cyclic(3)
 1

/* 0 0 5 0 0 6 4 0 264 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
\end{verbatim}
\( (x^{(y^t)*t^t-(x^2)}*6, (y^t*t-(y*x))^-4) >; \)
f1, g1, k1 := CosetAction(G, sub<G|x,y>);
CompositionFactors(g1);

\[
\begin{array}{c}
G \\
| Cyclic(2) \\
* \\
| A(1, 11) = L(2, 11) \\
* \\
| Cyclic(2) \\
1
\end{array}
\]

\(/* 0 0 5 0 0 5 5 0 342 */\)

\[
\begin{align*}
G^{\langle x, y, t \rangle} : = & \text{Group}^{\langle x, y, t \rangle} | x^{15}, y^{-2}, (x*y)^{-2}, t^{-2}, (t, y), (x-t)^{-5}, \\
&(x^{(y^t)*t^t-(x^2)}*5, (y^t*t^t-(y*x))^-5); \\
f1, g1, k1 := & \text{CosetAction}(G, sub<G|x,y>); \\
CompositionFactors(g1);
\end{align*}
\]

\[
\begin{array}{c}
G \\
| A(1, 19) = L(2, 19) \\
1
\end{array}
\]

\(/* 0 0 5 0 5 9 0 5 486 */\)

\[
\begin{align*}
G^{\langle x, y, t \rangle} : = & \text{Group}^{\langle x, y, t \rangle} | x^{15}, y^{-2}, (x*y)^{-2}, t^{-2}, (t, y), (x-t)^{-5}, \\
&(x^{(y^t)*t^t-(x^2)}*5, (x^{-2}+y^t*t^t-(t*x))^{-5}); \\
f1, g1, k1 := & \text{CosetAction}(G, sub<G|x,y>); \\
CompositionFactors(g1);
\end{align*}
\]

\[
\begin{array}{c}
G \\
| Alternating(5) \\
* \\
| Cyclic(3) \\
* \\
| Cyclic(3) \\
* \\
| Cyclic(3) \\
* \\
| Cyclic(3) \\
1
\end{array}
\]

\(/* 0 0 5 0 6 0 7 3 504 */\)

\[
\begin{align*}
G^{\langle x, y, t \rangle} : = & \text{Group}^{\langle x, y, t \rangle} | x^{15}, y^{-2}, (x*y)^{-2}, t^{-2}, (t, y), (x-t)^{-5}, \\
&(x^{(y^2)*t^t-(x^2)}*6, (y^t*t^t-(y*x))^-7, (x^2+y^t*t^t-(t*x))^-3); \\
f1, g1, k1 := & \text{CosetAction}(G, sub<G|x,y>); \\
CompositionFactors(g1);
\end{align*}
\]

\[
\begin{array}{c}
G \\
CompositionFactors(g1);
\end{array}
\]
G\langle x, y, t \rangle := \text{Group}\langle x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x^{-t})^5, \\
(x^{-t} y)^8, (x^{-t} y \ast t \ast (x^{-t}))^4, (y^{-t} \ast t \ast (y \ast x))^{-5} \rangle;
\text{f1, g1, k1 := CosetAction}(G, \langle x, y \rangle);
\text{CompositionFactors}(g1);
\text{CompositionFactors}(g1);

G
| Cyclic(2)
| *
| Alternating(7)
| 1

G\langle x, y, t \rangle := \text{Group}\langle x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x^{-t})^5, \\
(x^{-t} y)^8, (x^{-t} y \ast t \ast (x^{-t}))^4, (y^{-t} \ast t \ast (y \ast x))^{-5} \rangle;
\text{f1, g1, k1 := CosetAction}(G, \langle x, y \rangle);
\text{CompositionFactors}(g1);
\text{CompositionFactors}(g1);

G
| A(1, 11) = L(2, 11)
Cyclic(2)  
*  
Cyclic(5)  
1  
/* 0 0 5 0 4 0 0 9 684 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^4,(x^-2*y^t*(t^-x))^9>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 19) = L(2, 19)  
1  
/* 0 0 3 0 10 4 0 0 1320 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^3,  
(x^-y)^10,(x^-y*t^y*(x^-2))^4>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 11) = L(2, 11)  
*  
| Cyclic(2)  
|  
| Cyclic(3)  
1  
/* 0 0 5 0 6 6 0 3 1512 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^6,(x^-y*t^y*(x^-2))^6,(x^-2*y^t*(t^-x))^3>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| Alternating(7)  
*  
| Cyclic(3)  
1  
/* 0 0 5 0 4 0 0 9 684 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^4,(x^-2*y^t*(t^-x))^9>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 19) = L(2, 19)  
1  
/* 0 0 3 0 10 4 0 0 1320 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^3,  
(x^-y)^10,(x^-y*t^y*(x^-2))^4>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 11) = L(2, 11)  
*  
| Cyclic(2)  
|  
| Cyclic(3)  
1  
/* 0 0 5 0 6 6 0 3 1512 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^6,(x^-y*t^y*(x^-2))^6,(x^-2*y^t*(t^-x))^3>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| Alternating(7)  
*  
| Cyclic(3)  
1  
/* 0 0 5 0 4 0 0 9 684 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^4,(x^-2*y^t*(t^-x))^9>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 19) = L(2, 19)  
1  
/* 0 0 3 0 10 4 0 0 1320 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^3,  
(x^-y)^10,(x^-y*t^y*(x^-2))^4>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| A(1, 11) = L(2, 11)  
*  
| Cyclic(2)  
|  
| Cyclic(3)  
1  
/* 0 0 5 0 6 6 0 3 1512 */  
G<x,y,t>::=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,  
(x^-y)^6,(x^-y*t^y*(x^-2))^6,(x^-2*y^t*(t^-x))^3>;  
f1,g1,k1:=CosetAction(G,sub<G|x,y>);  
CompositionFactors(g1);  
   G  
| Cyclic(2)  
|  
| Alternating(7)  
*  
| Cyclic(3)  
1
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,(t,y),(x^t)^5,
(x^y^2*t^y)^4,(x^y^t*t^y)^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
  | Cyclic(2)
  *
  | A(1, 29) = L(2, 29)
  1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,(t,y),(x^t)^5,
(y^t*t^y)^4,(x^2*y^t*(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
  | A(1, 31) = L(2, 31)
  *
  | Cyclic(2)
  1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,(t,y),(x^t)^5,
(x^y^2*t^y)^4,(x^y^t*t^y)^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

G
  | Cyclic(2)
  *
  | Alternating(5)
  *
  | Cyclic(2)
  *
  | Cyclic(5)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  *
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);

G
| Cyclic(2)
| Alternating(6)
| Alternating(5)
1

G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^-t)^5,
(x^(y^2)*t^y)^6,(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);

G
| Cyclic(2)
| A(1, 61) = L(2, 61)
1
Bibliography


