

6-2014

## MONOID RINGS AND STRONGLY TWO-GENERATED IDEALS

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MONOID RINGS AND STRONGLY TWO-GENERATED IDEALS

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

Brittney Salt

June, 2014

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## ABSTRACT

This paper determines whether monoid rings with the two-generator property have the strong two-generator property. Dedekind domains have both the two-generator and strong two-generator properties. How common is this? Two cases are considered here: the zero-dimensional case and the one-dimensional case for monoid rings. Each case is looked at to determine if monoid rings that are not PIRs but are two-generated have the strong two-generator property. Full results are given in the zero-dimensional case, however only partial results have been found for the one-dimensional case.

## ACKNOWLEDGEMENTS

I would like give a special thanks to Dr. J. Paul Vicknair for all of his help and input in putting this paper together. I would also like to thank Dr. Charles Stanton for his guidance during this process. Your help was greatly appreciated.

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# Chapter 1

## Introduction

Dedekind Domains are important to commutative algebra, but came about in algebraic number theory. Commonly, Dedekind domains are defined as integral domains with the special property that every nonzero proper ideal factors into a product of prime ideals. Another common definition of a Dedekind domain is an integrally closed Noetherian domain where every nonzero prime ideal is a maximal ideal. However, other equivalent definitions of Dedekind domains are given as Theorems 37.1 and 37.8 in [Gil72]. Since Dedekind domains are Noetherian, their ideals are finitely generated. In fact, there is a bound on the number of generators an ideal can have. By Theorem 8.5.1 in [AW04], the maximum number of generators is two. By studying the proof, one sees that the method used is taking a nonzero element, say  $\alpha \in I$ , where  $I$  is an ideal, and finding the second generator,  $\beta$ , also in  $I$ . Thus  $I = \langle \alpha, \beta \rangle$ . However, this is the very definition of the strong two-generator property. If  $\forall \alpha \neq 0 \in I, \exists \beta \in I \ni I = \langle \alpha, \beta \rangle$ , then an ideal has the strong two-generator property. Now, if every ideal in a ring has this property, then the ring is said to have the strong two-generator property. Since this is true of every ideal in Dedekind domains, then Dedekind domains are said to have the strong two-generator property.

Much work has been done on domains with the strong two-generator property, but not much has been done on rings with zero-divisors with the strong two-generator property. We look at monoid rings to see which rings have the strong two-generator property. Monoid rings are semigroups with the identity property. We consider three sets:  $\mathbb{S}_1$ ,  $\mathbb{S}_{1.5}$ , and  $\mathbb{S}_2$ , where  $\mathbb{S}_1$  is the set of all monoid rings that have the one-generator

property, or are principal ideal rings (PIR),  $\mathbb{S}_{1.5}$  is the set of all monoid rings that are strongly two-generated, and  $\mathbb{S}_2$  is the set of all monoid rings that are two-generated. We know  $\mathbb{S}_1 \subseteq \mathbb{S}_{1.5} \subseteq \mathbb{S}_2$ . What we want to know is if there are any monoid rings that are two-generated and not PIR's, but are strongly two-generated. In symbols, are there any monoid rings that are in both  $\mathbb{S}_2 \setminus \mathbb{S}_1$  and  $\mathbb{S}_{1.5}$ .

It is known that rings with the two-generator property have Krull dimension at most 1. In order to look at monoid rings with the strong two-generator property, one needs to have a characterization of monoid rings with the two-generator property. These characterizations can be found in [OV92] as Theorem 4.1, and in [ORV92] as Theorems 2.6, 2.7, and 3.1 for the zero-dimensional and one-dimensional cases. They are also stated here as Theorem 3.1 in Chapter 3 and Theorem 4.9, 4.1, and 4.11 in Chapter 4, respectively, for convenience.

Chapter 2 gives definitions and preliminary theorems that are needed in order to understand the theorems and proofs used throughout. Chapter 3 gives the result that there are no monoid rings that are in both  $\mathbb{S}_2 \setminus \mathbb{S}_1$  and  $\mathbb{S}_{1.5}$  for the zero-dimensional case. This is stated in Theorem 3.9. Chapter 4 gives the result that for the monoid rings with the two-generator property as described in Theorem 2.7 in [ORV92], the only ones with the strong two-generator property are those that are also PIRs. This is summarized in Theorem 4.8. For the monoid rings described in Theorems 2.6 and 3.1 in [ORV92], only partial results are given, and they are given as Theorems 4.10 and 4.12. Lastly, Chapter 5 is a discussion of the remaining cases, what is known about them and the problems they portray, and other problems left to be looked at.

All rings will be assumed to be commutative with identity, all groups will be abelian, and all monoids are cancellative and abelian. Lastly,  $\mathbb{Q}$  denotes the set of all rational numbers,  $\mathbb{Z}$  denotes the set of all integers, and  $\mathbb{Z}_+$  denotes the set of all nonnegative integers.



## Chapter 2

# Preliminaries

A few definitions are presented to understand the terms. Also, a few preliminary theorems that are used throughout chapters 3 and 4 are stated here. The first definition is the main focus of this paper.

**Definition 2.1** (Monoid Ring). *Let  $S$  be a monoid and  $R$  be a commutative ring with identity. Then  $R[S]$  is a monoid ring. Elements of  $R[S]$  will be written as  $r_1x^{s_1} + \dots + r_nx^{s_n}$ , where  $r_1, \dots, r_n \in R$  and  $s_1, \dots, s_n \in S$ . If  $S$  is a group, then  $R[S]$  is a group ring.*

Monoid rings are similar to polynomial rings, where the coefficients come from the ring  $R$ , and the exponents come from the monoid  $S$ . Consider the group ring  $Z[Z/5Z]$  for example. An example of an element from this ring would be  $a = 3 + 2x - 5x^2 + 0x^3 + x^4$ . The coefficients, 3, 2, -5, 0, and 4 come from the set of integers,  $Z$ , and the exponents 0, 1, 2, 3, and 4 come from the group  $Z/5Z$ . Also, addition and multiplication of elements of  $R[S]$  are performed just as one would for polynomials. For example, take  $a, b \in Z[Z/2Z]$ , where  $a = 3 + x$  and  $b = 4 - 3x$ . Then  $a + b = (3 + x) + (4 - 3x) = 7 - 2x$ , and  $a \cdot b = (3 + x)(4 - 3x) = 12 - 9x + 4x - 3 = 9 - 5x$ . In the example of  $a \cdot b$ , the  $x^2$  term becomes  $x^0$  since the exponents come from the group  $Z/2Z = \{0, 1\}$ .

The next definition is a basic term used throughout the chapters as well.

**Definition 2.2** (Noetherian Ring). *A ring  $R$  where every ascending chain of ideals terminates is called a Noetherian ring.*

Another equivalent definition seen often states that a ring  $R$  where every ideal

is finitely generated is a Noetherian ring. A few well-known examples of Noetherian rings include the monoid rings  $Z[X]$  and  $F[X, Y]$ , where  $F$  is a field.

The next two definitions are related to Noetherian rings in a unique way.

**Definition 2.3** (Krull Dimension). *The Krull dimension of a commutative ring  $R$  is the maximum length of all chains of prime ideals in  $R$ .*

For example, both  $Z$  and  $F[X]$ , where  $F$  is a field, have Krull dimension 1, meaning the maximum length of all chains of prime ideals in these rings is 1. For  $Z$ , every prime number is a generator of a prime ideal. Thus, there are infinitely many prime ideals, however the longest chain is one. So, look at  $\langle 2 \rangle$ . It is a prime ideal.  $\langle 0 \rangle$ , which is also a prime ideal, sits below  $\langle 2 \rangle$ . Then, the length of this chain is 1. The same is true of every nonzero prime ideal. The chain includes the nonzero prime ideal and  $\langle 0 \rangle$ . So  $Z$  has Krull dimension 1. The same is true of  $K[X]$ .

An example of a ring with Krull dimension 0 is  $Z/6Z$ . This ring has only two prime ideals,  $\langle 2 \rangle$  and  $\langle 3 \rangle$ . They have no other prime ideals in their chain, so they each have length 0. In this case, the zero ideal is not prime, since  $2 \cdot 3 = 0$ .

An example of a ring with Krull dimension 2 is  $K[X, Y]$ , where  $K$  is a field. An example of a chain of prime ideals of maximum length is  $\langle 0 \rangle \subset \langle X \rangle \subset \langle X, Y \rangle$ . An example of a ring with Krull dimension 3 is  $K[X, Y, Z]$  where  $K$  is a field. An example of a chain of prime ideals of maximum length is  $\langle 0 \rangle \subset \langle X \rangle \subset \langle X, Y \rangle \subset \langle X, Y, Z \rangle$ . It is now easy to see how to find examples of rings with arbitrarily large Krull dimension.

Another similar ring is called an Artinian ring. These rings deal with descending chains of ideals, instead of ascending chains of ideals.

**Definition 2.4** (Artinian Ring). *A ring  $R$  where every descending chain of ideals terminates is called an Artinian ring.*

$Z/6Z$  is an example of an Artinian ring as well as a Noetherian ring. Its two prime ideals,  $\langle 2 \rangle$  and  $\langle 3 \rangle$ , are also maximal ideals. A special property of Artinian rings is that they are Noetherian rings with Krull dimension 0, as seen in this ring.

The next two definitions are special properties of some of the rings looked at in Chapter 4.

**Definition 2.5** (Nilpotent). *An element  $r$  of a ring  $R$  is called nilpotent if there exists some positive integer  $n$  such that  $r^n = 0$ .*

**Definition 2.6** (Nilradical). *The nilradical  $N$  of a commutative ring  $R$  is the ideal that consists of all nilpotent elements of  $R$ .*

Next, we define principal ideal rings.

**Definition 2.7** (PIR). *A ring  $R$  is a principal ideal ring, (PIR), if every ideal is generated by a single element from the ideal.*

An example of a PIR is the ring of integers,  $Z$ . Every nonzero ideal can be generated by the smallest positive integer in the ideal. However, the polynomial ring  $Z[X]$  is not a PIR because the ideal  $\langle 2, X \rangle$  cannot be generated by just a single element. Note that the polynomial ring  $Z[X]$  is the monoid ring  $Z[Z_+]$ . Another example that is a PIR is  $K[X]$ , where  $K$  is a field. The ideals in this polynomial ring are generated by the elements of least degree in that ideal. However,  $K[X, Y]$ , where  $K$  is a field, is not a PIR, since the ideal  $\langle X, Y \rangle$  cannot be generated by just a single element.

The main reason we are looking at monoid rings and the strong two-generator property is because of Dedekind domains. A few definitions were given in Chapter 1, but a more formal definition is now given.

**Definition 2.8** (Dedekind Domain). *An integrally closed integral domain  $D$  that is Noetherian, where each nonzero prime ideal of  $D$  is a maximal ideal, is called a Dedekind domain.*

Dedekind domains first came about in algebraic number theory, as mentioned before. A Dedekind domain is a ring of integers from an algebraic number field. An algebraic number field is a finite field extension of  $Q$ . A specific example of a Dedekind domain can be found in [AW04] as Theorem 5.4.2.  $O_K$  is a Dedekind domain where  $K = Q(\sqrt{m})$ , for  $m$  a unique squarefree integer, and  $K$  is a quadratic extension of  $Q$ . Elements of  $O_K$  satisfy monic polynomials with coefficients from  $Z$ . Then  $O_K$  is the ring of integers of  $K$ , and  $O_K = Z + Z\sqrt{m}$ , if  $m \not\equiv 1 \pmod{4}$ , and  $O_K = Z + Z(\frac{1+\sqrt{m}}{2})$  if  $m \equiv 1 \pmod{4}$ .

Some properties of Dedekind domains are included below.

**Definition 2.9** (Two-Generated). *An ideal  $I$  that is generated by two elements is called a two-generated ideal. A ring  $R$  in which every ideal of  $R$  is two-generated is said to have the two-generator property.*

All PIRs are two-generated rings. However, the converse is not true. An example of a two-generated ring that is not a PIR is  $K[X^2, X^n]$ , where  $K$  is a field, and  $n$  is a positive odd integer. This is not a PIR because it contains the ideal  $\langle X^2, X^n \rangle$ , which cannot be generated by a single element. This ring is an example of a monoid ring  $R[S]$  with the two-generator property where  $S = \langle 2, n \rangle$ .

**Definition 2.10** (Strongly Two-Generated). *Let  $\alpha \neq 0$ .  $\alpha$  is a strong two-generator if  $\forall \alpha$  where  $\alpha \in I, \exists \beta \in I \ni I = \langle \alpha, \beta \rangle$ . An ideal  $I$  is said to be strongly two-generated if every nonzero element of  $I$  is a strong two-generator. A ring  $R$  in which every ideal of  $R$  is strongly two-generated is said to have the strong two-generator property.*

For more information on rings with the strong two-generator property, see [LM88].

Dedekind domains have the two-generator and strong two-generator properties, which is why we chose to study the strong two-generator property in greater depth.

The next definition is used in a few proofs in chapters 3 and 4.

**Definition 2.11** (Augmentation Map and Ideal). *If  $R$  is a commutative ring and  $S$  is a monoid or semigroup, then the augmentation map is a ring homomorphism  $\phi : R[S] \rightarrow R$  defined by mapping  $\phi(\sum r_i X^{s_i}) = \sum r_i$ . The augmentation ideal  $I$  of a monoid ring  $R[G]$  is the kernel of the augmentation map  $\phi$ .*

Thus, the augmentation ideal is everything that gets mapped to zero under the augmentation map. Gilmer takes a look at  $R[X, S]$  and points out that the augmentation ideal is generated by  $\{rX^a - rX^b \mid r \in R \text{ and } a, b \in S\}$ . See page 75 of [Gil84].

Now, something worth mentioning is when  $S$  is a cancellative monoid, there does exist a smallest group, up to an isomorphism, that contains it, which is called the quotient group, and is denoted by  $G(S)$ . It is constructed in the same manner as one constructs  $Z$  from  $Z_+$ . See page 6 of [Gil84] for further details.

The next lemma and theorems are needed for most of the proofs in Chapters 3 and 4. Their proofs are provided here, in order to shorten the lengths of the proofs in chapters 3 and 4.

**Lemma 2.12.** *If  $f : R \rightarrow S$  is a ring homomorphism and  $J$  is an ideal of  $S$ , then  $f^{-1}(J)$  is an ideal of  $R$ .*

*Proof.* First,  $f^{-1}(J) \subseteq R$ . Now let  $r_1, r_2 \in f^{-1}(J) \Rightarrow f(r_1) \in J$  and  $f(r_2) \in J$  by definition. Then,  $f(r_1) + f(r_2) \in J$  since  $J$  is an ideal. So,  $f(r_1 + r_2) \in J$  by definition of a ring homomorphism. This says  $r_1 + r_2 \in f^{-1}(J)$ . Next, let  $r \in R, r_1 \in f^{-1}(J)$ . By definition,  $f(r_1) \in J$ . Look at  $f(r_1 r) = f(r_1)f(r) \in J$  by definition of a ring homomorphism. This says that  $r_1 r \in f^{-1}(J)$ . Thus,  $f^{-1}(J)$  is an ideal in  $R$ .  $\square$

The next two theorems show that an onto ring homomorphism preserves certain properties. One shows that the one-generator property is preserved; in other words, an onto ring homomorphism maps a PIR to a PIR. The other theorem shows that the two-generator property is preserved. They are proven in similar fashions.

**Theorem 2.13.** *Let  $f : R \rightarrow S$  be an onto ring homomorphism. If  $R$  is a PIR, then  $S$  is a PIR.*

*Proof.* To show  $S$  is a PIR, start with  $J \subseteq S$ , where  $J$  is an ideal. Then  $f^{-1}(J)$  is also an ideal in  $R$  by Lemma 2.12. Since  $R$  is a PIR, then  $f^{-1}(J)$  is principal. So,  $\exists r \in R \ni \langle r \rangle = f^{-1}(J)$ . Now, we will show  $\langle f(r) \rangle = J$ . We know  $f(r) \in J \Rightarrow \langle f(r) \rangle \subseteq J$ . Take  $s \in J$ . Then  $\exists r_1 \in R \ni f(r_1) = s$  by definition of onto. This implies  $r_1 \in f^{-1}(J) \Rightarrow r_1 \in \langle r \rangle \Rightarrow r_1 = ar$  for some  $a \in R$ . Now  $s = f(r_1) = f(ar) = f(a)f(r) \Rightarrow s \in \langle f(r) \rangle \Rightarrow \langle f(r) \rangle = J \Rightarrow S$  is a PIR.  $\square$

**Theorem 2.14.** *Let  $f : R \rightarrow S$  be an onto ring homomorphism. If  $R$  has the two-generator property, then  $S$  has the two-generator property.*

*Proof.* Let  $J \subseteq S$ , where  $J$  is an ideal.  $f^{-1}(J) \subseteq R$  is an ideal in  $R$ . Since  $R$  has the two-generator property, then  $\exists c, d \in R \ni f^{-1}(J) = \langle c, d \rangle$ . If  $j \in J$ , then  $\exists r_1 \in R \ni f(r_1) = j$  by definition of onto. This implies  $r_1 \in f^{-1}(J) \Rightarrow r_1 = pc + qd$  for some  $p, q \in R$ . Now  $j = f(r_1) = f(pc + qd) = f(pc) + f(qd) = f(p)f(c) + f(q)f(d) \Rightarrow j \in \langle f(c, d) \rangle \Rightarrow \langle f(c, d) \rangle = J \Rightarrow S$  has the two-generator property.  $\square$

The next theorem shows that an onto ring homomorphism also preserves the strong two-generator property, as seen below.

**Theorem 2.15.** *Let  $f : R \rightarrow S$  be an onto ring homomorphism. If  $R$  has the strong two-generator property, then  $S$  has the strong two-generator property.*

*Proof.* Let  $J \subseteq S$ , where  $J$  is an ideal. Since  $f^{-1}(J)$  is an ideal in  $R$  that has the strong two-generator property, then  $\forall a \in f^{-1}(J), a \neq 0, \exists b \in R \ni f^{-1}(J) = \langle a, b \rangle$ . Take an arbitrary  $s \neq 0 \in J$ . Then  $\exists a \neq 0 \in f^{-1}(J) \ni f(a) = s$ . Then  $\exists b \in R \ni f^{-1}(J) = \langle a, b \rangle$  since  $R$  has the strong two-generator property. Since  $f$  is onto,  $J = f(f^{-1}(J)) = \langle f(a), f(b) \rangle = \langle s, f(b) \rangle$ , which says that for an arbitrary nonzero  $s \in J$ ,  $\exists t \in S$  (where  $f(b) = t$ )  $\ni J = \langle s, t \rangle$ . Hence,  $S$  has the strong two-generator property.  $\square$

A special property of rings with the strong two-generator property is given in the next theorem. This property is used to show rings are not strongly two-generated if the ring  $R/\langle a \rangle, \forall a \in R, a \neq 0$ , is not a PIR. This result is important because it is sometimes easier to show that  $\exists a \neq 0 \in R \ni R/\langle a \rangle$  is not a PIR, than to show directly that it is not strongly two-generated.

**Theorem 2.16.**  *$R$  is strongly two-generated if and only if  $R/\langle a \rangle$  is a PIR  $\forall a \in R, a \neq 0$ .*

*Proof.* ( $\Rightarrow$ ) Let  $a \in R, a \neq 0$ . We will show  $R/\langle a \rangle$  is a PIR. Let  $f : R \rightarrow R/\langle a \rangle$  be the natural homomorphism (which is onto). Choose  $J \subseteq R/\langle a \rangle$ , where  $J$  is an ideal.  $f^{-1}(J) \subseteq R$  is also an ideal.  $R$  is strongly two-generated, so  $\exists b \in R$  such that  $f^{-1}(J) = \langle a, b \rangle$ . Claim:  $J = \langle f(b) \rangle$ . Clearly,  $\langle f(b) \rangle \subseteq J$ . Take  $f(c) \in J$ . Then  $c \in f^{-1}(J)$  implies  $c = ap + bq$  for some  $p, q \in R$ . Apply  $f$  to obtain  $f(c) = f(ap + bq) = f(ap) + f(bq) = f(a)f(p) + f(b)f(q)$ . In this mapping,  $a$  maps to 0, so this becomes  $f(c) = f(b)f(q)$  so that  $f(c) \in \langle f(b) \rangle$ . Thus,  $J = \langle f(b) \rangle$ , implying that  $R/\langle a \rangle$  is a PIR.

( $\Leftarrow$ ) Let  $I$  be a nonzero proper ideal of  $R$  and let  $a \neq 0 \in I$ . We will find  $b \in R$  such that  $I = \langle a, b \rangle$ . Consider  $f(I)$ .  $f(I)$  is an ideal in  $R/\langle a \rangle$  since  $f$  is onto. So  $\exists \bar{b} \in R/\langle a \rangle$  such that  $f(I) = \langle \bar{b} \rangle$ . For this part of the proof, it makes more sense to use the bar notation as opposed to the  $f$ -notation. Claim:  $I = \langle a, b \rangle$ . Clearly,  $a \in I$  (given) and  $b$  maps to  $\bar{b} \in f(I)$  so  $b \in I$ . Now take  $c \neq 0 \in I$ . Consider  $f(c)$  or  $\bar{c}$ .  $\bar{c} \in f(I) = \langle \bar{b} \rangle$ . So  $\bar{c} = \bar{b}\bar{t}$  for some  $\bar{t} \in R/\langle a \rangle$ . So  $\overline{c - bt} = \bar{0} \Rightarrow c - bt \in \langle a \rangle \Rightarrow c - bt = as$  for some  $s \in R$ . Then  $c = bt + as \Rightarrow c \in \langle a, b \rangle$ . Thus,  $I = \langle a, b \rangle$ . Therefore,  $R$  has the strong two-generator property.  $\square$

Now, we take a look at what happens in the zero-dimensional case and the one-dimensional case. In the zero-dimensional case, we focus on Theorem 4.1 from [OV92]. In this Theorem,  $R = R_1 \oplus \dots \oplus R_s$ . What we did to show that these cases that are two-generated are not strongly two-generated, is we mapped  $R$  to a homomorphic image

of  $R$  that is just one piece of  $R$ , say  $R_i$ , from the direct sum. Then, by Theorem 2.15, if  $R_i[S]$  is not strongly two-generated, then  $R[S]$  cannot be strongly two-generated. To make things simple,  $R$  is used in place of  $R_i$ .

For the one-dimensional case, we first focus on Theorem 2.7 from [ORV92]. Again,  $R$  is written as a product of  $R_i$ 's, and again, we look at a homomorphic image of  $R$  that focuses on a single piece of  $R$ . It is also referred to as just  $R$ , and not  $R_i$ . The same idea used for the zero-dimensional case is used here, where if  $R_i[G]$  is not strongly two-generated, then  $R[G]$  cannot be strongly two-generated by Theorem 2.15. The same method is used on the cancellative monoid  $S$ . In [ORV92], Theorem 2.7 gives  $S$  as a direct sum. This time, we map  $R[S]$  by an onto homomorphism to  $R[S_i]$ , where  $S_i$  is one of the summands of  $S$ , and we look at  $R[S_i]$  to help determine if  $R[S]$  has the strong two-generator property or not. Finally, we also consider cases of Theorems 2.6 and 3.1 of [ORV92]. In addition to the techniques mentioned above, we also use Theorem 2.16 from this chapter.

## Chapter 3

# Zero-Dimensional Case

As mentioned, the focus of this chapter will be on Theorem 4.1 from [OV92], which is listed below as Theorem 3.1 for convenience.

**Theorem 3.1.** *Let  $R$  be an artinian ring and  $G$  a finite abelian group.  $R[G]$  has the two-generator property if and only if  $R = R_1 \oplus \dots \oplus R_s$  where for each  $i$ ,  $(R_i, M_i)$  is a local artinian ring which has the two-generator property subject to:*

(i) *Assume  $(R_i, M_i)$  is a principal ideal ring (maybe a field),  $p$  a prime integer which divides the order of  $G$  and  $p \in M_i$ .*

(a) *If  $p$  is odd, then  $G_p$  is cyclic; furthermore, if  $M_i^2 \neq 0$ , then  $G_p \cong Z/pZ$  and  $pR_i = M_i$ .*

(b) *If  $p = 2$ , then  $G_p \cong Z/p^j Z \oplus Z/2^k Z$  where*

(1)  $j \leq 1$  if  $M_i = 0$

(2)  $j = 0$  and  $k = 1$  if  $M_i^2 \neq 0$

(3)  $j = 0$  otherwise.

(ii) *The order of  $G$  is a unit in any  $R_i$  which is not a principal ideal ring.*

Now, each case must be looked at to determine whether these monoid rings are strongly two-generated. First, a result that is heavily relied on in the following theorems is Corollary 5.3 from [Nag75]. This corollary, when translated to ring theory, states that an ideal  $I$  that has three generators, say  $I = \langle a, b, c \rangle$ , in a two-generated local ring must be two-generated, and the two generators can be chosen from  $a, b$ , and  $c$ .



**Theorem 3.2.** *Let  $R$  be a local Artinian PIR where the maximal ideal of  $R$  contains  $p$ , where  $p$  is an odd prime integer. Assume the square of the maximal ideal of  $R$  is zero. Let  $G = Z/p^iZ$ . Then  $R[G]$  is a local ring with the two-generator property but not the strong two-generator property.*

*Proof.* We know  $R[G]$  is local by Corollary 19.2 in [Gil84], and has the two-generator property by Theorem 3.1. Let  $M \subseteq R[G]$  be the maximal ideal of  $R[G]$ . Then  $M = \langle r, 1 - x^g \rangle$ , where  $rR$  is the maximal ideal of  $R$ , and  $\langle 1 - x^g \rangle$  is the maximal ideal of  $G$ . Then  $M^2 = \langle r(1 - x^g), (1 - x^g)^2 \rangle$ . Take  $r - rx^g \in M^2 \subseteq M$ . If  $M$  is strongly two-generated, then  $\exists t \in R[G] \ni M = \langle r - rx^g, t \rangle$ . By Corollary 5.3 of [Nag75], one of the generators must be either  $r$  or  $1 - x^g$ , so there are two cases.

Case 1:  $M = \langle r, r - rx^g \rangle$ . We will show that this case fails. Since  $1 - x^g \in M, \exists a_0 + a_1x + \dots + a_nx^n, b_0 + b_1x + \dots + b_mx^m \in R[G]$  with

$$1 - x^g = r(a_0 + a_1x + \dots + a_nx^n) + (r - rx^g)(b_0 + b_1x + \dots + b_mx^m).$$

Thus  $1 = rc$ , where  $c$  is the sum of the constant terms when  $r$  is factored out. This implies  $r$  is a unit in  $R$ , which is a contradiction. Thus,  $M \neq \langle r, r - rx^g \rangle$ . Now, we will show that Case 2 fails also.

Case 2:  $M = \langle 1 - x^g, r - rx^g \rangle$ . Since  $r \in M, \exists f, h \in R[G]$  with

$$\begin{aligned} r &= (1 - x^g)f + (r - rx^g)h \\ &= (1 - x^g)f + (r(1 - x^g))h \\ &= (1 - x^g)(f + rh). \end{aligned}$$

Now apply the Augmentation Map and observe that  $r = 0(a)$ , where  $a \in R$ . Hence, we get a contradiction. Thus,  $M \neq \langle 1 - x^g, r - rx^g \rangle$ . Therefore,  $M$  is not strongly two-generated, meaning  $R[G]$  is not strongly two-generated.  $\square$

Since  $R[G]$  is local, it has a unique maximal ideal  $M$ . Then, since we know  $M$  is two-generated and we have the two generators, we needed to determine if  $M$  was strongly two-generated. We squared  $M$ , and took an element in  $M^2$  not in  $M$  to try as a generator. If  $M$  was strongly two-generated, then the generators of  $M$  would be one of the original generators of  $M$  together with the new element chosen from  $M^2$ . Both of the original generators of  $M$  were tried as generators with the new element from  $M^2$ . Since

neither one worked, then  $M$  is not strongly two-generated. Now, this same method will be used to yield the same result in Theorem 3.3.

Theorem 3.2 takes care of the first part of part (i)(a) of Theorem 3.1. Next, we take a look at the second part of part (i)(a) of Theorem 3.1: the case where the square of the maximal ideal is nonzero.

**Theorem 3.3.** *Let  $p$  be an odd prime. Assume  $R$  is a local Artinian PIR with maximal ideal  $pR$  where  $p^2 \neq 0$ . Let  $G = Z/pZ$ . Then  $R[G]$  is a local ring with the two-generator property but not the strong two-generator property.*

*Proof.* We know  $R[G]$  is local by Corollary 19.2 in [Gil84], and has the two-generator property by Theorem 3.1. Let  $M \subseteq R[G]$  be the maximal ideal of  $R[G]$ . Then  $M = \langle p, 1 - x \rangle$ , where  $\langle 1 - x \rangle$  is the maximal ideal of  $G$ . Take  $p - px \in M^2 \subseteq M$ . If  $M$  is strongly two-generated, then  $\exists t \in R[G] \ni M = \langle p - px, t \rangle$ . By Corollary 5.3 of [Nag75], one of the generators must be either  $p$  or  $1 - x$ , so there are two cases.

Case 1:  $M = \langle p, p - px \rangle$ . We will show that this cannot be. Since  $1 - x \in M, \exists a_0 + a_1x + \dots + a_nx^n, b_0 + b_1x + \dots + b_mx^m \in R[G]$  with

$$1 - x = p(a_0 + a_1x + \dots + a_nx^n) + (p - px)(b_0 + b_1x + \dots + b_mx^m).$$

Thus,  $1 = pc$ , where  $c$  is the sum of all the constant terms when  $p$  is factored out. This implies  $p$  is a unit in  $R$ , which is a contradiction. Thus,  $M \neq \langle p, p - px \rangle$ . Now, we will show that Case 2 fails also.

Case 2:  $M = \langle 1 - x, p - px \rangle$ . Since  $p \in M, \exists f, g \in R[G]$  with

$$\begin{aligned} p &= (1 - x)f + (p - px)g \\ &= (1 - x)f + (p(1 - x))g \\ &= (1 - x)(f + pg). \end{aligned}$$

Now, apply the Augmentation Map and observe that  $p = 0(a)$ , where  $a \in R$ . Hence, we get a contradiction. Thus  $M \neq \langle 1 - x, p - px \rangle$ . Therefore,  $M$  is not strongly two-generated, meaning  $R[G]$  is not strongly two-generated.  $\square$

Now we take a look at the case where  $R$  is a field. This is part (i)(b)(1) of Theorem 3.1.

**Theorem 3.4.** *Let  $F$  be a field of characteristic 2 and  $G = Z/2Z \oplus Z/2^kZ$ . Then  $F[G]$  is a local group ring with the two-generator property but not the strong two-generator property.*

*Proof.* First, look at  $F[Z/2Z][Z/2^kZ]$ , where  $F$  is local. Then, by Corollary 19.2 in [Gil84],  $F[Z/2Z]$  is also local. Since  $F[Z/2Z]$  is local, then again by Corollary 19.2,  $F[Z/2Z][Z/2^kZ]$  is local [Gil84]. Well,  $F[Z/2Z][Z/2^kZ] = F[Z/2Z \oplus Z/2^kZ] = F[G]$ . Hence,  $F[G]$  is local. Also,  $F[G]$  has the two-generator property by Theorem 3.1. Now, we know the maximal ideal of  $F[G]$  is  $M = \langle 1 - x^{(1,0)}, 1 - x^{(0,1)} \rangle$ . We will show  $M$  cannot be strongly two-generated, therefore showing  $F[Z/2Z \oplus Z/2^kZ]$  is not strongly two-generated. Now consider  $M^2 = \langle 1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)}, 1 + x^{(0,2)} \rangle$ . Take  $1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)} \in M^2 \subseteq M$ . By Corollary 5.3 of [Nag75],  $1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)}$  is one of the generators of  $M$  if  $M$  is strongly two-generated. Then the other generator must be either  $1 - x^{(0,1)}$  or  $1 - x^{(1,0)}$ . So we consider both cases.

Case 1:  $M = \langle 1 - x^{(0,1)}, 1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)} \rangle$ . Since  $1 - x^{(1,0)} \in M$ ,

$$1 - x^{(1,0)} = (1 - x^{(0,1)})a + (1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)})b, \quad (3.1)$$

with  $a, b \in F[Z/2Z \oplus Z/2^kZ]$ , where  $a = a_{00} + a_{10}x^{(1,0)} + a_{11}x^{(1,1)} + a_{12}x^{(1,2)} + \dots + a_{12^{k-1}}x^{(1,2^{k-1})} + a_{01}x^{(0,1)} + a_{02}x^{(0,2)} + \dots + a_{02^{k-1}}x^{(0,2^{k-1})}$  and  $b = b_{00} + b_{10}x^{(1,0)} + b_{11}x^{(1,1)} + b_{12}x^{(1,2)} + \dots + b_{12^{k-1}}x^{(1,2^{k-1})} + b_{01}x^{(0,1)} + b_{02}x^{(0,2)} + \dots + b_{02^{k-1}}x^{(0,2^{k-1})}$ .

Multiplying the right hand side out in equation (3.1) above yields:

$$\begin{aligned} & b_{00} + b_{10}x^{(1,0)} + b_{01}x^{(0,1)} + b_{11}x^{(1,1)} + b_{02}x^{(0,2)} + b_{12}x^{(1,2)} + \dots + \\ & \quad b_{02^{k-1}}x^{(0,2^{k-1})} + b_{12^{k-1}}x^{(1,2^{k-1})} + \\ & b_{10} + b_{00}x^{(1,0)} + b_{11}x^{(0,1)} + b_{01}x^{(1,1)} + b_{12}x^{(0,2)} + b_{02}x^{(1,2)} + \dots + \\ & \quad b_{12^{k-1}}x^{(0,2^{k-1})} + b_{02^{k-1}}x^{(1,2^{k-1})} + \\ & b_{02^{k-1}} + b_{12^{k-1}}x^{(1,0)} + b_{00}x^{(0,1)} + b_{10}x^{(1,1)} + b_{01}x^{(0,2)} + b_{11}x^{(1,2)} + \dots + \\ & \quad b_{02^{k-2}}x^{(0,2^{k-1})} + b_{12^{k-2}}x^{(1,2^{k-1})} + \\ & b_{12^{k-1}} + b_{02^{k-1}}x^{(1,0)} + b_{10}x^{(0,1)} + b_{00}x^{(1,1)} + b_{11}x^{(0,2)} + b_{01}x^{(1,2)} + \dots + \\ & \quad b_{12^{k-2}}x^{(0,2^{k-1})} + b_{02^{k-2}}x^{(1,2^{k-1})} + \end{aligned}$$

$$\begin{aligned}
& a_{00} + a_{10}x^{(1,0)} + a_{01}x^{(0,1)} + a_{11}x^{(1,1)} + a_{02}x^{(0,2)} + a_{12}x^{(1,2)} + \dots + \\
& \quad a_{02^k-1}x^{(0,2^k-1)} + a_{12^k-1}x^{(1,2^k-1)} + \\
& a_{02^k-1} + a_{12^k-1}x^{(1,0)} + a_{00}x^{(0,1)} + a_{10}x^{(1,1)} + a_{01}x^{(0,2)} + a_{11}x^{(1,2)} + \dots + \\
& \quad a_{02^k-2}x^{(0,2^k-1)} + a_{12^k-2}x^{(1,2^k-1)}
\end{aligned}$$

Now the constant terms must equal 1 and the coefficients in front of  $x^{(1,0)}$  must also equal 1, where the rest of the coefficients must be zero. We get the following  $2^{k+1}$  equations:

$$\begin{aligned}
b_{00} + b_{10} + b_{02^k-1} + b_{12^k-1} + a_{00} + a_{02^k-1} &= 1, \\
b_{10} + b_{00} + b_{12^k-1} + b_{02^k-1} + a_{10} + a_{12^k-1} &= 1, \\
b_{01} + b_{11} + b_{00} + b_{10} + a_{01} + a_{00} &= 0, \\
b_{11} + b_{01} + b_{10} + b_{00} + a_{11} + a_{10} &= 0, \\
&\vdots \\
b_{02^k-1} + b_{12^k-1} + b_{02^k-2} + b_{12^k-2} + a_{02^k-1} + a_{02^k-2} &= 0, \\
b_{12^k-1} + b_{02^k-1} + b_{12^k-2} + b_{02^k-2} + a_{12^k-1} + a_{02^k-2} &= 0.
\end{aligned}$$

(Note: There will always be  $2^{k+1}$  equations depending on what  $k$  is. This is true because there are  $2^k$  elements in  $Z/2^kZ$ . Since the first coordinate in the pair  $(a, b)$ , or  $a$ , comes from  $Z/2Z$ , there are only two options for  $a$ : 0 or 1. Thus, there is an equation for  $x^{(0, c_i)}$  for every  $c_i \in Z/2^kZ$ . There are exactly  $2^k$  elements in there, now add  $x^{(1, c_i)}$ , which doubles the number of equations from  $2^k$  to  $2 \cdot 2^k = 2^{k+1}$ .)

Now multiply all the odd equations by 1 and the even equations by 0, then add to obtain:

$$\begin{aligned}
2b_{00} + 2b_{10} + 2b_{02^k-1} + 2b_{12^k-1} + 2b_{01} + 2b_{11} + \dots + 2a_{00} + 2a_{01} + \dots + 2a_{02^k-1} \\
= 1 + 0 + 0 + \dots + 0
\end{aligned}$$

which implies  $0 = 1$ , a contradiction. So,  $M \neq \langle 1 - x^{(0,1)}, 1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)} \rangle$ .

Case 2:  $M = \langle 1 - x^{(1,0)}, 1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)} \rangle$ . We will show that this case fails also.

Since  $1 - x^{(0,1)} \in M$ , again we get our equation

$$1 - x^{(0,1)} = (1 - x^{(1,0)})a + (1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)})b,$$

where  $a, b$  are defined as above. This time, when we get the  $2^{k+1}$  equations, the pattern will have to be adjusted to get our contradiction. Before, the order of the exponents went

as follows:

$(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3)$ , etc. For this case, we arrange the exponents:  $(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (0, 3), (1, 2), (1, 3)$ , etc. The new order yields the following  $2^{k+1}$  equations:

$$\begin{aligned}
b_{00} + b_{10} + b_{02^{k-1}} + b_{12^{k-1}} + a_{00} + a_{10} &= 1, \\
b_{01} + b_{11} + b_{00} + b_{10} + a_{01} + a_{11} &= 1, \\
b_{10} + b_{00} + b_{12^{k-1}} + b_{02^{k-1}} + a_{10} + a_{00} &= 0, \\
b_{11} + b_{01} + b_{10} + b_{00} + a_{11} + a_{01} &= 0, \\
&\vdots \\
b_{02^{k-2}} + b_{12^{k-2}} + b_{02^{k-3}} + b_{12^{k-3}} + a_{02^{k-2}} + a_{12^{k-2}} &= 0, \\
b_{02^{k-1}} + b_{12^{k-1}} + b_{02^{k-2}} + b_{12^{k-2}} + a_{02^{k-1}} + a_{12^{k-1}} &= 0, \\
b_{12^{k-2}} + b_{02^{k-2}} + b_{12^{k-3}} + b_{02^{k-3}} + a_{12^{k-2}} + a_{02^{k-2}} &= 0, \\
b_{12^{k-1}} + b_{02^{k-1}} + b_{12^{k-2}} + b_{02^{k-2}} + a_{12^{k-1}} + a_{02^{k-1}} &= 0.
\end{aligned}$$

Now multiply all the odd equations by 1 and the even equations by 0, then add to obtain:

$$\begin{aligned}
2b_{00} + 2b_{10} + 2b_{02^{k-1}} + 2b_{12^{k-1}} + \dots + 2b_{02^{k-2}} + 2b_{12^{k-2}} + 2b_{02^{k-3}} + 2b_{12^{k-3}} + \\
2a_{00} + 2a_{10} + \dots + 2a_{02^{k-2}} + 2a_{12^{k-2}} \\
= 1 + 0 + 0 + \dots + 0,
\end{aligned}$$

which implies  $0 = 1$ , a contradiction. Thus,  $M \neq \langle 1 - x^{(1,0)}, 1 - x^{(1,0)} - x^{(0,1)} + x^{(1,1)} \rangle$ . So,  $M$  is not strongly two-generated, therefore  $F[G]$  is not strongly two-generated.  $\square$

This theorem is proven slightly differently from the first two. Again, we square the maximal ideal to find a generator and try each of the generators of  $M$  along with the previously chosen one. Again, we look at both cases. However, whereas before for one case we could look at the augmentation map, this time the augmentation map will not help. For both cases, we look at what the coefficients must be in the given equations. This one is a little more involved, but we eventually come to the same conclusion as before: that  $R[G]$  is not strongly two-generated.

For the next theorem, we are now proving the case of part (i)(b)(2) of Theorem 3.1.

**Theorem 3.5.** *Let  $R$  be a local Artinian PIR where  $2$  is in the maximal ideal of  $R$  and the square of the maximal ideal of  $R$  is not zero. Let  $G = Z/2Z$ . Then  $R[G]$  is a local ring with the two-generator property but not the strong two-generator property.*

*Proof.* Again, by Corollary 19.2,  $R[G]$  is local [Gil84]. It also has the two-generator property by Theorem 3.1. Let  $M \subseteq R[G]$  be the maximal ideal of  $R[G]$ . Then  $M = \langle r, 1 - x \rangle$ , where  $rR$  is the maximal ideal out of  $R$ , and  $\langle 1 - x \rangle$  is the maximal ideal out of  $G$ . Then  $M^2 = \langle r^2, r(1 - x) \rangle$ . Take  $r(1 - x) \in M^2 \subseteq M$ . If  $M$  is strongly two-generated, then  $\exists t \in R[G] \ni M = \langle r(1 - x), t \rangle$ . By Corollary 5.3 of [Nag75], one of the generators must be either  $r$  or  $1 - x$ , so there are two cases.

Case 1:  $M = \langle r, r(1 - x) \rangle$ . We will show that this case fails. Since  $1 - x \in M, \exists a_0 + a_1x, b_0 + b_1x \in R[G]$  with

$$1 - x = r(1 - x)(a_0 + a_1x) + r(b_0 + b_1x).$$

Thus,  $1 = r(a_0 - a_1 + b_0)$ . This implies  $r$  is a unit in  $R$ , which is a contradiction. Thus,  $M \neq \langle r, r(1 - x) \rangle$ . Now we will show that Case 2 fails also.

Case 2:  $M = \langle 1 - x, r(1 - x) \rangle$ . Since  $r \in M, \exists f, g \in R[G]$  with

$$\begin{aligned} r &= (1 - x)f + (r(1 - x))g \\ &= (1 - x)(f + rg) \end{aligned}$$

Now apply the Augmentation Map and observe that  $r = 0(a)$ , where  $a \in R$ . Hence, we get a contradiction. Thus,  $M \neq \langle 1 - x, r(1 - x) \rangle$ . Therefore,  $M$  is not strongly two-generated, meaning  $R[G]$  is not strongly two-generated.  $\square$

The proof of Theorem 3.5 should be no surprise, as it is similar to the previous proofs already given. The same is true of the following proof of Theorem 3.6.

**Theorem 3.6.** *Let  $R$  be a local Artinian PIR where  $2$  belongs to the maximal ideal of  $R$  and the square of the maximal ideal of  $R$  is zero. Let  $G = Z/2^kZ$ . Then  $R[G]$  is a local ring with the two-generator property but not the strong two-generator property.*

*Proof.*  $R[G]$  is local by Corollary 19.2, and it has the two-generator property by Theorem 3.1 [Gil84]. Let  $rR$  be the maximal ideal out of  $R$ , where  $r^2 = 0$ . Then  $M = \langle r, 1 - x \rangle$ , is the maximal ideal out of  $R[G]$  because  $R[G]/M$  is a field. Then  $M^2 =$

$\langle r(1-x), 1-2x+x^2 \rangle$ . Take  $r(1-x) \in M^2 \subseteq M$ . If  $M$  is strongly two-generated, then  $\exists t \in R[G] \ni M = \langle r(1-x), t \rangle$ . By Corollary 5.3 of [Nag75], one of the generators must be either  $r$  or  $1-x$ , so there are two cases.

Case 1:  $M = \langle r, r(1-x) \rangle$ . We will show that this case fails. Since  $1-x \in M, \exists a_0 + a_1x + a_2x^2 + \dots + a_{2^k-1}x^{2^k-1}, b_0 + b_1x + b_2x^2 + \dots + b_{2^k-1}x^{2^k-1} \in R[G]$  with

$$\begin{aligned} 1-x &= r(1-x)(a_0 + a_1x + a_2x^2 + \dots + a_{2^k-1}x^{2^k-1}) + \\ &\quad r(b_0 + b_1x + b_2x^2 + \dots + b_{2^k-1}x^{2^k-1}). \end{aligned}$$

Thus,  $1 = r(a_0 - b_{2^k-1} + b_0)$ . This implies  $r$  is a unit in  $R$ , which is a contradiction.

Thus,  $M \neq \langle r, r(1-x) \rangle$ . Now we will show that Case 2 fails also.

Case 2:  $M = \langle 1-x, r(1-x) \rangle$ . Since  $r \in M, \exists f, g \in R[G]$  with

$$\begin{aligned} r &= (1-x)f + (r(1-x))g \\ &= (1-x)(f + rg) \end{aligned}$$

Now apply the Augmentation Map and observe that  $r = 0(a)$ , where  $a \in R$ . Hence, we get a contradiction. Thus,  $M \neq \langle 1-x, r(1-x) \rangle$ . Therefore,  $M$  is not strongly two-generated, meaning  $R[G]$  is not strongly two-generated.  $\square$

The last part of Theorem 3.1, part (ii), requires the following Lemma in order to prove it.

**Lemma 3.7.** *Let  $R$  be a commutative ring with identity, and let  $G$  be a finite cyclic group of order  $m+1$  with generator  $g$ . Let  $I = \langle 1-x^g, 1-x^{2g}, 1-x^{3g}, \dots, 1-x^{ng} \rangle$ . Then  $I$  is the Augmentation Ideal (AI) and  $AI = \langle 1-x^g \rangle$ .*

*Proof.* Clearly,  $I \subseteq AI$ . Let  $f \in AI$ , where  $f = a_0 + a_1x^g + a_2x^{2g} + \dots + a_nx^{ng}$ , meaning  $a_0 + a_1 + a_2 + \dots + a_n = 0$ . Well,

$$\begin{aligned} 0-f &= -f = (a_0 + a_1 + a_2 + \dots + a_n) - (a_0 + a_1x^g + a_2x^{2g} + \dots + a_nx^{ng}) \\ &= a_1(1-x^g) + a_2(1-x^{2g}) + \dots + a_n(1-x^{ng}). \end{aligned}$$

Thus,  $f \in I$ , where  $AI = \langle 1-x^g, 1-x^{2g}, 1-x^{3g}, \dots, 1-x^{ng} \rangle$ . Now, since all the entries in  $AI$  are multiples of  $1-x^g$ , for example  $1-x^{2g}$  can be expressed as  $(1-x^g)(1+x^g)$ , and  $1-x^{3g}$  can be expressed as  $(1-x^g)(1+x^g+x^{2g})$ , etc., then  $AI$  can be written as  $\langle 1-x^g \rangle$ .  $\square$

Lastly, Theorem 3.8 takes care of the final piece of Theorem 3.1, or part (ii).

**Theorem 3.8.** *Let  $p$  be an odd prime integer. For  $R$  a local Artinian ring with the two-generator property but not a PIR, let  $G = Z/p^k Z$  be a finite Abelian group with the order of  $G$  a unit in  $R$ . Then  $R[G]$  has the two-generator property but not the strong two-generator property.*

*Proof.* First, we know that  $R[G]$  has the two-generator property by Theorem 3.1. Now, we also know that  $AI = \langle 1 - x^g \rangle$  by Lemma 3.8. Also,  $AI$  is the kernel of the Augmentation Map. Look at  $R[G]$  and mod out by the kernel to obtain  $R[G]/\langle 1 - x^g \rangle$ . Since the Augmentation Map is onto, then by the First Isomorphism Theorem, [Gal10],  $R[G]/\langle 1 - x^g \rangle \cong R$ . But  $R$  is not a PIR, and by Theorem 2.16,  $R[G]$  is not strongly two-generated.  $\square$

Theorems 3.2-3.6, and 3.8 leave us with Theorem 3.9, which summarizes what has been done in this chapter.

**Theorem 3.9.** *The monoid rings from Theorem 3.1, which are not PIRs, do not have the strong two-generator property.*

Theorem 3.9 has already been proven in the proofs of Theorems 3.2-3.6, and 3.8. The proofs of these theorems are all similar, in that we take an element out of the square of the maximal ideal, and try to find another generator. Each time, we get a contradiction. Using the terminology from Chapter 1, Theorem 3.9 can be stated as  $\mathbb{S}_2 \setminus \mathbb{S}_1 \cap \mathbb{S}_{1.5} = \phi$ . Now, we need to see if the same is true of the one-dimensional case.



## Chapter 4

# One-Dimensional Case

Our main focus of the one-dimensional case is Theorem 2.7 from [ORV92], which is listed below as Theorem 4.1. We show that the monoid rings of this theorem which are not PIRs do not have the strong two-generator property. We also look at Theorems 2.6 and 3.1 from [ORV92]. These will be listed later in the chapter as Theorems 4.9 and 4.11. However, only partial results have been obtained for these two theorems, as these cases are more difficult than the ones from Theorem 4.1. Our results are summarized in Theorems 4.8, 4.10, and 4.12.

**Theorem 4.1.** *Let  $R$  be a commutative ring with nilradical  $N$  and let  $S$  be a cancellative monoid whose quotient group  $G(S)$  has torsion-free rank one, say  $G(S) = Z \oplus H$ , where  $H$  is a finite group of order  $m = 2^k m_1$  with  $m_1$  odd. If  $H \subseteq S$  then  $R[S]$  has the two-generator property if and only if the following two statements hold,*

(i)  *$R$  is an Artinian principal ideal ring,  $N^2 = 0$ ,  $m_1$  is a unit in  $R$ , and if  $k \geq 1$  then each local summand of  $R$  having characteristic  $2^j$  with  $j > 0$  is a field.*

(ii) *One of the following holds:*

(a)  *$S \cong Z \oplus H$ , and if 2 divides the characteristic of  $R$  then  $k \leq 1$ ;*

(b)  *$S \cong Z_+ \oplus H$ , and if 2 divides the characteristic of  $R$  then  $k \leq 1$ ;*

(c)  *$S \cong T \oplus H$ , where  $T$  is a submonoid of  $Z_+ \setminus \{1\}$  containing 2,  $R$  is a product of fields, and if one of these fields has characteristic 2 then  $k = 0$ .*

Again, there are many cases to consider, and each case is broken down into its own theorem. Whereas before, the same method was used to prove each theorem in the

zero-dimensional case, the one-dimensional case uses many different techniques, and relies more heavily on previous results. The first theorem uses the First Isomorphism Theorem from [Gal10] to look at a simpler monoid ring. Once mapped to the simpler case, we use previous results to end up with the end result that this monoid ring cannot be strongly two-generated. Since Theorem 4.1 requires both (i) and (ii) to be satisfied, part (i) is given in the statement of each theorem to satisfy that case. Then, we look at the different cases of (ii). Now Theorem 4.2 proves part (ii)(a).

**Theorem 4.2.** *Let  $R$  be an Artinian local ring, not a field or a finite direct sum of fields, with nilradical  $N$  where  $N^2 = 0$ . Let  $S \cong Z \oplus H$  be a cancellative monoid, where  $H$  is a finite group of order  $m = 2^k m_1$ , for  $m_1$  odd and  $m$  is a unit of  $R$ . If  $k \geq 1$ , then each local summand of  $R$  having characteristic  $2^j$  with  $j > 0$  is a field, and if 2 divides the characteristic of  $R$ , then  $k \leq 1$ .  $R[S]$  has the two-generator property but not the strong two-generator property.*

*Proof.* First, we know that  $R[S]$  is two-generated by Theorem 4.1. Also,  $H$  is abelian by assumption, thus by Theorem 11.1 in [Gal10], let  $H = G_1 \oplus G_2 \oplus \dots \oplus G_i$ , where each  $G_j$  is a cyclic group. Look at any summand of this decomposition of  $H$ , say  $G$ . By passing to a homomorphic image, we look at  $R[Z \oplus G] = R[Z][G]$ . Since  $G$  is cyclic, let  $G = \langle g \rangle$ . Then take the ideal  $\langle 1 - x^g \rangle \in R[Z][G]$ . This is the augmentation ideal by Lemma 3.7. Hence, it is the kernel of the augmentation map. Then,  $R[Z][G]/\langle 1 - x^g \rangle \cong R[Z]$  by the First Isomorphism Theorem. By Theorem 2.16,  $R[Z]$  must be a PIR. By Theorem 18.10 in [Gil84],  $R$  must be a finite direct sum of fields, which is a contradiction. So,  $R[S]$  cannot be strongly two-generated.  $\square$

Again, multiple results were needed in order to complete this proof. Since parts (a) and (b) of Theorem 4.1 above are so similar, their proofs are very similar as well. Hence, Theorem 4.3 is proven the same way as Theorem 4.2.

**Theorem 4.3.** *Let  $R$  be an Artinian local ring, not a field or a finite direct sum of fields, with nilradical  $N$  where  $N^2 = 0$ . Let  $S \cong Z_+ \oplus H$  be a cancellative monoid, where  $H$  is a finite group of order  $m = 2^k m_1$ , for  $m_1$  odd and  $m$  is a unit of  $R$ . If  $k \geq 1$ , then each local summand of  $R$  having characteristic  $2^j$  with  $j > 0$  is a field, and if 2 divides the characteristic of  $R$ , then  $k \leq 1$ .  $R[S]$  has the two-generator property but not the strong two-generator property.*

*Proof.* Again, we know  $R[S]$  is two-generated by Theorem 4.1, and  $H$  is abelian by assumption. Thus by Theorem 11.1 of [Gal10], let  $H = G_1 \oplus G_2 \oplus \dots \oplus G_i$ , where each  $G_j$  is a cyclic group. Look at any summand of this decomposition of  $H$  as before, say  $G$  again. By passing to a homomorphic image, we look at  $R[Z_+ \oplus G] = R[Z_+][G]$ . Since  $G$  is cyclic, let  $G = \langle g \rangle$ . Then take the ideal generated by  $\langle 1 - x^g \rangle \in R[Z_+][G]$ . This is the augmentation ideal by Lemma 3.7. Hence, it is the kernel of the augmentation map. Then,  $R[Z_+][G]/\langle 1 - x^g \rangle \cong R[Z_+]$  by the First Isomorphism Theorem. By Theorem 2.16,  $R[Z_+]$  must be a PIR. By Theorem 18.10 in [Gil84],  $R$  must be a finite direct sum of fields, which is a contradiction. So,  $R[S]$  cannot be strongly two-generated.  $\square$

The last piece of Theorem 4.1, part (ii)(c), is listed below. This proof again maps the monoid ring to a simpler ring under an onto ring homomorphism to determine whether or not it is strongly two-generated. This time, only one previous result is needed.

**Theorem 4.4.** *Let  $R$  be a finite direct sum of fields. Let  $S \cong T \oplus H$ , where  $T$  is a submonoid of  $Z_+ \setminus \{1\}$  containing 2,  $H$  is a finite group of order  $m = 2^k m_1$ , for  $m_1$  odd,  $m_1$  is a unit of  $R$ , and if one of these fields has characteristic 2, then  $k = 0$ . Then  $R[G]$  has the two-generator property but not the strong two-generator property.*

*Proof.* First, we know  $R[S]$  is two-generated by Theorem 4.1. Now let  $R = F_1 \oplus F_2 \oplus \dots \oplus F_r$ , where each  $F_i$  is a field. So  $R[S] = (F_1 \oplus F_2 \oplus \dots \oplus F_r)[S]$ . Consider the homomorphic image  $F[S]$  of  $(F_1 \oplus F_2 \oplus \dots \oplus F_r)[S]$ , where  $F = F_j$  for some  $j$ . Now  $F[S] = F[T \oplus H] = F[T][H]$ , and pass to the homomorphic image  $F[T]$ . By Theorem 14 of [Pet94], this cannot be strongly two-generated. So,  $R[S]$  cannot be strongly two-generated.  $\square$

Theorems 4.2 and 4.3 take care of the case where  $R$  is not a field or a finite direct sum of fields, and Theorem 4.4 takes care of part (ii)(c) entirely. Theorem 4.5 will take care of the case where  $R$  is a field or a finite direct sum of fields where the order of  $H$  is a unit of  $R$ . Its proof treats parts (ii)(a) and (b) and comes directly from Theorem 19.13 in [Gil84].

**Theorem 4.5.** *Let  $R$  be a field or a finite direct sum of fields. Let  $S \cong S_0 \oplus H$  be a cancellative monoid, where  $S_0 = Z$  or  $Z_+$ , and  $H$  is a finite group of order  $m = 2^k m_1$ , for  $m_1$  odd and  $m$  a unit of  $R$ . Then  $R[S]$  is a PIR.*

*Proof.* Since  $m$  is a unit of  $R$ , then  $R[S] = R[S_0 \oplus H]$  is a PIR by Theorem 19.13 [Gil84].  $\square$

Since our main focus is to determine which monoid rings are two-generated not PIRs, but are strongly two-generated, the case of Theorem 4.5 is done once it is proven to be a PIR. It is no longer part of the monoid rings we care to look at.

Now we consider the case where  $R$  is a field or a finite direct sum of fields, and the order of  $H$  is not a unit of  $R$ . Our assumption from Theorem 4.1 is that the order of  $H$  is  $2^k m_1$ , where  $m_1$  is odd and  $m_1$  is a unit of  $R$ . This means we are considering the case where  $k > 0$  and 2 is not a unit of  $R$ . This implies that one of the summands of  $R$ , say  $F$ , has characteristic 2. But from (ii) (a) and (b) of Theorem 4.1, 2 dividing the characteristic of  $R$  implies  $k \leq 1$ . Thus,  $k = 1$ . Hence, Theorems 4.6 and 4.7 treat these remaining cases for (ii) (a) and (b), respectively. In the following proof, we map a complicated monoid ring, under a few onto homomorphisms to  $F[Z/2Z][Z]$ , where  $F$  is a field of characteristic 2. The elements of this ring can be thought of as polynomials with the addition of negative exponents. For example,  $a_{-n}Y^{-n} + a_{-(n-1)}Y^{-(n-1)} + \dots + a_{-1}Y^{-1} + a_0 + a_1Y + a_2Y^2 + \dots + a_mY^m$ , where the coefficients are elements from  $F[Z/2Z]$ .

**Theorem 4.6.** *Let  $R$  be a finite direct sum of fields. Let  $S \cong Z \oplus H$  be a cancellative monoid, and  $H$  be a finite group of order  $m = 2m_1$ , for  $m_1$  odd and 2 not a unit of  $R$ . Then  $R[S]$  is not strongly two-generated.*

*Proof.* Since 2 is not a unit of  $R$ , and  $R$  is a finite direct sum of fields, then the characteristic of at least one summand of  $R$  must be 2. Choose  $F$  to be a summand of characteristic 2. Then map  $R[S]$  under an onto homomorphism to  $F[S] = F[Z \oplus H]$ . Now write  $H = Z/2Z \oplus H_1$ , where the order of  $H_1$  is  $m_1$ . Again, map  $F[Z \oplus H]$  under an onto homomorphism to  $F[Z \oplus Z/2Z \oplus H_1] = F[Z \oplus Z/2Z][H_1]$ . Under another onto homomorphism, map  $F[Z \oplus Z/2Z][H_1]$  onto  $F[Z \oplus Z/2Z] = F[Z/2Z][Z] \cong F[Z/2Z][Y, Y^{-1}]$ , with the latter being an overring of the polynomial ring  $F[Z/2Z][Y]$ . Now, define  $\psi$  to be the onto ring homomorphism that maps  $F[Z/2Z][Y, Y^{-1}]$  to  $F[Z/2Z][Z/2Z]$ , where  $Y$  and  $Y^{-1}$  both map to  $X^{(0,1)}$ . Consider the element  $1 + Y^2$ .  $\psi(1 + Y^2) = 1 + (X^{(0,1)})^2 = 1 + X^{(0,2)} = 1 + X^{(0,0)} = 1 + 1 = 2 = 0$ . Thus,  $1 + Y^2 \in \ker(\psi)$ . We will now show that  $\langle 1 + Y^2 \rangle = \ker(\psi)$ . Take  $f \in \ker(\psi)$ . Then  $\exists n \ni Y^n f \in F[Z/2Z][Y]$ , where  $F[Z/2Z][Y]$  is a polynomial ring. Then  $\exists g, h \in F[Z/2Z][Y] \ni Y^n f = (1 + Y^2)g + h$ , where  $\deg(h) < 2$ .

Then  $h = a_0 + a_1Y$ . So since  $f \in \ker(\psi)$  and  $(1 + Y^2) \in \ker(\psi)$ , then  $h$  must map to 0 to which implies  $h = a_0 + a_1X^{(0,1)} = 0$ . Thus,  $a_0 = 0$  and  $a_1 = 0$ . Thus,  $Y^n f = (1 + Y^2)g \Rightarrow Y^n f \in \langle 1 + Y^2 \rangle$ . But  $Y^n$  is a unit in  $F[Z/2Z][Y, Y^{-1}]$ , so  $f \in \langle 1 + Y^2 \rangle$ . Thus, we have shown  $\langle 1 + Y^2 \rangle = \ker(\psi)$ . Now  $F[Z/2Z][Y, Y^{-1}]/\langle 1 + Y^2 \rangle \cong F[Z/2Z][Z/2Z]$  by the First Isomorphism Theorem. Since  $F[Z/2Z][Z/2Z] = F[Z/2Z \oplus Z/2Z]$ ,  $F$  is a field of characteristic 2, and  $Z/2Z \oplus Z/2Z$  is not a cyclic 2-group, then by Theorem 19.14 in [Gil84],  $F[Z/2Z][Z/2Z]$  is not a PIR. Then by Theorem 2.16,  $F[Z/2Z][Z/2Z]$  is not strongly two-generated. Hence,  $R[S]$  is not strongly two-generated.  $\square$

The proof of Theorem 4.7 is very similar to the proof of Theorem 4.6.

**Theorem 4.7.** *Let  $R$  be a finite direct sum of fields. Let  $S \cong Z_+ \oplus H$  be a cancellative monoid, and  $H$  be a finite group of order  $m = 2m_1$ , for  $m_1$  odd and 2 not a unit of  $R$ . Then  $R[S]$  is not strongly two-generated.*

*Proof.* Since 2 is not a unit of  $R$ , and  $R$  is a finite direct sum of fields, then the characteristic of at least one of the summands of  $R$  must be 2. Choose  $F$  to be a summand of characteristic 2. Then map  $R[S]$  under an onto homomorphism to  $F[S] = F[Z_+ \oplus H]$ . Now write  $H = Z/2Z \oplus H_1$ , where the order of  $H_1$  is  $m_1$ . Then,  $F[Z_+ \oplus H] = F[Z_+ \oplus Z/2Z \oplus H_1] = F[Z_+ \oplus Z/2Z][H_1]$ . Under another onto homomorphism, map  $F[Z_+ \oplus Z/2Z][H_1]$  to  $F[Z_+ \oplus Z/2Z] = F[Z/2Z][Z_+] \cong F[Z/2Z][Y]$ , with the latter being a polynomial ring. Define  $\psi$  to be the onto homomorphism that maps  $F[Z/2Z][Y]$  to  $F[Z/2Z][Z/2Z]$ , where  $Y$  maps to  $1 - X^{(0,1)}$ . Then  $Y^2$  maps to  $1 - 2X^{(0,1)} + X^{(0,2)} = 1 - 0 + X^{(0,0)} = 1 + 1 = 2 = 0$ . So  $\langle Y^2 \rangle \subseteq \ker(\psi)$ . We will now show that  $\langle Y^2 \rangle = \ker(\psi)$ . Take  $f \in \ker(\psi)$ , where  $f = r_0 + r_1Y + r_2Y^2 + \dots + r_nY^n$ . Since  $\langle Y^2 \rangle \subseteq \ker(\psi)$ ,  $r_2Y^2 + \dots + r_nY^n$  maps to 0 under  $\psi$ . This means  $f$  maps to  $r_0 + r_1(1 - X^{(0,1)}) = r_0 + r_1 - r_1X^{(0,1)}$ . Since  $f$  maps to 0, then  $r_1$  has to be 0, making  $r_0 = 0$  as well. Also,  $r_2Y^2 + \dots + r_nY^n = 0$  since  $\langle Y^2 \rangle \subseteq \ker(\psi)$ . Thus,  $f \in \langle Y^2 \rangle$  which implies  $\langle Y^2 \rangle = \ker(\psi)$ . Now,  $F[Z/2Z][Y]/\langle Y^2 \rangle \cong F[Z/2Z][Z/2Z]$  by the First Isomorphism Theorem. Since  $F[Z/2Z][Z/2Z] = F[Z/2Z \oplus Z/2Z]$ ,  $F$  is a field of characteristic 2, and  $Z/2Z \oplus Z/2Z$  is not a cyclic 2-group, then by Theorem 19.14 in [Gil84],  $F[Z/2Z][Z/2Z]$  is not a PIR. Then by Theorem 2.16,  $F[Z/2Z][Z/2Z]$  is not strongly two-generated. Hence,  $R[S]$  is not strongly two-generated.  $\square$

Theorems 4.2-4.7 take care of all the needed cases to prove Theorem 4.8.

**Theorem 4.8.** *The monoid rings from Theorem 4.1, which are not PIRs, do not have the strong two-generator property.*

Using the terminology from Chapter 1, Theorem 4.8 can be stated as  $\mathbb{S}_2 \setminus \mathbb{S}_1 \cap \mathbb{S}_{1.5} = \phi$ .

Now, we look at Theorem 2.6 from [ORV92], which we state as Theorem 4.9.

**Theorem 4.9.** *Let  $R$  be a commutative ring with nilradical  $N$ , and let  $T$  be a cancellative torsion-free monoid. Then  $R[T]$  has the two-generator property if and only if  $R$  is an Artinian principal ideal ring with  $N^2 = 0$ , and  $T$  is isomorphic to either  $Z_+$ ,  $Z$ , or a submonoid of  $Z_+ \setminus \{1\}$  containing 2, and in the last case  $R$  is a finite direct product of fields.*

We will only prove the last case of Theorem 4.9.

**Theorem 4.10.** *Let  $R$  be a finite direct sum of fields with nilradical  $N$ , where  $N^2 = 0$  and let  $T$  be isomorphic to a submonoid of  $Z_+ \setminus \{1\}$  containing 2 be a cancellative torsion-free monoid. Then  $R[T]$  has the two-generator property, but not the strong two-generator property.*

*Proof.*  $R[T]$  has the two-generator property by Theorem 4.9. Again,  $T = \langle 2, n \rangle$ , for  $n$  an odd positive integer. Let  $R = F_1 \oplus F_2 \oplus \dots \oplus F_s$ , where each  $F_i$  is a field. So  $R[T] = (F_1 \oplus F_2 \oplus \dots \oplus F_s)[\langle 2, n \rangle]$ . Map to  $F[\langle 2, n \rangle]$  under an onto ring homomorphism, where  $F$  is one of the summands of  $R$ . Then, by the proof of Theorem 4.4,  $F[\langle 2, n \rangle]$  is not strongly two-generated. Thus,  $R[T]$  is not strongly two-generated.  $\square$

Theorem 4.10 was proven using Pettersson's result [Pet94]. The other two cases, where  $T$  is isomorphic to  $Z_+$  or  $Z$ , are more difficult to prove.

Lastly, we will give partial results of Theorem 3.1 in [ORV92], stated below as Theorem 4.11.

**Theorem 4.11.** *Let  $R$  be a ring and let  $S \subseteq Z_+ \oplus H$  be a monoid with quotient group  $G(S) = Z \oplus H$ , where  $H$  is a finite abelian group not contained in  $S$ . Then  $R[S]$  has the two-generator property if and only if the following hold:*

(i)  $R = R_1 \oplus \dots \oplus R_s$ , where each  $R_i$  is a field.

(ii)  $S = S_1 \oplus K$ ,  $K$  a finite abelian group of order  $m$  and either  $m = 0$  or  $m$  is a unit in each of the  $R_i$ .

(iii)  $S_1$  is one of the following submonoids of  $Z_+ \oplus (Z/2^j Z)$  with  $j \geq 1$ :

- (a)  $\langle (1, \bar{0}), (n, \bar{1}), (0, \bar{2}) \rangle$  for some  $n \geq 1$ ;
- (b)  $\langle (1, \bar{1}), (2n, \bar{1}), (0, \bar{2}) \rangle$  for some  $n \geq 1$ ;
- (c)  $\langle (1, \bar{1}), (2n + 1, \bar{0}), (0, \bar{2}) \rangle$  for some  $n \geq 1$ .

(iv) If some  $R_i$  has characteristic 2, then  $j = 1$ , (and so  $(0, \bar{2}) = 0$ ). Thus  $S_1$  is one of the following submonoids of  $Z \oplus (Z/2Z)$ :

- (a)  $\langle (1, \bar{0}), (n, \bar{1}) \rangle$  for some  $n \geq 1$ ;
- (b)  $\langle (1, \bar{1}), (2n, \bar{1}) \rangle$  for some  $n \geq 1$ ;
- (c)  $\langle (1, \bar{1}), (2n + 1, \bar{0}) \rangle$  for some  $n \geq 1$ .

Again, only a partial result will be given on Theorem 4.11. All cases need to hold in Theorem 4.11, but we look at part (ii), where  $K \neq 0$ , and get the rest of the cases for (iii) and (iv), and show that  $R[S]$  is not strongly two-generated.

**Theorem 4.12.** *Let  $R$  be a finite direct sum of fields, and let  $S = S_1 \oplus K$ , for  $K$  a finite abelian group of order  $m \neq 0$ ,  $m$  a unit in each of the  $R_i$ ,  $G(S) = Z \oplus H$ , where  $H$  is a finite abelian group not contained in  $S$ , and  $S_1$  defined as it is in Theorem 4.13 parts (iii) and (iv). Then  $R[S]$  has the two-generator property but not the strong two-generator property.*

*Proof.* First, we know  $R[S]$  has the two-generator property by Theorem 4.11. Now write  $K = H + H_1$ , where  $H$  is cyclic. Now map under an onto homomorphism from  $R[S] = R[S_1 \oplus K] = R[S_1 \oplus H \oplus H_1] = R[S_1 \oplus H][H_1]$  to  $R[S_1 \oplus H]$ . Let  $I = \langle 1 - X^g \rangle$ , where  $g$  is the generator of  $H$ . Then  $R[S_1][H]/I \cong R[S_1]$  by the First Isomorphism Theorem. Hence, by Theorem 2.16,  $R[S_1]$  is a PIR. Then by Theorem 19.13 in [Gil84],  $R$  is a direct sum of fields and  $S_1$  has to be isomorphic to one of two groups. However,  $S_1$  is a monoid not a group, so we get a contradiction. Thus,  $R[S_1]$  cannot be strongly two-generated, meaning  $R[S]$  cannot be strongly two-generated.  $\square$

What still remains is the case where  $K = 0$ . Then we get the cases where  $S_1$  is a submonoid of all the different pieces from parts (iii) and (iv).

In conclusion, with Theorems 4.8, 4.10, and 4.12, we have found that monoid rings in the one-dimensional case that are two-generated not PIRs, are not strongly two-generated. However, as mentioned above, there remain a few cases from Theorems 4.9 and 4.11 that still need to be checked. It may be that the rest of the unsolved cases are strongly two-generated, though we doubt it. This is left for the reader to try.



## Chapter 5

# Conclusion

It was mentioned that Dedekind domains are two-generated, when in fact, after looking at the proof found in [AW04] more carefully, it is seen that they are also strongly two-generated. We wondered how common this property was amongst rings with zero-divisors, and found out that it is not very common. In fact, in the zero-dimensional case, it does not occur for any monoid rings that are not already PIRs. The methods used for proving these cases were all similar, and all dealt with mapping the original monoid ring to a simpler ring under an onto homomorphic image. If the homomorphic image was not strongly two-generated, then the original ring cannot be strongly two-generated either by Theorem 2.15.

The problem lies with the one-dimensional case. It is not clear how common this property is for these monoid rings. Some cases were proven to not be strongly two-generated. Others were proven to be PIRs, which we do not care to look at since we want two-generated monoid rings, not PIRs. However, there remain a few cases that still need to be checked. It is unclear what the best strategy is to prove the remaining cases that have not been taken care of. The usual strategy used in the zero-dimensional case, mapping from the original monoid ring onto a simpler one, does not work for the one-dimensional case. For example, for the remaining cases of Theorem 4.11 we have a monoid inside a direct sum, which is harder to work with. That is why only partial results have been given in the one-dimensional case for two of the three theorems we considered from [ORV92].

We considered rings with the strong two-generator property. This required that

all the nonzero elements were strong two-generators. What about changing the focus to elements of a ring, in particular determining which elements are strong two-generators? We close with a problem for future consideration.

**Problem 5.1.** *For the monoid rings with the two-generator property that are not PIRs, what are the elements that are strong two-generators?*

Theorem 14 from [Pet94], stated below as Theorem 5.2, provides an answer in a particular case.

**Theorem 5.2.** *Let  $B = K[X^2, X^n]$ , where  $K$  is a field, and  $m = (X^2, X^n)$ , where  $n$  is an odd positive integer greater than 2. Let  $p$  be a non-zero polynomial in  $B$ . Then  $p$  is a strong two-generator if  $p \notin m^2$ .*

Even though  $B$  from Theorem 5.2 above is a subring of a polynomial ring, it can also be thought of as  $K[S]$ , where  $S = \langle 2, n \rangle$ , for  $n$  an odd positive integer. The strong two-generators for the case where  $n = 3$  is found in [Cha90]. Now, will there be a similar result in regards to determining the elements that are strong two-generators for the monoid rings we have examined in Chapters 3 and 4? Namely, will there be a maximal ideal  $M$  so that the strong two-generators are those elements belonging to  $M$  but not  $M^2$ ?

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