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A KLEINIAN APPROACH TO FUNDAMENTAL REGIONS

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A Kleinian Approach to Fundamental Regions

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Joshua Lee Hidalgo

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This thesis takes a Kleinian approach to hyperbolic geometry in order to illustrate the importance of discrete subgroups and their fundamental domains (fundamental regions). A brief history of Euclid’s Parallel Postulate and its relation to the discovery of hyperbolic geometry be given first. We will explore two models of hyperbolic \( n \)-space: \( U^n \) and \( B^n \). Points, lines, distances, and spheres of these two models will be defined and examples in \( U^2 \), \( U^3 \), and \( B^2 \) will be given. We will then discuss the isometries of \( U^n \) and \( B^n \). These isometries, known as Möbius transformations, have special properties and turn out to be linear fractional transformations when in \( U^2 \) and \( B^2 \). We will then study a bit of topology, specifically the topological groups relevant to the group of isometries of hyperbolic \( n \)-space, \( I(H^n) \). Finally we will combine what we know about hyperbolic \( n \)-space and topological groups in order to study fundamental regions, fundamental domains, Dirichlet domains, and quotient spaces. Using examples in \( U^2 \), we will then illustrate how useful fundamental domains are when it comes to visualizing the geometry of quotient spaces.
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Chapter 1

Introduction

Euclid’s Parallel Postulate is historically one of the most researched and studied writings of mathematics. Mathematicians tried proving this particular postulate and have failed over the years. Euclid himself used it very sparingly in proving various theorems in his book, *Elements*. This is because, unlike Euclid’s other postulates, the Parallel Postulate cannot be proved using his other axioms. Of all the equivalent statements of the Parallel Postulate, Playfair’s axiom, named after the Scottish mathematician John Playfair, is probably the most well-known. It essentially states that given any straight line \( l \) and a point \( p \) not on \( l \), there exists a unique straight line \( m \) which passes through \( p \) and never intersects \( l \), no matter how far they are extended. As shown in Figure 1.1.

Let us analyze Playfair’s version of Euclid’s Parallel Postulate. The postulate gives two interesting claims: one is that parallel lines exist and the other is that the parallel line is unique. These two claims are crucial to Euclidean geometry and Euclidean \( n \)-space, \( E^n \). *Hyperbolic geometry* differs from Euclidean geometry in that there exist more than
one line, say $n$ and $m$, through $p$ that do not intersect $l$. Implying that the parallel line through $p$ is no longer unique.

![Diagram of hyperbolic parallel axiom](image)

Figure 1.2: Hyperbolic Parallel Axiom

The subject of hyperbolic geometry was fine-tuned in the 19th century and expanded on by such mathematicians as Lobachevskii, Beltrami, Klein, Poincaré, and Riemann. Non-Euclidean geometry has had several innovative consequences. It gives models of geometry different from Euclidean geometry that are just as consistent. *Hyperbolic geometry* is one of the results of non-Euclidean geometry. (This introduction cited [Gre93].)
Chapter 2

Models of Hyperbolic Space

In this chapter, we will look at hyperbolic geometry and how it can be studied using two common models: the open upper-half-space, $U^n$, and open ball, $B^n$. A model for hyperbolic geometry is a definition of points, lines, and distances that satisfies the axioms. In particular, the hyperbolic parallel axiom is satisfied in each of these models. (Throughout this chapter we cite [Bon09], [Bra99], and [Rat94].)

2.1 Open Upper-Half-Space Model, $U^n$

In this section we describe the open upper-half-space, $U^n$, and how its Euclidean properties can be used to describe hyperbolic geometry. We will be looking at points, metric, ideal points, lines, planes, and spheres of $U^n$. Examples in $U^2$ and $U^3$ and familiar subsets of Euclidean space will help us describe these properties of $U^n$. Points in $U^n$ can be described using real or complex numbers. Upper half-space is generally defined in the following way:

$$U^n = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_n > 0\}.$$

We can identify the upper half-plane, $U^2$, as the set

$$\{z \in \mathbb{C} \mid \text{Im}z > 0\}.$$

The upper half-space, $U^3$, can be defined as the set of quaternions

$$\{z + tj \mid z \in \mathbb{C} \text{ and } t > 0\}$$
or more commonly

\[ \{ q = a + bi + cj \mid a, b, c \in \mathbb{R} \text{ and } c > 0 \} \]

where \( i^2 = j^2 = -1 \).

**Example 2.1.1.** In \( U^2 \), \( a = i \).

\[ U^2 \]

\[ \bullet \]

\[ a \]

\[ \rightarrow \]

\[ E^3 \]

Figure 2.1: Point of \( U^2 \)

**Example 2.1.2.** In \( U^3 \), \( a = i + j \).

\[ U^3 \]

\[ \rightarrow \]

\[ E^3 \]

Figure 2.2: Point of \( U^3 \)
Now let's look at some representations of the metric for $U^n$, $d_U$. One metric is given by

$$\cosh d_U(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n},$$

where $|x - y|$ is the standard Euclidean distance, $d_E$. It is known that hyperbolic arc length of a curve $\gamma$ is different from Euclidean arc length. For $U^n$, it is given by the formula

$$\frac{|dx|}{x_n}.$$

In $U^2$, this definition implies arc length can be calculated as follows. Let $\gamma$ be the differentiable vector valued function

$$t \mapsto (x(t), y(t)), \text{ such that } a \leq t \leq b.$$

Then

$$l_{U^2}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

This metric provides a way to measure lengths of curves. We will illustrate this with an example.

**Example 2.1.3.** Suppose $\gamma$ is parametrized by

$$t \mapsto (0, t), \text{ such that } a \leq t \leq b,$$

as shown in the following figure.

![Figure 2.3: Parametrized Curve in $U^2$](image)
Then
\[ l_{U^2}(\gamma) = \int_a^b \frac{\sqrt{(0)^2 + (1)^2}}{t} dt = \int_a^b \frac{1}{t} dt = \ln(b) - \ln(a) = \ln \left( \frac{b}{a} \right). \]

Before we talk about what lines look like in \( U^n \), we will first look what it means to be an ideal point. Generally, ideal points are points at infinity that are not included in the space but may be included in its Euclidean closure. We can think of these ideal points as \( \hat{E}^{n-1} = E^{n-1} \cup \{\infty\} \), where \( \hat{E}^n = E^n \cup \{\infty\} \) is the extended Euclidean plane. These points have special properties that may effect points in the space. In \( U^2 \), ideal points are the real line (or the x-axis) and the point at infinity. Now we will look at hyperbolic lines in \( U^n \) and how they differ from Euclidean lines.

**Definition 2.1.4.** Hyperbolic lines of \( U^n \) are geodesics of \( U^n \).

The following examples show hyperbolic lines, geodesics, in \( U^2 \) and \( U^3 \). There are two types: Euclidean semicircles (semispheres) centered on the real line (plane) or Euclidean vertical rays (planes) originating on the real line (plane). If we think of vertical rays (planes) as semicircles (semispheres) centered at \( \infty \), then all hyperbolic lines are in fact semicircles (semispheres) centered at an ideal point.

**Example 2.1.5.** In \( U^2 \), let \( l_1 = \{z = e^{i\theta} \mid 0 < \theta < \pi\} \) and \( l_2 = \{x = 4 \mid y > 0\} \).

![Figure 2.4: Hyperbolic Lines/Geodesics of \( U^2 \)](image-url)
Note, with this definition of a hyperbolic line, that the hyperbolic parallel axiom is satisfied. Both lines $n$ and $m$ through $p$ do not intersect to the line $l$.

![Figure 2.5: Hyperbolic Parallele Axiom in $U^2$](image)

**Example 2.1.6.** Planes in $U^3$, let $l_1 = \{(1, \theta, \phi) \mid 0 < \theta \leq 2\pi \text{ and } 0 \leq \phi < \frac{\pi}{2}\}$ and $l_2 = \{(x_1, 4, x_3) \mid x_1 \in \mathbb{R} \text{ and } x_3 > 0\}$. See figure 2.5 below.

![Figure 2.6: Hyperbolic Planes/Geodesics of $U^3$](image)

Now that we have looked at hyperbolic lines in $U^n$, lets take a look at spheres (circles) in $U^n$.

**Definition 2.1.7.** The hyperbolic sphere of $U^n$, with center $a$ and radius $r > 0$, is defined to be the set $S_U(a, r) = \{x \in U^n : d_U(a, x) = r\}$. 
This is shown in $U^2$ in the following example. Take note that every hyperbolic sphere of $U^2$ is a Euclidean circle in $U^2$, though the hyperbolic radius and hyperbolic center are different from the Euclidean radius and the Euclidean center of the circle.

**Example 2.1.8.** In $U^2$, let $S_U(2 + 2i, 1) = \{x \in U^2 : d_U(2 + 2i, x) = 1\}$.

![Figure 2.7: Hyperbolic Sphere of $U^2$](image)

**Example 2.1.9.** In $U^3$, let $S_U(2i + 2j, 1) = \{x \in U^3 : d_U(2i + 2j, x) = 1\}$.

![Figure 2.8: Hyperbolic Sphere of $U^3$](image)
Lastly we will look at horospheres of $U^n$ and look at what they may look like in $U^2$ and $U^3$.

**Definition 2.1.10.** Let $b$ be an ideal point of $U^2$. Horospheres are Euclidean spheres tangent, at $b$, to $\hat{E}^{n-1}$ or horizontal planes (when $b$ is at $\infty$) in $U^n$. The point $b$ is called the “center” of the horosphere.

In $U^2$ and $U^3$, horospheres are shown as Euclidean circles (spheres) tangent to the real axis (plane) and Euclidean lines (planes) parallel to the real axis (plane), this is actually a sphere centered at $\infty$, respectively. It is important to note that the point $b$, as described in the definition, is not in the open half-space but is in the set of ideal points of the open half-space. In the Euclidean sense, $b$ is on the sphere. So the hyperbolic center of a horosphere is very different from the Euclidean center.

**Example 2.1.11.** In $U^2$, $S_1 = x + 3i$ (a circle with center at $\infty$) and $S_2 = (i, 1) = \{x \in U^2 : d_E(i, x) = 1\}$.

![Figure 2.9: Horosphere of $U^2$](image)

**Example 2.1.12.** In $U^3$, $S_1 = 2j$ (a sphere with center at $\infty$) and $S_2 = (i + j, 1) = \{x \in U^3 : d_E(i + j, x) = 1\}$. See Figure 2.10.
This subsection will focus on our second representation of Hyperbolic Geometry, the open-ball \( B^n \). Similar to \( U^n \), \( B^n \), can be described using familiar Euclidean properties.

We will again look at points, metrics, ideal points, lines, planes, and spheres of \( B^n \). Examples of these properties will be shown using \( B^2 \) and familiar subsets of Euclidean space. Let us first look at points in \( B^n \). Let \( B^n \) be the unit open-ball in Euclidean n-space, so that

\[
B^n = \{ x \in \mathbb{E}^n \mid |x| < 1 \}.
\]

We will look at \( B^2 \) for our examples. We can define \( B^2 \) to be the unit disk. The unit disk can be represented in two ways, both having their advantages:

\[
\{ z \mid |z| < 1 \}
\]

or

\[
\{ (x, y) \mid x^2 + y^2 < 1 \}.
\]

**Example 2.2.1.** In \( B^2 \), \( a = \frac{i}{2} \). See Figure 2.11.

We will define a metric on \( B^n \) via

\[
\cosh d_B(x, y) = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}
\]
where $|x - y|$ again is the standard Euclidean distance. Similar to $U^n$, hyperbolic arc length of a curve $\gamma$ in $B^n$ is different from Euclidean arc length. For $B^n$, it is given by the formula 

$$\frac{2|dx|}{1 - |x|^2}.$$ 

In $B^2$, the hyperbolic distance between two points $z_1$ and $z_2$ is defined by 

$$d_B(z_1, z_2) = \tanh^{-1}\left(\frac{|z_2 - z_1|}{1 - \overline{z_1}z_2}\right).$$ 

**Example 2.2.2.** Let $a = \frac{1}{3}i$ and $b = \frac{1}{2}$. Then 

$$d(b, a) = d\left(\frac{1}{2}, \frac{1}{3}i\right) = \tanh^{-1}\left(\frac{1}{\frac{1}{2} - \frac{1}{3}i} \cdot \frac{1}{2}\right) \approx 0.6819.$$ 

\[\square\]
It is worth mentioning that above metric and the space $B^n$ creates the conformal ball model of $H^n$. Since the model is conformal, angles in $B^n$ are the same as Euclidean angles. In other words, the angle between two geodesic rays is the Euclidean angle between the tangents to the rays. Now for $B^n$, ideal points are on the boundary of $B^n$. Next let us take a look at hyperbolic lines of $B^n$. We have a similar definition of hyperbolic lines of $B^n$ as we do in $U^n$.

**Definition 2.2.3.** Hyperbolic lines of $B^n$ are geodesics of $B^n$.

These are either arcs of circles orthogonal to the unit circle or diameters of $B^2$. If the line is a diameter of $B^2$, then this is just an arc of a generalized circle with the center at the point of infinity. In this model, each hyperbolic line is part of a Euclidean generalized circle which is orthogonal to the boundary of $B^n$ and are in $B^n$. The following example gives two images of these geodesics in $B^2$.

**Example 2.2.4.** In $B^2$, let $l_1 = \{(x, y) \cap B^2 | y = x\}$ and $l_2 = \{(x, y) \cap B^2 | x^2 + y^2 - 4x + 1 = 0\}$.

![Figure 2.13: Hyperbolic Lines/Geodesics of $B^2$](image)

Note, with this definition of a hyperbolic line in $B^2$, that the hyperbolic parallel axiom is satisfied. Both lines $n$ and $m$ through $p$ do not intersect to the line $l$. This is shown in Figure 2.14.
Next, we will look at hyperbolic spheres of $B^n$. Similar to how we examined spheres of $U^n$, hyperbolic spheres of $B^n$ are Euclidean spheres, only with different centers and radii.

**Definition 2.2.5.** The hyperbolic sphere of $B^n$ with center $a$ and radius $r > 0$, is defined to be the set

$$ S_B(a, r) = \{ x \in B^n \mid d_B(a, x) = r \}. $$

The following example shows this in $B^2$.

**Example 2.2.6.** In $B^2$, Let $S = \left( \frac{1}{2} + \frac{1}{2}i, \frac{1}{3} \right) = \{ x \in B^2 \mid d_B\left(\frac{1}{2} + \frac{1}{2}i, x\right) = \frac{1}{4} \}$. Observe once again, $S$ is also an Euclidean circle just with different center and radius.
This section will end with the definition and an example of horospheres of $B^n$. The example will be given in $B^2$.

**Definition 2.2.7.** A *horosphere* centered at $b \in S^{n-1}$ is an Euclidean sphere tangent, at $b$, to $S^{n-1}$ in $B^n$.

**Example 2.2.8.** A horosphere is sometimes called a horocycle when in two dimensions. Here, in $B^2$, the horocycle $S$ is tangent to the boundary of $B^2$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

![Figure 2.16: Horocycle of $B^2$](image)
Chapter 3

Isometries of $U^n$ and $B^n$

Christian Felix Klein (1849-1926) was a German mathematician who introduced a new take on geometry and how different geometries could be classified. His work in the 1870’s allows us to approach a geometry not as a set of axioms, and their consequences, but as a space together with a group of transformations acting on the space. Any properties remaining invariant by the group transformations would become the geometric properties of that space. If a transformation is not changing the properties of the shapes in the space, then we might be looking at the group of isometries of that space. It was Kleins view that connected the mathematics of group theory and geometry. This is the approach we will take while studying the metric spaces $U^n$ and $B^n$. In this section we will investigate inversions, the group of Möbius transformations, and their relationship to hyperbolic geometry by looking at examples in $U^2$ and $B^2$. (Throughout this chapter we cite [Bon09], [Bra99], and [Rat94].)

3.1 Inversion

Before talking about Möbius transformations, it is important to learn about the type of transformation known as inversion. In summary, inversion is a reflection in a line or a circle. These inversions map points from one side of a sphere of $E^n$ to the other. Let us first look at hyperplanes of $E^n$ defined as $(n - 1)$-planes of $E^n$ such that

$$P(a, t) = \{x \in E^n \mid a \cdot x = t\}.$$
In this definition, \( \vec{a} \) is a unit vector in \( E^n \), which is normal to \( P(a,t) \). Moreover, \( t \) is the scalar multiple of \( a \) that lies on \( P(a,t) \). The following two examples give us a visual of hyperplanes of \( E^2 \) and \( E^3 \), respectively.

**Example 3.1.1.** Let \( P \) be the hyperplane \( P((1,0), 2) = \{(x,y) \in E^2 \mid (1,0) \cdot (x,y) = 2\} \) in \( E^2 \).

\[ E^2 \quad \begin{array}{c|c}
2 & P((1,0),2) \\
1 & \\
1 & \end{array} \]

Figure 3.1: Hyperplane of \( E^2 \)

**Example 3.1.2.** Let \( P \) be the general hyperplane \( P(a,t) \) in \( E^3 \).

\[ E^3 \]

Figure 3.2: Hyperplane of \( E^3 \)
Now that we know what a hyperplane is, let’s look at a *reflection* in a hyperplane. These *reflections* turn out to be Euclidean reflections as we all know them! These next two examples are *reflections* in hyperplanes of $E^2$ and $E^3$, respectively.

**Example 3.1.3.** Let $P$ be the hyperplane $P((1, 0), 2)$ in $E^2$ and $\rho(x)$ be the reflection of $x$ in $P((1, 0), 2)$.

![Figure 3.3: Reflection in a Hyperplane](image)

**Example 3.1.4.** Let $P$ be the general hyperplane $P(a, t)$ in $E^3$ and $\rho$ be the reflection of $x$ in $P(a, t)$.

![Figure 3.4: Reflection in a General Hyperplane of $E^3$](image)
Generally, we will define a reflection $\rho$ of $E^n$ in the plane $P(a, t)$ by the formula

$$\rho(x) = x + (t - a \cdot x)a.$$ 

As mentioned above, inversions are reflections in lines or circles. Up until now we have only taken a look at reflections in a line (or hyperplane). Now let us take a look at inversions in a Euclidean circle. Let $C$ be a circle with center $O$ and radius $r$, and let $A$ be any point other than $O$. If $A'$ is on the ray $\overrightarrow{OA}$ and satisfies the equation $OA \cdot OA' = r^2$, then we call $A'$ the inverse of $A$ with respect to the circle $C$ where $C$ is the circle of inversion. The transformation

$$\sigma(A) = A'$$

is known as inversion in $C$.

![Figure 3.5: Inversion in a circle C](image)

Suppose we have the sphere $S(a, r)$ of $E^n$ such that $S(a, r) = \{ x \in E^n \mid \|x - a\| = r \}$, then we will use the following formula for inversion in a sphere:

$$\sigma(x) = a + \left(\frac{r}{\|x - a\|}\right)^2 (x - a).$$

The term inversion will now mean either reflection in a hyperplane or inversion in a sphere. We will now give an example showing two inversions in $U^2$.

**Example 3.1.5.** Let $\sigma_1$ be the inversion in $l_1$ and $\sigma_2$ be the inversion in $l_2$. See Figure 3.6. Notice that the inversion in $l_1$ sends points within the arc to points outside the arc and the inversion in $l_2$ is the honest-to-goodness Euclidean reflection about $l_2$. 

▲
3.2 Möbius Transformations of $U^n$

In this section we will investigate Möbius transformations of $U^n$ by looking at examples in $U^2$. We will also look at the algebraic properties of these Möbius transformations in this space. First let us define a sphere of $\Sigma$ of $\hat{E}^n$ to be either a Euclidean sphere $S(a,r)$ or an extended plane $\hat{P}(a,t) = P(a,t) \cup \infty$. Note that $\hat{P}(a,t)$ is topologically a sphere. We will first define a Möbius transformation.

**Definition 3.2.1.** A Möbius transformation of $\hat{E}^n$ is a finite composition of inversions of $\hat{E}^n$ in spheres.

It turns out Möbius transformations of $U^n$ are Möbius transformations of $\hat{E}^n$ that leave $U^n$ unchanged. If we use the complex definition of $U^2$, these Möbius transformations turn out to be isometries of $U^2$ known as linear fractional transformations.

**Theorem 3.2.2.** The isometries of the $U^2$ are linear fractional transformations of the form:

1. $\phi(z) = \frac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$,

2. and $\phi(z) = \frac{cz + d}{az + b}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Now since inversions are just reflections and it is a well known fact that every isometry in $E^n$ can be made up of a composition (or product) of reflections, then it is safe to say every isometry of $H^n$ is a composition of at most $n + 1$ inversions. This well-known result can be proven by showing such linear fractional transformations can be
written as a composition of inversions. Rather than provide a proof, we will illustrate with an example that the composition of two particular inversions result in a linear fractional transformation. Let us take a look at the inversions in Example 3.1.5 and find a Möbius transformation, \( \phi \), such that \( \phi = \sigma_2 \circ \sigma_1 \). First we will assign some values. Let \( l_1 = \{ z = x + yi \mid |z| = 1 \text{ and } y > 0 \} \). Then the inversion in \( l_1 \) is defined as \( \sigma_1(z) = \frac{1}{\bar{z}} \).

Also let \( l_2 = \{ z = 4 + yi \mid y > 0 \} \). Then the inversion in \( l_2 \) is defined as \( \sigma_2(z) = -\bar{z} + 8 \).

Now since \( \phi = \sigma_2 \circ \sigma_1 \), this implies \( \phi(z) = \frac{8z - 1}{z} \). Notice that this transformation satisfies the definition of linear fractional transformations, \( ad - bc = (8)(0) - (-1)(1) = 1 \).

The Möbius transformation, \( \phi \), is shown in the following image.

![Figure 3.7: Möbius Transformation, \( \phi \), of \( U^2 \)](image)

### 3.3 Möbius Transformations of \( B^n \)

The linear fractional transformation \( \phi(z) = \frac{-z + i}{z + i} \) induces an isometry from \( U^2 \) to \( B^2 \). This mapping, known as the standard transformation, is useful to take what we know about \( U^2 \) and find out what it looks like in \( B^2 \). We will now look at Möbius transformations of \( B^n \). In this subsection we will investigate Möbius transformations of \( B^n \) by looking at examples in \( B^2 \). We will also look at the algebraic properties of these transformations in this space. Inversions in \( B^n \) are no different than inversions of \( U^n \): reflections in planes or inversions in spheres. Similar to \( U^n \), we will use inversion to denote both reflections and inversions. We will look at examples of two different types of inversions of \( B^2 \).
Example 3.3.1. Let $\sigma_1$ be the inversion in $l_1$ and $\sigma_2$ be the inversion in $l_2$.

Figure 3.8: Inversions of $B^2$

Now it turns out M"{o}bius transformations of $B^n$ are M"{o}bius transformations of $\hat{E}^n$ that leave $B^n$ unchanged. If we use the complex definition of $B^2$, these M"{o}bius transformations are also isometries.

Theorem 3.3.2. The isometries of the $B^2$ are linear fractional transformations of the form:

1. $\phi(z) = \frac{az + b}{bz + \bar{a}}$ with $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$,

2. and $\phi(z) = \frac{a\bar{z} + b}{b\bar{z} + \bar{a}}$ with $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$.

These can be derived by using the mapping $\phi(z) = \frac{-z + i}{z + i}$ on the linear fractional transformations of $U^2$. Let us now take a look at the inversions in Example 20 and find a M"{o}bius transformation, $\phi$, such that $\phi = \sigma_2 \circ \sigma_1$. We will do this in the same fashion as finding $\phi$ in the previous subsection. Let $l_1 = \{(x, y) \cap B^2 \mid -1 < x < 1 \text{ and } y = 0\}$. Then the inversion in $l_1$ is defined as $\sigma_1(z) = \bar{z}$. Now let $l_2 = \{(x, y) \cap B^2 \mid x^2 + y^2 - 2\sqrt{2} + 1 = 0\}$. Then the inversion in $l_2$ is defined as $\sigma_2(z) = \frac{\sqrt{2}z - 1}{\bar{z} - \sqrt{2}}$. Now since $\phi = \sigma_2 \circ \sigma_1$, this implies $\phi(z) = \frac{\sqrt{2}z - 1}{z - \sqrt{2}}$.
that this transformation satisfies the definition of linear fractional transformations of $B^2$, $|\sqrt{2}|^2 - |-1|^2 = 1$. The Möbius tranformation, $\phi$, is shown in the following image.

![Figure 3.9: Möbius Transformation, $\phi$, of $B^2$](image)

There are different types of Möbius transformations. In the next section, we will classify these transformations and explain their properties with examples in $U^2$ and $B^2$.

### 3.4 Parabolic, Hyperbolic, and Elliptic Transformations

Here, we will look at different Möbius transformations and the behavior they may have. There are three types of Möbius transformations: parabolic, hyperbolic, and elliptic. The difference bewteen these transformations are the number of fixed points and their locations. To help explain, we will observe the Möbius transformations of $U^2$. First of all, it is obvious that the identity map $\phi(z) = z$ fixes all the points, but there are other maps that fix other particular points. To find these fixed points we will have to solve the equation

$$\phi(z) = \frac{az + b}{cz + d} = z.$$ 

Using the Quadratic Formula, we get the resulting solution:

$$z = \frac{a - d \pm \sqrt{(d - a)^2 + 4bc}}{2c}.$$  

This result gives us three cases depending on the determinant, $(d - a)^2 + 4cb$. Recall, however, that one of the properties of these Möbius transformations is $ad - bc = 1$. Using
substition, we have a new form for our determinant: \((a + d)^2 - 4\). Now we will describe
the difference between the three transformations mentioned above. If \((a + d)^2 = 4\) we
have a parabolic transformation which gives a unique fixed real number \(z = \frac{a - d}{2c}\). If
\((a + d)^2 > 4\) then we have a hyperbolic transformation giving two real fixed points. If
\(0 < (a + d)^2 < 4\) then we have an elliptic transformation which has two complex fixed
points. Generally, a transformation is said to be

1. parabolic if \(\phi\) fixes no points of \(U^n\) or \(B^n\) and fixes a unique point in their boundaries;

2. hyperbolic if \(\phi\) fixes no points of \(U^n\) or \(B^n\) and fixes two unique points in their
   boundaries;

3. elliptic if \(\phi\) fixes at least one point of \(U^n\) or \(B^n\).

The rest of this section will demonstrate various Möbius transformations of both \(U^2\) and
\(B^2\).

Example 3.4.1. Let \(\phi(z) = z - 4\) and \(z \in R\) such that \(R = \{z = x + iy \mid 1 < x < 2, y > 0\}\). Then \(\phi(z)\) is as shown below. The transformation \(\phi\) is parabolic in \(U^2\) because it

![Figure 3.10: Parabolic Transformation of \(U^2\)](image)

only fixes the point at infinity.

Example 3.4.2. Let \(\phi(z) = \frac{(122 + 45i)z + (-88 - 60i)}{(-88 + 60i)z + (122 - 45i)}\). Then \(\phi(z)\) is a composition of
inversions in \(l_1\) and \(l_2\) as shown below in Figure 3.11. The transformation \(\phi\) is parabolic
in \(B^2\) because it only fixes the common point of lines \(l_1\) and \(l_2\) on the boundary of \(B^2\).
Example 3.4.3. Let $\phi(z) = \frac{1}{2}z$ and $z \in R$ such that $R = \{re^{i\theta} | 1 < r < 2, 0 < \theta < \pi\}$. Then $\phi(z)$ is as shown below in Figure 3.12. The transformation $\phi$ is hyperbolic in $U^2$ because it only fixes the point at infinity and the origin.

Example 3.4.4. Let $\phi(z)(z) = \frac{(91 + 9i)z + (-63 - 12i)}{(-63 + 12i)z + (91 - 9i)}$. Then $\phi(z)$ is as shown in Figure 3.13. The transformation $\phi$ is hyperbolic in $B^2$ because it fixes two points on the boundary of $B^2$. These points turn out to be the end points, $a$ and $b$, of the unique common perpendicular, $l_3$, of $l_1$ and $l_2$ as shown in Figure 3.14.
Example 3.4.5. Let $\phi(z) = \frac{1}{z}$ and $z \in R$ such that $R = \{r e^{i\theta} \mid 0 < r < 1, 0 < \theta < \pi\}$. Then $\phi(z)$ is as shown in Figure 3.15. The transformation $\phi$ is elliptic in $U^2$ because it only fixes the arc of the circle-of-inversion.
Example 3.4.6. Let $\phi(z) = \frac{(247 + 174i)z + (-100 - 232i)}{(-100 + 232i) + (247 - 174i)}$. Then $\phi(z)$ is as shown below in Figure 3.16. The transformation $\phi$ is elliptic in $B^2$ because it only fixes the point at which $l_1$ and $l_2$ intersect.

![Figure 3.16: Elliptic Transformation of $B^2$](image)
Chapter 4

Topogical Groups

In this chapter we will study a bit about topological groups and their structures related to hyperbolic n-space. We will also look into discrete groups and discontinuous groups. This chapter will definitely prepare us to think about geometry in the same way Klein did, a group acting on a space. In Chapter 1 we described two models of hyperbolic n-space while Chapter 2 talked about isometries of hyperbolic n-space. We will begin to look at the set of isometries of $H^n$ as a group. (Throughout this chapter we cite [Rat94].)

4.1 Topological Groups

In this section, we will define transformational groups, topological groups, and quotient topological groups. First let us define a transformational group.

Definition 4.1.1. A transformational group on a set $X$ is a family $\Gamma$ of bijections $\gamma : X \to X$ such that

1. if $\gamma$ and $\gamma'$ are in $\Gamma$, their composition is also in $\Gamma$;
2. the group $\Gamma$ contains the identity map $Id_X$;
3. if $\gamma$ is in $\Gamma$, its inverse map $\gamma^{-1}$ is also in $\Gamma$.

Generally, a set of isometries from a metric space $X$ to itself, together with multiplication defined by composition, forms a group $I(X)$, called the group of isometries of $X$. This notation implies $I(H^n)$ is the group of isometries of hyperbolic $n$-space. With this in mind, we will now define a topological group and give some examples.
Definition 4.1.2. A topological group is a group $G$ that is also a tolopogical space such that the multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ in $G$ are continuous functions.

Example 4.1.3. The real $n$-space $\mathbb{R}^n$ with the operation of vector addition is a topological group.

Example 4.1.4. The complex $n$-space $\mathbb{C}^n$ with the operation of vector addition is a topological group.

Example 4.1.5. The projective special linear group $\text{PSL}(2, \mathbb{R})$ is a topological group.

It is worth mentioning a few things about quotient topological groups. Suppose $H$ is a subgroup of $G$, a topologicial group. Then the coset space $G/H = \{gH \mid g \in G\}$ is equipped with the quotient topology. We will define the quotient map by $\pi : G \to G/H$.

Theorem 4.1.6. If $N$ is a normal subgroup of a topological group $G$, then $G/N$, with the quotient topology is a topogical group.

Example 4.1.7. The projective general linear group $\text{PGL}(n, \mathbb{C}) = \text{GL}(n, \mathbb{C})/N$, where $N$ is the normal subgroup $\{kI \mid k \in \mathbb{R}^*\}$ is a quotient topological group.

4.2 Discrete Subgroups

The next group that we will investigate is the discrete group. We will also define a group action and a give an example of an orbit in $U^2$. Discrete subgroups of continuous groups will be of particular interest. After defining the discrete group we will provide some examples of discrete subgroups.

Definition 4.2.1. A discrete group is a topological group $\Gamma$ all of whose points are open. This means a group $\Gamma$ is a discrete group so long as $\Gamma$ equipped with the discrete topology.

Example 4.2.2. The integeres $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$. (Q is not)
Example 4.2.3. $\mathbb{Z}[i] = \{m + ni : m, n \in \mathbb{Z}\}$ is a discrete subgroup of $\mathbb{C}$.

Example 4.2.4. The modular group $PSL(2n, \mathbb{Z}) = SL(2n, \mathbb{Z})/\{\pm I\}$ is a discrete subgroup of $PSL(2n, \mathbb{R})$.

Let us now define what it means for a group to act on a set. With this definition we move closer to Klein’s view of geometry as explained in his work.

**Definition 4.2.5.** A group $G$ acts on a set $X$ if and only if there is a function from $G \times X$ to $X$, written $(g, x) \mapsto gx$, such that for all $g, h \in G$ and $x \in X$, we have

1. $1 \cdot x = x$ and
2. $g(hx) = (gh)x$.

A function from $G \times X$ to $X$ satisfying (1) and (2) is called an action of $G$ on $X$.

We now essentially have what we need to study Klein’s program: knowledge of groups and actions. Let $G$ be a group acting on a set $X$ and let $x \in X$. We define the following terms:

1. the subgroup $G_x = \{g \in G \mid gx = x\}$ of $G$ is called the stabilizer of $x \in G$;
2. the subset $Gx = \{gx \mid g \in G\}$ of $X$ is called the $G$-orbit through $x$;
3. if $\phi : G/G_x \to Gx$ by $\phi(gG_x) = gx$, then $\phi$ is a bijection and therefore the index $G_x$ in $G$ is the cardinality of the orbit $G_x$. This is in reference to the familiar Orbit-Stabilizer Theorem.

**Example 4.2.6.** Let $\phi(z) = \frac{1}{2}z$ and $G = \langle \phi \rangle$ be the cyclic group generated by $\phi$. Then the orbit of the point $p$ generated by $\langle \phi \rangle$ is shown in the image below in Figure 4.1. It consists of all scalar multiples of $p$ by a power of two.
4.3 Discontinuous Groups

Finally we take a look at another interesting type of group: the discontinuous

groups. Let us first define a discontinuous action and a homeomorphism.

Definition 4.3.1. A group $\Gamma$ acts discontinuously on a topological space $X$ if and only

if $\Gamma$ acts on $X$ and for each compact subset $K$ of $X$, the set $K \cap gK$ is nonempty for only

finitely many $g$ in $\Gamma$.

The previous action in the last example is discontinuous. In the previous example, the group $\langle \phi \rangle$ acts discontinuously on $U^2$ because we can take a compact subset $K$ of $U^2$, and there will only be a finite amount of points from the orbit of $p$ in $K \cap gK$, for $g \in \Gamma$. Any compact subset $K$ of $U^2$ will be closed and bounded. Therefore there will be a maximum and minimum distance from $K$ to $(0,0)$. Let $M=$ maximum distance and $m=$ minimum distance. As long as $2^N m > M$, $\phi^N(K)$ will be farther from $(0,0)$ than $K$, so $\phi^N(K) \cap K = \phi$. Similarly, as long as $2^n M < m$, we will have $\phi^n(K) \cap K = \phi$. Hence only finitely many elements $g$ in $\langle \phi \rangle$ have $gK \cap K \neq \phi$. The following example, however, is not of a discontinuous action.

Example 4.3.2. Let $\phi(z) = rz$ for all $r \in \mathbb{R}$ and $\Gamma = \langle \phi \rangle$ be the cyclic group generated by $\phi$. In this case, the orbit of a point $p$ is the ray originating from the origin through $p$ (not including the origin of course) as shown below in Figure 4.2. The action that $\Gamma$ acts on $U^2$ is not discontinuous because there will be an infinite amount of $g \in \Gamma$ for the nonempty set $K \cap gK$ where $K$ is a compact subset of $U^2$. Note that $\Gamma$ is a subgroup consisting of all hyperbolic isometries fixing the origin and the point at infinity. This will

Figure 4.1: Orbit of $U^2$
be used again in the next chapter.

\[
\begin{array}{c}
U^2 \\
-2 & -1 & 0 & 1 & 2 & E^1 \\
p
\end{array}
\]

Figure 4.2: Discontinuous Group Action

**Definition 4.3.3.** Let \( X \) and \( Y \) be topological spaces; let \( f : X \to Y \) be bijection. If both the function \( f \) and the inverse function \( f^{-1} : Y \to X \) are continuous, then \( f \) is called a *homeomorphism*.

We will can now define what it means to be a *discontinuous group*.

**Definition 4.3.4.** A group \( G \) of homeomorphisms of a topological space \( X \) is *discontinuous* if and only if \( G \) acts discontinuously on \( X \).

It turns out that *discrete groups* and *discontinuous groups* are equivalent when considering \( I(H^n) \). Discrete subgroups are defined soley in terms of a topology on the group. Discontinuous subgroups are defined in terms of the action of a group on a set. For \( I(H^n) \), these two definitions coincide.
Chapter 5

Geometry: Fundamental Regions

We have now come to our final chapter. We will define fundamental regions, fundamental domains, and Dirichlet domains and make relationships between these concepts. To help visualize these concepts, examples in $U^2$ will be given using familiar points, lines, regions, and actions. We will end by seeing how fundamental domains characterize discrete subgroups of $I(U^2)$ and how they can help us visualize group actions of $I(U^2)$. Throughout this chapter we cite [Bon09] and [Rat94].

5.1 Fundamental Regions and Fundamental Domains

Let us start by first defining a fundamental region.

**Definition 5.1.1.** A subset $R$ of a metric space $X$ is a fundamental region for a group $\Gamma$ of isometries of $X$ if and only if

1. the set $R$ is open in $X$;
2. the members of $\{gR : g \in \Gamma\}$ are mutually disjoint; and
3. $X = \cup \{g\bar{R} : g \in \Gamma\}$. Here, $\bar{R}$ is the closure of $R$.

**Definition 5.1.2.** A subset $D$ of a metric space $X$ is a fundamental domain for a group $\Gamma$ of isometries of $X$ if and only if $D$ is a connected fundamental region for $\Gamma$.

**Example 5.1.3.** Let $\phi(z) = \frac{1}{2}z$ and $\Gamma = \langle \phi \rangle$ be the group action generated by $\phi$ in $U^2$. Then the region $R = \{re^{i\theta} | 1 < r < 2, 0 < \theta < \pi\}$ is a Fundamental Domain for $\Gamma$. 
Similar to Example 1, we can derive $\phi^n(z) = \left(\frac{1}{2}\right)^n z$.

Figure 5.1: Fundamental Domain, $R$

(1) $R$ is open in $U^2$ by the way $R$ is defined above.

(2) All the members of $\{gR : g \in \Gamma\}$ are mutually disjoint. This is true since the boundaries of $R$ are not in $R$ and $\Phi^n(R)$ is between circles with radius $\left(\frac{1}{2}\right)^{n-1}$ and $\left(\frac{1}{2}\right)^n$. We can see this in the following figure.

Figure 5.2: Mutually Disjoint $gR$ for all $g \in \Gamma$

(3) $U^2 = \bigcup\{g\bar{R} : g \in \Gamma\}$. This is true since the only pieces missing are circles with radii $\left(\frac{1}{2}\right)^n$ which are in $\bar{R}$. We can see this in the following figure.
Example 5.1.4. Let $\phi(z) = \frac{1}{2}z$ and $\Gamma = \langle \phi \rangle$ be the group action generated by $\phi$ in $U^2$. Let $R_1 = \{z = re^{i\theta} \mid 1 < r < 2, \frac{\pi}{2} < \theta < \pi\}$ and $R_2 = \{z = x + iy \mid 1 < x < 2, y > 0\}$. Then the region $R = R_1 \cup R_2$ is a fundamental region for $\Gamma$. We can derive $\Phi^n(z) = \left(\frac{1}{2}\right)^n z$. 

(1) $R$ is open in $U^2$ by the way $R$ is defined above.

(2) All the members of $\{gR : g \in \Gamma\}$ are mutually disjoint. This is true since the boundaries of $R$ are not in $R$ and images tessellate. We can see this in Figure 5.5.

(3) $U^2 = \cup\{g\bar{R} : g \in \Gamma\}$. This is true since the boundaries of $R$ are in $\bar{R}$. We can see
this in Figure 5.6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure55.png}
\caption{Mutually Disjoint $gR$ for all $g \in \Gamma$, $R$ as a Fundamental Region}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure56.png}
\caption{$U^2 = \bigcup g\bar{R}$ for all $g \in \Gamma$, $R$ as a Fundamental Region}
\end{figure}

\textbf{Theorem 5.1.5}. If a group $\Gamma$ of isometries of a metric space $X$ has a fundamental region, then $\Gamma$ is a discrete subgroup of the groups of isometries of $X$, $I(X)$.

\textbf{Proof}: Let $x$ be a point of a fundamental region $R$ for a group of isometries $\Gamma$ of a metric space $X$. Then the evaluation map

$$\varepsilon : \Gamma \to \Gamma x,$$

defined by $\varepsilon(g) = gx$, is continuous. Now the point $x$ is open in $\Gamma x$, since $R \cap \Gamma x = \{x\}$. This is true by (2) of the definition of a fundamental region. Moreover, the stabilizer
Γₙ is trivial, again by the definition of a fundamental region. Now suppose that \{1\} is open. Let \(g\) be in \(Γ\). Then left multiplication by \(g\) is a homeomorphism of \(Γ\). Since homeomorphism’s map open sets to open sets, \(g\{1\} = \{g\}\) is open in \(Γ\). This implies \(1 = \varepsilon^{-1}(x)\) is open in \(Γ\) by how \(ε\) is defined above. Therefore \(Γ\) is discrete.  

5.2 Dirichlet Domains

Now that we have looked at fundamental regions and fundamental domains, we will study what it means to be a Dirichlet domain. Let \(Γ\) be a discontinuous group of isometries of the metric space \(H^n\), and let \(u\) be a point of \(H^n\) whose stabalizer \(Γ_u\) is trivial. For each \(g \neq 1 \in Γ\), define

\[ H_g(u) = \{ x \in X \mid d(x, u) < d(x, gu) \}. \]

It turns out, \(H_g(u)\) is the open half-space of \(H^n\) containing \(u\) whose boundary is the perpendicular bisector of the geodesic segment joining \(u\) to \(gu\). The Dirichlet domain \(D(u)\) for \(Γ\), with center \(u\), is either \(H^n\) if \(Γ\) is trivial or

\[ D(u) = \cap \{ H_g(u) \mid g \neq 1 \in Γ \} \]

if \(Γ\) is nontrivial.

Example 5.2.1. Let \(X\) be the ideal hyperbolic square below in Figure 5.7. If \(Γ = \langle φ_1(z) = \frac{z + 1}{\bar{z} + 2}, φ_2(z) = \frac{z - 1}{\bar{z} - 2} \rangle\), then the orbit of \(i \in X\) generated by \(Γ\) is as shown in Figure 5.8. Now by finding all the perpendicular bisectors of the segments \([i, φ(i)]\) for \(φ \in Γ\) (Figure 5.9), we derive the Dirichlet domain, \(D(i)\), shown below in Figure 5.10.

![Figure 5.7](image_url)

![Figure 5.8](image_url)

![Figure 5.9](image_url)

![Figure 5.10](image_url)

Theorem 5.2.2. Let \(D(u)\) be the Dirichlet domain, with center \(u\), for a discrete subgroup \(Γ\) of \(I(H^n)\), then \(D(u)\) is a locally finite fundamental domain for \(Γ\).

(The proof of this theorem could be found in [Rat94].)

Thus, discrete subgroups of \(I(H^n)\) are precisely those which admit a fundamental domain.
Figure 5.7: Ideal Hyperbolic Square

Figure 5.8: Orbit of \( i \) in the Ideal Hyperbolic Square

Figure 5.9: Perpendicular Bisectors of \([i, \phi(i)]\)
Figure 5.10: Dirichlet Domain, $D(i)$
5.3 Visualizing Quotient Spaces

Now, using what we know about the quotient topology and group actions, an informal definition of a *quotient space* will be given. Let $\Gamma$ be the topological group acting on a space $X$. If we say points on $X$ are equivalent if and only if they lie in the same orbit, then $X/\Gamma$ is the *quotient space* of $X$ under the action of $\Gamma$. Fundamental domains are useful because they allow us to intuitively visualize $X/\Gamma$. The following examples are of well known fundamental domains: the torus (using $E^2$) and the punctured torus (using $U^2$).

**Example 5.3.1.** First, let $X = E^2$ and $\Gamma$ be the subgroup of $I(E^2)$ consisting of translations by vectors with integral components. In other words, let

$$\phi_{m,n} : E^2 \rightarrow E^2$$

be defined by

$$\phi_{m,n} : (x, y) \rightarrow (x + m, y + n),$$

then $\Gamma = \langle \phi_{m,n} \rangle$ where $m, n \in \mathbb{Z}$. The orbit of $(0, 0)$ under the action of $\Gamma$ is the “lattice” $\{(m, n) \mid m, n \in \mathbb{R} \}$, shown below in Figure 5.11. Similarly, the orbit of any point will be a lattice. The set $R = \{(x, y) \mid 0 < x, y < 1\}$ in $X$ forms a fundamental domain for the group $\Gamma$ (as mentioned above). In fact, $\phi_{m,n}$ translates $X$ to be an open-unit-square

![Figure 5.11: Lattice of points in $E^2$](image)
with \((m, n)\) as the lower left corner. Now \(R\) is open in \(X\) and \(R \cap gR\) is the empty for all \(g \in \Gamma\) that are not the identity satisfying properties (1) and (2) of fundamental domains, and \(\bar{R}\) is the closed unit square and tessellates \(X\). So \(x = \cup g\bar{R}\) satisfying property (3) of fundamental domains. Thus, the quotient space \(X/\Gamma\) can be visualized in the following way:

Suppose the orbit of \((0, 0)\) in \(X\) in the above example is not included. Then \(X/\Gamma\) becomes the punctured torus below in Figure 2.13. Topologically, this is the same as the punctured torus in the next example.
Example 5.3.2. Let $X = U^2$ and $\Gamma$ be the subgroup of $I(U^2)$ generated by $\phi_1$ and $\phi_2$ where

$$\phi_{1,2} : U^2 \rightarrow U^2$$

such that

$$\phi_1(z) = \frac{z + 1}{z + 2} \text{ and } \phi_2(z) = \frac{z - 1}{-z + 2}.$$ 

The orbit of $i$ under the action of $\Gamma$ can be seen in a previous example (Ex. 38). The ideal hyperbolic square is a fundamental domain. Let $R \in X$ be the hyperbolic square and $\Gamma = \langle \phi_1, \phi_2 \rangle$. Then the quotient space $X/\Gamma$ gives us the hyperbolic punctured torus shown in Figure 5.14. Note, the “boundary curve” of $X/\Gamma$, the punctured point, is represented in the fundamental domain by the four hyperbolic lines $l_1$, $l_2$, $l_3$, and $l_4$ (Figure 5.15). As they move closer to the ideal points, the total length of these arcs moves closer to 0. This means the length of the curve around the punctured point in $X/\Gamma$ tends to zero as you move towards infinity.

Figure 5.14: Hyperbolic Punctured Torus

Figure 5.15: Punctured Point of the Hyperbolic Punctured Torus
Chapter 6

Conclusion

It was Klein who thought of studying a geometry as a space with a group acting on the space. In our past two examples, we looked at the geometry of the Euclidean torus as the set $\mathbb{E}^2$ together with the group $\Gamma = \langle \phi_{m,n} \rangle$ acting on it and the geometry of the hyperbolic punctured torus as the set $U^2$ together with the group $\Gamma = \langle \phi_1, \phi_2 \rangle$ acting on it. Thus we see that the algebraic description of Klein also provides us with a geometric model. In this paper we first looked at hyperbolic $n$-space by studying two different models: the open- upper-half space and the open-bal-modell. Then we looked at the isometries of these spaces and the topological properties used so that these groups of isometries can act on these spaces. With this knowledge, we can now study the geometry of quotient spaces both by looking at discrete subgoups and by looking at their fundamental domains.
Bibliography


