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The Irreducible Representations of D2n

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A Thesis

Presented to the

Faculty of

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In Partial Fulfillment

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Master of Arts

in

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by

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Abstract

Irreducible representations of a finite group over a field are important because all representations of a group are direct sums of irreducible representations. Maschke tells us that if ϕ is a representation of the finite group G of order n on the m-dimensional space V over the field K of complex numbers and if U is an invariant subspace of ϕ, then U has a complementary reducing subspace W.

The objective of this thesis is to find all irreducible representations of the dihedral group D_{2n}. The reason we will work with the dihedral group is because it is one of the first and most intuitive non-abelian group we encounter in abstract algebra. I will compute the representations and characters of D_{2n} and my thesis will be an explanation of these computations. When $n = 2k + 1$ we will show that there are $k + 2$ irreducible representations of D_{2n}, but when $n = 2k$ we will see that D_{2n} has $k + 3$ irreducible representations. To achieve this we will first give some background in group, ring, module, and vector space theory that is used in representation theory. We will then explain what general representation theory is. Finally we will show how we arrived at our conclusion.

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Chapter 1

Background

In order to *understand group representation* theory, or *linear representation theory*, it is important to be well versed in many areas of abstract algebra: group theory, ring theory, module theory, and vector space theory. In this chapter we will highlight many of the results that are needed to understand group representation theory. We begin by looking at group theory.

1.1 Group Theory

We start this section by defining a group.

Definition 1.1.1. *A* group *is an ordered pair* $(G, \cdot)$, *where* G *is a set and* $\cdot$ *is a binary operation on* G *satisfying the following axioms:*

(i) $\forall a, b \in G$ *we have that* $a \cdot b \in G$,

(ii) $\cdot$ *is associative,*

(iii) $\exists e \in G$, *called an identity in* G *such that* $\forall a \in G$ *we have* $a \cdot e = e \cdot a = a$,

(iv) $\forall a \in G$ *there exists an element* a^{-1} *of* G, *called an inverse of* a, *such that* $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Note that we usually write ab instead of $a \cdot b$ when dealing with multiplcative groups. A group G is called *abelian* if $ab = ba$ $\forall a, b \in G$. We also say that a group G is $finite$ if G is a finite set. If $H \subseteq G$ then we say H is a subgroup of G, denoted by

$H \leq G$, if H is a group under $\cdot$. A quick way to determine if we have a group is the following test:

Proposition 1.1.1 (One Step Subgroup Test)**.** *A nonempty subset H of the group G is a subgroup of G iff $xy^{-1} \in H\ \forall x, y \in H$.*

Given a group G and an element $g \in G$ we define the *cyclic group* generated by g as follows: $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$. Note the following fact:

Proposition 1.1.2. *A cyclic group is abelian.*

It will be important for us to determine when two groups "look the same" and we will say that two groups that "look the same" are isomorphic. Let us define this more precisely by first defining the notion of a homomorphism:

Definition 1.1.2 (Group Homomorphism)**.** *Let $(G, \cdot)$ and $(H, *)$ be groups. Then a map $\phi : G \to H$ is called a* group homomorphism *iff $\phi(x \cdot y) = \phi(x) * \phi(y)\ \forall x, y \in G$.*

The *kernel* of a group homomorphism $\phi : G \to H$ is the set $ker\phi = \{g \in G : \phi(g) = e_H\}$. The *image* of a group homomorphism ϕ is given by the set $im\phi = \{\phi(g) : g \in G\}$. We say that a group homormorphism ϕ is a *group isomorphism* if ϕ is a group homomorphism that is also bijective. We note the following useful result.

Proposition 1.1.3. *Let $\phi : G \to H$ be a group homomorphism. Then $ker\phi \leq G$ and $im\phi \leq H$.*

A group has special subgroups called *normal subgroups* that allow us to create *quotient groups*. Before we define this notion let us define what a coset is. Then we will define the order of a group and give an important result called $Lagrange's\ Theorem$ which relates the order of a group with the order of its subgroups.

Definition 1.1.3. *Let $H \leq G$. For any $a \in G$ define $aH = \{ah : h \in H\}$. The set aH is called the* left coset *of H in G containing a.*

Definition 1.1.4. *The* order *of a group G is the cardinality of the group G and is denoted by $|G|$.*

Theorem 1.1.1 (Lagrange's Theorem)**.** *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.*

Now let $N \leq G$. We say that N is *normal* in G if $\forall x \in N$ and $\forall g \in G$ we have that $gxg^{-1} \in N$ and we denote this by writing $N \triangleleft G$. The following is a major result that highlights the importance of normal subgroups.

Theorem 1.1.2 (Quotient Groups)**.** *Let G be a group and let $N \triangleleft G$. Then the set $G/N = \{gN : g \in G\}$ is a group under the operation $(aN)(bN) = abN\ \forall a, b \in G$. We call G/N a* quotient group *of G.*

The following is an important result that gives the connection between group homomorphisms and quotient groups.

Theorem 1.1.3 (First Isomorphism Theorem)**.** *Let $\phi : G \to H$ be a group homomorphism. Then $ker\phi \triangleleft G$ and $G/ker\phi \cong im\phi$.*

Note then that if the kernel of ϕ is trivial then ϕ is one-to-one, or *injective*. Also if $im\phi = H$ and $ker\phi = 1_G$ then ϕ is a group isomorphism. Also note that if $N \triangleleft G$, then the map $G \to G/N$ defined by $g \mapsto gN$ is a surjective group homomorphism with kernel N. This map is called the *natural projection* of G onto G/N. Also note that the left cosets of H in G for any subgrop H of G *partition* G, i.e., we can write G as a disjoint union of its left cosets. Another way to partition G is through its *conjugacy classes*, which we define next.

Definition 1.1.5. *The* conjugacy class *of $g \in G$ is the set $[g] = \{xgx^{-1} | x \in G\}$.*

We also note that if G is abelian, then for any $x \in G$ we have that $[x] = \{x\}$ since if y is conjugate to x in G, then $y = gxg^{-1}$ for some $g \in G$. But G is abelian so $gxg^{-1} = gg^{-1}x = x$, i.e. $y = x$. We finish by discussing *group actions* and a special action that plays a role in linear representation theory.

Definition 1.1.6 (Group Action)**.** *A* group action *of a group G on a set A is a map from $G \times A \to A$, written as $g \cdot a$, for all $g \in G$ and $a \in A$ satisfying the following properties:*

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 \cdot g_2) \cdot a,\ \forall g_1, g_2 \in G, \forall a \in A$, *and*

(ii) $1_G \cdot a = a\ \forall a \in A$.

Definition 1.1.7 (Permutation Representation)**.** *Let G be a group acting on a nonempty set A. For each $g \in G$ the map $\sigma_g : A \to A$ defined by $\sigma_g : a \mapsto g \cdot a$ is a permutation* *of A. (This $g \cdot a$ is the map from $G \times A$ to A in the group action def above) There is a homomorphism associated to an action of G on A: ϕ: G to S_A defined by $\phi(g) = \sigma_g$, called the permutation representation.*

In particular we note that G can act on itself by left multiplication. This will help us define a linear representation of G called the *regular representation* of G in Chapter 2.

1.2 Ring Theory

In Chapter 2 we use the notion of a *group ring* which we define in this section. Hence it is important to have some basic definitions from ring theory such as subrings, ideals, and ring homomorphisms. Moreover, the basic homomorphisms theorems of the theory of groups can be extended to the theory of rings and we list those here. Furthermore, special concepts such as integral domain, division ring, and fields are an important part of linear representation theory and thus reviewed here.

Definition 1.2.1. *A* ring *R is a set together with two binary operations + and $\cdot$ (called addition and multiplication) satisfying the following axioms:*

(i) *(R,+) is an abelian group,*

(ii) *$\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and*

(iii) *the distributive laws hold in R: for all a,b,c $\in$ R $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$.*

We say that a ring R is *commutative* if multiplication is commutative, i.e. $ab = ba$ $\forall a, b \in R$. A ring R is said to have an *identity* (or contain a 1) if there is an element $1 \in R$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in R$. A subset S of R is called a *subring* of R if S is a ring under the same two operations of the ring R. We obtain a result similar to the One Step Subgroup Test.

Proposition 1.2.1 (Subring Test)**.** *A nonempty subset S of the ring R is a subring of R iff $\forall a, b \in S$ we have that $a - b \in S$ and $ab \in S$.*

Example of rings include the set of integers, $\mathbb{Z}$, the set of rationals $\mathbb{Q}$, the set of reals, $\mathbb{R}$, and the set of complex numbers, $\mathbb{C}$, all under ordinary addition and multiplication. The set of $n \times n$ matrices with entries in a ring R, denoted by $M_n(R)$, is a ring under matrix multiplication and matrix addition. As in group theory we have a notion of ring homomorphisms.

Definition 1.2.2 (Ring Homomorphism)**.** *Let R and S be rings. A* ring homomorphism *is a map $\phi : R \to S$ satisfying the following:*

(i) *$\phi(a+b) = \phi(a) + \phi(b)$ $\forall a, b \in R$, i.e. ϕ is a group homomorphism on the additive group $(R, +)$, and*

(ii) *$\phi(ab) = \phi(a)\phi(b)$ $\forall a, b \in R$*

The *kernel* of a ring homomorphism $\phi : R \to S$ is the set of elements in R that map to 0 in S, denoted by $ker\phi = \{a \in R : \phi(a) = 0_S\}$. The image of ϕ, denoted by $im\phi$ or $\phi(R)$ is the set $\{\phi(r) : r \in R\}$ A *ring isomorphism* is a bijective ring homomorphism. We note the following important result:

Proposition 1.2.2. *Let R and S be rings and let $\phi : R \to S$ be a ring homomorphism.*

(i) The kernel of ϕ is a subring of R.

(ii) The image of ϕ is a subring of S.

We have a notion similar to normal subgroups which we define next.

Definition 1.2.3. *Let R be a ring, let I be a subset of R, and let $r \in R$.*

1. *$rI = \{ra | a \in I\}$ and $Ir = \{ar | a \in I\}$*

2. *A subset I of R is a* left ideal *of R if*

   (i) *I is a subring of R, and*

   (ii) *I is closed under left multiplication by elements of R,i.e.,$rI \subseteq I$ for all $r \in R$.*

   Similarly I is a right ideal *if (i) holds and in place of (ii) we have the following:*

   (ii)' *I is closed under right multiplication by elements of R,i.e.,$Ir \subseteq I$ for all $r \in R$.*

3. *A subset I that is both a left and right ideal is called an* ideal, *(or a* two-sided ideal*) of R.*

Note for commutative rings the notion of left, right, two-sided ideal coincide. Moreover we say an ideal I of R is a proper ideal if and only if I does not coincide with R, i.e., $I \subsetneq R$. Now define the set $R/I = \{r + I : r \in R\}$. Since R is an abelian group under addition then R/I is a group under addition. It can be shown that the following two operations are well defined for all $r, s \in R$:

(a) $(r + I) + (s + I) = (r + s) + I$, and

(b) $(r + I)(s + I) = (rs) + I$.

It can also be shown that set R/I under the above operations satisfies the ring axioms, thus R/I is a ring, called the *quotient ring* of R by I. We also get a result that is analogous to the First Isomorphism Theorem for Groups.

Theorem 1.2.1. (The First Isomorphism Theorem for Rings)

1. *If $\phi : R \to S$ is a homomorphism of rings, then the kernel of ϕ is an ideal of R, the image of ϕ is subring of S and $R/ker\phi$ is isomorphic as a ring to $\phi(R)$.*

2. *If I is any ideal of R, then the map $R \to R/I$ defined by $r \mapsto r+I$ is a surjective ring homomorphism with kernel I (this homomorphism is called the natural projection of R onto R/I). Thus every ideal is the kernel of a ring homomorphism and vice versa.*

We define a few special rings and then go over rings that play a special role in linear representation theory. First we start by defining zero divisors.

Definition 1.2.4. *Let R be a ring.*

1. *A nonzero element a of R is called a* zero divisor *if there is a nonzero element $b \in R$ such that either $ab = 0$ or $ba = 0$.*

2. *Assume R has an identity $1 \neq 0$. An element u of R is called a* unit *in R if there is some v in R such that $uv = vu = 1$. The set of units in R is denoted $R^{\times}$.*

Definition 1.2.5. *A commutative ring with identity $1 \neq 0$ is called an* integral domain *if it has no zero divisors.*

Another special ring is the following:

Definition 1.2.6 (Division Ring)**.** *A ring R with identity 1, where $1 \neq 0$, is called a* division ring*, if every nonzero element $a \in R$ has a multiplicative inverse, i.e., there exists a $b \in R$ such that $ab = ba = 1$. A commutative division ring is called a field.*

The following ring plays an important part in representation theory.

Definition 1.2.7 (Group Ring)**.** *Let R be an arbitrary ring with identity $1 \neq 0$ and let G=$\{g_1, g_2, \cdots, g_n\}$ be any finite group with group operation written multiplicatively. The* group ring *of G over R is the set of all formal sums $\sum_{i=1}^{n} \alpha_i g_i$ where $\alpha \in R$ with component wise addition and multiplication $(\alpha g_i)(\beta h_i) = (\alpha\beta)(g_i h_i)$ (where α and β are multiplied in R and $g_i h_i$ is the product in G) extended to sums via the distributive law.*

Note that for a finite group G and a ring R, the group ring RG is not an integral domain since for any $g \in G$ such that $g \neq 1$ we have that $g^n - 1 = (g-1)(g^{n-1} + \cdots g + 1) = 0$ in RG. We finish this section with a few last definitions.

Definition 1.2.8 (Characteristic of a ring)**.** *The* characteristic *of a ring R is the smallest positive integer n such that $1 + 1 + \cdots + 1 = 0$(n times) in R; if no such integer exists the characteristic of R is 0. The characteristic of R is denoted by char(R).*

Definition 1.2.9 (Center of a Ring)**.** *Let R be a ring. Let the* center *of a ring be denoted by $Z(R) = \{a \in R : ax = xa, \forall x \in R\}$*

Definition 1.2.10. *Let A be a subset of R.*

1. *Let (A) denote the smallest ideal of R containing A, called the* ideal generated *by A. It can be shown that (A) is the intersection of all ideals of R that contain A.*

2. *Let RA denote the set of all finite sums of elements of the form ra with $r \in R$ and $a \in A$ i.e., $RA = \{\sum_{i=1}^{n} r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{Z}^+\}$ (where the convention is $RA = 0$ if $A = \emptyset$). We can similarly define AR and RAR. Note that we can show that RA is the left ideal generated by A, AR is tje right ideal generated by A, and $RAR = (A)$. In particular if R is commutative then $RA = AR = (A)$.*

3. *An ideal generated by a single element $a \in R$ is called a* principal ideal *and is denoted by (a).*

4. *An ideal generated by a finite set, say $\{a_1, a_2, \ldots, a_m\}$ is called a* finitely generated ideal *and is denoted by $(a_1, a_2, \ldots, a_m)$.*

Definition 1.2.11. *Let R be a nonzero ring.*

1. *A nonzero element e in a ring R is called* idempotent *if $e^2 = e$.*

2. *Idempotents e_1 and e_2 are said to be* orthogonal *if $e_1e_2 = e_2e_1 = 0$.*

3. *An idempotent e is said to be* primitive *if it cannot be written as a sum of two (commuting) orthogonal idempotents.*

4. *The idempotent e is called a* primitive central idempotent *if $e \in Z(R)$ and e cannot be written as a sum of two orthogonal idempotents in the ring Z(R).*

1.3 Module Theory

In Chapter 2 we will see that every representation of a group G over a field K is in one-to-one correspondence with a left module over the group ring KG. Thus in this section we will highlight the elementary aspects of the theory of modules.

Definition 1.3.1. *Let R be a ring (not necessarily commutative nor with a 1). A* left R-module *or a* left module *over R is a set M together with*

1. *a binary operation + on M under which M is an abelian group, and*

2. *an action of R on M (that is, a map $R \times M \to M$) denoted by r.m, for all $r \in R$ and for all $m \in M$ that satisfies*

   (a) *(r+s).m=r.m+s.m, for all $r, s \in R, m \in M$*

   (b) *(rs)m=r(sm), $r, s \in R, m \in M$*

   (c) *r.(m+n)=r.m+r.n,, $r \in R, m, n \in M$*

   If the ring has a 1 we impose the additional axiom:

   (d) *1.m=m, for all $m \in M$*

The descriptor "left" in the above definition indicates that the ring elements appear on the left. "Right" R-modules can be defined analogously. If the ring R is commutative and M is a left R- module then we can make M into a right R-module by defining $m.r = r.m$ for $m \in M, r \in R$. If R is not commutative, axiom 2(b) in general will not hold with this definition, thus not every left R-module is a right R-module. The term "module" in this thesis will always mean "left module". Modules satisfying 2(d) are called *unital modules.* Analogously to group and ring theory we have a notion of submodules and a test for submodules.

Definition 1.3.2 (R-submodule)**.** *Let R be a ring and let M be an R-module. An* R-submodule *of M is a subgroup N of M which is closed under the action of ring elements, i.e., $r.n \in N$, for all $r \in R$, $n \in N$.*

Proposition 1.3.1 (Submodule Criterion)**.** *Let R be a ring and let M be an R-module. A subset N of M is a submodule of M if*

1. *$N \neq 0$, and*

2. *$x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.*

We say that an R-module M is *irreducible* or *simple* if $M \neq 0$ and if 0 and M are the only submodules of M.

Definition 1.3.3. *Let R be a ring and let M and N be R-modules.*

1. *A map $\phi : M \to N$ is an* R-module homomorphism *if it respects the R-module structures of M and N, i.e.,*

   (a) *$\phi(x + y) = \phi(x) + \phi(y)$, for all $x, y \in M$ and*

   (b) *$\phi(rx) = r\phi(x)$, for all $r \in R, x \in M$*

2. *An R-module homomorphism is an* isomorphism *(of R-modules) if it is both injective and surjective. The modules M and N are said to be isomorphic, denoted $M \cong N$, if there is some module isomorphism $\phi : M \to N$.*

3. *If $\phi : M \to N$ is an R-module homomorphism, then denote the* kernel *of ϕ by $ker\phi = \{m \in M : \phi(m) = 0\}$. Denote the* image *of ϕ by $\phi(M) = \{\phi(m) : m \in M\}$. It can be shown that $ker\phi$ is a submodule of M and that $\phi(M)$ is a submodule of N.*

4. *Let M and N be R-modules and define $Hom_R(M, N)$ to be the set of all R-module homomorphisms from M into N.*

A couple of notes. First of all, if R is a field, R-module homomorphisms are called linear transformations. We will formally define these in the next section. Secondly, when R is commutative then $Hom_R(M, M)$ is an R-algebra. We will also see in Chapter 2 that the group ring KG is K-algebra, thus we define in general what an R-algebra is and and R-algebra homomorphism.

Definition 1.3.4 (R-algebra)**.** *Let R be a commutative ring with identity. An* R-algebra *is a ring A with identity together with a ring homomorphism $f : R \to A$ mapping $1_R \mapsto 1_A$ such that the subring $f(R)$ of A is contained in the center of A.*

Definition 1.3.5 (R-algebra Homomorphism)**.** *If A and B are two R-algebras, an R-*algebra homomorphism *(or isomorphism) is a ring homomorphism (isomorphism, respectively) $\phi : A \to B$ mapping 1_A to 1_B such that $\phi(r.a) = r.\phi(a)$ $\forall r \in R$ and $\forall a \in A$.*

Definition 1.3.6. *The ring $Hom_R(M, M)$ is called the* endomorphism ring *of M and will be denoted by $End_R(M)$. Elements of $End_R(M)$ are called endomorphisms.*

When R is commutative there is a natural map(projection) from R to $End_R(M)$ given by $r \mapsto rId_M$ where the latter endomorphism is just multiplication by r on M. The image of R is contained in the center of $End_R(M)$ so if R has an identity, $End_R(M)$ is an $R - algebra$.

Proposition 1.3.2. *Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive,abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by $r.(x + N) = (r.x) + N$ for all $r \in R, x + N \in M/N$. The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.*

As in group and ring theory we have a First Isomorphism theorem for Modules.

Theorem 1.3.1 (First Isomorphism theorem for Modules)**.** *Let M, N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Then $ker\phi$ is a submodule of M and $M/ker\phi \cong \phi(M)$.*

We now describe the generation of modules and the direct sum of modules.

Definition 1.3.7. *Let M be an R-module and let $N_1, N_2, \ldots, N_n$ be submodules of M.*

1. *The sum of* $N_1, N_2, \ldots, N_n$ *is the set of all finite sums of elements from the sets* N_i*:* $\{\sum_{i=1}^{m} a_i : a_i \in N_i\ \forall 1 \leq i \leq m\}$*. We denote this sum by* $N_1 + N_2 + \cdots + N_n$*.*

2. *For any subset* A *of* M *let* $RA = \{\sum_{i=1}^{m} r_i a_i : r_i \in R, a_i \in A, m \in \mathbb{Z}^+\}$ *(where the convention is* $RA = 0$ *if* $A = \emptyset$*). We call* RA *the* submodule of M generated *by* A*. If* N *is a submodule of* M *(possibly* $N = M$*) and* $N = RA$*, for some subset* A *of* M*, then we call* A *the* set of generators *or* generating set *for* N*, and we say that* N *is* generated *by* A*.*

3. *A submodule* N *of* M *(possibly* $N = M$*) is* finitely generated *if there is some finite subset* A *of* M *such that* $N = RA$*.*

Definition 1.3.8. *Let* $M_1, M_2, \ldots, M_n$ *be a collection of* R*-modules. The collection of* n*-tuples* $(m_1, m_2, \ldots, m_n)$ *where* $m_i \in M_i\ \forall 1 \leq i \leq n$ *with addition and action of* R *defined componentwise is called the* direct product *of* $M_1, M_2, \ldots, M_n$ *and is denoted by* $M_1 \oplus M_2 \oplus \ldots \oplus M_n$*.*

The next proposition allows us to find out when a sum of submodules of M is isomorphic to a direct product of these submodules.

Proposition 1.3.3. *Let* $N_1, N_2, \ldots, N_k$ *be submodules of the* R*-module* M*. Then the following are equivalent:*

1. *The map* $\pi : N_1 \oplus N_2 \oplus \ldots \oplus N_k \to N_1 + N_2 + \ldots + N_k$ *defined by*

$$\pi(a_1, a_2, \ldots, a_k) = a_1 + a_2 + \cdots + a_k$$

is an isomorphism (of R*-modules):* $N_1 + N_2 + \ldots + N_k \cong N_1 \oplus N_2 \oplus \ldots \oplus N_k$*.*

2. $N_j \cap (N_1 + N_2 + \cdots + N_{j-1} + N_j + \cdots + N_k) = 0$ *for all* $j \in \{1, 2, \ldots k\}$*.*

3. *Every* $x \in N_1 + N_2 + \ldots + N_k$ *can be written uniquely in the form* $a_1 + a_2 + \cdots + a_k$ *with* $a_i \in N_i\ \forall 1 \leq i \leq k$*.*

Finally we will see that the group rings that we work with will satisfy something called the *descending chain condition*.

Definition 1.3.9. *An* R*-module* M *is said to satisfy the* descending chain condition*(D.C.C.) on* R*-submodules iff every properly descending chain of submodules of* M*,* $N_1 \supset N_2 \supset \cdots N_k \supset \cdots$*, contains only a finite number of elements.*

1.4 Vector Space Theory

We can now use the language of module theory to define a vector space V over a field K. We say that a set V is a vector space over the field F if it is an F-module. Since a field is commutative we know that left modules and right modules coincide so it is not necessary to distinguis between the two, although by convention our scalars are written on the left. In this section then we highlight some definitions and theorems that are quite useful when looking at vector spaces.

Let K be a field and let V be a vector space over K. We call a map $f : V \to V$ a *linear transformation* if f is an F-module homomorphism. We denote by $GL(V)$ the set of all bijective linear transformations from V to V. We have that $GL(V)$ is a group under composition, called the *general linear group*.

A subset S of V is called a set of linearly independent vectors if the equation $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$ where $a_i \in K$ and $v_i \in V$ $\forall 1 \leq i \leq k$ implies that $a_1 = a_2 = \cdots = a_k = 0$. We say that a set B of vectors in V *span* V if every vector in V can be written as a K-linear combination of a finite number of vectors in B. A *K-basis* for a vector space V is an ordered set of linearly independent vectors that span V. If V has a finite K-basis B then the cardinality of B is called the dimension of V over K and is denoted as $dim_K V$. The dimension of V is finite if the basis B is finite, otherwise we say that the V is infinite dimensional.

We note that if $dim_K V = n < \infty$ then $V \cong K^n$ and in particular any two finite dimensional vector spaces are isomorphic if they have the same dimension over K. A subset W of V is called a *subspace* of V if it is a K-vector space. We also note the following result:

Proposition 1.4.1. *Let V be a vector space over K and let W be a subspace over V. Then V/W is a K-vector space and*

$$dim_K V = dim_K W + dim_K V/W.$$

An important area of study in vector space theory is the study of the different "invariants" of a linear transformation, including its eigenvectors, eigenspaces, and invariant subspaces. These are concepts that are definitely used in the study of linear representation. So we will define those here.

Let K be a field, V be a K-vector space and $\phi \in Hom_K(V, V)$. Then a nonzero element $v \in V$ is called an *eigenvector* of ϕ corresponding to the *eigenvalue* λ if and only if $\phi(v) = \lambda v$. We say that a nontrivial subspace U of V is an *eigenspace* of ϕ corresponding to λ if $\phi(u) = \lambda u \ \forall u \in U$. Finally we say that a subspace U of V is *invariant* under ϕ if $\phi(u) \in U \ \forall u \in U$. Note that an eigenspace of ϕ is invarant under ϕ.

Now let K be a field, V, W be finite dimensional K-vector spaces, $B = \{b_1, \ldots, b_m\}$ be a basis for V, $E = \{e_1, \ldots, e_n\}$ be a basis for W, and $\phi \in Hom_K(V, W)$. Then we can define a matrix associated to ϕ with respect to B and E, denoted by $M_B^E[\phi]$ as follows. Let $M[\phi]_B^E = (a_{ij})$ where $\phi(b_j) = \sum_{i=1}^{n} a_{ij} e_i \ \forall 1 \leq j \leq m$.

Thus to find the eigenvalues of a linear transformation $\phi \in Hom_K(V, V)$ we proceed by solving the equation $\det(A - \lambda I) = 0$, where A is the associated matrix of ϕ with respect to some basis B of V. We can then proceed to find the eigenspace of ϕ associated to an eigenvalue λ by solving the equation $(A - \lambda I)v = 0$.

We say that two $n \times n$ matrices A and B are *similar* if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. Two linear transformations $\phi, \psi : V \to V$ are called *similar* if there exists an invertible linear transformation $\rho : V \to V$ such that $\rho^{-1}\phi\rho = \psi$. We say a matrix A is *diagonalizable* if it is similar to a diagonal matrix.

An $n \times n$ matrix with λ along the diagonal and 1 along the first superdiagonal is called the *Jordan block of size k with eigenvalue* λ and is depicted below:

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix},$$

where the blank entries are all zero. We say that a matrix is in *Jordan canonical form* if it is a block diagonal matrix with Jordan blocks along the diagonal, i.e., it is of the following form:

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_t \end{pmatrix}.$$

An important result in vector space theory states that every $n \times n$ matrix is similar to a matrix in Jordan canonical form.

Finally we define a positive definite, *hermitian symmetric form*, orthogonal and orthonormal vectors, and inner products.

Definition 1.4.1. *Let V be a $\mathbb{C}$-vector space. A map $f : V \times V \to \mathbb{C}$ is said to be a positive definite,* hermitian symmetric form *on V iff the following conditions hold for all $u, v, w \in V$ and for all $\alpha \in \mathbb{C}$.*

(i) $f(u, v) \geq 0$ *and* $f(u, u) = 0$ *iff* $u = 0$,

(ii) $f(u, v) = \overline{f(v, u)}$,

(iii) $f(\alpha u, v) = \alpha f(u, v)$, *and*

(iv) $f(u, v + w) = f(u, v) + f(u, w)$.

A mapping f that satisfies the properties above is also called an *inner product* on V and we usually denote $f(u, v)$ by just (u, v). A $\mathbb{C}$-vector space V that has an inner product is called an *inner product space*.

Definition 1.4.2. *A subset $\{v_1, \ldots v_k\}$ of the n-dimensional inner product space V is called* orthogonal *iff $(v_i, v_j) = 0$ whenever $i \neq j$. It is called* normal *iff $(v_i, v_i) = 1$ for all $1 \leq i \leq k$. The set is called* orthonormal *iff $(v_i, v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for all $1 \leq i, j \leq k$.*

Chapter 2

General Representation Theory

This chapter is an introduction to the representation theory of finite groups, which is a branch of mathematics that studies algebraic structures by "representing" their elements as linear transformations of vector spaces and modules over these algebraic structures. In this project the structure will be the dihedral group and I will be representing its elements as linear transformations of vector spaces, but working with the invertible matrices will be much easier. These linear transformations will provide information about the groups themselves. Finally, we note that modules are the "representation objects" for rings in the sense that the axioms for an R-module specify a "ring action" of R on some abelian group M which preserves the abelian group structure of M. So in order to discuss the representations of a group we need some terminology. Hence the definitions and theorems below.

2.1 Linear Representations and KG-modules

In this section we would like to establish an important connection between linear representations of a group G and KG-modules, thus we start with the following definitions.

Definition 2.1.1. *Let G be a finite group, let K be a field, and let V be a K-vector space.*

1. *A* linear representation *of G is any group homomorphism from G into $GL(V)$. The* degree *of the representation is the dimension of V.*

2. *Let $n \in \mathbb{Z}^+$. A* matrix representation *of G is any homomorphism from G into $GL_n(K)$, where $GL_n(K)$ denotes the group of invertible matrices.*

3. *A linear or matrix representation is* faithful *if it is injective.*

Recall that from vector space theory we know that if V is a finite dimensional vector space of dimension n, then by fixing a basis of V we can obtain a matrix representation of any linear transformation from $V$to V. We actually obtain an isomorphism from $GL(V)$ to $GL_n(K)$. Therefore any representation of G on a finite dimensional vector space gives us a matrix representation and vice versa.

As we stated in Section 1.2, the *group ring* of G over K is the set of all formal sums of the form $\sum_{g\in G} \alpha_g g$ where $\alpha_g \in K$ with componentwise addition and multiplication $(\alpha_g)(\beta h) = (\alpha\beta)(gh)$ (where α and β are multiplied in K and gh is the product in G) extended to sums via the distributive law.

Example 2.1.1. Let $G = \langle g \rangle$ be the cyclic group of order $n \in \mathbb{Z}^+$ and let K be a field. Then we obtain the group ring $K\langle g \rangle = \{\sum_{i=0}^{n-1} \alpha_i g^i | \alpha_i \in K\}$. If we let $\psi : K[x] \to K\langle g \rangle$ be the map where $x^k \mapsto g^k$ where $k \geqslant 0$ and extend ψ K-linearly, then ψ is a ring homomorphism, ψ is surjective, and ker$\psi = \langle x^n - 1 \rangle$. Therfore, by the first Isomorphism theorem for rings we have a special relationship between the group ring $K\langle g \rangle$ and the infinite dimensional polynomial ring $K[x]$ which states that $K\langle g \rangle \cong K[x]/\langle x^n - 1 \rangle$.

Example 2.1.2. Let $G = D_8$ and $K = \mathbb{R}$. An example of addition and multiplication in $\mathbb{R}D_8$ where $\alpha, \beta \in \mathbb{R}D_8$, $\alpha = r + r^2 - 2s$, and $\beta = -3r^2 + rs$ is the following:

$$\begin{aligned} \alpha + \beta &= (r + r^2 - 2s) + (-3r^2 + rs) \\ &= r + r^2 - 2s - 3r^2 + rs \\ &= r - 2r^2 - 2s + rs, \end{aligned}$$

and

$$\begin{aligned} \alpha\beta &= (r + r^2 - 2s)(-3r^2 + rs) \\ &= -3r^3 + r^2 s - 3 + r^3 s + 6r^2 s - 2r^3 \\ &= -3 - 5r^3 + 7r^2 s + r^3 s, \end{aligned}$$

Moreover, $\mathbb{R}$ is in the center of $\mathbb{R}D_8$, i.e., $\mathbb{R} \subset Z(\mathbb{R}D_8)$ and since $1 < |D_8| < \infty 1$ then $\mathbb{R}D_8$ is not an integral domain by our discussion in Section 1.2 (see discussion after Definition 1.2.7).

We note the following results concerning group rings.

Proposition 2.1.1. *KG is a commutative ring if and only if G is abelian.*

Proposition 2.1.2. *KG is a vector space over K with the elements of G as a basis..*

Proposition 2.1.3. *Since K is in the center of KG then KG is a K-algebra.*

The most important result that allows us view representations of a group G over a field K as a KG-modules is the following. We let (V, ϕ) denote the representation $\phi : G \to GL(V)$.

Theorem 2.1.1. *There is a bijection between KG-modules and pairs (V,ϕ):*

$$\left\{ V \text{ is a } KG\text{-module} \right\} \longleftrightarrow \left\{ \begin{array}{c} V \text{ is a } K\text{-vector space} \\ \text{and} \\ \phi : G \to GL(V) \end{array} \right\}$$

Proof. Assume that $\phi : G \to GL(V)$ is a representation of V over K. Let $G = \{g_1, g_2, \dots g_n\}$. We make V into a KG-module by defining the following action of G on V and extending linearly: $g.v = \phi(g)(v)\ \forall g \in G$. Thus the action of an element in KG on V is given by

$$\left(\sum_{i=1}^{n} a_i g_i\right).v = \sum_{i=1}^{n} a_i \phi(g_i)(v) \qquad \forall \sum_{i=1}^{n} a_i g_i \in KG, v \in V.$$

Checking that the V is a KG-module requires some work but we will check a special case of 2(b) given in the axioms of modules in Section 1.2. Let $g_i, g_j \in G$. Then note that

$$\begin{aligned} (g_i g_j).v &= \phi(g_i g_j)(v) && \text{by definition of the action} \\ &= (\phi(g_i) \circ \phi(g_j))(v) && \text{since } \phi \text{ is a group homomorphism} \\ &= \phi(g_i)(\phi(g_j)(v)) && \\ &= g_i.(g_j.v) && \text{by definition of the action.} \end{aligned}$$

We can extend this argument by linearity and show that any element of KG satisfies axiom 2(b). Similarly we can show that the other axioms hold.

Conversely, suppose now that we have a KG-module V. Since K is a subring of KG we can restrict the action of KG to K and see that we get that V is a K-module, i.e. a K-vector space. We define the following map $\phi : G \to GL(V)$ sending $g \mapsto \phi_g$ where we define $\phi_g : V \to V$ as follows: for all $v \in V$, let $\phi_g(v) = g.v$. We claim that ϕ_g is an invertible linear transformation and the ϕ is a group homomorphism.

Let $\alpha \in K$ and let $u, v \in V$. Then $\phi_g(\alpha u + v) = g.(\alpha u + v) = \alpha g.u + g.v = \alpha\phi_g(u) + \phi(v)$. We also see that $(\phi_{g^{-1}} \circ \phi_g)(v) = \phi_{g^{-1}}(\phi_g(v)) = \phi_{g^{-1}}(g.v) = g^{-1}.(g.v) = (g^{-1}g).v = v$ for all $v \in V$. Thus $\phi_{g^{-1}} \circ \phi_g$ is the identity transformation. Similarly $\phi_g \circ \phi_{g^{-1}}$ is also the identity transformation. Finally we show that ϕ is a group homomorphism. Note that for any $v \in V$ and that for any $g, h \in G$ we get that $(\phi_g \circ \phi_h)(v) = g.(h.v) = (gh).v = \phi_{gh}(v)$. Thus we see that $\phi(g) \circ \phi(h) = \phi(gh)$. □

Thus, giving a representation $\phi : G \to GL(V)$ on a vector space V over K is equivalent to giving a KG-module. Under this correspondence we shall say that the KG-module V affords the representation ϕ of G.

We now extend the notion of invariant spaces of a transformation.

Definition 2.1.2 (G-Stable)**.** *If V is a KG-module affording a representation ϕ, then the subspace U of V is called G-*invariant *or G-*stable *if $\phi_g(u) \in U$ for all $g \in G$, for all $u \in U$ (i.e., $g \cdot u \in U$ for all $g \in G$, for all $u \in U$). In fact the KG-submodules of V are precisely the G-stable subspaces of V.*

The following result is used often in our project.

Proposition 2.1.4. *If $h : G \to H$ is any group homomorphism and $\phi : H \to GL(V)$ is a representation of H with representation space V, then the composition $\phi \circ h : G \to GL(V)$ is a representation of G.*

Proof. First of all since h and ϕ are both group homomorphisms then their composition is also a group homomorphism. Thus by definition $\phi \circ h$ is a linear representation of G. □

We end this section by recalling a definition from Section 1.3 and giving other relevant definitions and a proposition that we will find useful.

Definition 2.1.3. *Let R be a ring and let M be a nonzero R-module.*

1. *The module M is said to be* irreducible *(or* simple*) if its only submodules are 0 and M; otherwise M is called reducible.*

2. *The module M is said to be* indecomposable *if M cannot be written as $M_1 \oplus M_2$ where M_1 and M_2 are modules (or submodules of M.*

3. *The module M is said to be* completely reducible *if its direct sum of irreducible submodules.*

4. *A representation is called* irreducible, reducible, indecomposable, decomposable *or* completely reducible *according to whether the KG-module affording it has the corresponding property.*

5. *If M is a completely reducible R-module, any direct summand of M is called a* constituent *of M (i.e., N is a constituent of M is there is a submodule N' of M such that $M = N \oplus N'$.*

Proposition 2.1.5.

All degree one representations are irreducible(or simple), indecomposable, and completely reducible.

2.2 Examples of KG-modules

Since there is a one-to-one correspondence between KG-modules and linear representations of G, in this section we give examples of KG-modules and the representation that is afforded by each of them .

Example 2.2.1 (Trivial Module)**.** Let V be a one-dimensional K-vector space (i.e. V is a K-module). We make V into a KG-module by defining the following the ring action: $g \cdot v = gv = v \ \forall v \in V$. So this module affords a linear representation $T : G \to GL(V)$ where $g \mapsto T(g) = T_g = I_V \ \forall g \in G$. The corresponding matrix representation of G is the following $T : G \to GL_1(K) = K^X = K \setminus \{0\} = \{a | a \in K\}$, the multiplicative group where $\dim_K V = \deg T = 1$. T is not faithful since the ker $T = G$. Furthermore, since the trivial representation is 1-dimensional then it is irreducible(or simple), indecomposable, and completely reducible.

Example 2.2.2. Let V be any n-dimensional K-vector space for $n \in \mathbb{Z}^+$ with basis $\{e_1, \cdots, e_n\}$. Again V is K-module but we can make V into a KS_n-module by defining a ring action $\sigma_i \cdot e_j = e_{\sigma_i(j)} \forall \sigma_i \in S_n$ and $\forall i, j \in \{1, \cdots, n\}$ and extending linearly. Thus, V affords a linear representation $\psi : S_n \to GL(V)$ where $\sigma_i \mapsto \psi(\sigma_i) = \psi_{\sigma_i}$ and $\psi_{\sigma_i} : V \to V$ which is defined as $\psi_{\sigma_i}(e_j) = e_{\sigma_i(j)} \forall \sigma_i \in S_n$ and $\forall i, j \in \{1, \cdots, n\}$. Finally since $\ker\psi = 1_{S_n}$ then ψ is fatithful.

Note that V is reducible since it contains the following two proper, nonzero submodules:

$$\begin{aligned} N &= \{\alpha_1 e_1 + \cdots + \alpha_n e_n | \alpha_1 = \cdots = \alpha_n\} \\ &= \{\alpha_1(e_1 + \cdots + e_n)\} \\ &= span_K\{(e_1 + \cdots + e_n)\}, \end{aligned}$$

and

$$I = \{\alpha_1 e_1 + \cdots + \alpha_n e_n | \alpha_1 + \cdots + \alpha_n = 0\}.$$

Moreover since N is 1-dimensional, then it is irreducible, indecomposable, and completely reducible. However, if the characteristic of the field K does not divide n, then I is also irreducible. Since N and I are KS_n submodules of V then they are $S_n - stable$.

Example 2.2.3 (Regular Representation)**.** Let G be a group of order $|G| = n$ and let V be the vector space $KG = span_K\{g_1 \cdots g_n\}$. Note V is a KG-module by defining the following ring action and extending K-linearly: $g \cdot g_i = gg_i\ \forall g_i, g \in G$. Thus V affords a representation of G given by $\mathcal{R} : G \to GL(KG)$ where $g \mapsto \mathcal{R}_g$ and $\mathcal{R}_g(g_i) = g \cdot g_i = gg_i$ $\forall g_i, g \in G$. Note that $dim_K(KG) = |G|$ and $\mathcal{R}$ is faithful since $ker\mathcal{R} = 1_G$. If we define

$$\begin{aligned} N &= \{\alpha_1 g_1 + \cdots + \alpha_n g_n | \alpha_1 = \cdots = \alpha_n\} \\ &= \{\alpha_1(g_1 + \cdots g_n)\} \\ &= span_K\{(g_1 + \cdots + g_n)\}, \end{aligned}$$

and

$$I = \{\alpha_1 g_1 + \cdots + \alpha_n g_n | \alpha_1 + \cdots + \alpha_n = 0\}$$

then N is 1-dimensional which implies N is irreducible, indecomposable, and completely reducible. Note: If we define the linear functional $\phi_i : V \to K$ by $\sum_{i=1}^{n} a_i g_i \mapsto \sum_{i=1}^{n} a_i$, $a_i \in K$ then $\ker \phi_i = I$. From linear algebra we know $V = \ker \phi_i \oplus \text{ im } \phi_i$. Since the $\dim(\text{im } \phi) = 1$ and $\dim(KG) = |G|$ we see that dim ($\ker\phi_i$) is forced to be $n-1$. Since N and I are KG submodules of KG then they are $G-stable$. In fact N and I happen to be 2-sided ideals of KG. Finally, the regular representation is reducible when $|G| > 1$. However later we will determine the conditions under which the regular representation is completely reducible and how we can decompose it into a direct sum of submodules, i.e. $KG = U_1 \oplus \cdots U_n$ where $U_1 \oplus \cdots U_n$ where U_i, $1 \leqslant i \leqslant n$ are irreducible.

In many situations we will see that it is easier to specify an explicit "matrix representation" of a group G rather than to exhibit a KG-module. Therefore below are examples of matrix representations of a group G.

Example 2.2.4. Let $G = \langle g \rangle \cong \mathbb{Z}_n$, where $|G| = n$ and let $\zeta = e^{\frac{2\pi k}{n}} = cos(\frac{2\pi k}{n}) + sin(\frac{2\pi k}{n})$. Then $h : G \to K^x = GL_1(K)$ where $g^i \mapsto \zeta^i$ is a matrix representation of degree 1. Here h is faithful $\iff$ ζ is a primitive nth root of 1. Moreover, since deg(h)=1 then it is irreducible(or simple), indecomposable, and completely reducible. In fact any homomorphism of G into the multiplicative group K^x is irreducible(or simple), indecomposable, and completely reducible.

Example 2.2.5. A matrix representation of the dihedral group with presentation
$D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$ is given by $\phi : D_{2n} \to GL_2(\mathbb{R})$ where
$r \mapsto R = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}$ and $s \mapsto S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. More specifically, R is the matrix of linear transformations which rotates the xy-plane about the origin in counter clockwise by $\frac{2\pi}{n}$ radians and S reflects the plane about the $y = x$. These matrices act as symmetries of an n-gon and satisfy the relations of D_{2n}. Thus this representation is faithful. This representation is irreducible for $n \geqslant 3$ because there are no D_{2n}-stable 1-dimensional subspaces since a rotation by $2\pi/n$ sends no line in $\mathbb{R}^2$ to itself. In other words, we cannot find a subspace U of $\mathbb{R}^2$ such that $\phi_r(u) \in U$ and $\phi_s(u) \in U$ for all $u \in U$.

Note: A linear representation would be $\phi : D_{2n} \to GL(\mathbb{R}^2)$ where $r \mapsto \phi_r$ and $s \mapsto \phi_s$, where $\phi_r : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi_r(x, y) = \left(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}, x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n}\right)$ and $\phi_s : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi_s(x, y) = (-y, x)$

Example 2.2.6. The matrix representation of the Quaternion group with presentation $Q_8 = \langle i, j | i^4 = j^4 = 1, i^2 = j^2, i^{-1}ji = j^{-1}\rangle$ is given by $\phi : Q_8 \to GL_2(\mathbb{C})$ where $i \mapsto I = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$ and $j \mapsto J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
This representation is faithful since I and J satisfy the same relations of Q_8. Hence, $Q_8 \cong \langle I, J\rangle \leqslant GL_2(\mathbb{C})$.

We know that ϕ is irreducible if and only if the only Q_8-stable subspaces of $V = \mathbb{C}^2$ are 0 and $\mathbb{C}^2$. Moreover, we know the invariant subspaces of V under ϕ_i are the eigenspaces $\epsilon_{\sqrt{-1}} = span_{\mathbb{C}}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $\epsilon_{-\sqrt{-1}} = span_{\mathbb{C}}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, with corresponding eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. But, neither of these spaces are invariant under ϕ_j. Therefore, ϕ is irreducible.

Example 2.2.7. Let $\mathbb{H}$ be the real Hamilton quaternions ring, i.e. $\mathbb{H} = span_{\mathbb{R}}\{1, i, j, k\}$. We know that $\mathbb{H}$ is an $\mathbb{R}$-vector which implies $\mathbb{H}$ is an $\mathbb{R}$-module. But we want to make $\mathbb{H}$ into a $\mathbb{R}Q_8$-module. So we define the ring action by left multipication $i \cdot x = ix$ and $j \cdot x = jx$ $\forall x \in \mathbb{H}$. Thus, this module affords a linear representation $\mathcal{R} : Q_8 \to GL(\mathbb{H})$ by $i \mapsto \mathcal{R}_i$ and $j \mapsto \mathcal{R}_j$ where $\mathcal{R}_i : \mathbb{H} \to \mathbb{H}$ where x $\mapsto i \cdot x$ and $\mathcal{R}_j : \mathbb{H} \to \mathbb{H}, x \mapsto j \cdot x$ $\forall x \in \mathbb{H}$.

Note that we have the left regular representation of Q_8, i.e, $\mathcal{R} : Q_8 \to GL(\mathbb{R}Q_8) = GL(\mathbb{H})$. Now we can easily write out the explicit matrices of each of the elements of Q_8 with respect to the basis $\{1, i, j, k\}$.

$\mathcal{R}_i(1) = i \cdot 1 = i = 0 \cdot 1 + 1 \cdot i + 0 \cdot j + 0 \cdot k = i$

$\mathcal{R}_i(i) = i \cdot i = i^2 = -1 = -1 \cdot 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k = k$

$\mathcal{R}_i(j) = i \cdot j = ij = k = 0 \cdot 1 + 0 \cdot i + 0 \cdot j + 1 \cdot k = i$

$\mathcal{R}_i(k) = i \cdot k = ik = -j = 0 \cdot 1 + 0 \cdot i - 1 \cdot j + 0 \cdot k = -j$

For each linear transformation there is an associated matrix representation, $\mathcal{M}_i$ given by $\mathcal{M}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Similarly we can obtain $\mathcal{M}_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$.

This linear action gives a homomorphism of Q_8 into $GL_4(\mathbb{R})$. Therefore, the matrix representation of the Quaternion group is
$\mathcal{R} : Q_8 \to GL_4(\mathbb{R})$ where

$$i \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ and } j \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

This representation is irreducible, however, if we extend the field to the complex numbers then it becomes reducible since there exists a complex matrix P such that $P^{-1}\phi(g)P$ is a direct sum of 2×2 block matrices for all $g \in Q_8$. Therefore an irreducible representation over a field K may become reducible when the field is extended. In fact "reducible" representations are those with a corresponding matrix representation whose matrices are in block triangular form, whereas decomposable representations are in block diagonal form.

Definition 2.2.1 (Equivalent Linear Representations)**.** *Two representations of G are* equivalent *(or* similar*) if the KG-modules affording them are isomorphic modules. Representations which are not equivalent are called inequivalent.*

Definition 2.2.2 (Equivalent Matrix Representations)**.** *Two representations ϕ and ψ are* equivalent *if there is a fixed invertible matrix P such that $P\phi(g)P^{-1} = \psi(g)$ for all $g \in G$.*

Finally we present a result that will have important consequences, as we will see in the next section.

Theorem 2.2.1 (Maschke)**.** *Let G be a finite group and let K be a field whose characteristic does not divide $|G|$. If V is any KG-module and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$ (i.e., every submodule is a direct summand.)*

Corollary 2.2.2. *If G is a finite group and K is a field whose characteristic does not divide $|G|$, then every finite generated KG-module is completely reducible(equivalently, every K-representation of G of finite degree is completely reducible)*

Corollary 2.2.3. *Let G be a finite group, let K be a field whose characteristic does not divide $|G|$ and let $\phi : G \to GL(V)$ be a representation of G of finite degree. Then there*

is a basis of V such that for each $g \in G$ the matrix $\phi(g)$ with respect to this basis is block diagonal: $\begin{pmatrix} \phi_1(g) & & & \\ & \phi_2(g) & & \\ & & \ddots & \\ & & & \phi_m(g) \end{pmatrix}$ *where ϕ_i is an irreducible matrix representation of G,*
$1 \leqslant i \leqslant m$.

2.3 Irreducible Representations of D_8

In the last section we discussed a very important correspondence between KG-modules and pairs (V, ϕ), where V is a vector space over K and ϕ is a representation of G. That is, there is a bijection between KG-modules and representations of G or we say a KG module affords a representation. In this section we will let $K = \mathbb{C}$, unless otherwise stated, and we will compute the irreducible representations of D_8. Along the way we will discuss characters of a representation and note that a representation of G affords a character. We will then see that there is a correspondence between these characters and equivalence classes of complex representations. We learn that the complex representations of finite groups (specifically D_8) are characterized by their characters. Finally, we discuss other useful properties of characters.

One of the most important theorems in section that I will use in my project will be the following:

Theorem 2.3.1. *Let G be a finite group. Then:*

1. $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ *as $\mathbb{C}$-modules.*

2. *$\mathbb{C}G$ has exactly r distinct isomorphism types of irreducible modules and these have complex dimensions $n_1, n_2, \cdots, n_r$ (and so G has exactly r inequivalent irreducible complex representation of the corresponding degrees).*

3. $\sum_{i=1}^{r} n_i^2 = |G|$

4. *r equals the number of conjugacy classes in G.*

We will use this theorem now to find the complex irreducible representations of D_8. Recall a presentation of D_8 is $\langle r, s | r^4 = s^2 = 1, rs = sr^{-1} \rangle$, i.e. $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, and a matrix representation of D_8 is given by $\phi : D_8 \to GL_2(\mathbb{R})$ where $r \mapsto R = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $s \mapsto S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. More specifically, R is the matrix of a linear transformation which rotates the xy-plane about the origin in a counter-clockwise direction by $\pi/2$ radians and S is the matrix of a linear transformation that reflects the plane about the line $y = x$. These matrices act as symmetries of a square and satisfy the relations of D_8. We can also consider D_8 as the group of permutations of a regular 4-gon or even just of its 4 vertices since the dihedral group is a subgroup of the symmetric group. This representation is faithful. Furthermore, this representation is irreducible because there are no D_8-stable 1-dimensional subspaces since a rotation by $\pi/2$ sends no line in $\mathbb{R}^2$ to itself. In other words, $\nexists U \subset \mathbb{R}^2$ such that $\phi_r(u) \in U$ and $\phi_s(u) \in U$ for all $u \in U$. Later we will also show that this representation is irreducible by looking at its character.

Now using the fact that ϕ is a homomorphism, the matrices $\phi(g)$ for all $g \in D_8$ are the following:

- $\phi(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is given above.
- $\phi(r^2) = \phi(r)\phi(r) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\phi(r^3) = \phi(r^2)\phi(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\phi(1) = \phi(r^4) = \phi(r^2)\phi(r^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\phi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is also given above.
- $\phi(sr) = \phi(s)\phi(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\phi(sr^2) = \phi(s)\phi(r^2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

- $\phi(sr^3) = \phi(s)\phi(r^3) = \phi(s)\phi(r^2)\phi(r) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Note that $D_8 \cong \langle R, S \rangle \leqslant GL_2(\mathbb{R})$ and that ϕ is a degree 2 representation of D_8. But D_8 has five conjugacy classes so by Theorem 1 there should be four more representations we can find. Specifically we want to find the "irreducible complex" representations of D_8. To help us find them we define some new terminology and recall a proposition from Section 2.1.

Definition 2.3.1. *Let G be a group and let x, $y \in G$. Then*

1. $[x, y] = x^{-1}y^{-1}xy$ *is the called the* commutator *of x and y.*
2. $G^{'} = \langle [x, y] | x, y \in G \rangle$ *is the subgroup of G generated by commutators of elements of G. That is, the* commutator subgroup *of G.*

Recall Proposition 2.1.4, if $\rho : G \to H$ is any group homomorphism and ϕ is a representation of H then the composition $\rho \circ \phi : G \to GL(V)$ is a representation of G. This result in addtion to the following corollaries of Theorem 2.3.1 will help us find all the degree 1 representations of D_8.

Corollary 2.3.2. *Let A be a finite abelian group. Every irreducible complex representation of A is 1-dimensional (i.e. is a homomorphism from $A \to \mathbb{C}^X$) and A has $|A|$ inequivalent irreducible complex representations. Furthermore, every finite dimensional complex matrix representation of A is equivalent to a representation into a group of diagonal matrices.*

Corollary 2.3.3.

The number of inequivalent (irreducible) degree 1 complex representations of any finite group G equals $|G/G^{'}|$ where $G^{'}$ is the commutator.

Proposition 2.3.1. *Let G be a group, x, $y \in G$, let $H \leqslant G$. Then $G/G^{'}$ is the largest abelian quotient of G in the sense that if $H \lhd G$ and G/H is abelian, then $G^{'} \leqslant H$.*

Hence these results give us the following plan of attack. We will find the commutator subgroup H of D_8. Then we will find the degree one representations $\phi : D_8/H \to$

$GL(\mathbb{C})$ guaranteed by the proposition and corollary above. Then since we have the natural homomorphism denoted by

$$\pi : D_8 \to D_8/H$$

we can compose ϕ with π to obtain the representation $\phi \circ \pi : D_8 \to GL(\mathbb{C})$.

We claim that $H = \{1, r^2\}$ is the commutator subgroup of D_8. We know that H is a normal subgroup of D_8 so we can construct the quotient group $D_8/H = \{H, rH, sH, srH\}$. We note that $Z(D_8/H) = D_8/H$. Therefore D_8/H is abelian and has order 4. Note that the only quotient group of D_8 that has larger order is $D_8/\{0\} \cong D_8$ which is not abelian. Thus by Proposition 2.3.1, H must be the commutator of D_8. Since D_8/H is a finite abelian group then by Corollary 2.3.2 every irreducible complex representation of D_8/H is 1-dimensional, i.e, $\phi : D_8/H \to GL(\mathbb{C})$. Moreover D_8/H has $|D_8/H| = 4$ inequivalent irreducible complex representations. These are the following, where we only define the representations on the generators of D_8:

$\overline{\phi_1} : D_8/H \to GL_1(\mathbb{C})$, where $rH \mapsto 1$ and $sH \mapsto 1$

$\overline{\phi_2} : D_8/H \to GL_1(\mathbb{C})$, where $rH \mapsto 1$ and $sH \mapsto -1$

$\overline{\phi_3} : D_8/H \to GL_1(\mathbb{C})$, where $rH \mapsto -1$ and $sH \mapsto 1$

$\overline{\phi_4} : D_8/H \to GL_1(\mathbb{C})$, where $rH \mapsto -1$ and $sH \mapsto -1$

It can be shown that these mappings are group homomorphisms that are not faithful. But we want representations of D_8, not D_8/H. So we will use Proposition 2.1.4 to find the representations of D_8. Let us denote $\bar{\phi_i} = \phi_i \circ \pi$ where $1 \leqslant i \leqslant 4$ and we have

- $\phi_1 = \phi_1 \circ \pi : D_8 \to GL_1(\mathbb{C})$ and $\bar{\phi}_1(r) = (\phi_1 \circ \pi)(r) = \phi_1(\pi(r)) = \phi_1(rH_1) = 1$ and $\bar{\phi}_1(s) = (\phi_1 \circ \pi)(s) = \phi_1(\pi(s)) = \phi_1(sH_1) = 1$. Therefore the representation is $\bar{\phi}_1 : D_8 \to GL_1(\mathbb{C})$ where $r \mapsto 1$ and $s \mapsto 1$

- $\phi_2 = \phi_2 \circ \pi : D_8 \to GL_1(\mathbb{C})$ and $\bar{\phi}_2(r) = (\phi_2 \circ \pi)(r) = \phi_2(\pi(r)) = \phi_2(rH_1) = 1$ and $\bar{\phi}_2(s) = (\phi_2 \circ \pi)(s) = \phi_2(\pi(s)) = \phi_2(sH_1) = -1$. Therefore the representation is $\bar{\phi}_2 : D_8 \to GL_1(\mathbb{C})$ where $r \mapsto 1$ and $s \mapsto -1$

- $\phi_3 = \phi_3 \circ \pi : D_8 \to GL_1(\mathbb{C})$ and $\bar{\phi}_3(r) = (\phi_3 \circ \pi)(r) = \phi_3(\pi(r)) = \phi_3(rH_1) = -1$ and $\bar{\phi}_3(s) = (\phi_3 \circ \pi)(s) = \phi_3(\pi(s)) = \phi_3(sH_1) = 1$. Therefore the representation is $\bar{\phi}_3 : D_8 \to GL_1(\mathbb{C})$ where $r \mapsto -1$ and $s \mapsto 1$

- $\bar{\phi}_4 = \phi_4 \circ \pi : D_8 \to GL_1(\mathbb{C})$ and $\bar{\phi}_4(r) = (\phi_4 \circ \pi)(r) = \phi_4(\pi(r)) = \phi_4(rH_1) = -1$ and $\bar{\phi}_4(s) = (\phi_4 \circ \pi)(s) = \phi_4(\pi(s)) = \phi_4(sH_1) = -1$. Therefore the representation is $\bar{\phi}_4 : D_8 \to GL_1(\mathbb{C})$ where $r \mapsto -1$ and $s \mapsto -1$

From the problem above which we described before, we already know the fifth representation of D_8 which is $\phi_5 : D_8 \to GL_2(\mathbb{C})$, where $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Again, since D_8 has five conjugacy classes, then by Theorem 2.3.1 we know $\mathbb{C}D_8$ has exactly five distinct isomorphism types of irreducible modules which have complex dimensions 1, 1, 1, 1, and 2. Equivalently we can say D_8 has exactly five inequivalent irreducible complex representations of the corresponding degrees. But most importantly each representation afforded by the $\mathbb{C}D_8$-module affords a character, which we will define now.

Definition 2.3.2 (Character of a Representation)**.** *If ϕ is a representation of G afforded by the KG-module V, the character of ϕ is the function $\chi : G \to K$ defined by $\chi(g) = tr(\phi(g))$, where the $tr(\phi(g))$ is the trace of the matrix $\phi(g)$ with respect to some basis of V (i.e., the sum of the diagonal entries of that matrix). The character is called irreducible or reducible according to whether the representation is irreducible or reducible, respectively. The degree of a character is the degree of the representation affording it.*

Definition 2.3.3 (Principal Character)**.** *The character of the trivial representation is the function $\chi(g) = 1$ for all $g \in G$.*

Recall the conjugacy class of of $g \in G$ is $[g] = \{xgx^1 | x \in G\}$. We know that D_8 has the following five conjugacy classes: $[1], [r], [s], [r^2]$, and $[sr]$. We show below that the character of a representation of G is a class function, i.e., a function from G into K which is constant on the conjugacy classes of G.

Lemma 2.3.1. *Let ϕ be a representation of G of degree n with character χ. Then χ is a class function.*

Proof. Since $\phi(g^{-1}xg) = \phi(g)^{-1}\phi(x)\phi(g)$ for all $g, x \in G$, we see that by taking the trace of both sides we get the following:

$$\begin{aligned} tr[\phi(g^{-1}xg)] &= tr[\phi(g^{-1})\phi(x)\phi(g)] \\ &= tr(\phi(x)) \\ \\ \Rightarrow \chi(g^{-1}xg) &= \chi(g) \end{aligned}$$

□

The character table of D_8 is the 5 ×5 table of character values where the conjugacy classes,1, s, r, r^2, sr are located along the top row and the list of irreducible characters down the first column. The entry in the table in row χ_i and column g_j is $\chi_i(g_j)$. Moreover, by convention we make the principal character the first row, the identity the first column, and we list the characters in increasing order by degrees. Finally, the character table of a finite group is unique up to a permutation of its rows and columns. Note that we have already found where each $\phi_i, (i = 1...5)$ sends the generators r and s, thus it is easy to compute $\chi_i(r)$ and $\chi_i(s)$ for all $1 \leq i \leq 5$. Thus we can place each of these values in columns three and four of the character table, but we still need to fill in the entries for columns one, two, and five. Thus, we need to find the value of the characters χ_i for all $1 \leq i \leq 5$ evaluated at $1, r^2$, and sr. So we use the fact that $\phi_i(g)$ is a group homomorphism to help us fill in the last 15 entries.

To fill column one, we compute the following:

$\chi_1(1) = tr(\phi_1(1)) = 1$

$\chi_2(1) = tr(\phi_2(1)) = 1$

$\chi_3(1) = tr(\phi_3(1)) = 1$

$\chi_4(1) = tr(\phi_4(1)) = 1$

$\chi_5(1) = tr(\phi_5(1)) = 2.$

We call this first column the identity column. Since the first four representations are of degree 1 then by definition they are irreducible representations (or simple modules).

classes:	1	r^2	s	r	sr
sizes:	1	1	2	2	2
χ_1	1				
χ_2	1				
χ_3	1				
χ_4	1				
χ_5	2				

To fill in the second column, we compute the following:

$$\chi_1(r^2) = tr(\phi_1(r^2)) = tr.[\phi_1(r)\phi_1(r)] = tr[1 \cdot 1] = tr[1] = 1$$

$$\chi_2(r^2) = tr(\phi_2(r^2)) = tr[\phi_2(r)\phi_2(r)] = tr[1 \cdot 1] = tr[1] = 1$$

$$\chi_3(r^2) = tr(\phi_3(r^2)) = tr[\phi_3(r)\phi_3(r)] = tr[-1 \cdot -1] = tr[1] = 1$$

$$\chi_4(r^2) = tr(\phi_4(r^2)) = tr[\phi_4(r)\phi_4(r)] = tr[-1 \cdot -1] = tr[1] = 1$$

$$\chi_5(r^2) = tr[\phi_5(r^2)] = tr[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}] = tr[\begin{pmatrix} -1 & -0 \\ 0 & -1 \end{pmatrix}] = -2$$

classes:	1	r^2	s	r	sr
sizes:	1	1	2	2	2
χ_1	1	1			
χ_2	1	1			
χ_3	1	1			
χ_4	1	1			
χ_5	2	-2			

Finally we use the following computations to fill in the fifth column and our work from before to fill in the columns for conjugacy classes $[r]$ and $[s]$.

$$\chi_1(sr) = tr(\phi_1(sr)) = tr[\phi_1(s)\phi_1(r)] = tr[1 \cdot 1] = tr[1] = 1$$

$$\chi_2(sr) = tr(\phi_2(sr)) = tr[\phi_2(s)\phi_2(r)] = tr[-1 \cdot 1] = tr[-1] = -1$$

$$\chi_3(sr) = tr(\phi_3(sr)) = tr[\phi_3(s)\phi_3(r)] = tr[1 \cdot -1] = tr[-1] = -1$$

$$\chi_4(sr) = tr(\phi_4(sr)) = tr[\phi_4(s)\phi_4(r)] = tr[-1 \cdot -1] = tr[1] = 1$$

$$\chi_5(sr) = tr(\phi_5(sr)) = tr[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}] = tr[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}] = 0$$

classes:	1	r^2	s	r	sr
sizes:	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Let us keep exploring the structure of the $\mathbb{C}D_8$ group ring ($\mathbb{C}-$algebra). We recall the following definition from Section 1.2 and we state Wedderburn's theorem, another result that has far reaching consequences for our group ring.

Definition 2.3.4. *Let R be a nonzero ring.*

1. *A nonzero element e in a ring R is called idempotent if $e^2 = e$.*

2. *Idempotents e_1 and e_2 are said to be orthogonal if $e_1e_2 = e_2e_1 = 0$.*

3. *An idempotent element e is said to be primitive if it cannot be written as a sum of two (commuting) orthogonal idempotents.*

4. *The idempotent e is called a primitive central idempotent if $e \in Z(R)$ and e cannot be written as a sum of two orthogonal idempotents in the ring Z(R).*

Theorem 2.3.4 (Wedderburn's)**.** *Let R be a nonzero ring with 1 (not necessarily commutative). Then the following are equivalent:*

1. *every R-module is projective*

2. *every R-module is injective*

3. *every R-module is completely reducible*

4. *the ring R considered as a left R-module is a direct sum: $R \cong L_1 \oplus L_2 \oplus \cdots L_n$, where each L_i is a simple module (i.e., a simple left ideal) with $L_i = Re_i$, for some $e_i \in R$ with*

   (i) $e_ie_j = 0$ *if* $i \neq j$

   (ii) $e_i^2 = e_i$ *for all* i

   (iii) $\sum_{i=1}^{n} e_i = 1$

5. *as rings, R is isomorphic to a direct product of matrix rings over division rings, i.e., $R \cong R_1 \times R_2 \times \cdots \times R_r$ where R_j is a two-sided ideal of R and R_j is isomorphic to the ring of all $n_j \times n_j$ matrices with entries in a division ring $\Delta_j, j = 1, 2, \cdots, r$. The integer r, the integers n_j, and the division ring Δ_j (up to isomorphism) are uniquely determined by R.*

Now since we are working over the field $\mathbb{C}$, which is an algebraically closed field, it turns out that each of the division rings in Wedderburn's theorem is actually isomorphic to $\mathbb{C}$. Thus in part (5) of Wedderburn's theorem we get that

$$R \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

Hence another useful way of thinking of the elements of R is as $n \times n$ (block diagonal) matrices of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}$$

where $A_i \in M_{n_i}(\mathbb{C})$ for $i = 1, 2, \ldots, r$ and $n = \sum_{i=1}^{r} n_i$.

We say that a ring R satisfying any of the (equivalent) properties in Wedderburn's Theorem is *semisimple* with minimum condition(D.C.C). Note that an R-module Q is *injective* if and only if whenever Q is a submodule of any R-module M, then M has a submodule N such that $M = Q \oplus N$. Now let's recall Maschke's Theorem:

Theorem 2.3.5 (Maschke)**.** *Let G be a finite group and let K be a field whose characteristic does not divide $|G|$. If V is any KG-module and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$ (i.e., every submodule is a direct summand.)*

Thus we see that group ring KG where $|G|$ does not divide the characteristic of K is injective and thus obtain the following corollary to Wedderburn's Theorem:

Corollary 2.3.6. *If G is a finite group and K is a field whose characteristic does not divide $|G|$, then the group algebra KG is a semisimple ring.*

So as a ring we see that $\mathbb{C}D_8 \cong M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C})$ and as a vector space over $\mathbb{C}$

$$\mathbb{C}D_8 \cong span_{\mathbb{C}} \left\{ \begin{pmatrix} \phi_1(g) & 0 & 0 & 0 & 0 \\ 0 & \phi_2(g) & 0 & 0 & 0 \\ 0 & 0 & \phi_3(g) & 0 & 0 \\ 0 & 0 & 0 & \phi_4(g) & 0 \\ 0 & 0 & 0 & 0 & \phi_5(g) \end{pmatrix} : g \in D_8 \right\}. \tag{2.1}$$

Thus we identify the identity in $\mathbb{C}D_8$ with the 6×6 identity matrix contained on the right hand side of (2.1). Thus we can see that $\mathbb{C}D_8$ has five primitive orthogonal idempotents. These are given by $z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $z_3 =$ $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $z_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $z_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Hence we can identify the left ideals of $\mathbb{C}D_8$ with $L_i = \mathbb{C}D_8 z_i$ for all $1 \leq i \leq 5$ which gives us the irreducible representations of $\mathbb{C}D_8$. Note then, that these left ideals give a complete set of isomorphism classes of irreducible $\mathbb{C}D_8$-modules. So as a left module over itself we see that $\mathbb{C}D_8 \cong L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5 \oplus L_5$. We also know that any left $\mathbb{C}D_8$-module can be written as a direct sum of irreducible $\mathbb{C}D_8$-modules isomorphic to these five. Hence we have described the structure of $\mathbb{C}D_8$ as a group ring and the structure of all left $\mathbb{C}D_8$-modules.

2.4 Character Theory

Now many times it is difficult to compute or even write down the representations for large groups. Yet we are able to calculate numerical invariants that allow us to characterize the similarity classes of representations of our groups. We we have already introduced these numerical invariants of linear representations, called *characters*, and now we explore some of their properties.

When calculating the irreducible representations of $\mathbb{C}D_8$ we did not need to use the properties of characters, but sometimes it is easier to calculate the characters of irreducible representations of a group by exploiting some nice properties of characters. We have already seen that characters are class functions. We now note another one: since the trace of any $n \times n$ identity matrix is n we see that for any character $\chi : G \to K$, $\chi(e_g)$ is equal to the degree of the representation ϕ associated with χ, where e_g is the identity element in G. Also note that since $tr(AB) = tr(BA)$ for any matrices, then whenever A is invertible we get that $tr(A^{-1}BA) = tr(B)$. Hence, we obtain the following theorem.

Theorem 2.4.1. *Equivalent representations have the same character.*

Proof. Let ϕ_1 and ϕ_2 be equivalent linear representations of the finite group G with representation spaces V_1 and V_2 respectively. Let us show that ϕ_1 and ϕ_2 have the same character. By assumption, there exists a isomorphism $h \in Hom_K(V_1, V_2)$ such that

$$h\phi_1(g)h^{-1} = \phi_2(g), g \in G.$$

Now let $\chi_1(g)$ be the character of ϕ_1. Thus

$$\chi_1(g) = tr[\phi_1(g)] \quad = \quad tr[h^{-1}\phi_2(g)h] = tr[\phi_2(g)hh^{-1}] = tr[\phi_2(g)] = \chi_2(g).$$

Therefore, $\chi_1(g) = \chi_2(g)$ □

Now suppose $\mathbb{C}G = M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, where r denotes the number of conjugacy classes of G. Let $L_1, L_2, \ldots, L_r$ denote the non-isomorphic irreducible representations of G and let χ_i be the corresponding character, called the *irreducible character*, for all $1 \leq i \leq r$. Since any left $\mathbb{C}G$-module is isomorphic to a direct sum of irreducible left $\mathbb{C}G$-modules we see that its associated character is a sum of the characters associated to the irreducible representations. In fact, it can be shown that $\chi_1, \chi_2, \ldots, \chi_r$ are a basis for the space of all complex class functions.

Now to establish other useful properties of characters we will define the following product, which turns out to be a Hermitian inner product on the space of class functions:

$$(\theta, \psi) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\psi(g)}.$$

Here the bar denotes complex conjugation. It can be shown that the product $(-,-)$ is Hermitian, that is for all $\alpha, \beta \in \mathbb{C}$ the following properties are satisfied:

(a) $(\alpha\theta_1 + \beta\theta_2, \psi) = \alpha(\theta_1, \psi) + \beta(\theta_2, \psi)$,

(b) $(\theta, \alpha\psi_1 + \beta\psi_2) = \alpha(\theta, \psi_1) + \beta(\theta, \psi_2)$, and

(c) $(\theta, \psi) = \overline{(\psi, \theta)}$.

With respect to this product it can be shown that the irreducible characters of G form an orthonormal basis on the space of class functions, i.e. not only do the characters form a basis, but also $(\chi_i, \chi_j) = \delta_{ij}$. The following propositions and theorems allow us to prove this fact but in addition provide a useful orthogonality relation on the irreducible characters.

Proposition 2.4.1. *Let $z_1, \ldots, z_r$, be the orthogonal primitive central idempotents in $\mathbb{C}G$ labelled in such a way that z_i acts as the identity on the irreducible $\mathbb{C}G$-module M_i, an let χ_i be the character afforded by M_i. Then*

$$z_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1})g.$$

Proof. Let $z = z_i \in \mathbb{C}G$. Then $z = \sum_{g \in G} \alpha_g g$. Let ϕ be the regular representation of G whose character is ρ,

$$\rho : G \to \mathbb{C}\rho(g) = tr(\phi(g))$$

where $tr(\phi(g)) = 0$ if $g \neq 1$ or $tr(\phi(g)) = |G|$ if $g = 1$.

Recall

$$\begin{aligned} \mathbb{C}G &\cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times M_{n_r}(\mathbb{C}) \\ &\cong n_1 M_1 \oplus n_2 M_2 \oplus \cdots \oplus n_r M_r \end{aligned}$$

where M_i are left $\mathbb{C}G$-modules (simple ideals), $1 \leqslant i \leqslant r$. But we know that every M_i affords a χ_i. Therefore,

$$\rho = n_1\chi_1 + n_2\chi_2 + \cdots + n_r\chi_r$$

And since $n_i = tr(1) = \chi_i(1)$ we have

$$\begin{aligned} \rho &= \chi_1(1)\chi_1 + \chi_2(1)\chi_2 + \cdots + \chi_r(1)\chi_r \\ &= \sum_{j=1}^{r} \chi_j(1)\chi_j \end{aligned}$$

So our goal is to show that the coefficient $\alpha_g = \frac{\chi_j(1)}{|G|}\chi_j(g^{-1})$ and we have

$$\rho(zg^{-1}) = \sum_{j=1}^{r} \chi_j(1)\chi_j(zg^{-1})$$

So let $z = \sum_{h \in G} \alpha_h h$. Then

$$\begin{aligned} \Rightarrow \quad zg^{-1} &= \sum_{h \in G} \alpha_h h(g^{-1}) \\ \Rightarrow \quad \rho(zg^{-1}) &= \rho(\sum_{h \in G} \alpha_h h(g^{-1})) \\ &= \sum_{h \in G} \alpha_h \rho(hg^{-1}) \\ &= \alpha_g \rho(1) \\ &= \alpha_g |G|. \end{aligned}$$

Also note that $\rho(zg^{-1}) = \sum_{j=1}^{r} \chi_j(1)\chi_j(zg^{-1}) \Rightarrow \alpha_g|G| = \sum_{j=1}^{r} \chi_j(1)\chi_j(zg^{-1})$. Let ϕ_j be an irreducible representation afforded by M_j where $1 \leqslant j \leqslant r$. We know that representation problems are simplified by noting that every linear representation of G (a group homomorphism) defines a corresponding representation of the the group algebra $\mathbb{C}G$ (a linear transformation). With that said we have

$$\phi^* : \mathbb{C}G \to End(M_j) \Rightarrow \phi_j(zg^{-1}) = \phi_j(z) \cdot \phi_j(g^{-1}).$$

where ϕ^* is a linear transformation. Thus,

$$\phi^* : \mathbb{C}G \to End(M_j) \Rightarrow \phi_j(zg^{-1}) = \phi_j(z) \cdot \phi_j(g^{-1}).$$

where ϕ^* is a linear transformation. Thus,

$$\begin{aligned}
\chi_j(zg^{-1}) &= tr(\phi_j(zg^{-1})) \\
&= \begin{cases} tr(0) & \text{if } j \neq i \\ tr(\phi_i(g^{-1})) & \text{if } j = i \end{cases} \\
&= \begin{cases} 0 & \text{if } j \neq i \\ \chi_i(g^{-1}) & \text{if } j = i \end{cases} \\
&= \delta_{ij}\chi_i(g^{-1})
\end{aligned}$$

So we can see that $\chi_j(zg^{-1}) = \delta_{ij}\chi_i(g^{-1})$.Finally, we have that

$$\begin{aligned}
\textstyle\sum_{j=1}^{r} \chi_j(1)\chi_j(zg^{-1}) &= \alpha_g|G| \\
\Rightarrow \quad \textstyle\sum_{j=1}^{r} \chi_j(1)\delta_{ij}\chi(g^{-1}) &= \alpha_g|G| \\
\Rightarrow \quad \alpha_g &= \tfrac{1}{|G|}\chi_i(1)\overline{\chi_i(g^{-1})}.
\end{aligned}$$

□

Proposition 2.4.2. *If χ is any character of G then $\chi(g)$ is the sum of roots of 1 in $\mathbb{C}$ and $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.*

Proof. Let ϕ be a representation of G whose character is χ. That is,

$$\begin{aligned}
\phi : G &\to GL(V) \\
g &\mapsto \phi(g)
\end{aligned}$$

and

$$\begin{aligned}
\chi : G &\to K, \\
\chi(g) &= tr(\phi(g))
\end{aligned}$$

Let $g \in G$ be a fixed element where $|g| = n$ and let $m(x)$ denote the minimal polynomial of g acting on V. Since $m(x)|x^n - 1$ (where $x^n - 1$ is the cyclotomic polynomial in $\mathbb{C}$), then $m(x)$ has distinct roots $\Rightarrow$ $\phi(g)$ is diagonalizable with n^{th} roots of 1 on the diagonal which implies that there exist some square matrix A such that

$$\phi(g) = A^{-1} \begin{pmatrix} \zeta_1 & & & \\ & \zeta_2 & & \\ & & \ddots & \\ & & & \zeta_r \end{pmatrix} A$$

$$\Rightarrow tr(\phi(g)) = tr \begin{pmatrix} \zeta_1 & & & \\ & \zeta_2 & & \\ & & \ddots & \\ & & & \zeta_r \end{pmatrix}$$

$$\Rightarrow tr(\phi(g)) = \zeta_1 + \cdots + \zeta_r.$$

Therefore $\phi(g)$ is the sum of roots of 1 in $\mathbb{C}$. Furthermore, is ζ is a root of 1in $\mathbb{C}$, then $\zeta^{-1} = \bar{\zeta}$. Therefore, the inverse of a diagonal matrix with roots of 1 on the diagonal is the diagonal matrix with the complex conjugates of those roots of 1 on the diagonal. That is,

$$\begin{pmatrix} \zeta_1 & & & \\ & \zeta_2 & & \\ & & \ddots & \\ & & & \zeta_r \end{pmatrix}^{-1}$$

where $i = 1, 2, \ldots, r$.

Finally since the complex conjugates of a sum is the sum of the complex conjugates, i.e., $\bar{A} + \bar{B} = \overline{A + B}$, then

$$\chi(g^{-1}) = tr[\phi(g^{-1})] = \overline{tr[\phi(g^{-1})]} = \overline{\chi(g)}.$$

□

Theorem 2.4.2. *Let G be a finite group and $\chi_1, \cdots, \chi_r$ be the irreducible characters of G over $\mathbb{C}$. Then with respect to the inner product $(-,-)$ defined above we have*

$$(\chi_i, \chi_j) = \delta_{ij}$$

and the irreducible characters are an orthonormal basis for the space of class functions. In particular, if θ is any class function then

$$\theta = \sum_{i=1}^{r} (\theta, \chi_i)\chi_i.$$

This is usually called thhe First Orthogonality Relation for Group Characters. We finish this chapter by stating one more definition and two results that will aid greatly in computing the irreducible representations of D_{2n}.

Definition 2.4.1 (Norm of a Class function)**.** *For θ any class function on G the norm of θ is $(\theta, \theta)^{1/2}$ and will be denoted by $||\theta||$*

Proposition 2.4.3 (Irreducibilty criterion)**.** *If ϕ is a complex representation of G affording χ then ϕ is irreducible if and only if the norm is 1.*

Finally if θ and ψ are any two characters of representations of G we can simplify the computations of (θ, ψ) as follows: If $\kappa_1, \ldots, \kappa_r$ are the conjugacy classes of G with sizes $d_1, \ldots, d_r$ and representatives $g_1, \ldots g_r$ respectively, then the value $\theta(g_i)\overline{\psi(g_i)}$ appears d_i times in the sum for (θ, ψ), once for each element of κ_i. Collecting these terms gives us the following:

$$(\theta, \psi) = \frac{1}{|G|} \sum_{i=1}^{r} d_i \theta(g_i)\overline{\psi(g_i)},$$

a sum only over the conjugacy classes of G. In particular we obtain the following:

$$||\theta||^2 = (\theta, \theta) = \frac{1}{|G|} \sum_{i=1}^{r} d_i |\theta(g_i)|^2.$$

Chapter 3

The Representation Theory of D_{2n}

3.1 Introduction

Before we prove the main result of this section let us explore a bit. In section 2.3 we saw that the linear representations of $\rho : D_8 \to GL_2(\mathbb{C})$ given by $r \mapsto R = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $s \mapsto S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is of degree 2 and irreducible. Now let $\theta = \frac{2\pi}{n}$ and observe the following.

Lemma 3.1.1. *There exists a matrix $A \in GL_2(\mathbb{C})$ such that $R = A \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} A^{-1}$.*

Proof. By solving the the characteristic equation $det(R - \lambda I) = 0$, we can see that

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\begin{aligned}
\Rightarrow \quad & \cos^2\theta - 2\lambda\cos\theta + \lambda^2\theta = 0 \\
\Rightarrow \quad & \lambda^2\theta - 2\lambda\cos\theta + 1 = 0 \\
\Rightarrow \quad & \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\
& = \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} \\
& = \cos\theta \pm \sqrt{-\sin^2\theta}
\end{aligned}$$

Since for $n \geqslant 3$ we have that $0 \leqslant \theta \leqslant \pi \Rightarrow \sin\theta > 0$. Hence $\lambda = \cos\theta \pm i\sin\theta \Rightarrow \lambda = e^{\pm\theta}$. Computing the eigenspace associated with $\lambda_1 = e^{i\theta}$ we see that

$$\left[\begin{array}{cc|c} \cos\theta - \lambda_1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda_1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & i\sin\theta & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This implies then that $-ix\sin\theta - y\sin\theta = 0 \Rightarrow \sin\theta(ix+y) = 0 \Rightarrow y = -ix$ Thus we see that $E_{\lambda_1} = span\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$. Similarly we obtain that $E_{\lambda_2} = span\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$. Direct computation show that

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = R.$$

Thus, $A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. □

Corollary 3.1.1. *Let $\rho_\theta : D_{2n} \rightarrow GL_2(\mathbb{C})$ be given by $r \mapsto R' = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $s \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then ρ_θ and ρ are equivalent representations of D_{2n}.*

Proof. By Theorem 2.4.1 it is sufficient to show that the characters on the generators is the same. Clearly the trace of S for both representations is the same and since R is similar to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ then by our lemma their traces are the same. □

In Section 2.3 we explored D_8 (n=4). Now let us explore D_{10} (n=5) so that we can motivate how we found the representations of D_{2n} in general. By the results in section 3.2 later in this chapter we know that D_{10} has four conjugacy classes. We list the classes here:

$$\begin{aligned} [1] &= \{1\}, \\ [r] &= \{r, r^{-1}\} = \{r, r^4\}, \\ [r^2] &= \{r^2, r^{-2}\} = \{r^2, r^3\}, \\ [s] &= \{s, sr, sr^2, sr^3, sr^4\}, \end{aligned}$$

From the results in section 3.3 later in this chapter we know the commutator subgroup of D_{10} is $D_{10}^{'} = \{1, r, r^2, r^3, r^4\}$ which gives us 2 degree 1 representations

$$\begin{aligned} \phi_1 : \quad & D_{10} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto 1 \end{aligned}$$

and

$$\begin{aligned} \phi_2 : \quad & D_{10} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto -1 \end{aligned}$$

in a process similar to how we analyzed D_8 in section 2.3. We know from example 2.2.5 and from the corollary above (3.3.1) that one degree two irreducible representations of D_{10} is given by

$$\rho_1 : D_{10} \to GL_2(\mathbb{C})$$

where

$$r \mapsto R^{'} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, s \mapsto S, \theta = \frac{2\pi}{5}.$$

Hence we have 3 irreducible representations of D_{10}; we just need one more. We note that $R^{'}$ is a rotation of the xy-plane by $2\pi/5$ radians. Could it be possible that if we rotate the pane by $4\pi/5$ radians that we get another irreducible representation of D_{2n}, i.e.,

$$\rho_2 : D_{10} \to GL_2(\mathbb{C}),$$

$$r \mapsto R^{'} = \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix}, s \mapsto S$$

another degree two representation. First note that $r^5 = 1$ and

$$\begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix}^5 = \begin{pmatrix} (e^{2i\theta})^5 & 0 \\ 0 & (e^{-2i\theta})^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $e^{2i\theta}$ and $e^{-2i\theta}$ are fifth roots of unity. Hence if we send

$$sr^m \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix}^m$$

we see that ρ_2 is an injective homomorphsim, i.e., a faithful representation of D_{10}. But is it irreducible? From the last result in Section 2.4, Proposition 2.4.3, we just need to show that the norm of the character afforded by ρ_2 is equal to 1. We use the following table to organize our computations.

Conjugacy Class $[a]$	Size of Conj. Class	$\chi_2(a)$
$[1]$	1	2
$[r]$	2	$2\cos\left(\frac{4\pi}{5}\right)$
$[r^2]$	2	$2\cos\left(\frac{8\pi}{5}\right)$
$[s]$	5	0

A quick calculation shows that

$$\begin{aligned}
(\chi_2, \chi_2) &= \frac{1}{10}[1 \cdot \chi_2(1)^2 + 2 \cdot \chi_2(r)^2 + 2\chi_2(r^2)^2 + 5\chi_2(s)^2] \\
&= \frac{1}{10}[4 + 2 \cdot 4cos^2(\frac{4\pi}{5}) + 2 \cdot 4cos^2(\frac{8\pi}{5}) + 0] \\
&= \frac{4}{10}[1 + 2cos^2(\frac{4\pi}{5}) + 2cos^2(\frac{8\pi}{5})] \\
&= \frac{2}{5}(2.5) \\
&= 1
\end{aligned}$$

Thus ρ_2 is irreducible. But now the question becomes, is the representation equivalent to either θ_1, θ_2, or ρ_1. By dimension of the representation space we know that ρ_2 cannot be equivalent to θ_1, or θ_2. But what about ρ_1? By theorem 2.4.1 we see that they would be equivalent if they had the same character, but

$$\chi_1(r) = 2\cos\left(\frac{2\pi}{5}\right) \neq 2\cos\left(\frac{4\pi}{5}\right) = \chi_2(r)$$

So ρ_1 and ρ_2 cannot be equivalent. Thus we have found all four irreducible representations of D_{10}. To find the representations of D_{2n} in general we explored D_{2n} for certain values of n; $n = 4, 6, 8$ to get an idea on how to attack the representations when n is even and $n = 3, 5, 7$ to understand better the case of when n is odd. My results are summarized in the next three sections.

3.2 Conjugacy Classes of D_{2n}

The main result of section 3.2 is the following:

Theorem 3.2.1. *Let* $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle = \{1, r, r^2, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}\}$, *where* $n \in \mathbb{Z}^+, n \geqslant 3$.

1. *If n is even, i.e., $n = 2k$ for $k \geqslant 2$, then D_{2n} has $k + 3$ conjugacy classes.*
2. *If n is odd, i.e., $n = 2k + 1$ for $k \geqslant 1$, then D_{2n} has $k + 2$ conjugacy classes.*

To prove this, we first need to prove the following lemmas

Lemma 3.2.1. *Let* $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$ *such that* $n \in \mathbb{Z}^+, n \geqslant 3$. *Then* $r^m s = sr^{-m}$ *for all* m *in* $\mathbb{Z}$.

Proof. By induction on m. If $m = 1$ then $rs = sr^{-1}$ by our presentation. Let $m > 1$ and suppose $r^m s = sr^{-m}$. We need to show $r^{m+1}s = sr^{-(m+1)}$. We start with the left hand side and use our assumptions to get the right hand side.

$$\begin{aligned} r^{m+1}s &= r^m \cdot r \cdot s \\ &= r^m(r \cdot s) \\ &= r^m(sr^{-1}) \\ &= (r^m s)r^{-1} \\ &= (sr^{-m})r^{-1} \\ &= sr^{-m-1} \\ &= sr^{-(m+1)} \end{aligned}$$

$\square$

Lemma 3.2.2. *If $x \in D_{2n}$ such that $x \neq r^m$ for any $0 \leqslant m \leqslant n$, then $rx = xr^{-1}$.*

Proof. If $x \in D_{2n}$ such that $x \neq r^m$ for any $0 \leqslant m \leqslant n$, then $x = sr^m$ for some $0 \leqslant m \leqslant n$. Thus,

$$\begin{aligned} rx &= r(sr^m) \\ &= (rs)r^m \\ &= (sr^{-1})r^m \\ &= s(r^{-1}r^m) \\ &= s(r^m r^{-1}) \\ &= (sr^m)r^{-1} \\ &= xr^{-1} \end{aligned}$$

□

Lemma 3.2.3. *Let $n = 2k, k \in \mathbb{Z}^+, k \geqslant 2$, then the conjugacy classes of D_{2n} are*

$$\begin{aligned} [1] &= \{1\}, \\ [r] &= \{r, r^{-1}\}, \\ [r^2] &= \{r^2, r^{-2}\}, \\ &\vdots \\ [r^{k-1}] &= \{r^{k-1}, r^{-(k+1)}\}, \\ [r^k] &= \{r^k\} \\ [sr] &= \{sr^{2b-1} : b = 1, \cdots k\}, \\ [sr^2] &= \{sr^{2b} : b = 1, \cdots k\}. \end{aligned}$$

Proof. We know that if $x \in Z(D_{2n})$,then $[x] = \{x\}$. Thus since $Z(D_{2n}) = \{1, r^k\}$ we see that $[1] = \{1\}$ and $[r^k] = \{r^k\}$.

Claim 1: $[r^m] = \{r^m, r^{-m}\}$ for all $1 \leqslant m \leqslant k-1$.

"$\supseteq$" We know $r^m \in [r^m]$ by definition. So all that is left to show is that $r^{-m} \in [r^m]$, i.e., $r^{-m} = y^{-1}r^m y$ for some $y \in D_{2n}$. By the previous lemma

$$\begin{aligned} rx &= xr^{-1} \\ \Rightarrow r^{-1} &= x^{-1}rx \\ \Rightarrow r^{-m} &= (x^{-1}rx)^m \\ &= (x^{-1}rx)(x^{-1}rx)\cdots(x^{-1}rx) \\ &= x^{-1}r^m x \\ \Rightarrow & \quad r^{-m} \in [r^m]. \end{aligned}$$

" $\subseteq$" Let $g \in [r^m] \Rightarrow g = x^{-1}r^m x$ for some $x \in D_{2n}$. We need to show that $g \in \{r^m, r^{-m}\}$.

Case 1 Suppose $x = r^t$ for some $0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} g &= (r^t)^{-1}r^m r^t \\ &= r^{-t}r^m r^t \\ &= r^m \in \{r^m, r^{-m}\} \end{aligned}$$

Case 2 Suppose $x \neq r^t$ for all $0 \leqslant t \leqslant n$. Then $x = sr^t$ for some $0 \leqslant t \leqslant n-1$. Thus,

$$\begin{aligned} g &= (sr^t)^{-1}r^m(sr^t) \\ &= r^{-t}sr^m sr^t \\ &= r^{-t}ssr^{-m}r^t \\ &= r^{-t}r^{-m}r^t \\ &= r^{-m} \in \{r^m, r^{-m}\} \end{aligned}$$

Claim 2: $[sr] = \{sr^{2b-1} : b = 1, \cdots k\}$.

" $\supseteq$ " By induction on b. Let $b = 1$. Then $sr^{2(1)-1} = sr^1 = sr \in [sr]$ by definition. Let $1 < b < k$ and suppose $sr^{2b-1} \in [sr]$. This means $sr^{2b-1} = x^{-1}srx$ for some $x \in D_{2n}$. We will show that $sr^{2(b+1)-1} = sr^{2b+1} \in [sr]$. Again by lemma

3.2.2 we have

$$\begin{aligned} r(sr^{2b+1}) &= (sr^{2b+1})r^{-1} \\ \Rightarrow sr^{2(b+1)} &= r^{-1}sr^{2b+1}r^{-1} \\ &= r^{-1}sr^{2b} \\ &= r^{-1}(sr^{2b-1+1}) \\ &= r^{-1}(sr^{2b-1}r) \\ &= r^{-1}(x^{-1}srx)r \\ &= (xr)^{-1}(sr)(xr) \in [sr] \end{aligned}$$

" $\subseteq$ " Let $x \in [sr] \Rightarrow x = y^{-1}(sr)y$ for some $y \in D_{2n}$.

Case 1 Let $y = r^m, 0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} x &= r^{-m}(sr)r^m \\ &= (r^{-m}s)r^{m+1} \\ &= sr^m r^{m+1} \\ &= sr^{2m+1} \end{aligned}$$

Now $2m+1$ is odd $\Rightarrow 2m+1(mod 2k) = 2b-1$ for some $1 \leqslant b \leqslant k$. Thus $x \in \{sr^{2b-1} | b = 1, \cdots, k\}$.

Case 2 Let $y = sr^m, 0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} x &= (sr^m)^{-1}sr(sr^m) \\ &= r^{-m}s^{-1}srsr^m \\ &= r^{-m}sr^{-1}r^m \\ &= sr^m r^{m-1} \\ &= sr^{2m-1} \end{aligned}$$

Again $2m-1$ is odd $\Rightarrow (2m-1)mod(2k) = 2b-1$ for some $1 \leqslant b \leqslant k$. Thus $x \in \{sr^{2b-1} | b = 1, \cdots, k\}$.

Claim 3: $[sr^2] = \{sr^{2b} : b = 1, \cdots k\}$

" $\supseteq$ " By induction on b. Let $b = 1$. Then $sr^2 \in [sr^2]$ by definition. Let $1 \leqslant b \leqslant k$ and suppose $sr^{2b} \in [sr]$. This means $sr^{2b} = x^{-1}(sr)x$ for some $x \in D_{2n}$. We will show that $sr^{2(b+1)} = sr^{2b+2} \in [sr]$. Once again by the previous lemma we have that

$$\begin{aligned} r(sr^{2b+2}) &= (sr^{2b+2})r^{-1} \\ \Rightarrow sr^{2b+2} &= r^{-1}sr^{2b}r^{2}r^{-1} \\ &= r^{-1}(x^{-1}srx)r \\ &= (xr)^{-1}(sr)(xr) \in [sr] \end{aligned}$$

" $\subseteq$ " Let $x \in [sr^2] \Rightarrow x = y^{-1}(sr^2)y$ for some $y \in D_{2n}$.

Case 1 Let $y = r^m, 0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} x &= r^{-m}(sr^2)r^m \\ &= sr^m r^{m+2} \\ &= sr^{2m+2} \end{aligned}$$

Since $2m + 2mod(2k)$ is even $\Rightarrow x = sr^{2b}$ for some $1 \leqslant b \leqslant k \Rightarrow x \in \{sr^{2b}|b = 1, \cdots, k\} \Rightarrow x \in \{sr^{2b} : b = 1, \cdots k\}$

Case 2 Let $y = sr^m, 0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} x &= (sr^m)^{-1}(sr^r)(sr^m) \\ &= r^{-m}s^{-1}sr^2sr^m \\ &= r^{-m}sr^{-2}r^m \\ &= sr^m r^{m-2} \\ &= sr^{2m-2} \end{aligned}$$

Again since $2m - 2mod(2k)$ is even $\Rightarrow x = sr^{2b}$ for some $1 \leqslant b \leqslant k$.

□

By counting the conjugacy classes listed in this lemma, we see that part (a) of Theorem 1 is proved. Part (b) is proved similarly by proving the following lemma:

Lemma 3.2.4. *If $n = 2k+1, n \in \mathbb{Z}^+, k \geqslant 1$, then the conjugacy classes of D_{2n} are*

$[1] = \{1\}, [r] = \{r, r^{-1}\}, [r^2] = \{r^2, r^{-2}\}, \cdots, [r^{k-1}] = \{r^{k-1}, r^{-(k+1)}\}, [r^k] = \{r^k, r^{-k}\}, [s] = \{sr^b|b = 1, \cdots n\}$.

3.3 Commutator Subgroup of D_{2n}

From our discussions in Section 2.3, we know that if we can find the commutator subgroup of D_{2n}, D'_{2n}, then this will help us in finding $|D_{2n}/D'_{2n}|$ degree 1 irreducible representations of D_{2n}. Thus the goal of this section is to prove the following theorem:

Theorem 3.3.1. *Let* $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1}\rangle$*. Then* $D_{2n}^{'} = \langle r^2\rangle$*.*

First we prove the following lemmas

Lemma 3.3.1. $\langle r^2\rangle \lhd D_{2n}$*.*

Proof. Let $x \in D_{2n}, y \in \langle r^2\rangle$. We need to show $xyx^{-1} \in \langle r^2\rangle$. Since $y \in \langle r^2\rangle$, then $y = (r^2)^b = r^{2b}$ for some $b \in \mathbb{Z}$.

Case 1 Suppose $x = r^m, 0 \leqslant m \leqslant n-1$. Then $xyx^{-1} = r^m r^{2b} r^{-m} = r^{2b} \in \langle r^2\rangle$.

Case 2 Suppose $x = sr^m, 0 \leqslant m \leqslant n-1$. Then

$$\begin{aligned} xyx^{-1} &= sr^m r^{2b}(sr^m)^{-1} \\ &= sr^m r^{2b} r^{-m} s \\ &= sr^{2b} s \\ &= ssr^{-2b} \\ &= r^{-2b} \\ &= (r^2)^{-b} \in \langle r^2\rangle \end{aligned}$$

Thus $\langle r^2\rangle \lhd D_{2n}$

□

Lemma 3.3.2. $D_{2n}/\langle r^2\rangle$ *is abelian.*

Proof. We prove this using two cases, when n is odd and when n is even.

Case 1 Suppose n is odd. If n is odd, then since 2 is relatively prime to n we see that

$$\begin{aligned} & \langle r^2\rangle &= \{1, r, r^2, \cdots, r^{n-1}\} \\ \Rightarrow\ & |\langle r^2\rangle| &= n \\ \Rightarrow\ & [D_{2n} : \langle r^2\rangle] &= \tfrac{|D_{2n}|}{|\langle r^2\rangle|} = \tfrac{2n}{n} = 2 \\ \Rightarrow\ & D_{2n}/\langle r^2\rangle &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

$\Rightarrow D_{2n}/\langle r^2\rangle$ is abelian.

Case 2 Suppose n is even. Then $\langle r^2\rangle = \{1, r, r^2, r^4, \cdots, r^{n/2}\} \Rightarrow |\langle r^2\rangle| = n/2$. Thus, $[D_{2n} : \langle r^2\rangle] = \frac{|D_{2n}|}{|\langle r^2\rangle|} = 2n/(n/2) = 4$. Since all groups of order 4 are abelian then $D_{2n}/\langle r^2\rangle$ is abelian

□

We now prove Theorem 3.3.1 above:

Proof. We proceed by set inclusion.

"⊇" We know $D^{'}_{2n} = \{[x, y]|x, y \in D_{2n}\}$. Since $r, s \in D_{2n}$ then $[s, r] \in D^{'}_{2n}$. Note that

$$\begin{aligned} [s, r] &= s^{-1}r^{-1}sr \\ &= s^{-1}srr \\ &= r^2 \end{aligned}$$

Hence $r^2 \in D_{2n'} \Rightarrow \langle r^2\rangle \leqslant D^{'}_{2n}$

"⊆" By lemma 3.3.1 we know that $\langle r^2\rangle \lhd D_{2n}$, thus we can construct the quotient group $D_{2n}/\langle r^2\rangle$. By lemma 3.3.2 we know that $D_{2n}/\langle r^2\rangle$ is abelian. Since the commutator $D^{'}_{2n}$ is the smallest subgroup of D_{2n} such that $D_{2n}/D_{2n'}$ is abelian this means that $D^{'}_{2n} \leqslant \langle r^2\rangle$.

Therefore, $D^{'}_{2n} = \langle r^2\rangle$. □

Corollary 3.3.2. *If n is odd, then D_{2n} has two degree 1 (irreducible) representations.*

Proof. Let $H = D^{'}_{2n} = \langle r^2\rangle$. By lemma 2 we know that $D_{2n}/H \cong \mathbb{Z}/2\mathbb{Z}$. Note also that $rH = H$ since $r \in H$. Thus $D_{2n}/H = \{H, sH\}$. So two representations of D_{2n}/H are given by the following:

$$\begin{aligned} \overline{\phi_1} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto 1 \\ & sH \mapsto 1 \end{aligned}$$

$$\begin{aligned} \overline{\phi_2} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto 1 \\ & sH \mapsto -1 \end{aligned}$$

Since $\pi : D_{2n} \to D_{2n}/H, x \mapsto xH$ is a group homomorphism, then $\phi_1 = \overline{\phi_1} \circ \pi : D_{2n} \to \mathbb{C}$ and $\phi_2 = \overline{\phi_2} \circ \pi : D_{2n} \to \mathbb{C}$ are representations of D_{2n}. These are given by

$$\begin{aligned} \phi_1 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto 1 \end{aligned}$$

$$\begin{aligned} \phi_2 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto -1 \end{aligned}$$

□

Corollary 3.3.3. *If n is even, then D_{2n} has four degree 1 (irreducible) representations.*

Proof. Let $H = D_{2n}^{'} = \langle r^2 \rangle$. By lemma 2 we know that D_{2n}/H is an abelian group of order 4. In fact $D_{2n}/H = \{H, rH, sH, srH\}$. Since $(srH)^2 = r^2H = H, (sH)^2 = s^2H = H$, and $(srH)^2 = srsrH = ssr^{-1}rH = H$ we see that $D_{2n}/H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We find the following four representations of D_{2n}/H.

$$\begin{aligned} \overline{\phi_1} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto 1 \\ & sH \mapsto 1 \end{aligned} \qquad \begin{aligned} \overline{\phi_2} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto 1 \\ & sH \mapsto -1 \end{aligned}$$

$$\begin{aligned} \overline{\phi_3} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto -1 \\ & sH \mapsto 1 \end{aligned} \qquad \begin{aligned} \overline{\phi_4} : \quad & D_{2n}/H \to \mathbb{C} \\ & rH \mapsto -1 \\ & sH \mapsto -1 \end{aligned}$$

Proceeding as we did in the example of D_8 in Section 2.3, we get the following degree 1 representations of D_{2n}:

$$\begin{aligned} \phi_1 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto 1 \end{aligned} \qquad \begin{aligned} \phi_2 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 \\ & s \mapsto -1 \end{aligned}$$

$$\begin{aligned} \phi_3 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto -1 \\ & s \mapsto 1 \end{aligned} \qquad \begin{aligned} \phi_4 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto -1 \\ & s \mapsto -1 \end{aligned}$$

□

3.4 Irreducible Representations of D_{2n}

The goal of this last section, and indeed of this thesis is to prove the following theorem, and its analogue for when n is odd:

Theorem 3.4.1. *Let* $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$ *such that* $n = 2k, n \in \mathbb{Z}^+, k \geqslant 2$. *Then the irreducible representations of* D_{2n} *are given by the following:*

$$\begin{aligned} \phi_1 :\ & D_{2n} \to \mathbb{C} & \phi_2 :\ & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 & & r \mapsto 1 \\ & s \mapsto 1 & & s \mapsto -1 \end{aligned}$$

$$\begin{aligned} \phi_3 :\ & D_{2n} \to \mathbb{C} & \phi_4 :\ & D_{2n} \to \mathbb{C} \\ & r \mapsto -1 & & r \mapsto -1 \\ & s \mapsto 1 & & s \mapsto -1 \end{aligned}$$

and for all $1 \leq m \leq k-1$

$$\begin{aligned} \rho_m :\ & D_{2n} \to GL_2(\mathbb{C}) \\ & r \mapsto \begin{pmatrix} e^{\frac{2m\pi i}{n}} & 0 \\ 0 & e^{-\frac{2m\pi i}{n}} \end{pmatrix} \\ & s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Proof. Note $\phi_1, \phi_2, \phi_3, \phi_4$ come from our analysis of commutator subgroup of D_{2n} in the prior section. If we can show that all the ρ_m's are irreducible and inequivalent then by Theorem 2.3.1 and Thereom 3.2.1 we have a list of $k+3$ irreducible and inequivalent representations which means we have found them all since D_{2n} has $k+3$ conjugacy classes when n is even. First we will show that ρ_m, for all $1 \leqslant m \leqslant k-1$, is irreducible. We will use character theory to prove this. Let $\chi_m : D_{2n} \to \mathbb{C}$ be the character afforded by ρ_m. If we can show that $||\chi_m|| = 1$ then ρ_m is irreducible by Proposition 2.4.3. We know that $e^{\frac{2m\pi i}{n}}$ and $e^{-\frac{2m\pi i}{n}}$ are n^{th} roots of unity for all $1 \leqslant m \leqslant k-1$ since

$$(e^{\pm\frac{2m\pi i}{n}})^n = 1$$

Let $\omega_m = e^{-\frac{2m\pi i}{n}}$ and note that $\overline{\omega_m} \neq 1 \ \forall 1 \leq m \leq k-1$. Before we proceed with the rest of this proof let us prove the following lemma.

Lemma 3.4.1. *If $\omega \neq 1$ is an nth root of unity then $\omega^{n-1} + \cdots + \omega + 1 = \sum_{i=0}^{n-1} \omega^i = 0$.*

Proof. For any n^{th} root of unity ω, we have the following:

$$\omega^n - 1 = 0 \Rightarrow (\omega - 1)(\omega^{n-1} + \cdots + w + 1) = 0.$$

Since $\omega \neq 1$ this implies that $\omega^{n-1} + \cdots + \omega + 1 = 0$. □

Since none of the ω_m's are equal to 1 then $\sum_{i=0}^{n-1} \omega_m^i = 0$. Also since $\omega_m^2 \neq 1 \Rightarrow \sum_{i=0}^{n-1} \omega^{2i} = 0$ for all $1 \leqslant m \leqslant k-1$. We should also note that for all $1 \leq t \leq k$ we have that $\overline{\omega_m}^t = \omega_m^{n-t}$, and in particular $\overline{\omega_m}^k = \omega_m^k$.

Now let us examine the values of χ_m on the conjugacy classes of D_{2n} by looking at the following table. The sizes of each class and the classes come from the Theorem 3.2.3.

Conjugacy Class $[a]$	Size of Conj. Class	$\chi_m(a)$
$[1]$	1	2
$[r]$	2	$\omega_m + \overline{\omega_m}$
$[r^2]$	2	$\omega_m^2 + \overline{\omega_m}^2$
$\vdots$	$\vdots$	$\vdots$
$[r^{k-1}]$	2	$\omega_m^{k-1} + \overline{\omega_m}^{k-1}$
$[r^k]$	1	$\omega_m^k + \overline{\omega_m}^k$
$[s]$	k	0
$[sr]$	k	0

The computations for $\chi_m(a)$ are simplified since the matrices associated to r are diagonal. Let d_i denote the size of the conjugacy class $[r^i]$ $\forall 0 \leq i \leq k$. Computing the norm of χ_m we get

$$
\begin{aligned}
(\chi_m, \chi_m) &= \frac{1}{2n}\left[\sum_{j=0}^{k}\left(d_j\chi_m(r^j)^2\right) + k\chi_m(s)^2 + k\chi_m(sr)^2\right] \\
&= \frac{1}{2n}\left[2^2 + 2\sum_{j=1}^{k-1}\left(\omega_m^j + \overline{\omega_m}^j\right)^2 + \left(\omega_m^k + \overline{\omega_m}^k\right)^2\right] \\
&= \frac{1}{2n}\left[4 + 2\sum_{j=1}^{k-1}\left(\omega_m^{2j} + \omega_m^{2(n-j)}\right) + 2\sum_{j=1}^{k-1}2 + 4\omega_m^{2k}\right] \\
&= \frac{1}{2n}\left[4 + 4(k-1) + 2\sum_{j=1}^{k-1}\omega_m^{2j} + 2\sum_{j=k+1}^{n-1}\omega_m^{2j} + 2 + 2\omega_m^{2k}\right] \\
&= \frac{1}{2n}\left[4k + 2\sum_{j=0}^{n-1}\omega_m^{2j}\right] \\
&= \frac{1}{4k}(4k+0) \\
&= 1.
\end{aligned}
$$

Thus, ρ_m is irreducible for all $1 \leqslant m \leqslant k-1$. Finally, let us to show these representations are inequivalent. We know two representations are equivalent if and only if their characters are the same. Note that for all $1 \leqslant m \leqslant k-1$.

$$
\begin{aligned}
\chi_m(r) &= \omega_m + \overline{\omega_m} \\
&= e^{\frac{2m\pi i}{n}} + e^{-\frac{2m\pi i}{n}} \\
&= \cos(\frac{m\pi}{k}) + i\sin(\frac{m\pi}{k}) + \cos(\frac{m\pi}{k}) - i\sin(\frac{m\pi}{k}) \\
&= 2\cos(\frac{m\pi}{k}).
\end{aligned}
$$

Since $1 \leqslant m \leqslant k-1$ then $0 \leqslant \frac{\pi}{k} \leqslant \frac{m\pi}{k} \leqslant \frac{(k-1)\pi}{k} \leqslant \pi$ and since $\cos x$ is one-to-one on the interval $(0,\ 2\pi) \Rightarrow \chi_m(r) \neq \chi_l(r)$ whenever $m \neq l$ with $1 \leqslant m, l \leqslant k-1$. Hence each of the ρ_m's are inequivalent. This concludes the proof of our theorem.

□

We state and now prove the case when n is odd.

Theorem 3.4.2. *Let $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1}\rangle$ such that $n = 2k+1, n \in \mathbb{Z}^+, k \geqslant 1$. Then the irreducible representations of D_{2n} are given by the following:*

$$\begin{aligned} \phi_1 : \quad & D_{2n} \to \mathbb{C} & \qquad \phi_2 : \quad & D_{2n} \to \mathbb{C} \\ & r \mapsto 1 & & r \mapsto 1 \\ & s \mapsto 1 & & s \mapsto -1 \end{aligned}$$

and for all $1 \leq m \leq k$

$$\begin{aligned} \rho_m : \quad & D_{2n} \to GL_2(\mathbb{C}) \\ & r \mapsto \begin{pmatrix} e^{\frac{2m\pi i}{n}} & 0 \\ 0 & e^{-\frac{2m\pi i}{n}} \end{pmatrix} \\ & s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Proof. Again ϕ_1, ϕ_2 come from our analysis of commutator subgroup of D_{2n} in section 3.3. If we can show that all the ρ_m's are irreducible and inequivalent then by Theorem 2.3.1 we have a list of $k+2$ irreducible and inequivalent representations which means we have found them all since D_{2n} has $k+2$ conjugacy classes when n is odd. First we will show that ρ_m is irreducible for all $1 \leqslant m \leqslant k$. Again we create a table that contains the conjugacy classes, sizes of conjugacy classes, and the value of χ_m, where χ_m is the character associated to ρ_m.

Conjugacy Class $[a]$	Size of Conj. Class	$\chi_m(a)$
$[1]$	1	2
$[r]$	2	$\omega_m + \overline{\omega_m}$
$[r^2]$	2	$\omega_m^2 + \overline{\omega_m}^2$
$\vdots$	$\vdots$	$\vdots$
$[r^k]$	2	$\omega_m^k + \overline{\omega_m}^k$
$[s]$	n	0

Computing the norm of χ_m we get the following:

$$
\begin{aligned}
(\chi_m, \chi_m) &= \frac{1}{2n}\left[\sum_{j=0}^{k}\left(d_j\chi_m(r^j)^2\right) + n\chi_m(s)^2\right] \\
&= \frac{1}{2n}\left[2^2 + 2\sum_{j=1}^{k}\left(\omega_m^j + \overline{\omega_m}^j\right)^2\right] \\
&= \frac{1}{2n}\left[4 + 2\sum_{j=1}^{k}\left(\omega_m^j + \omega_m^{n-j}\right)^2\right] \\
&= \frac{1}{2n}\left[4 + 2\sum_{j=1}^{k}\left(2 + 2\omega_m^{2j} + \omega_m^{2(n-j)}\right)\right] \\
&= \frac{1}{2n}\left[4 + 4k + 2\sum_{j=1}^{n-1}\omega_m^{2j}\right] \\
&= \frac{1}{4k+2}\left[4k + 2 + 2\sum_{j=0}^{n-1}\omega_m^{2j}\right] \\
&= 1.
\end{aligned}
$$

Thus, ρ_m is irreducible for all $1 \leqslant m \leqslant k$. Finally note again that for all $1 \leqslant m \leqslant k$.

$$\chi_m(r) \quad = \quad 2\cos(\frac{m\pi}{k}).$$

Since $1 \leqslant m \leqslant k$ then $0 \leqslant \frac{\pi}{k} \leqslant \frac{m\pi}{k} \leqslant \frac{k\pi}{k} \leqslant \pi$ and since $\cos x$ is one-to-one on the interval $(0,\ 2\pi) \Rightarrow \chi_m(r) \neq \chi_l(r)$ whenever $m \neq l$ with $1 \leqslant m, l \leqslant k$. Hence each of the ρ_m's are inequivalent. $\square$

Chapter 4

Conclusion

Having completed this project I am now interested in the representation theory over $\mathbb{C}$ of other well known non-abelian groups that I have been introduced to in my studies. For example, the quaternion group defined by $Q = \langle x, y : x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle$, would be another relatively easy non-ablian group that I can study. I would probably start by finding the conjugacy classes in Q and the commutator of Q to mimic what I have learned from computing the representation theory of D_n. I am also interested in finding the representation theory of S_n, the symmetric group on a set of n elements. If $A = \{1, 2, ..., n\}$ then the group S_n contains all the possible permutations of the set A. We have results that describe the conjugacy classes of S_n and again I would try to compute the commutator subgroup of S_n. In both of these examples I hope to be able to use character theory so that I can better understand its uses.

Yet another possiblity would be to study the representations of these groups over other fields, especially finite fields. I would need to understand these fields and their algebraic closure to be able to use many of the results that I know thus far. Finally, I am interested in the applications of representation theory to other branches of learning.

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