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Different Conceptions of Lines and Points in the Context of Graphing

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Background and Framework

Lakoff and Núñez (2000) asserted that individuals conceptualize abstract concepts, via a mechanism called *conceptual metaphor*, in terms of concrete concepts. They stated that conceptual metaphor allows people to conceive one thing (e.g., numbers as abstract concepts) as if it were another (e.g., points on a number line). For example, as also pointed out by David et al. (2019), using the conceptual metaphor for each value of the pair, students conceive points on a graph in a Cartesian plane as a pair of values each of which map into the physical positions as points on each axis of the plane (Lakoff & Núñez, 2000).

There are two different conceptualizations of graphs in this perspective: (1) one includes thinking of a graph representing a path of a continuously moving object in the plane and (2) the other includes thinking of a graph as a collection of points (as a pair of values each of which map into the physical positions on each axis of the plane). According to Lakoff and Núñez (2000), this distinction relies on the distinction between two very different conceptualization of space—*naturally continuous space* and *the-set-of-points conception of space*. This distinction was based in the difference between classical mathematics (prior to the mid-nineteenth century) and modern mathematics (after Descartes' invention of analytic geometry). They claimed that naturally continuous space is "our normal conceptualization" because we function in everyday world by using our body and brain unconsciously. On the other hand, the-Set-of-Points conception of space is consciously constructed for particular purposes (e.g., to meet certain axioms such as dimensionality and curvature). They considered the second one as a reconceptualization of the first, via conceptual metaphor (i.e., "Spaces Are Set of Points" metaphor, p. 263). This reconceptualization process is called "the program of discretization" (p. 261) that is a strategy to replace naturally continuous space with infinite set of points for the sake of arithmetization, symbolization, and formalization to move from *intuitive* to *rigorous* mathematics.

Lakoff and Núñez laid out the inconsistent features of each conceptualization of space as follows:

> In one, space, lines, and planes *exist independently of* points, while in the other they *are constituted by points*. In one, properties are inherent; in the other they are assigned by relations and functions. In one, the entities are inherently spatial in nature; in the other, they are not. (p. 265)

For example, a circle is a spatial object independent of the points (like the path traced by a moving point) in a space in terms of the natural continuous space. Whereas a circle does not exist without considering points that constitute itself in terms of the Set-of-Points conception of space. In this conceptualization, a circle is a subset of points in a space, with particular relations to one another (i.e., a circle is a set of points in a plane that are a given distance [i.e., radius length] from a given point [i.e., center]).

Since there is a rich history of different conceptions of lines and points generated by Aristotle, Euclid and Descartes, among many others—as discussed in Lakoff & Núñez (2000)—we would expect to see important and potentially related distinctions with students' thinking, and this study suggests as such. A few researchers have investigated how students conceived points and lines in the context of graphing. For example, Mansfield (1985) reported that some secondary and undergraduate students believed there is no points on a line until they are visually marked on a graph. Similarly, Kerslake (1981) reported that about 89% of secondary students (N=1798) did not conceive of infinitely many points on a line. When asked how many points are on a straight line that goes through three points, several students answered "three" or the number of places where the line and a coordinate grid intersect. Although there were some students being aware that there were many points (e.g., "lots" or "hundreds" p. 123) on a line, Kerslake reported that their conception included “the physical constraints of actually drawing the points” (p. 123). In another study, when the teacher asked students if they have drawn a point on line segment that they drew on paper, Thompson (2002) reported that most students answered “no.” Thompson stated that students tend to think they don’t create any points when drawing a segment; creating a point “requires more of a dotting motion” (p. 207). In this study, I found a student whose meanings for points on a line was similar to that of the students reported by these aforementioned researchers. I contribute to these findings by discussing how such meanings had implications on students graphing activities.

Methodology and Data Sources

Participants. The data for my study were gathered in sets of teaching experiments (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. The participants for my study are sixth (age 11) and seventh-grade students (age 12) who were recruited on a volunteer basis with their parents’ permission. Every student who volunteered for the study was accepted, which resulted in a sample of 4 seventh graders (Ella, Dave, Mate, and Zane) and 2 sixth graders (Mike and Naya). All seventh-grade students were African American, and all sixth-grade students were White. In this paper, I focus on Zane and Mike since their meanings for the lines and points provided an opportunity for me to compare two different conceptualizations of a line as discussed in the background section.

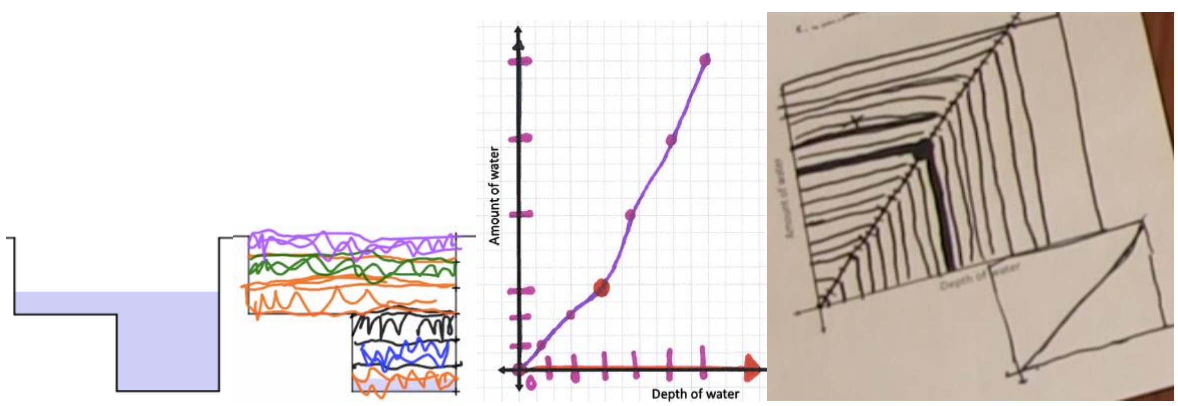
Data Sources and Analysis. Zane and Mike participated in 16 and 14 teaching sessions, respectively, each of which last for approximately one hour. Data sources included video and transcripts of each session that captured the participants’ exact words, gestures, and drawings. Students’ written work, screen recordings from a tablet device, and notes taken by research members who participated in the sessions were also collected as data sources. I conducted a conceptual analysis in order to understand students' verbal explanations and actions and develop viable models of *their* mathematics (Steffe & Thompson, 2000). My analysis relied on the generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to develop working models of their thinking based on their observable and audible behaviors.

Tasks. In this paper, I report data from Zane and Mike's activity in the Swimming Pool Task (SPT) adapted from Swan (1985). I presented students a dynamic diagram of a pool (see Figure 1a and https://bit.ly/SPT111 for a dynamic illustration), where they could fill or drain the pool by dragging a point on a given slider. I designed the task to support students in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool. I asked students to sketch a graph that shows the relationship between AoW and DoW as the pool fills up.

Results

Zane started his graphing activity by drawing tick marks on each axis (see Figure 1c). He wanted to make sure to insert each tick marks to represent AoW and DoW as he partitioned in the animation (see Figure 1b). For example, he spaced out the tick marks on the vertical axes in order to represent bigger increments of AoW ("twice as big") at the top half of the pool. After inserting tick marks, Zane plotted points for each related tick marks correspondingly, then he connected those points with line segments on the plane. When asked to explain how his graph showed the relationship between AoW and DoW in the pool, he described a point moving up and down along the line as representing both quantities' increases and decreases at the same time, saying "the dot represents both [AoW and DoW]."

Mike started his graphing activity by drawing a line upward from left to right (see Figure 1c). He also used vertical and horizontal segments on the plane to represent the quantities' magnitudes. I don’t have evidence that he imagined the measurements of these lengths. Instead, he attended to an overall change in the magnitudes of AoW and DoW. He said “I drew a straight line, or straight diagonal line from the origin this time because depth of water and amount of water both increase.” Pointing to the vertical and horizontal segments on his graph, he added “like, say the depth of water is getting deeper, it is because the amount of water is getting larger.” I infer that Mike drew a straight line upward from left to right because he initially determined gross (i.e., directional) covariational relationship (e.g., “The more the water, the deeper it gets”) between AoW and DoW in the pool. Mike also placed marks on his graph (see his tick marks where the vertical and horizontal segments meet on his graph in Figure 1d). When asked what the tick marks represented, he said “it represents the depth of the water and the amount of water.” For example, the tick mark at the top right side of his graph referred to a moment in the animation when the pool is full (i.e., magnitudes of both AoW and DoW has their maximums). Similarly, the tick mark at the origin referred to a moment in the animation when the pool is empty (i.e., AoW and DoW are zero).



(a) (b) (c) (d)

Figure 1. (a) SPT, (b) Zane’s partitioning activity, (c) Zane’s final graph, and (d) Mike’s first draft in SPT

So far, I demonstrated Zane and Mike’s graphing activities and their meanings of a single point on their graph. Next, I classify two different ways of conceiving a graph (e.g., a line drawn on the plane) when students represent a relationship between two varying quantities: (1) a line as a path or direction of movement of a dot on the plane, and (2) a line as a trace consisting of infinitely many points, each of which showing the relationship between two quantities. I describe and illustrate those types below.

Line as a path or direction of movement of a dot on the plane. I asked Zane whether his graph (see Figure 1c) showed every single moment of how the two quantities varied in the situation, Zane claimed no because one would need to stop the swimming pool animation and plot an additional point in order to show the desired moment and state of the quantities. I infer that, for Zane, his line did not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. When questioned what the line segments that he drew in between dots meant to him, Zane responded that the line shows "where the dots go." By go, he meant a dot moving from one plotted point to the next plotted point, but not in a way that preserved an invariant relationship between those two points.

Given these interpretations of Zane's meanings for points and lines, I hypothesized that he did not conceive a line consisting of infinitely many points. To test this hypothesis, I showed him an animation on a tablet device (see https://bit.ly/SPT123) and asked: "is this trace [see the animation in the previous link] showing us the relationship between depth of water and amount of water for this pool?" He replied "no" and struggled to make sense of what the animation was showing, which suggested that he did not perceive the animation as a simulation of his prior graphing activity on paper seen in Figure 1c. We asked Zane whether those dots on his paper are "part of the line on the computer." Zane replied, "there is only one dot," pointing to the animating dot that produced the trace in the animation. When asked "is there any other dots on this graph?" he shook his head. Moreover, he interpreted his graph (Figure 1c) as having more dots than the one produced in the animation, commenting that his graph is better because "mine have more dots." Zane also claimed that he could not construct his graph in the same way as the animation did due to physical constraints of human, saying, "well, I cannot do that, because, like, can you do dots and dots [*tapping his right index finger very fast along his graph*] and trace it?" This is an additional contraindication that he conceived of graphing a line as a way to represent infinitely many points.

Line as a trace consisting of infinitely many points. Recall that Mike's meaning of the points (i.e., tick marks) on his graph included a representation of both AoW and DoW and he was able to illustrate how each point on his graph referred to a certain instantiation of the pool animation. To get insights into his meaning of the line independent of the physical tick marks that he added on, I told Melvin to imagine another person whose graph included only a line without tick marks or dots visually plotted on the line (see the line graph that he drew below his original graph in Figure 1d). I asked him whether that line showed every single moment of how the two quantities varied in the situation. He said, "she [*referring to the imaginary person*] gets all the, like, everything in the animation" because "the line is basically made of tick marks." He then added "or she gets nothing because there is like no tick marks [on the line] and it [*referring to the line*] is not made of tick marks or anything, and she has nothing to represent." For Mike, a line can show every single moment of the animation under the assumption that line is made of tick marks. Melvin was also aware that if we assume the line is not made of tick marks (just a line), then the line itself shows nothing in relation to the pool situation. He said, "assuming that she meant that [*referring to the assumption that the line is made of tick marks*], I guess she would get all the, like, every exact little parts of the animation but if she was just drawing a line for a line, then I don't know if she would." This suggested that Mike—under the assumption—is able to envision a point as an abstract object that represents two quantities' magnitudes or values, and envision a graph (e.g., a line) as composed of points—although they are not visually plotted, each of which represent two quantities' magnitudes or values.

Moreover, Mike connected his conception of a line to the idea of representing numbers in a number line. He said,

> You know like on a number line [*drawing a number line on paper and inserting the large tick marks*], like the big tick marks, right [*pointing to the larger tick marks on the number line*]. And then, to represent like the half, smaller tick marks [*adding smaller tick marks in between the large tick marks*]. Like, 1, 2, 3, 4 [*labeling the large tick marks with integers*] and like one half, one and a half, two and a half [*labelling the small tick marks on the number line*]. And you can even do like fourths, like a quarter, like one-fourth. You can have, like, it could be really small fractions.

I asked him "How small?" He answered "I mean there is no limit" implying that there are infinitely many fractions that could be represented on a number line.

Figure 2. Mike's number line to illustrate there is infinitely many points on a line

Discussion

In this study, I found a student, Zane, who envisioned points as a circular *dot* that simultaneously represents two quantities' magnitudes or values and envisioned that points on a graph (e.g., a line) do not exist until they are physically and visually plotted. Therefore, he conceived the line as representing a direction of movement of a dot on a coordinate plane. On the contrary, Mike's meaning of a line included a record of two covarying quantities with the result of the trace consisting of infinitely many points, each of which represents both quantities' measures. For Mike, the visually available dots on the line are used for communication purposes to talk about which part of the graph refers to which part of the animation. That is, he can visually plot additional points on the line to indicate particular AoW and DoW in relation to a particular moment in the situation. The line without the visual points already shows every single moment of the situation; however, someone who comes in after the construction of graph may not immediately know which part of the graph refers to the which instantiations of the pool situation depicted in Figure 1b. This distinction in my work suggests as such and further research is necessary to have in depth investigation of students' meanings of points and lines in the light of this historical trail of the different conceptions of the line.

There is a good chance that Zane's meaning for a line might be related to his meaning constructed outside a graphing context prior to the study. Paul Klee—a famous painter and a scholar in German art schools (e.g., Bauhaus and Kunstakademie Düsseldorf) whose ideas inspired many school curriculum—defined that "a line is a dot that takes a walk" (Ringe, 2021). Moreover, in the National Core Arts Standards, a line is defined as "the one-dimensional path of a dot through space used by artists to control the viewer's eye movement; a thin mark made by a pencil, pen, or brush" (Washington Arts K–12 Learning Standards, 2017; p. 136). These definitions are compatible with my model of Zane's meaning of a line (i.e., "a line shows where the dots go"). Given the results provided by this study and other researchers (e.g., Manfield, 1985; Kerslake, 1981) regarding students' understanding of lines and points, it is important that we take into account students' meanings for lines and points that they might learn outside of a math class and investigate how those meanings may influence learning math concepts that includes lines and points.

Future research is needed to understand if the conception of a line as a path or direction of movement of a dot on the plane could create any implications or difficulties in students' learning of other topics, such as linear equations in the context of graphing or line of best fit in the context of a scatterplot. For example, maybe different conceptualization of a line has different implications on the nature of students' approaches in solving systems of linear equations. Maybe, students who don't imagine a line as including infinitely many points tend to use algebraic approaches more than graphical approaches when solving linear equations. Moreover, further research is necessary to investigate how a student who conceive a line as a path could develop a conception of a line as consist of infinitely many points and/or conception of points comprising a line.

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