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Toroidal Embeddings and Desingularization

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Toroidal Embeddings and Desingularization

A Thesis

Presented to the

Faculty of

California State University,

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

Algebraic geometry is the study of solutions in polynomial equations using objects and shapes. Differential geometry is based on surfaces, curves, and dimensions of shapes and applying calculus and algebra. Desingularizing the singularities of a variety plays an important role in research in algebraic and differential geometry. Toroidal Embedding is one of the tools used in desingularization. Therefore, Toroidal Embedding and desingularization will be the main focus of my project. In this paper, we first provide a brief introduction on Toroidal Embedding, then show an explicit construction on how to smooth a variety with singularity through Toroidal Embeddings.

Acknowledgements

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Chapter 1

Introduction

1.1 Introduction

An affine variety is the zero set of a family of polynomials in $\mathbb{C}^n$. It often has singularities. To be able to desingularize the singularities of a variety is the key to many research in algebraic and differential geometry. There are different techniques in desingularization. Toroidal Embedding is one of them. In this paper, we will first introduce the Toroidal Embedding, then apply it to smooth a variety with singularity.

First, we introduce some definitions in algebraic and differential geometry, and recall some theorems we need. For the further details on these definitions and theorems, see [GH] and [H].

1.2 Definitions

Definition 1.1: Diffeomorphism

Let

$$f(x_1, x_2, \ldots, x_n) = \big(f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)\big)$$

be a mapping between two open sets of $\mathbb{R}^n$.

f is said to be a C^r mapping if each f_i can be differentiated up to r times,

$$i = 1, 2, \ldots, n.$$

f is said to be C^∞-differentiable if it is C^∞ mapping.

f is called a diffeomorphism if f is a bijection, and f and f^{-1} are both C^∞-differentiable.

Definition 1.2: Holomorphism

Let $f(z) : U \to \mathbb{C}$ be a function of one complex variable, where U is an open set of $\mathbb{C}$,

$$f(z) = u(x, y) + iv(x, y)$$

$f(z)$ is said to be holomorphic, if $f'(z)$ exists for every

$$z_0 = (x_0, y_0) \in U,$$

Where

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

$$= -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

A function $f(z_1, z_2, \ldots, z_n)$ of n complex variables is said to be holomorphic if $f(z_1, z_2, \ldots, z_n)$ is holomorphic in each of variables.

Definition 1.3: Manifold

Let X be a topological space satisfying the Hansdorff separation axiom.

A differentiable structure on X of dimension n is a collection of open charts $\{(U_i, \phi_i)\}, i$ is ranging in some index I, satisfying the following conditions

(i) $X = \bigcup_{i \in I} U_i$

(ii) Each of ϕ_i is a bijection of U_i onto an open set of $\mathbb{R}^n$.

(iii) $\phi_j \cdot \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is a diffeomorphism.

X is called a $\mathbb{R}$-manifold of dimension n if X admits a differentiable structure of dim $n, \{(U_i, \phi_i), i \in I\}$.

Definition 1.4: Complex Manifold

A complex manifold M is a differentiable manifold admitting an open covering

$\{U_\alpha\}$ and coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha \cdot \phi_\beta^{-1}$ a biholomorphism from

$\phi_\beta(U_\alpha \cap U_\beta)$ to $\phi_\alpha(U_\alpha \cap U_\beta)$.

Example 1.1 Real Manifold

Let $X = \{(x, y) : x^2 + y^2 = 1\}$, a subset of $\mathbb{R}^2$

$$U_1 = \{(x, y) : -1 < x < 1, y = \sqrt{1 - x^2}\},$$

$$U_2 = \{(x, y) : -1 < x < 1, y = -\sqrt{1 - x^2}\},$$

$$U_3 = \{(x, y) : -1 < y < 1, x = \sqrt{1 - y^2}\},$$

$$U_4 = \{(x, y) : -1 < y < 1, x = -\sqrt{1 - y^2}\}$$

$$\phi_1 : U_1 \to \mathbb{R},$$

defined by

$$\phi_1(x, y) = x$$

$$\phi_3 : U_3 \to \mathbb{R},$$

defined by

$$\phi_3(x, y) = y$$

ϕ_2 and ϕ_4 can be defined similarly. Then

$$\phi_1 \cdot \phi_1^{-1} : \phi_1(U_1), \to \ \phi_1(U_1),$$

$$(\phi_1 \cdot \phi_1^{-1})(t) = t$$

$$\phi_3 \cdot \phi_1^{-1} : \phi_1(U_1 \cap U_3) \to \phi_3(U_1 \cap U_3)$$

$$(\phi_3 \cdot \phi_1^{-1})(t) = \phi_3(\phi^{-1}(t)) = \phi_3(t, \sqrt{1 - t^2}) = \sqrt{1 - t^2},$$

where

$$0 < t < 1$$

$\{(U_i, \phi_i), i = 1, 2, 3, 4\}$ is a differential structure on X.

Therefore, X is a $\mathbb{R}$-manifold of dimension 1.

Example 1.2 Complex Manifold The unit two-sphere S^2, which is the subset of $\mathbb{R}^3$, defined by

$$x^2 + y^2 + z^2 = 1$$

is a complex manifold. One can use stereographic projection from the North Pole to the real plane $\mathbb{R}^2$ with coordinates X, Y given by

$$(X, Y) = \Big(\frac{x}{1 - z}, \frac{y}{1 - z}\Big).$$

This can be done for any point except the North Pole itself (corresponding to $z = 1$). To include the North Pole, we introduce a second chart, in which we stereographically project from the South Pole:

$$(U, V) = \Big(\frac{x}{1 + z}, \frac{y}{1 + z}\Big),$$

which holds for any point on S^2 except for the South Pole(at$z = -1$). In both patches, we can now define complex coordinates

$$Z = X + iY, \;\; \bar{Z} = X - iY, \;\; W = U - iV, \;\; \bar{W} = U + iV,$$

and show that on the overlap of the two patches, the transition function is holomorphic. Indeed, on the overlap we compute that

$$W = \frac{1}{Z}.$$

This expression relates the coordinates W to Z in a holomorphic way. Hence the two-sphere is a complex manifold which can be identified with $\mathbb{C} \cup \infty$.

Definition 1.5: Jacobian

(i). Holomorphic Jacobian

Let $U \subset \mathbb{C}^n$ be an open set of $\mathbb{C}^n$ and let $f : U \to \mathbb{C}^n$ be a holomorphic mapping, that is, $f = (f_1, f_2, \ldots, f_n)$ with each f_j holomorphic. Let $w_j = f_j(z)$, where $z = (z_1, \ldots, z_n)$. The Holomorphic Jacobian of f is the matrix

$$J_{\mathbb{C}} f = \frac{\partial(w_1, \ldots, w_n)}{\partial(z_1, \ldots, z_n)}$$

$$= \begin{bmatrix} \frac{\partial w_1}{\partial z_1} & \cdots & \frac{\partial w_1}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial w_n}{\partial z_1} & \cdots & \frac{\partial w_n}{\partial z_n} \end{bmatrix}$$

Recall: For a complex valued function $f(z) = u + iv$ of a complex variable $z = x + iy$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big)$$

$$= \frac{1}{2}\Big[\big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\big) + i\big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\big)\Big]$$

and

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

(ii). Real Jacobian

Let

$$z_j = x_j + iy_j,$$

$$w_k = u_k + iv_k,$$

$$j = 1, \ldots, n,$$

$$k = 1, \ldots, n.$$

The Real Jacobian of f is the matrix

$$J_{\mathbb{R}} f = \frac{\partial(u_1, v_1, \ldots, u_n, v_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)}$$

$$= \begin{bmatrix} \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial x_n}, \frac{\partial u_1}{\partial y_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial v_n}{\partial x_1}, \frac{\partial v_n}{\partial y_1} & \cdots & \frac{\partial v_n}{\partial x_n}, \frac{\partial v_n}{\partial y_n} \end{bmatrix}$$

One can prove that

$$det J_{\mathbb{R}} f = det J_{\mathbb{C}} f \cdot \overline{det J_{\mathbb{C}} f}.$$

Definition 1.6: Variety over $\mathbb{C}$

1. X is called an analytic variety if X is the common zero locus of a collection of holomorphic functions of n variables, i.e

$$X = \{(z_1, z_2, \ldots, z_n); f_1(z_1, \ldots, z_n) = 0, \ldots, f_k(z_1, \ldots, z_n) = 0\} \subset \mathbb{C}^n, k \leq n,$$

where each f_i is a holomorphic function.

$$\bar{z}_0 = (z_1, \ldots, z_n) \in X$$

is called a non-singular point if rank $[J_{\mathbb{C}}(f)(\bar{z}_0)] = k$,

(i.e. there is a $k \times k$ sub-matrix A of $J_{\mathbb{C}}(f)(z_0)$, such that $det A \neq 0$.)

where

$$f = (f_1, \ldots, f_k)$$

Otherwise, $\bar{z}_0$ is called a singular point.

If $X_{sig} = \{$all singular point of $X\}$,then X_{sig} as a submanifold of $\mathbb{C}^n$.

2. X is called affine variety if X is the common zero locus of a collection of polynomial in

$$\mathbb{C}[z_1, \ldots, z_n].$$

3. X is called an algebraic variety (projective variety) if X is the common zero locus of a collection of homogenous polynomials.

1.3 Theorems

Theorem 1.1: Implicit Function Theorem

Let $(f_1, f_2, \ldots, f_k)$ be a holomorphic function of n complex variable $(z_1, \ldots, z_n), k \leq n$. If

$$det(J_{\mathbb{C}}(f)_k(z_0)) \neq 0,$$

where

$$J_{\mathbb{C}}(f)_k = \frac{\partial(f_1, \ldots, f_k)}{\partial(z_1, \ldots, z_k)}$$

and

$$z_0 = (z_1^0, z_2^0, \ldots, z_n^0)$$

then there exist holomorphic function $\phi_1, \ldots, \phi_k$ of $n-k$ variables such that in a neighborhood of z_0,

$$f_1(z_1, \ldots, z_n) = f_2(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0$$

if and only if

$$z_j = \phi_j(z_{k+1}, \ldots, z_n),$$

$$j = 1, \ldots, k.$$

Theorem 1.2: Inverse Function Theorem

Let $U \subset \mathbb{C}^n$ be an open set of $\mathbb{C}^n$ and $f : U \to \mathbb{C}^n$ be a holomorphic mapping with $detJ_{\mathbb{C}}f(z_0) \neq 0, z_0 \in U$. Then f is one-to-one in a neighborhood of z_0, and f^{-1} is holomorphic at $f(z_0)$.

Chapter 2

Toroidal Embeddings

Toroidal Embeddings

We first give a brief introduction on toroidal embedding. For the further details on the subject , see [N].

Let T be an n-dimensional complex torus, i.e., $T = (\mathbf{C}^*)^n$.

Definition.

(1) A torus embedding of T is an algebraic variety X such that

(a) X contains T as a Zariski open dense subset;

(b) T acts on X extending the natural action on itself defined by translation.

(2) A morphism between torus embedding X and X' is a map $f : X \to X'$ such that the following diagram commutes,

$$\begin{array}{ccc} T & \longrightarrow & X \\ \big\downarrow & & f\big\downarrow \\ X' & = & X' \end{array}$$

We can describe the torus embedding combinatorially.

Let $T = (\mathbf{C}^*)^n = Spec(\mathbf{C}[T_1, T_1^{-1}, ..., T_n, T_n^{-1}])$ as a scheme. For a commutative ring R, $Spec(R)$ is the set of all prime ideals of R. Let $M = Hom(T, \mathbf{C}^*)$. M is called the character Group of T. Then $M \simeq \mathbf{Z}^n$ with the following mapping,

for $r = (r_1, r_2, ...r_n) \in \mathbf{Z}^n$, $\chi^r \in M$,

where $\chi^r(t_1, t_2, ..., t_n) = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$.

Let $N = Hom(\mathbf{C}^*, T)$. N is called the group of one-parameter subgroup in T. Then $N \simeq \mathbf{Z}^n$ with the following mapping, for $a = (a_1, a_2, ..., a_n) \in \mathbf{Z}^n, \lambda_a \in N$,

where $\lambda_a(t) = (t^{a_1}, t^{a_2}, ..., t^{a_n})$.

M and N are dual to each other under the pairing $\langle , \rangle : M \times N \to \mathbf{Z}$,

$$\langle r, a \rangle = \Sigma_{i=1}^n r_i a_i.$$

Notice that $\chi^r(\lambda_a(t)) = t^{\langle r,a \rangle}$ for $r \in M, a \in N$, and $t \in \mathbf{C}^*$.

If we identify χ^r with the monomial $\Pi_{i=1}^n T_i^{r_i}$, for a subsemigroup S of M containing 0, then $\mathbf{C}[\chi^r]_{r \in S}$ is a subring of $\mathbf{C}[M] = \mathbf{C}[T_1, T_1^{-1}, ..., T_n, T_n^{-1}]$. Let $N_c = N \otimes \mathbf{C}$, then $N_C \simeq \mathbf{C}^n$ and $T = Spec(\mathbf{C}[M]) = N_C/N$.

Let σ be a convex rational polyhedral cone in $N_R = N \otimes \mathbf{R} = \mathbf{R}^n$ not containing a line. Then

$$\sigma = \{a \in N_R ; \langle r_i, a \rangle \geq 0, i = 1, ..., k, r_i \in M\}.$$

The dual of σ ,

$$\hat{\sigma} = \{r \in M_R ; \langle r, a \rangle \geq 0, \text{for all} \quad a \in \sigma\},$$

is the cone in M_R.

X_σ is defined to be $Spec(\mathbf{C}[\hat{\sigma} \cap M])$. Then X_σ is an affine normal torus embedding of T through $Spec(\mathbf{C}[M]) \subset Spec(\mathbf{C}[\hat{\sigma} \cap M])$.

Let $\{r_1, ..., r_m\}$ be a subset of M which generates $\hat{\sigma} \cap M$, i.e.,

$$\hat{\sigma} \cap M = \mathbf{Z}^+ r_1 + \cdots + \mathbf{Z}^+ r_m.$$

($m \geq n$, since σ does not contain a line)

Then

$$X_\sigma = Spec(\mathbf{C}[\chi^{r_1}, ..., \chi^{r_m}]) \subset \mathbf{C}^m.$$

The embedding of T into X_σ is defined by

$$i : T \to \mathbf{C}^m, i(\mathbf{t}) = (\chi^{r_1}(\mathbf{t}), ..., \chi^{r_m}(\mathbf{t})),$$

where $\mathbf{t} = (t_1, ..., t_n) \in T$.

X_σ is the scheme-theoretic closure of $i(T)$ in $\mathbf{C}^m$. T acts on X_σ as

$$\mathbf{t} \cdot x = (\chi^{r_1}(\mathbf{t})x_1, ..., \chi^{r_m}(\mathbf{t})x_m)$$

for $\mathbf{t} \in T, x = (x_1, ..., x_m) \in X_\sigma$. X_σ can be decomposed as the disjoint union of T-orbits under this action, and

$$\{T - \text{orbits} \;\; \text{in} \; X_\sigma\} \longleftrightarrow \{\text{all} \;\; \text{faces} \;\; \text{of} \; \sigma\}.$$

If τ is a face of σ (we will write it as $\tau < \sigma$), let $N(\tau)$ be the subset $\{r_i; \langle r_i, \rangle|_\tau = 0\}$ of $\{r_1, ..., r_m\}$, and O_τ be the T-orbit in X_σ corresponding to τ. Then

$$O_\tau = \{(x_1, ..., x_m) \in X_\sigma; \; x_i \neq 0 \text{ iff } r_i \in N(\tau)\},$$

and

$$dim\tau + dimO_\tau = dimT = n$$

$$O_0 = T.$$

A finite rational partial polyhedral decomposition of $N_{\mathbf{R}}$ is a finite collection $\Sigma = \{\sigma_i\}$ such that

(i) the face of σ is in Σ if $\sigma \in \Sigma$;

(ii) $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j for $\sigma_i, \sigma_j \in \Sigma$.

For a finite rational partial polyhedral decomposition Σ, we can patch all $X_{\sigma_i}, \sigma_i \in \Sigma$ together to form a normal torus embedding of T, X_Σ, by the T-orbits. In fact, if $\tau < \sigma$, then $X_\tau \subset X_\sigma$ and the inclusion $X_\tau \to X_\sigma$ is an open immersion in the following diagram,

$$\begin{array}{ccc} T & = & T \\ \downarrow & & \downarrow \\ X_\tau & \longrightarrow & X_\sigma. \end{array}$$

X_Σ is the disjoint union of all T-orbits in X_Σ.

X_σ is smooth if and only if σ is regular, i.e.,σ is generated by a part of a $\mathbf{Z}$-basis of N. X_Σ is smooth if and only if each member σ of Σ is regular. For a non-regular σ, one can find a finite rational polyhedral decomposition Σ of σ such that each member of Σ is regular. Then X_Σ will be a smooth variety which is a blowing-up of X_σ at its singularity.

Chapter 3

Desingularization

Consider $S = \{(x, y, z) \in \mathbf{C^3}; F(x, y, z) = z^3 - xy = 0\}$, then S is an analytic hypersurface in $\mathbf{C^3}$. Since $J_{\mathbb{C}}F = [-y, -x, 3z^2]$. It is clear that $\mathbf{0} = (0, 0, 0)$ is the only singular point of S, i.e. $S_0 = S \backslash \{\mathbf{0}\}$ is a complex manifold. S is the natural completion of S_0 in $\mathbf{C^3}$, yet singular. In algebraic geometry, it is known that one can obtain a canonical smooth completion $\overline{S_0}$ of S_0 by blowing up the singular point $\mathbf{0}$. In this project, we are going to explicitly construct $\overline{S_0}$, by using toroidal embeddings.

Let $\sigma = \{(x, y) \in \mathbf{R^2}; x - y \geq 0, -x + 4y \geq 0\}$. σ is called a rational convex polyhedral cone.

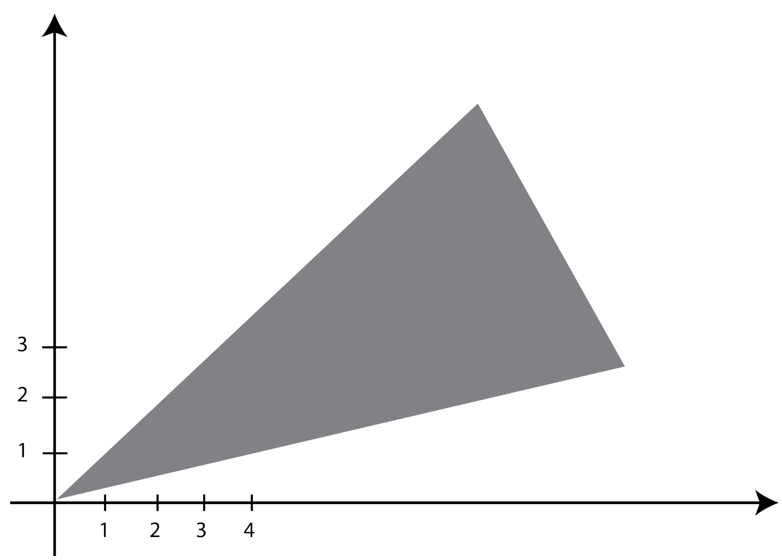

Figure 3.1: σ

Notice that $\sigma = \mathbf{R}^+\{(4,1),(1,1)\}$, where $\mathbf{R}^+$ is the set of all non-negative real numbers. Let $<,>$ denote the usual inner product in $\mathbf{R^2}$. The dual of σ is defined as the following,

$$\hat{\sigma} = \{r \in \mathbf{R}^2; < \vec{r}, \vec{a} > \geq 0, \forall a \in \sigma\}.$$

$$\hat{\sigma} = \{\vec{r} = (r_1, r_2); r_1 a_1 + r_2 a_2 \geq 0, \forall \vec{a} = (a_1, a_2) \in \sigma\}.$$

Let

$$\vec{r}_1 = (1, -1)$$

and

$$\vec{r}_2 = (-1, 4)$$

Claim:

$$\hat{\sigma} = \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$$

Proof: We will prove it by showing two inclusions, $\mathbf{R}^+\{\vec{r}_1, \vec{r}_2\} \subset \hat{\sigma}$ and $\hat{\sigma} \subset \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$.

(i) First, we prove $\mathbf{R}^+\{\vec{r}_1, \vec{r}_2\} \subset \hat{\sigma}$

By the definition of σ, it is clear that

$$\vec{r}_1 \in \hat{\sigma}$$

and

$$\vec{r}_2 \in \hat{\sigma}.$$

Therefore,

$$\mathbf{R}^+\{\vec{r}_1, \vec{r}_2\} \subset \hat{\sigma}$$

(ii) Second, we prove $\hat{\sigma} \subset \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$.

For any

$$\vec{r} = (r_1, r_2) \in \hat{\sigma},$$

then

$$< \vec{r}, \vec{a} > \geq 0, \forall \vec{a} \in \sigma.$$

Since $\{\vec{r}_1\}, \{\vec{r}_2\}$ forms a basis for $\mathbf{R^2}$, then

$$\vec{r} = \alpha\vec{r}_1 + \beta\vec{r}_2,$$

for some $\alpha \in \mathbf{R}$, and $\beta \in \mathbf{R}$.

For

$$\vec{a}_1 = (1, 1) \in \sigma,$$

$$< \vec{r}_1, \vec{a}_1 >= 0,$$

$$< \vec{r}_2, \vec{a}_1 >= 3,$$

therefore,

$$< \vec{r}, \vec{a}_1 >= 3\beta \geq 0.$$

It implies that

$$\beta \geq 0.$$

Similarly,

$$\vec{a}_2 = (4, 1) \in \sigma,$$

$$< \vec{r}_1, \vec{a}_1 >= 3,$$

$$< \vec{r}_2, \vec{a}_2 >= 0,$$

then

$$< \vec{r}, \vec{a}_2 >= 3\alpha \geq 0.$$

It implies that

$$\alpha \geq 0$$

Then

$$\vec{r} \in \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$$

Therefore,

$$\hat{\sigma} \subset \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$$

We have proved that

$$\hat{\sigma} = \mathbf{R}^+\{\vec{r}_1, \vec{r}_2\}$$

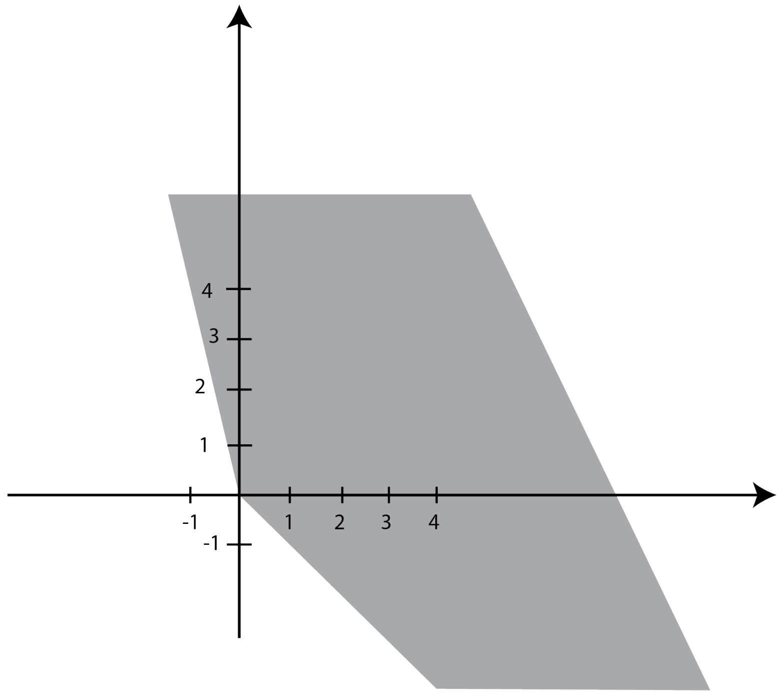

Figure 3.2: $\hat{\sigma}$

Let $\hat{\sigma}_{\mathbf{Z}}$ denote the set the all integer points of $\hat{\sigma}$, i.e., $\hat{\sigma}_{\mathbf{Z}} = \hat{\sigma} \cap \mathbf{Z}^2$, where $\mathbf{Z}$ = the set of integers. Then

$$\hat{\sigma}_{\mathbf{Z}} = \mathbf{Z}^+\{r_1 = (1, -1), r_2 = (-1, 4), r_3 = (0, 1)\},$$

i.e. $\{r_1, r_2, r_3\}$ forms a $\mathbf{Z}^+$-basis for $\hat{\sigma}_{\mathbf{Z}}$

Each r_i corresponds to a monomial m_{r_i} in $\mathbf{C}[T_1, T_1^{-1}, T_2, T_2^{-1}]$,

$$m_{r_1} = T_1 T_2^{-1}, m_{r_2} = T_1^{-1} T_2^4, m_{r_3} = T_2.$$

They induce a map

$$i_\sigma : T \to \mathbf{C}^3,$$

where, $T = (\mathbf{C}^*)^2$, the complex torus of dim 2.

$$i_\sigma(t_1, t_2) = (t_1 t_2^{-1}, t_1^{-1} t_2^4, t_2) =: (x, y, z).$$

It is clear that $i_\sigma(T) \subset S$. In fact,

$$S = Spec(\mathbf{C}[m_{r_1}, m_{r_2}, m_{r_3}]),$$

the Scheme-theoretic closure of $i_\sigma(T)$ in $\mathbf{C^3}$. We will denote $Spec(\mathbf{C}[m_{r_1}, m_{r_2}, m_{r_3}])$ by X_σ. It is called the Torus embedding associated with a rational convex polyhedral cone σ.

It is known that, from the general theory of Toroidal Embeddings, X_ϕ is smooth if and only if ϕ is regular, for any rational convex polyhedral cone ϕ of $\mathbf{R^2}$. ϕ is called regular if $\phi \cap \mathbf{Z}^2$ can be generated by a $\mathbf{R}^+$-basis of ϕ.

The cone σ above is not regular because the $\mathbf{R}^+$-basis $\{(4,1),(1,1)\}$ can not generate $\sigma \cap \mathbf{Z^2}$.

Hence, $X_\sigma = S$ has a singularity at $0 = (0,0,0)$

Now we consider the sub-cones and their faces of σ.

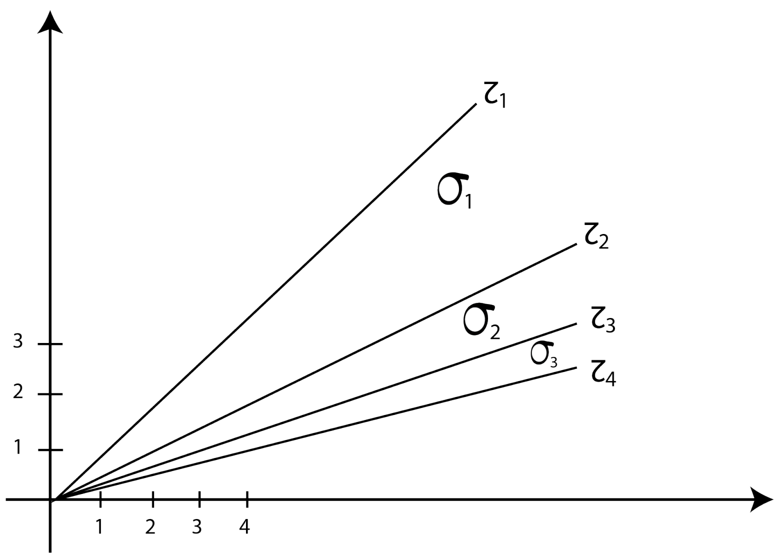

Figure 3.3: The sub-cones and their faces of σ

$\sigma_1 = \mathbf{R}^+\{(2,1),(1,1)\}$
$\sigma_2 = \mathbf{R}^+\{(2,1),(3,1)\}$
$\sigma_3 = \mathbf{R}^+\{(3,1),(4,1)\}$
$\tau_1 = \mathbf{R}^+\{(1,1)\}$
$\tau_2 = \mathbf{R}^+\{(2,1)\}$
$\tau_3 = \mathbf{R}^+\{(3,1)\}$
$\tau_4 = \mathbf{R}^+\{(4,1)\}$

They are all regular. $\Sigma = \{0, \tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \sigma_2, \sigma_3\}$ is called a regular rational convex polyhedral decomposition of σ.

For each cone in Σ, there is a Torus embedding of T associated to it. We are going to construct some of them here in detail.

Considering the sub-cone σ_1

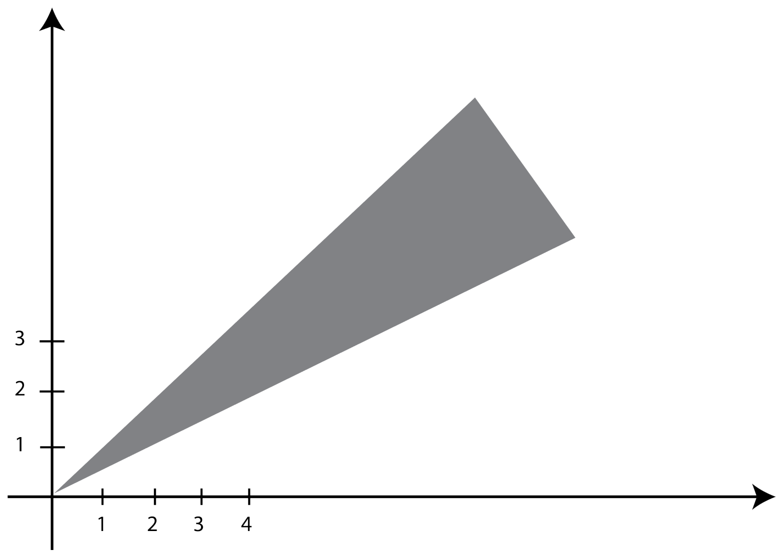

Figure 3.4: σ_1

It can be verified that the dual

$$\hat{\sigma}_1 =: \{r \in \mathbf{R^2}, < r, a > \geq 0, \forall a \in \sigma_1\}$$

$$= \mathbf{R}^+\{(1,-1), (-1,2)\}.$$

$$\hat{\sigma}_1 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(1,-1), (-1,2)\}.$$

(So, σ_1 is regular.)

Therefore, the embedding $i_{\sigma_1} : T \to \mathbf{C}^2$ is given by

$$i_{\sigma_1}(t_1, t_2) = (t_1 t_2^{-1}, t_1^{-1} t_2^2).$$

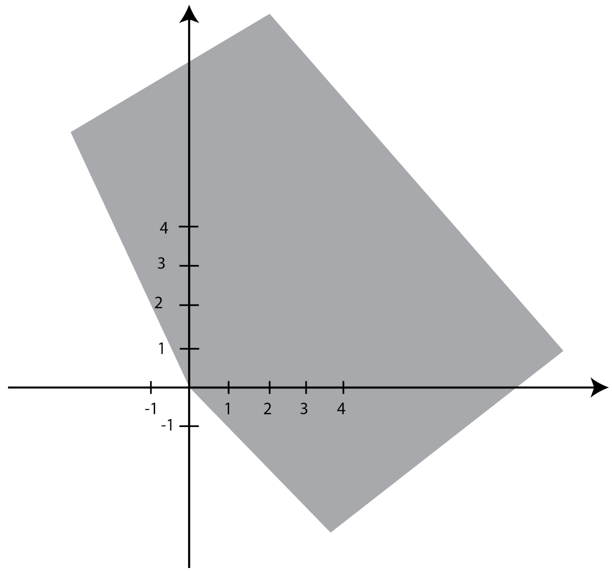

Figure 3.5: $\hat{\sigma}_1$

Let X_{σ_1} denote the closure of $i_{\sigma_1}(T)$ in $\mathbf{C^3}$, then

$$X_{\sigma_1} = \mathbf{C}^2.$$

Considering the sub-cone σ_2

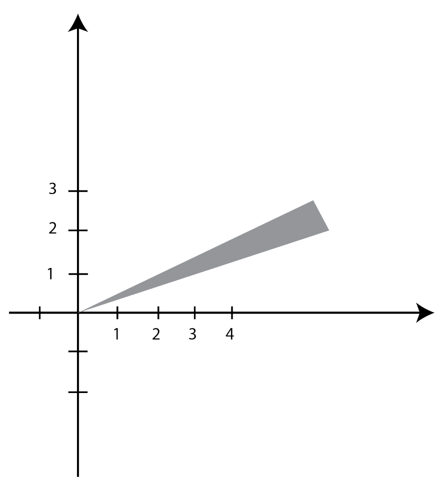

Figure 3.6: σ_2

It can be verified that the dual

$$\hat{\sigma}_2 =: \{r \in \mathbf{R^2}, < r, a > \geq 0, \forall a \in \sigma_2\}$$

$$= \mathbf{R}^+\{(1, -2), (-1, 3)\}.$$

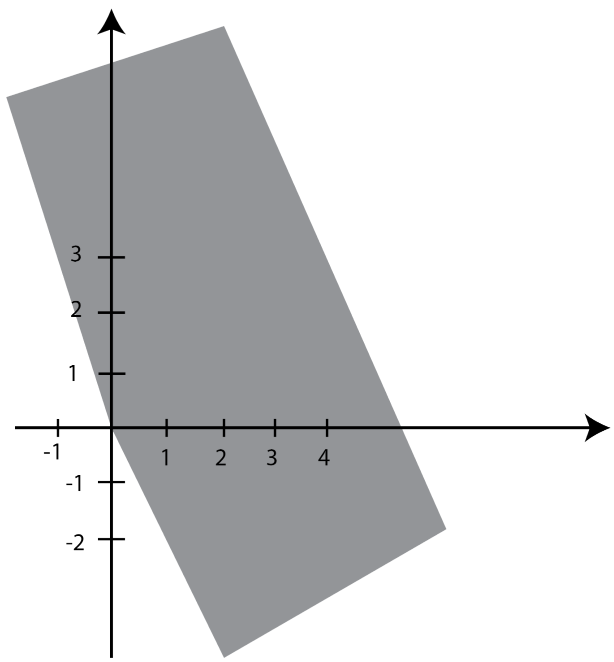

Figure 3.7: $\hat{\sigma}_2$

$$\hat{\sigma}_2 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(1, -2), (-1, 3)\}.$$

(So, σ_2 is regular.)

Therefore, the embedding $i_{\sigma_2} : T \to \mathbf{C}^2$ is given by

$$i_{\sigma_2}(t_1, t_2) = (t_1 t_2^{-2}, t_1^{-1} t_2^3).$$

Let X_{σ_2} denote the closure of $i_{\sigma_2}(T)$ in $\mathbf{C^3}$, then

$$X_{\sigma_2} = \mathbf{C}^2.$$

Considering the sub-cone σ_3

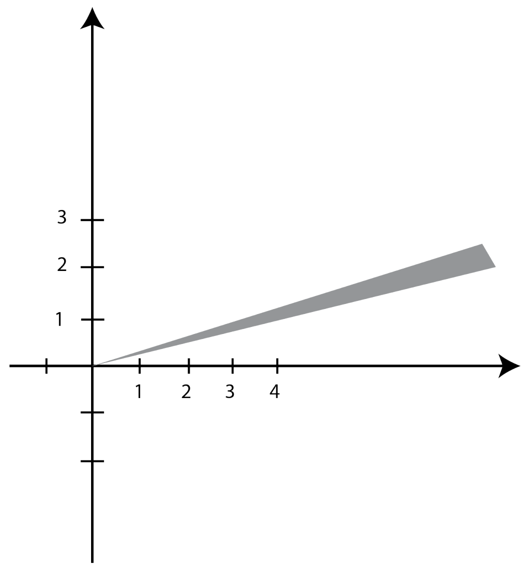

Figure 3.8: σ_3

It can be verified that the dual

$$\hat{\sigma}_3 =: \{r \in \mathbf{R^2}, < r, a > \geq 0, \forall a \in \sigma_3\}$$

$$= \mathbf{R}^+\{(1, -3), (-1, 4)\}.$$

$$\hat{\sigma}_3 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(1, -3), (-1, 4)\}.$$

(So, σ_3 is regular.)

Therefore, the embedding $i_{\sigma_3} : T \to \mathbf{C}^2$ is given by

$$i_{\sigma_3}(t_1, t_2) = (t_1 t_2^{-3}, t_1^{-1} t_2^4).$$

For the face $\tau_1 = \mathbf{R}^+\{(1, 1)\}$ of σ,

it is found that

$$\hat{\tau}_1 = \mathbf{R}^+\{(-1, 1), (1, -1), (0, 1)\},$$

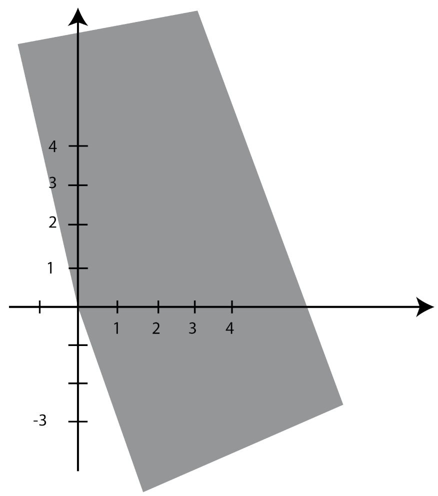

Figure 3.9: $\hat{\sigma}_3$

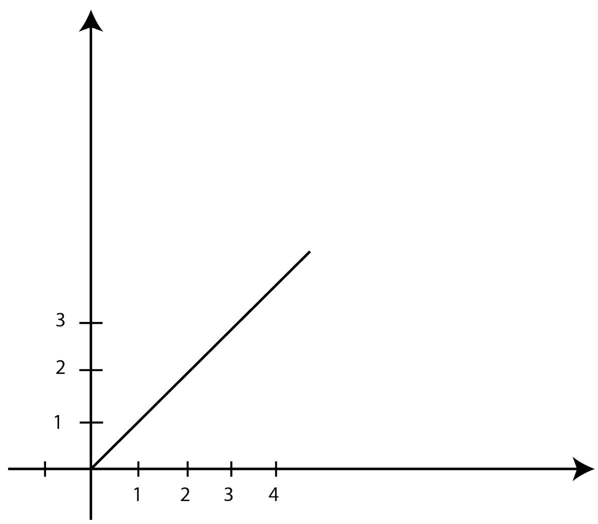

Figure 3.10: τ_1

$$\hat{\tau}_1 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(-1,1),(1,-1),(0,1)\},$$

and i_{τ_1} defines an embedding of T into $\mathbf{C}^3$, where

$$i_{\tau_1}(t_1,t_2) = (t_1 t_2^{-1}, t_1^{-1} t_2, t_2).$$

Hence, $X_{\tau_1} = \{(x,y,z) \in \mathbf{C}^3, xy = 1\}$ is a smooth surface in $\mathbf{C}^3$.

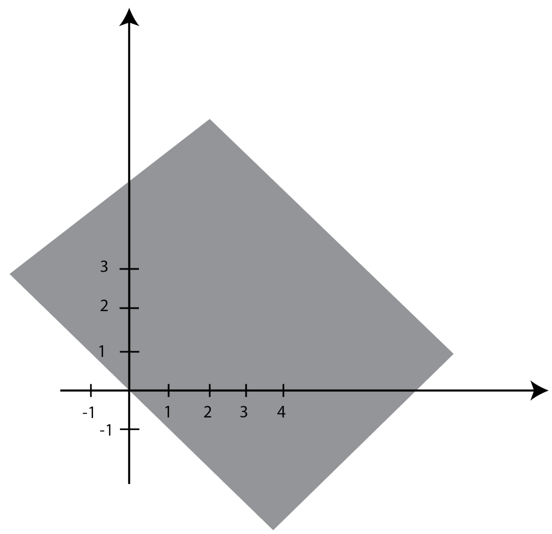

Figure 3.11: $\hat{\tau}_1$

For the face $\tau_2 = \mathbf{R}^+\{(2,1)\}$ of σ,

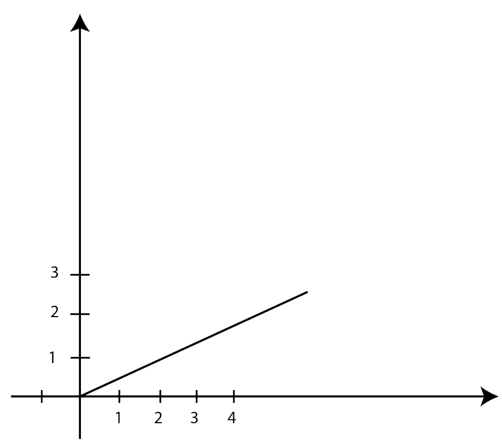

Figure 3.12: τ_2

it is found that

$$\hat{\tau}_2 = \mathbf{R}^+\{(-1,2),(1,-2),(1,0)\},$$

$$\hat{\tau}_2 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(-1,2),(1,-2),(1,0)\},$$

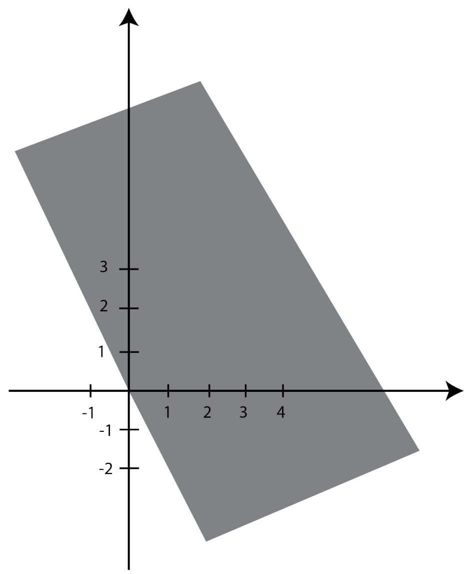

Figure 3.13: $\hat{\tau}_2$

and i_{τ_2} defines an embedding of T into $\mathbf{C}^3$, where

$$i_{\tau_2}(t_1, t_2) = (t_1^{-1} t_2^2, t_1 t_2^{-2}, t_1).$$

Hence, $X_{\tau_2} = \{(x, y, z) \in \mathbf{C}^3, xy = 1\}$ is a smooth surface in $\mathbf{C}^3$.

For the face $\tau_3 = \mathbf{R}^+\{(3, 1)\}$ of σ,

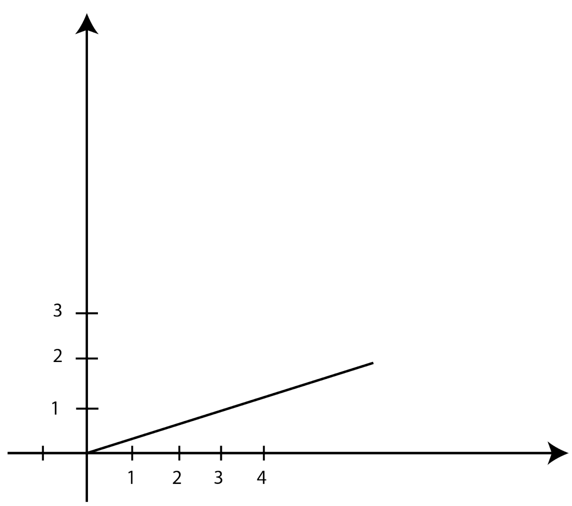

Figure 3.14: τ_3

it is found that

$$\hat{\tau}_3 = \mathbf{R}^+\{(-1,3),(1,-3),(1,0)\},$$

$$\hat{\tau}_3 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(-1,3),(1,-3),(1,0)\},$$

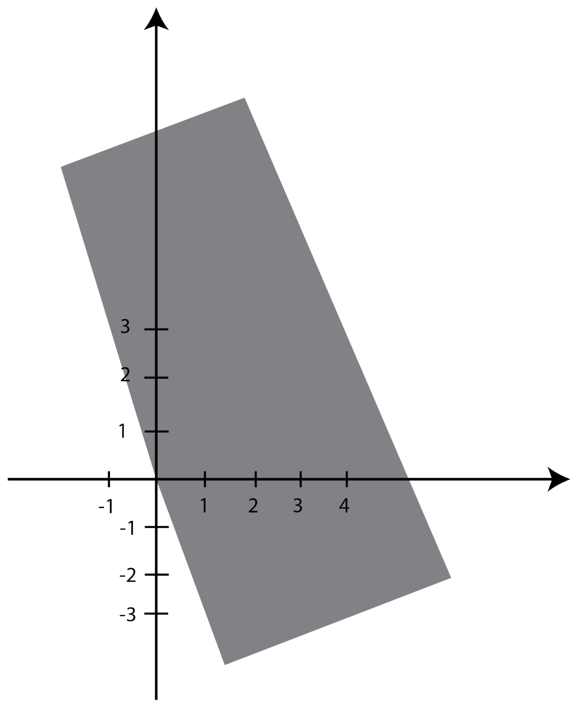

Figure 3.15: $\hat{\tau}_3$

and i_{τ_3} defines an embedding of T into $\mathbf{C}^3$, where

$$i_{\tau_3}(t_1,t_2) = (t_1^{-1}t_2^3, t_1t_2^{-3}, t_1).$$

Hence, $X_{\tau_3} = \{(x,y,z) \in \mathbf{C}^3, xy = 1\}$ is a smooth surface in $\mathbf{C}^3$.

For the face $\tau_4 = \mathbf{R}^+\{(4,1)\}$ of σ,
it is found that

$$\hat{\tau}_4 = \mathbf{R}^+\{(-1,4),(1,-4),(1,0)\},$$

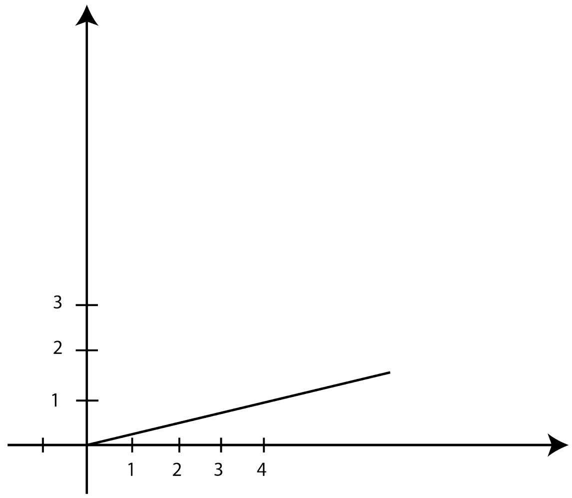

Figure 3.16: τ_4

$$\hat{\tau}_4 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{(-1,4),(1,-4),(1,0)\},$$

and i_{τ_4} defines an embedding of T into $\mathbf{C}^3$, where

$$i_{\tau_4}(t_1,t_2) = (t_1^{-1}t_2^4, t_1t_2^{-4}, t_1).$$

Hence, $X_{\tau_4} = \{(x,y,z) \in \mathbf{C}^3, xy = 1\}$ is a smooth surface in $\mathbf{C}^3$.

X_{τ_2}, X_{τ_3} and X_{τ_4} will be the same type of surface as X_{τ_1}.

We are now going to show how can these X_{τ_i} and X_{σ_j} be patched together through the "orbit decompositions".

Consider the sub-cone σ_1 of σ first. It is clear that

$$\hat{\sigma} \subset \hat{\sigma}_1$$

and

$$\hat{\sigma} \cap \mathbf{Z}^2 \subset \hat{\sigma}_1 \cap \mathbf{Z}^2.$$

Recall that

$$\hat{\sigma}_1 \cap \mathbf{Z}^2 = \mathbf{Z}^+\{s_1, s_2\}$$

and

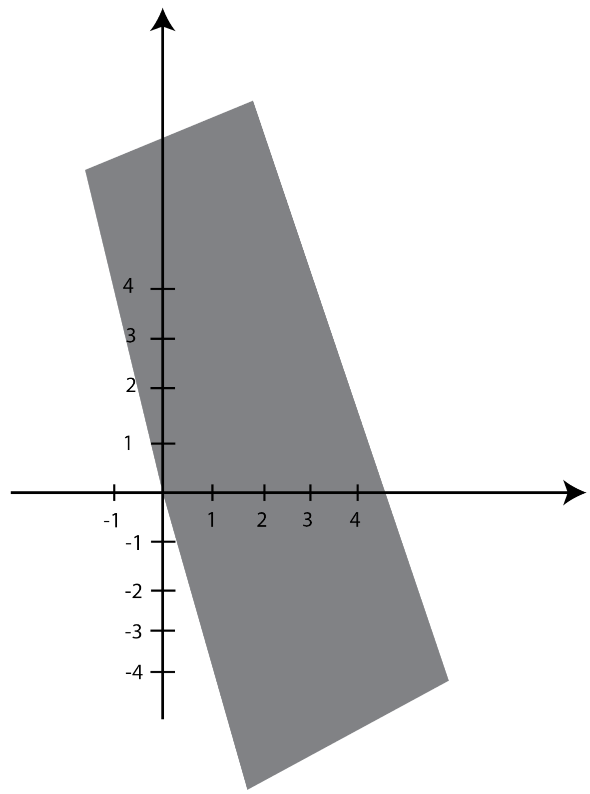

Figure 3.17: $\hat{\tau}_4$

$$\hat{\sigma} \cap \mathbf{Z}^2 = \mathbf{Z}^+\{r_1, r_2, r_3\},$$

where $s_1 = (1, -1)$ and $s_2 = (-1, 2)$. Therefore, the set $\{s_1, s_2\}$ should generates the set $\{r_1, r_2, r_3\}$ over $\mathbf{Z}^+$. In fact,

$$r_1 = s_1, r_2 = s_1 + 2s_2, r_3 = s_1 + s_2.$$

Their monomials are related as in the following,

$$m_{r_1} = m_{s_1}, m_{r_2} = m_{s_1} m_{s_2}^2, m_{r_3} = m_{s_1} m_{s_2}.$$

They induce a homomorphism α_1 from X_{σ_1} to X_σ.

$\alpha_1(u, v) = (u, uv^2, uv) =: (x, y, z)$.

It can be verified that α_1 is 1-1 on $X_\sigma \cap \{z \neq 0\}$, and the following diagram commutes.

$$\begin{array}{ccc} T & \longrightarrow & X_{\sigma_1} \\ \alpha \big\downarrow & & \alpha_1 \big\downarrow \\ X_\sigma & = & X_\sigma \end{array}$$

On the other hand, each X_* has a canonical decomposition in terms of all possible faces of the cone. For instance,

$X_\sigma = O_0 \cup O_{\tau_1} \cup O_{\tau_4} \cup O_\sigma$, where

$O_0 = \{(x, y, z); z^3 = xy, xyz \neq 0\} \cong (\mathbf{C}^*)^2 = T$

$O_{\tau_1} = \{(x, 0, 0); x \neq 0\} \cong \mathbf{C}^*$

$O_{\tau_4} = \{(0, y, 0); y \neq 0\} \cong \mathbf{C}^*$

$O_\sigma = \{(0, 0, 0)\}$

This is called the orbits decompositions of X_σ.

We explain in the following how the orbit associated to a face of σ, for instance O_{τ_1}, is determined.

Recall that the embedding

$$i_\sigma : T \to X_\sigma \subset \mathbf{C}^3$$

is given by the monomials $\{m_{r_1}, m_{r_2}, m_{r_3}\}$.

$$i_\sigma(t_1, t_2) = (x, y, z)$$

where

$$x = m_{r_1}(t_1, t_2), y = m_{r_2}(t_1, t_2), z = m_{r_3}(t_1, t_2),$$

and

$$\{r_1 = (1, -1), r_2 = (-1, 3), r_3 = (0, 1)\} \subset \hat{\sigma}.$$

Notice that $r_1 = 0$ on τ_1, r_2 and r_3 are non-zero on τ_1. Hence, the orbit O_{τ_1} is determined by the conditions

$$x \neq 0, \quad y = z = 0.$$

Similarly, $X_{\sigma_1} = \mathbf{C}^2 = O_0^1 \cup O_{\tau_1}^1 \cup O_{\tau_2}^1 \cup O_{\sigma_1}$, where

$O_0^1 = \{(u, v), uv \neq 0\} \cong (\mathbf{C}^*)^2 = T$

$O_{\tau_1}^1 = \{(u, 0); u \neq 0\} \cong \mathbf{C}^*$

$O_{\tau_2}^1 = \{(0, v); v \neq 0\} \cong \mathbf{C}^*$

$O_{\sigma_1} = \{(0, 0)\}$

σ and σ_1 have common faces τ_1 and $\mathbf{0}$ (the origin). The induced homomorphism α_1 becomes an isomorphism when it is restricted to O_0^1 or $O_{\tau_1}^1$. In the mean time, α_1 will collapse $O_{\tau_2}^1 \cup O_{\sigma_1}$ into a single point O_σ. This is illustrated in the following diagrams.

$$
\begin{array}{ccccccccccccc}
X_{\sigma_1} & = & O_0^1 & \cup & O_{\tau_1}^1 & \cup & O_{\tau_2}^1 & & \cup & O_{\sigma_1} & & & \\
\downarrow & & \Downarrow & & \Downarrow & & & \searrow & & \swarrow & & & \\
X_\sigma & = & O_0 & \cup & O_{\tau_1} & \cup & & & O_\sigma & & \cup & O_{\tau_4} &
\end{array}
$$

The orbit decompositions for X_{σ_2} are stated in the following,

$$X_{\sigma_2} = O_0^2 \cup O_{\tau_2}^2 \cup O_{\tau_3}^2 \cup O_{\sigma_2}.$$

The induced homomorphism α_2 from X_{σ_2} to X_σ will collapse $O_{\tau_2}^2 \cup O_{\tau_3}^2 \cup O_{\sigma_2}$ into a single point O_σ,while $\{\alpha_2 : O_0^2 \to O_0\}$ is isomorphism.

$$
\begin{array}{ccccccccccc}
X_{\sigma_2} & = & O_0^2 & \cup & O_{\tau_2}^2 & \cup & O_{\tau_3}^2 & \cup & O_{\sigma_2} \\
\downarrow & & \Downarrow & & & \searrow & & \swarrow & \\
X_\sigma & = & O_0 & \cup & O_{\tau_1} & \cup & O_\sigma & \cup & O_{\tau_4}
\end{array}
$$

Each X_{τ_i} has an orbit decompositions with two orbits only. For instance,

$$X_{\tau_2} = O'_0 \cup O'_{\tau_2}.$$

Since τ_2 is a face of σ_1, there is an induced immersion from X_{τ_2} into X_{σ_1} which will isomorphically map O'_0 onto O_0^1, and O'_{τ_2} onto $O_{\tau_2}^1$. Therefore, under the isomorphisms, we may write

$$X_{\tau_2} = O'_0 \cup O'_{\tau_2}$$

and

$$X_{\sigma_1} = O'_0 \cup O'_{\tau_1} \cup O'_{\tau_2} \cup O_{\sigma_1}.$$

Similarly,

$$X_{\sigma_2} = O'_0 \cup O'_{\tau_2} \cup O'_{\tau_3} \cup O_{\sigma_2}.$$

$$X_{\sigma_3} = O'_0 \cup O'_{\tau_3} \cup O'_{\tau_4} \cup O_{\sigma_3}.$$

$$X_{\tau_1} = O'_0 \cup O'_{\tau_1},$$

$$X_{\tau_3} = O'_0 \cup O'_{\tau_3}$$

and

$$X_{\tau_4} = O'_0 \cup O'_{\tau_4}$$

Let $X_\Sigma = \bigsqcup_{i=1}^{3} X_{\sigma_i} \bigsqcup_{j=1}^{4} X_{\tau_j}$ be the union of X_{σ_i} and X_{τ_j}, patching through the orbits. Hence, X_Σ has an orbit decomposition as the following,

$$X_\Sigma = O'_0 \cup O'_{\tau_1} \cup O'_{\tau_3} \cup O'_{\tau_2} \cup O'_{\tau_4} \cup O_{\sigma_1} \cup O_{\sigma_2} \cup O_{\sigma_3}.$$

X_Σ is a smooth variety, since $\Sigma = \{0, \tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \sigma_2, \sigma_3\}$ is a regular rational convex polyhedral decomposition of σ. There is a homomorphism β from X_Σ to X_σ, which will be an isomorphism on O'_0, O'_{τ_1} and O'_{τ_4} respectively,

$$O'_0 \cong O_0$$

$$O'_{\tau_1} \cong O_{\tau_1}$$

$$O'_{\tau_3} \cong O_{\tau_4}.$$

β will map $O'_{\tau_2} \cup O'_{\tau_3} \cup O_{\sigma_1} \cup O_{\sigma_2} \cup O_{\sigma_3}$ into the single point O_σ. Therefore, we have the following diagrams.

$$\begin{array}{ccccccccc} X_\Sigma & = & O'_0 & \cup & O'_{\tau_1} & \cup & O'_{\tau_4} & \cup & D \\ \downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \downarrow \\ X_\sigma & = & O_0 & \cup & O_{\tau_1} & \cup & O_{\tau_4} & \cup & O_\sigma \end{array}$$

where

$$D = O'_{\tau_2} \cup O'_{\tau_3} \cup O_{\sigma_1} \cup O_{\sigma_2} \cup O_{\sigma_3}$$

Notice that $O_\sigma = \{\mathbf{0}\}$, the singular point of S.

We have constructed a smooth completion X_Σ of the surface S with a mapping

$$\beta : X_\Sigma \longrightarrow X_\sigma = S$$

.

β is the blowing-up of S at its singular point $\mathbf{0}$.

$$\beta^{-1}(\mathbf{0}) = O'_{\tau_2} \cup O'_{\tau_3} \cup O_{\sigma_1} \cup O_{\sigma_2} \cup O_{\sigma_3},$$

is the exceptional divisor.

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